ALGEBRAS OVER EQUIVARIANT SPHERE SPECTRA

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ABSTRACT. We study the category of algebras over the sphere G-spectrum of a compact Lie group G. A priori, this category depends on which representations appear in the underlying universe on which G-spectra are indexed, but we prove that different universes give rise to equivalent categories of point-set level algebras. The relevant change of universe functors are defined on categories of modules over sphere spectra and induce the classical change of universe functors (which are not equivalences!) on passage to stable homotopy categories. In particular, we show how to construct equivariant algebras from nonequivariant algebras by change of universe. This gives a reservoir of equivariant examples to which recently developed algebraic techniques in stable homotopy theory can be applied.

1. Introduction

In [4], Kriz, Mandell, and the authors developed a theory of highly structured ring, module, and algebra spectra. Although that paper was written nonequivariantly, its first stated result was the blanket assertion that all of the rest of its general theoretical results apply verbatim to G-spectra for a compact Lie group G. An exposition from the equivariant point of view will appear in [10]. We here begin to explore such highly structured equivariant spectra.

The ground category for the theory of [4] (see also [3]) is the category $\mathscr{M}_S$ of S-modules, where S is the sphere spectrum; its derived category $\mathscr{D}_S$ is obtained by inverting the weak equivalences and is equivalent to the classical stable homotopy category. The ground category we are most interested in is the category of S_G-modules, where S_G is the sphere G-spectrum indexed on a complete G-universe; its derived category is equivalent to the Lewis-May equivariant stable homotopy category [8] (see also [6]).

A G-universe is a countably infinite dimensional real inner product space U with an action of G through linear isometric isomorphisms such that U is the colimit of its finite dimensional representations V, the trivial representation occurs in U, and each irreducible representation that occurs in U occurs infinitely often. For a universe U, a G-spectrum E indexed on U is a collection of based G-spaces EV and suitably compatible homeomorphisms $EV \longrightarrow \Omega^{W-V}EW$ for $V \subset W$, where $W-V$ is the orthogonal complement of V in W. We let $G\mathscr{S}U$ denote the category of G-spectra indexed on U and let $hG\mathscr{S}U$ be its homotopy category. The associated "stable homotopy category" $\bar{h}G\mathscr{S}U$ is obtained from $hG\mathscr{S}U$ by adjoining inverses to the weak equivalences.

We obtain different stable homotopy categories of G-spectra depending on which representations occur in U. In fact, Lewis [7] recently proved that the stable homotopy categories of G-spectra indexed on universes U and U' are equivalent if and only if the orbits G/H that embed in U are the same as the orbits that embed in U'.

If only the trivial representation occurs in U, then we say that U is a trivial universe and we obtain what are called naive G-spectra. In particular, any nonequivariant spectrum may be regarded as a naive G-spectrum with trivial G-action. If all irreducible representations occur in U, then we say that U is a complete universe and we obtain what are called genuine G-spectra, or simply G-spectra. We obtain a trivial universe U^G from a universe U by passing to fixed points.

A crucial feature of the equivariant world is change of universe. A G-linear isometry $f : U \longrightarrow U'$ induces a point-set level change of universe functor $f_* : G\mathscr{S}U \longrightarrow G\mathscr{S}U'$, which in turn induces a functor $f_* : \bar{h}G\mathscr{S}U \longrightarrow \bar{h}G\mathscr{S}U'$ on stable homotopy categories. For example, there is a change of universe functor $i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$ associated with the inclusion $i : U^G \longrightarrow U$. When G is complete, it assigns a genuine G-spectrum to a naive G-spectrum. The point-set level functors f_* do not preserve highly structured ring, module, and algebra spectra. We shall obtain new point-set level change of universe functors that do preserve highly structured ring, module, and algebra spectra and that become equivalent to the functors f_* on passage to stable homotopy categories.

We let S_U denote the sphere G-spectrum indexed on a G-universe U. When U is a fixed given complete G-universe, we write $S_G = S_U$ and $S = S_{U^G}$. Thus S_G is the genuine sphere G-spectrum and S is the nonequivariant sphere spectrum regarded as a naive G-spectrum with trivial action by G. Let $G\mathscr{M}_{S_U}$ denote the category of S_U-modules and let $hG\mathscr{M}_{S_U}$ be its homotopy category. Let $G\mathscr{D}_{S_U}$ be the derived category of S_U-modules; it is obtained from $hG\mathscr{M}_{S_U}$ by adjoining inverses to the weak equivalences, which are the maps of S_U-modules that are weak equivalences of underlying G-spectra. We view the category $\mathscr{M}_S$ of nonequivariant S-modules as the full subcategory of G-trivial modules in the category $G\mathscr{M}_S$ of equivariant S-modules.

The categories $G\mathscr{D}_{S_U}$ and $\bar{h}G\mathscr{S}U$ are equivalent, and Lewis's result shows how this category depends on U. In contrast, we shall prove the startling fact that the point-set level categories $G\mathscr{M}_{S_U}$, and also their homotopy categories $hG\mathscr{M}_{S_U}$, are independent of U, up to natural equivalence of categories. There is no contradiction: the relevant change of universe functors do not preserve weak equivalences and so do not pass to equivalences of derived categories. The point-set level equivalences of categories preserve rings, modules, and algebras.

Recall that a functor between symmetric monoidal categories is said to be monoidal if it preserves the given products and unit objects up to isomorphisms that are suitably compatible with the respective unity, associativity, and commutativity isomorphisms. Such a functor necessarily preserves monoids, commutative monoids, and objects with actions by monoids. In the category $G\mathscr{M}_{S_U}$ of S_U-modules, the monoids are the S_U-algebras R and the objects with action by R are the R-modules M. These notions are defined in terms of maps of S_U-modules $S_U \longrightarrow R$, $R \wedge_{S_U} R \longrightarrow R$, and $R \wedge_{S_U} M \longrightarrow M$ such that the usual diagrams commute. See [4, II§§3-4] for discussion and for comparison with the earlier definitions of A_∞ and E_∞ ring G-spectra; the discussion is given nonequivariantly, but it applies verbatim equivariantly. We shall prove the following formal result.

Theorem 1.1. *Let U and U' be G-universes. There is a monoidal equivalence of categories*

$$I_U^{U'} : G\mathscr{M}_{S_U} \longrightarrow G\mathscr{M}_{S_{U'}}.$$

Therefore, if R is an S_U-algebra and M is an R-module, then $I_U^{U'}R$ is an $S_{U'}$-algebra and $I_U^{U'}M$ is an $I_U^{U'}R$-module.

By Lewis's result, the functor $I_U^{U'}$ cannot induce an equivalence of derived categories in general. The following result shows that, if there is a G-linear isometry $f : U \longrightarrow U'$, then the functor on derived categories induced by $I_U^{U'}$ becomes equivalent to $f_* : \bar{h}G\mathscr{S}U \longrightarrow \bar{h}G\mathscr{S}U'$ when we forget the module structures. Thus our new point-set level change of universe equivalences pass to derived categories to give new models for the homotopical change of universe functors between derived categories. The details depend on the Quillen model category structures that the theory of [4] assigns to all categories in sight. We use the term "q-cofibrant" for Quillen cofibrant objects in a given category.

Theorem 1.2. *Let $f : U \longrightarrow U'$ be a G-linear isometry. Then there is a natural map $\alpha : f_*M \longrightarrow I_U^{U'}M$ of G-spectra indexed on U' that is a homotopy equivalence for every S_U-module M in a class $\bar{\mathscr{E}}_{S_U}$ of S_U-modules that includes all q-cofibrant S_U-modules, all q-cofibrant S_U-algebras, and all q-cofibrant commutative S_U-algebras.*

Replacing the pair (U, U') by the pair (U^G, U) for a given complete G-universe U and replacing f by the inclusion $i : U^G \longrightarrow U$, we obtain the following special cases of the previous two theorems.

Corollary 1.3. *There is a monoidal equivalence of categories*

$$I_{U^G}^U : G\mathscr{M}_S \longrightarrow G\mathscr{M}_{S_G}.$$

Therefore, if R is an S-algebra and M is an R-module, then $I_{U^G}^U R$ is an S_G-algebra and $I_{U^G}^U M$ is an $I_{U^G}^U R$-module.

Corollary 1.4. *There is a natural map $\alpha : i_*M \longrightarrow I_{U^G}^U M$ of genuine G-spectra that is a homotopy equivalence for every S-module M in a class $\bar{\mathscr{E}}_S$ of S-modules that includes all q-cofibrant S-modules, all q-cofibrant S-algebras, and all q-cofibrant commutative S-algebras.*

We conclude that $I_{U^G}^U$ gives a point-set level model for the change of universe functor $i_* : \bar{h}G\mathscr{S}U^G \longrightarrow \bar{h}G\mathscr{S}U$ that carries highly structured naive G-spectra (or nonequivariant spectra) to highly structured genuine G-spectra. This has considerable constructive power. A major gap in equivariant stable homotopy theory is that equivariant infinite loop space theory has not yet been developed for compact Lie groups: we do not have a recognition principle that allows us to construct G-spectra, let alone highly structured G-spectra, from space level data. Our results partially rectify this by showing how to construct highly structured G-spectra from highly structured spectra.

In a companion paper [1], Benson and Greenlees use the following special case of the construction to study the ordinary cohomology of the classifying space BG. For an S_G-algebra R_G and a G-space X, the function G-spectrum $F(X_+, R_G)$ is an S_G-algebra, commutative if R_G is so, with product induced by the product of R_G and the diagonal on X [4, VII.2.10]. Let Σ_G^∞ denote the suspension G-spectrum functor.

Corollary 1.5. *If R is a commutative S-algebra, then $F(EG_+, I_{U^G}^U R)$ is a commutative S_G-algebra. If R is q-cofibrant, then $F(EG_+, I_{U^G}^U R)$ represents R-Borel* *cohomology on based G-spaces X. Precisely,*

$$[\Sigma^\infty(EG\times_G X)_+, R]^* \cong [\Sigma^\infty_G X_+, F(EG_+, I^U_{U^G}R)]^*_{S_G}.$$

Proof. On the left, we understand maps in the classical stable homotopy category. On the right, we understand maps in the derived category of S_G-modules, which is equivalent to the Lewis-May stable homotopy category of G-spectra; we denote maps in the latter category by $[-,-]^*_G$. The first of the following isomorphisms is standard (e.g. [5, 0.3, 0.7]). The second follows from the natural isomorphism of functors $i_*\circ\Sigma^\infty \cong \Sigma^\infty_G$, the commutation of the suspension G-spectrum functor with smash products with G-spaces, and adjunction. The third is a direct consequence of Corollary 1.4 and the cited equivalence of categories.

$$\begin{aligned}[\Sigma^\infty(EG\times_G X)_+, R]^* &\cong [i_*\Sigma^\infty(EG\times X)_+, i_*R]^*_G\\ &\cong [\Sigma^\infty_G X_+, F(EG_+, i_*R)]^*_G\\ &\cong [\Sigma^\infty_G X_+, F(EG_+, I^U_{U^G}R)]^*_{S_G}. \quad \square\end{aligned}$$

Benson and Greenlees apply this with R taken to be the Eilenberg-MacLane spectrum Hk associated to a commutative ring k. Such Eilenberg-MacLane spectra are commutative S-algebras by multiplicative infinite loop space theory [9] and the theory of [4].

2. Formal results on change of universe

All change of universe functors are obtained as examples of twisted half-smash products. Let U and U' be G-universes and let $\mathscr{I}(U,U')$ be the space of linear isometries $U \longrightarrow U'$, with G acting by conjugation. Let A be a G-space and let $\alpha : A \longrightarrow \mathscr{I}(U,U')$ be a G-map. The twisted half-smash product is a functor $G\mathscr{S}U \longrightarrow G\mathscr{S}U'$, written $A\ltimes E$ on objects; this is an abuse of notation since the functor depends on α and not just A. Different α give rise to equivalent functors on passage to derived categories. See [4, I§§2,3] for a summary of the properties of this functor and [8, Ch. VI] for details of its construction. A new and simpler construction has been obtained recently by Cole [2].

A G-linear isometry $f : U \longrightarrow U'$ may be regarded as a G-map $\{*\} \longrightarrow \mathscr{I}(U,U')$. The corresponding twisted half-smash product functor is denoted $f_* : G\mathscr{S}U \longrightarrow G\mathscr{S}U'$. The existence of such a G-linear isometry ensures that $\mathscr{I}(U,U')$ is a G-contractible G-space [8, II.1.5], and it follows that different choices of f give rise to equivalent functors on passage to stable homotopy categories. These are the standard change of universe functors. However, there is a more canonical choice, namely $\mathscr{I}(U,U')\ltimes E$. For reasonable G-spectra, namely tame ones, the G-equivalence $\{*\} \longrightarrow \mathscr{I}(U,U')$ determined by f induces a homotopy equivalence of G-spectra $f_*E \longrightarrow \mathscr{I}(U,U')\ltimes E$ [4, I.2.5]; see also [2].

We now sketch the equivariant versions of the basic definitions of [4, I, II]. The details are identical with those given there. Fix a universe U. There is a monad $\mathbb{L}$ on the category $G\mathscr{S}U$ such that $\mathbb{L}E = \mathscr{I}(U,U)\ltimes E$. The unit and product of $\mathbb{L}$ are induced by the inclusion of the identity isometry in $\mathscr{I}(U,U)$ and by the composition product $\mathscr{I}(U,U)\times\mathscr{I}(U,U) \longrightarrow \mathscr{I}(U,U)$. An $\mathbb{L}$-spectrum is an algebra over the monad $\mathbb{L}$, and a map of $\mathbb{L}$-spectra is a map of algebras over $\mathbb{L}$.

Given two $\mathbb{L}$-spectra M and N, their operadic smash product is the $\mathbb{L}$-spectrum

$$M\wedge_{\mathscr{L}} N = \mathscr{I}(U\oplus U, U)\ltimes_{\mathscr{I}(U,U)\times\mathscr{I}(U,U)} (M\wedge N).$$

Here $\wedge$ on the right is the external smash product $G\mathscr{S}U \times G\mathscr{S}U \longrightarrow G\mathscr{S}(U \oplus U)$. The construction is made precise by a coequalizer diagram based on the evident right action of $\mathscr{I}(U,U) \times \mathscr{I}(U,U)$ on $\mathscr{I}(U \oplus U, U)$ and a left action of $\mathscr{I}(U,U) \times \mathscr{I}(U,U)$ on $M \wedge N$ induced by the actions of $\mathscr{I}(U,U)$ on M and N; the details are just like those in the following definition. The term "operadic" refers to the operad $\mathscr{L}$ in the category of G-spaces whose jth G-space is $\mathscr{L}(j) = \mathscr{I}(U^j, U)$. The operadic smash product is associative and commutative, and there is a natural weak equivalence $\lambda : S_U \wedge_{\mathscr{L}} M \longrightarrow M$.

An S_U-module is an $\mathbb{L}$-spectrum M for which λ is an isomorphism, and a map of S_U-modules is a map of $\mathbb{L}$-spectra between them. For S_U-modules M and N, $M \wedge_{\mathscr{L}} N$ is again an S_U-module and is denoted $M \wedge_{S_U} N$. This smash product of S_U-modules is commutative, associative, and unital, with unit S_U. That is, the category $G\mathscr{M}_{S_U}$ of S_U-modules is symmetric monoidal under its smash product. We obtain the homotopy category $hG\mathscr{M}_{S_U}$ by identifying homotopic maps of S_U-modules, and we obtain the derived category $G\mathscr{D}_{S_U}$ by adjoining formal inverses to the maps of S_U-modules that are weak equivalences as maps of G-spectra . This is made rigorous by CW-approximation. We emphasize that all of this applies to any G-universe U.

With motivation exactly as in the definition of the smash product over $\mathscr{L}$, this leads us inexorably to the following version of our change of universe functors.

Definition 2.1. Fix universes U and U' and write $\mathbb{L}$ and $\mathbb{L}'$ for the respective monads in $G\mathscr{S}U$ and $G\mathscr{S}U'$ and $\mathscr{L}$ and $\mathscr{L}'$ for the respective operads of G-spaces. For an $\mathbb{L}$-spectrum M, define an $\mathbb{L}'$-spectrum $I_U^{U'} M$ by

$$I_U^{U'} M = \mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} M.$$

That is, $I_U^{U'} M$ is the coequalizer displayed in the diagram

$$\mathscr{I}(U,U') \ltimes (\mathscr{I}(U,U) \ltimes M) \underset{\mathrm{id}\ltimes\xi}{\overset{\gamma\ltimes\mathrm{id}}{\rightrightarrows}} \mathscr{I}(U,U') \ltimes M \longrightarrow I_U^{U'} M.$$

Here $\xi : \mathscr{I}(U,U) \ltimes M \longrightarrow M$ is the given action of $\mathbb{L}$ on M. We regard $\mathscr{I}(U,U') \times \mathscr{I}(U,U)$ as a space over $\mathscr{I}(U,U')$ via the composition product

$$\gamma : \mathscr{I}(U,U') \times \mathscr{I}(U,U) \longrightarrow \mathscr{I}(U,U'),$$

and there results a natural isomorphism

$$\mathscr{I}(U,U') \ltimes (\mathscr{I}(U,U) \ltimes M) \cong (\mathscr{I}(U,U') \times \mathscr{I}(U,U)) \ltimes M.$$

This makes sense of the map $\gamma \ltimes \mathrm{id}$ in the diagram. The required left action of $\mathscr{I}(U',U')$ on $I_U^{U'} M$ is induced by the composition product

$$\gamma : \mathscr{I}(U',U') \times \mathscr{I}(U,U') \longrightarrow \mathscr{I}(U,U'),$$

which induces a natural map of coequalizer diagrams on passage to twisted half-smash products.

We need the following easy observation, in which we do not restrict to universes.

Lemma 2.2. *Let U, U', and U'' be G-inner product spaces. Assume that either U is nonequivariantly isomorphic to U' or U' is nonequivariantly isomorphic to U''.*

Then the diagram

$$\mathscr{I}(U',U'')\times\mathscr{I}(U',U')\times\mathscr{I}(U,U') \underset{\mathrm{id}\times\gamma}{\overset{\gamma\times\mathrm{id}}{\rightrightarrows}} \mathscr{I}(U',U'')\times\mathscr{I}(U,U') \xrightarrow{\ \gamma\ } \mathscr{I}(U,U'')$$

is a split coequalizer of spaces and therefore a coequalizer of G-spaces. Thus

$$\mathscr{I}(U,U'')\cong\mathscr{I}(U',U'')\times_{\mathscr{I}(U',U')}\mathscr{I}(U,U').$$

Proof. Define maps

$$h:\mathscr{I}(U,U'')\longrightarrow\mathscr{I}(U',U'')\times\mathscr{I}(U,U')$$

and

$$k:\mathscr{I}(U',U'')\times\mathscr{I}(U,U')\longrightarrow\mathscr{I}(U',U'')\times\mathscr{I}(U',U')\times\mathscr{I}(U,U')$$

as follows. If $s:U\longrightarrow U'$ is an isomorphism, define

$$h(f)=(f\circ s^{-1},s)\quad\text{ and }\quad k(g',g)=(g',g\circ s^{-1},s).$$

If $t:U'\longrightarrow U''$ is an isomorphism, define

$$h(f)=(t,t^{-1}\circ f)\quad\text{ and }\quad k(g',g)=(t,t^{-1}\circ g',g).$$

In the first case, $\gamma\circ h=\mathrm{id}$, $(\mathrm{id}\times\gamma)\circ k=\mathrm{id}$, and $(\gamma\times\mathrm{id})\circ k=h\circ\gamma$. In the second case, $\gamma\circ h=\mathrm{id}$, $(\gamma\times\mathrm{id})\circ k=\mathrm{id}$, and $(\mathrm{id}\times\gamma)\circ k=h\circ\gamma$. In either case, this proves that we have a split coequalizer of spaces. Since coequalizers of G-spaces are created in the underlying category of spaces, it follows that we have a coequalizer of G-spaces, although not necessarily a split one. □

Let $G\mathscr{S}U[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra. Write Σ^∞_U for the suspension G-spectrum functor from the category $G\mathscr{T}$ of based G-spaces to $G\mathscr{S}U$. This functor takes values in $G\mathscr{S}U[\mathbb{L}]$ and in fact in $G\mathscr{M}_{S_U}$ [4, II.1.2].

Proposition 2.3. *Let U, U', and U'' be G-universes. Consider the functors*

$$I_U^{U'}:G\mathscr{S}U[\mathbb{L}]\longrightarrow G\mathscr{S}U'[\mathbb{L}']\quad\textit{ and }\quad\Sigma^\infty_U:G\mathscr{T}\longrightarrow G\mathscr{S}U[\mathbb{L}].$$

(i) *$I_U^{U'}\circ\Sigma^\infty_U$ is naturally isomorphic to $\Sigma^\infty_{U'}$.*
(ii) *$I_{U'}^{U''}\circ I_U^{U'}$ is naturally isomorphic to $I_U^{U''}$.*
(iii) *I_U^U is naturally isomorphic to the identity functor.*

Therefore the functor $I_U^{U'}$ is an equivalence of categories with inverse $I_{U'}^U$. Moreover, the functor $I_U^{U'}$ is continuous and satisfies $I_U^{U'}(M\wedge X)\cong(I_U^{U'}M)\wedge X$ for $\mathbb{L}$-spectra M and based G-spaces X. In particular, it is homotopy preserving, and $I_U^{U'}$ and $I_{U'}^U$ induce inverse equivalences of homotopy categories.

Proof. For (i), we view $G\mathscr{T}$ as the category of "G-spectra indexed on $\{0\}$" and have that $(\Sigma^\infty_U)(X)$ is isomorphic to $\mathscr{I}(\{0\},U)\ltimes X$. By [4, I.2.2(ii)] and the previous lemma, we find

$$(I_U^{U'}\circ\Sigma^\infty_U)(X)\cong[\mathscr{I}(U,U')\times_{\mathscr{I}(U,U)}\mathscr{I}(\{0\},U)]\ltimes X\cong\mathscr{I}(\{0\},U')\ltimes X\cong(\Sigma^\infty_{U'})(X).$$

For (ii), [4, I.2.2(ii)] and the previous lemma give that

$$(I_{U'}^{U''}\circ I_U^{U'})(M)\cong[\mathscr{I}(U',U'')\times_{\mathscr{I}(U',U')}\mathscr{I}(U,U')]\ltimes_{\mathscr{I}(U,U)}M\cong I_U^{U''}M.$$

Part (iii) is trivial since the relevant coequalizer splits. The topologies on the Hom sets of our categories are discussed in [4, VII§2]. The claimed continuity is easily checked, and commutation with smash products with spaces follows from the analogous property of twisted half-smash products [4, I.2.2(iv)]. □

These conclusions pass from $\mathbb{L}$-spectra to $G\mathscr{M}_{S_U}$-modules to give the following elaboration of Theorem 1.1. Remember that the smash product of S_U-modules is their smash product as $\mathbb{L}$-spectra.

Theorem 2.4. *The following statements hold.*

(i) $I_U^{U'} S_U$ *is canonically isomorphic to* $S_{U'}$.

(ii) *For* $\mathbb{L}$*-spectra* M *and* N*, there is a natural isomorphism*

$$\omega : I_U^{U'}(M \wedge_{\mathscr{L}} N) \cong (I_U^{U'} M) \wedge_{\mathscr{L}'} (I_U^{U'} N).$$

(iii) *The following diagram commutes for all* $\mathbb{L}$*-spectra* M*:*

$$\begin{tikzcd} I_U^{U'}(S_U \wedge_{\mathscr{L}} M) \arrow[rr, "\omega"] \arrow[dr, "I_U^{U'}\lambda"'] & & S_{U'} \wedge_{\mathscr{L}'} (I_U^{U'} M) \arrow[dl, "\lambda"] \\ & I_U^{U'} M. & \end{tikzcd}$$

(iv) M *is an* S_U*-module if and only if* $I_U^{U'} M$ *is an* $S_{U'}$*-module.*

Therefore the functors $I_U^{U'}$ *and* $I_{U'}^{U}$ *restrict to inverse monoidal equivalences of categories between* $G\mathscr{M}_{S_U}$ *and* $G\mathscr{M}_{S_{U'}}$ *that induce inverse monoidal equivalences of categories between* $hG\mathscr{M}_{S_U}$ *and* $hG\mathscr{M}_{S_{U'}}$.

Proof. The isomorphism in (i) is obtained by applying (i) of the previous proposition to the space S^0. For (ii), [4, I.2.2(ii)], Lemma 2.2, and a slight generalization of its analog [4, I.5.4] give that

$$\begin{aligned} & I_U^{U'}(M \wedge_{\mathscr{L}} N) \\ & \qquad = \mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} [\mathscr{I}(U^2,U) \ltimes_{\mathscr{I}(U,U)^2} (M \wedge N)] \\ & \qquad \cong [\mathscr{I}(U,U') \times_{\mathscr{I}(U,U)} \mathscr{I}(U^2,U)] \ltimes_{\mathscr{I}(U,U)^2} (M \wedge N) \\ & \qquad \cong \mathscr{I}(U^2,U') \ltimes_{\mathscr{I}(U,U)^2} (M \wedge N) \end{aligned}$$

and

$$\begin{aligned} & (I_U^{U'} M) \wedge_{\mathscr{L}'} (I_U^{U'} N) \\ & \qquad = \mathscr{I}((U')^2,U') \ltimes_{\mathscr{I}(U',U')^2} (\mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} M) \wedge (\mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} N) \\ & \qquad \cong [\mathscr{I}((U')^2,U') \times_{\mathscr{I}(U',U')^2} \mathscr{I}(U,U')^2] \ltimes_{\mathscr{I}(U,U)^2} (M \wedge N) \\ & \qquad \cong \mathscr{I}(U^2,U') \ltimes_{\mathscr{I}(U,U)^2} (M \wedge N). \end{aligned}$$

Now (iii) is an exercise from the definition of λ in [4, I.8.3] and (iv) follows from (iii) since λ is an isomorphism if and only if $I_U^{U'}\lambda$ is an isomorphism. ◻

3. Homotopical results on change of universe

Obviously, the functor $I_U^{U'}$ cannot preserve weak equivalences in general. To obtain a functor $G\mathscr{D}_{S_U} \longrightarrow G\mathscr{D}_{S_{U'}}$, we take an S_U-module M, construct a weak equivalence $\Gamma M \longrightarrow M$, where ΓM is a CW S_U-module, and then apply the functor $I_U^{U'}$. This loses the good formal properties that we have just discussed. Moreover, we see that passage to derived categories in this fashion cannot preserve composition of functors. Since the functor on derived categories induced by I_U^U is still the identity, we cannot have $I_{U'}^U \circ I_U^{U'} \cong I_U^U : G\mathscr{D}_{S_U} \longrightarrow G\mathscr{D}_{S_U}$, as the case $U' = U^G$ makes abundantly clear. The point is that, on the level of derived categories,

$(I^U_{U'} \circ I^{U'}_U)(X)$ means $I^U_{U'}(\Gamma I^{U'}_U \Gamma X)$, and, because of the reapproximation in the middle, this need not and generally will not be equivalent to $I^U_U \Gamma X$.

Suppose given a G-linear isometry $f : U \longrightarrow U'$ and consider the composite

$$\alpha : f_* M \longrightarrow \mathscr{I}(U,U') \ltimes M \longrightarrow \mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} M = I^{U'}_U M,$$

where M is an $\mathbb{L}$-spectrum. As we have already pointed out, the first arrow is a homotopy equivalence when M is tame, for example, when M has the homotopy type of a G-CW spectrum. We shall prove that the second arrow, and therefore α, is a homotopy equivalence for a large class of S_U-modules.

Fix U for the moment. For a G-spectrum X and $j \geq 0$, define an $\mathbb{L}$-spectrum $D_j X$ by

$$D_j X = \mathscr{L}(j) \ltimes_{\Sigma_j} X^j, \quad \text{where} \quad \mathscr{L}(j) = \mathscr{I}(U^j, U).$$

By convention, $D_0 X = S_U$ for any X; $D_1 X = \mathscr{I}(U,U) \ltimes X$ is the free $\mathbb{L}$-spectrum generated by X. If M is a CW $\mathbb{L}$-spectrum, then M is homotopy equivalent to $\mathbb{L}X$ for some G-CW spectrum X [4, I.4.7]. The functor $S_U \wedge_{\mathscr{L}} (?)$ converts $\mathbb{L}$-spectra to S_U-modules [4, II.1.3]. If M is a CW S_U-module, then M is homotopy equivalent to $S_U \wedge_{\mathscr{L}} (\mathbb{L}X)$ for some G-CW spectrum X [4, II.1.9].

We repeat the following definition from [4, VII.6.4].

Definition 3.1. Let $\mathscr{E}_{S_U}$ be the collection of S_U-modules of the form

$$S_U \wedge_{\mathscr{L}} D_j X,$$

where X is any G-spectrum of the homotopy type of a G-CW spectrum and $j \geq 0$. Let $\bar{\mathscr{E}}_{S_U}$ be the closure of $\mathscr{E}_{S_U}$ under finite $\wedge_{S_U}$-products, wedges, pushouts along cofibrations, colimits of countable sequences of cofibrations, and homotopy equivalences.

Clearly all S_U-modules of the homotopy types of G-CW S_U-modules are in $\bar{\mathscr{E}}_{S_U}$. This class also contains an S_U-algebra weakly equivalent to any given S_U-algebra. In fact, there are Quillen model structures on the categories of S_U-algebras and of commutative S_U-algebras. Every S_U-algebra or commutative S_U-algebra is weakly equivalent to one which is q-cofibrant in the relevant model structure. The following result is part of the equivariant version of [4, VII.6.5].

Theorem 3.2. *The underlying S_U-module of a q-cofibrant S_U-algebra or commutative S_U-algebra is in $\bar{\mathscr{E}}_{S_U}$. Therefore any cell module over a q-cofibrant S_U-algebra or q-cofibrant commutative S_U-algebra is in $\bar{\mathscr{E}}_{S_U}$.*

The following result is part of the equivariant version of [4, VII.6.6].

Proposition 3.3. *The underlying G-spectrum of any S_U-module in $\bar{\mathscr{E}}_{S_U}$ has the homotopy type of a G-CW spectrum.*

Proof of Theorem 1.2. We begin by considering an $\mathbb{L}$-spectrum $D_j X$ for a G-CW spectrum X. In this case, we see by Lemma 2.2 that the second arrow in the definition of α is

$$\begin{aligned} \mathscr{I}(U,U') \ltimes (\mathscr{I}(U^j,U) \ltimes_{\Sigma_j} X^j) &\cong [\mathscr{I}(U,U') \times \mathscr{I}(U^j,U)] \ltimes_{\Sigma_j} X^j \\ &\longrightarrow [\mathscr{I}(U,U') \times_{\mathscr{I}(U,U)} \mathscr{I}(U^j,U)] \ltimes_{\Sigma_j} X^j \cong \mathscr{I}(U^j,U') \ltimes_{\Sigma_j} X^j. \end{aligned}$$

The existence of the G-linear isometry $f : U \longrightarrow U'$ ensures that $\mathscr{I}(U^j,U')$ and $\mathscr{I}(U,U') \times \mathscr{I}(U^j,U)$ are universal principal (G,Σ_j)-bundles [8, II.2.11, VII.1.3] and thus that $\gamma : \mathscr{I}(U,U') \times \mathscr{I}(U^j,U) \longrightarrow \mathscr{I}(U^j,U')$ is a $(G \times \Sigma_j)$-homotopy equivalence. It follows from the equivariant version of [4, I.2.5] (see also [2]) that the displayed map is a G-homotopy equivalence. We claim that we can replace D_jX by $S_U \wedge_{\mathscr{L}} D_jX$ in this argument. In fact,

$$S_U \wedge_{\mathscr{L}} D_jX \cong \tilde{\mathscr{L}}(j) \ltimes X^j,$$

where

$$\tilde{\mathscr{L}}(j) = \mathscr{I}(U^2, U) \times_{\mathscr{I}(U,U)\times\mathscr{I}(U,U)} \mathscr{I}(0, U) \times \mathscr{I}(U^j, U).$$

By the equivariant version of [4, XI.2.2],

$$\gamma : \tilde{\mathscr{L}}(j) \longrightarrow \mathscr{L}(j) = \mathscr{I}(U^j, U)$$

is a $G \times \Sigma_j$-homotopy equivalence, and the claim follows. Theorem 1.2 follows in view of the way that $\bar{\mathscr{E}}_{S_U}$ is obtained from the $S_U \wedge_{\mathscr{L}} D_jX$. $\square$

References

[1] D. J. Benson and J. P. C. Greenlees. Commutative algebra for cohomology rings of classifying spaces of compact Lie groups. Preprint. 1995.

[2] M. Cole. Twisted half-smash products. In preparation.

[3] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory. In “A handbook of algebraic topology”, edited by I. M. James. Elsevier Science. 1995.

[4] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. Preprint 1995.

[5] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoirs Amer. Math. Soc. Vol 113, No 543. 1995.

[6] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. In “A handbook of algebraic topology”, edited by I. M. James. Elsevier Science. 1995.

[7] L. G. Lewis, Jr. Change of universe functors in equivariant stable homotopy theory. Preprint, 1995.

[8] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics Volume 1213. 1986.

[9] J. P. May. Multiplicative infinite loop space theory. J. Pure and Applied Algebra 26(1983), 1–69.

[10] J. P. May. Equivariant homotopy and cohomology theory. NSF-CBMS Regional Conference Monograph. Expected to appear in 1996.

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