JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 140 (1999) 23–32

Stabilization as a CW approximation

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Communicated by E.M. Friedlander; received 24 June 1997

Abstract

This paper describes a peculiar property of the category of S-modules constructed by the author, Kriz, Mandell, and May: the full subcategory of suspension spectra (all of which are S-modules) forms a precise copy of the category of topological spaces. Consequently, the "classical" homotopy category of S-modules with morphisms the actual homotopy classes of maps contains a copy of unstable homotopy theory. Stabilization and stable homotopy are induced by CW approximation as S-modules. One consequence is that CW complexes whose suspension spectra are CW S-modules must be contractible.

MSC: 55P42

This paper offers somewhat belated proofs of the following theorems, which have by now become "well-known to the experts" [6]. Let $\mathscr{M}_S$ be the category of S-modules constructed in [1]; $\mathscr{M}_S$ is a symmetric monoidal closed category, and also a topological closed model category whose "derived" category $\mathscr{D}_S$ (obtained by inverting the weak equivalences) is equivalent to Boardman's stable category as a symmetric monoidal category. We will *not* refer to $\mathscr{D}_S$ as the homotopy category of $\mathscr{M}_S$, instead reserving that term for the "classical" homotopy category $h\mathscr{M}_S$, which has the same objects as $\mathscr{M}_S$ and morphisms defined by

$$h\mathscr{M}_S(M,N) := \pi_0 \mathscr{M}_S(M,N).$$

In this paper all spaces and homotopies are based. We work in the category $\mathscr{T}$ of compactly generated weak Hausdorff spaces. Given any space X, there is a natural S-module structure on the suspension spectrum $\Sigma^\infty X$, and this provides us with a continuous functor $\Sigma^\infty : \mathscr{T} \to \mathscr{M}_S$.

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0022-4049/99/$ - see front matter
PII: S0022-4049(99)00044-4

Theorem 1. *For any based spaces X and Y, the map*

$$\Sigma^\infty : \mathscr{T}(X,Y) \to \mathscr{M}_S(\Sigma^\infty X, \Sigma^\infty Y)$$

is a homeomorphism.

This result may seem unsettling at first, since $\Sigma^\infty X$ is supposed to represent the stabilization of X, and nothing has been stabilized! We see immediately that $h\mathscr{M}_S(\Sigma^\infty X, \Sigma^\infty Y)$ is isomorphic to the *unstable* homotopy classes $[X,Y]$. However, it is precisely this lack of stabilization that prevents the functor Σ^∞ from contradicting the non-existence results of Hastings [2] and Lewis [3]. Hastings's Corollary 2a of [2] does not apply, since our Σ^∞ does not even reach the Spanier–Whitehead category. Lewis's axiom A5 of [3] is violated, since the right adjoint of Σ^∞ is given explicitly by $\mathscr{M}_S(\Sigma^\infty S^0, \_)$, and Theorem 1 then shows that "Ω^∞"$\Sigma^\infty X \cong X$, rather than "Ω^∞"$\Sigma^\infty X \simeq QX$, as required. (We will make no further use of this right adjoint of Σ^∞.)

The problem, of course, is that the weak equivalences have not been inverted. All objects of $\mathscr{M}_S$ are fibrant in the model category structure, so this amounts precisely to a lack of CW (i.e., cofibrant) approximation. Let $\gamma : \Gamma\Sigma^\infty X \to \Sigma^\infty X$ be any CW approximation of $\Sigma^\infty X$ in $\mathscr{M}_S$. We have the following reassuring result.

Theorem 2. *For based spaces X and Y where X is homotopic to a CW complex, the composite*

$$\mathscr{T}(X,Y) \xrightarrow{\Sigma^\infty} \mathscr{M}_S(\Sigma^\infty X, \Sigma^\infty Y) \xrightarrow{\gamma^*} \mathscr{M}_S(\Gamma\Sigma^\infty X, \Sigma^\infty Y)$$

induces the usual stabilization map $[X,Y] \to \{X,Y\}$ on passage to homotopy classes.

This provides conclusive evidence that very few of the S-modules $\Sigma^\infty X$ are themselves cofibrant. Say that an S-module M is *resolvent* if it is homotopic (in the sense of being isomorphic in $h\mathscr{M}_S$) to a cofibrant S-module. The following result follows from one proved by (in alphabetical order) Mike Hopkins, Norio Iwase, John Klein, and Nick Kuhn in response to a question I posted on the algebraic topology discussion list.

Theorem 3. *If X is a based space based homotopy equivalent to a CW complex, and $\Sigma^\infty X$ is a resolvent S-module, then X is contractible.*

Overall, the situation is one that is familiar in algebra, but perhaps not so much so in stable homotopy. There is a good notion of homotopy in $\mathscr{M}_S$, and consequently a homotopy category $h\mathscr{M}_S$, but it is very far from the stable category. The inversion of weak equivalences works a profound change in transforming $h\mathscr{M}_S$ into $\mathscr{D}_S$ – it results in stabilization itself.

We turn now to the proofs, and begin by reviewing the basic definitions of the theory of S-modules. More details can be found in [1].

We fix our attention on a particular "universe" $\mathscr{U}$, which the reader is welcome to consider as $\mathbb{R}^\infty$. By a *spectrum* E we mean a space EV for each finite dimensional subspace V of $\mathscr{U}$, together with homeomorphisms $EV \cong \Omega^W E(V \oplus W)$ whenever $V \perp W$, subject to an evident associativity diagram. Given a based space Y, we write $\Sigma^\infty Y$ for the spectrum whose Vth space is $\operatorname{colim}_{W \perp V} \Omega^W \Sigma^{V \oplus W} Y$; we write S for $\Sigma^\infty S^0$. We write $\mathscr{S}\mathscr{U}$ for the category of spectra. Next, we write $\mathscr{L}(1)$ for the space of linear isometries from $\mathscr{U}$ to itself (it is the first space in the linear isometries operad on $\mathscr{U}$, which explains the notation.) Using the twisted half-smash product of [5], there is a monad $\mathbb{L}$ in $\mathscr{S}\mathscr{U}$ defined by $\mathbb{L}E = \mathscr{L}(1) \ltimes E$. Using the twisted function spectrum right adjoint to the twisted half-smash product, $\mathbb{L}$ has a right adjoint $\mathbb{L}^*$ defined by $\mathbb{L}^* E = F[\mathscr{L}(1), E)$. As the right adjoint of a monad, it is a comonad, and the algebras over $\mathbb{L}$ can be identified with the coalgebras over $\mathbb{L}^*$. Either one defines the category of $\mathbb{L}$-*spectra*, written $\mathscr{S}[\mathbb{L}]$. It supports a coherently commutative and associative smash product written $\wedge_{\mathscr{L}}$. An S-*module* is an $\mathbb{L}$-spectrum satisfying a unital condition that will be unimportant in this paper; it is satisfied by the spectra $\Sigma^\infty Y$ with their natural $\mathbb{L}$-spectrum structure described below. The category $\mathscr{M}_S$ of S-modules is simply the full subcategory of S-modules in $\mathscr{S}[\mathbb{L}]$.

Given a space Y, the spectrum $\Sigma^\infty Y$ has the structure of an $\mathbb{L}$-spectrum with the structure map

$$\xi : \mathscr{L}(1) \ltimes \Sigma^\infty Y \cong \Sigma^\infty(\mathscr{L}(1)_+ \wedge Y) \xrightarrow{p*} \Sigma^\infty Y,$$

where $p : \mathscr{L}(1)_+ \to S^0$ is the collapse map. We will be mostly interested in its dual $\mathbb{L}^*$-structure given by the adjoint map

$$\hat{\xi} : \Sigma^\infty Y \to F[\mathscr{L}(1), \Sigma^\infty Y).$$

We show first that Theorems 2 and 3 follow from Theorem 1.

Proof of Theorem 2. Since X has the homotopy type of a CW complex, we may CW approximate $\Sigma^\infty X$ as an $\mathbb{L}$-spectrum by applying $\mathbb{L}$ and as an S-module by applying $\Gamma = S \wedge_{\mathscr{L}} \mathbb{L}$; this is a consequence of Lemma I.5.4 of [5] and Theorems I.4.6 and II.1.9 of [1]. We have the $\mathbb{L}$-spectrum structure map $\xi : \mathbb{L}\Sigma^\infty X \to \Sigma^\infty X$ and the unit map $\lambda : \Gamma\Sigma^\infty X = S \wedge_{\mathscr{L}} \mathbb{L}\Sigma^\infty X \to \mathbb{L}\Sigma^\infty X$. The composite $\xi \circ \lambda$ is the CW approximation map γ. Since λ is a homotopy equivalence by [1], II.1.9(iv), it follows that ξ and γ induce equivalent maps on passage to π_0. We have a commutative diagram

$$\begin{tikzcd}
\mathscr{T}(X,Y) \arrow[rr, "\eta_*"] \arrow[d, "\Sigma^\infty"] & & \mathscr{T}(X, \Omega^\infty\Sigma^\infty Y) \arrow[d, "\cong"] \\
\mathscr{M}_S(\Sigma^\infty X, \Sigma^\infty Y) \arrow[r, "\xi^*"] & \mathscr{S}[\mathbb{L}](\mathbb{L}\Sigma^\infty X, \Sigma^\infty Y) \arrow[r, "\cong"] & \mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y)
\end{tikzcd}$$

for any based space Y, and since Σ^∞ is a homeomorphism by Theorem 1, the desired result now follows on passage to π_0. $\square$

The proof of Theorem 3 now follows directly.

Proof of Theorem 3. By Theorem 1, the left vertical arrow in the above diagram is a homeomorphism. Since $\Sigma^\infty X$ is resolvent, γ is a homotopy equivalence, and therefore so is ξ. It follows that ξ^* in the above diagram is an isomorphism on passage to π_0. Therefore η_* is an isomorphism on passage to π_0. This in turn implies that X is contractible; we give a modification of an argument due to John Klein.

First, X must be connected, because otherwise

$$\eta_* : [X, S^0] \to [X, \Omega^\infty \Sigma^\infty S^0]$$

cannot be onto. Next, let X_+ be X with a new disjoint basepoint. Then $\Omega^\infty\Sigma^\infty(X_+) \simeq \Omega^\infty\Sigma^\infty(X \vee S^0)$, since X is equivalent to a space with a non-degenerate basepoint. We get

$$[X,X] \cong [X, X \vee S^0] \cong [X, \Omega^\infty\Sigma^\infty(X \vee S^0)] \cong [X, \Omega^\infty\Sigma^\infty(X_+)] \cong [X, X_+] \cong \{*\}.$$

Therefore X is contractible. $\square$

We begin the proof of Theorem 1 with a reduction.

Lemma 4. *Theorem* 1 *follows from the special case in which* $X = S^0$.

Proof. $\mathscr{M}_S(\Sigma^\infty X, \Sigma^\infty Y)$ is by definition the equalizer of two maps

$$\mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y) \to \mathscr{S}\mathscr{U}(\Sigma^\infty X, F[\mathscr{L}(1), \Sigma^\infty Y)),$$

the first induced by the structure map $\hat{\xi} : \Sigma^\infty Y \to F[\mathscr{L}(1), \Sigma^\infty Y)$, and the second given by the composite

$$\begin{aligned}\mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y) &\to \mathscr{S}\mathscr{U}(F[\mathscr{L}(1), \Sigma^\infty X), F[\mathscr{L}(1), \Sigma^\infty Y))\\ &\to \mathscr{S}\mathscr{U}(\Sigma^\infty X, F[\mathscr{L}(1), \Sigma^\infty Y)),\end{aligned}$$

where the second map is induced by the structure map of $\Sigma^\infty X$. Call this composite θ. The $(\Sigma^\infty, \Omega^\infty)$ adjunction gives us a commutative square

$$\begin{tikzcd}
\mathscr{T}(X, \Omega^\infty\Sigma^\infty Y) \arrow[r, shift left, "\Omega^\infty\hat{\xi}_*"] \arrow[r, shift right, "\theta'"'] \arrow[d, "\cong"] & \mathscr{T}(X, \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y)) \arrow[d, "\cong"] \\
\mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y) \arrow[r, shift left, "\hat{\xi}_*"] \arrow[r, shift right, "\theta"'] & \mathscr{S}\mathscr{U}(\Sigma^\infty X, F[\mathscr{L}(1), \Sigma^\infty Y))
\end{tikzcd}$$

where θ' is defined so that the square with lower horizontal arrows commutes. By naturality and the Yoneda Lemma, θ' is induced by a map

$$j : \Omega^\infty \Sigma^\infty Y \to \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y)$$

which will be identified explicitly below. We wish to show that in the total diagram

$$\begin{tikzcd}
\mathscr{T}(X,Y) \arrow[r, "\eta_*"] \arrow[d, "\Sigma^\infty"'] \arrow[dr, "\Sigma^\infty"] & \mathscr{T}(X, \Omega^\infty \Sigma^\infty Y) \arrow[r, shift left, "\Omega^\infty \hat{\xi}_*"] \arrow[r, shift right, "\theta'"'] \arrow[d, "\cong"] & \mathscr{T}(X, \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y)) \arrow[d, "\cong"] \\
\mathscr{M}_S(\Sigma^\infty X, \Sigma^\infty Y) \arrow[r, "\imath"] & \mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y) \arrow[r, shift left, "\hat{\xi}_*"] \arrow[r, shift right, "\theta"'] & \mathscr{S}\mathscr{U}(\Sigma^\infty X, F[\mathscr{L}(1), \Sigma^\infty Y))
\end{tikzcd}$$

the top row is an equalizer and the left square commutes. Since the bottom row is an equalizer, this will force the left vertical arrow to be an isomorphism, establishing Theorem 1. The lower triangle in the left square commutes simply because $\Sigma^\infty : \mathscr{T}(X,Y) \to \mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y)$ factors through $\mathscr{M}_S$; this is what defines the left vertical arrow. The top triangle in the left square commutes since η is the unit of the $(\Sigma^\infty, \Omega^\infty)$ adjunction. All that remains is to show that the top row is an equalizer. In the special case $X = S^0$, we are assuming the left vertical arrow is an isomorphism, so the top row is forced to be an equalizer which we can display explicitly as

$$Y \xrightarrow{\eta} \Omega^\infty \Sigma^\infty Y \underset{j}{\overset{\Omega^\infty \hat{\xi}}{\rightrightarrows}} \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y)).$$

Since hom-functors preserve equalizers, the general case now follows by applying $\mathscr{T}(X, —)$ to this equalizer. □

We are left with the proof of Theorem 1 in the special case $X = S^0$, which we now address. We need to identify explicitly the map $j : \Omega^\infty \Sigma^\infty Y \to \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y)$ described in the proof above. We use the fact that the structure map $\hat{\xi}$ on $\Sigma^\infty X$ is by definition the composite

$$\Sigma^\infty X \xrightarrow{\eta} F[\mathscr{L}(1), \mathscr{L}(1) \ltimes \Sigma^\infty X) \cong F[\mathscr{L}(1), \Sigma^\infty(\mathscr{L}(1)_+ \wedge X))$$
$$\xrightarrow{p_*} F[\mathscr{L}(1), \Sigma^\infty X)$$

and the commutative diagram

$$\begin{tikzcd}
\mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y) \arrow[r] \arrow[d, "p^*"] & \mathscr{S}\mathscr{U}(F[\mathscr{L}(1), \Sigma^\infty X), F[\mathscr{L}(1), \Sigma^\infty Y)) \arrow[d, "p^*"] \\
\mathscr{S}\mathscr{U}(\Sigma^\infty(\mathscr{L}(1)_+ \wedge X), \Sigma^\infty Y) \arrow[r] \arrow[d, "\cong"] & \mathscr{S}\mathscr{U}(F[\mathscr{L}(1), \Sigma^\infty(\mathscr{L}(1)_+ \wedge X)), F[\mathscr{L}(1), \Sigma^\infty Y)) \arrow[d, "\cong"] \\
\mathscr{S}\mathscr{U}(\mathscr{L}(1) \ltimes \Sigma^\infty X, \Sigma^\infty Y) \arrow[r] \arrow[dr, "\cong"'] & \mathscr{S}\mathscr{U}(F[\mathscr{L}(1), \mathscr{L}(1) \ltimes \Sigma^\infty X), F[\mathscr{L}(1), \Sigma^\infty Y)) \arrow[d, "\eta^*"] \\
& \mathscr{S}\mathscr{U}(\Sigma^\infty X, F[\mathscr{L}(1), \Sigma^\infty Y))
\end{tikzcd}$$

to rewrite θ as

$$\mathscr{S}\mathscr{U}(\Sigma^\infty X, \Sigma^\infty Y) \xrightarrow{p^*} \mathscr{S}\mathscr{U}(\Sigma^\infty(\mathscr{L}(1)_+ \wedge X), \Sigma^\infty Y)$$

$$\cong \mathscr{S}\mathscr{U}(\mathscr{L}(1) \ltimes \Sigma^\infty X, \Sigma^\infty Y) \cong \mathscr{S}\mathscr{U}(\Sigma^\infty X, F[\mathscr{L}(1), \Sigma^\infty Y)).$$

Specializing again to $X = S^0$ and using the $(\Sigma^\infty, \Omega^\infty)$ adjunction, this displays j as the composite

$$\Omega^\infty \Sigma^\infty Y \xrightarrow{p^*} F(\mathscr{L}(1)_+, \Omega^\infty \Sigma^\infty Y) \cong \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y).$$

We must show that $\eta : Y \to \Omega^\infty \Sigma^\infty Y$ is an equalizer for j and $\Omega^\infty \hat{\xi}$. We do this in three steps: first, show that η composed with either map gives the same result, second, show that η is a closed inclusion, which implies it is a topological equalizer if it is a set-theoretic equalizer, and finally, show that it is a set-theoretic equalizer.

For the first step, a chase around the diagram

$$\begin{tikzcd}
\mathscr{S}\mathscr{U}(\Sigma^\infty Y, \Sigma^\infty Y) \arrow[r, "\cong"] \arrow[d, "p^*"] & \mathscr{T}(Y, \Omega^\infty \Sigma^\infty Y) \arrow[d, "p^*"] \\
\mathscr{S}\mathscr{U}(\Sigma^\infty(\mathscr{L}(1)_+ \wedge Y), \Sigma^\infty Y) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathscr{T}(Y, F(\mathscr{L}(1)_+, \Omega^\infty \Sigma^\infty Y)) \arrow[dd, "\cong"] \\
\mathscr{S}\mathscr{U}(\mathscr{L}(1) \ltimes \Sigma^\infty Y, \Sigma^\infty Y) \arrow[d, "\cong"] & \\
\mathscr{S}\mathscr{U}(\Sigma^\infty Y, F[\mathscr{L}(1), \Sigma^\infty Y)) \arrow[r, "\cong"] & \mathscr{T}(Y, \Omega^\infty F[\mathscr{L}(1), \Sigma^\infty Y))
\end{tikzcd}$$

starting at $\mathrm{id}_{\Sigma^\infty Y}$ gives the desired equality: note that $\hat{\xi}$ is reached in the lower right corner.

The second step consists of a sequence of lemmas taken mainly from Gaunce Lewis's unpublished 1978 dissertation [4].

Lemma 5. *A map of spaces is an equalizer if and only if it is a closed inclusion.*

Proof. Clearly any equalizer is a closed inclusion, by construction. Conversely, let $f : A \to B$ be a closed inclusion. Then f is an equalizer of the two maps $B \to B/f(A)$, one the canonical map, the other the trivial (basepoint) map. □

It follows immediately that a closed inclusion that is a set-theoretic equalizer of two continuous maps is a topological equalizer of the two maps.

Lemma 6. *The unit map $u : Y \to \Omega\Sigma Y$ is a closed inclusion.*

Proof. I am indebted to Gaunce Lewis for the following proof, which corrects a flaw in his dissertation.

Let $\sigma : S^0 \to S^1$ send the non-basepoint to $\frac{1}{2} \in S^1$. Then smashing with σ gives a natural closed inclusion $\sigma : Y \to \Sigma Y$. Next, observe that the composite

$$\Sigma Y \xrightarrow{\Sigma u} \Sigma\Omega\Sigma Y \xrightarrow{e} \Sigma Y$$

is the identity, where the evaluation map e is the counit of the (Σ, Ω) adjunction. Consequently Σu is a closed inclusion, being the inclusion of a retract. (This is a basic property of compactly generated weak Hausdorff spaces.) We now have a commutative square

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "\sigma"'] & \Omega\Sigma Y \arrow[d, "\sigma"] \\ \Sigma Y \arrow[r, "\Sigma u"] & \Sigma\Omega\Sigma Y \end{tikzcd}$$

in which all arrows except u are known to be closed inclusions. The composite $\Sigma u \circ \sigma$ is therefore a closed inclusion, and thus an equalizer. Since $\sigma : \Omega\Sigma Y \to \Sigma\Omega\Sigma Y$ is monic (being an equalizer) and $\sigma \circ u$ is an equalizer, it follows that u is an equalizer, and thus a closed inclusion. □

From Lemma 5 and the fact that Ω is a right adjoint, we deduce that if f is a closed inclusion of based spaces, so is Ωf. Lemma 6 now implies that all the maps in the system

$$Y \xrightarrow{u} \Omega\Sigma Y \xrightarrow{\Omega u} \Omega^2\Sigma^2 Y \to \cdots$$

are closed inclusions.

Lemma 7. *The map into the colimit, $\eta : Y \to \Omega^\infty\Sigma^\infty Y$, is a closed inclusion.*

Proof. Since η is clearly injective, it suffices to show that it is a closed map. Let $u^k : Y \to \Omega^k\Sigma^k Y$ and $\eta_k : \Omega^k\Sigma^k Y \to \Omega^\infty\Sigma^\infty Y$ be the canonical maps, and let $C \subset Y$ be a closed subset. Then $\eta_k^{-1}(\eta(C)) = u^k(C)$, which is closed in $\Omega^k\Sigma^k Y$ from above.

This is precisely the criterion for $\eta(C)$ to be closed in $\Omega^\infty\Sigma^\infty Y$, so η is a closed map. □

It now suffices to show that η is the set-theoretic equalizer of the two maps, and for this purpose, we consider an arbitrary linear isometry $f\in\mathscr{L}(1)$. Then the inclusion $\iota_f:\{f\}\to\mathscr{L}(1)$ induces a map $\iota_f^*:F[\mathscr{L}(1),\Sigma^\infty Y)\to F[\{f\},\Sigma^\infty Y)=f^*\Sigma^\infty Y$. Define ξ_f as the composite

$$\Sigma^\infty Y\xrightarrow{\hat{\xi}}F[\mathscr{L}(1),\Sigma^\infty Y)\xrightarrow{\iota_f^*}f^*\Sigma^\infty Y.$$

We can consider the composite maps

$$\Omega^\infty\Sigma^\infty Y\xrightarrow{p^*}F[\mathscr{L}(1)_+,\Omega^\infty\Sigma^\infty Y)\cong\Omega^\infty F[\mathscr{L}(1),\Sigma^\infty Y)\xrightarrow{\Omega^\infty\iota_f^*}\Omega^\infty f^*\Sigma^\infty Y$$

and

$$\Omega^\infty\xi_f:\Omega^\infty\Sigma^\infty Y\xrightarrow{\Omega^\infty\hat{\xi}}\Omega^\infty F[\mathscr{L}(1),\Sigma^\infty Y)\xrightarrow{\Omega^\infty\iota_f^*}\Omega^\infty f^*\Sigma^\infty Y,$$

and it suffices to show that $\eta:Y\to\Omega^\infty\Sigma^\infty Y$ is the joint equalizer of all such pairs. The first map is the set-theoretic equality, valid for all spectra,

$$\Omega^\infty E=E(\{0\})=E(f(\{0\}))=(f^*E)(\{0\})=\Omega^\infty f^*E,$$

applied when $E=\Sigma^\infty Y$. We wish to describe the second map, $\Omega^\infty\xi_f$, equally explicitly. We have already shown that

$$Y\xrightarrow{\eta}\Omega^\infty\Sigma^\infty Y\xrightarrow{p^*}F(\mathscr{L}(1)_+,\Omega^\infty\Sigma^\infty Y)\cong\Omega^\infty F[\mathscr{L}(1),\Sigma^\infty Y)$$

coincides with

$$Y\xrightarrow{\eta}\Omega^\infty\Sigma^\infty Y\xrightarrow{\Omega^\infty\hat{\xi}}\Omega^\infty F[\mathscr{L}(1),\Sigma^\infty Y),$$

so composing with $\Omega^\infty\iota_f:\Omega^\infty F[\mathscr{L}(1),\Sigma^\infty Y)\to\Omega^\infty f^*\Sigma^\infty Y$ shows that

$$Y\xrightarrow{\eta}\Omega^\infty\Sigma^\infty Y=\Omega^\infty f^*\Sigma^\infty Y$$

coincides with

$$Y\xrightarrow{\eta}\Omega^\infty\Sigma^\infty Y\xrightarrow{\Omega^\infty\xi_f}\Omega^\infty f^*\Sigma^\infty Y.$$

We may therefore take the adjoint of

$$Y\xrightarrow{\eta}\Omega^\infty\Sigma^\infty Y=\Omega^\infty f^*\Sigma^\infty Y$$

to arrive at

$$\xi_f:\Sigma^\infty Y\to f^*\Sigma^\infty Y$$

and apply Ω^∞ to get our second map,

$$\Omega^\infty\xi_f:\Omega^\infty\Sigma^\infty Y\to\Omega^\infty f^*\Sigma^\infty Y.$$

Using this description, we can compute $\Omega^\infty \xi_f$ explicitly as follows. Let ϕ be an arbitrary element of $\Omega^\infty \Sigma^\infty Y$, and suppose $\phi_1 : S^V \to Y \wedge S^V$ represents ϕ. Then $\Omega^\infty \xi_f(\phi)$ is represented by the composite

$$S^{fV} \xrightarrow{f^{-1}} S^V \xrightarrow{\phi_1} Y \wedge S^V \xrightarrow{1\wedge f} Y \wedge S^{fV}.$$

The proof of Theorem 1 is completed by proving the following lemma.

Lemma 8. *Let* $\phi \in \Omega^\infty \Sigma^\infty Y$ *and suppose* $\Omega^\infty \xi_f(\phi) = \phi$ for all $f \in \mathscr{L}(1)$. *Then* $\phi \in \operatorname{im} \eta$.

Proof. Let ϕ be represented by $\phi_1 : S^V \to Y \wedge S^V$, and choose W orthogonal to V with the same dimension. We can choose an $f \in \mathscr{L}(1)$ such that $f(V) = W$ and $f(W) = V$. Then ϕ is also represented by the map ϕ_2 given by the composite

$$\phi_2 : S^{V\oplus W} \cong S^V \wedge S^W \xrightarrow{\phi_1 \wedge 1_{S^W}} Y \wedge S^V \wedge S^W \cong Y \wedge S^{V\oplus W}.$$

Since $f(V \oplus W) = V \oplus W$, $\Omega^\infty \xi_f(\phi)$ is represented by

$$\phi_3 : S^{V\oplus W} \xrightarrow{f^{-1}} S^{V\oplus W} \xrightarrow{\phi_1 \wedge 1} Y \wedge S^{V\oplus W} \xrightarrow{1\wedge f} Y \wedge S^{V\oplus W}.$$

Since f switches V and W, we see that ϕ_3 is given by the composite

$$S^{V\oplus W} \cong S^W \wedge S^V \xrightarrow{\phi_f \wedge 1_{S^V}} Y \wedge S^W \wedge S^V \cong Y \wedge S^{V\oplus W}$$

for some map $\phi_f : S^W \to Y \wedge S^W$. Our assumption implies that this composite coincides with $\phi_1 \wedge 1_{S^W}$, which represents ϕ.

Next, we show that if $\phi \neq *$, then ϕ_1 sends non-basepoints to non-basepoints. Let

$$(v, w) \in S^{V\oplus W} \setminus \{\infty\} \cong V \oplus W,$$

and suppose $\phi_1(v) = *$. Then

$$\phi_2[v, w] = (\phi_1 \wedge 1_{S^W})[v, w] = [*, w] = *$$

for all $w \in W$, but also $\phi_2[v, w] = \phi_3[v, w] = [\phi_f(w), v]$, so since $v \neq \infty$, $\phi_f(w) = *$ for all $w \in W$. Therefore, $\phi_1 = *$, since ϕ_f is just a conjugate of ϕ_1 by f, and so $\phi = *$ as well.

Since $\phi = *$ is in the image of η, we may now assume $\phi \neq *$, and since ϕ_1 was arbitrary, any representative of ϕ sends non-basepoints to non-basepoints. Consequently,

$$\phi_2 : S^{V\oplus W} \setminus \{\infty\} \to (Y \wedge S^{V\oplus W}) \setminus \{*\},$$

that is,

$$\phi_2 : V \oplus W \to (Y \setminus \{*\}) \times (V \oplus W).$$

The coordinate functions may be written

$$\phi_2(v, w) = (\phi_1' v, \phi_1'' v, w)$$

by the first characterization of ϕ_2 as $\phi_1 \wedge 1_{S^W}$. But this coincides with ϕ_3, which has coordinate functions

$$\phi_3(v, w) = (\phi_3' w, v, \phi_3'' w),$$

where $\phi_3' w \in Y \backslash \{*\}$, $\phi_3'' w \in W$, and neither depends on v. From $\phi_2 = \phi_3$ we conclude that

(1) $\phi_1' v$ is constant, and

(2) $\phi_1'' v = v$.

This precisely characterizes ϕ_1 as representing an element of the image of η, so $\phi \in \operatorname{im} \eta$. $\square$

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