

CHROMATIC FRACTURE CUBES

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ABSTRACT. In this note, we construct a general form of the chromatic fracture cube, using a convenient characterization of the total homotopy fiber, and deduce a decomposition of the $E(n)$ -local stable homotopy category.

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1. INTRODUCTION

The chromatic fracture square can be interpreted as the chromatic analogue of the arithmetic pullback square

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \prod_p \mathbb{Z}_p \end{array}$$

and thus expresses a fundamental local-to-global principle in stable homotopy theory. As such, it has been repeatedly used to first study problems in an appropriate local context and to then reassemble the results. Most noticeably, this approach is used in the construction and study of the spectrum of topological modular forms, see [Beh].

The goal of this note is to construct a higher dimensional version of the chromatic fracture square, well-known to the experts, which allows to explicitly decompose the $E(n)$ -localization of a spectrum into its $K(t)$ -local pieces for $0 \leq t \leq n$. Moreover, we obtain a decomposition of the $E(n)$ -local category into diagram categories of $K(t)$ -local categories, for $0 \leq t \leq n$. In fact, we work in a slightly more abstract setting, see Proposition 3.2, Theorem 4.4, and Theorem 5.5 for precise statements of our results.

The proof uses the characterization of the total homotopy fiber of an n -cube as a right adjoint; since we do not know of a published reference for this fact, we include

the argument. This then allows to easily deduce the iterative construction of the total homotopy fiber, as can be found for instance in [Goo92].

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2. CUBICAL HOMOTOPY PRELIMINARIES

Let \mathcal{C} be a pointed ∞ -category with finite limits, and let $\text{Fun}(\mathcal{P}(T), \mathcal{C})$ be the category of n -cubes in \mathcal{C} . Here, T is a fixed set with n elements and $\mathcal{P}(T)$ is its poset of subsets, ordered by inclusion. We will also use $\mathcal{P}_{\neq \emptyset}(T)$ for the poset of non-empty subsets of T . If $\mathcal{F} \in \text{Fun}(\mathcal{P}(T), \mathcal{C})$, then the total homotopy fiber $\text{tfib}(\mathcal{F})$ of \mathcal{F} is the fiber of the natural map $f: \mathcal{F}_{\emptyset} \rightarrow \lim(\mathcal{F}|_{\mathcal{P}_{\neq \emptyset}(T)}) = \lim_{S \neq \emptyset} \mathcal{F}(S)$. Recall that \mathcal{F} is said to be Cartesian if f is an equivalence; this implies that $\text{tfib}(\mathcal{F})$ is contractible. If \mathcal{C} is stable the converse is true: $\text{tfib}(\mathcal{F})$ being contractible implies that \mathcal{F} is Cartesian.

Proposition 2.1. *The total (homotopy) fiber of a cubical diagram is right adjoint to the functor*

$$\mathcal{C} \xrightarrow{\text{in}_{\emptyset}} \text{Fun}(\mathcal{P}(T), \mathcal{C})$$

sending an element $X \in \mathcal{C}$ to the n -cube with initial vertex X and the terminal object $*$ elsewhere.

Proof. Let $\tilde{\mathcal{F}}$ be an n -cube that is a limit cone over the diagram $\mathcal{F}_{\neq \emptyset} := \mathcal{F}|_{\mathcal{P}_{\neq \emptyset}(T)}$; note that there is a canonical map $\mathcal{F} \rightarrow \tilde{\mathcal{F}}$. We thus get for $X \in \mathcal{C}$:

$$\begin{aligned} \text{Hom}(X, \text{tfib}(\mathcal{F})) &= \text{Hom}(\text{in}_{\emptyset}(X), \text{in}_{\emptyset}(\text{tfib}(\mathcal{F}))) \\ &= \text{Hom}(\text{in}_{\emptyset}(X), \text{fib}(\mathcal{F} \rightarrow \tilde{\mathcal{F}})) \\ &= \text{fib}(\text{Hom}(\text{in}_{\emptyset}(X), \mathcal{F}) \rightarrow \text{Hom}(\text{in}_{\emptyset}(X), \tilde{\mathcal{F}})) \\ &= \text{Hom}(\text{in}_{\emptyset}(X), \mathcal{F}), \end{aligned}$$

because, as we will show, $\text{Hom}(\text{in}_{\emptyset}(X), \tilde{\mathcal{F}})$ is contractible.

Indeed, letting $T_+ := \{*\} \cup T$, we can identify $\text{Hom}(\text{in}_{\emptyset}(X), \mathcal{F})$ with the space of $(n+1)$ -cubes $\mathcal{G} \in \text{Fun}(\mathcal{P}(T_+), \mathcal{C})$, such that \mathcal{G} restricts to $\text{in}_{\emptyset}(X)$ on $\{S \subseteq T_+ : * \notin S\}$ and restricts to $\tilde{\mathcal{F}}$ on $\{S \subseteq T_+ : * \in S\}$. Now, since $\tilde{\mathcal{F}}$ is Cartesian, these $(n+1)$ -cubes \mathcal{G} are determined up to contractible choice by their restrictions to $\{S \subseteq T_+ : S \neq \{*\}\}$; and the space of these possible restrictions is contractible by the presence of the terminal objects in $\text{in}_{\emptyset}(X)$. \square

As a consequence of this proposition we obtain an easy proof of the fact that the total homotopy fiber can be computed by taking fibers in all the edges of a cube in a fixed direction and then taking total fibers of the resulting cube. More precisely, let $\mathcal{F} \in \text{Fun}(\mathcal{P}(T), \mathcal{C})$ be an n -cube and $T' \subseteq T$. For any $S' \in \mathcal{P}(T')$, we can consider the cube $\mathcal{F}_{T \setminus T', S'} \in \text{Fun}(\mathcal{P}(T \setminus T'), \mathcal{C})$ given by $S \mapsto \mathcal{F}(S \cup S')$. This yields a new cube $\text{tfib}_{T'}(\mathcal{F}) \in \text{Fun}(\mathcal{P}(T'), \mathcal{C})$ whose value on $S' \subseteq T'$ is $\text{tfib}(\mathcal{F}_{T \setminus T', S'})$.

Corollary 2.2. *With notation as above, there is a natural equivalence $\text{tfib}(\mathcal{F}) = \text{tfib}(\text{tfib}_{T'}(\mathcal{F}))$ for any $T' \subseteq T$*

Proof. Observe that $\text{in}_\emptyset: \mathcal{C} \rightarrow \text{Fun}(\mathcal{P}(T), \mathcal{C})$ can be decomposed as follows

$$\mathcal{C} \xrightarrow{\text{in}_\emptyset} \text{Fun}(\mathcal{P}(T'), \mathcal{C}) \xrightarrow{\text{in}_\emptyset} \text{Fun}(\mathcal{P}(T \setminus T'), \text{Fun}(\mathcal{P}(T'), \mathcal{C})) \xrightarrow{\sim} \text{Fun}(\mathcal{P}(T), \mathcal{C}),$$

so the same is true for the right adjoint tfib . \square

Corollary 2.3. *If \mathcal{C} is stable and $\mathcal{F} \in \text{Fun}(\mathcal{P}(T), \mathcal{C})$ is a Cartesian n -cube and $T' = \{t\} \subseteq T$ a subset of size 1, then the fiber of $(n-1)$ -cubes*

$$\text{fib}(\mathcal{F}_{T \setminus \{t\}, \emptyset} \rightarrow \mathcal{F}_{T \setminus \{t\}, \{t\}})$$

is also Cartesian.

Proof. This follows immediately from Corollary 2.2 and the fact that a cube in a stable category is Cartesian if and only if its total homotopy fiber is contractible. \square

In the same spirit, but moving away from total fibers, we give a formula for inductively computing limits of partial n -cubes $\mathcal{G}: \mathcal{P}_{\neq \emptyset}(T) \rightarrow \mathcal{C}$.

Proposition 2.4. *Let $\mathcal{G}: \mathcal{P}_{\neq \emptyset}(T) \rightarrow \mathcal{C}$ be a partial cube in \mathcal{C} . Let $t \in T$ be arbitrary and set $T' = T \setminus \{t\}$. Then there is a pullback square:*

$$\begin{array}{ccc} \lim_{S \in \mathcal{P}_{\neq \emptyset}(T)} \mathcal{G}(S) & \longrightarrow & \lim_{S \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(S) \\ \downarrow & & \downarrow \\ \mathcal{G}(\{t\}) & \longrightarrow & \lim_{S \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(\{t\} \cup S) \end{array}$$

Proof. Let $\mathcal{Q} = \mathcal{P}_{\neq \emptyset}(\{a, b\}) \times \mathcal{P}_{\neq \emptyset}(T')$ and $F: \mathcal{Q} \rightarrow \mathcal{P}_{\neq \emptyset}(T)$ be the map of posets defined by $F(\{a\}, I) = \{t\}$, $F(\{b\}, I) = I$ and $F(\{a, b\}, I) = \{t\} \cup I$. We will show that F is (homotopy) initial. It suffices to check that for each $I \in \mathcal{P}_{\neq \emptyset}(T)$, the comma category $F \downarrow I$ is contractible. In all cases the comma category is a subposet of \mathcal{Q} :

- If $t \notin I$, $F \downarrow I$ is the subposet $\{(\{b\}, J) \in \mathcal{Q} : J \subseteq I\}$. This is contractible because it has a largest element, namely $(\{b\}, I)$.
- If $t \in I$, $F \downarrow I$ is the subposet $\mathcal{R} = \{(K, J) \in \mathcal{Q} : K = \{a\} \text{ or } J \subseteq I\}$. We can write \mathcal{R} as a pushout of posets:

$$\begin{array}{ccc} \{\{a\}\} \times \mathcal{P}_{\neq \emptyset}(I) & \longrightarrow & \{\{a\}\} \times \mathcal{P}_{\neq \emptyset}(T') \\ \downarrow & & \downarrow \\ \mathcal{P}_{\neq \emptyset}(\{a, b\}) \times \mathcal{P}_{\neq \emptyset}(I) & \longrightarrow & \mathcal{R} \end{array}$$

The geometric realizations of the three posets besides \mathcal{R} are cubes (of dimensions $|I|$, $n-1$ and $2+|I|$) and the top horizontal and left vertical maps realize to a face inclusion. This shows \mathcal{R} is contractible.

Since F is initial, $\lim \mathcal{G}$ can be computed as $\lim(\mathcal{G} \circ F)$. This limit we compute as an iterated limit:

$$\lim_{(K, J) \in \mathcal{Q}} \mathcal{G}(F(K, J)) = \lim_{K \in \mathcal{P}_{\neq \emptyset}(\{a, b\})} \left(\lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(F(K, J)) \right).$$

To conclude we identify the three terms in the pullback with the ones in the statement of the proposition:

- (1) $\lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(F(\{a\}, J)) = \lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(\{1\}) \cong \mathcal{G}(\{1\})$, since the indexing category $\mathcal{P}_{\neq \emptyset}(T')$ is contractible.
- (2) $\lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(F(\{b\}, J)) = \lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(J)$.
- (3) $\lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(F(\{a, b\}, J)) = \lim_{J \in \mathcal{P}_{\neq \emptyset}(T')} \mathcal{G}(\{1\} \cup J)$.

□

3. THE CHROMATIC FRACTURE CUBE

Let Sp be the stable ∞ -category of spectra [Lur14], and denote Bousfield localization [Bou79] at a spectrum E by L_E , with associated category Sp_E of E -local spectra. Two spectra E and F are said to be Bousfield equivalent if $L_E = L_F$; in this case, we write $\langle E \rangle = \langle F \rangle$. An endofunctor on Sp will be called E -local if it takes values in the category of E -local spectra. If we are given a collection $\{F(1), F(2), \dots, F(n)\}$ of spectra, for any set $S = \{i_1, \dots, i_k\}$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$, we write L_S for the composite $L_{F(i_1)} \dots L_{F(i_k)}$. The general form of the chromatic fracture cube takes the following form, generalizing the fracture square in [Bau].

Construction 3.1. Suppose $\{F(1), \dots, F(n)\}$ is any collection of spectra. We inductively define an n -cube $\mathcal{F} : \mathcal{P}(\{1, \dots, n\}) \rightarrow \mathrm{End}(\mathrm{Sp})$, whose vertices will turn out to be given by $S \mapsto L_S$, as follows:

- If $n = 1$, the cube is simply the natural localization morphism $\mathrm{id} \rightarrow L_{F(1)}$.
- For $n > 1$, inductively construct the cube \mathcal{F}' on $\mathcal{P}(\{2, \dots, n\})$, and get the full n -cube \mathcal{F} as the morphism of $(n-1)$ -cubes $\mathcal{F}' \rightarrow L_{F(1)}\mathcal{F}'$ given by the naturality of the localization morphisms $\mathrm{id} \rightarrow L_{F(1)}$.

Proposition 3.2. *Suppose $\{F(1), \dots, F(n)\}$ is a collection of spectra such that $L_{F(j)}L_{F(i)} = 0$ for all $j > i$. If $\mathcal{F} \in \mathrm{Fun}(\mathcal{P}(\{1, \dots, n\}), \mathrm{End}(\mathrm{Sp}))$ is the n -cube given by Construction 3.1 then*

$$\lim_{S \neq \emptyset} \mathcal{F}(S) = L_E,$$

where E is any spectrum Bousfield equivalent to $F(1) \oplus \dots \oplus F(n)$.

Proof. Let $P = \lim_{S \neq \emptyset} \mathcal{F}(S)$ be the limit with legs $f_i : P \rightarrow L_{F(i)}$. The localization maps $\eta_i : L_E \rightarrow L_{F(i)}L_E = L_{F(i)}$ induce a natural map from L_E to this limit,

$$\eta : L_E \longrightarrow P.$$

Since $F(i)$ -locals are clearly E -local and locality is preserved under limits, P is E -local. It therefore suffices to show that η is an $F(i)$ -equivalence for all $1 \leq i \leq n$. To this end, fix i and consider the commutative triangle

$$\begin{array}{ccc} L_E & \xrightarrow{\eta} & P \\ & \searrow \eta_i & \downarrow f_i \\ & & L_{F(i)}. \end{array}$$

Because η_i is an $F(i)$ -equivalence by definition, we only need to show that so is $f_i : P \rightarrow L_{F(i)}$. To this end, apply L_{F_i} to the limit cube $\tilde{\mathcal{F}}$ with P as initial vertex, $\tilde{\mathcal{F}}(S) = \mathcal{F}(S)$ for $S \neq \emptyset$, and the natural maps. This yields a cube $L_{F_i}\tilde{\mathcal{F}} \in \mathrm{Fun}(\mathcal{P}(\{1, \dots, n\}), \mathrm{End}(\mathrm{Sp}))$ with three properties:

- It is again Cartesian as L_{F_i} is exact.

- $L_{F(i)}\tilde{\mathcal{F}}(S) = 0$ whenever S contains an element smaller than i , because $L_{F(i)}L_{F(k)} = 0$ for $k < i$.
- For $S \neq \emptyset$ and $i \notin S$, the edges from S to $S \cup \{i\}$ are all equivalences since either both vertices are 0 by the previous point or, if $\min(S) < i$, the edge $\mathcal{F}(S) \rightarrow \mathcal{F}(S \cup \{i\})$ of \mathcal{F} is equivalent to the localization map $\mathcal{F}(S) \rightarrow L_{F(i)}\mathcal{F}(S)$ by construction, so applying $L_{F(i)}$ will yield an equivalence.

By Corollary 2.3, taking fibers in the $\{i\}$ direction produces a Cartesian $(n-1)$ -cube, which by the third item above is actually just $\text{in}_{\emptyset}(\text{fib}(L_{F(i)}P \rightarrow L_{F(i)}))$. Therefore, $L_{F(i)}P \cong L_{F(i)}$ and the claim follows. \square

Example 3.3. The Morava K -theories $K(0), \dots, K(n)$ satisfy the conditions of Proposition 3.2, since $K(i) \otimes K(j) = 0$ whenever $i \neq j$. In particular, if $i < j$, then there is a pullback square

$$\begin{array}{ccc} L_{K(i) \oplus K(j)} & \longrightarrow & L_{K(j)} \\ \downarrow & & \downarrow \\ L_{K(i)} & \longrightarrow & L_{K(i)}L_{K(j)}. \end{array}$$

Similarly, for $F(1) = E(n-1)$ and $F(2) = K(n)$ we get the usual chromatic fracture square

$$\begin{array}{ccc} L_{E(n)} & \longrightarrow & L_{K(n)} \\ \downarrow & & \downarrow \\ L_{E(n-1)} & \longrightarrow & L_{E(n-1)}L_{K(n)} \end{array}$$

using the identity of Bousfield classes $\langle E(n-1) \oplus K(n) \rangle = \langle E(n) \rangle$.

4. A DESCRIPTION OF THE CATEGORY OF LOCAL OBJECTS

As in the previous section, let $\{F(1), \dots, F(n)\}$ be a collection of spectra such that $L_{F(j)}L_{F(i)} = 0$ for all $j > i$, fixed for the remainder of the section, and let $T = \{1, \dots, n\}$. We will inductively construct a category \mathcal{C}_S of partial cubes $\mathcal{P}_{\neq \emptyset}(S)$ for $S \subset T$, and prove that \mathcal{C}_T is equivalent to Sp_E for any spectrum E with the same Bousfield class as $F(1) \oplus \dots \oplus F(n)$.

Construction 4.1. For $\emptyset \neq S \subset T$ we let \mathcal{C}_S be the full subcategory of the diagram category $\text{Fun}(\mathcal{P}_{\neq \emptyset}(S), \text{Sp})$ spanned by certain partial cubes \mathcal{G} chosen as follows:

- If $S = \{i\}$ is a singleton, we take all \mathcal{G} such that the unique value of \mathcal{G} , namely $\mathcal{G}(S)$, is $F(i)$ -local.
- If $S = \{i\} \cup S'$ where $i = \min(S) \notin S'$, we take all \mathcal{G} such that:
 - (1) $\mathcal{G}' := \mathcal{G}|_{\mathcal{P}_{\neq \emptyset}(S')}$ belongs to $\mathcal{C}_{S'}$, and
 - (2) if we think of $\mathcal{G}|_{\mathcal{P}(S) \setminus \{\emptyset, \{i\}\}}$ as a morphism between diagrams of shape $\mathcal{P}_{\neq \emptyset}(S')$, namely $\mathcal{G}' \rightarrow \mathcal{G}'|_{\{U \subseteq S': i \in U\}}$, this morphism is the natural morphism $\mathcal{G}' \rightarrow L_{F(i)}\mathcal{G}'$.

Remark 4.2. From the definitions it is clear that for any $\mathcal{G} \in \mathcal{C}_T$ we have that:

- For any $S \subseteq T$, $\mathcal{G}(S)$ is $F(\min(S))$ -local.
- For every k , $\mathcal{G}|_{\{S \subseteq T: \max(S) = k\}}$ is just the cube from Construction 3.1 applied to $\mathcal{G}(\{k\})$.

This construction becomes much clearer with an example.

Example 4.3. The ∞ -category $\mathcal{C}_{\{1,2,3\}}$ is equivalent to the full subcategory of $\text{Fun}(\mathcal{P}_{\neq\emptyset}(\{1,2,3\}), \text{Sp})$ on the diagrams of the form:

$$\begin{array}{ccccc}
 & & & & X_3 \\
 & & & \swarrow \eta_2 & \downarrow \eta_1 \\
 X_2 & \xrightarrow{f_{23}} & L_{F(2)}X_3 & & \\
 \downarrow \eta_1 & & \downarrow \eta_1 & & \\
 & X_1 & \xrightarrow{f_{13}} & L_{F(1)}X_3 & \\
 & \swarrow f_{12} & & \swarrow L_{F(1)}\eta_2 & \\
 L_{F(1)}X_2 & \xrightarrow{L_{F(1)}f_{23}} & L_{F(1)}L_{F(2)}X_3 & &
 \end{array}$$

where X_i is $F(i)$ -local for $i = 1, 2, 3$ and the morphisms labeled η_i are the natural maps $Y \rightarrow L_{F(i)}Y$. Notice that the diagram is determined just by $X_1, X_2, X_3, f_{12}, f_{13}, f_{23}$ and a homotopy showing the bottom square commutes.

As a simpler example, the ∞ -category $\mathcal{C}_{\{1,2\}}$ is the category of cospans of the form of the left face of the above partial cube. Those cospans are determined by X_1, X_2 and f_{12} .

Theorem 4.4. *If E is any spectrum Bousfield equivalent to $F(1) \oplus \cdots \oplus F(n)$, the ∞ -category \mathcal{C}_T constructed above is equivalent to Sp_E . Moreover, the following functors are mutually inverse equivalences:*

- $\lim: \mathcal{C}_T \rightarrow \text{Sp}_E$ given by $\mathcal{G} \mapsto \lim_{S \neq \emptyset} \mathcal{G}(S)$, and
- $\mathcal{F}_{\neq\emptyset}: \text{Sp}_E \rightarrow \mathcal{C}_T$ given by $X \mapsto \mathcal{F}[X]|_{\mathcal{P}_{\neq\emptyset}(T)}$ where \mathcal{F} is the n -cube of functors from Construction 3.1 and $\mathcal{F}[X]$ denotes the cube obtained by applying those functors to X .

Proof. First of all let us show those functors are well defined. This is clear for $\mathcal{F}_{\neq\emptyset}$ by construction. For \lim it is because all spectra in the image of a given partial cube \mathcal{G} are $F(i)$ -local for some i and thus also E -local.

Now we will show the functors are mutually inverse. First, for any E -local spectrum X , the canonical map $\lim(\mathcal{F}_{\neq\emptyset}[X]) \rightarrow X$ (coming from the diagram $\mathcal{F}[X]$, which is a cone over $\mathcal{F}_{\neq\emptyset}[X]$) is an equivalence by Proposition 3.2.

For the other composite, let $\mathcal{G} \in \mathcal{C}_T$ and let $P = \lim \mathcal{G}$. We must show that $\mathcal{F}_{\neq\emptyset}(P) \cong \mathcal{G}$. Extend \mathcal{G} to a Cartesian cube $\tilde{\mathcal{G}}$ with $\tilde{\mathcal{G}}(\emptyset) = P$. First we show that $L_{F(i)}P \cong \mathcal{G}(\{i\})$, and moreover, that the $\emptyset \rightarrow \{i\}$ edge in the n -cube $\tilde{\mathcal{G}}$ is the localization map $P \rightarrow L_{F(i)}P$ under this equivalence.

To this end, apply $L_{F(i)}$ to $\tilde{\mathcal{G}}$. For any $i \notin S \subseteq T$ consider the edge of $L_{F(i)}\tilde{\mathcal{G}}$ from S to $S \cup \{i\}$. There are three cases:

- If $S = \emptyset$, the edge has the form $L_{F(i)}P \rightarrow L_{F(i)}\mathcal{G}(\{i\})$. This is the morphism we wish to show is an equivalence.
- If $\min(S) < i$, we have $L_{F(i)}\mathcal{G}(S) = L_{F(i)}\mathcal{G}(S \cup \{i\}) = 0$, because both $\mathcal{G}(S)$ and $\mathcal{G}(S \cup \{i\})$ are $F(\min(S))$ -local. Therefore these edges are equivalences.
- If $\min(S) > i$, then the edge $\mathcal{G}(S) \rightarrow \mathcal{G}(\{i\} \cup S)$ is the natural map $\mathcal{G}(S) \rightarrow L_{F(i)}\mathcal{G}(S)$ because this edge is contained in $\mathcal{G}|_{\{U \subseteq T: \max(U) = \max(S)\}}$ which is

a cube obtained from Construction 3.1 applied to $\mathcal{G}(\{\max(S)\})$. This edge, of course, becomes an equivalence after applying $L_{F(i)}$.

Now the cube $L_{F(i)}\tilde{\mathcal{G}}$ is Cartesian because $\tilde{\mathcal{G}}$ was and $L_{F(i)}$ is exact. This means that taking fibers in the direction of $\{i\}$ must lead to a Cartesian $(n-1)$ -cube. By the above case analysis, that $(n-1)$ -cube is simply $\text{in}_\emptyset(\text{fib}(L_{F(i)}P \rightarrow L_{F(i)}\mathcal{G}(\{i\})))$, from which we conclude that $\text{fib}(L_{F(i)}P \rightarrow L_{F(i)}\mathcal{G}(\{i\}))$ is 0 and thus $L_{F(i)}P \rightarrow L_{F(i)}\mathcal{G}(\{i\})$ is an equivalence, as desired.

At this point we are close to showing that $\tilde{\mathcal{G}}$ and $\mathcal{F}(P)$ are equivalent n -cubes: we have shown that they have equivalent objects at all vertices and also many of the maps agree, but we have not shown for example that the map $\mathcal{G}(\{1\}) \rightarrow \mathcal{G}(\{1, i\})$ is $L_{F(1)}(P \rightarrow \mathcal{G}(\{i\}))$.

To conclude, consider taking fibers of $\tilde{\mathcal{G}}$ in the $\{1\}$ direction to get an $(n-1)$ -cube $\tilde{\mathcal{G}}'$. From the argument above and the exactness of $L_{F(1)}$, we know that this $(n-1)$ -cube vanishes if we apply $L_{F(1)}$ to it, so that the n -cube $\tilde{\mathcal{G}}$ when thought of as a map of $(n-1)$ -cubes $\tilde{\mathcal{G}}|_{\{S:1 \notin S\}} \rightarrow \mathcal{G}|_{\{S:1 \in S\}}$ is just $L_{F(1)}$ -localization.

Now applying $L_{F(2)}$ to $\tilde{\mathcal{G}}$, we get a Cartesian n -cube whose “bottom” face, $L_{F(2)}\tilde{\mathcal{G}}|_{\{S:1 \in S\}}$, vanishes. This means the top face, $L_{F(2)}\tilde{\mathcal{G}}|_{\{S:1 \notin S\}}$, is also Cartesian and we can recursively apply the argument of the previous paragraph to conclude that $\tilde{\mathcal{G}} \cong \mathcal{F}(P)$. \square

Remark 4.5. The special case $n = 2$ of the previous result appears as Remark 7 in [Lur10, Lecture 23].

5. A DECOMPOSITION OF THE CATEGORY OF LOCAL OBJECTS

As in the previous section, let $\{F(1), \dots, F(n)\}$ be a collection of spectra such that $L_{F(j)}L_{F(i)} = 0$ for all $j > i$, and let $T = \{1, \dots, n\}$. In this section we will describe a partial n -cube of ∞ -categories whose limit is Sp_E where E is Bousfield equivalent to $F(1) \oplus \dots \oplus F(n)$. To do that we will need some combinatorial preliminaries; an illustration of the construction can be found below in Example 5.4.

Definitions 5.1. For $\emptyset \neq S \subseteq S' \subseteq T$, define $\alpha(S)$, $\beta(S, S')$ and $\theta_{S, S'}$:

$$\begin{aligned} \alpha(S) &= \{U \subseteq T : S \subseteq U, \min(S) = \min(U)\} \\ \beta(S, S') &= \{V \subseteq [\min(S'), \min(S)] : S' \cap [\min(S'), \min(S)] \subseteq V\} \\ \theta_{S, S'} : \alpha(S') &\rightarrow \beta(S, S') \times \alpha(S) \\ U &\mapsto (U \cap [\min(S'), \min(S)], U \cap [\min(S), n]) \end{aligned}$$

where we have repurposed traditional interval notation to denote intervals of integers.

We will regard $\alpha(S)$ and $\beta(S, S')$ as posets, ordering them by inclusion, which makes $\theta_{S, S'}$ a map of posets. Notice that $\alpha(S)$ and $\beta(S, S')$ are isomorphic to posets of all subsets of some set, so that diagrams of shape $\alpha(S)$ or $\beta(S, S')$ are cubical diagrams.

Construction 5.2. We construct a partial n -cube of ∞ -categories $\mathcal{G} : \mathcal{P}_{\neq \emptyset}(T) \rightarrow \text{Cat}_\infty$ as follows:

- The vertices are given by $\mathcal{G}(S) = \text{Sp}_{F(\min(S))}^{\alpha(S)} := \text{Fun}(\alpha(S), \text{Sp}_{F(\min(S))})$.

- For $S \subseteq S'$, the functor $\mathcal{G}(S \subseteq S') : \mathcal{G}(S) \rightarrow \mathcal{G}(S')$ is given by the composite

$$\begin{aligned} \text{Fun}(\alpha(S), \text{Sp}_{F(\min(S))}) &\xrightarrow{\iota \circ -} \text{Fun}(\alpha(S), \text{Sp}) \\ &\xrightarrow{\mathcal{F}|_{\beta(S, S') \circ -}} \text{Fun}(\alpha(S), \text{Fun}(\beta(S, S'), \text{Sp}_{F(\min(S'))})) \\ &\xrightarrow{\cong} \text{Fun}(\beta(S, S') \times \alpha(S), \text{Sp}_{F(\min(S'))}) \\ &\xrightarrow{- \circ \theta} \text{Fun}(\alpha(S'), \text{Sp}_{F(\min(S'))}) \end{aligned}$$

where $\iota : \text{Sp}_{F(\min(S))} \rightarrow \text{Sp}$ denotes the natural inclusion functor and \mathcal{F} is essentially the cube of functors from Construction 3.1 restricted to $\beta(S, S') \subseteq \mathcal{P}(T)$: this restriction is a functor $\beta(S, S') \rightarrow \text{End}(\text{Sp})$, which we can think of instead as a functor $\text{Sp} \rightarrow \text{Fun}(\beta(S, S'), \text{Sp})$ and then replace Sp in the target by $\text{Sp}_{F(\min(S'))}$ since all $V \in \beta(S, S')$ satisfy $\min(V) = \min(S')$.

Remark 5.3. Unwinding the definitions in Constructions 5.2 and 3.1, we see that for, $\emptyset \neq S \subseteq S'$, the functor $\mathcal{G}(S) \rightarrow \mathcal{G}(S')$ sends a cube $X : \alpha(S) \rightarrow \text{Sp}_{F(\min(S))}$ to the cube $X' : \alpha(S') \rightarrow \text{Sp}_{F(\min(S'))}$ given on vertices by

$$X'(U) = L_{U \cap [\min(S'), \min(S)]} X(U \cap [\min(S), n]).$$

Notice that if $\min(S') < \min(S)$, the formula shows $X'(U)$ is $L_{F(\min(S'))}$ -local, as it should be. Also, when $\min(S') = \min(S)$, there is no localization at all and X' is simply the restriction of X to the face $\alpha(S') \subseteq \alpha(S)$ (this inclusion does not hold when $\min(S) \neq \min(S')$).

This definition also becomes much clearer with an example:

Example 5.4. Let $n = 3$. The diagram \mathcal{G} looks like:

$$\begin{array}{ccccc} & & \text{Sp}_E & \xrightarrow{\quad} & \text{Sp}_{F(3)}^{\{3\}} \\ & \swarrow & \downarrow & & \downarrow \\ \text{Sp}_{F(2)}^{\{2, 23\}} & & & \xrightarrow{\quad} & \text{Sp}_{F(2)}^{\{23\}} \\ & \downarrow & \downarrow & & \downarrow \\ & & \text{Sp}_{F(1)}^{\{1, 12, 13, 123\}} & \xrightarrow{\quad} & \text{Sp}_{F(1)}^{\{13, 123\}} \\ & \downarrow & \downarrow & & \downarrow \\ \text{Sp}_{F(1)}^{\{12, 123\}} & & & \xrightarrow{\quad} & \text{Sp}_{F(1)}^{\{123\}} \end{array}$$

where we have used shorthand for the elements of the various $\alpha(S)$: $\underline{13}$ denotes the set $\{1, 3\}$, for example.

Though $\mathcal{G}(\emptyset)$ is not defined, we have put Sp_E in that corner, since Theorem 5.5 will show that this produces a Cartesian cube for any E which is Bousfield equivalent to $F(1) \oplus F(2) \oplus F(3)$.

A square in $\mathrm{Sp}_{F(1)}^{\{\underline{1}, \underline{12}, \underline{13}, \underline{123}\}}$ should be thought of as being

$$\begin{array}{ccc} L_{F(1)}X & \longrightarrow & L_{F(1)}L_{F(3)}X \\ \downarrow & & \downarrow \\ L_{F(1)}L_{F(2)}X & \longrightarrow & L_{F(1)}L_{F(2)}L_{F(3)}X \end{array}$$

for some spectrum X , and the two functors out of $\mathrm{Sp}_{F(1)}^{\{\underline{1}, \underline{12}, \underline{13}, \underline{123}\}}$ are projection to the faces

$$L_{F(1)}L_{F(2)}X \longrightarrow L_{F(1)}L_{F(2)}L_{F(3)}X$$

and

$$L_{F(1)}L_{F(3)}X \longrightarrow L_{F(1)}L_{F(2)}L_{F(3)}X .$$

The vertical functor $\mathrm{Sp}_{F(3)}^{\{\underline{3}\}} \rightarrow \mathrm{Sp}_{F(1)}^{\{\underline{13}, \underline{123}\}}$ sends a $F(3)$ -local spectrum X to the morphism $L_{F(1)}X \rightarrow L_{F(1)}L_{F(2)}X$ you get by applying $L_{F(1)}$ to the localization map $X \rightarrow L_{F(2)}X$.

Our decomposition of the category of $\bigoplus_{i=1}^n F(i)$ -local objects can now be stated as follows.

Theorem 5.5. *If E is any spectrum Bousfield equivalent to $F(1) \oplus \dots \oplus F(n)$, then there is an equivalence of stable presentable ∞ -categories*

$$\mathrm{Sp}_E \xrightarrow{\sim} \lim_{S \neq \emptyset} \mathcal{G}(S)$$

where \mathcal{G} is given by Construction 5.2.

Proof. For any $S \in T$, observe that there is a natural functor

$$\Phi_S: \mathcal{C}_T \longrightarrow \mathrm{Sp}_{F(\min(S))}^{\alpha(S)},$$

sending a partial T -cube \mathcal{X} to the cube given by restriction, $\Phi_S(\mathcal{X}) = \mathcal{X}|_{\alpha(S)}$. The collection of these functors induces a natural functor

$$\Phi: \mathcal{C}_T \longrightarrow \lim_{S \neq \emptyset} \mathcal{G}(S)$$

and it suffices to show that this is an equivalence by Theorem 4.4.

We will now argue by induction on n , the case $n = 1$ being trivial. Proposition 2.4 applied with $t = 1$ shows that there is a pullback diagram

$$\begin{array}{ccc} \lim_{S \neq \emptyset} \mathcal{G}(S) & \longrightarrow & \lim_{S \in \alpha(\underline{1}) \setminus \{\underline{1}\}} \mathcal{G}(S \setminus \underline{1}) \\ \downarrow & & \downarrow \\ \mathrm{Sp}_{F(1)}^{\alpha(\underline{1})} & \longrightarrow & \lim_{S \in \alpha(\underline{1}) \setminus \{\underline{1}\}} \mathcal{G}(S) \end{array}$$

(where again $\underline{1} = \{1\}$) and the inductive hypothesis gives $\lim_{S \in \alpha(\underline{1}) \setminus \{\underline{1}\}} \mathcal{G}(S \setminus \underline{1}) = \mathcal{C}_{\{2, \dots, n\}}$. Also, since all the edges in the bottom face of \mathcal{G} are restrictions:

$$\begin{aligned} \lim_{S \in \alpha(\underline{1}) \setminus \{\underline{1}\}} \mathcal{G}(S) &= \lim_{S \in \alpha(\underline{1}) \setminus \{\underline{1}\}} \mathrm{Sp}_{F(1)}^{\alpha(S)} \\ &= \mathrm{Sp}_{F(1)}^{\mathrm{colim}_{S \in \alpha(\underline{1}) \setminus \{\underline{1}\}} \alpha(S)} \\ &= \mathrm{Sp}_{F(1)}^{\alpha(\underline{1}) \setminus \{\underline{1}\}}, \end{aligned}$$

so it suffices to show that the following commutative square is a pullback:

$$\begin{array}{ccc} \mathcal{C}_T & \xrightarrow{p} & \mathcal{C}_{\{2,\dots,n\}} \\ f \downarrow & & \downarrow L_{F(1)} \\ \mathrm{Sp}_{F(1)}^{\alpha(\underline{1})} & \xrightarrow{q} & \mathrm{Sp}_{F(1)}^{\alpha(\underline{1}) \setminus \{\underline{1}\}}. \end{array}$$

This is intuitively clear: objects of the pullback can be described by giving a diagram $\mathcal{Y} \in \mathcal{C}_{\{2,\dots,n\}}$, a diagram $\mathcal{X} \in \mathrm{Sp}_{F(1)}^{\alpha(\underline{1})}$, and an equivalence $\mathcal{X}|_{\alpha(\underline{1}) \setminus \{\underline{1}\}} \rightarrow L_{F(1)}\mathcal{Y}$. That data clearly assembles to make a diagram in $\mathcal{Z} \in \mathcal{C}_T$ with top (partial) face $\mathcal{Z}|_{\mathcal{P}(\{2,\dots,n\})} = \mathcal{Y}$ and bottom face $\mathcal{Z}|_{\alpha(\underline{1})} = \mathcal{X}$.

More formally, first note that it is easy to check that the horizontal maps p and q in the above diagram are coCartesian fibrations and that the left vertical map f preserves coCartesian morphisms. Therefore, we can make use of [Lur09, 2.4.4.4] to reduce the claim to checking that, for every $\mathcal{Y} \in \mathcal{C}_{\{2,\dots,n\}}$, the fiber over \mathcal{Y} in the horizontal direction are equivalent via f , i.e.,

$$(\mathcal{C}_T)\mathcal{Y} \xrightarrow[\sim]{f} (\mathrm{Sp}_{F(1)}^{\alpha(\underline{1})})_{L_{F(1)}\mathcal{Y}}.$$

Let us first consider the case $n = 2$. The fiber over $\mathcal{Y} = Y \in \mathrm{Sp}_{F(2)}$ is given by the full subcategory of $\mathcal{C}_{\{1,2\}}$ on object of shape

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & L_{F(1)}Y, \end{array}$$

which are thus determined by the bottom morphism $X \rightarrow L_{F(1)}Y$. This category is then easily seen to be equivalent to the fiber $(\mathrm{Sp}_{F(1)}^{\alpha(\underline{1})})_{L_{F(1)}Y}$, hence the claim holds for $n = 2$. Now we can explain how to deduce the general case from here. We have a commutative diagram of fiber sequences:

$$\begin{array}{ccccc} (\mathcal{C}_T)\mathcal{Y} & \longrightarrow & \mathcal{C}_T & \longrightarrow & \mathcal{C}_{\{2,\dots,n\}} \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ (\mathrm{Sp}_E)\mathrm{lim} \mathcal{Y} & \longrightarrow & \mathrm{Sp}_E & \longrightarrow & \mathrm{Sp}_{F(2) \oplus \dots \oplus F(n)} \\ \downarrow \sim & \swarrow f| & \downarrow & & \downarrow L_{F(1)} \\ (\mathrm{Sp}_{F(1)}^{\mathcal{P}(\underline{1})})\mathrm{lim} L_{F(1)}\mathcal{Y} & \longrightarrow & \mathrm{Sp}_{F(1)}^{\mathcal{P}(\underline{1})} & \xrightarrow{\mathrm{ev}_{\underline{1}}} & \mathrm{Sp}_{F(1)} \\ \downarrow \sim & & \downarrow & & \downarrow \mathrm{lim} \\ (\mathrm{Sp}_{F(1)}^{\alpha(\underline{1})})_{L_{F(1)}\mathcal{Y}} & \longrightarrow & \mathrm{Sp}_{F(1)}^{\alpha(\underline{1})} & \longrightarrow & \mathrm{Sp}_{F(1)}^{\alpha(\underline{1}) \setminus \{\underline{1}\}}. \end{array}$$

where the top right square is a pullback by Theorem 4.4, the middle right one is by the induction hypothesis applied to the pair $(F(1), F(2) \oplus \dots \oplus F(n))$, and the bottom right one is by construction. The claim follows. \square

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