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Math. Annalen 151, 150—186 (1963)

Group-Like Structures in General Categories II
Equalizers, Limits, Lengths *

Dedicated to Professor Reinhold Baer on the occasion of his sixtieth birthday

By

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Introduction

In this paper we continue the study initiated in [1] in which familiar algebraic concepts are studied within the framework of a general category. In [1] our main emphasis was on the question of the definition of a multiplication in this general setting, on the general categorical form of the group axioms, and on the implications of assuming certain objects in a category to be group-like. In this paper we concern ourselves more with certain auxiliary features and concepts associated traditionally with the category of sets which play an important role in group theory. Our main object in this paper is to define these concepts in a categorical fashion and to show how, with the use of some of these concepts, a generalization of the notion of multiplication may be introduced; in the third paper of the series we will consider more explicitly the relation of these concepts to the assumption of multiplicative structure in certain objects of the category.

The nature of the generalization referred to may be indicated by reference to the category $\mathfrak{G}$ of groups and homomorphisms. A multiplication satisfying[1]) Axiom I (an $\underline{\mathrm{H}}$-structure) in this category is a homomorphism $\mu : G \times G \to G$ such that $\mu\varkappa = \bar{d} : G * G \to G$, where $\varkappa : G * G \to G \times G$ is the canonical map from free (inverse) to direct product and $\bar{d} : G * G \to G$ is the "folding" map $\langle 1, 1\rangle$. It turns out then (see [1] or [2]) that G admits an $\underline{\mathrm{H}}$-structure if and only if it is commutative. It would be idle to attempt to generalize the concept of a multiplication by demanding a map $\mu : G \times G \times \cdots \times G$ (n factors) $\to G$ with $\mu\varkappa = \bar{d}$ since, as may be readily proved in any DI-category, such a multiplication exists if and only if the ordinary multiplication exists. However, we may factorize $\varkappa : G_1 * \cdots * G_n \to G_1 \times \cdots \times G_n$ in $\mathfrak{G}$ as

$$G_1 * \cdots * G_n = G^n \xrightarrow{\varkappa^{n-1}} G^{n-1} \to \cdots \to G^2 \xrightarrow{\varkappa^1} G^1 = G_1 \times \cdots \times G_n$$

(factorization (F) of Theorem 4.4) and demand in the case $G_1 = \cdots = G_n = G$ that there exist a map $\mu : G^{n-1} \to G$ such that $\mu\varkappa^{n-1} = \bar{d}$. Of course, if $n = 2$

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* This research was partly supported by the U. S. Department of Army through its European Research Office.

[1]) This Axiom asserts the existence of two-sided unity.

this is just the original concept of an $\underline{\mathrm{H}}$-structure. It is a satisfactory feature of this notion of an $\underline{\mathrm{H}}_n$-structure that (see [2]) G admits an $\underline{\mathrm{H}}_n$-structure if and only if the nilpotency class of G is less than n. The first two sections of the present paper lay the necessary foundations for the definition of $\underline{\mathrm{H}}_n$-structures in general categories.

We could then develop the study of $\underline{\mathrm{H}}_n$-structures from the point of view of imposing axioms on such structures analogous to those imposed on $\underline{\mathrm{H}}$-($=\underline{\mathrm{H}}_2$)-structures in [1], and of studying the nature of primitive maps of $\underline{\mathrm{H}}_n$-objects. This development (which has been carried out in special cases by Berstein, Hilton and Peterson) is surely worth carrying out in a general category; but in this paper our interest has centred on the narrower question of the *existence* of such structures on given objects of a category. This leads to the notion of the *length* of an object which is defined and investigated in section 5 of the paper.

A further remark about the category $\mathfrak{G}$ may be helpful in illuminating a modification of the notion of length (namely, to that of *weak length*) which is suggested in section 5. Let K be the kernel of $\varkappa : G * G \to G \times G$. Then we may replace the condition that there be a homomorphism μ with $\mu\varkappa = \bar{d} : G * G \to G$ by the condition that $\bar{d}$ annihilate the kernel of $\varkappa$. In the category $\mathfrak{G}$, in which $\varkappa$ is an epimorphism, this condition is, of course, equivalent to the existence of μ; indeed the existence of an $\underline{\mathrm{H}}_n$-structure $\mu : G^{n-1} \to G$ is equivalent to the condition that $\bar{d}$ annihilate the kernel of $\varkappa^{n-1} : G^n \to G^{n-1}$. However, in general categories the two conditions are not equivalent; it remains true that the existence of μ implies the annihilator condition but the opposite implication is not generally valid. We are thus led to a notion of weak length and the implication that the weak length of an object is never greater than its length. The notion of weak length has also proved fruitful in topology.

In order to have the concepts of length and weak length available in abstract *homotopy theory*, it is necessary to introduce a further refinement of the concepts. Broadly speaking we wish to replace the notion of strict equality which appears in the statements $\mu\varkappa^{n-1} = \bar{d}$, $\mu k = 0$ (where $k : K \to G^n$ embeds ker $\varkappa^{n-1}$ in G^n) by that of homotopy. We could approach this question simply by considering a *classifying* functor $\mathfrak{C} \to \mathfrak{C}_h$ which places each map of $\mathfrak{C}$ in an appropriate equivalence class called a homotopy class, demanding only the compatibility of the classification with the law of composition of maps in $\mathfrak{C}$. However, we have preferred to model ourselves on the situation in a topological category and have therefore adopted Kan's notion of a *category with homotopy*, elaborating it only by considering both left and right homotopy systems (see section 6). In this way we have been able to establish, by category-theoretical arguments, the homotopy invariance of the factorization (F) and of its dual (F') under certain very general assumptions on the homotopy systems in question. The notions of length and weak length generalize in an obvious way to those of *homotopy length* and *weak homotopy length*. It should be made clear, however, that the concept of homotopy length for objects of $\mathfrak{C}$ is not just the concept of length for the category $\mathfrak{C}_h$. For the factorization on which the concept of homotopy length is based is carried out in $\mathfrak{C}$, not in $\mathfrak{C}_h$; indeed it may well be [as it is for the category $\mathfrak{T}$ of based spaces (of the based homotopy type of CW-complexes) and based maps] that the factorization is defined in $\mathfrak{C}$ but not in $\mathfrak{C}_h$. Thus the definition involves simultaneously the maps of the category themselves and their homotopy classification[2]).

We have stated that, from the point of view of this paper, the notions of equalizer, intersection, union, and direct and inverse limit discussed in the first two sections are in the nature of necessary preparation for the subsequent notions of length and weak length. However, we believe that they are basic to any development of category theory, and they will recur very frequently and prominently in paper III of the series and in subsequent contributions to abstract homotopy theory. Of course, we make great use of categorical duality throughout this paper so that every notion turns up in twin guises; we have naturally not insisted on always being explicit about dual formulations.

1. Equalizers

Let $\mathfrak{C}$ be an arbitrary category with zero-maps. The concept of left (right) equalizer of a finite collection of maps $f_1, f_2, \ldots, f_n$ between the same objects A, B of $\mathfrak{C}$, which will be introduced in this section, is in close relation to the notions of kernel and intersection (cokernel and union), familiar in various special categories. The definitions are as follows:

Given $f_1, f_2, \ldots, f_n \in H(A, B)$, a map $k: K \to A$ is called a *left equalizer* of $f_1, f_2, \ldots, f_n$ if

(1.1) (i) $f_1 k = f_2 k = \cdots = f_n k$;

(ii) for any X in $\mathfrak{C}$ and $\xi \in H(X, A)$, $f_1 \xi = f_2 \xi = \cdots = f_n \xi$ implies the unique factorization $\xi = k\xi'$, $\xi' \in H(X, K)$.

A map $c: B \to C$ is called a *right equalizer* of $f_1, f_2, \ldots, f_n$ if

(1.2) (i) $cf_1 = cf_2 = \cdots = cf_n$;

(ii) for any Y in $\mathfrak{C}$ and $\eta \in H(B, Y)$, $\eta f_1 = \eta f_2 = \cdots = \eta f_n$ implies the unique factorization $\eta = \eta' c$, $\eta' \in H(C, Y)$.

The definitions are illustrated by the diagram

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\vdots" below] & B \arrow[r, "c"] \arrow[dr, "\eta"'] & C \arrow[d, dashed, "\eta'"] \\ X \arrow[u, dashed, "\xi'"] \arrow[ur, "\xi"'] & & & Y \end{tikzcd}$$

Obviously, left and right equalizers are dual to each other; properties of one of them are obtained from those of the other "by duality" and not always mentioned in the following.

Proposition 1.3. *Left equalizers are monomorphisms, right equalizers epimorphisms*[2a]).

[2]) It happens that the case $n = 2$ is exceptional; H-structures in $\mathfrak{C}_h$ are nothing other than homotopy-$\underline{\mathrm{H}}$-structures in $\mathfrak{C}$.

[2a]) In view of terminological differences appearing in the literature we wish to emphasize that the terms "monomorphism" and "epimorphism" are used in agreement with [9], see also [1]: A map $f: X \to Y$ in the category $\mathfrak{C}$ is a monomorphism if, for all $Z \in \mathfrak{C}$ and maps $g, h: Z \to X$ in $\mathfrak{C}$, $fg = fh$ implies $g = h$; and dually for epimorphisms.

Proof. The second assertion is, by duality, equivalent to the first. To prove the first, consider, for a map $\xi' \in H(X, K)$, $k\xi' \in H(X, A)$; since $f_1 k\xi' = f_2 k\xi' = \cdots = f_n k\xi'$, the factorization $k\xi'$ is unique, i.e., $k\xi' = k\xi''$ implies $\xi' = \xi''$. Thus k is monomorphic.

Note that, conversely, k being monomorphic implies that the factorization in (1.1) (ii) is unique. In the postulate (ii) the uniqueness of ξ' can therefore be replaced by the requirement that k be a monomorphism; and dually, in (1.2), the uniqueness of η' by the requirement that c be an epimorphism.

Proposition 1.4. *If $k: K \to A$ and $k': K' \to A$ are two left equalizers of $f_1, f_2, \ldots, f_n$, there is a unique equivalence $h: K \to K'$ such that $k' = kh$.* (The dual statement for right equalizers is left to the reader.)

Proof. From $f_1 k' = \cdots = f_n k'$ it follows that $k' = kh$, with a unique $h: K' \to K$. Similarly $k = k'h'$, hence $k = kh'h$. Since k is a monomorphism, this implies $h'h = 1$, and in the same way one proves $hh' = 1$, so that h is an equivalence.

The left equalizer of $f_1, f_2, \ldots, f_n$, if it exists, is thus, to the greatest possible extent, uniquely determined; we denote it by $\lambda(f_1, \ldots, f_n)$, and the right equalizer by $\varrho(f_1, \ldots, f_n)$. We shall say that $\mathfrak{C}$ is a *category with left (right) equalizers*, if for any two objects A, B in $\mathfrak{C}$ and any finite collection of maps $f_1, f_2, \ldots, f_n \in H(A, B)$ left (right) equalizers do exist; examples of such categories are given later (§ 3). It will follow from the next proposition that equalizers exist if they exist for any *two* maps $f_1, f_2 \in H(A, B)$. For a single map $f \in H(A, B)$, $\lambda(f)$ is of course simply the identity 1_A of A.

Proposition 1.5. *Let $f_1, f_2, \ldots, f_n \in H(A, B)$, $n \geqq 2$. If the equalizers $\lambda(f_1, \ldots, f_{n-1}) = k$ and $\lambda(f_1 k, f_n k) = l$ exist, then kl is the left equalizer of $f_1, f_2, \ldots, f_n$.*

Proof. In the diagram

$$L \xrightarrow{l} K \xrightarrow{k} A \overset{\longrightarrow}{\underset{\vdots}{\longrightarrow}} B$$

we have $f_1 k = f_2 k = \cdots = f_{n-1} k$ and $f_1 kl = f_n kl$, so that $f_1 kl = f_2 kl = \cdots = f_n kl$. Since l and k are monomorphisms, so is kl. Now, for any $\xi: X \to A$ such that $f_1 \xi = f_2 \xi = \cdots = f_n \xi$ one has $\xi = k\xi': X \to K \to A$, hence $f_1 k\xi' = f_n k\xi'$, and therefore $\xi' = l\xi''$, whence $\xi = kl\xi''$.

We now give a list of various important properties of equalizers, to be used throughout this paper and number III of the series.

(1) $\lambda(f_1, f_2, \ldots, f_n)$ and $\varrho(f_1, f_2, \ldots, f_n)$ are independent of the numbering of the maps. If two of the maps are equal, one of them can be omitted without changing the equalizers.

(2) The equalizer $\lambda(f, 0)$, determined by $f: A \to B$ up to a unique equivalence is usually called the *kernel of* f, written ker f: it is a map $k: K \to A$ such that $fk = 0$ and all ξ with $f\xi = 0$ admit a unique factorization $\xi = k\xi'$.

The equalizer $\varrho(g, 0)$ is usually called the *cokernel of* f, written coker f:, it is a map $c: B \to C$ such that $cf = 0$ and all η with $\eta f = 0$ admit a unique factorization $\eta = \eta' c$.

Thus, in a category with left (right) equalizers any map f has a (unique) kernel k (cokernel c). Kernels are monomorphisms, cokernels epimorphisms. If kernels and cokernels exist, any map $f: A \to B$ can be uniquely factored through the cokernel of its kernel and through the kernel of its cokernel.

(3) An object Z in $\mathfrak{C}$ with $1_Z = 0$ is called a *zero-object*; for such an object Z and for any X in $\mathfrak{C}$, $H(Z, X)$ and $H(X, Z)$ consist each of a single element 0. Any two zero-objects Z, Z' in $\mathfrak{C}$ are equivalent, the equivalence being given by $0: Z \to Z'$.

If for $A \in \mathfrak{C}$ the equalizer $\lambda(1_A, 0) = k: K \to A$ (i.e., the kernel $\ker 1_A$) exists, *K is a zero-object*. Indeed we have

$$k 1_K = k = 1_A k = 0k = 0 = k 0_K ,$$

where 0_K is the zero-map $K \to K$; this implies $1_K = 0_K$. Similarly the equalizer $\varrho(1_A, 0)$, i.e. coker 1_A, yields a zero-object.

Thus existence of left (or right) equalizers in $\mathfrak{C}$ implies existence of zero-objects.

(4) Given maps $f_1, f_2, \ldots, f_n \in H(A, B)$ the set of maps ξ into A such that $f_1\xi = f_2\xi = \cdots = f_n\xi$ is a "*right ideal*" in $\mathfrak{C}$; i.e., it is not empty and contains with any ξ all its right multiples $\xi\varphi$. This ideal may be called the "*left equalizer ideal*" of $f_1, f_2, \ldots, f_n$. It always exists; however, the existence of $\lambda(f_1, f_2, \ldots, f_n) = k$ means (i) *that it is a principal right ideal, and* (ii) *that it is generated by a monomorphism.* The property (ii) implies that the generator is determined up to a canonical equivalence. — Similar remarks apply to the right equalizer $\varrho(f_1, f_2, \ldots, f_n)$ and, in particular, to ker f and coker f; the "kernel ideal" and the "cokernel ideal" always exist and can replace ker f and coker f in various arguments.

It should be remarked that in the category $\mathfrak{T}_h$ (of topological spaces and homotopy classes of maps) equalizer ideals are principal, but do not possess a canonical generator.

(5) A rather trivial generalization of the notion of equalizer will be useful in the following; we formulate it here for left equalizers only. Given a collection of maps $f_{ij}: A \to B_i$, $i = 1, \ldots, m$; $j = 1, \ldots, n_i$ of an object A to several object B_i, we may understand by the left equalizer of the system f_{ij} a map $k: K \to A$ such that

$$f_{i1} k = \cdots = f_{in_i} k , \qquad \text{for } i = 1, \ldots, m$$

and that the factorization property corresponding to (ii) in (1.1) holds.

Proposition 1.6. *If $\mathfrak{C}$ is a* D*-category (a category with direct products) and if in $\mathfrak{C}$ ordinary left equalizers exist, then they also exist in the above generalized sense.*

Proof. By repeating, if necessary, some of the f_{ij} for fixed i, we may assume that all n_i are equal, $n_i = n$, $i = 1, \ldots, m$. Then we write f_j for the maps $\{f_{1j}, f_{2j}, \ldots, f_{mj}\}: A \to B_1 \times B_2 \times \cdots \times B_m$ for $j = 1, \ldots, n$. As one may easily check, the equalizer $\lambda(f_1, f_2, \ldots, f_m): K \to A$ has all the required properties, and is unique. Dually, in an I-category (a category with inverse products) in which ordinary right equalizers exist, they also exist in the generalized sense referring to a system of maps $f_{ij}: A_i \to B$, $i = 1, \ldots, m$; $j = 1, \ldots, n_i$ of several objects A_i into B.

(6) *Naturality.* Equalizers are "natural" (i.e., functors between obvious categories). For let Φ be a map of the diagram

$$A \overset{f_1}{\underset{f_n}{\rightrightarrows}} B \quad \text{into} \quad A' \overset{f'_1}{\underset{f'_n}{\rightrightarrows}} B'$$

given by two maps $\alpha: A \to A'$, $\beta: B \to B'$ such that for each f'_i there is an f_j with $\beta f_j = f'_i \alpha$; then there is a unique map $\varphi: K \to K'$ between the two equalizers such that

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\varphi"'] & A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
K' \arrow[r, "k'"] & A' \arrow[r, shift left] \arrow[r, shift right] & B'
\end{tikzcd}$$

is commutative: $\alpha k = k' \varphi$. The proof is immediate from the definition.

Notice as a special case that a map Φ from the map f to the map f' (that is, a pair of maps α, β such that $f' \alpha = \beta f$) induces a map φ from $\ker f$ to $\ker f'$.

We may formulate the naturality of equalizers more generally in the following proposition; we shall need this extra generality in subsequent applications.

Proposition 1.7. *Assume several maps* $\alpha^{(1)}, \alpha^{(2)}, \ldots: A \to A'$ *and* $\beta^{(1)}, \beta^{(2)}, \ldots: B \to B'$ *given such that*

(i) *for each* f'_i *there are an* $\alpha^{(p)}$, $\beta^{(q)}$ *and* f_j *with* $\beta^{(q)} f_j = f'_i \alpha^{(p)}$, *and*

(ii) $\alpha^{(1)} k = \alpha^{(2)} k = \ldots$, *and* $\beta^{(1)} f_j k = \beta^{(2)} f_j k = \cdots$.

Then there is a unique $\varphi: K \to K'$ *such that* $\alpha^{(p)} k = k' \varphi$ *(for all p).*

For putting $\xi = \alpha^{(p)} k$, we have

$$f'_i \xi = \beta^{(q)} f_j k$$

which is independent of i; hence there is a φ with $\xi = k' \varphi$, and φ is unique.

(7) In applications it will be useful to know that a functor T preserves equalizers. Exactly as for the preservation of direct or inverse products, there is a simple criterion under which this is the case: namely that T possess an *adjoint* S in the sense of Kan [8], cf. [1], § 6. This is made precise in the following statement.

Proposition 1.8. *Let* T *be a covariant functor from the category* $\mathfrak{D}$ *to the category* $\mathfrak{C}$. *If* T *possesses a left-adjoint functor* $S: \mathfrak{C} \to \mathfrak{D}$, *then* T *preserves left equalizers.*

Proof. Let $\eta = \eta_{XY}$ be the adjugant of T and S; i.e., the natural equivalence $H(SX, Y) \to H(X, TY)$ postulated in the definition of left-adjointness (cf. [1], § 6). For simplicity, we consider two maps $f, g \in H(A, B)$ in $\mathfrak{D}$ and assume that they have an equalizer $\lambda(f, g) = k: K \to A$. We have to prove that $Tk: TK \to TA$ is the left equalizer of Tf, $Tg: TA \to TB$ in $\mathfrak{C}$.

We first note that

$$Tf \circ Tk = T(f \circ k) = T(g \circ k) = Tg \circ Tk\,.$$

Furthermore, let $\xi : X \to TA$ in $\mathfrak{C}$ be such that $Tf \circ \xi = Tg \circ \xi$; writing $\eta^{-1}(\xi) = \xi_1 : SX \to A$, we have

$$Tf \circ \xi = Tf \circ \eta(\xi_1) = \eta(f \circ \xi_1) = \eta(g \circ \xi_1)\,,$$

hence $f \circ \xi_1 = g \circ \xi_1$, which implies $\xi_1 = k \circ \xi_1'$ with a unique $\xi_1' \in H(SX, K)$. We put $\eta(\xi_1') = \xi' \in H(X, TK)$ in $\mathfrak{C}$ and obtain the unique factorization

$$\xi = \eta(\xi_1) = \eta(k \circ \xi_1') = Tk \circ \eta(\xi_1') = Tk \circ \xi'.$$

Thus Tk has all the required properties. Notice that the existence of $\lambda(Tf, Tg)$ is not assumed in $\mathfrak{C}$, but *established* from that of $\lambda(f, g)$ in $\mathfrak{D}$.

Dually, a *covariant functor which possesses a right-adjoint preserves right equalizers.* It is easy to formulate and prove the corresponding statements for contravariant functors (left equalizers are transformed into right ones or vice-versa). – We further remark that by the above method one easily proves that a covariant functor T which has a left-adjoint preserves monomorphisms (a right-adjoint, epimorphisms); a contravariant functor transforms one into the other.

In general, a (covariant) functor T will of course not preserve equalizers. However, there is always a *natural transformation τ of $T\lambda$ into λT*, as follows. For two maps $f, g : A \to B$, let $[K; k] = \lambda(f, g)$ and $[\tilde{K}; \tilde{k}] = \lambda(Tf, Tg)$. There is a unique map $\tau_{f,g} : TK \to \tilde{K}$ such that

$$\tilde{k} \circ \tau_{f,g} = Tk : TK \to \tilde{K} \to TA\,.$$

For Tk satisfies $Tf \circ Tk = Tg \circ Tk$ and thus factors uniquely through $\tilde{K}$ as $Tk = \tilde{k} \circ \tau_{f,g}$. – Moreover, if Φ maps f, g into f', g' (cf. (6)), with induced maps $\varphi : K \to K' = \lambda(f', g')$, and $\tilde{\varphi} : \tilde{K} \to \tilde{K}' = \lambda(Tf', Tg')$, one has

$$\tilde{\varphi} \circ \tau_{f,g} = \tau_{f',g'} \circ T_\varphi\,,$$

i.e. a commutative square

$$\begin{tikzcd} TK \arrow[r, "T\varphi"] \arrow[d, "\tau_{f,g}"'] & TK' \arrow[d, "\tau_{f',g'}"] \\ \tilde{K} \arrow[r, "\tilde{\varphi}"] & \tilde{K}' \end{tikzcd}$$

The proof of this is immediate from the definitions.

(8) Equalizers and products. The following statement is an immediate consequence of the definitions (we omit the proof).

Proposition 1.9. *The direct product of left equalizers is the equalizer of the direct product, the inverse product of right equalizers the equalizer of the inverse product.* More precisely, with restriction to the first case and to equalizers of two maps: Given maps $f_i, g_i : A_i \to B_i$, $i = 1, 2$, with left equalizers $\lambda(f_i, g_i)$

$= k_i : K_i \to A_i$, the map $k_1 \times k_2 : K_1 \times K_2 \to A_1 \times A_2$ is the left equalizer of $f_1 \times f_2$ and $g_1 \times g_2 : A_1 \times A_2 \to B_1 \times B_2$.

Remark. The statement is in general not true for the direct product of right equalizers or the inverse product of left equalizers. (It will be shown, however, in paper III of this series, that it holds, in a certain sense, in primitive categories.) Examples:

1) In the category $\mathfrak{G}$ of groups and homomorphisms, the kernel of $f_1 * f_2$: $A_1 * A_2 \to B_1 * B_2$ is in general not the free product $K_1 * K_2$ of the two kernels of f_1 and f_2.

2) In the category M of based sets, the cokernel of $f : A \to B$ is the quotient set $B/f(A)$ of B modulo the relation $f(a) = o$ for all $a \in A$ (cf. § 3). Obviously the cokernel of $f_1 \times f_2 : A_1 \times A_2 \to B_1 \times B_2$ is not in general the set $B_1/f_1(A_1) \times \times B_2/f_2(A_2)$.

(9) *Equalizers of primitive maps* (cf. [1], § 4). Let (A, m_A) and (B, m_B) be $\underline{\mathrm{M}}$-objects in $\mathfrak{C}$, and f, g primitive maps $A \to B$, $k : K \to A$ their left equalizer. In the diagram

$$\begin{tikzcd}
K \times K \arrow[r, "k \times k"] \arrow[d, dashed, "m_K"'] & A \times A \arrow[r, shift left, "f \times f"] \arrow[r, shift right, "g \times g"'] \arrow[d, "m_A"'] & B \times B \arrow[d, "m_B"] \\
K \arrow[r, "k"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

naturality of equalizers (1.7) yields a unique map m_K such that $k m_K = m_A(k \times k)$; in other words, there is a unique $\underline{\mathrm{M}}$-structure in K for which k is primitive. Moreover any of the axioms I, II, IV of $\underline{\mathrm{M}}$-structures (cf. [1], § 4) if valid for m_A is also satisfied by m_K. The three proofs being very similar, we here describe only the case of axiom I: It asserts that $m_A\{1_A, 0\} = m_A\{0, 1_A\} = 1_A$. In the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "\{1,0\}"'] & A \arrow[d, "\{1,0\}"] \\
K \times K \arrow[r, "k \times k"] \arrow[d, "m_K"'] & A \times A \arrow[d, "m_A"] \\
K \arrow[r, "k"] & A
\end{tikzcd}$$

we have $k m_K\{1_K, 0\} = m_A\{1_A, 0\} k = k$; hence, k being a monomorphism, $m_K\{1_K, 0\} = 1_K$. Similarly $m_K\{0, 1_K\} = 1_K$.

We summarize as follows.

Proposition 1.10. *Let $k: K \to A$ be the left equalizer of primitive maps $A \to B$ relative to $\underline{\mathrm{M}}$-structures m_A and m_B. There is a unique* M*-structure m_K in K with respect to which k is primitive. If m_A satisfies axiom I, II or IV, so does m_K.*

With regard to axiom III (existence of inverse) the situation is slightly different; if III is combined with I and II and if all three hold both in A and B, then m_K satisfies I, II and III. In other words (using the terminology of [1], §4).

Proposition 1.11. *If in* (1.10) *both m_A and m_B are $\underline{\mathrm{G}}$-structures, then m_K is also a $\underline{\mathrm{G}}$-structure.*

Proof. Let s_A and s_B be the inverses in A and B. We recall (Prop. 4.16 of [1]) that if $f: A \to B$ is primitive, then $f s_A = s_B f$. Now in the diagram

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, dashed, "s_K"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "s_A"] & B \arrow[d, "s_B"] \\
K \arrow[r, "k"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

naturality of equalizers yields a map s_K with $k s_K = s_A k$. Then

$$0 = k + s_A k = k + k s_K = k(1 + s_K)\,,$$

since k is primitive. Thus $1 + s_K = 0$, since k is a monomorphism, so s_K is the inverse in K.

The duals of (1.10) and (1.11) constitute the analogous statements for right equalizers of primitive maps relative to $\overline{\mathrm{M}}$-structures.

2. Intersections and unions. Limits

The concept of intersection (union) familiar, e.g., in the category of sets will be generalized in such a way that it applies to an arbitrary D-category with left equalizers $\mathfrak{C}$ (I-category with right equalizers). The general notion refers to an arbitrary collection of maps between various objects in $\mathfrak{C}$, but we prefer to formulate it first for the case of two maps α_1, α_2 from two objects A_1, A_2 to the same object B.

Given $\alpha_1: A_1 \to B$ and $\alpha_2: A_2 \to B$, we consider $f_1 = \alpha_1 p_1$ and $f_2 = \alpha_2 p_2$, both $\in H(A_1 \times A_2, B)$, where p_1, p_2 denote as usual the projections of $A_1 \times A_2$ onto A_1 and A_2. The equalizer $\lambda(f_1, f_2) = \lambda(\alpha_1 p_1, \alpha_2 p_2) = k: K \to A_1 \times A_2$ has the properties:

(i) $\alpha_1 p_1 k = \alpha_2 p_2 k$.

(ii) For any $\xi: X \to A_1 \times A_2$ such that $\alpha_1 p_1 \xi = \alpha_2 p_2 \xi$ one has $\xi = k \xi'$ with a unique $\xi': X \to K$.

Writing $k = \{k_1, k_2\}$ and $\xi = \{\xi_1, \xi_2\}$ in components, i.e. $p_1 k = k_1 : K \to A_1$ etc., this can be formulated as follows, and illustrated by the diagram:

$$\begin{tikzcd}
 & & A_1 \arrow[dr, "\alpha_1"] & \\
X \arrow[r, "\xi'"] \arrow[urr, "\xi_1"] \arrow[drr, "\xi_2"'] & K \arrow[ur, "k_1"] \arrow[dr, "k_2"'] & & B \\
 & & A_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

(2.1) (i) $\alpha_1 k_1 = \alpha_2 k_2$.

(ii) $\alpha_1 \xi_1 = \alpha_2 \xi_2$ implies the unique factorization $\xi_1 = k_1 \xi'$, $\xi_2 = k_2 \xi'$.

Given α_1, α_2, a system $[K; k_1, k_2]$ with the properties (2.1) will be called the *intersection of* α_1 *and* α_2. It is determined by α_1 and α_2 up to a unique equivalence of K (cf. Prop. 1.4). In a D-category $\mathfrak{C}$, the uniqueness of the factorization (ii) is equivalent to $\{k_1, k_2\}$ being a monomorphism.

Proposition 2.2. *Let* $[K; k_1, k_2]$ *be the intersection of* $\alpha_1 : A_1 \to B$ *and* $\alpha_2 : A_2 \to B$. *If* α_1 *is a monomorphism, then* $k_2 : K \to A_2$ *is also a monomorphism.*

Proof. Let $\xi', \xi'' : X \to K$ be such that $k_2 \xi' = k_2 \xi''$. From $\alpha_2 k_2 \xi' = \alpha_1 k_1 \xi'$ and the same equation for ξ'' we then conclude that $\alpha_1 k_1 \xi' = \alpha_1 k_1 \xi''$; since α_1 is a monomorphism, this implies $k_1 \xi' = k_1 \xi''$. By (2.1) (ii), $k_j \xi' = k_j \xi''$ for $j = 1, 2$ imply $\xi' = \xi''$; hence k_2 is a monomorphism.

Proposition 2.3. *In a D-category* $\mathfrak{C}$ *the intersection exists for any two maps* $\alpha_1 : A_1 \to B$, $\alpha_2 : A_2 \to B$, *if and only if* $\mathfrak{C}$ *has left equalizers.*

Proof. It has already been shown that left equalizers yield the intersection of α_1 and α_2. Conversely, assume that intersections exist in $\mathfrak{C}$, and let f_1, f_2 be two maps $A \rightrightarrows B$. They have an intersection $[K; k_1, k_2]$ — not to be confounded with the equalizer we are looking for! —, corresponding to the diagram

$$\begin{tikzcd}
 & A \arrow[dr, "f_1"] & \\
K \arrow[ur, "k_1"] \arrow[dr, "k_2"'] & & B \\
 & A \arrow[ur, "f_2"'] &
\end{tikzcd}$$

We now put $k = \{k_1, k_2\} : K \to A \times A$ and consider the intersection $[K'; k_1', k_2']$ of d and k, d being as usual the diagonal map $\{1, 1\} : A \to A \times A$. In the diagram

$$\begin{tikzcd}
 & A \arrow[dr, "d"] & & A \arrow[dr, "f_1"] & \\
K' \arrow[ur, "k_1'"] \arrow[dr, "k_2'"'] & & A \times A \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & B \\
 & K \arrow[ur, "k"'] & & A \arrow[ur, "f_2"'] &
\end{tikzcd}$$

we then have, for $j = 1, 2$,

$$f_j p_j d k_1' = f_j p_j k k_2' ;$$

since $p_j k = k_j$, $p_j d = 1$, this means $f_j k_1' = f_j k_j k_2'$. But $f_1 k_1 = f_2 k_2$, whence

$$f_1 k_1' = f_2 k_1' .$$

In order to show that k_1' is the equalizer of f_1 and f_2, it remains to check that k_1' fulfills (ii) of (1.1). Let $\xi : X \to A$ be such that $f_1 \xi = f_2 \xi$. By (ii) of (2.1) there is a $\xi' : X \to K$ such that $\xi = k_1 \xi' = k_2 \xi'$, which can be written

$$k\xi' = \{k_1, k_2\}\xi' = \{\xi, \xi\} = d\xi\,.$$

Hence, again by (ii) of (2.1), there is a $\xi'' : X \to K'$ such that $\xi = k_1' \xi''$ (and $\xi' = k_2' \xi''$). Thus we have the required factorization $\xi = k_1' \xi''$; and since k is a monomorphism, so is k_1' (Prop. 2.2).

We will also need the following relation between kernels, intersections, and direct products. The proof will be left to the reader.

Proposition 2.4. *Let $\alpha_1 : A \to B_1$ and $\alpha_2 : A \to B_2$ be maps in the D-category $\mathfrak{C}$ with kernels $k_1 : K_1 \to A$ and $k_2 : K_2 \to A$ respectively, and let $[K; l_1, l_2]$ be the intersection of k_1 and k_2. Then the map $k = k_1 l_1 = k_2 l_2 : K \to A$ is the kernel of $\alpha = \{\alpha_1, \alpha_2\} : A \to B_1 \times B_2$.*

We turn rapidly to the dual concept: *The union of two maps $\beta_i : A \to B_i$, $i = 1, 2$ is a system $[C; c_1, c_2]$ of two maps $c_i : B_i \to C$, $i = 1, 2$, such that in the diagram*

$$\begin{tikzcd}
 & B_1 \arrow[dr, "c_1"] \arrow[drr, "\eta_1"] & & \\
A \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & & C \arrow[r, "\eta'"] & Y \\
 & B_2 \arrow[ur, "c_2"'] \arrow[urr, "\eta_2"'] & &
\end{tikzcd}$$

(i) $c_1 \beta_1 = c_2 \beta_2$ and

(ii) for any two maps $\eta_j : B_j \to Y$ with $\eta_1 \beta_1 = \eta_2 \beta_2$ there is a unique $\eta' : C \to Y$ with $\eta_j = \eta' c_j$, $j = 1, 2$.

If the union exists, it is determined up to a unique equivalence of C. In an I-category $\mathfrak{C}$, the uniqueness of η' in (ii) above is equivalent to $c = \langle c_1, c_2 \rangle : B_1 * B_2 \to C$ being an *epimorphism*. The result dual to (2.2) states that if β_1 is an epimorphism, so is c_2; the result dual to (2.3), that in an I-category $\mathfrak{C}$ unions exist for any β_1, β_2 if and only if $\mathfrak{C}$ possesses right equalizers. The dual of (2.4) is a relation between cokernels, unions and inverse products; the precise formulation is left to the reader.

The examples below (§ 3) will show that certain familiar constructions in homotopy theory and in group theory fall under these generalized notions of union and intersection. These examples will also prepare the ground for the later developments generalizing group-like structures to *more than two factors* which constitute a main objective of this paper. However, before passing to these examples, we discuss the general concept of intersection and union for arbitrary collections of maps in $\mathfrak{C}$, called direct and inverse limits, of which equalizers as well as the intersection and union above are special cases; we also investigate their naturality and functorial behaviour.

The collections of maps of $\mathfrak{C}$ for which we define these concepts will be called *aggregates* in $\mathfrak{C}$ or $\mathfrak{C}$-aggregates[3]); a $\mathfrak{C}$-aggregate $\mathscr{A}$ is, by definition, a subcategory of $\mathfrak{C}$ with the understanding that the 0-maps need not belong to $\mathscr{A}$ (but, of course, the identity maps must), and that objects of $\mathfrak{C}$ can be

[3]) A $\mathfrak{C}$-aggregate is essentially a diagram in the sense of KAN [8]. KAN, however, specifies a model category $\mathfrak{V}$ and defines a diagram as a functor $K : \mathfrak{V} \to \mathfrak{C}$. Thus his definition of a limit is formulated quite differently from ours, although the basic idea is the same. However, an important difference arises in that we consider maps between $\mathfrak{C}$-aggregates (see Proposition 2.6) which are much more general than natural transformations of functors $\mathfrak{V} \to \mathfrak{C}$.

repeated arbitrarily often (i.e., $\mathscr{A}$ is a subcategory of $\dot{\mathfrak{C}}$, the category obtained from $\mathfrak{C}$ by arbitrary repetition of objects). Although we are here interested in finite aggregates only, the definitions and general properties are not limited to that case.

Definition 2.5. *The inverse limit* $\varprojlim \mathscr{A}$ *of a* $\mathfrak{C}$*-aggregate* $\mathscr{A}$ *is a system* $[K; k_A]$ *consisting of an object* K *of* $\mathfrak{C}$ *and maps* $k_A : K \to A$, *one for each*[4]) $A \in \mathscr{A}$, *such that*

(i) $f k_A = k_B$ *for all* A, B *and* $f : A \to B$ *in* $\mathscr{A}$;

(ii) *If* $\xi_A : X \to A$ *is a system of maps in* $\mathfrak{C}$, *one for each* $A \in \mathscr{A}$, *with* $f\xi_A = \xi_B$ *for all* A, B *and* $f : A \to B$ *in* $\mathscr{A}$, *then there is a unique* $\xi' : X \to K$ *with* $\xi_A = k_A \xi'$ *for all* $A \in \mathscr{A}$.

The *direct limit* $\varinjlim \mathscr{A}$ is defined dually as a system $[C; c_A]$, $c_A : A \to C$ with $c_B f = c_A$ for all maps $f : A \to B$ of $\mathscr{A}$ and with the factorization property corresponding to (ii). — $\varprojlim \mathscr{A}$ and $\varinjlim \mathscr{A}$, if they exist, are unique up to a canonical equivalence of K or C respectively.

Remarks. (1) Let $f_1, f_2, \ldots \in H(A, B)$; the left equalizer $\lambda(f_1, f_2, \ldots)$ and $\varprojlim \mathscr{A}$ of the aggregate $\mathscr{A}$ consisting of A, B, $f_1, f_2, \ldots, 1_A, 1_B$ coincide in the following sense. $k : K \to A$ being the left equalizer, the maps $k_A = k$ and $k_B = f_1 k = f_2 k = \cdots$ constitute $\varprojlim \mathscr{A}$; and conversely, if $\varprojlim \mathscr{A} = [K; k_A, k_B]$, the map $k_A : K \to A$ is the left equalizer of $f_1, f_2, \ldots$

The proof is immediate from the definitions. Similarly right equalizers and $\varinjlim \mathscr{A}$ coincide. In the same sense, for two maps $\alpha_1 : A_1 \to B$, $\alpha_2 : A_2 \to B$, the intersection of α_1 and α_2 and $\varprojlim \mathscr{A}$ of the corresponding aggregate coincide; and for $\beta_1 : A \to B_1$, $\beta_2 : A \to B_2$, the union of β_1 and β_2 and $\varinjlim \mathscr{A}$ of the corresponding $\mathscr{A}$. We will often use the terms "union" and "intersection" for $\varprojlim$ and $\varinjlim$, in agreement with these facts and with standard terminology in particular special cases.

(2) If $\mathscr{A}$ contains a finite number of objects $A_1, \ldots, A_m$, we write k_i for k_{A_i}, $i = 1, \ldots, m$, in $\varprojlim \mathscr{A}$. If $\mathfrak{C}$ is a D-category, the map $\tilde{k} = \{k_1, \ldots, k_m\} : K \to A = A_1 \times \cdots \times A_m$ is obviously a *monomorphism*, and this is equivalent to the uniqueness of ξ' in (ii) above. This statement can be refined in the following sense; we will make use of this refinement later.

Proposition 2.6. *Let the objects of the aggregate* $\mathscr{A}$ *be* $A_1, \ldots, A_m$ *and let* $\varprojlim \mathscr{A} = [K; k_i]$. *Suppose there exists* l, $1 \leq l \leq m$, *such that, to each* $i > l$ *there is a* $j \leq l$ *and a map* $f : A_j \to A_i$ *in* $\mathscr{A}$. *Then* $k = \{k_1, \ldots, k_l\}$ *is a monomorphism.*

Proof. Assume $k\xi' = k\xi''$; this means $k_j \xi' = k_j \xi''$, $j = 1, \ldots, l$. For each $i > l$, choose j and $f : A_j \to A_i$ in $\mathscr{A}$. Then $k_i = f k_j$, so that $k_i \xi' = k_i \xi''$. Thus $k_j \xi' = k_j \xi''$, $j = 1, \ldots, m$, so that $\xi' = \xi''$.

Dually, in an I-category, the map $c = \langle c_1, \ldots, c_m \rangle : A_1 * \cdots * A_m \to C$ connected with $\varinjlim$ is an *epimorphism*, with the same comments as above.

(3) Similar statements are, of course, valid for arbitrary aggregates if we work in a category $\mathfrak{C}$ with infinite direct and inverse products. In the same way, the following theorem could easily be carried over to the infinite case.

[4]) If the object $A \in \mathfrak{C}$ is repeated in $\mathscr{A}$, there is one k_A for each copy of A!

Theorem 2.7. *In a* D*-category* $\mathfrak{C}$, *the inverse limit exists for all finite aggregates if and only if* $\mathfrak{C}$ *admits left equalizers. In an* I*-category* $\mathfrak{C}$ *the direct limit exists for all finite aggregates if and only if* $\mathfrak{C}$ *admits right equalizers.*

Proof. In view of Remark (1) above we only have to prove the if-part. We thus assume that in $\mathfrak{C}$ (finite) left equalizers exist.

Left $\mathscr{A}$ be a finite aggregate, consisting of objects $A_1, \ldots, A_m$ and maps $f_{ij}, f'_{ij}, \ldots : A_i \to A_j$ defined for some values of i and j (including in any case all $i = j$, $f_{ii} = 1$). We put $S = A_1 \times \cdots \times A_m$ and consider $f_{ij} p_i, f'_{ij} p_i, \ldots : S \to \to A_i \to A_j$, for all maps into A_j belonging to $\mathscr{A}$. There is at least *one* such map into A_j, namely p_j. In the diagram

$$\begin{tikzcd}
 & & A_1 \\
K \arrow[r, dashed, "k"] & S \arrow[ur, shift left] \arrow[ur] \arrow[ur, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & A_2 \\
 & & A_m
\end{tikzcd}$$

let $k : K \to S$ be the left equalizer in the slightly generalized sense of (1.7) and $k = \{k_1, \ldots, k_m\}$. Then

$$f_{ij} k_i = f_{ij} p_i k = p_j k = k_j : K \to A_j \,.$$

Hence (i) of definition (2.3) is fulfilled, and the factorization property (ii) follows immediately from that of the equalizers.

The *naturality of limits* will be expressed by the next statement (which we formulate for $\varprojlim$ only). For this we have to consider a map $\Phi : \mathscr{A} \to \mathscr{A}'$ of one $\mathfrak{C}$-aggregate into another, i.e. a system of maps[4a] $\varphi_{AA'}$ given for some $A \in \mathscr{A}$, $A' \in \mathscr{A}$ and such that certain squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi_{AA'}"'] & B \arrow[d, "\varphi_{BB'}"] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

are commutative (f in $\mathscr{A}$, f' in $\mathscr{A}'$). Naturality of limits holds for maps which fulfil special conditions and which we call *essential.* We formulate the precise definition of an essential map $\Phi : \mathscr{A} \to \mathscr{A}'$ in the course of stating the next proposition.

Proposition 2.8. *Let* $\mathscr{A}, \mathscr{A}'$ *be* $\mathfrak{C}$*-aggregates with* $\varprojlim \mathscr{A} = [K; k_A]$, $\varprojlim \mathscr{A}' = [K'; k'_{A'}]$. *If* Φ *is an essential map* $\mathscr{A} \to \mathscr{A}'$, *i.e. if*

(i) *for each* $f' : A' \to B'$ *in* $\mathscr{A}'$, *there is at least one commutative square* $f' \varphi_{AA'} = \varphi_{BB'} f$ with $\varphi_{AA'}, \varphi_{BB'} \in \Phi$; *and*

(ii) $\varphi_{AA'} k_A$ *is independent of* $A \in \mathscr{A}$ *and of* $\varphi_{AA'}$,

then there exists a unique $\varphi : K \to K'$ *such that* $\varphi_{AA'} k_A = k'_{A'} \varphi$ *for all* $A' \in \mathscr{A}'$. —

[4a]) The notation is not intended to imply that there is only one map in Φ from A to A'.

We write $\varphi = \Phi_*$; then for two essential maps $\Phi : \mathscr{A} \to \mathscr{A}', \Phi' : \mathscr{A}' \to \mathscr{A}''$ one has $(\Phi'\Phi)_* = \Phi'_*\Phi_*$.

The statement is illustrated by the diagram

$$\begin{tikzcd} K \arrow[r, dashed, "k_A"] \arrow[d, "\varphi"'] & A \arrow[r, "f"] \arrow[d, "\varphi_{AA'}"'] & B \arrow[d, "\varphi_{BB'}"] \\ K' \arrow[r, "k'_{A'}"] & A' \arrow[r, "f'"] & B' \end{tikzcd}$$

We note that for each $A' \in \mathscr{A}'$ there is at least one $\varphi_{AA'}$: it suffices to take in (i) $A' = B'$, $f' = 1_{A'}$. According to (ii), we thus have, for each A', a well-defined map $\xi_{A'} = \varphi_{AA'}\, k_A : K \to A'$.

Proof of 2.8. The maps $\xi_{A'}$ above fulfil, for any $f' : A' \to B'$ in $\mathscr{A}'$,

$$f'\xi_{A'} = f'\varphi_{AA'}\, k_A = \varphi_{BB'}\, f k_A$$

for some $B \in \mathscr{A}$ and $f : A \to B$; but $f k_A = k_B$ by definition of $\varprojlim \mathscr{A}$, hence

$$f'\xi_{A'} = \varphi_{BB'}\, k_B = \xi_{B'}\,.$$

Therefore, there is a unique $\varphi : K \to K'$ with $\xi_{A'} = k'_{A'}\varphi$, i.e.,

$$\varphi_{AA'}\, k_A = k'_{A'}\,\varphi$$

for all A' and all $\varphi_{AA'}$. — The statement concerning $\Phi'\Phi$ follows from the uniqueness of $(\Phi'\Phi)_*$.

Finally, it need not be emphasized that *functors preserving direct and inverse limits* are of special importance in applications. As before there is a simple sufficient criterion for this property of a functor, namely to admit an adjoint. We formulate the statement explicitly for covariant functors T, leaving the formulation for contravariant functors to the reader [cf. the remarks after the proof of (1.8)]. For a functor T from $\mathfrak{D}$ to $\mathfrak{C}$ and a $\mathfrak{D}$-aggregate $\mathscr{A}$, the images TA and Tf of the objects and maps of $\mathscr{A}$ evidently form a $\mathfrak{C}$-aggregate $T\mathscr{A}$. The following proposition generalizes Proposition 1.8.

Proposition 2.9. *Let T be a covariant functor from $\mathfrak{D}$ to $\mathfrak{C}$. If T possesses a left-adjoint $S : \mathfrak{C} \to \mathfrak{D}$ (a right-adjoint), then T preserves inverse limits (direct limits). More precisely, let $\mathscr{A}$ be a $\mathfrak{D}$-aggregate having an inverse limit $\varprojlim \mathscr{A} = [K; k_A]$; if T has a left-adjoint then $T\mathscr{A}$ has in $\mathfrak{C}$ an inverse limit $\varprojlim T\mathscr{A} = [TK; Tk_A]$.*

Proposition 2.9 is easily proved directly (for a D-category $\mathfrak{D}$ with left equalizers it can, of course, be deduced from the preservation of direct products and from (1.6), together with the fact that $\varprojlim$ can be expressed by left equalizers).

In general, covariant functors T will, of course, not preserve inverse limits. However, there is always a *natural transformation* τ of $T\varprojlim$ into $\varprojlim T$, as follows.

Proposition 2.10. *Let T be covariant functor $\mathfrak{D} \to \mathfrak{C}$ and $\mathscr{A}$ an aggregate in $\mathfrak{D}$. There is a unique map $\tau_{\mathscr{A}}$ of $T \varprojlim \mathscr{A} = [TK;\, Tk_A]$ into $\varprojlim T\mathscr{A} = [\tilde{K};\, \tilde{k}_{TA}]$, i.e., a map $\tau_{\mathscr{A}} : TK \to \tilde{K}$ such that $\tilde{k}_{TA} \circ \tau_{\mathscr{A}} = Tk_A$ for all $A \in \mathscr{A}$. For any essential map Φ of $\mathscr{A}$ into a $\mathfrak{D}$-aggregate $\mathscr{A}'$, with $\Phi_* = \varphi : K \to K'$ and $(T\Phi)_* = \tilde{\varphi} : \tilde{K} \to \tilde{K}'$, one has $\tilde{\varphi} \circ \tau_{\mathscr{A}} = \tau_{\mathscr{A}'} \circ T\varphi$.*

Proof. For all maps $f_{AB}: A \to B$ of $\mathscr{A}$, we have $Tf_{AB} \circ Tk_A = Tk_B$; hence there is a unique $\tau_{\mathscr{A}} : TK \to \tilde{K}$ such that $\tilde{k}_{TA} \tau_{\mathscr{A}} = Tk_A$ for all A. To prove the second part of the statement, let $\Phi : \mathscr{A} \to \mathscr{A}'$ be given by maps $\varphi_{AA'}$, $A \in \mathscr{A}$, $A' \in \mathscr{A}'$; for each of these $\varphi_{AA'}$ we have

$$\varphi_{AA'} \circ k_A = k'_{A'} \circ \varphi$$

and

$$T\varphi_{AA'} \circ \tilde{k}_{TA} = \tilde{k}'_{TA'} \circ \tilde{\varphi} .$$

Now $\tilde{k}'_{TA'} \circ \tilde{\varphi} \circ \tau_{\mathscr{A}} = T\varphi_{AA'} \circ \tilde{k}_{TA} \tau_{\mathscr{A}} = T\varphi_{AA'} \circ Tk_A = Tk'_{A'} \circ T\varphi = \tilde{k}'_{TA'} \circ \tau_{\mathscr{A}'} \circ T\varphi$, for all $A' \in \mathscr{A}$, which implies.

$$\tilde{\varphi} \circ \tau_{\mathscr{A}} = \tau_{\mathscr{A}'} \circ T\varphi .$$

In a similar way, one establishes the following useful relation. Let R and T be covariant functors $\mathfrak{D} \to \mathfrak{C}$, ϱ and τ the corresponding transformations of $R \varprojlim$ into $\varprojlim R$ and of $T \varprojlim$ into $\varprojlim T$ respectively, and b a natural transformation of R into T. For a $\mathfrak{D}$-aggregate $\mathscr{A}$, we obtain from b a map $b_{\mathscr{A}} : R\mathscr{A} \to T\mathscr{A}$ given by $b_A : RA \to TA$ for all $A \in \mathscr{A}$. This map $b_{\mathscr{A}}$ is essential; let $(b_{\mathscr{A}})_*$ be the induced map of $\varprojlim R\mathscr{A}$ into $\varprojlim T\mathscr{A}$.

Proposition 2.11. *Given covariant functors $R, T : \mathfrak{D} \to \mathfrak{C}$ and a natural transformation b of R into T, one has, for any $\mathfrak{D}$-aggregate $\mathscr{A}$, $(b_{\mathscr{A}})_* \circ \varrho_{\mathscr{A}} = \tau_{\mathscr{A}} \circ b_{\varprojlim \mathscr{A}}$:*

$$\begin{tikzcd}
R \lim \mathscr{A} \arrow[r, "b_{\lim \mathscr{A}}"] \arrow[d, "\varrho_A"'] & T \lim \mathscr{A} \arrow[d, "\tau_{\mathscr{A}}"] \\
\lim R\mathscr{A} \arrow[r, "(b_{\mathscr{A}})_*"'] & \lim T\mathscr{A}
\end{tikzcd}$$

In particular, if $\mathfrak{D} = \mathfrak{C}$ and $R = I$ (identity functor), one has

$$(b_{\mathscr{A}})_* = \tau_{\mathscr{A}} \circ b_{\varprojlim \mathscr{A}} .$$

We finally note that the analogues of (1.9) – (1.10) hold for inverse limits (direct limits): The direct product of $\varprojlim$ is $\varprojlim$ of the direct product maps. And $\varprojlim$ of an aggregate $\mathscr{A}$ consisting of primitive maps, relative to $\underline{\mathrm{M}}$-structures, is primitive; i.e., $[K;\, k_A]$ being the limit, there is a unique $\underline{\mathrm{M}}$-structure in K for which the k_A are primitive; etc. These facts can again be established directly, or deduced from (1.9)–(1.10) together with the fact that direct products of primitive maps (rel. to $\underline{\mathrm{M}}$-structures) are primitive.

3. Examples of intersections and unions

Example 1. The category $\mathfrak{S}$ of based sets.

The left equalizer $\lambda(f_1, f_2, \ldots, f_n)$ of an arbitrary collection of element maps $f_j : A \to B$ is the "coincidence set" of these maps; i.e., the subset K of A consisting of those elements $a \in A$ for which $f_1(a) = f_2(a) = \cdots = f_n(a)$. More precisely, $\lambda(f_1, f_2, \ldots, f_n) = k$ is the embedding map $K \to A$. In particular, the kernel $\ker f = \lambda(f, 0)$ of $f : A \to B$ is the subset of A sent by f into the base element $o \in B$.

The right equalizer $\varrho(f_1, f_2, \ldots, f_n)$ is the quotient set C of B modulo the equivalence relation $[f_1(a) = f_2(a) = \cdots = f_n(a)$ for all $a \in A]$; more precisely, the natural epimorphism of B onto C. The cokernel of $f : A \to B$ is the quotient set of B modulo the relation $[f(a) = o$ for all $a \in A]$.

The "intersection" $[K; k_1, k_2]$ of two maps $\alpha_1 : A_1 \to B$, $\alpha_2 : A_2 \to B$ in $\mathfrak{S}$ is the subset K of $A_1 \times A_2$ consisting of those pairs (a_1, a_2), $a_1 \in A_1$, $a_2 \in A_2$ for which $\alpha_1(a_1) = \alpha_2(a_2)$, together with the two maps $k_j : K \to A_j$, $j = 1, 2$, given by $(a_1, a_2) \to a_j$. If in particular α_1, α_2 are embedding maps $A_1 \subset B$, $A_2 \subset B$, then there is a one-to-one map of K onto the ordinary intersection set $A_1 \cap A_2$, under which K can be identified with $A_1 \cap A_2$, the maps k_1, k_2 corresponding to the embeddings into A_1 and A_2:

$$\begin{tikzcd} & A_1 \arrow[dr, "\alpha_1"] & \\ A_1 \cap A_2 \arrow[ur, "k_1"] \arrow[dr, "k_2"'] & & B \\ & A_2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

The "union" $[C; c_1, c_2]$ of two maps $\beta_1 : A \to B_1$ and $\beta_2 : A \to B_2$ is the quotient set C of $B_1 \vee B_2$ (the union of B_1 and B_2 with identified base element o) modulo the relation $[\beta_1(a) = \beta_2(a)$ for all $a \in A]$, together with the two natural maps $c_j : B_j \to B_1 \vee B_2 \to C$, $j = 1, 2$. If β_1 and β_2 are embedding maps $A \subset B_1$ and $A \subset B_2$, then C can be identified with the ordinary union $B_1 \cup B_2$, and c_j with the embeddings $B_j \subset B_1 \cup B_2$, $j = 1, 2$:

$$\begin{tikzcd} & B_1 \arrow[dr, "c_1"] & \\ A \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & & B_1 \cup B_2 . \\ & B_2 \arrow[ur, "c_2"'] & \end{tikzcd}$$

Example 2. The category $\mathfrak{T}$ of based topological spaces and continuous maps.

Left and right equalizers, and unions and intersections in $\mathfrak{T}$ are, of course, described as in $\mathfrak{S}$, except for the induced topologies to be taken into account (subset topology; identification topology). More specifically, if B_1, B_2 are subsets of a space B, α_j the embedding maps of B_j in B and β_j the embedding maps of $B_1 \cap B_2$ into B_j, $j = 1, 2$, then $B_1 \cap B_2$ is the "intersection" of α_1 and α_2 and there is a one-to-one map of the "union" of β_1 and β_2 onto $B_1 \cup B_2$; this map is a homeomorphism if, for example, B_1 and B_2 are both closed (or open) sets in B.

It may be worthwile to add the following remarks on the intersections and unions thus obtained.

(3.1) Consider the diagram for the intersection $[K; k_1, k_2]$ of α_1 and α_2:

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_2"'] & A_1 \arrow[d, "\alpha_1"] \\ A_2 \arrow[r, "\alpha_2"] & B \end{tikzcd}$$

If α_1 is a fibre map with fibre F, then so is $k_2 : K \to A_2$. The map k_2 is usually called "the fibre map with base space A_2 *induced* from the fibre map $\alpha_1 : A_1 \to B$ by the map α_2 of A_2 into B". We recall from above that K is the subspace of $A_1 \times A_2$ consisting of all pairs (a_1, a_2) with $\alpha_1(a_1) = \alpha_2(a_2)$, and that $k_2(a_1, a_2) = a_2$.

(3.2) Consider the diagram for the union $[C; c_1, c_2]$ of β_1 and β_2:

$$\begin{tikzcd} A \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] & B_1 \arrow[d, "c_1"] \\ B_2 \arrow[r, "c_2"'] & C \end{tikzcd}$$

If β_2 is a cofibre map with cofibre F, then so is $c_1 : B_1 \to C$. The map c_1 is called "the cofibre map with cobase B_1 induced from the cofibre map $\beta_2 : A \to B_2$ by the map β_1 of A into B_1." We recall from above that C is the quotient space of $B_1 \vee B_2$ modulo the relation $[\beta_1(a) = \beta_2(a)$ for all $a \in A]$, i.e. obtained from $B_1 \vee B_2$ by identifying images of the same $a \in A$.

We omit the proofs of (3.1) and (3.2) which are straightforward. We mention the important example of (3.1) where $A_1 = EB$ is the path-space of B, the fibre of α_1 being the loop space ΩB, and k_2 *the fibre map with fibre* ΩB induced by $\alpha_2 : A_2 \to B$; and the example of (3.2) where B_2 is the cone CA over A, the cofibre of β_2 the suspension ΣA, and c_1, *the cofibre map with cofibre* ΣA *induced by* $\beta_1 : A \to B_1$ (C is usually called the space obtained by *attaching a cone* CA to B_1, by means of the map $\beta_1 : A \to B_1$).

Example 3. The category $\mathfrak{G}$ of groups and homomorphisms.

The left equalizer $\lambda(f_1, f_2, \ldots, f_n)$ of the homomorphisms $f_j : A \to B$, $j = 1, \ldots, n$, is (the embedding in A of) the *coincidence subgroup* of the f_j. The right equalizer $\varrho(f_1, f_2, \ldots, f_n)$ is the reduction of B modulo the normal subgroup generated by the elements $f_j(a) f_{j+1}(a)^{-1}$, for all $a \in A$ and all values $0 < j < n$.

The "intersection" $[K; k_1, k_2]$ of two homomorphisms $\alpha_1 : A_1 \to B$, $\alpha_2 : A_2 \to B$ is the "subdirect" product $K \subset A_1 \times A_2$ consisting of all (a_1, a_2) with $\alpha_1(a_1) = \alpha_2(a_2)$, together with the projections into A_1 and A_2. The "union" $[C; c_1, c_2]$

of two homomorphisms $\beta_1 : A \to B_1$, $\beta_2 : A \to B_2$ is the "free product of B_1 and B_2 with amalgamated images of A," i.e., the free product $B_1 * B_2$ modulo the normal subgroup generated by all $\beta_1(a)\,\beta_2(a)^{-1}$, $a \in A$.

Example 4. The category $\mathfrak{A}$ of Abelian groups and homomorphisms. Equalizers are the same as in $\mathfrak{G}$. The intersection of $\alpha_1 : A_1 \to B$ and $\alpha_2 : A_2 \to B$ is the same as in $\mathfrak{G}$; the union of $\beta_1 : A \to B_1$ and $\beta_2 : A \to B_2$ is equal to the *Abelianized* union in $\mathfrak{G}$.

A series of examples of a more general nature is given in the next section, first for arbitrary categories and then in the special cases $\mathfrak{S}$ and $\mathfrak{G}$.

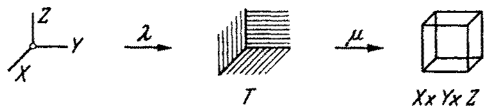
4. The canonical factorization

In a DI-category $\mathfrak{C}$, there is for any system $X_1, X_2, \ldots, X_n$ of n objects a "canonical map" $\varkappa$ from the inverse product $X_1 * X_2 * \cdots * X_n$ to the direct product $X_1 \times X_2 \times \cdots \times X_n$, given by

$$\varkappa = \langle \iota_1, \ldots, \iota_n \rangle = \{\pi_1, \ldots, \pi_n\}$$

(cf. [1], 3.34). Here ι_j denotes the map $\{0, \ldots, 1, \ldots, 0\} : X_j \to X_1 \times \cdots \times X_n$, all components being 0 except for the j^{th}; and π_j the map $\langle 0, \ldots, 1, \ldots, 0 \rangle :$ $: X_1 * X_2 * \cdots * X_n \to X_j$, $j = 1, \ldots, n$. In the present section, this map $\varkappa$ will be factorized, in two dual ways, through a sequence of "*intermediate products*" between the inverse and the direct which will be obtained as unions (or dually intersections) of maps arising naturally from the construction of direct and inverse products.

We first illustrate the factorization in a simple example. In the category $\mathfrak{T}$ of based topological spaces and continuous maps, let X, Y, Z be three spaces; the subspace of $X \times Y \times Z$ consisting of those points with at least two "coordinates" equal to the base-point o is the inverse product $X * Y * Z$, and its natural embedding in $X \times Y \times Z$ is the canonical map $\varkappa$. The subspace T of $X \times Y \times Z$ consisting of those points with at least one "coordinate" equal to the base-point o contains $X * Y * Z$; if the embedding of $X * Y * Z$ in T is denoted by λ and the embedding of T in $X \times Y \times Z$ by μ, we obviously have $\varkappa = \mu\lambda$:



In view of the generalization we have in mind we give a different description of T. For that purpose, we consider the various embeddings given in components

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[ddr, "a'"'] & X \times Y \arrow[dr, "d"] & \\
Y \arrow[ur, "b"] \arrow[r, "b'"'] & Y \times Z \arrow[r, "e"] & X \times Y \times Z \\
Z \arrow[uur, "c"] \arrow[r, "c'"'] & X \times Z \arrow[ur, "f"'] &
\end{tikzcd}$$

by $a = \{1, 0\}$, $a' = \{1, 0\}$, $b = \{0, 1\}$ etc.; $d = \{p_1, p_2, 0\}$, $e = \{0, p_1, p_2\}$, $f = \{p_1, 0, p_2\}$. Then we take the "union" of the six maps a, a', b, b', c, c';

more precisely, these maps together with the respective identities form an aggregate $\mathscr{A}$, and we take $\varinjlim \mathscr{A} = [T; \xi_1, \xi_2, \ldots, \xi_6]$ where $\xi_1 : X \times Y \to T$, $\xi_2 : Y \times Z \to T$, $\xi_3 : X \times Z \to T$ play the essential rôle, while $\xi_4 : X \to T$ is just $\xi_1 a = \xi_3 a'$ etc. It is easily seen that this T is the same as above; indeed it is the quotient space of $(X \times Y) * (Y \times Z) * (X \times Z)$ modulo the relation $[a(x) = a'(x)$ for all $x \in X$, $b(y) = b'(y)$ for all $y \in Y$, etc.], i.e., it is obtained by identifying $X \times o$ in $X \times Y$ and $X \times Z$ etc.

This definition of T applies, of course, to any DI-category $\mathfrak{C}$ with right equalizers (direct limits), the maps a, a', b, b', c, c' and their union being well-defined. Moreover, since $da = fa' = \{1, 0, 0\}$, $db = eb' = \{0, 1, 0\}$, $ec = fc' = \{0, 0, 1\}$, there is unique $\mu : T \to X \times Y \times Z$ such that

$$\mu \xi_1 = d; \quad \mu \xi_2 = e, \quad \mu \xi_3 = f .$$

If we denote by λ the map $\langle \xi_1 a, \xi_2 b', \xi_3 c' \rangle : X * Y * Z \to T$, we have

$$\mu \lambda = \langle \mu \xi_1 a, \mu \xi_2 b', \mu \xi_3 c' \rangle = \langle da, eb', fc' \rangle = \langle \iota_1, \iota_2, \iota_3 \rangle = \varkappa ,$$

and we thus obtain the factorization of $\varkappa$ as $X * Y * Z \xrightarrow{\lambda} T \xrightarrow{\mu} X \times Y \times Z$.

We now pass to the general case of n objects $X_1, X_2, \ldots, X_n$ of $\mathfrak{C}$ and to the description of various "intermediate products". For completeness this description will be given in more generality than would be necessary for the factorization of $\varkappa$ and for the applications made in the next section. We prefer to introduce the notations and the whole set-up in the situation *dual* to that given above (for $n = 3$); this factorization, in $\mathfrak{G}$ is referred to in the introduction.

Notations. $X_1, \ldots, X_n$ are n fixed objects of $\mathfrak{C}$, numbered in a definite way; they may be different from each other or equal. We use "strings", i.e. ordered subsets of the ordered set of integers $N = (1, 2, \ldots, n)$; let $J = (j_1, j_2, \ldots, j_r)$ be such a string, its number r of elements being denoted by $|J|$. For any $J \subset N$, $0 \leq |J| \leq n$, let X_J be the inverse product $X_{j_1} * \cdots * X_{j_r}$ (the zero-object if $r = 0$); X_N is $X_1 * \cdots * X_n$. For any $K \subset J$, $K = (k_1, k_2, \ldots, k_s)$, let π^J_K be the natural map $X_J \to X_K$ given in components by

$$\pi^J_K = \langle a_1, \ldots, a_r \rangle \quad \text{with} \quad a_\nu = \begin{cases} 0 & \text{if } j_\nu \notin K \\ q_\mu & \text{if } j_\nu = k_\mu \in K \end{cases}$$

π^J_J is the identity map of X_J.

Lemma 4.1. *Let $L \subset K \subset J$; then $\pi^K_L \pi^J_K = \pi^J_L$.*

Proof. We write in components

$$\pi^J_K = \langle a_1, \ldots, a_r \rangle, \quad \pi^K_L = \langle b_1, \ldots, b_s \rangle, \quad \pi^J_L = \langle c_1, \ldots, c_r \rangle ,$$

$r = |J| \geq s = |K|$. Then

$$\pi^K_L \pi^J_K = \langle b_1, \ldots, b_s \rangle \langle a_1, \ldots, a_r \rangle = \langle \langle b_1, \ldots, b_s \rangle a_1, \ldots, \langle b_1, \ldots, b_s \rangle a_r \rangle .$$

By definition we have $\langle b_1, \ldots, b_s \rangle a_\nu = 0$ if $j_\nu \notin K$, $= \langle b_1, \ldots, b_s \rangle q_\mu = b_\mu$ if $j_\nu = k_\mu \in K$; the second possibility gives $\langle b_1, \ldots, b_s \rangle a_\nu = 0$ if $k_\mu \notin L$, $= q_\lambda$ if $k_\mu = l_\lambda \in L$. Hence

$$\langle b_1, \ldots, b_s \rangle a_\nu = \begin{cases} 0 & \text{if } j_\nu \notin L \\ q_\lambda & \text{if } j_\nu = l_\lambda \in L \end{cases}$$

which is precisely c_ν, $\nu = 1, \ldots, r$.

In a DI-category $\mathfrak{C}$ with left equalizers, *intermediate-direct products* of $X_1, \ldots, X_n$ are now defined as intersections (inverse limits) of the maps π_J^K, as follows.

Definition 4.2. Let $X_1, \ldots, X_n$ be objects of $\mathfrak{C}$; r, s two integers with $0 \leqq s \leqq r \leqq n$; $\mathscr{A}^{r,s}$ the aggregate consisting of all X_J and π_K^J with $|J|, |K| = r$ or $\leqq s$. The *intermediate-direct product of* $X_1, \ldots, X_n$ *of type* r, s is defined as $\varprojlim \mathscr{A}^{r,s} = [X^{r,s}; \xi_J^{r,s}]$. It is an object $X^{r,s}$ together with maps $\xi_J = \xi_J^{r,s}$: $: X^{r,s} \to X_J$, for $|J| = r$ or $\leqq s$, such that

$$\pi_K^J \, \xi_J = \xi_K \quad \text{for all } K \subset J \text{ with } |J|, \qquad |K| = r \text{ or } \leqq s,$$

and that any system of maps $\eta_J : Y \to X_J$ with the property $\pi_K^J \, \eta_J = \eta_K$ can be uniquely factored through $X^{r,s}$ as $\eta_J = \xi_J \eta'$.

From the naturality theorem for limits (2.8) it is easily seen that $X^{r,s}$ is a *covariant functor* of the n variables $X_1, \ldots, X_n$. We will often call $X^{r,s}$ the intermediate-direct product, leaving the ξ_J implicit. Clearly all the maps ξ_J are determined by those with $|J| = r$. The map $\tilde{\xi}$ of $X^{r,s}$ into the direct product of all X_J, with components ξ_J, is a *monomorphism*, and so is the map ξ of $X^{r,s}$ into the direct product of the X_J with $|J| = r$ (cf. Prop. 2.6). We note some special cases:

(1) $X^{n,r}$, for any r, $0 \leqq r \leqq n$, is the inverse product $X_N = X_1 * \cdots * X_n$.

(2) $X^{r,0}$, for any r, $0 \leqq r \leqq n$, is the direct product of all X_J with $|J| = r$. For example, $X^{1,0} = X_1 \times \cdots \times X_n$; $X^{0,0}$ is the zero-object.

(3) $X^{r,r}$, for any r, $0 \leqq r \leqq n$, is the same as $X^{r,r-1}$, by the definition above. It is the $\varprojlim$ of the aggregate consisting of all π_K^J with $|J|$ and $|K| \leqq r$. These products will appear in the factorization of $\varkappa : X^{n,n} \to X^{1,1}$; *we will simply write* X^r *for* $X^{r,r}$, $r = 0, \ldots, n$. Thus $X^n = X_N = X_1 * \cdots * X_n$, $X^1 = X_1 \times \cdots \times X_n$.

Proposition 4.3. *Let* $X^{r,s}$ *and* $X^{r',s'}$ *be two intermediate-direct products with* $r \leqq s'$. *Then there is a unique map* $\gamma : X^{r',s'} \to X^{r,s}$ *such that* $\xi_J \, \gamma = \pi_J^{J'} \, \xi'_{J'}$ *for all* $J' \supset J$, $|J'| = r'$, $|J| = r$, *where* ξ_J *stands for* $\xi_J^{r,s}$, $\xi'_{J'}$ *for* $\xi_{J'}^{r',s'}$:

$$\begin{tikzcd} X^{r',s'} \arrow[r, "\xi'_{J'}"] \arrow[d, "\gamma"'] & X_{J'} \arrow[d, "\pi_J^{J'}"] \\ X^{r,s} \arrow[r, "\xi_J"] & X_J \end{tikzcd}$$

The maps γ *are transitive, i.e., if* $\gamma' : X^{r'',s''} \to X^{r',s'}$ *and* $\gamma'' : X^{r'',s''} \to X^{r,s}$, $s'' \geqq r'$, $s' \geqq r$, *then* $\gamma'' = \gamma \, \gamma'$.

Proof. The assertion follows immediately from the naturality of inverse limits, Proposition 2.8: γ is induced by the map Φ of $\mathscr{A}^{r',s'}$ to $\mathscr{A}^{r,s}$ given by all $\pi_J^{J'}$, $|J'| = r'$ or $\leqq s'$, $|J| = r$ or $\leqq s$. For each map π_K^J of $\mathscr{A}^{r,s}$ there is at least one commutative square $\pi_K^J \, \pi_J^{J'} = \pi_K^{K'} \, \pi_{K'}^{J'}$, and the condition (ii) of (2.8) is fulfilled, since $\pi_J^{J'} \, \xi_{J'}^{r',s'} = \xi_J^{r',s'}$ for all $|J'| = r'$, $|J| = r$ or $\leqq s$, (since $r \leqq s'$, $\pi_J^{J'}$ belongs to $\mathscr{A}^{r',s'}$!) is independent of J'. Hence Φ is essential, and thus there is a unique map $\gamma = \Phi_*$ such that $\xi_J \gamma = \pi_J^{J'} \xi'_{J'}$ for all $\xi'_{J'}$ belonging to $\mathscr{A}^{r',s'}$ and all ξ_J to $\mathscr{A}^{r,s}$; in particular, for all $|J'| = r'$, $|J| = r$, and this is sufficient to ensure uniqueness of γ. The "transitivity" follows from the functorial behaviour of Φ_* [last part of (2.8)].

For example, the map γ of X^n to X^1 is the canonical map $\varkappa$. In the following, we pay special attention to the maps $\gamma: X^r \to X^{r-1}$ and denote them by $\varkappa^{r-1}$, $r = 2, 3, \ldots, n$. According to the transitivity in (4.3), the composition $\varkappa^1\varkappa^2 \ldots \varkappa^{n-1}$ is precisely $\varkappa$.

Theorem 4.4. *The canonical map $\varkappa$ of $X^n = X_1 * \cdots * X_n$ into $X^1 = X_1 \times \cdots \times X_n$ can be factored through the intermediate-direct products X^r of $X_1, \ldots, X_n$ as $\varkappa = \varkappa^1\varkappa^2 \ldots \varkappa^{n-1}$, $n \geqq 2$:*

$$\text{(F)} \qquad X^n \xrightarrow{\varkappa^{n-1}} X^{n-1} \xrightarrow{\varkappa^{n-2}} X^{n-2} \longrightarrow \cdots \longrightarrow X^2 \xrightarrow{\varkappa^1} X^1 .$$

By dualizing the definition (4.2), intermediate-inverse products ${}^{r,s}X$, $0 \leqq r \leqq s \leqq n$ of $X_1, \ldots, X_n$ are obtained, as follows. For a string $J \subset N$, $J = (j_1, \ldots, j_r)$, the direct product ${}_JX = X_{j_1} \times \cdots \times X_{j_r}$ is considered, and for $J \subset K$ the map ${}^K_J\iota : {}_JX \to {}_KX$ with components $\{a_1, \ldots, a_s\}$, $s = |K|$, is given by $a_\nu = 0$ if $k_\nu \notin J$, $= p_\mu$ if $k_\nu = j_\mu \in J$; ${}^K_J\iota$ "embeds" ${}_JX$ in ${}_KX$. For $0 \leqq r \leqq s \leqq n$, ${}^{r,s}X$ is then defined as the direct limit $\varinjlim {}^{r,s}\mathscr{A}$ of the aggregate ${}^{r,s}\mathscr{A}$ consisting of all ${}^K_J\iota$ with $|J|$ and $|K| = r$ or $\geqq s$; ${}^{r,s}X$ is provided with maps ${}_K\xi = {}^{r,s}_K\xi : {}_KX \to {}^{r,s}X$ for all $|K| = r$ or $\geqq s$, satisfying ${}_K\xi \circ {}^K_J\iota = {}_J\xi$ (and with the unique factorization property). The map $\tilde{\xi}$ of the inverse product of all ${}_JX$ to ${}^{r,s}X$, with components ${}_J\xi$ is an epimorphism, and so is the map ξ of the inverse product of all ${}_JX$ with $|J| = r$ to ${}^{r,s}X$. By naturality, transitive maps $\gamma : {}^{r,s}X \to {}^{r',s'}X$ are uniquely defined for $s \leqq r'$; the map ${}^{r-1,r-1}X \to {}^{r,r}X$, in short ${}^{r-1}X \to {}^rX$, is written ${}^{r-1}\varkappa$, and the map ${}^1X \to {}^nX$ is again the canonical map $\varkappa$ of ${}^1X = X_1 * \cdots * X_n$ into ${}^nX = X_1 \times \cdots \times X_n$.

Theorem 4.5. *The canonical map $\varkappa$ of ${}^1X = X_1 * \cdots * X_n$ into ${}^nX = X_1 \times \cdots \times X_n$ can be factored through the intermediate-inverse products ${}^rX$ of $X_1, \ldots, X_n$ as $\varkappa = {}^{n-1}\varkappa \ldots {}^2\varkappa\, {}^1\varkappa$, $n \geqq 2$*

$$\text{(F}'\text{)} \qquad {}^1X \xrightarrow{{}^1\varkappa} {}^2X \longrightarrow \cdots \longrightarrow {}^{n-2}X \xrightarrow{{}^{n-2}\varkappa} {}^{n-1}X \xrightarrow{{}^{n-1}\varkappa} {}^nX .$$

If we return to the special case of three objects X, Y, Z dealt with at the beginning of this section, we note that ${}^2X$ is just T. We will generally adopt the notation $\overline{T}$ or $\overline{T}(X_1, \ldots, X_n)$ for ${}^{n-1}X$ associated with n objects $X_1, \ldots, X_n$, and $\underline{T}$ for X^{n-1} in the dual construction.

Remark. Both factorizations can be completed by adding a map $\varkappa^0$ (or ${}^0\varkappa$ respectively); namely, $\varkappa^0: X^1 \to X^0$ and ${}^0\varkappa : {}^0X \to {}^1X$. We recall that X^0 and ${}^0X$ are zero-objects.
In the case $n = 1$ only these maps $\varkappa^0$ and ${}^0\varkappa$ are available.

As an example we describe the factorizations (F) and (F′) in the *category of based sets* $\mathfrak{S}$. An element a of the product set $A_1 \times A_2 \times \cdots$ will as usual be given by its "components" $a = (a_1, a_2, \ldots)$, $a_i \in A_i$; an element of $A_1 * A_2 * \cdots$ by $a = (a_1, a_2, \ldots)$ with at most one $a_i \neq o \in A_i$, i.e., we use the construction of $A_1 * A_2 * \cdots$ as a subset of $A_1 \times A_2 \times \cdots$, $\varkappa$ being simply the inclusion map.

Given n sets $X_1, \ldots, X_n$ the object ${}^rX$ of the factorization (F') is the set consisting of those n-tuples $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ with at most r components $\neq o$, the maps ${}^rk : {}^rX \to {}^{r+1}X$ being the corresponding inclusions (cf. the explicit description, for $n = 3$, at the beginning of § 4; the proof for general n is similar and left to the reader). Thus the factorization $\varkappa = {}^{n-1}\varkappa \ldots {}^2\varkappa\, {}^1\varkappa$ is *non-trivial* (for $n > 2$) and all ${}^r\varkappa$ are *monomorphisms.*

The dual factorization (F) however, is *trivial* in $\mathfrak{S}$ in the following sense:

(4.6) *In* $\varkappa = \varkappa^1 \varkappa^2 \ldots \varkappa^{n-1}$ *each* $\varkappa^r$, $r > 1$, *is an equivalence* $X^{r+1} \to X^r$. *Hence all* X^r, $2 \leqq r \leqq n$ *can be identified with* $X_1 * \cdots * X_n$, *and in that sense* $\varkappa^1 = \varkappa$.

Proof of (4.6). From the definitions it follows easily that X^r is the following subset of the direct product of all X_J, $|J| = r$: An element of X_J, $J = (j_1, \ldots, j_r)$, is an r-tuple $(x_{j_1}, \ldots, x_{j_r})$, $x_{j_\nu} \in X_{j_\nu}$, with *at most one* $x_j \neq o$; an element of X^r is a system of such r-tuples, one for each J,

$$((x_{j_1}, \ldots, x_{j_r}), \quad (x'_{k_1}, \ldots, x'_{k_r}), \ldots)$$

with components $x_{j_\nu} = x'_{k_\lambda}$ if $j_\nu = k_\lambda$. In other words, an element of X^r is a system of r-tuples $(x_{j_1}, \ldots, x_{j_r})$ of the type described, one for each J with $|J| = r$, of the form

$$((x_{j_1}, \ldots, x_{j_r}), \quad (x_{k_1}, \ldots, x_{k_r}), \ldots)$$

In this system $x_{j_1} \neq o$, for example, implies $x_{j_\nu} = o$ for $\nu = 2, \ldots, r$, and more generally $x_{k_\nu} = o$ for all $k_\nu \neq j_1$. I.e., if $x_{j_1} \neq o$, all x_i, $i \neq j_1$, are o *(provided that* $r \geqq 2$*)*. Thus the system is simply given by an n-tuple *of representatives* $x_i \in X_i$, $i = 1, \ldots, n$ *with at most one* $x_i \neq o$, i.e., an element of $X_1 * \cdots * X_n$. The elements of X^r and those of X^{r+1} are in one-to-one correspondence and differ only by the arrangement (with repetitions) of one and the same n-tuple $x_1, \ldots, x_n$ in the double bracketing $((\ldots), (\ldots), \ldots)$, the correspondence being established precisely by the map $\varkappa^r$.

In the category $\mathfrak{G}$ of groups, the factorizations (F) and (F′) are described explicitly in [2]. It turns out that $\varkappa = \varkappa^1 \varkappa^2 \ldots \varkappa^{n-1}$ is non-trivial and that all $\varkappa^r$ are epimorphisms; and that $\varkappa = {}^{n-1}\varkappa \ldots {}^2\varkappa\, {}^1\varkappa$ is trivial in the sense that all ${}^r\varkappa : {}^rX \to {}^{r+1}X$, $r \geqq 2$, are equivalences (isomorphisms), so that all ${}^rX$ can be identified with $X_1 \times \cdots \times X_n$ and ${}^1\varkappa$ with $\varkappa$. The curious duality between $\mathfrak{S}$ and $\mathfrak{G}$ appearing in these examples will find an explanation in paper III of this series, in connection with primitive categories. — A study of the occurrence of triviality in the factorization of $\varkappa$, for general categories, can be found in [4].

We finally remark that in the category $\mathfrak{T}$ of based topological spaces the situation is exactly the same as in $\mathfrak{S}$, the only additional feature being the natural topologies involved in the constructions and the continuity of the various maps.

§ 5. Length of an object

We consider, in the canonical factorization (F) (Theorem 4.4) of $\varkappa : X_1 * \cdots * X_n \to X_1 \times \cdots \times X_n$ the map $\varkappa^{n-1} : X_1 * \cdots * X_n \to X^{n-1}$ and write σ for $\varkappa^{n-1}$, $\underline{T}$ or $\underline{T}(X_1, \ldots, X_n)$ for X^{n-1}; and similarly, in (F') (Theorem 4.5), τ for ${}^{n-1}\varkappa$ and $\overline{T}$ for ${}^{n-1}X$. Note that for $n = 1$, both $\underline{T}$ and $\overline{T}$ are equal to the zero-object 0 of $\mathfrak{C}$. For $n = 2$, both σ and τ are equal to $\varkappa$ ($\underline{T} = X_1 \times X_2$, $\overline{T} = X_1 * X_2$). We assume now that $X_1 = X_2 = \cdots = X_n = X$, and recall that an $\underline{\mathrm{H}}$-object in $\mathfrak{C}$ is an object X together with a "multiplication" $m: X_1 \times X_2 \to X$ such that $m\varkappa = \langle m\{1, 0\}, m\{0, 1\}\rangle = \langle 1, 1\rangle = \bar{d}$, the folding map $X_1 * X_2 \to X$; and that an $\overline{\mathrm{H}}$-object is an object X with a "comultiplication" $m: X \to X_1 * X_2$ such that $\varkappa m = \{1, 1\} = \underline{d}$, the diagonal map $X \to X_1 \times X_2$. Using $\sigma: X_1 * \cdots * X_n \to \underline{T}$ and $\tau: \overline{T} \to X_1 \times \cdots \times X_n$ above, these concepts of a multiplicative or comultiplicative structure in X are generalized to an arbitrary number $n \geqq 1$ of factors, as follows.

Definition 5.1. *In a* DI-*category* $\mathfrak{C}$ *with left equalizers, an* $\underline{\mathrm{H}}_n$*-structure in* $X \in \mathfrak{C}$ *is a map* $m: \underline{T}(X, \ldots, X) \to X$ *such that* $m\sigma = \bar{d} = \langle 1, \ldots, 1\rangle$:

$$X_1 * \cdots * X_n \xrightarrow{\sigma} \underline{T} \xrightarrow{m} X .$$

In a DI-*category* $\mathfrak{C}$ *with right equalizers, an* $\overline{\mathrm{H}}_n$*-structure in* $X \in \mathfrak{C}$ *is a map* $m: X \to \overline{T}(X, \ldots, X)$ *such that* $\tau m = \underline{d} = \{1, \ldots, 1\}$:

$$X \xrightarrow{m} \overline{T} \xrightarrow{\tau} X_1 \times \cdots \times X_n .$$

An $\underline{\mathrm{H}}_2$-structure m is thus the same as an $\underline{\mathrm{H}}$-structure. An $\underline{\mathrm{H}}_n$-object is a pair consisting of an object X and an $\underline{\mathrm{H}}_n$-structure m in X. Similarly for the dual concepts. An $\underline{\mathrm{H}}_1$-object is an object X and a map $m: 0 \to X$ such that $X \xrightarrow{\sigma} 0 \xrightarrow{m} X$ is equal to $\bar{d} = 1$, i.e., an object with $1 = 0$, and dually; *thus the only* $\underline{\mathrm{H}}_1$*-objects and* $\overline{\mathrm{H}}_1$*-objects in* $\mathfrak{C}$ *are the zero-objects.*

In the following we concentrate on the question whether a given object X admits an $\underline{\mathrm{H}}_n$-structure (an $\overline{\mathrm{H}}_n$-structure), rather than on the $\underline{\mathrm{H}}_n$- and $\overline{\mathrm{H}}_n$-objects themselves. A fundamental fact in that context is given by the following theorem.

Theorem 5.2. *For each* $n \geqq 1$, *if* X *admits an* $\underline{\mathrm{H}}_n$*-structure, it also admits an* $\underline{\mathrm{H}}_{n+1}$*-structure.*

The case $n = 1$ is trivial: zero-objects admit $\underline{\mathrm{H}}_n$-structures for all n. We assume $n \geqq 2$; in order to establish the theorem, we refer to the construction of $\underline{T} = X^{n-1}$ with respect to n objects $X_1, \ldots, X_n$ of $\mathfrak{C}$, given in § 4 and to the notations used there such as $N = (1, \ldots, n)$, $J = (j_1, \ldots, j_r) \subset N$, X_J, $\pi^J_K: X_J \to$ $\to X_K$ for $K \subset J$, $\xi_J: \underline{T} \to X_J$ for $|J| = r \leq n - 1$. For the same construction of X^n *with respect to* $n + 1$ *objects* $X_0, X_1, \ldots, X_n$ *of* $\mathfrak{C}$ *with* $X_0 = X_1$, we use the same notations with a dash: $N' = (0, 1, \ldots, n)$, $J' \subset N'$, $X_{J'}$, $\pi^{J'}_{K'}$ for $K' \subset J'$, $\underline{T}' = X^n$, $\xi'_{J'}$: $\underline{T}' \to X_{J'}$, σ': $X_0 * X_1 * \cdots * X_n \to \underline{T}'$. We first prove the following lemma.

Lemma 5.3. *Let* φ *be the map* $\langle q_1, 1\rangle: X_{N'} = X_0 * X_N \to X_N$. *There exists a map* $\psi: \underline{T}' \to \underline{T}$ *such that* $\psi\sigma' = \sigma\varphi$.

Proof. As a first step we define a map $a_J: X_0 \to X_J$, $J \subset N$, by $a_J = 0$ if $j_1 > 1$, $= q_1$ if $j_1 = 1$ (q_1 with respect to $X_J = X_1 * X_{j_2} * \cdots$). Then we take for $K = (k_1, \ldots, k_s) \subset J$ the map

$$\pi^J_K a_J = \langle c_1, \ldots, c_r\rangle a_J: X_0 \to X_J \to X_K ,$$

where $c_\nu = 0$ if $j_\nu \notin K$, $= q_\lambda$ if $j_\nu = k_\lambda \in K$. Thus $\pi_K^J a_J = 0$ if $j_1 > 1$, $= c_1$ if $j_1 = 1$; i.e., in the second case, $= 0$ if $k_1 > 1$, $= q_1$ if $k_1 = 1$. This means $\pi_K^J a_J = 0$ if $k_1 > 1$, $= q_1$ if $k_1 = 1$, which is precisely the map a_K:

$$\pi_K^J a_J = a_K \,. \tag{5.4}$$

Next we take the aggregates $\mathscr{A}$ and $\mathscr{A}'$ from which $\underline{T}$ and $\underline{T}'$ are defined as $\varprojlim$, and the map Φ of $\mathscr{A}'$ to $\mathscr{A}$ given by maps $\varphi_J^{J'}: X_{J'} \to X_J$, as follows: $\varphi_J^{J'}$ is defined for $J' = (0, j_1, \ldots, j_r) = 0 \cup J$ only; as a map of $X_{J'} = X_0 * X_J$ to X_J it is given by $\varphi_J^{J'} = \langle a_J, 1\rangle$. Note that $\varphi_N^{N'} = \varphi$ in the lemma. The maps $\varphi_J^{J'}$ fulfill the conditions (i) and (ii) of the naturality theorem (2.6), i.e., Φ is essential:

$$\begin{tikzcd}
\underline{T}' \arrow[r, "\xi'_{J'}"] \arrow[d, dashed, "\psi"] & X_{J'} \arrow[r, "\pi_{K'}^{J'}"] \arrow[d, "\varphi_J^{J'}"] & X_{K'} \arrow[d, "\varphi_K^{K'}"] \\
\underline{T} \arrow[r, "\xi_J"'] & X_J \arrow[r, "\pi_K^J"'] & X_K
\end{tikzcd}$$

(i) For $J' = 0 \cup J$, $K' = 0 \cup K$, the map $\pi_{K'}^{J'}: X_0 * X_J \to X_0 * X_K$ can be written as $\langle q_1, q_2\pi_K^J\rangle$. For any π_K^J of $\mathscr{A}$, taking these J', K', we find a commutative square

$$\varphi_K^{K'}\pi_{K'}^{J'} = \langle a_K, 1\rangle\langle q_1, q_2\pi_K^J\rangle = \langle a_K, \pi_K^J\rangle : X_0 * X_J \to X_J$$

and

$$\pi_K^J\varphi_J^{J'} = \pi_K^J\langle a_J, 1\rangle = \langle \pi_K^J a_J, \pi_K^J\rangle = \langle a_K, \pi_K^J\rangle$$

by (5.4), hence $\varphi_K^{K'}\pi_{K'}^{J'} = \pi_K^J\varphi_J^{J'}$.

(ii) For a given J, there is only *one* map $\varphi_J^{J'}: X_{J'} \to X_J$. Thus $\varphi_J^{J'}\xi'_{J'}$ depends on J only.

According to (2.6) there is a unique map $\Phi_* = \psi: \underline{T}' \to \underline{T}$ satisfying

$$\xi_J\psi = \varphi_J^{J'}\xi'_{J'}$$

for all J with $|J| \leqq n-1$. Now, by definition of $\sigma: X_1 * \cdots * X_n \to \underline{T}$, we have $\xi_J\sigma = \pi_J^N$ for all J with $|J| \leqq n-1$ and similarly $\xi'_{J'}\sigma' = \pi_{J'}^{N'}$, hence

$$\xi_J\sigma\varphi = \pi_J^N\varphi = \pi_J^N\varphi_N^{N'} = \varphi_J^{J'}\pi_{J'}^{N'} = \varphi_J^{J'}\xi'_{J'}\sigma' = \xi_J\psi\sigma' .$$

By the unique factorization property of $\varprojlim$ this implies $\sigma\varphi = \psi\sigma'$.

Proof of theorem 5.2. In the diagram

$$\begin{tikzcd}
X_0 * X_1 * \cdots * X_n \arrow[r, "\sigma'"] \arrow[d, "\varphi"] & \underline{T}' \arrow[r, "m'"] \arrow[d, "\psi"] & X \arrow[d, "1"] \\
X_1 * \cdots * X_n \arrow[r, "\sigma"] & \underline{T} \arrow[r, "m"] & X
\end{tikzcd}$$

we assume $X_0 = X_1 = \cdots = X_n = X$, and m to be an $\underline{\mathrm{H}}_n$-structure in X, $m\sigma = \bar{d}$. We put

$$m' = m\psi;$$

then $m'\sigma' = m\psi\sigma' = m\sigma\varphi = \bar{d}\varphi = \bar{d}\langle q_1, 1\rangle : X_0 * X_N \to X_N \to X$. This map is equal to $\langle \bar{d}q_1, \bar{d}\rangle = \langle 1, \bar{d}\rangle = \bar{d}' : X_{N'} \to X$, and hence m' is an $\underline{\mathrm{H}}_{n+1}$-structure in X. — Note that m' is obtained functorially from m.

On the basis of theorem 5.2 and its dual, non-negative integers (possibly ∞) $\underline{l}(X)$ and $\bar{l}(X)$, called *lengths*, can now be attached to any $X \in \mathfrak{C}$, as follows.

Definition 5.5. *$\underline{l}(X) < n$ if and only if X admits an $\underline{\mathrm{H}}_n$-structure; $\bar{l}(X) < n$ if and only if X admits an $\overline{\mathrm{H}}_n$-structure.*

$\underline{l}(X) = n$ means $\underline{l}(X) < n + 1$ but $\underline{l}(X)$ not $< n$; and $\underline{l}(X) = \infty$ if there is no integer n such that $\underline{l}(X) < n$. And similarly for $\bar{l}(X)$. In particular, $\underline{l}(X) = 0$ or $\bar{l}(X) = 0$ means that X is a zero-object; $\underline{l}(X) = 1$ means that X admits an $\underline{\mathrm{H}}$-structure ($= \underline{\mathrm{H}}_2$-structure) without being a zero-object, and similarly for $\bar{l}(X) = 1$.

Examples: (1) *In the category $\mathfrak{S}$ of based sets, $\underline{l}(X)$ is always <2* (any set can be given an $\underline{\mathrm{H}}$-structure); *and $\bar{l}(X) = 0$ for a one-element set and otherwise $\bar{l}(X) = \infty$.* — To prove the last assertion, consider, for $n \geq 2$, the map $\tau : \bar{T} \to \to X_1 \times X_2 \times \cdots \times X_n$; here $\bar{T}$ is the subset of the cartesian product consisting of those elements for which at least one coordinate $x_i = o \in X_i$ and τ is the embedding. Then given a map $m : X \to \bar{T}$ with $\tau m = \underline{d}$ ($X_1 = \cdots = X_n = X$), we have $\tau m(x) = (x, x, \ldots, x) \in X_1 \times X_2 \times \cdots \times X_n$; the i-th coordinate being $= o$, we thus have $x = o$. In other words, $\bar{l}(X) < n$ implies $X = o$.

(2) In the category $\mathfrak{T}$ of based topological spaces, $\underline{l}(X)$ is always <3; and $\bar{l}(X) = 0$ for a one-point space and otherwise $\bar{l}(X) = \infty$. — The first assertion follows from the fact that in $\mathfrak{T}$ (as in $\mathfrak{S}$) the factorization (F) is trivial for $n \geq 3$, i.e., $X^r = X^n$, $2 \leq r \leq n$, and $\varkappa^1 = \varkappa$ (see the end of § 4); hence $\underline{T}(X) = X^n$, $\sigma = 1$, so that $m = \bar{d} : X^n \to X$ is an $\underline{\mathrm{H}}_n$-structure for $n \geq 3$. As one knows, there are spaces X which do not admit an $\underline{\mathrm{H}}_2$-structure, i.e., with $\underline{l}(X) = 2$. — The proof of the second assertion is exactly as in example (1) above.

By replacing the strict notion of length by a *homotopy* notion, we may obtain interesting invariants in $\mathfrak{T}$ analogous to $\bar{l}$. The general categorical framework for this refinement of the notion of length is laid in § 6.

(3) In the category $\mathfrak{G}$ of groups, $\underline{l}(X) = 1$ for the non-trivial Abelian groups, and $\bar{l}(X) = 1$ for the non-trivial free groups. As established in [2], $\underline{l}(X)$ is the nilpotency class of the group X, and $\bar{l}(X) \leq 2$ for all groups. [This last assertion is an immediate consequence of the "triviality" of the canonical factorization (F′) for $n > 2$, i.e. $\bar{T} = {}^{n-1}X = X_1 \times \cdots \times X_n$, $\tau = 1$; an $\overline{\mathrm{H}}_n$-structure in X is thus simply given by $m = \underline{d}$.]

Before discussing general properties of the lengths $\underline{l}(X)$ and $\bar{l}(X)$, we introduce further numerical invariants of objects X in a general category, called *weak lengths* and denoted by $\underline{wl}(X)$ and $\overline{wl}(X)$. They are closely related to $\underline{l}(X)$ and $\bar{l}(X)$ respectively but in some cases easier to handle (no structure map m being involved).

As above, we consider for $X_1 = X_2 = \cdots = X_n = X$ the map $\sigma : X_1 * \cdots * X_n \to \underline{T}$; let $\ker\sigma = k : K \to X_1 * \cdots * X_n$ be its kernel. If X admits an $\underline{\mathrm{H}}_n$-structure $m : \underline{T} \to X$ ($m\sigma = \bar{d}$), one has $\bar{d}k = m\sigma k = 0 : K \xrightarrow{k} X_1 * \cdots * X_n \xrightarrow{\bar{d}} X$.

Definition 5.6. $\underline{wl}(X) < n$, *if and only if* $\bar{d}k = 0$.

It is plain that $\underline{l}(X) < n$ implies $\underline{wl}(X) < n$, but that the converse is not true in general.

Proposition 5.7. *If* $\underline{wl}(X) < n$, *then* $\underline{wl}(X) < n + 1$.

This justifies the Definition 5.6 and yields a non-negative integer $\underline{wl}(X)$, possibly ∞, exactly as for $\underline{l}(X)$. The dual definition and proposition yield $\overline{wl}(X)$. Obviously one has

Proposition 5.8. $\underline{wl}(X) \leqq \underline{l}(X)$ *and* $\overline{wl}(X) \leqq \bar{l}(X)$.

Proof of 5.7. We use, in addition to the notations above, those appearing in the proof of Theorem 5.2 and referring to the constructions made for $n + 1$, instead of n, objects $X_0, X_1, \ldots, X_n$. In particular we make use of the map $\varphi : X_0 * X_1 * \cdots * X_n \to X_1 * \cdots * X_n$ and of $\psi : \underline{T}' \to \underline{T}$ such that $\sigma\varphi = \psi\sigma'$ (Lemma 5.3) and $\bar{d}\varphi = \bar{d}' : X_0 * X_1 * \cdots * X_n \to X$. Let $k' : K' \to X_0 * * X_1 * \cdots * X_n$ be the kernel of $\sigma' : X_0 * X_1 * \cdots * X_n \to \underline{T}'$. From $\sigma\varphi k' = \psi\sigma' k' = 0$ we obtain a (unique) $\gamma : K' \to K$ such that $\varphi k' = k\gamma$. Assuming now $\underline{wl}(X) < n$, i.e. $\bar{d}k = 0$, it follows that

$$\bar{d}' k' = \bar{d}\varphi k' = \bar{d}k\gamma = 0 ,$$

i.e., $\underline{wl}(X) < n + 1$.

Remarks. (1) The map $k : K \to X_1 * \cdots * X_n$ on which the definition of weak length rests has been defined as the kernel of $\sigma : X_1 * \cdots * X_n \to \underline{T}$. There is an equivalent description of k without using $\underline{T}$; we give it here in the following terms (referring to the notations used in § 4).

Proposition 5.9. *Let* $h_J : H_J \to X_1 * \cdots * X_n$ *be the kernel of* $\pi_J^N : X_1 * \cdots * X_n \to X_J$, $|J| = n - 1$, *and* $[K; k_J]$ *the intersection (the inverse limit) of all these kernels* h_J. *Then the map* $\tilde{k} : \tilde{K} \to X_1 * \cdots * X_n$ *given by* $\tilde{k} = h_J k_J$ *(independent of* J*) is the kernel of* σ.

Proof. Let P be the direct product of the X_J with $|J| = n - 1$, let $\pi : X_1 * \cdots * X_n \to P$ be the map with components π_J^N and let $\xi : \underline{T} \to P$ be the map with components ξ_J. Then ξ is a monomorphism by Proposition 2.6 and $\xi\sigma = \pi$; thus $\ker\pi = \ker\sigma$, and the proposition follows from Proposition 2.4 (more pricesely, its analogue for n, instead of 2, maps).

(2) Other "kernels" in $X_1 * \cdots * X_n$ can be chosen either by universal procedures or by special ones applying to particular categories; these yield similar numerical invariants. For example, a very general concept of nilpotency class can be introduced in this way (compare [3]).

There are special circumstances under which *weak length and length coincide.* This is obviously the case if $\bar{d}k = 0$ implies the factorization $\bar{d} = m\sigma : X_1 * \cdots * X_n \to \underline{T} \to X$. For example, such a conclusion with a unique m holds *if* σ *is the cokernel of its kernel* k. But it is, of course, not necessary in that connection to assume the existence of cokernels in the category; it would be enough to look at the cokernel ideal of k (the right annihilator)[5]), and the conclusion holds

[5]) The *right* annihilator of a map g is the (left) ideal consisting of those maps h which annihilate g when g stands to the *right* ($hg = 0$).

if it is the principal left ideal generated by σ. Maps with that property are called "normal"[6]); we avoid the introduction of two dual types of normality for arbitrary maps and define only the concept of normal epimorphism (or dually, normal monomorphism) without reference to the existence of kernels, or cokernels, as follows.

Definition 5.10. In an arbitrary category $\mathfrak{C}$, an epimorphism $f: X \to Y$ is called *normal* if the following property holds: Any map h of X into some Z such that $hg = 0$ for all maps g into X with $fg = 0$, can be (uniquely) factored through f:

$$h = h'f : X \to Y \to Z .$$

(I.o.w., if the right annihilator of the left annihilator of f is the ideal generated by f.)

For example, any cokernel is a normal epimorphism. — We do not enunciate the dual definition of a normal monomorphism; any kernel is a normal monomorphism. We note that in the category $\mathfrak{G}$ of groups all epimorphisms are normal (they are cokernels), but not all monomorphisms (the embedding of a non-normal subgroup into a group is not a normal monomorphism); and that in the category $\mathfrak{S}$ of sets all monomorphisms are normal, but not all epimorphisms. This duality between the categories $\mathfrak{G}$ and $\mathfrak{S}$ will again find its explanation in part III of this series.

Theorem 5.11. *If $\sigma: X_1 * \cdots * X_n \to \underline{T}$ is a normal epimorphism, $\underline{wl}(X) = \underline{l}(X)$. If $\tau: \overline{T} \to X_1 \times \cdots \times X_n$ is a normal monomorphism, $\overline{wl}(X) = \overline{l}(X)$.*

We now list some general properties of lengths and weak lengths valid in arbitrary categories (in which the appropriate equalizers exist).

Theorem 5.12. *If Y dominates X, i.e., if there exist maps $f: X \to Y$ and $g: Y \to X$ such that $gf = 1_X$, then $\underline{l}(X) \leqq \underline{l}(Y)$ and $\overline{l}(X) \leqq \overline{l}(Y)$.*

Proof. We write X^n for $X_1 * \cdots * X_n$ with $X_1 = \cdots = X_n = X$, f^n for the map $f * \cdots * f: X^n \to Y^n$, $\underline{T}(X)$ for X^{n-1}, $\underline{T}(f)$ for the map $\underline{T}(X) \to \underline{T}(Y)$ induced by f, σ_X for $\sigma: X^n \to \underline{T}(X)$, $m_Y: \underline{T}(Y) \to Y$ for an $\underline{\mathrm{H}}_n$-structure given in $Y (m_Y \sigma_Y = \overline{d}_Y: Y^n \to Y)$. In the diagram

$$\begin{tikzcd} X^n \arrow[r, "\sigma_X"] \arrow[d, "f^n"] & \underline{T}(X) \arrow[r, "m_X"] \arrow[d, "\underline{T}(f)"] & X \arrow[d, shift right, "f"'] \\ Y^n \arrow[r, "\sigma_Y"] & \underline{T}(Y) \arrow[r, "m_Y"] & Y \arrow[u, shift right, "g"'] \end{tikzcd}$$

the first square is commutative; we put $m_X = g m_Y \underline{T}(f)$. Then

$$m_X \sigma_X = g m_Y \underline{T}(f) \sigma_X = g m_Y \sigma_Y f^n = g \overline{d}_Y f^n .$$

Now $\overline{d}_Y f^n = f \overline{d}_X$ is immediate to verify; hence

$$m_X \sigma_X = g f \overline{d}_X = \overline{d}_X .$$

Thus m_X is an $\underline{\mathrm{H}}_n$-structure in X, i.e., $\underline{l}(Y) < n$ implies $\underline{l}(X) < n$. — The inequality $\overline{l}(X) \leqq \overline{l}(Y)$ is obtained by the dual proof, using the *same* assumption $gf = 1$ (instead of permuting X and Y and making the dual assumption!).

[6]) For the concept of *normality* the reader is referred to [5] and [9].

A similar proof would yield the same inequalities for $\underline{wl}$ and $\overline{wl}$. However, for these weak lengths, we have a stronger result.

Theorem 5.13. *If there exists a map $f\colon X \to Y$ with $\ker f = 0$, then $\underline{wl}(X) \leqq \underline{wl}(Y)$ and $\overline{wl}(X) \leqq \overline{wl}(Y)$.*

Note that $gf = 1$ implies $\ker f = 0$.

Proof of (5.13). We write $\ker \sigma_X = k_X\colon K_X \to X^n$. In the diagram

$$\begin{tikzcd} K_X \arrow[r, "k_X"] \arrow[d, "\gamma"'] & X^n \arrow[r, "\sigma_X"] \arrow[d, "f^n"] & \underline{T}(X) \arrow[d, "\underline{T}(f)"] \\ K_Y \arrow[r, "k_Y"] & Y^n \arrow[r, "\sigma_Y"] & \underline{T}(Y) \end{tikzcd}$$

the second square is commutative. Hence $\sigma_Y f^n k_X = \underline{T}(f)\sigma_X k_X = 0$; thus there is a unique $\gamma\colon K_X \to K_Y$ such that $f^n k_X = k_Y \gamma$. Now we assume $\underline{wl}(Y) < n$, i.e., $\bar{d}_Y k_Y = 0$. Then

$$f\bar{d}_X k_X = \bar{d}_Y f^n k_X = \bar{d}_Y k_Y \gamma = 0 .$$

f having 0-kernel, this implies $\bar{d}_X k_X = 0$, i.e., $\underline{wl}(X) < n$.

Corollary 5.14. *For any A and $B \in \mathfrak{C}$, one has $\underline{l}(A) \leqq \underline{l}(A \times B)$ and $\underline{l}(A) \leqq \underline{l}(A * B)$; the same inequalities hold for $\bar{l}$, $\underline{wl}$ and $\overline{wl}$.*

For A is dominated by $A \times B$ ($p_1\{1, 0\} = 1_A$), and by $A * B$ ($\langle 1, 0\rangle q_1 = 1_A$).

Theorem 5.15. *For any A and $B \in \mathfrak{C}$, one has*

$$\underline{l}(A \times B) = \max(\underline{l}(A), \underline{l}(B))$$
$$\bar{l}(A * B) = \max(\bar{l}(A), \bar{l}(B)) .$$

Proof. We only give the proof of the first assertion. The notations are as above. We assume $\underline{l}(A) < n$ and $\underline{l}(B) < n$, the $\underline{\mathrm{H}}_n$-structure maps being m_A and m_B. In the diagram

$$\begin{tikzcd} (A \times B)^n \arrow[r, "\sigma_{A \times B}"] \arrow[d, "p_1^n"'] & \underline{T}(A \times B) \arrow[r, "m_{A \times B}"] \arrow[d, "T(p_1)"'] & A \times B \arrow[d, "p_1"] \\ A^n \arrow[r, "\sigma_A"] & \underline{T}(A) \arrow[r, "m_A"] & A \end{tikzcd}$$

we put $m_{A \times B} = \{m_A \underline{T}(p_1), m_B \underline{T}(p_2)\}\colon \underline{T}(A \times B) \to A \times B$; thus the diagram is commutative, and so is the analoguous one with B^n, $\underline{T}(B)$, B in the second row, p_1 being replaced by p_2.

Then

$$\begin{aligned} m_{A \times B}\sigma_{A \times B} &= \{m_A \underline{T}(p_1)\sigma_{A \times B}, m_B \underline{T}(p_2)\sigma_{A \times B}\} \\ &= \{m_A \sigma_A p_1^n, m_B \sigma_B p_2^n\} = \{\bar{d}_A p_1^n, \bar{d}_B p_2^n\} \\ &= \{p_1 \bar{d}_{A \times B}, p_2 \bar{d}_{A \times B}\} = \{p_1, p_2\}\bar{d}_{A \times B} = \bar{d}_{A \times B} . \end{aligned}$$

Thus $m_{A \times B}$ is an $\underline{\mathrm{H}}_n$-structure in $A \times B$, and $\underline{l}(A \times B) < n$. Therefore $\underline{l}(A \times B) \leqq \max(\underline{l}(A), \underline{l}(B))$; together with (5.14) this establishes the theorem. — Note that the dual proof yields the result for $\bar{l}(A * B)$, but *not* for $\bar{l}(A \times B)$ or $\underline{l}(A * B)$.

Now let $\mathfrak{C}$ and $\mathfrak{C}'$ be DI-categories and let $T : \mathfrak{C} \to \mathfrak{C}'$ be a functor preserving inverse limits. We consider the factorization (F) of Theorem 4.4 for both $\mathfrak{C}$ and $\mathfrak{C}'$; then, for a family of objects $X_1, \ldots, X_n$ of $\mathfrak{C}$, T induces natural transformations $\varphi^r : (TX)^r \to T(X^r)$ such that the diagram

$$\begin{tikzcd} (TX)^r \arrow[r, "\varkappa_{TX}^{r-1}"] \arrow[d, "\varphi^r"'] & (TX)^{r-1} \arrow[d, "\varphi^{r-1}"] \\ T(X^r) \arrow[r, "T\varkappa_X^{r-1}"'] & T(X^{r-1}) \end{tikzcd} \tag{5.16}$$

is commutative (this is shown in the next section, in the special context of homotopy systems, but the argument is quite general[7])). In particular, $\varphi^n = \langle Tq_1, \ldots, Tq_n \rangle : TX_1 * \cdots * TX_n \to T(X_1 * \cdots * X_n)$. We use (5.16) in the special case $r = n$ to prove

Theorem 5.17. *Let $T : \mathfrak{C} \to \mathfrak{C}'$ be a functor from the DI-category $\mathfrak{C}$ to the DI-category $\mathfrak{C}'$ preserving inverse limits. Then, for each $X \in \mathfrak{C}$, $\underline{l}(TX) \leq \underline{l}(X)$, $\underline{wl}(TX) \leq \underline{wl}(X)$.*

Proof. We assume the existence of a map $m : X^{n-1} \to X$ such that $m\sigma = \bar{d} : X^n \to X$, where $\sigma = \varkappa^{n-1}$. Then we have the diagram

$$\begin{tikzcd} (TX)^n \arrow[r, "\sigma_{TX}"] \arrow[d, "\varphi^n"'] & (TX)^{n-1} \arrow[r, "m_{TX}"] \arrow[d, "\varphi^{n-1}"] & TX \arrow[d, "1"] \\ T(X^n) \arrow[r, "T\sigma_X"] & T(X^{n-1}) \arrow[r, "Tm_X"] & TX \end{tikzcd}$$

and we set $m_{TX} = Tm_X \circ \varphi^{n-1} : (TX)^{n-1} \to TX$. Then $m_{TX} \circ \sigma_{TX} = Tm_X \circ T\sigma_X \circ \varphi^n = T\bar{d}_X \circ \varphi^n$. But $\varphi^n = \langle Tq_1, \ldots, Tq_n \rangle$, so $T\bar{d}_X \circ \varphi^n = \langle T\bar{d}_X \circ Tq_1, \ldots, T\bar{d}_X \circ Tq_n \rangle = \langle T(\bar{d}_X q_1), \ldots, T(\bar{d}_X q_n) \rangle$. Further $\bar{d} q_i = 1$ and $T1 = 1$, so $T\bar{d}_X \circ \varphi^n = \langle 1, \ldots, 1 \rangle = \bar{d}_{TX}$. This shows that $\underline{l}(TX) \leq \underline{l}(X)$.

To prove the second assertion of the theorem, we set $\ker \sigma_X = k : K \to X^n$, $\ker \sigma_{TX} = l : L \to (TX)^n$. Since T preserves inverse limits, $\ker T\sigma_X = Tk : TK \to T(X^n)$; further $T\sigma_X \circ \varphi^n \circ l = \varphi^{n-1} \circ \sigma_{TX} \circ l = 0$, so that there exists $\varrho : L \to TK$ with $Tk \circ \varrho = \varphi^n \circ l$. We have the diagram

$$\begin{tikzcd} L \arrow[r, "l"] \arrow[d, "\varrho"'] & (TX)^n \arrow[r, "\bar{d}_{TX}"] \arrow[d, "\varphi^n"] & TX \arrow[d, "1"] \\ TK \arrow[r, "Tk"'] & T(X^n) \arrow[r, "T\bar{d}_X"'] & TX \end{tikzcd}$$

where the commutativity of the right-hand square was proved above. Then $\bar{d}_{TX} \circ l = T\bar{d}_X \circ \varphi^n \circ l = T\bar{d}_X \circ Tk \circ \varrho = T(\bar{d}_X \circ k) \circ \varrho$. Thus if $\bar{d}_X \circ k = 0$ it follows that $\bar{d}_{TX} \circ l = 0$ and the theorem is proved.

[7]) It matters not at all that the functor P considered in § 6 is actually from $\mathfrak{C}$ to itself, whereas here the range of the functor T need not coincide with its domain.

We close this section with a theorem which states a consequence of imposing a limitation on the length of objects.

Theorem 5.18. *Let A, $B \in \mathfrak{C}$, let (B, m) be an $\underline{\mathrm{H}}$-object of $\mathfrak{C}$ and let $\bar{l}(A) < 3$. Then the structure induced in $H(A, B)$ from that in B is associative.*

Proof. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\bar{d}"] \arrow[dr] & A \times A \times A \arrow[r, "f_1 \times f_2 \times f_3"] & B \times B \times B \arrow[r, shift left, "m\circ(m\times 1)"] \arrow[r, shift right, "m\circ(1\times m)"'] & B. \\
& \bar{T}(A) \arrow[u, "\tau"] \arrow[r, "\bar{T}(f_1, f_2, f_3)"] & \bar{T}(B) \arrow[u, "\tau"] &
\end{tikzcd}$$

We wish to show that the two horizontal routes from A to B are the same. In view of the commutativity of the diagram it is plainly sufficient to show that

(5.19) $$m \circ (m \times 1) \circ \tau = m \circ (1 \times m) \circ \tau;$$

but (5.19) is equivalent to the three assertions

(5.20) $$m \circ (m \times 1) \circ \iota_{ij} = m \circ (1 \times m) \circ \iota_{ij}, \qquad (i, j) = (1, 2), (1, 3), (2, 3),$$

where ι_{ij} embeds $B \times B$ in $B \times B \times B$ as the product of the i^{th} and j^{th} factor. Now (5.20) is a straightforward consequence of the fact that m is an $\underline{\mathrm{H}}$-structure. Thus the theorem is completely proved.

Remarks. (i) Recall (Theorem 4.17 of [1]) that $H(A, B)$ is commutative if $\bar{l}\,(A) < 2$.

(ii) If the $\underline{\mathrm{H}}$-object B is a quasi-group in $\mathfrak{C}$ then we are content to suppose that $\overline{wl}\,(A) < 3$, and conclude that $H(A, B)$ is a group.

§ 6. Homotopy systems and homotopy length

In this section we follow Kan [6] in describing a framework in which, for general categories $\mathfrak{C}$, the notion of homotopy between two maps $f, g : X \to Y$ can be defined. It is, of course, patterned after the homotopy concept in topology, so as to include this case and various other examples. The homotopy concept in a general category $\mathfrak{C}$ will allow us to introduce further integers attached to the objects of $\mathfrak{C}$, called homotopy lengths and weak homotopy lengths, which are "homotopy type invariants". In the category $\mathfrak{T}$ of (based) topological spaces and continuous maps, with the usual homotopy concept, the "Lusternik-Schnirelmann category" $\operatorname{cat} X$ of a space X (if defined according to G. W. Whitehead [11]) is such a homotopy length; its definition is exactly as that of $\bar{l}(X)$ in $\mathfrak{T}$, except that the characteristic property of the structure map m, $\tau m = \underline{d} : X \xrightarrow{m} \bar{T} \xrightarrow{\tau} X_l \times \cdots \times X_n$ is replaced by "τm homotopic to $\underline{d}$". The corresponding weak homotopy length ($w \operatorname{cat} X$) is defined by using "$c\underline{d}$ homotopic to 0" instead of $c\underline{d} = 0$, c being the cokernel of τ. (NB. The construction of $\bar{T}$ is made in $\mathfrak{T}$, not in $\mathfrak{T}_h$; in $\mathfrak{T}_h$ equalizers do not exist in general.)

Definition 6.1. (see [6]). *A left homotopy system $S = [Z; t, b, p]$ in the category $\mathfrak{C}$ is a system consisting of*

(i) *a covariant* I-*functor $Z : \mathfrak{C} \to \mathfrak{C}$ called the "cylinder functor", and*

(ii) *three natural transformations* $t: I \to Z$, $b: I \to Z$ *and* $p: Z \to I$ *(*I *being the identity functor) satisfying* $pt = pb = 1: I \to I$.

Relative to a left homotopy system S, a map $F: ZA \to B$ is called a (left) *homotopy between the maps* Fb_A *and* $Ft_A: A \to ZA \to B$. Two maps $f, g: A \to B$ are called *homotopic*, $f \sim g$, if there is a homotopy $F: ZA \to B$ such that $Fb_A = f$ and $Ft_A = g$. Note that *for any map* $f: A \to B$, *we have* $f \sim f$, the homotopy being $F = fp_A$. To ensure further properties of the relation $f \sim g$ such as symmetry and transitivity, suitable axioms must be imposed on S, in addition to (ii) above (e.g., symmetry is obtained by means of a natural transformation $r: Z \to Z$ satisfying $rt = b$, $rb = t$). In the context of this paper we are not so interested in the nature (and number) of the natural transformations attached to a homotopy system as in the cylinder functor itself. We propose to return to these finer questions of homotopy in a later publication.

If the above homotopy relation between f and g is *not* an equivalence relation, one usually considers the *equivalence relation it generates*; for simplicity we make here the convention that the symbol $f \sim g$ shall be interpreted to designate this equivalence relation. Then, e.g., transitivity can be used wherever it is necessary (in fact this will here be the case only in one instance, namely in the proof of 6.9). Moreover, as shown in the next proposition, the *classification of maps* of $\mathfrak{C}$ thus obtained is compatible with composition; hence a category $\mathfrak{C}_h$ is obtained whose objects are the objects of $\mathfrak{C}$ and whose maps are the equivalence classes of maps of $\mathfrak{C}$, with the induced composition.

Proposition 6.2. *The relation* $f \sim g$ *relative to a left homotopy system* S *is compatible with the composition of maps in* $\mathfrak{C}$; *i.e., if* $f \sim g: A \to B$, *and if* $\alpha: A' \to A$ *and* $\beta: B \to B'$ *are arbitrary maps, then* $f\alpha \sim g\alpha$ *and* $\beta f \sim \beta g$.

Proof. It is sufficient to consider a single homotopy $F: ZA \to B$ between f and g. Then (a) $F' = F \circ Z(\alpha): ZA' \to ZA \to B$ is a homotopy between $f\alpha$ and $g\alpha$, and (b) $F'' = \beta \circ F: ZA \to B \to B'$ is a homotopy between βf and βg.

To prove (a) we use the equation

$$Z(\alpha) \circ b_{A'} = b_A \circ \alpha$$

which holds since b is a natural transformation $I \to Z$. Then

$$F' \circ b_{A'} = F \circ Z(\alpha) \circ b_{A'} = F \circ b_A \circ \alpha = f\alpha\,,$$

and similarly

$$F' \circ t_{A'} = F \circ Z(\alpha) \circ t_{A'} = F \circ t_A \circ \alpha = g\alpha\,.$$

(b) simply follows from

$$F'' b_A = \beta F b_A = \beta f \quad \text{and} \quad F'' t_A = \beta F t_A = \beta g\,.$$

Dually a right homotopy system in $\mathfrak{C}$ is defined as a left homotopy system in the dual category $\mathfrak{C}^*$. Explicitly:

Definition 6.3. *A right homotopy system* $S = [P; t, b, p]$ *in* $\mathfrak{C}$ *is a system consisting of*

(i) *a covariant* D-*functor* $P: \mathfrak{C} \to \mathfrak{C}$ *called the "path functor", and*

(ii) *three natural transformations* $t: P \to I$, $b: P \to I$ *and* $p: I \to P$ *satisfying* $tp = bp = 1: I \to I$.

A homotopy, relative to a right homotopy system, is a map $F: A \to PB$; $f \sim g: A \to B$ means that there is a homotopy $F: A \to PB$ such that $b_B F = f$ and $t_B F = g$. All the remarks above, including Proposition 6.2 apply as well to a right homotopy system and the corresponding homotopy relation.

Examples. (1) In any category $\mathfrak{C}$ there is a trivial left homotopy system consisting of $Z = I$, and $t = b = p = 1: I \to I$. The corresponding relation $f \sim g$ holds if and only if $f = g$; thus $\mathfrak{C}_h = \mathfrak{C}$.

(2) The usual homotopy system in $\mathfrak{T}$ is given by $ZX = X \times [0, 1]/o \times [0, 1]$; $t_X(x) = x \times (1)$ for all $x \in X$, $b_X(x) = x \times (0)$, $p_X(x, t) = x$ for $0 \leqq t \leqq 1$.

There is, in $\mathfrak{T}$, a right homotopy system given by $PX =$ path space of X (function space of all maps $\omega: [0, 1] \to X$, with the constant map at $o \in X$ as base point), $t_X(\omega) = \omega(1)$, $b_X(\omega) = \omega(0)$, $p_X(x) =$ constant path at $x \in X$. The corresponding homotopy relation $f \sim g$ coincides with that given by the left homotopy system above: for there is a natural one-to-one correspondence between the two kinds of homotopies $H(ZA, B)$ and $H(A, PB)$ compatible with the natural transformations t, b and p. In other words, the functor P is *right-adjoint* to Z; from this it automatically follows that Z preserves inverse products and right equalizers, and that P preserves direct products and left equalizers. We return below to "adjoint homotopy systems" in general.

(3) In the category of group complexes (c.s.s. groups) Kan's notion of "homotopy through homomorphisms" or "loop homotopy" (see [7]) is obtained from a right homotopy system in which the path functor P is essentially given by $PX =$ function space $X^{[0,1]}$.

Definition 6.4. *A left homotopy system S is called faithful if its cylinder functor Z preserves right equalizers (direct limits); a right homotopy system, if its path functor P preserves left equalizers (inverse limits).*

We recall [cf. Prop. (1.8)] that if Z has a *right-adjoint* then it certainly preserves right equalizers (and inverse products); this sufficient condition for *a* homotopy system to be faithful is available in many cases. Moreover, if Z has a right-adjoint P, this D-functor P can be used as path functor of a right homotopy system. $S = [P; \tilde{t}, \tilde{b}, \tilde{p}]$ – which of course will be faithful – as follows.

Let $\eta: H(X, PY) \to H(ZX, Y)$ be the adjugant (the natural one-to-one correspondence) of Z and P. We define $\tilde{t}, \tilde{b}, \tilde{p}$ as the "adjoint" transformations of t, b, p; e.g., $\tilde{t}: P \to I$ is defined by the condition

$$f \circ \tilde{t}_X = \eta(P(f)) \circ t_{PX} \qquad \text{for all} \quad f: X \to Y,$$

which is equivalent (putting $f = 1: X \to X$) to

$$\tilde{t}_X = \eta(1_{PX}) \circ t_{PX}.$$

Similarly

$$\tilde{b}_X = \eta(1_{PX}) \circ b_{PX}.$$

The transformation $\tilde{p}: I \to P$ is defined by the condition

$$\eta(\tilde{p}_X f) = p_X \circ Z(f) \qquad \text{for all} \quad f: Y \to X$$

which is equivalent to $\eta(\tilde{p}_X) = p_X \circ 1_{ZX} = p_X$, or

$$\tilde{p}_X = \eta^{-1}(p_X) .$$

Using the naturality of the transformation t and η we then have

$$\begin{aligned} \tilde{t}_X \tilde{p}_X &= \eta(1_{PX}) \circ t_{PX} \circ \tilde{p}_X = \eta(1_{PX}) \circ Z(\tilde{p}_X) \circ t_X \\ &= \eta(1_{PX}\tilde{p}_X) \circ t_X = \eta(\tilde{p}_X) \circ t_X = p_X t_X = 1_X . \end{aligned}$$

Hence $\tilde{t}\tilde{p} = 1 : I \to I$ and similarly $\tilde{b}\tilde{p} = 1$. We summarize

Proposition 6.5. *If the cylinder functor Z of the left homotopy system $S = [Z; t, b, p]$ has a right-adjoint P, then S is faithful, and P together with the adjoint transformations $\tilde{t}, \tilde{b}, \tilde{p}$ yields a faithful right homotopy system $\tilde{S} = [P; \tilde{t}, \tilde{b}, \tilde{p}]$.*

We now consider, exactly as in § 5, assuming $\mathfrak{C}$ to be a DI-category with left equalizers, the map $\sigma : X_1 * \cdots * X_n \to \underline{T}$ for $X_1 = \cdots = X_n$ and its kernel $k : K \to X_1 * \cdots * X_n$. Using the homotopy relation $f \sim g$ in $\mathfrak{C}$ with reference to a fixed (left or right) homotopy system S we define homotopy length $\underline{hl}(X)$ and weak homotopy length $\underline{whl}(X)$ as follows.

Definition 6.6. *$\underline{hl}(X) < n$ if and only if X admits a homotopy $\underline{\mathrm{H}}_n$-structure, i.e., a map $m : \underline{T} \to X$ such that $m\sigma \sim \bar{d}$. Furthermore $\underline{whl}(X) < n$ if and only if $0 \sim \bar{d}k$.*

To justify these definitions, we have to show that (i) $\underline{hl}(X) < n$ implies $\underline{hl}(X) < n + 1$ and (ii) $\underline{whl}(X) < n$ implies $\underline{whl}(X) < n + 1$. The proofs are as those of (5.2) and (5.7), except that in one instance the equality sign is replaced by $\sim$. In detail:

(i) Using the notations of the proof of (5.2), we have

$$m'\sigma' = m\psi\sigma' = m\sigma\varphi \sim \bar{d}\varphi = \bar{d}' : X_0 * \cdots * X_n \to X .$$

(ii) Using the notations of the proof of (5.7), we have

$$0 \sim \bar{d}k\gamma = \bar{d}\varphi k' = \bar{d}'k' .$$

Note that the only property of the homotopy relation used in this context is that it is compatible with composition of maps. The same remark applies to the following propositions listing *inequalities* for $\underline{hl}$ and $\underline{whl}$.

It is plain that the integers $\underline{hl}(X)$ and $\underline{whl}(X)$ defined by (6.6) fulfil

$$\underline{hl}(X) \leqq \underline{l}(X) \quad \text{and} \quad \underline{whl}(X) \leqq \underline{wl}(X) .$$

Moreover we have

Proposition 6.7. $\underline{whl}(X) \leqq \underline{hl}(X)$.

For, if there is a map m with $m\sigma \sim \bar{d}$, one has $0 = m\sigma k \sim \bar{d}k$.

Definition 6.8. We say that Y *homotopy-dominates* X if there are maps $f : X \to Y$ and $g : Y \to X$ with $gf \sim 1_X$. – We say that X and Y are *homotopy-equivalent*, if there are maps $f : X \to Y$ and $g : Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$.

Theorem 6.9. *If Y homotopy-dominates X, then* (i) $\underline{hl}(X) \leqq \underline{hl}(Y)$ *and* (ii) $\underline{whl}(X) \leqq \underline{whl}(Y)$.

Corollary 6.10. *If X and Y are homotopy-equivalent, then $\underline{hl}(X) = \underline{hl}(Y)$ and $\underline{whl}(X) = \underline{whl}(Y)$. (Homotopy invariance of homotopy lengths.)*

Proof of 6.9. This proof follows exactly that of (5.12), to which we refer without any further explanation.

(i) Assuming that there is a map $m_Y : \underline{T}(Y) \to Y$ with $m_Y \sigma_Y \sim \bar{d}_Y$, we put $m_X = g m_Y \underline{T}(f)$; then

$$m_X \sigma_X = g m_Y \underline{T}(f)\, \sigma_X = g m_Y \sigma_Y f^n \sim g \bar{d}_Y f^n = g f \bar{d}_X \sim \bar{d}_X .$$

(ii) Assuming $0 \sim \bar{d}_Y k_Y$, we have

$$0 \sim \bar{d}_Y k_Y \gamma = \bar{d}_Y f^n k_X = f \bar{d}_X k_X , \quad \text{hence} \quad 0 \sim g f \bar{d}_X k_X \sim \bar{d}_X k_X .$$

We now formulate the analogue of Theorem 5.13 on $\underline{whl}$, using the concept of a map f *with zero homotopy kernel*; i.e., the kernel ideal of the class of f in $\mathfrak{C}_h$ is the zero-ideal. In other words, $f: X \to Y$ having zero homotopy kernel means that for any map $g: A \to X$ with $0 \sim fg$ one has $0 \sim g$.

Theorem 6.11. *If there exists a map $f: X \to Y$ with zero homotopy kernel then $\underline{whl}(X) \leqq \underline{whl}(Y)$.*

Proof (cf. the proof of 5.13). We assume $\underline{whl}(Y) < n$, i.e. $0 \sim \bar{d}_Y k_Y$. Then

$$0 \sim \bar{d}_Y k_Y \, \gamma = f \bar{d}_X k_X ;$$

hence, f having zero homotopy kernel, $0 \sim \bar{d}_X k_X$, i.e., $\underline{whl}(X) < n$.

Theorem 6.12. *For any A and $B \in \mathfrak{C}$ one has*

$$\underline{hl}(A \times B) = \max\,(\underline{hl}(A), \underline{hl}(B)) .$$

Proof (cf. Proof of 5.15). Assuming the existence of m_A and m_B with $m_A \sigma_A \sim \bar{d}_A$ and $m_B \sigma_B \sim \bar{d}_B$, we put

$$m_{A \times B} = \{m_A \underline{T}(p_1), m_B \underline{T}(p_2)\};$$

then

$$m_{A \times B} \sigma_{A \times B} = \{m_A \sigma_A p_1^n, m_B \sigma_B p_2^n\} \sim \{\bar{d}_A p_1^n, \bar{d}_B p_2^n\} = \{p_1 \bar{d}_A, p_2 \bar{d}_B\} = \bar{d}_{A \times B} .$$

(Here we have used the fact that, F, F' being homotopies $ZX \to A$, $ZX \to B$ respectively, with $F b_X = f, F t_X = g, F' b_X = f', F' t_X = g'$, $\{F, F'\}: ZX \to A \times B$ is a homotopy with $\{F, F'\}\, b_X = \{f, f'\}$ and $\{F, F'\}\, t_X = \{g, g'\}$; in other words, that $f \sim g : X \to A$, $f' \sim g' : X \to B$ implies $\{f, f'\} \sim \{g, g'\} : X \to A \times B$.)

Thus $\underline{hl}(A \times B) \leqq \max\,(\underline{hl}(A), \underline{hl}(B))$. On the other hand, $\underline{hl}(A) \leqq$ $\leqq \underline{hl}(A \times B)$, since $A \times B$ dominates A.

The dual definitions, for $\overline{hl}(X)$ and $\overline{whl}(X)$ and the duals of all the above inequalities or equalities are obvious.

We now proceed to a further main objective of this section, namely to show that not only the homotopy lengths, but *the full canonical factorizations* (F) *and* (F′) *of* $\varkappa : X_1 * \cdots * X_n \to X_1 \times \cdots \times X_n$ (cf. § 4) are *homotopy invariant*. For this not only the homotopy relation, but the homotopy system S itself is needed.

We first consider the homotopy behaviour of the factorization (F) with respect to a left homotopy system $S = [Z; t, b, p]$ in $\mathfrak{C}$. By assumption, Z is an I-functor; moreover one has the natural transformation ζ of $Z \varprojlim$ into $\varprojlim Z$ of Prop. 2.8. Given n objects $X_1, \ldots, X_n$ of $\mathfrak{C}$, the X^r appearing in (F) are inverse limits of $\mathfrak{C}$-aggregates $\mathscr{A}^{r,r}$ consisting of inverse products X_J of $X_1, \ldots, X_n$ (cf. § 4). In the same way we form from $ZX_1, \ldots, ZX_n$ inverse products $(ZX)_J$ which can be naturally identified with $Z(X_J)$, and aggregates $(Z\mathscr{A})^{r,r}$ which can be identified with $Z(\mathscr{A}^{r,r})$; thus $(ZX)^r = \varprojlim (Z\mathscr{A})^{r,r} = \varprojlim Z(\mathscr{A}^{r,r})$. Similarly, if $\Phi : \mathscr{A}^{r,r} \to \mathscr{A}^{r-1,r-1}$ is the map inducing $\varkappa^{r-1} = \varkappa_X^{r-1} : X^r \to X^{r-1}$, the map $\varkappa_{ZX} : (ZX)^r \to (ZX)^{r-1}$ is induced by $Z\Phi : Z\mathscr{A}^{r,r} \to Z\mathscr{A}^{r-1,r-1}$. The natural transformation ζ then yields a sequence of maps $\zeta^r : Z(X^r) \to (ZX)^r$, $r = 1, 2, \ldots, n$ such that the diagram

$$\begin{tikzcd} Z(X^r) \arrow[r, "Z(\varkappa_X^{r-1})"] \arrow[d, "\zeta^r"'] & Z(X^{r-1}) \arrow[d, "\zeta^{r-1}"] \\ (ZX)^r \arrow[r, "\varkappa_{ZX}^{r-1}"] & (ZX)^{r-1} \end{tikzcd}$$

is commutative; in other words, the ζ^r constitute a map of $Z(F)$ into $F(Z)$.

Now let $Y_1, \ldots, Y_n$ be n further objects, and $F_i : ZX_i \to Y_i$ left homotopies, $i = 1, \ldots, n$; they induce maps $F^r : (ZX)^r \to Y^r$, $r = 1, \ldots, n$.

Then $\tilde{F}^r = F^r \zeta^r$ are homotopies, i.e. maps $Z(X^r) \to Y^r$ with the properties

(i) $\varkappa_Y^{r-1} \circ \tilde{F}^r = \tilde{F}^{r-1} \circ Z(\varkappa_X^{r-1}) : Z(X^r) \to Y^r$

(ii) $\tilde{F}^r \circ b_{X^r} = (F \circ b_X)^r : X^r \to Y^r$,

where $(F \circ b_X)^r$ stands for the map induced by $F_1 \circ b_{X_1}, \ldots, F_n \circ b_{X_n}$.

Proof of (i): $\varkappa^{r-1}$ is natural, hence

$$F^{r-1} \circ \varkappa_{ZX}^{r-1} = \varkappa_Y^{r-1} \circ F^r.$$

Thus $\varkappa_Y^{r-1} \circ \tilde{F}^r = \varkappa_Y^{r-1} \circ F^r \circ \zeta^r = F^{r-1} \circ \varkappa_{ZX}^{r-1} \circ \zeta^r = F^{r-1} \circ \zeta^{r-1} \circ Z(\varkappa_X^{r-1})$
$= \tilde{F}^{r-1} \circ Z(\varkappa_X^{r-1})$.

Proof of (ii). The natural transformation b yields maps $b_{X_i} : X_i \to ZX_i$ and thus induced map $(b_X)^r : X^r \to (ZX)^r$. According to (2.9) $\zeta^r \circ b_{X^r} = (b_X)^r$, whence

$$\tilde{F}^r \circ b_{X^r} = F^r \circ \zeta^r \circ b_{X^r} = F^r \circ (b_X)^r = (F \circ b_X)^r .$$

(ii) holds, of course, also for the transformation t. We summarize the results as follows, writing $f_i = F_i b_{X_i}$ and $g_i = F_i t_{X_i}$.

Theorem 6.13. *If for $i = 1, 2, \ldots, n$ the maps f_i and $g_i : X_i \to Y_i$ are homotopic, with respect to a left homotopy system in $\mathfrak{C}$, then the induced maps f^r and $g^r : X^r \to Y^r$ are also homotopic; the homotopies $\tilde{F}^r : Z(X^r) \to Y^r$ are obtained canonically from the homotopies $F_i : ZX_i \to Y_i$ between f_i and g_i as $\tilde{F}^r = F^r \circ \zeta^r$, and are compatible with the maps $\varkappa^{r-1}$* (i.e., $\varkappa_Y^{r-1} \circ \tilde{F}^r = \tilde{F}^{r-1} \circ Z(\varkappa_X^{r-1})$).

Corollary 6.14. (*Homotopy invariance of* (F) *with respect to a left homotopy system*). *If $f_i : X_i \to Y_i$, $i = 1, \ldots, n$ are homotopy equivalences, then the induced maps $f^r : X^r \to Y^r$ are also homotopy equivalences.*

By duality (6.13) and (6.14) hold for the dual factorization (F′) of $\varkappa : X_1 * \cdots * X_n \to X_1 \times \cdots \times X_n$ with respect to a *right* homotopy system.

The results (6.13) and (6.14) on the factorization (F) also hold for a right homotopy system $S = [P; t, b, p]$ unter the additional assumption that S

be *faithful*, i.e. *that P preserves inverse limits*. The proof proceeds as before by establishing a sequence of maps $\varphi^n : (PX)^r \to P(X^r)$ such that the diagram

$$\begin{array}{ccc} (PX)^r & \xrightarrow{\varkappa_{PX}^{r-1}} & (PX)^{r-1} \\ \Big\downarrow{\scriptstyle\varphi^r} & & \Big\downarrow{\scriptstyle\varphi^{r-1}} \\ P(X^r) & \xrightarrow{P\varkappa_X^{r-1}} & P(X^{r-1}) \end{array}$$

is commutative, and then by defining a (right) homotopy $\tilde{F}^r : X^r \to P(Y^r)$ as $\tilde{F}^r = \varphi^r F^r : X^r \to (PY)^r \to P(Y^r)$, where F^r is the map induced by right homotopies $F_i : X_i \to PY_i$, $i = 1, \ldots, n$. – In order to get the φ^r one first uses the natural transformation which maps the inverse product of $PX_1, \ldots, PX_n$ into $P(X_1 * \cdots * X_n)$, given by $\langle Pq_1, \ldots, Pq_n \rangle$, to obtain natural maps $(PX)_J \to P(X_J)$ and thus an (essential) map of $(P\mathscr{A})^{r,r}$, i.e. the aggregate formed with $PX_1, \ldots, PX_n$ instead of $X_1, \ldots, X_n$, into $P(\mathscr{A}^{r,r})$. Then $\varprojlim(P\mathscr{A})^{r,r} = (PX)^r$ is mapped naturally into $\varprojlim P(\mathscr{A}^{r,r})$ which by assumption can be identified with $P \varprojlim \mathscr{A}^{r,r} = P(X^r)$. Without repeating the further details, we state the analogues of (6.13) and (6.14) in short as follows:

Theorem 6.15. *The factorization* (F) *is homotopy invariant with respect to a faithful right homotopy system.*

Examples of homotopy-lengths. (1) In the category $\mathfrak{T}$ of based topological spaces, $\overline{hl}(X)$ is the (based) "Lusternik-Schnirelmann category" cat X defined according to G. W. Whitehead [11], and $\overline{whl}(X)$ the "weak category" w cat X[8]. There exist examples of spaces for which $\overline{hl}(X)$ and $\overline{whl}(X)$ have prescribed integer values (while $\overline{l}(X) = \infty$ for non-trivial spaces, cf. example (2) in § 5). The dual length $\underline{hl}(X)$ in $\mathfrak{T}$ is always < 3 ($\underline{hl}(X) = 0$ for contractible spaces, $\underline{hl}(X) = 1$ for non-contractible H-spaces, $\underline{hl}(X) = 2$ otherwise); this follows from $\underline{hl}(X) \leqq \underline{l}(X) < 3$, cf. example (2) in § 5.

(2) In the category of group complexes (css-groups) and homomorphisms, with Kan's concept of homotopy through homomorphisms[9]), $\underline{hl}(X)$ and $\underline{whl}(X)$ are not limited as in $\mathfrak{T}$; thus *non-trivial numerical homotopy invariants* can be obtained by using functors which lead from $\mathfrak{T}$ to the category of group complexes.

(3) Conversely, functors passing from the category of groups $\mathfrak{G}$ to $\mathfrak{T}$ yield numerical invarients $\overline{hl}$ or $\overline{whl}$ of groups which give more information than $\overline{l}$ and $\overline{wl}$ in $\mathfrak{G}$ itself. For example, Eilenberg and Ganea have considered the invariant cat $K[\pi, 1]$ of the group π.

Let $\mathfrak{C}$ and $\mathfrak{C}'$ be categories and let $S = [Z; t, b, p]$, $S' = [Z'; t', b', p']$ be left homotopy systems in $\mathfrak{C}$, $\mathfrak{C}'$ respectively. A functor $T : \mathfrak{C} \to \mathfrak{C}'$ will be said to *preserve homotopy* if $TZ = Z'T$ and, for each $X \in \mathfrak{C}$, $T(t_X) = t'_{TX}$,

[8]) Cat and w cat are here renormalized so that they take the value 0 on contractible spaces (instead of 1).

[9]) See [7]. A concept of $\underline{hl}(X)$ and $\underline{whl}(X)$ for c.s.s. groups was suggested by Hilton [Homotopy theory and duality, Cornell (1958)]; it has recently been shown that $\underline{whl}$ coincides in this category with nilpotency.

$T(b_X) = b'_{TX}$, $T(p_X) = p'_{TX}$. Notice in particular that if S' is the trivial left homotopy system then T preserves homotopy if and only if $TZ = T$ and $T(t_X) = T(b_X) = T(p_X) = 1_{TX}$. A similar notion is available for right homotopy systems.

Proposition 6.16. *Let $\mathfrak{C}$ and $\mathfrak{C}'$ be categories and S, S' left homotopy systems in $\mathfrak{C}$, $\mathfrak{C}'$ respectively. Let $T : \mathfrak{C} \to \mathfrak{C}'$ be a homotopy-preserving functor. Then $Tf \sim Tg : TA \to TB$ if $f \sim g : A \to B$.*

Proof. It is plainly sufficient to consider a single homotopy $F : ZA \to B$ between f and g. Then it is immediately verified that $TF : Z'TA = TZA \to TB$ is a homotopy from Tf to Tg.

Theorem 6.17. *Let $T : \mathfrak{C} \to \mathfrak{C}'$ be a functor from the* DI*-category $\mathfrak{C}$ to the* DI*-category $\mathfrak{C}$ preserving inverse limits. Suppose further that $\mathfrak{C}$ and $\mathfrak{C}'$ are furnished with left (right) homotopy systems and that T preserves homotopy. Then, for each $X \in \mathfrak{C}$, $\underline{hl}(TX) \leqq \underline{hl}(X)$, $\underline{whl}(TX) \leqq \underline{whl}(X)$.*

The proof is the obvious small modification of that of Theorem 5.17 and uses Proposition 6.16.

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(Received October 30, 1962)