AN ATIYAH-HIRZEBRUCH SPECTRAL SEQUENCE FOR KR-THEORY

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Contents

1. Introduction

In recent years much attention has been given to a certain spectral sequence relating motivic cohomology to algebraic K-theory [Be, BL, FS, V3]. This spectral sequence takes on the form

$$H^p(X, \mathbb{Z}(-\tfrac{q}{2})) \Rightarrow K^{p+q}(X),$$

where the $H^s(X; \mathbb{Z}(t))$ are the bi-graded motivic cohomology groups, and $K^n(X)$ denotes the algebraic K-theory of X. It is useful in our context to use topologists' notation and write $K^n(X)$ for what K-theorists call $K_{-n}(X)$. The above spectral sequence is the analog of the classical Atiyah-Hirzebruch spectral sequence relating ordinary singular cohomology to complex K-theory, in a way that is explained further below.

It is well known that there are close similarities between motivic homotopy theory and the equivariant homotopy theory of $\mathbb{Z}/2$-spaces (cf. [HK1, HK2], for example). In fact there is even a forgetful map of the form

$$(\text{motivic homotopy theory over } \mathbb{R}) \to (\mathbb{Z}/2\text{-equivariant homotopy theory}),$$

discussed in [MV, Section 3.3] and [DI, Section 5]. Our aim in this paper is to construct the analog of the above motivic spectral sequence in the $\mathbb{Z}/2$-equivariant context. The spectral sequence takes on the form

$$H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}}) \Rightarrow KR^{p+q}(X),$$

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where the analog of algebraic K-theory is Atiyah's KR-theory [At]. The analog of motivic cohomology is $RO(G)$-graded Eilenberg-MacLane cohomology, with coefficients being the constant Mackey functor $\underline{\mathbb{Z}}$. The indexing conventions have been chosen for their analogy with the motivic situation, and will be elucidated further in just a moment.

The fact is that constructing the above spectral sequence is not at all difficult, and there are many ways it could be done. In equivariant topology one has so many tools to work with that the arguments end up being very simple. Unfortunately most of these tools are not yet available in the motivic context. This paper tries to develop the spectral sequence in a way that might eventually work motivically, and which accentuates the basic properties of the spectral sequence. We introduce certain 'twisted' Postnikov section functors, and use these to construct a tower for the equivariant space $\mathbb{Z} \times BU$ in which the layers are equivariant Eilenberg-MacLane spaces. The homotopy spectral sequence for the tower is essentially what we're looking for—although technically speaking this only produces half the spectral sequence, and to get the other half we must stabilize. The approach here is similar to the one advocated in [V2], but was worked out independently (in fact there are several differences, one being that [V2] takes place in the stable category).

We'll now explain the methods of the paper in more detail, starting with our basic notation. Recall that every real vector space V with an involution gives rise to a $\mathbb{Z}/2$-sphere S^V by taking its one-point compactification. If $\mathbb{R}$ and $\mathbb{R}_-$ denote the one-dimensional vector spaces with trivial and sign involutions, respectively, then any V will decompose as $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$ for some p and q. So the spheres S^V form a bi-graded family, and when V is as above we'll use the notation

$$S^V = S^{p+q,q}.$$

Here the first index is the topological *dimension* of the sphere, and the second index is called the *weight*. Note that when $V = \mathbb{C}^n$ (regarded as a real vector space with the conjugation action) then $S^V = S^{2n,n}$; in particular, $\mathbb{C}\mathrm{P}^1 \cong S^{\mathbb{C}} = S^{2,1}$. The reader should be warned that this differs from the bi-graded indexing introduced in [At] and later used in [AM, Ar].

Recall from [LMS] that to give an equivariant spectrum E is to give an assignment $V \mapsto E_V$ together with suspension maps $\Sigma^W E_V \to E_{V\oplus W}$ which are compatible in a certain sense. One then has cohomology groups $E^V(X)$ for any representation V, and when V is as above we will likewise write $E^V(X) = E^{p+q,q}(X)$ to correspond with our bi-graded indexing of the spheres. This 'motivic indexing' is quite suggestive, and ends up being a useful convention.

For the group $\mathbb{Z}/2$, every real representation is contained in a $\mathbb{C}^n$ for a large enough value of n. The definition of equivariant spectra can then be streamlined a bit by only giving the assignment $\mathbb{C}^n \mapsto E_{(2n,n)} = E_n$ together with structure maps $S^{2,1} \wedge E_n \to E_{n+1}$. This is the approach first taken in [AM]—albeit with different indexing conventions, as mentioned above—and was later used in [V1, J]. We will treat spectra this way throughout the paper.

Our first example of such an object is the KR spectrum. The space $\mathbb{Z}\times BU$ has an obvious $\mathbb{Z}/2$-action coming from complex conjugation on the unitary group U. From another perspective, one could model $\mathbb{Z}\times BU$ by the infinite complex Grassmannian, again with the action of complex conjugation. The reduced canonical line bundle over $\mathbb{C}\mathrm{P}^1$ is classified by an equivariant map $S^{2,1} = \mathbb{C}\mathrm{P}^1 \to \mathbb{Z} \times BU$, and so one gets $S^{2,1} \wedge (\mathbb{Z} \times BU) \to \mathbb{Z} \times BU$ by using the multiplication in $\mathbb{Z} \times BU$. So we have a $\mathbb{Z}/2$-spectrum in which every term is $\mathbb{Z} \times BU$, and this is called the KR spectrum. In fact, it is an Omega-spectrum: equivariant Bott periodicity shows that the maps $\mathbb{Z} \times BU \to \Omega^{2,1}(\mathbb{Z} \times BU)$ are equivariant weak equivalences, or that $KR^{s,t}(X) \cong KR^{s-2,t-1}(X)$. The reference for this fact is [At].

The second spectrum we will need is the equivariant Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$. The easiest way to construct this, by analogy with the non-equivariant case, is to consider the spectrum $\mathbb{C}^n \mapsto \mathrm{AG}(S^{2n,n})$. Here $\mathrm{AG}(X)$ denotes the free abelian group on the space X, given a suitable topology. The structure maps are the obvious ones, induced in the end by the isomorphisms $S^{2,1} \wedge S^{2n,n} \cong S^{2n+2,n+1}$. It is proven in [dS] that this spectrum represents Eilenberg-MacLane cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$, and that it is an Omega-spectrum. The nth space $\mathrm{AG}(S^{2n,n})$ is therefore an equivariant Eilenberg-MacLane space, and will be denoted $K(\mathbb{Z}(n), 2n)$. This is the last of the basic notation needed to describe our results.

Our goal in this paper will be to construct certain functors P_{2n} on the category of $\mathbb{Z}/2$-spaces, which are analogs of the classical Postnikov section functors. Roughly speaking, $P_{2n}X$ will be built from X by attaching cones on all maps from spheres 'bigger than' $S^{2n,n}$. There are different possible choices for what is meant by this, for which we refer the reader to Section 3.

As was pointed out above there is a Bott map $\beta \colon S^{2,1} \to \mathbb{Z} \times BU$ which classifies the reduced canonical line bundle over $\mathbb{C}\mathrm{P}^1$; let β^n denote its nth power $S^{2n,n} \to \mathbb{Z} \times BU$. Applying Postnikov section functors gives the induced map $P_{2n}(S^{2n,n}) \to P_{2n}(\mathbb{Z} \times BU)$. The main goal of this paper is the following:

Theorem 1.1. *There are Postnikov functors P_{2n} on the category of $\mathbb{Z}/2$-spaces with the properties that*

(a) $P_{2n}(S^{2n,n})$ is weakly equivalent to $K(\mathbb{Z}(n), 2n)$, and

(b) $P_{2n}(S^{2n,n}) \xrightarrow{\beta^n} P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU)$ is a homotopy fiber sequence.

Corollary 1.2. *The tower*

$$\cdots \longrightarrow P_4(\mathbb{Z} \times BU) \longrightarrow P_2(\mathbb{Z} \times BU) \longrightarrow P_0(\mathbb{Z} \times BU)$$

has the following properties:

- *(i) The homotopy fiber F_n of the map $P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU)$ is an equivariant Eilenberg-MacLane space $K(\mathbb{Z}(n), 2n)$.*
- *(ii) The Adams operation $\psi^k \colon \mathbb{Z} \times BU \to \mathbb{Z} \times BU$ induces a self-map of the tower, whose action on F_n coincides with the multiplication-by-k^n map on the Eilenberg-MacLane space $K(\mathbb{Z}(n), 2n)$.*

Looking at the homotopy spectral sequence of the above tower then gives the following:

Corollary 1.3. *There is a fringed spectral sequence $E_2^{p,q} \Rightarrow KR^{p+q,0}(X)$ where $E_2^{p,q} = H^{p,-\frac{q}{2}}(X; \underline{\mathbb{Z}})$ when $p + q \leq 0$ and q is even, and $E_2^{p,q} = 0$ otherwise. The spectral sequence converges conditionally for $p + q < 0$, is multiplicative, and has an action of the Adams operations ψ^k in which ψ^k acts on $E_2^{p,q}$ as multiplication by $k^{-\frac{q}{2}}$.*

One can also stabilize the spectral sequence to avoid the awkward truncation, but then one loses the action of the Adams operations. To this end, we let W_n denote the homotopy fiber of $\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU)$. In non-equivariant topology the W_n's are the connective covers of $\mathbb{Z}\times BU$, and are also the spaces in the Ω-spectrum for connective K-theory bu. The following result shows the same for the $\mathbb{Z}/2$-case:

Proposition 1.4. *There are weak equivalences $W_n \to \Omega^{2,1}W_{n+1}$, unique up to homotopy, making the diagrams*

$$\begin{array}{ccc} W_n & \longrightarrow & \Omega^{2,1}W_{n+1} \\ \downarrow & & \downarrow \\ \mathbb{Z} \times BU & \longrightarrow & \Omega^{2,1}(\mathbb{Z} \times BU) \end{array}$$

commute (where the bottom map is the Bott periodicity map).

The corresponding $\mathbb{Z}/2$-spectrum whose nth object is W_n will be denoted $\boldsymbol{kr}$ and called the **connective $\boldsymbol{KR}$-spectrum**.

Theorem 1.5. *There is a 'Bott map' $\beta\colon \Sigma^{2,1}kr \to kr$ with the following properties:*
(a) The cofiber of β is $H\underline{\mathbb{Z}}$;
(b) The telescope of the tower

$$\cdots \to \Sigma^{2,1}kr \to kr \to \Sigma^{-2,-1}kr \to \Sigma^{-4,-2}kr \to \cdots$$

is weakly equivalent to the spectrum KR (where each map in the tower is the obvious suspension or desuspension of β);
(c) The homotopy inverse limit of the above tower is contractible.

The above tower of course yields a spectral sequence for computing $KR^*(X)$ for any $\mathbb{Z}/2$-space X, which could be considered the Bockstein spectral sequence for the map β:

Theorem 1.6. *For any $\mathbb{Z}/2$-space X, there is a conditionally convergent, multiplicative spectral sequence of the form $H^{p,-\frac{q}{2}}(X,\underline{\mathbb{Z}}) \Rightarrow KR^{p+q,0}(X)$.*

This spectral sequence is interesting even when X is a point, in which case it converges to the groups KO^*; it is drawn in detail in section 6.4. Also, note that there is really a whole family of spectral sequences of the form

$$H^{p,r-\frac{q}{2}}(X,\underline{\mathbb{Z}}) \Rightarrow KR^{p+q,r}(X),$$

but these can all be shifted back to the case $r = 0$ by using Bott periodicity $KR^{s,t}(X) = KR^{s+2,t+1}(X)$.

1.7. **Acknowledgments.** Most of the results in this paper were taken from the author's MIT doctoral dissertation [D1]. The author would like to thank his thesis advisor Mike Hopkins, and would also like to acknowledge very helpful conversations with Gustavo Granja. The final year of this research was generously supported by a Sloan Dissertation Fellowship.

Since there has been a long delay between [D1] and the appearance of this paper, a brief history of related work might be in order. Very shortly after [D1] was written, Friedlander and Suslin released [FS] which constructed the more interesting motivic spectral sequence, using very different methods. In early 2000 the paper [V2] was released, outlining via conjectures a homotopy-theoretic approach similar to the one given here (but working in the stable category, and using a different definition of the Postnikov sections). These ideas were developed a little further in [V3]. Sometime in 2000-2001 Hopkins and Morel also announced proofs of results along these lines, although the details have yet to appear. At the end of 2002, the paper [V4] proved a stable result similar to Theorem 1.1(a) in the motivic context, over fields of characteristic zero. An analog of Theorem 1.1 for the unstable motivic category has never been claimed or proven, as far as I know.

1.8. **Organization of the paper.** The paper has been written with a good deal of exposition, partly because the literature on these subjects is not always so clear. Sections 2 and 3 set down the necessary background, in particular giving the constructions of equivariant Postnikov functors. In these sections we often work over an arbitrary finite group, because it is easier to understand the ideas in this generality. In this context everything is graded by orthogonal G-representations, as is standard from [LMS]. When specializing to the $\mathbb{Z}/2$ case we always translate into the motivic (p,q)-indexing. Section 2 also recalls the basic facts we will need about the theory $H^{*,*}(-;\underline{\mathbb{Z}})$.

The real work takes place in section 4, where we analyze the Postnikov tower for $\mathbb{Z}\times BU$. Section 5 discusses the basic properties of the associated spectral sequence, most of which follow immediately from the way the tower was constructed. Section 6 is concerned with passing to the stable case. Section 7, which goes back to being very expository, deals with the 'étale' version of the spectral sequence and the analog of the Quillen-Lichtenbaum conjecture. Finally, in section 8 we give the proof of Theorem 1.1(a).

2. Background

2.1. **Basic setup.** Throughout this paper we will be working in the world of equivariant homotopy theory over a finite group G (usually with $G = \mathbb{Z}/2$). Unless otherwise indicated, 'space' means 'equivariant space' and 'map' means 'equivariant map'. If X and Y are spaces, then $[X, Y]$ denotes the set of equivariant homotopy classes of maps. When H is a subgroup of G, $[X, Y]^H$ is the set of H-equivariant homotopy classes of H-equivariant maps; in particular, $[X, Y]^e$ is the set of non-equivariant homotopy classes. The phrase 'weak equivalence' means 'equivariant weak equivalence': this refers to a map $X \to Y$ such that $X^H \to Y^H$ is an ordinary weak equivalence for every subgroup $H \subseteq G$.

2.2. **Connectivity.** Let V be an orthogonal G-representation. Waner [W, Section 2] introduced the notion of an equivariant space being V*-connective*, generalizing the non-equivariant notion of n-connectivity. The key observation is that one can make sense of the set $[S^{V+k} \wedge G/H_+, X]_*$ not just for $k \geq 0$, but for $k \geq -|V^H|$ (here, and elsewhere, $|W|$ denotes the real dimension of the vector space W). If V_H denotes V regarded as an H-representation, then there is a decomposition $V_H = V(H) \oplus V^H$, where $V(H)$ is the orthogonal complement of the fixed space V^H. One then considers the chain of equalities

$$[S^{V+k} \wedge G/H_+, X]_* \cong [S^{V_H+k}, X]_*^H \cong [S^{V(H)+|V^H|+k}, X]_*^H$$

and observes that the right-hand set makes sense for $k \geq -|V^H|$. We can therefore take this as a *definition* for the left-hand set when k is negative.

A pointed G-space X is called ***V*-connected** if $[S^{V+k} \wedge G/H_+, X]_* = 0$ for all subgroups H and all $0 \geq k \geq -|V^H|$. Waner proved that this is equivalent to requiring that X^H is $|V^H|$-connected for all subgroups H. This result eventually appeared, in expanded form, in [Lw2, Lemma 1.2].

The following result of Lewis [Lw3, Lemma 3.7] will be used often:

Lemma 2.3. *Suppose $V \supseteq 1$ (the trivial representation), and let X and Y be pointed G-spaces which are both $(V-1)$-connected. Then a map $X \to Y$ is a weak equivalence if and only if for every $k \geq 0$ and every subgroup H it induces an isomorphism $[S^{V+k} \wedge G/H_+, X]_* \cong [S^{V+k} \wedge G/H_+, Y]_*$.*

2.4. **Eilenberg-MacLane spectra.**
When G is a finite group, let $Or(G)$ denote the *orbit category* of G—the full subcategory of G-spaces whose objects are the orbits G/H. Recall that a *Mackey functor* for G is a pair of functors (M^*, M_*) from $Or(G)$ to Abelian groups having the properties that

(a) M^* is contravariant and M_* is covariant;
(b) $M^*(G/H) = M_*(G/H)$ for all H;
(c) For every $t : G/H \to G/H$ one has $t_* \circ t^* = id$;
(d) The double coset formula holds.

We will not write down what the last condition means in general, but see [M, XIX.3].

The importance of Mackey functors is that if E is an equivariant spectrum and X is any pointed space, then the assignment $G/H \mapsto [\Sigma^\infty(G/H_+ \wedge X), E]$ has a natural structure of a Mackey functor. In the case $X = S^V$, this Mackey functor is denoted $\underline{\pi}_V(E)$ or $\underline{E}^{-V}$.

In the case $G = \mathbb{Z}/2$ the orbit category is quite simple, having the form

$$\overset{t}{\circlearrowleft}\; \mathbb{Z}/2 \xrightarrow{\;i\;} e$$

where $it = i$ and $t^2 = id$. It follows that a Mackey functor for $\mathbb{Z}/2$ consists of Abelian groups $M(\mathbb{Z}/2)$ and $M(e)$ together with restriction and transfer maps

$$M(e) \underset{i_*}{\overset{i^*}{\rightleftarrows}} M(\mathbb{Z}/2), \qquad M(\mathbb{Z}/2) \underset{t_*}{\overset{t^*}{\rightleftarrows}} M(\mathbb{Z}/2)$$

satisfying the following conditions:

(i) (Contravariant functoriality) $(t^*)^2 = id$ and $t^*i^* = i^*$;
(ii) (Covariant functoriality) $(t_*)^2 = id$ and $i_*t_* = i_*$;
(iii) $t_* \circ t^* = id$;
(iv) (Double Coset formula) $i^* \circ i_* = id + t^*$.

We will specify a Mackey functor for $\mathbb{Z}/2$ by specifying the diagram

$$\overset{t^*}{\circlearrowleft}\; M(\mathbb{Z}/2) \underset{i^*}{\overset{i_*}{\rightleftarrows}} M(e).$$

Example 2.5.

(a) The Mackey functor we will be most concerned with is the *constant coefficient* Mackey functor $\underline{\mathbb{Z}}$:

$$\overset{id}{\circlearrowleft}\; \mathbb{Z} \underset{id}{\overset{2}{\rightleftarrows}} \mathbb{Z}.$$

Such a Mackey functor exists over any finite group G, and for any abelian group in place of $\mathbb{Z}$: the restriction maps are all identities, and the transfer maps $M(G/H) \to M(G/K)$ are multiplication by the index $[K : H]$.

(b) We define $\underline{\mathbb{Z}}^{op}$ to be

$$\overset{id}{\circlearrowleft}\; \mathbb{Z} \underset{2}{\overset{id}{\rightleftarrows}} \mathbb{Z}.$$

(c) The *Burnside ring* Mackey functor $\underline{A}$ is the one for which $A(G/H)$ is the Burnside ring of H. For $G = \mathbb{Z}/2$ this is

$$\overset{id}{\circlearrowleft}\; \mathbb{Z} \underset{i^*}{\overset{i_*}{\rightleftarrows}} \mathbb{Z} \oplus \mathbb{Z}.$$

where $i^*(a, b) = a + 2b$ and $i_*(a) = (0, a)$.

Every Mackey functor M has an associated $RO(G)$-graded cohomology theory denoted $V \mapsto H^V(-; M)$ (where V runs over all orthogonal G-representations), which is uniquely characterized by the properties that

- $H^n(G/H;M) = \begin{cases} M(G/H) & \text{if } n=0, \\ 0 & \text{otherwise,} \end{cases}$
  (here n denotes the trivial representation of G on $\mathbb{R}^n$), and
- the restriction maps $H^0(G/K;M) \to H^0(G/H;M)$ induced by $i\colon G/H \to G/K$ coincide with the maps i^* in the Mackey functor.

The transfer maps of the Mackey functor will coincide with the transfer maps in this cohomology theory (or with the pushforward maps in the associated *homology* theory). Details are in [M, Chap. IX.5].

2.6. **Eilenberg-MacLane spaces.**
When M is a Mackey functor, the Vth space in the Ω-spectrum for HM is called an Eilenberg-MacLane space of type $K(M,V)$. Such spaces are $(V-1)$-connected, and have the properties that $[S^{V+k} \wedge G/H_+, K(M,V)] = 0$ for $k > 0$ and the Mackey functor $G/H \mapsto [S^V \wedge G/H_+, K(M,V)]$ is isomorphic to M. See [Lw3, Definition 1.4] for this characterization.

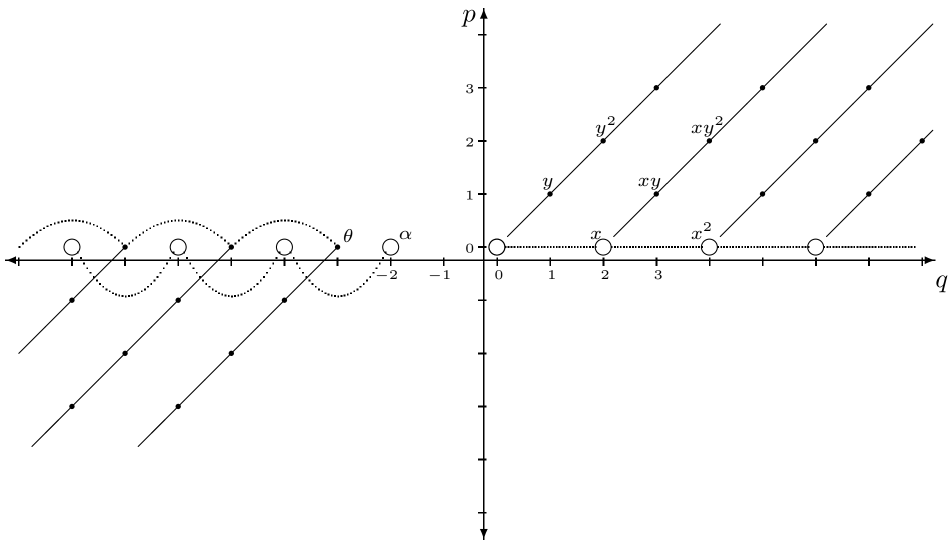
When $G = \mathbb{Z}/2$ and $V = \mathbb{R}^p \oplus (\mathbb{R}_-)^q$, we will usually adopt the motivic notation $K(M,V) = K(M(q), p+q)$. Likewise $H^V(-;M)$ will be written as either $H^{p+q,q}(-;M)$ or $H^{p+q}(-;M(q))$, usually the former.

2.7. **The theory** $H^{*,*}(X;\underline{\mathbb{Z}})$**.**
In this section we set down the basic facts about the cohomology theory $H^*(-;\underline{\mathbb{Z}})$ (which we'll sometimes write $H\underline{\mathbb{Z}}$). We will need to know its coefficient groups $H^{p,q}(pt;\underline{\mathbb{Z}})$ and $H^{p,q}(\mathbb{Z}/2;\underline{\mathbb{Z}})$, their ring structures, and the transfer and restriction maps between them. These things have certainly been computed many times over the years, although it's hard to find a precise reference. The corresponding facts about the theory $H^*(-;\underline{A})$ can be found in [Lw1, Thms. 2.1,4.3], where they are attributed to Stong. The necessary information about $H^*(-;\underline{\mathbb{Z}})$ can be deduced from these with a little bit of work, although it turns out to be much easier to avoid $H^*(-;\underline{A})$ altogether. The corresponding information about $H^*(-;\underline{\mathbb{Z}}/2)$ is in [HK1, Prop. 6.2], and again one can deduce the integral analogs with a little bit of work. An interesting computation of the *positive* part of the coefficient ring can be found in [LLM, Thm. 4.1]; this gives explicit cycles representing each element. In any case, the ultimate conclusions are listed in the theorem below. Although the results are not new, we have included proofs in Appendix B for the reader's convenience.

Theorem 2.8.
(a) The abelian group structure of $H^{*,*}(pt;\underline{\mathbb{Z}})$ is

$$H^{p,q}(pt;\underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2 & \text{if } p-q \text{ is even and } q \geq p > 0; \\ \mathbb{Z} & \text{if } p=0 \text{ and } q \text{ is even;} \\ \mathbb{Z}/2 & \text{if } p-q \text{ is odd and } q+1 < p \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

These groups are shown in the following picture, where hollow circles denote $\mathbb{Z}$'s and solid dots represent $\mathbb{Z}/2$'s (note that the p-axis is the vertical one):



(An easy way to keep track of the grading is to remember that $y \in H^{1,1}$ and $x \in H^{0,2}$).

(b) The multiplicative structure is completely determined by the properties that

(i) It is commutative;

(ii) The solid lines in the above diagram represent multiplication by the class $y \in H^{1,1}$;

(iii) The dotted lines represent multiplication by $x \in H^{0,2}$ (but note that only a representative set of dotted lines have been drawn);

(iv) $x\alpha = 2$.

In particular, the subring consisting of $H^{p,q}$ where $p, q \geq 0$ is the polynomial algebra $\mathbb{Z}[x, y]/(2y)$.

(c) The ring $H^{*,*}(\mathbb{Z}/2; \underline{\mathbb{Z}})$ is isomorphic to $\mathbb{Z}[u, u^{-1}]$, where u has degree $(0, 1)$.

(d) The Mackey functor $\underline{H\mathbb{Z}}^{0,2n}$ is $\underline{\mathbb{Z}}$ when $n \geq 0$ and $\underline{\mathbb{Z}}^{op}$ when $n < 0$ (see Example 2.5 for notation).

Remark 2.9. For computations it's often convenient to give every element of $H^{*,*}(pt)$ a name in terms of x, y, and θ. For instance, α can be named as $\frac{2}{x}$, and the class in degree $(-2, -7)$ can be named $\frac{\theta}{xy^2}$.

Remark 2.10. If $q > 0$ and you look in degree $(1 - q, -q)$ and read vertically upwards, you are seeing the groups $\tilde{H}^*_{sing}(\mathbb{R}P^{q-1})$. If you look in degree $(q - 1, q)$ and read vertically downwards, you are seeing the groups $\tilde{H}_*(\mathbb{R}P^{q-1})$. The connection is explained in detail in Appendix B.

Remark 2.11. It's worth pointing out what aspects of the above picture are similar to the motivic setting, and which are not. In the motivic setting one has that $H^{p,q}_{mot}(pt; \underline{\mathbb{Z}}) = 0$ for $q < 0$, but notice that this is *not* the case for the above $\mathbb{Z}/2$-equivariant theory. This difference is tied to the fact that classical algebraic K-theory is connective, whereas KO-theory is not. The Beilinson-Soulé conjecture is that $H^{p,q}_{mot}(pt; \underline{\mathbb{Z}}) = 0$ when $q > 0$ and $p < 0$, which is clearly satisfied in our $\mathbb{Z}/2$-world. The non-zero motivic cohomology groups of a point should correspond to the groups lying in the first quadrant of the above diagram, with the same vanishing line. The motivic groups lying *along* this line are the Milnor K-theory groups.

The above result tells us everything about the Eilenberg-MacLane spaces $K(\mathbb{Z}(n), 2n)$. Here are a couple of the main points:

Corollary 2.12.
(a) Non-equivariantly, $K(\mathbb{Z}(n), 2n)$ is a $K(\mathbb{Z}, 2n)$.
(b) The induced action of $\mathbb{Z}/2$ on $\pi_{2n}(K(\mathbb{Z}(n), 2n)) = \mathbb{Z}$ is multiplication by $(-1)^n$.
(c) The fixed set $K(\mathbb{Z}(n), 2n)^{\mathbb{Z}_2}$ has the homotopy type of either

$$K(\mathbb{Z}, 2n) \times K(\mathbb{Z}/2, 2n-2) \times K(\mathbb{Z}/2, 2n-4) \times \cdots \times K(\mathbb{Z}/2, n) \qquad (n \text{ even,})$$

or

$$K(\mathbb{Z}/2, 2n-1) \times K(\mathbb{Z}/2, 2n-3) \times \cdots \times K(\mathbb{Z}/2, n) \qquad (n \text{ odd}).$$

Proof. A theorem of [dS] identifies $K(\mathbb{Z}(n), 2n)$ with $AG(S^{2n,n})$, the free abelian group generated by $S^{2n,n}$. So both $K(\mathbb{Z}(n), 2n)$ and its fixed set are topological abelian groups, hence products of Eilenberg-MacLane spaces. The homotopy groups can be read off of $H^{*,*}(pt; \underline{\mathbb{Z}})$ and $H^{*,*}(\mathbb{Z}/2; \underline{\mathbb{Z}})$. This proves (a) and (c).

Note that $[S^{2n,n}, K(\mathbb{Z}(n), 2n)]^e \cong [S^{2n,n} \wedge \mathbb{Z}/2_+, K(\mathbb{Z}(n), 2n)]_* \cong H^{0,0}(\mathbb{Z}/2)$, and we know the group action on the latter is trivial (because we know the Mackey functor $\underline{H}^{0,0}$). The group $\pi_{2n}K(\mathbb{Z}(n), 2n)$ may be written $[S^{2n,0}, K(\mathbb{Z}(n), 2n)]^e$, and this differs from the above in the replacement of $S^{2n,n}$ by $S^{2n,0}$. On the former sphere, the automorphism coming from the $\mathbb{Z}/2$ action has degree $(-1)^n$ (complex conjugation on $\mathbb{C}^n$ reflects n real coordinates). This proves (b). □

Remark 2.13. The homotopy fixed set $K(\mathbb{Z}(n), 2n)^{h\mathbb{Z}_2}$ will also be a topological abelian group, and hence a generalized Eilenberg-MacLane space. The spectral sequence for computing homotopy groups of a homotopy limit collapses, and shows that $K(\mathbb{Z}(n), 2n)^{h\mathbb{Z}/2}$ is either

$$K(\mathbb{Z}, 2n) \times K(\mathbb{Z}/2, 2n-2) \times K(\mathbb{Z}/2, 2n-4) \times \cdots \times K(\mathbb{Z}/2, 0) \qquad (n \text{ even}),$$

or

$$K(\mathbb{Z}/2, 2n-1) \times K(\mathbb{Z}/2, 2n-3) \times \cdots \times K(\mathbb{Z}/2, 1) \qquad (n \text{ odd}).$$

So the actual fixed set is a truncation of the homotopy fixed set. This observation reappears in section 7.

3. Equivariant Postnikov-section functors

In this section we define two types of equivariant Postnikov section functors, denoted $\mathbb{P}_V$ and P_V, and list their basic properties.

To begin this section we work in the context of an arbitrary finite group G. The category $\mathcal{T}op_G$ denotes the category of G-spaces which are compactly-generated and weak Hausdorff, with equivariant maps. We will eventually specialize to the case $G = \mathbb{Z}/2$, but for the present it is just as easy to work in greater generality. There is a model category structure on $\mathcal{T}op_G$ analagous to the usual one on $\mathcal{T}op$.

3.1. **Generalities.** Recall that a space A is said to be **small with respect to closed inclusions** if it has the property that for any sequence of closed inclusions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow Z_2 \hookrightarrow \cdots$$

the canonical map $\operatorname{colim}_i \mathcal{T}op_G(A, Z_i) \to \mathcal{T}op_G(A, \operatorname{colim} Z_i)$ is an isomorphism. Every compact Hausdorff space is small in this sense. Let CA denote the cone on A, and recall that $[X, Y]$ denotes *unpointed* homotopy classes of maps.

Let $\mathcal{A}$ be a set of well-pointed spaces, all of which are compact Hausdorff. The pointedness can be ignored for the moment, but will be needed later. We will say that a space Z is **$\mathcal{A}$-null** if it has the property that the maps $[*, Z] \to [\Sigma^n A, Z]$ (induced by $\Sigma^n A \to *$) are isomorphisms, for all $n \geq 0$ and all $A \in \mathcal{A}$. This is equivalent to saying that every map $\Sigma^n A \to Z$ extends over the cone.

For a space X one can construct a new space $P_{\mathcal{A}}(X)$ with the following properties:

(1) There is a natural map $X \to P_{\mathcal{A}} X$;
(2) $P_{\mathcal{A}} X$ is $\mathcal{A}$-null;
(3) If Z is an $\mathcal{A}$-null space and $X \to Z$ is a map, then there is a lifting

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \nearrow & \\ P_{\mathcal{A}} X & & \end{array}$$

and this lifting is unique up to homotopy.

The functors $P_{\mathcal{A}}(X)$ are called **nullification functors** in [F]. They are examples of Bousfield localization functors, for which an excellent reference is [H, Chapters 3,4]. In our context we construct them as follows: For any space Y, let $F_{\mathcal{A}}(Y)$ be defined by the pushout square

$$\begin{array}{ccc} \coprod_\sigma \Sigma^n A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \coprod_\sigma C(\Sigma^n A) & \longrightarrow & F_{\mathcal{A}} Y, \end{array}$$

where σ runs over all maps $\Sigma^n A \to Y$ (for all $A \in \mathcal{A}$, and $n \geq 0$). One then considers the sequence of closed inclusions

$$X \hookrightarrow F_{\mathcal{A}} X \hookrightarrow F_{\mathcal{A}} F_{\mathcal{A}} X \hookrightarrow F_{\mathcal{A}} F_{\mathcal{A}} F_{\mathcal{A}} X \hookrightarrow \cdots$$

and $P_{\mathcal{A}}(X)$ is defined to be the colimit. It is routine to check, using basic obstruction theory, that this construction has the required properties.

Remark 3.2. We will often use the observation that if $\mathcal{A} \subseteq \mathcal{A}'$ then one has a canonical map $P_{\mathcal{A}} X \to P_{\mathcal{A}'} X$.

In terms of the localization of model categories (cf. [H]), we are localizing $\mathcal{T}op_G$ at the set of maps $\{A \to * | A \in \mathcal{A}\}$ and $P_{\mathcal{A}}$ is the localization functor. This uses the fact that the objects in $\mathcal{A}$ are well-pointed. As a consequence, the functors $P_{\mathcal{A}} X$ have the standard properties one would expect from a localization functor. We omit the proof of the following result: such properties can be found in [H] in complete generality, or in [F] for the category of spaces. In this case they are also easy to prove directly by standard arguments.

Proposition 3.3.

(a) Let $X \to Y$ and $X \to Z$ be maps, where $X \to Y$ is a cofibration. If Z is $\mathcal{A}$-null and $P_{\mathcal{A}}X \to P_{\mathcal{A}}Y$ is a weak equivalence, then there is a lift

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & \nearrow & \\ Y & & \end{array}$$

and this lift is unique up to homotopy.

(b) Let $X : \mathcal{C} \to \mathcal{T}op_G$ be a diagram. Then the natural map

$$P_{\mathcal{A}}(\operatorname{hocolim}_{\alpha} X_\alpha) \longrightarrow P_{\mathcal{A}}(\operatorname{hocolim} P_{\mathcal{A}} X_\alpha)$$

is a weak equivalence.

(c) If $X \to Y \to Z$ is a homotopy cofiber sequence and $P_{\mathcal{A}}X$ is contractible, then $P_{\mathcal{A}}Y \to P_{\mathcal{A}}Z$ is a weak equivalence.

3.4. **The functors $\mathbb{P}$ and P.** The most basic examples of nullification functors are the ordinary Postnikov section functors (when $G = \{e\}$): given a non-equivariant space X one forms P_nX by killing off all homotopy groups above dimension n. In the language of the previous section $P_nX = P_{\mathcal{A}_n}X$, where $\mathcal{A}_n$ is the set $\{S^{n+1}, S^{n+2}, \cdots\}$. In fact we would get a homotopy equivalent space by just taking $\mathcal{A}_n$ to be $\{S^{n+1}\}$, but we have arranged things so that $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$ because our formalism then gives natural maps $P_{\mathcal{A}_{n+1}}X \to P_{\mathcal{A}_n}X$.

When one wants to introduce Postnikov section functors for G-spaces several possibilities present themselves. One thing to note is that when we kill off all maps from a space A, we would also like to be killing off maps from all spaces $A \wedge Z$. In the nonequivariant setting this is automatic, because Z can be built from spheres and therefore $A \wedge Z$ is built from suspensions of A. In the equivariant setting we have to explicitly build this into the theory, by making sure that whenever we kill off a space A we also kill off $A \wedge G/H_+$ for all subgroups H.

If V is a representation of G then one Postnikov functor we can consider is $P_{\mathcal{A}}(X)$ where $\mathcal{A}$ is the set $\{S^{V+n} \wedge G/H_+ | n > 0, H \leq G\}$. One can get the same result by doing the following, and for functorial reasons it is somewhat better: Let

$$\tilde{\mathcal{A}}_V = \{S^W \wedge G/H_+ \mid W \supseteq V + 1, H \leq G\},$$

and define $\mathbb{P}_V X = P_{\tilde{\mathcal{A}}_V}(X)$. This definition guarantees that if $V \subseteq U$ then there are natural maps $\mathbb{P}_U X \to \mathbb{P}_V X$.

On the other hand we can also do the following. Let

$$\mathcal{A}_V = \{S^W \wedge G/H_+ \mid W \supset V, H \leq G\},$$

and define $P_V X = P_{\mathcal{A}_V}(X)$. Again, whenever $V \subseteq U$ there are natural maps $P_U X \to P_V X$. Moreover, since $\tilde{\mathcal{A}}_V \subset \mathcal{A}_V$ one has maps $\mathbb{P}_V X \to P_V X$.

Remark 3.5. The difference between $\mathbb{P}_V$ and P_V shows up in the following way. Non-equivariantly, there are no non-trivial maps $S^n \to S^k$ when $k > n$. As a result, when one forms the Postnikov section P_nX one doesn't change the homotopy groups in low dimensions: $[S^m, X] \to [S^m, P_nX]$ is an isomorphism for $m \leq n$. In the equivariant theory, however, there can be many non-trivial maps $S^V \to S^W$ for $V \subset W$; so when forming a Postnikov section by killing off maps from large spheres, one may actually be creating *new* maps from smaller spheres.

What is true, however, is that if $V + 1 \subseteq W$—that is, if W contains V plus at least one copy of the trivial representation—then all equivariant maps $S^V \to S^W$ are null. This leads one to the functors $\mathbb{P}_V$ defined above, which are designed so that they don't change homotopy classes of maps from S^V and smaller spheres. The general rule is that the functors $\mathbb{P}_V$ are better behaved that P_V: they are easier to compute, and their properties (outlined below) closely resemble those of non-equivariant Postnikov sections. These are the same as the Postnikov functors in [M, II.1].

Proposition 3.6 (Properties of $\mathbb{P}$)**.** *If X is a pointed G-space X and V is a G-representation, the following are true:*

(a) The map $X \to \mathbb{P}_V X$ induces an isomorphism of the sets $[S^{k,0} \wedge G/H_+, -]_*$ for $0 \le k \le \dim V^H$, and an epimorphism for $k = \dim V^H + 1$.
(b) If W is a G-representation for which $\dim W^H \le \dim V^H$ for all subgroups $H \subseteq G$, then $[S^W, X]_* \to [S^W, \mathbb{P}_V X]_*$ is an isomorphism.
(c) The homotopy fiber of $\mathbb{P}_{V+1}X \to \mathbb{P}_V X$ is an Eilenberg-MacLane space of type $K(\underline{\pi}_{V+1}X, V)$.
(d) The homotopy limit of the sequence

$$\cdots \to \mathbb{P}_{V+2}X \to \mathbb{P}_{V+1}X \to \mathbb{P}_V X$$

is weakly equivalent to X.
(e) If V contains the regular representation of G, then the Postnikov section $\mathbb{P}_V(S^V)$ is an Eilenberg-MacLane space of type $K(\underline{A}, V)$ where $\underline{A}$ is the Burnside-ring Mackey functor.

Proof. The proof is completely standard, so we will only give a brief sketch. Suppose that X is a space, $S^W \wedge G/J_+ \to X$ is a map, and we attach the cone on this map to construct a new space X_1. The for any subgroup $H \le G$, X_1^H is obtained from X^H by attaching a cone on the map $(S^W \wedge G/J_+)^H \to X^H$. The domain of this map is a wedge of spheres of dimension $|W^H|$, and so $X^H \to X_1^H$ is $|W^H|$-connected. From these considerations (a) is immediate.

Part (b) follows from (a) and the fact that S^W has an equivariant CW-structure made up of cells $S^{k,0} \wedge G/H_+$ where $k \le \dim W^H$. Part (d) is also immediate from (a): the map $X \to \mathbb{P}_V X$ becomes highly connected on all fixed sets as V gets large.

For part (c), note first that for an arbitrary space X the object $\underline{\pi}_V X$ may not be a Mackey functor—it is instead a V*-Mackey functor* as defined in [Lw3, 1.2]. A characterization of Eilenberg-MacLane spaces is given in [Lw3, 1.4], and it is easy to use parts (a) and (b) to check that the homotopy fiber we're looking at has the properties listed there.

Part (d) is an immediate consequence of (c) and the well-known isomorphism of Mackey functors $\underline{\pi}_V(S^V) \cong \underline{A}$ (which follows from [M, IX.1.4,XVII.2]). □

Suppose now that $\mathcal{E}$ is an equivariant spectrum, and let E denote the 0th space of the corresponding Ω-spectrum. By applying the functors $\mathbb{P}_n$ we obtain a tower

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathbb{P}_2 E & \longrightarrow & \mathbb{P}_1 E & \longrightarrow & \mathbb{P}_0 E & \longrightarrow * \\
& \uparrow & & \uparrow & & \uparrow & \\
& K(\underline{\pi}_2 E, 2) & & K(\underline{\pi}_1 E, 1) & & K(\underline{\pi}_0 E, 0) &
\end{array}$$

The homotopy spectral sequence for maps from X into this tower gives the classical equivariant Atiyah-Hirzebruch spectral sequence $H^p(X;\underline{\mathcal{E}}^q) \Rightarrow \mathcal{E}^{p+q,0}(X)$ with a suitable truncation. Here $\underline{\mathcal{E}}^q$ denotes the Mackey functor $G/H \mapsto \mathcal{E}^q(G/H)$. To get the full spectral sequence one can look at the Postnikov towers for each $\mathcal{E}_V$ (the Vth space in the Ω-spectrum for $\mathcal{E}$) and note that the resulting spectral sequences can be pasted together.

Unfortunately, for $\mathcal{E} = KR$ this spectral sequence is not the one we're looking for. The above spectral sequence collapses when $X = *$ and gives no information, whereas the spectral sequence we're looking for is very non-trivial when $X = *$. In the context of algebraic K-theory, the analog of the above spectral sequence is the *Brown-Gersten spectral sequence* of [BG].

The functors P_V don't have the property that the homotopy fiber of $P_{V+1}X \to P_V X$ is necessarily an equivariant Eilenberg-MacLane space. For the record, here are the basic properties of P. The proofs are the same as for Proposition 3.6.

Proposition 3.7 (Properties of P)**.** *For any pointed G-space X and any G-representation V, the following are true:*

(a) The map $X \to P_V X$ induces an isomorphism of the sets $[S^{k,0} \wedge G/H_+, -]_*$ for $0 \le k < \dim V^H$, and an epimorphism for $k = \dim V^H$.

(b) If W is a G-representation for which $\dim W^H < \dim V^H$ for all subgroups $H \subseteq G$, then $[S^W, X]_* \to [S^W, \mathbb{P}_V(X)]_*$ is an isomorphism.

The main reason we care about the functors P_V is the following result:

Theorem 3.8. *When $G = \mathbb{Z}/2$ and V contains the trivial representation, the space $P_V(S^V)$ has the equivariant weak homotopy type of the Eilenberg-MacLane space $K(\underline{\mathbb{Z}}, V)$.*

The proof of this result is somewhat involved, and will be postponed until section 8. However, we can give some intuition for why it's true. If $V \supseteq \mathbb{C}$ one knows that $[S^V, S^V]_* = \mathbb{Z} \oplus \mathbb{Z}$ (the Burnside ring of $\mathbb{Z}/2$), and $[S^V, S^V]_* \to [S^V, \mathbb{P}_V(S^V)]_*$ is an isomorphism by Proposition 3.6(b). If one chooses the generators of $[S^V, S^V]_*$ appropriately, their difference factors through a 'Hopf map' $S^{V+\mathbb{R}_-} \to S^V$, where $\mathbb{R}_-$ is the sign representation of $\mathbb{Z}/2$. Since $P_V(S^{V+\mathbb{R}_-}) \simeq *$, this difference becomes null in $P_V(S^V)$ (note that $\mathbb{P}_V(S^{V+\mathbb{R}_-})$ is not contractible). So the two copies of $\mathbb{Z}$ in $[S^V, \mathbb{P}_V(S^V)]_*$ become identified in $[S^V, P_V(S^V)]_*$, and this is ultimately why $P_V(S^V)$ has the homotopy type of $K(\underline{\mathbb{Z}}, V)$ rather than $K(\underline{A}, V)$.

We'd like to justify the claim that the difference of generators factors through a Hopf map. First look at the case $V = \mathbb{C}$, so that $S^V = S^{2,1}$. Consider the degree map $[S^{2,1}, S^{2,1}]_* \to \mathbb{Z} \oplus \mathbb{Z}$ which sends a map f to the pair $(\deg f, \deg f^{\mathbb{Z}/2})$ (the degree, and the degree of the map restricted to the fixed set). This is injective, and the image consists of pairs (n, m) for which $n - m \equiv 0 \mod 2$. For the generators of $[S^{2,1}, S^{2,1}]_*$ we'll take the identity map, which has degree $(1, 1)$, and the complex conjugation map, which has degree $(-1, 1)$.

Now consider the Hopf projection $\mathbb{C}^2 - \{0\} \to P^1_{\mathbb{C}}$, which we may write as $p\colon S^{3,2} \to S^{2,1}$. Smashing the inclusion $S^{0,0} \hookrightarrow S^{1,1}$ with $S^{2,1}$ gives a map $j\colon S^{2,1} \hookrightarrow S^{3,2}$, and the degree of the composition pj is $(0, 2)$. So this composite is homotopic to the sum of our two chosen generators. When $V = \mathbb{C} \oplus W$, one takes this same argument and smashes everything in sight with S^W.

Another perspective on Theorem 3.8 is given in Section 8, where it is tied to the geometry of the infinite symmetric product construction.

4. The Postnikov tower for $\mathbb{Z} \times BU$

In this section we will consider the objects $P_{n\mathbb{C}}(\mathbb{Z} \times BU)$. In motivic indexing $P_{n\mathbb{C}}$ would be written $P_{(2n,n)}$, and we will abbreviate this as just P_{2n}. Our goal is the following

Theorem 4.1. *Let $\beta : S^{2,1} \to \mathbb{Z} \times BU$ be a map representing the Bott element in $\widetilde{KR}^{0,0}(S^{2,1})$, and let $\beta^n : S^{2n,n} \to \mathbb{Z} \times BU$ denote its nth power. Then*

$$P_{2n}(S^{2n,n}) \xrightarrow{\beta^n} P_{2n}(\mathbb{Z} \times BU) \longrightarrow P_{2n-2}(\mathbb{Z} \times BU)$$

is a homotopy fiber sequence.

Corollary 4.2. *There is a tower of homotopy fiber sequences*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_4(\mathbb{Z} \times BU) & \longrightarrow & P_2(\mathbb{Z} \times BU) & \longrightarrow & P_0(\mathbb{Z} \times BU) & \longrightarrow & * \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & K(\mathbb{Z}(2), 4) & & K(\mathbb{Z}(1), 2) & & \mathbb{Z} & & \end{array}$$

and the homotopy limit of the tower is $\mathbb{Z} \times BU$.

As explained in the introduction, one can prove the corollary using only the functors $\mathbb{P}$, and this is easier in the end. Writing $\mathbb{P}_{2n}$ for $\mathbb{P}_{n\mathbb{C}}$, we have the following:

Proposition 4.3. *The homotopy fiber of $\mathbb{P}_{2n}(\mathbb{Z} \times BU) \to \mathbb{P}_{2n-2}(\mathbb{Z} \times BU)$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}(n), 2n)$.*

Proof. Let $\mathbb{F}_n$ denote the homotopy fiber. From Proposition 3.6(a) it follows immediately that the fixed set of $\mathbb{F}_n$ is $(n-1)$-connected. Non-equivariantly $\mathbb{P}_{2n}$ has the homotopy type of the ordinary Postnikov section functor, and so $\mathbb{F}_n$ is $(2n-1)$-connected as a non-equivariant space (here we are using that $\mathbb{Z} \times BU$ has no odd homotopy groups). So in equivariant language $\mathbb{F}_n$ is $(n\mathbb{C}-1)$-connected (cf. section 2.2). By construction we have that $[S^{2n+k,n} \wedge \mathbb{Z}/2_+, \mathbb{F}_n]_* = 0 = [S^{2n+k,n}, \mathbb{F}_n]_*$ for all $k > 0$, and the long exact homotopy sequence shows that the Mackey functor $\underline{\pi}_{2n,n}(\mathbb{F}_n)$ is isomorphic to $\underline{\pi}_{2n,n}(\mathbb{Z} \times BU) \cong \underline{KR}^{2n,n}(pt) \cong \underline{\mathbb{Z}}$. Lewis's characterization of Eilenberg-MacLane spaces [Lw3, Def. 1.4] now shows that $\mathbb{F}_n$ is a $K(\mathbb{Z}(n), 2n)$. $\square$

For the remainder of the section we let F_n denote the homotopy fiber

$$F_n \longrightarrow P_{2n}(\mathbb{Z} \times BU) \longrightarrow P_{2n-2}(\mathbb{Z} \times BU).$$

The map $\beta^n : P_{2n}(S^{2n,n}) \to P_{2n}(\mathbb{Z} \times BU)$ becomes null when we pass to the space $P_{2n-2}(\mathbb{Z} \times BU)$, and therefore it lifts to F_n (and the lifting is unique up to homotopy). Our task is to show that this lifting is a weak equivalence. Both the domain and codomain are highly connected, and so we can use Lemma 2.3. Here is a restatement for the special case where $G = \mathbb{Z}/2$ and $V = \mathbb{C}^n$:

Lemma 4.4. *Let X and Y be pointed $\mathbb{Z}/2$-spaces with the properties that*

(i) $[S^{k,0}, X]_* = [S^{k,0}, Y]_* = 0$ *for* $0 \le k < n$, *and*
(ii) $[\mathbb{Z}/2_+ \wedge S^{k,0}, X]_* = [\mathbb{Z}/2_+ \wedge S^{k,0}, Y]_* = 0$ *for* $0 \le k < 2n$.

Then a map $X \to Y$ is an equivariant weak equivalence if and only if it induces isomorphisms

$$[S^{2n+k,n}, X]_* \xrightarrow{\cong} [S^{2n+k,n}, Y]_* \text{ and } [\mathbb{Z}/2_+ \wedge S^{2n+k,n}, X]_* \xrightarrow{\cong} [\mathbb{Z}/2_+ \wedge S^{2n+k,n}, Y]_*$$

for every $k \geq 0$.

We know by Theorem 3.8 that $P_{2n}(S^{2n,n})$ is a $K(\mathbb{Z}(n), 2n)$-space and therefore we know it's homotopy groups—these are precisely the groups $H^{p,q}(pt; \underline{\mathbb{Z}})$ and $H^{p,q}(\mathbb{Z}/2; \underline{\mathbb{Z}})$. So it's easy to see that $K(\mathbb{Z}(n), 2n)$ satisfies the conditions in the above lemma. The general strategy at this point would be to

(a) Show that F_n also satisfies the conditions of the lemma;
(b) Observe that $[S^{2n+k,n}, F_n]_* = 0 = [\mathbb{Z}/2_+ \wedge S^{2n+k,n}, F_n]_*$ for $k > 0$, for trivial reasons;
(c) Show that the map $P_{2n}(S^{2n,n}) \to F_n$ induces isomorphisms on $[S^{2n,n}, -]_*$ and $[\mathbb{Z}/2_+ \wedge S^{2n,n}, -]_*$;
(d) Use Lemma 4.4 to deduce that $P_{2n}(S^{2n,n}) \to F_n$ is a weak equivalence.

In fact this approach can be streamlined a bit by using the functors $\mathbb{P}$ as a crutch.

Lemma 4.5.

(a) Let X be a pointed $\mathbb{Z}/2$-space with the property that the forgetful map $[S^{2n,n}, X]_* \to [S^{2n}, X]^e_*$ is injective. Then the natural map $\mathbb{P}_{2n}X \to P_{2n}X$ is a weak equivalence.
(b) $\mathbb{P}_{2n}(\mathbb{Z} \times BU) \to P_{2n}(\mathbb{Z} \times BU)$ is a weak equivalence.

Proof. For (a) we only have to show that $[S^{2n+p,n+p}, \mathbb{P}_{2n}X] = 0$ for all $p > 0$; that is, we must show that $\mathbb{P}_{2n}X$ is null with respect to $\mathcal{A}_{(2n,n)}$, not just $\tilde{\mathcal{A}}_{(2n,n)}$ (see section 3.4). Consider the basic Puppe sequence $\mathbb{Z}/2_+ \to S^{0,0} \to S^{1,1} \to \mathbb{Z}/2_+ \wedge S^{1,0}$. Smashing with $S^{2n,n}$ yields

$$\mathbb{Z}/2_+ \wedge S^{2n,n} \to S^{2n,n} \to S^{2n+1,n+1} \to \mathbb{Z}/2_+ \wedge S^{2n+1,n}.$$

Mapping this sequence into $\mathbb{P}_{2n}X$ gives the top edge of the following diagram:

$$\begin{array}{ccccc}
 & & [S^{2n+1,n+1}, \mathbb{P}_{2n}X] & \longleftarrow & [\mathbb{Z}/2_+ \wedge S^{2n+1,n}, \mathbb{P}_{2n}X] = 0 \\
 & & \downarrow & & \\
[\mathbb{Z}/2_+ \wedge S^{2n,n}, \mathbb{P}_{2n}X] & \longleftarrow & [S^{2n,n}, \mathbb{P}_{2n}X] & & \\
\cong \uparrow & & \cong \uparrow & & \\
[\mathbb{Z}/2_+ \wedge S^{2n,n}, X] & \longleftarrow & [S^{2n,n}, X]. & &
\end{array}$$

The right-most group in the top row is zero just because of the definition of $\mathbb{P}_{2n}$, and Proposition 3.6(b) implies that the labelled vertical maps are isomorphisms. The map in the bottom row may be identified with the forgetful map

$$[S^{2n}, X]^e \leftarrow [S^{2n,n}, X],$$

and we have assumed that this is injective. It's now clear that $[S^{2n+1,n+1}, \mathbb{P}_{2n}X]$ must be zero.

Smashing the above Puppe sequence with $S^{2n+p,n+p}$ gives

$$S^{2n+p,n+p} \to S^{2n+p+1,n+p+1} \to \mathbb{Z}/2_+ \wedge S^{2n+p+1,n+p}.$$

By induction we know that $\mathbb{P}_{2n}X$ is null with respect to the first and third space, so it is also null with respect to the second. This finishes (a).

Proving (b) is of course just a matter of checking that $\mathbb{Z}\times BU$ has the property specified in (a). So we must check that the forgetful map

$$\mathbb{Z}=\widetilde{KR}^{\,0,0}(S^{2n,n})\to \tilde{K}^0(S^{2n})=\mathbb{Z}$$

is injective. But the map is easily seen to be an isomorphism, as β^n is an explicit generator for both the domain and target. □

Proof of Theorem 4.1. We must show that $j\colon P_{2n}(S^{2n,n})\to F_n$ is a weak equivalence. The equivalences $\mathbb{P}_{2n}(\mathbb{Z}\times BU)\to P_{2n}(\mathbb{Z}\times BU)$ induce equivalences $\mathbb{F}_n\to F_n$, and we already know $\mathbb{F}_n\simeq K(\mathbb{Z}(n),2n)$. Lemma 4.4 now implies that j must be a weak equivalence (one uses the fact that β^n is a generator for $KR^{2n,n}(pt)$). □

Proof of Corollary 4.2. This is just a a restatement, together with the fact that the holim of the tower is $\mathbb{Z}\times BU$. The latter follows from Proposition 3.7(a,b). □

5. Properties of the spectral sequence

If X is a $\mathbb{Z}/2$-space, then the associated homotopy spectral sequence for the tower of Corollary 4.2 has the form

$$H^{p,-\frac{q}{2}}(X;\underline{\mathbb{Z}})\Rightarrow [S^{-p-q,0}\wedge X_+,\mathbb{Z}\times BU]_*,$$

being confined to the quadrant $p,q\le 0$. This is an unstable version of the spectral sequence we're looking for. Producing the stable version is not difficult, as one can replace X by various suspensions $S^{a,b}\wedge X$ and use the periodicity of $\mathbb{Z}\times BU$ to get a 'family' of spectral sequences which patch together. We'll take another approach to this in the next section, and for now be content with analyzing the unstable case.

5.1. **Adams operations.**

There is a map of $\mathbb{Z}/2$-spaces $\psi^k:\mathbb{Z}\times BU\to\mathbb{Z}\times BU$ inducing the operation ψ^k on $KR^0(X)$, constructed out of the λ^i maps in the usual way. The functoriality of the constructions P_{2n} shows that ψ^k induces a self-map of the Postnikov tower for $\mathbb{Z}\times BU$, and therefore we get an action of the Adams operations on the spectral sequence. We must identify the action on the E_2-term:

Proposition 5.2. *The induced map $\psi^k:F_n\to F_n$ coincides with the multiplication by k^n map $K(\mathbb{Z}(n),2n)\to K(\mathbb{Z}(n),2n)$.*

Proof. If $\beta^n:S^{2n,n}\to\mathbb{Z}\times BU$ is the nth power of the Bott element, then we know the following diagram commutes:

$$\begin{array}{ccc} S^{2n,n} & \longrightarrow & \mathbb{Z}\times BU \\ {\scriptstyle\cdot k^n}\downarrow & & \downarrow{\scriptstyle\psi^k} \\ S^{2n,n} & \longrightarrow & \mathbb{Z}\times BU. \end{array}$$

This is just because we can compute

$$\psi^k(\beta^n)=(\psi^k\beta)^n=(k\beta)^n=k^n\beta^n.$$

Applying P_{2n} to the above diagram gives

$$\begin{array}{ccccc} S^{2n,n} & \longrightarrow & P_{2n}(S^{2n,n}) & \longrightarrow & P_{2n}(\mathbb{Z}\times BU) \\ \downarrow{\scriptstyle k^n} & & \downarrow{\scriptstyle P_{2n}(k^n)} & & \downarrow{\scriptstyle P_{2n}(\psi^k)} \\ S^{2n,n} & \longrightarrow & P_{2n}(S^{2n,n}) & \longrightarrow & P_{2n}(\mathbb{Z}\times BU). \end{array}$$

We have previously identified the map $F_n \to P_{2n}(\mathbb{Z}\times BU)$ with $P_{2n}(S^{2n,n}) \to P_{2n}(\mathbb{Z}\times BU)$, and so the argument may be completed by proving the following lemma. □

Lemma 5.3. *Let $k \in \mathbb{Z}$ and let $k : S^{2n,n} \to S^{2n,n}$ denote the map obtained by adding the identity to itself k times in the group $[S^{2n,n}, S^{2n,n}]_*$ (using the fact that $S^{2n,n}$ is a suspension). Then the localized map*

$$P_{2n}(k) : P_{2n}(S^{2n,n}) \to P_{2n}(S^{2n,n})$$

may be identified with the map $K(\mathbb{Z}(n), 2n) \to K(\mathbb{Z}(n), 2n)$ representing multiplication by k.

Proof. There are several ways one could do this. Write S for $S^{2n,n}$ and P for $P_{2n}(S)$. We of course have the diagram

$$\begin{array}{ccc} S & \longrightarrow & P \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle P(k)} \\ S & \longrightarrow & P. \end{array}$$

Using Proposition 3.3(a) it's easy to see that $[P, P]_* \to [S, P]_*$ is an isomorphism, and the arguments in Section 8 show that $[S, P] \cong \mathbb{Z}$ is generated by the localization map $S \to P$. This proves it. □

5.4. **The rational tower.**

Grassmannians have nice Schubert cell decompositions, which make it easy to compute $H^{*,*}(-)$. One of course finds that $H^{*,*}(BU) = H^{*,*}(pt)[c_1, c_2, \ldots]$ where c_i has degree $(2i, i)$. If we regard c_n as a map $BU \to K(\mathbb{Z}(n), 2n)$, then applying P_{2n} gives $P_{2n}(BU) \to P_{2n}K(\mathbb{Z}(n), 2n) = K(\mathbb{Z}(n), 2n)$ (the Eilenberg-MacLane space is already $\mathcal{A}_{(2n,n)}$-null). We claim the composite

$$K(\mathbb{Z}(n), 2n) = P_{2n}(S^{2n,n}) \xrightarrow{P(\beta^n)} P_{2n}(\mathbb{Z}\times BU) \xrightarrow{c_n} K(\mathbb{Z}(n), 2n)$$

is multiplication by $(n-1)!$. As in the last section, the argument comes down to knowing that $c_n(\beta^n)$ is $(n-1)!$ times the generator of $\tilde{H}^{2n,n}(S^{2n,n})$. This can be deduced via comparison maps to the nonequivariant groups, where the result is well-known (due to Bott, originally).

So we see that the inclusions of homotopy fibers $K(\mathbb{Z}(n), 2n) \to P_{2n}(\mathbb{Z}\times BU)$ are split rationally; hence the spectral sequence collapses rationally. If $\oplus_n H^{2n,n}(X)\otimes\mathbb{Q}$ is finite-dimensional, then $KR^0(X)\otimes\mathbb{Q}$ decomposes into eigenspaces of the Adams operations.

5.5. **Convergence.**

The homotopy spectral sequence for a bounded below tower is automatically conditionally convergent [Bd, Def. 5.10]. So if $RE_\infty = 0$ it converges strongly, by [Bd, 7.4] and the Milnor exact sequence.

5.6. **Multiplicativity.**

The proof of multiplicativity follows the same lines as the nonequivariant case, which is written up in detail in [D2]. We will only give an outline.

One starts by letting W_n be the homotopy fiber of $\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU)$, and these come with natural maps $W_{n+1} \to W_n$. The square

$$\begin{array}{ccc} \mathbb{Z} \times BU & \longrightarrow & P_{2n-2}(\mathbb{Z} \times BU) \\ \downarrow & & \downarrow \\ P_{2n}(\mathbb{Z} \times BU) & \longrightarrow & P_{2n-2}(\mathbb{Z} \times BU) \end{array}$$

gives us a map $W_n \to F_n$, where F_n is the homotopy fiber of the bottom map (which we know is a $K(\mathbb{Z}(n), 2n)$). Routine nonsense shows that $W_{n+1} \to W_n \to F_n$ is a homotopy fiber sequence, and so we have a tower

$$\begin{array}{ccccccccc} & & K(\mathbb{Z}(3),6) & & K(\mathbb{Z}(2),4) & & K(\mathbb{Z}(1),2) & & K(\mathbb{Z}(0),0) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & W_3 & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & W_0 = \mathbb{Z} \times BU \end{array}$$

with $\operatorname{holim} W_n \simeq *$. The spectral sequence for this tower is isomorphic to the spectral sequence for our Postnikov tower: in fact, there is a map of towers $\Omega P_*(\mathbb{Z} \times BU) \to W_*$ which induces weak equivalences on the fibers.

At this point the goal becomes to produce pairings $W_m \wedge W_n \to W_{n+m}$ which commute on-the-nose with the maps in the towers, and where $W_0 \wedge W_0 \to W_0$ is the usual multiplication on $\mathbb{Z} \times BU$. It is easy to produce pairings which commute *up to homotopy* with the maps in the towers, and then an obstruction theory argument shows that the maps can be rigidified. This is the standard argument, and has been written up in detail in [D2]. The only thing which requires much thought in the present context is carrying out the relevant *equivariant* obstruction theory, but this is not hard in the end. We omit the details because the argument is not particularly revealing.

The pairings $W_m \wedge W_n \to W_{m+n}$ now induce a multiplicative structure on the homotopy spectral sequence in the usual way; the reader is again referred to [D2].

5.7. **The weight filtration and the γ-filtration.**

The *weight filtration* on $KR^0(X) = [X, \mathbb{Z} \times BU]$ is the one defined by the above tower: $F^n KR^0(X)$ is defined to be the image of $[X, W_n]$ in $[X, \mathbb{Z} \times BU]$, or equivalently as the subgroup of $[X, \mathbb{Z} \times BU]$ consisting of all elements which map to 0 in $[X, P_{2n-2}(\mathbb{Z} \times BU)]$. This is a multiplicative filtration, and by Proposition 5.2 it has the property that if $x \in F^n KR^0(X)$ then $\psi^k x = k^n x \,(\text{mod } F^{n+1})$. If X is a space for which $H^{2n,n}(X) = 0$ for $n \ggg 0$ we know that ψ^k acts diagonally on $KR^0(X) \otimes \mathbb{Q}$, with eigenvalues k^0, k^1, k^2, etc. The tower shows that $F^n \otimes \mathbb{Q}$ coincides with the sum of the eigenspaces corresponding to k^i, for $i \geq n$.

In SGA6, Grothendieck introduced the *γ-filtration* on algebraic K^0, designed to be an algebraic substitute for the topological filtration induced by the classical Atiyah-Hirzebruch spectral sequence. For any rank 0 stable Real bundle ξ on a $\mathbb{Z}/2$-space X, one has elements $\gamma^i(\xi) \in KR^0(X)$ (see [Gr, Sec. 14] for a nice exposition). The γ-filtration is defined by letting F^n_γ be the subgroup of $KR^0(X)$ generated by all products $\gamma^{i_1}(\xi_1)\gamma^{i_2}(\xi_2)\cdots\gamma^{i_k}(\xi_k)$, where the ξ_i's are rank 0 stable bundles over X and $i_1 + i_2 + \ldots + i_k \geq n$. (So it is the smallest multiplicative filtration in which $\gamma^i(\xi)$ is in F^i). By playing around with the algebraic definitions of γ^i and ψ^k, one can see that $F^n_\gamma \otimes \mathbb{Q}$ also coincides with the sum of eigenspaces of ψ^k for the eigenvalues k^i, $i \geq n$ (an explanation can be found in [Gr, Sec. 14]).

Proposition 5.8. *For any $\mathbb{Z}/2$-space X one has $F^n_\gamma KR^0(X) \subseteq F^n KR^0(X)$. If $H^{2n,n}(X) = 0$ for $n \ggg 0$, this becomes an equality after tensoring with $\mathbb{Q}$.*

Proof. We have already discussed the agreement rationally, since both filtrations give eigenspace decompositions for the Adams operations. To understand the integral story, one regards γ^i as a map $BU \to \mathbb{Z} \times BU$ (or as an element of $KR^0(BU)$). If E is a Real bundle of dimension $i < n$ then one can see algebraically that $\gamma^n(E - i) = 0$. So γ^n is null on $BU(n-1)$, and hence factors through the homotopy cofiber $BU/BU(n-1)$. Both BU and $BU(n-1)$ are weakly equivalent to Grassmannians, and one finds that $H\underline{\mathbb{Z}}^{*,*}(BU) = H\underline{\mathbb{Z}}^{*,*}(pt)[c_1, c_2, \ldots]$ (where c_i has degree $(2i, i)$), and $H\underline{\mathbb{Z}}^{*,*}(BU(n-1)) = H\underline{\mathbb{Z}}^{*,*}[c_1, c_2, \ldots, c_{n-1}]$. So it follows that $H\underline{\mathbb{Z}}^{*,*}(BU/BU(n-1)) = H\underline{\mathbb{Z}}^{*,*}(pt)[c_n, c_{n+1}, \ldots]$. In particular, $H\underline{\mathbb{Z}}^{2i,i}(BU/BU(n-1)) = 0$ for $i < n$. Therefore the map $\gamma^n \colon BU/BU(n-1) \to \mathbb{Z} \times BU$ lifts to W_n in the tower, and any element $\gamma^n(E-i)$ belongs to $F^n KR^0(X)$. $\square$

6. Connective KR-theory

The final task is to stabilize the spectral sequence we produced in the previous section. That spectral sequence converged to $KR^{p+q}(X)$ only for $p + q < 0$, and we'd like to repair this deficiency. This is not at all difficult, and proceeds exactly as in the non-equivariant case. What we will do is construct a "connective" version of KR-theory, represented by a spectrum we'll call kr. There will be a homotopy cofiber sequence

$$\Sigma^{2,1} kr \xrightarrow{\beta} kr \longrightarrow H\underline{\mathbb{Z}},$$

and the Bockstein spectral sequence associated with the map β will give the stabilized version of the spectral sequence we've been considering.

As in section 5.6, W_n denotes the homotopy fiber of $\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU)$.

Proposition 6.1. *There are weak equivalences $W_n \to \Omega^{2,1} W_{n+1}$, unique up to homotopy, which commute with the Bott map in the following diagram:*

$$\begin{array}{ccc} W_n & \longrightarrow & \Omega^{2,1} W_{n+1} \\ \downarrow & & \downarrow \\ \mathbb{Z} \times BU & \longrightarrow & \Omega^{2,1}(\mathbb{Z} \times BU). \end{array}$$

Proof of Proposition 6.1. Consider the natural map $\alpha : \mathbb{Z} \times BU \to P_{2n}(\mathbb{Z} \times BU)$, and apply $\Omega^{2,1}(-)$. The fact that $X \in \mathcal{A}_{(2n-2,n-1)} \Rightarrow S^{2,1} \wedge X \in \mathcal{A}_{(2n,n)}$ shows that $\Omega^{2,1} P_{2n}(\mathbb{Z} \times BU)$ is $\mathcal{A}_{(2n-2,n-1)}$-null. By Proposition 3.3(a) this implies there is a lift

$$\begin{array}{ccc} \Omega^{2,1}(\mathbb{Z} \times BU) & \longrightarrow & \Omega^{2,1} P_{2n}(\mathbb{Z} \times BU) \\ \downarrow & \overset{l}{\nearrow} & \\ P_{2n-2}(\Omega^{2,1}(\mathbb{Z} \times BU)) & & \end{array}$$

and this lift is unique up to homotopy.

Now let $\beta : \mathbb{Z} \times BU \to \Omega^{2,1}(\mathbb{Z} \times BU)$ be the Bott map, and consider the diagram

$$\begin{array}{ccccc} W_n & \longrightarrow & \mathbb{Z} \times BU & \longrightarrow & P_{2n-2}(\mathbb{Z} \times BU) \\ & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle P\beta} \\ & & \Omega^{2,1}(\mathbb{Z} \times BU) & \longrightarrow & P_{2n-2}(\Omega^{2,1}(\mathbb{Z} \times BU)) \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle l} \\ \Omega^{2,1}W_{n+1} & \longrightarrow & \Omega^{2,1}(\mathbb{Z} \times BU) & \longrightarrow & \Omega^{2,1}P_{2n}(\mathbb{Z} \times BU) \end{array}$$

It follows that there is a map on the homotopy fibers $W_n \to \Omega^{2,1}W_{n+1}$ making the diagram commute. We need to show that this is a weak equivalence, and the procedure is one which should be familiar by now: we use Lemma 4.4.

Using Proposition 3.6(a,b) and Lemma 4.5, one shows that

- $[S^{k,0}, W_n]_* = 0$ for $0 \le k < n$,
- $[\mathbb{Z}/2_+ \wedge S^{k,0}, W_n]_* = 0$ for $0 \le k < 2n$, and
- the same is true with W_n replaced by $\Omega^{2,1}W_{n+1}$.

The definition of P_{2n-2} yields that the maps

$$[S^{2n+k,n}, W_n]_* \to [S^{2n+k,n}, \mathbb{Z} \times BU]_* \quad \text{and}$$
$$[\mathbb{Z}/2_+ \wedge S^{2n+k,n}, W_n]_* \to [\mathbb{Z}/2_+ \wedge S^{2n+k,n}, \mathbb{Z} \times BU]_*$$

are isomorphisms for $k \ge 0$, using the fact that $[S^{2n+k,n}, P_{2n-2}(\mathbb{Z} \times BU)] = 0$, etc. Then the square

$$\begin{array}{ccc} W_n & \longrightarrow & \mathbb{Z} \times BU \\ \downarrow & & \simeq \downarrow{\scriptstyle \beta} \\ \Omega^{2,1}W_{n+1} & \longrightarrow & \Omega^{2,1}(\mathbb{Z} \times BU) \end{array}$$

shows at once that $W_n \to \Omega^{2,1}W_{n+1}$ induces an isomorphism on $[S^{2n+k,n}, -]_*$ and $[\mathbb{Z}/2_+ \wedge S^{2n+k,n}, -]_*$ for $k \ge 0$. By Lemma 4.4, $W_n \to \Omega^{2,1}W_{n+1}$ is an equivalence. □

Definition 6.2. *Let kr be the equivariant spectrum consisting of the spaces $\{W_n\}$ and the maps $W_n \to \Omega^{2,1}W_{n+1}$ given by the above proposition. The object kr is called the* ***connective KR-spectrum****.*

The Ω-spectrum for $\Sigma^{2,1}kr$ has nth space equal to W_{n+1}, so the maps $W_{n+1} \to W_n$ give a 'Bott map' $\Sigma^{2,1}kr \to kr$. Corollary 4.2 identifies the homotopy fiber as $\Sigma^{-1,0}H\underline{\mathbb{Z}}$, which is equivalent to the homotopy cofiber being $H\underline{\mathbb{Z}}$. So we may form the tower of homotopy cofiber sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Sigma^{2,1}kr & \xrightarrow{\beta} & kr & \xrightarrow{\Sigma^{-2,-1}\beta} & \Sigma^{-2,-1}kr & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \Sigma^{2,1}H\underline{\mathbb{Z}} & & H\underline{\mathbb{Z}} & & \Sigma^{-2,-1}H\underline{\mathbb{Z}} & & \end{array}$$

The colimit of the spectra in the tower is clearly KR, and the homotopy inverse limit is contractible (these follow from thinking about the spaces in the Ω-spectra for everything in the tower). This gives a stable version of the spectral sequence we've been considering: for any space X we have $H^{p,-\frac{q}{2}}(X) \Rightarrow KR^{p+q,0}(X)$. It converges conditionally because the holim of the tower is contractible, and if $RE_\infty = 0$ then it converges strongly by [Bd, Thm 8.10] (in the language of that result, the condition '$W = 0$' is easily checked to hold).

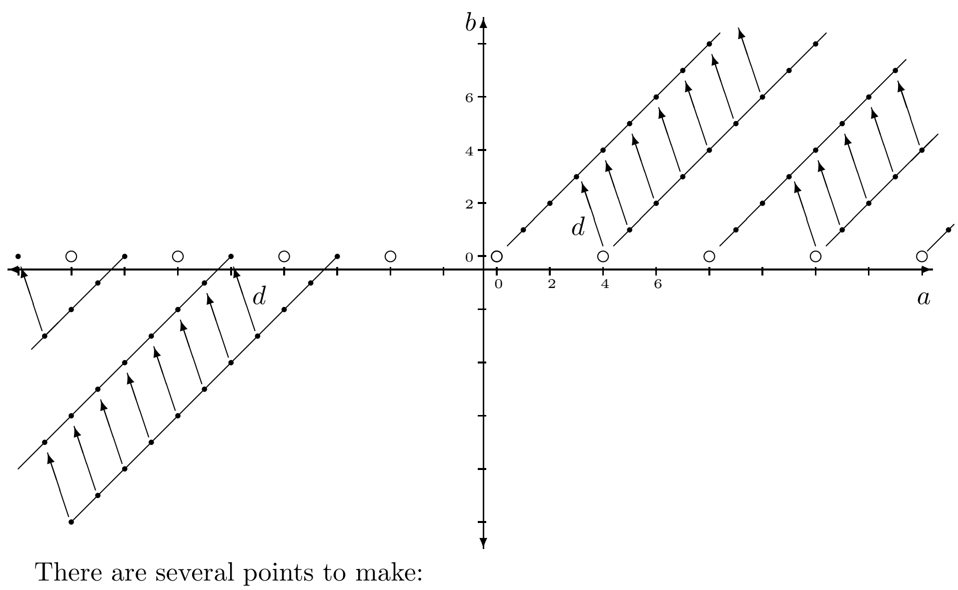
Remark 6.3. The Postnikov tower we've constructed—and its resulting spectral sequence—can be used to completely determine the homotopy groups of the spaces $P_n(\mathbb{Z}\times BU)$, and hence of W_n as well. In other words, we can completely determine the groups $kr^{*,*}(pt)$, and in fact the ring structure can also be deduced. At the moment, however, the answer doesn't seem to admit a simple description—in this sense it is somewhat like the ring $H^{*,*}(pt;\underline{\mathbb{Z}})$, only more complicated. It is *not* true that $kr^{*,*}(pt) \cong H^{*,*}(pt)[v]$, as one might naively guess based on the non-equivariant case. The reason essentially comes down to the fact that there are non-trivial differentials in the spectral sequence when $X = pt$ (see below). The paper [HK1] computes the much more complicated ring $M\mathbb{R}^{*,*}(pt)$, and their methods can be used to give $kr^{*,*}(pt)$ as well.

6.4. **The spectral sequence for $X = pt$.**

In the following diagram we draw the spectral sequence

$$H^p(pt;\mathbb{Z}(-\tfrac{q}{2})) \Rightarrow KR^{p+q}(pt) = KO^{p+q}(pt),$$

but using Adams indexing rather than the usual Serre conventions. In spot (a,b) we have drawn $H^{b,\frac{a+b}{2}}(pt)$, and the vertical line $a = N$ gives the associated graded of KO^{-N}. Said differently, the a-axis measures $-(p+q)$ and the b-axis measures p.



There are several points to make:

(a) Using the multiplicative properties of the spectral sequence, one only has to determine the two differentials labelled 'd'—all the others can be deduced from these. Since we know the groups $KO^*(pt)$, it's clear that these two differentials have to exist. (It would be nice to have a more intrinsic explanation, however).
(b) The spectral sequence collapses at the next page.
(c) The unstable spectral sequence of sections 4 and 5 is the part in the first quadrant. We can read off the action of the Adams operations on $KO^n(pt)$ for

$n \leq 0$ directly from the diagram, the 'weight lines' being along the antidiagonals: KO^0 is pure of weight 0, KO^{-1} is pure of weight 1, KO^{-2} and KO^{-4} are of weight 2, KO^{-8} is of weight 4, etc.

(d) Everything about the ring structure on KO^*, as well as the comparison map $KO^* \to K^*$, can be read off of this spectral sequence and the corresponding spectral sequence where $X = \mathbb{Z}/2$ (which lies entirely along the $b = 0$ line, and hence collapses). They can be deduced from our knowledge of $H^{*,*}(pt)$ and $H^{*,*}(\mathbb{Z}/2)$ provided by Theorem 2.8.

(e) The part of the spectral sequence in the first quadrant is known to topologists in another setting: it's the Adams spectral sequence for bo based on bu.

7. Étale analogs

The difference between algebraic K-theory and étale K-theory, or motivic cohomology and étale motivic cohomology, is very familiar in the motivic setting. In this section we play with similar ideas in the $\mathbb{Z}/2$ world. The analogs are well known, although the only source seems to be [MV, Section 3.3], which doesn't develop things in much detail. We use these ideas to give a proof of the classical fact that $K^{\mathbb{Z}/2} \simeq KO$ and $(\mathbb{Z} \times BU)^{h\mathbb{Z}/2} \simeq \mathbb{Z} \times BO$.

Let us return briefly to the setting where G is any finite group. By an **equivariant covering space** $E \to B$ we mean an equivariant map which, after forgetting the G-actions, is a covering space in the usual sense. Given such a map we may form its Čech complex $\check{C}(E)$, which is the simplicial space

$$E \leftleftarrows E \times_B E \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; E \times_B E \times_B E \;\overset{\leftleftarrows}{\leftleftarrows}\; \cdots$$

(where we have omitted the degeneracies for typographical reasons). In nonequivariant topology, the map $\operatorname{hocolim}_n \check{C}(E)_n \to B$ is a weak equivalence (cf. [DI, Cor. 1.3]). This is not true equivariantly, as the covering space $G \to *$ shows. In this case the realization of the Čech complex is precisely EG, and $EG \to *$ is not an equivariant equivalence. We will see that in some sense this turns out to be the only problem, though.

If Z is a G-space there is a natural map of G-spaces

$$F(B, Z) \to \operatorname*{holim}_n F(\check{C}(E)_n, Z)$$

(here $F(X, Y)$ is the usual mapping space, with its induced G-action). We will say the space Z satisfies **étale descent for the covering $\boldsymbol{E \to B}$** if this natural map is an equivariant weak equivalence. This is the same as requiring that the corresponding maps for the coverings $G/H \times E \to G/H \times B$ all be nonequivariant equivalences, where H ranges over the subgroups of G. If the phrase is not qualified, then "étale descent" means "étale descent for all covering spaces".

Using results of [H, Chaps. 3,4], we may localize the model category $\mathcal{T}op_G$ at the maps $\operatorname{hocolim} \check{C}(E) \to B$ where $E \to B$ ranges over the elements of the set $\{G/H \times G \to G/H \times * | H < G\}$. This produces a new model category structure which we'll denote $\mathcal{T}op_G^{et}$. As the following result shows, the fibrant objects are precisely the spaces which satisfy étale descent for all covering spaces.

Proposition 7.1.

(a) If Z is any G-space, then Z^{EG} satisfies étale descent.

(b) A map $X \to Y$ is a weak equivalence in $\mathcal{T}op_G^{et}$ iff it is a non-equivariant weak equivalence.
(c) For any space Z, the map $Z \to Z^{EG}$ is a fibrant replacement in $\mathcal{T}op_G^{et}$.

Proof. Let $X \to Y$ be a G-map which is also a nonequivariant equivalence, and assume that X and Y are cofibrant. Then $X \times EG \to Y \times EG$ is an equivariant equivalence, and therefore so is $F(Y \times EG, Z) \to F(X \times EG, Z)$ for any Z. By adjointness this map is the same as $F(Y, Z^{EG}) \to F(X, Z^{EG})$. In particular, if $E \to B$ is an equivariant covering space then by taking $X \to Y$ to be the map $\operatorname{hocolim} \check{C}(E) \to B$ (which is a nonequivariant equivalence by [DI, Cor. 1.3]) we find that Z^{EG} satisfies étale descent. This proves (a).

When forming $\mathcal{T}op_G^{et}$ we are localizing at maps which are nonequivariant equivalences. It follows from this that every equivalence in $\mathcal{T}op_G^{et}$ is a nonequivariant equivalence. For (b), we must show the other direction. Note that if Z is a fibrant object in $\mathcal{T}op_G^{et}$, then $Z \to Z^{EG}$ is an equivariant equivalence (this is étale descent for $G \to *$). If $X \to Y$ is a map between cofibrant objects, we may consider the diagram

$$\begin{array}{ccccc} F(Y,Z) & \xrightarrow{\sim} & F(Y,Z^{EG}) & \xrightarrow{\cong} & F(Y \times EG, Z) \\ \downarrow & & \downarrow & & \downarrow \\ F(X,Z) & \xrightarrow{\sim} & F(X,Z^{EG}) & \xrightarrow{\cong} & F(X \times EG, Z) \end{array}.$$

If $X \to Y$ was a nonequivariant equivalence then $X \times EG \to Y \times EG$ is an equivariant equivalence, and so the right vertical map is an equivalence as well. It follows that $F(Y,Z) \to F(X,Z)$ is an equivariant equivalence for every fibrant object Z in $\mathcal{T}op_G^{et}$, and therefore $X \to Y$ is an equivalence in $\mathcal{T}op_G^{et}$.

Part (c) is an immediate consequence of (a) and (b). □

We will call Z^{EG} the **étale localization** (or *Borel* localization) of the space Z, and we'll sometimes write it as Z_{et}. Note that étale localization preserves fiber sequences and homotopy limits.

Everything from our above discussion generalizes directly to spectra as well. We can talk about an equivariant spectrum $\mathcal{E}$ which satisfies étale descent—these correspond to what are usually called 'Borel cohomology theories' [M, p. 233]. If $\mathcal{E}$ is the $RO(G)$-graded spectrum given by $V \to \mathcal{E}_V$, its étale localization (or corresponding Borel theory) is the spectrum $\mathcal{E}_{et}$ given by $V \to \mathcal{E}_V^{EG}$. Note that if $\mathcal{E}$ was an Ω-spectrum then $\mathcal{E}_{et}$ is also an Ω-spectrum.

At this point we switch back to the $\mathbb{Z}/2$-setting, where we can write down the following two results. The first is an immediate consequence of Corollary 4.2, the second of Theorem 1.5.

Proposition 7.2. *There is a tower of homotopy fiber sequences*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & [P_4(\mathbb{Z} \times BU)]_{et} & \longrightarrow & [P_2(\mathbb{Z} \times BU)]_{et} & \longrightarrow & [P_0(\mathbb{Z} \times BU)]_{et} & \longrightarrow & * \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & K(\mathbb{Z}(2),4)_{et} & & K(\mathbb{Z}(1),2)_{et} & & \mathbb{Z}_{et} & & \end{array}$$

and the homotopy limit of the tower is $(\mathbb{Z} \times BU)_{et}$.

Proposition 7.3. *There is a tower of homotopy cofiber sequences in $\mathbb{Z}/2$-spectra of the form*

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Sigma^{2,1} kr_{et} & \xrightarrow{\beta} & kr_{et} & \xrightarrow{\Sigma^{-2,-1}\beta} & \Sigma^{-2,-1} kr_{et} & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & \Sigma^{2,1} H\underline{\mathbb{Z}}_{et} & & H\underline{\mathbb{Z}}_{et} & & \Sigma^{-2,-1} H\underline{\mathbb{Z}}_{et} & \end{array}$$

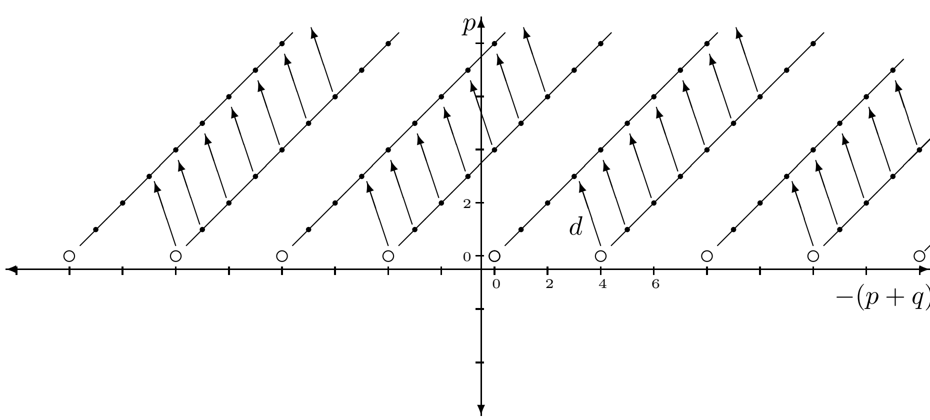
The homotopy colimit of the tower is KR_{et}, and the homotopy limit is contractible.

We want to analyze the spectral sequence which comes from the above tower, so to start with we need a knowledge of $H\underline{\mathbb{Z}}_{et}$. The theory $H\underline{\mathbb{Z}}_{et}$ has coefficient groups described as follows:

Proposition 7.4.
(a) $H\underline{\mathbb{Z}}_{et}^{*,*}(pt) = \mathbb{Z}[x, x^{-1}, y]/(2y)$ *where x has degree $(0,2)$ and y has degree $(1,1)$.*
(b) $H\underline{\mathbb{Z}}_{et}^{*,*}(\mathbb{Z}/2) = \mathbb{Z}[u, u^{-1}]$ *where u has degree $(0,1)$.*
(c) The Mackey functor $\underline{H\mathbb{Z}}_{et}^{0,2n}$ is equal to $\underline{\mathbb{Z}}$, for all n.
(d) The map $H\underline{\mathbb{Z}}^{*,*}(\mathbb{Z}/2) \to H\underline{\mathbb{Z}}_{et}^{*,*}(\mathbb{Z}/2)$ is an isomorphism. The map $H\underline{\mathbb{Z}}^{p,q}(pt) \to H\underline{\mathbb{Z}}_{et}^{p,q}(pt)$ is an isomorphism when $p < 2q$, and multiplication by 2 when $p = 0$ and $q < 0$ is even. These are the only degrees in which its possible to have a nonzero map.

The proof is given in Appendix B.

Proposition 7.3 gives rise to a spectral sequence $H\underline{\mathbb{Z}}_{et}^{p,-\frac{q}{2}}(X) \Rightarrow KR_{et}^{p+q,0}(X)$. The spectral sequence for $X = pt$ is drawn below, using the same indexing conventions as in section 6.4.



Note that there is a map from the spectral sequence of Theorem 1.6 to the one above (coming from the maps $H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}}_{et}$ and $KR \to KR_{et}$). Based on our discussion in section 6.4 we therefore know that the differential labelled 'd' has to exist—in fact, we know all the differentials in the first quadrant below the $y = x$ line. Multiplicativity then allows us to deduce what's happening in the rest of the spectral sequence, and that is what is shown above. As we see, the spectral sequence again converges to KO^*; but this time we've actually gained some information:

Corollary 7.5. *The natural map $KR \to KR_{et}$ is a weak equivalence—that is, KR satisfies étale descent.*

In very fancy language, this says 'the Quillen-Lichtenbaum conjecture holds for KR'.

Proof. The map is of course a nonequivariant equivalence, so we only have to analyze what happens on fixed sets—i.e., we study the map $\alpha\colon KR^{*,0}(pt) \to KR^{*,0}_{et}(pt)$. Consider the map of spectral sequences from $H\mathbb{Z}^{p,-\frac{q}{2}}(pt) \Rightarrow KR^{p+q,0}(pt)$ to the corresponding étale version. Using everything we know about both spectral sequences, as well as what Proposition 7.4(d) says about the map $H\underline{\mathbb{Z}}^{*,*}(pt) \to H\underline{\mathbb{Z}}^{*,*}_{et}(pt)$, it is easy to see that α is an isomorphism when $* \leq 0$, and also when $* = -4n$. The fact that α is a ring map then gives us the isomorphism in the remaining dimensions. □

Corollary 7.6. *If we consider $\mathbb{Z} \times BU$ as a nonequivariant space with its $\mathbb{Z}/2$ action, then $(\mathbb{Z} \times BU)^{h\mathbb{Z}/2} \simeq \mathbb{Z} \times BO$. Similarly, $K^{h\mathbb{Z}/2} \simeq KO$.*

Proof. This is just a translation of the previous corollary. The 0th space in the Ω-spectrum of KR is $\mathbb{Z} \times BU$, whereas the 0th space in the Ω-spectrum of KR_{et} is $(\mathbb{Z} \times BU)^{E\mathbb{Z}/2}$. Corollary 7.5 tells us that these spaces are equivariantly equivalent, and therefore they have weakly equivalent fixed sets. The proof that $K^{h\mathbb{Z}/2} \simeq KO$ follows the same lines, but takes place in the stable category. □

Corollary 7.6 (and 7.5, which is equivalent) is of course well known—a recent reference is [K]. In the end our proof is only slightly different from the classical proof which analyzes $\pi_* K^{h\mathbb{Z}/2}$ via the spectral sequence for π_* of a homotopy limit, making use of the map $KO \to K^{h\mathbb{Z}/2}$ (in fact this spectral sequence has the same form as the one drawn above). It may be worth summarizing our proof to exhibit the similarities. The spectral sequence relating $H\mathbb{Z}$ and KR, when applied to a point, gave us something converging to KO (the fixed set of KR). Because we knew the homotopy groups of KO, we could analyze the differentials in this spectral sequence. On the other hand we have a corresponding spectral sequence relating $H\mathbb{Z}_{et}$ and KR_{et}; when applied to a point, it converges to $K^{h\mathbb{Z}/2}$ (the fixed set of KR_{et}). There is a comparison map of spectral sequences, and our knowledge of the differentials in the first lets us deduce the differentials in the second.

8. Postnikov sections of spheres

In this final section we prove Theorem 3.8, which identifies the Postnikov section $P_V(S^V)$ with the Eilenberg-MacLane space $K(\underline{\mathbb{Z}}, V)$ over the group $G = \mathbb{Z}/2$. Theorem 1.1(a), which is all we really need in this paper, is the case where $V = \mathbb{C}^n$. This section takes place entirely in the context of $\mathbb{Z}/2$-spaces.

Suppose that V contains a copy of the trivial representation. It follows from [dS] and Corollary A.8 of the present paper that the infinite symmetric product $SP^\infty(S^V)$ is a model for $K(\underline{\mathbb{Z}}, V)$ (this actually works over any finite group, not just $G = \mathbb{Z}/2$). Using this, we have:

Lemma 8.1. *If $V \supseteq 1$ then $Sp^\infty(S^V) \to P_V(Sp^\infty(S^V))$ is a weak equivalence.*

Proof. We know by Theorem 2.8 that for any $r, s \geq 0$

$$\begin{aligned} [S^{V+r\mathbb{R}+s\mathbb{R}_-}, K(\underline{\mathbb{Z}}, V)]_* &= H^{-r-s,-s}(pt; \underline{\mathbb{Z}}) = 0 \quad \text{and,} \\ [\mathbb{Z}/2_+ \wedge S^{V+r\mathbb{R}+s\mathbb{R}_-}, K(\underline{\mathbb{Z}}, V)]_* &= H^{-r-s,-s}(\mathbb{Z}/2; \underline{\mathbb{Z}}) = 0 \end{aligned}$$

as long as r and s are not both zero. This shows that $Sp^\infty(S^V)$ is $\mathcal{A}_V$-null, which implies that the map in the statement of the lemma is a weak equivalence. □

Our goal is the following:

Theorem 8.2. *If $V \supseteq 1$ and $S^V \to Sp^\infty(S^V)$ is the obvious map, then*

$$P_V(S^V) \to P_V(Sp^\infty(S^V))$$

is a weak equivalence. Therefore $P_V(S^V) \simeq K(\mathbb{Z}, V)$.

Note that the second statement follows immediately from the first, in light of the above lemma. The proof of the first statement is based on a geometric analysis of the infinite symmetric product construction, and involves the following steps:

(1) For each $k \geq 2$ we produce a homotopy cofiber sequence

$$S^V \wedge \Big([V \otimes \tilde{R}] - 0\Big)/\Sigma_k \to SP^{k-1}(S^V) \to SP^k(S^V)$$

where $\tilde{R}$ is the reduced standard representation of the symmetric group Σ_k (defined below) equipped with the trivial $\mathbb{Z}/2$ action.

(2) We show that

$$P_V\Big(S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k\Big) \simeq *.$$

The key ingredient for this is a geometric analysis of the $\mathbb{Z}/2$–fixed sets of the orbit space $([V \otimes \tilde{R}] - 0)/\Sigma_k$.

(3) From (1) and (2) it follows that

$$P_V(SP^{k-1}(S^V)) \to P_V(SP^k(S^V))$$

is a weak equivalence for every $k \geq 2$, and passing to the limit yields the same for $P_V(S^V) \to P_V(Sp^\infty(S^V))$.

8.3. **Step 1.** Consider the filtration of $Sp^\infty(S^V)$ given by the finite symmetric products:

$$S^V \subseteq SP^2(S^V) \subseteq SP^3(S^V) \subseteq \cdots \subseteq Sp^\infty(S^V).$$

Recall that the **standard representation** of the symmetric group Σ_k is the space $R = \mathbb{R}^k$ where the group acts by permuting the standard basis elements. This contains a trivial, one-dimensional subrepresentation consisting of all vectors $(r, r, \ldots, r)$, and the **reduced standard representation** $\tilde{R}$ is the quotient of R by this subrepresentation. We regard R and $\tilde{R}$ as $\mathbb{Z}/2$-representations by giving them the trivial actions.

The following proposition was inspired by [JTTW, Thm. 2.3], which handled the case $k = 2$:

Proposition 8.4. *The inclusion $SP^{k-1}(S^V) \hookrightarrow SP^k(S^V)$ sits in a homotopy cofiber sequence of the form*

$$S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k \to SP^{k-1}(S^V) \to SP^k(S^V)$$

where $\tilde{R}$ denotes the reduced standard representation of Σ_k.

Proof. To save ink, write $B = B(V)$ and $S = S(V)$ for the unit ball and unit sphere in V. We begin with the relative homeomorphism

$$(B, S) \xrightarrow{\cong} (S^V, *).$$

Applying SP^k to these pairs gives a relative homeomorphism

$$(SP^k(B), Z/\Sigma_k) \xrightarrow{\cong} (SP^k(S^V), SP^{k-1}(S^V))$$

where Z is the space

$$(S \times B \times \cdots \times B) \cup (B \times S \times B \times \cdots \times B) \cup \cdots \cup (B \times \cdots \times B \times S) \subseteq B^k.$$

This says that there is a pushout square of the form

$$\begin{array}{ccc} Z/\Sigma_k & \longrightarrow & SP^{k-1}(S^V) \\ \downarrow & & \downarrow \\ SP^k(B) & \longrightarrow & SP^k(S^V). \end{array}$$

Since $SP^k(B)$ is clearly contractible, the desired cofiber sequence will follow if we can identify Z/Σ_k with $S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k$.

Z naturally includes into $(V \oplus \cdots \oplus V) - 0 = [V \otimes R] - 0$, and the assignment

$$Z \to S(V \oplus \cdots \oplus V), \qquad v \to \tfrac{v}{|v|}$$

is a homeomorphism which is both $\mathbb{Z}/2$- and Σ_k- equivariant. So we may identify the $\mathbb{Z}/2$-spaces Z/Σ_k and $S(V \otimes R)/\Sigma_k$.

Since R decomposes as $\mathbb{R} \oplus \tilde{R}$ (as both Σ_k- and $\mathbb{Z}/2$-representations), we have the corresponding decomposition $V \otimes R \cong V \oplus [V \otimes \tilde{R}]$. Lemma 8.5 below gives a homeomorphism

$$S\Big(V \oplus [V \otimes \tilde{R}]\Big)/\Sigma_k \cong S(V) * [S(V \otimes R)/\Sigma_k],$$

where $X * Y$ denotes the usual **join** of X and Y. It is a general fact (true in any model category) that for pointed spaces X and Y, the join $X * Y$ is weakly equivalent to $\Sigma(X \wedge Y)$. Because $V \supseteq 1$, both $S(V)$ and $S(V \otimes \tilde{R})$ have nonempty $\mathbb{Z}/2$-fixed sets, and therefore can be made pointed. So we finally conclude that

$$\begin{aligned} Z/\Sigma_k \cong S(V) * [S(V \otimes \tilde{R})/\Sigma_k] &\simeq S^1 \wedge S(V) \wedge [S(V \otimes \tilde{R})/\Sigma_k] \\ &\simeq S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k. \end{aligned}$$

□

Lemma 8.5. *Let V and W be orthogonal representations of $\mathbb{Z}/2$. Let G be a finite group acting $\mathbb{Z}/2$-equivariantly and orthogonally on W, and let G act on V trivially. Then there is a natural $\mathbb{Z}/2$-equivariant homeomorphism*

$$S(V) * [S(W)/G] \xrightarrow{\cong} S(V \oplus W)/G$$

where the space on the left denotes the join.

Proof. A point in the left-hand space can be represented by a triple $(v, t, [w])$ where $v \in S(V)$, $t \in [0,1]$, $w \in S(W)$, and $[w]$ denotes the G-orbit of w. We leave it to the reader to check that the map $(v, t, [w]) \mapsto [\sqrt{1-t} \cdot v \oplus \sqrt{t} \cdot w]$ is well-defined and a $\mathbb{Z}/2$-equivariant homeomorphism. □

8.6. **Step 2.**

Proposition 8.7. *Suppose $V = \mathbb{R}^p \oplus (\mathbb{R}_-)^q$, where $p \geq 1$. Let $X = ([V \otimes \tilde{R}] - 0)/\Sigma_k$.*

(a) The fixed set $X^{\mathbb{Z}/2}$ is path-connected for $k \geq 3$.

(b) When $k = 2$, $X^{\mathbb{Z}/2} \simeq \mathbb{R}P^{p-1} \amalg \mathbb{R}P^{q-1}$ (where the second summand is interpreted as $\emptyset$ when $q = 0$, and a point when $q = 1$). When $q \geq 1$ there exists a map $S^{1,1} \to X$ which on fixed sets induces an isomorphism on π_0.

Proof. The argument involves producing explicit paths in the fixed sets. As it's somewhat lengthy, we postpone it until the very end of the section. □

Corollary 8.8. *Consider the set $\mathcal{A}$ consisting of the objects*

- $S^{n,0},\ n \geq 1;$
- $S^{n,0} \wedge \mathbb{Z}/2_+,\ n \geq 1;$
- $S^{1,1}.$

Then the nullification $P_{\mathcal{A}}X$ at this set is contractible.

Proof. Let PX denote the nullification of X. To show PX is contractible we have only to check that $[S^{n,0}, X]_* = 0 = [S^{n,0} \wedge \mathbb{Z}/2_+, X]_*$ for all $n \geq 0$. For $n \geq 1$ this follows just from the definition of PX. We are therefore reduced to checking $n = 0$, which is the statement that both PX and $(PX)^{\mathbb{Z}/2}$ are path-connected (as non-equivariant spaces).

Clearly X is path connected. Attaching cones on maps cannot disconnect the space, so PX must also be path-connected.

Proposition 8.7 says that $X^{\mathbb{Z}/2}$ is path-connected for $k \geq 3$ (or $k = 2$ and $q = 0$), and attaching cones on maps cannot disconnect the fixed set. So $(PX)^{\mathbb{Z}/2}$ is again path-connected in this case.

When $k = 2$ and $q \geq 1$ the proposition says that $X^{\mathbb{Z}/2}$ has *two* path components, but they are linked by an $S^{1,1}$. Attaching a cone on this map will give a space whose fixed set is connected, and then reasoning as in the previous paragraph we find that PX will also have that property. □

Corollary 8.9. *The space $P_V(S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k)$ is contractible.*

Proof. Let $X = ([V \otimes \tilde{R}] - 0)/\Sigma_k$, and suppose we cone off a map $S^{1,0} \to X$ to make a space X':

$$S^{1,0} \to X \to X'.$$

Smashing with S^V gives a cofiber sequence

$$S^{V+1} \to S^V \wedge X \to S^V \wedge X',$$

and since $P_V(S^{V+1}) \simeq *$ it follows by Proposition 3.3(c) that

$$P_V(S^V \wedge X) \xrightarrow{\sim} P_V(S^V \wedge X').$$

In other words, we may cone off arbitrary maps $S^{1,0} \to X$ without effecting the homotopy type of $P_V(S^V \wedge X)$. The same reasoning shows we can cone off maps $S^{1,1} \to X$, $S^{n,0} \to X$, and $\mathbb{Z}/2_+ \wedge S^{n,0} \to X$ ($n \geq 1$) with the same result. So the conclusion is that

$$P_V(S^V \wedge X) \simeq P_V(S^V \wedge PX),$$

where PX denotes the nullification considered in Corollary 8.8. But that corollary says that PX is contractible, and so we're done. □

8.10. **Step 3.**

Proof of Theorem 8.2. We are to show that the map $S^V \to Sp^{\infty}(S^V)$ becomes a weak equivalence after applying P_V. We'll simplify $P_V(X)$ to just $P(X)$, and $SP^k(S^V)$ to just SP^k. Proposition 8.4 gives cofiber sequences

$$S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k \to SP^{k-1} \to SP^k,$$

and we have seen that

$$P(S^V \wedge ([V \otimes \tilde{R}] - 0)/\Sigma_k) \simeq *.$$

So Proposition 3.3(c) shows that $P(SP^{k-1}) \to P(SP^k)$ is a weak equivalence. Hence, one has a sequence of weak equivalences

$$P(S^V) \xrightarrow{\sim} P(SP^2) \xrightarrow{\sim} P(SP^3) \xrightarrow{\sim} \cdots$$

and therefore $P(S^V) \to \operatorname{hocolim}_k P(SP^k)$ is a weak equivalence as well.

Now look at the composite of the two maps

$$P(S^V) \longrightarrow P(\operatorname{hocolim} SP^k) \longrightarrow P(\operatorname{colim} SP^k) = P(Sp^\infty).$$

The middle object may be identified with $\operatorname{hocolim}_k P(SP^k)$ using Proposition 3.3(b) and some common sense, and so the first map is an equivalence. The second map is a weak equivalence because $\operatorname{hocolim} SP^k \to \operatorname{colim} SP^k$ was one. Hence, the composite is also a weak equivalence. □

8.11. Loose ends: The analysis of the fixed sets $[(V \otimes \tilde{R} - 0)/\Sigma_k]^{\mathbb{Z}/2}$.

The one thing still hanging over our heads is the

Proof of Proposition 8.7. Recall that $X = [V \otimes \tilde{R}] - 0/\Sigma_k$. Begin by decomposing V as a sum of irreducibles $V = U_0 \oplus U_1 \oplus \cdots \oplus U_n$ where $U_0 = 1$ (or course the only irreducible representations of $\mathbb{Z}/2$ are $\mathbb{R}$ and $\mathbb{R}_-$, but it's easiest to think in slightly more generality here). An element of X is represented by a coset $[u_0, \ldots, u_n]$ with $u_i \in U_i \otimes \tilde{R}$ and at least one u_i nonzero.

We begin with part (a), which says that when $k \geq 3$ the fixed set $X^{\mathbb{Z}/2}$ is path connected. First note that if $u = [u_0, \ldots, u_n] \in X^{\mathbb{Z}/2}$ and $u_i \neq 0$, then

$$t \mapsto [tu_0, tu_1, \ldots, tu_{i-1}, u_i, tu_{i+1}, \ldots, tu_n]$$

gives a path in $X^{\mathbb{Z}/2}$ from u to $[0, 0, \ldots, 0, u_i, 0, \ldots, 0]$. It follows that it suffices to prove the result when V is of the form $1 \oplus U$, where U is a (possibly 0) irreducible representation. (Recall that the result requires V to contain 1, which is why we don't reduce to $V = U$.) Since our group is $\mathbb{Z}/2$, we only have to worry about $V = \mathbb{R}$ and $V = \mathbb{R} \oplus \mathbb{R}_- = \mathbb{C}$.

The case $V = \mathbb{R}$ is trivial, because then $X^{\mathbb{Z}/2} = X = (\tilde{R} - 0)/\Sigma_k$, and $\tilde{R} - 0$ was path connected because $\dim \tilde{R} \geq 2$ (recall $k \geq 3$ here). So we are left to deal with $V = \mathbb{C}$. In this case $V \otimes \tilde{R}$ is the complex reduced standard representation, which we may identify with $\{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid \sum z_i = 0\}$. We will write $[z_1, \ldots, z_k]$ for the coset of $(z_1, \ldots, z_k)$ in $(V \otimes \tilde{R})/\Sigma_k$.

Let $A = \{[r_1, \ldots, r_k] \mid r_i \in \mathbb{R}\} \subseteq X^{\mathbb{Z}/2}$. Clearly A is path-connected, as $A \cong (\mathbb{R}^{k-1} - 0)/\Sigma_k$. We will show that any element of $X^{\mathbb{Z}/2}$ can be connected by a path to an element of A. If $[z_1, \ldots, z_k] \in X^{\mathbb{Z}/2}$ then there is a $\sigma \in \Sigma_k$ with the property that

$$(z_{\sigma(1)}, \ldots, z_{\sigma(k)}) = (\bar{z}_1, \ldots, \bar{z}_k).$$

By writing σ as a composite of disjoint cyclic permutations, it's easy to see that this can only happen if $(z_1, \ldots, z_k)$ has the form

$$(w_1, \bar{w}_1, \ldots, w_l, \bar{w}_l, r_1, \ldots, r_j)$$

up to permutation of the z's (where $r_i \in \mathbb{R}$). If all the w_i's are real, then our point is already in A and we can stop. So we can assume that $w_1 \notin \mathbb{R}$. Consider the path

$$t \mapsto [w_1 + f(t), \bar{w}_1 + f(t), tw_2, t\bar{w}_2, \ldots, tw_l, t\bar{w}_l, tr_1, \ldots, tr_j]$$

where

$$f(t) = -\frac{1}{2}(2Re(w_1) + 2tRe(w_2) + \ldots + 2tRe(w_l) + tr_1 + \ldots + tr_j)$$

(so $f(t)$ is the real number which makes the sum of the components zero in the previous expression). It's easy to see that this describes a path in $X^{\mathbb{Z}/2}$ connecting our original point with

$$[w_1 + f(0), \bar{w}_1 + f(0), 0, \ldots, 0],$$

and this point has the form $[bi, -bi, 0, \ldots, 0]$ for some nonzero $b \in \mathbb{R}$. Next we consider the path

$$t \mapsto [t + b(1-t)i, t - b(1-t)i, -2t, 0, 0, \ldots, 0]$$

(and here we use the fact that $k \geq 3$). This is a path in $X^{\mathbb{Z}/2}$ connecting $[bi, -bi, 0, \ldots, 0]$ with $[1, 1, -2, 0, \ldots, 0]$, the latter of which is in A. So this completes the proof that $X^{\mathbb{Z}/2}$ is path-connected when $V = \mathbb{C}$, and we are done with part (a).

Now we turn to part (b), which is the case $k = 2$. The reduced standard representation of Σ_2 is $\tilde{R} = \mathbb{R}$ with the Σ_2-action equal to multiplication by -1. If $V = \mathbb{R}^p \oplus \mathbb{R}^q_-$ then $X = [(\mathbb{R}^p \oplus \mathbb{R}^q_-) - 0]/\Sigma_2 \simeq \mathbb{R}P^{p+q-1}$. Using homogeneous coordinates $[r_1, \ldots, r_p, r_{p+1}, \ldots, r_{p+q}]$ on $\mathbb{R}P^{p+q-1}$, the $\mathbb{Z}/2$-action is the one which changes the signs on the final q coordinates. Hence

$$X^{\mathbb{Z}/2} = \{[r_1, \ldots, r_p, 0 \ldots, 0] \mid r_i \in \mathbb{R}\} \amalg \{[0, \ldots, 0, r_{p+1}, \ldots, r_{p+q}] \mid r_i \in \mathbb{R}\};$$

the first set is isomorphic to $\mathbb{R}P^{p-1}$, the second to $\mathbb{R}P^{q-1}$.

When $q \geq 1$, consider the map

$$\begin{array}{rcl} [(\mathbb{R} \oplus \mathbb{R}_-) - 0]/\pm 1 & \to & X \\ [r, s] & \mapsto & [r, 0, \ldots, 0, s, 0, \ldots, 0], \end{array}$$

where the s is placed in the qth spot. It's easy to check that $[(\mathbb{R}\oplus\mathbb{R}_-)-0]/\pm 1 \simeq S^{1,1}$, and that on fixed sets the map induces a bijection on the sets of path-components. This is what we wanted. □

Appendix A. Symmetric products and their group completions

The goal of this section is to show that $Sp^\infty(S^{2n,n})$ is equivariantly weakly equivalent to $\mathrm{AG}(S^{2n,n})$. The proof is not at all difficult, but requires a few lemmas. I would like to thank Gustavo Granja for an extremely helpful conversation about these results. In this section G is a fixed finite group.

Definition A.1.

(a) *Let $\mathcal{C}$ be a category with products and colimits, and let M be an abelian monoid object in $\mathcal{C}$. The **group completion** M^+ is the coequalizer of the maps*

$$M \times M \times M \rightrightarrows M \times M$$

$$(a, b, c) \mapsto (a, b), \qquad (a, b, c) \mapsto (a + c, b + c).$$

(b) *If K is a pointed simplicial set (or topological space), define $AG(K) = Sp^\infty(K)^+$.*

Remark A.2. In the above generality it is not true that M^+ will be an abelian group object in $\mathcal{C}$, or even a monoid object (so the term 'group completion' is somewhat of a misnomer). But this *is* the case when $\mathcal{C} = \mathcal{S}et$, and therefore also when $\mathcal{C} = s\mathcal{S}et$. It also holds when $\mathcal{C} = \mathcal{T}op$ and M is 'sufficiently nice'.

It's easy to see that AG is a functor, so that if K is a simplicial G-set (or a G-space) then $\mathrm{AG}(K)$ also has a G-action. If $\tilde{\mathbb{Z}}[S]$ denote the free abelian group on the pointed set S, where the basepoint is identified with the zero element, one may check that $AG(K)$ is isomorphic to the simplicial set $\tilde{\mathbb{Z}}[K]$.

Proposition A.3. *Let K be a pointed simplicial G-set with the property that K^H is path-connected for every subgroup $H \subseteq G$. Then the natural map $Sp^\infty(K) \to AG(K)$ is an equivariant weak equivalence.*

Proof. If M is a connected simplicial abelian monoid then [Q, Results Q1,Q2,Q4] show $M \to M^+$ induces an equivalence on integral homology. Since both M and M^+ are nilpotent spaces, the map is actually a weak equivalence. One can check, with only a little trouble, that $AG(K)^H$ is isomorphic to $[Sp^\infty(K)^H]^+$, and that $Sp^\infty(K)^H$ is path connected. So Quillen's result implies that $Sp^\infty(K)^H \to \mathrm{AG}(K)^H$ is a weak equivalence for every subgroup H. This completes the proof. □

The next step is to transport this result from G-simplicial sets to G-spaces. We start with two simple lemmas:

Lemma A.4. *Let K be a pointed simplicial set.*

(i) There are natural homeomorphisms $|SP^nK| \to SP^n|K|$, for $1 \leq n \leq \infty$.
(ii) If M is a simplicial abelian monoid, then there is a natural homeomorphism $|M^+| \to |M|^+$.
(iii) There is a natural homeomorphism $|AG(K)| \to AG(|K|)$.

Proof. The essential point is that $|-|$ commutes with colimits (being a left adjoint) and also finite products (making use of the fact that our topological spaces are compactly-generated and weak Hausdorff). Since SP^n $(n < \infty)$ is constructed by forming a finite product and then taking a colimit—namely, passing to Σ_n-orbits—realization will commute with SP^n. But then realization will also commute with Sp^∞, as Sp^∞ is defined as a colimit of the SP^n's. This proves (i).

The proof of (ii) is in the same spirit: M^+ is defined as a coequalizer of two products, so $|-|$ will commute with this construction. Part (iii) is an immediate consequence of (i) and (ii). □

Remark A.5. Again, since the above maps are all natural it follows that if K is a G-simplicial set then the maps are actually equivariant.

Lemma A.6. *If K is a G-simplicial set, the natural map $|K^H| \to |K|$ factors through the H-fixed set and gives a homeomorphism $|K^H| \to |K|^H$.*

Proof. The fact that the map factors though $|K|^H$ and that it is injective are immediate. So the content is that a cell of $|K|$ which is fixed by H must come from a simplex of K^H. But this is obvious. □

Proposition A.7. *Let X be a G-space of the form $|K|$ for some simplicial G-set K. If all the fixed sets X^H are path-connected, then the map $Sp^\infty(X) \to AG(X)$ is an equivariant weak equivalence.*

Proof. What must be shown is that for any subgroup $H \subseteq G$ the map $Sp^\infty(X)^H \to \mathrm{AG}(X)^H$ is a weak equivalence of topological spaces. Now $X = |K|$, and by the above lemmas we can commute the realization past the Sp^∞, the AG, and the fixed points. So we are left with showing that $|Sp^\infty(K)^H| \to |\mathrm{AG}(K)^H|$ is a weak equivalence. This was Proposition A.3. □

Corollary A.8. *For any V which contains a copy of the trivial representation, the map of $\mathbb{Z}/2$-spaces $Sp^\infty(S^V) \to AG(S^V)$ is an equivariant weak equivalence.*

Proof. It suffices to show that S^V is $\mathbb{Z}/2$-homeomorphic to a space of the form $|K|$. It's not hard to verify this for $V = \mathbb{R}$ and $V = \mathbb{R}_-$ by writing down an explicit K_1 and K_2. For a general $V = \mathbb{R}^p \oplus (\mathbb{R}_-)^q$ we have

$$\begin{aligned} S^V &\cong (S^{1,0} \wedge \ldots \wedge S^{1,0}) \wedge (S^{1,1} \wedge \ldots \wedge S^{1,1}) \\ &\cong |K_1| \wedge \ldots \wedge |K_1| \wedge |K_2| \wedge \ldots \wedge |K_2| \cong |K_1 \wedge \ldots \wedge K_1 \wedge K_2 \wedge \ldots \wedge K_2|. \end{aligned}$$

□

Appendix B. Computations of coefficient groups

Here we give the proofs of Theorem 2.8 and Proposition 7.4, which compute the coefficient rings of $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{Z}}_{et}$. As remarked earlier, these computations are routine among equivariant topologists—our only goal is to provide a reference for the nonexpert.

B.1. $H\mathbb{Z}$ **computations.** For any pointed $\mathbb{Z}/2$-spaces X and Y there is an isomorphism $[\mathbb{Z}/2_+ \wedge X, Y]_* \to [X, Y]_*^e$ obtained by restricting via the inclusion $\{0\} \hookrightarrow \mathbb{Z}/2$. So for any equivariant spectrum E there are isomorphisms $E^{p,q}(\mathbb{Z}/2) \to E_e^p(pt)$ where E_e is the nonequivariant spectrum obtained by forgetting the group actions. If E has a product, these isomorphisms give ring maps. It follows immediately that $H^{*,*}(\mathbb{Z}/2) = \mathbb{Z}[u, u^{-1}]$ where u has degree $(0,1)$ (in effect, u is just a placeholder for the second index).

For $H^{*,*}(pt)$ we first recall that $H^{p,0}(pt)$ is known by the definition of Eilenberg-MacLane cohomology—it is 0 when $p \neq 0$, and $\mathbb{Z}$ when $p = 0$. For any space X the groups $H^{p,0}(X)$ and $H_{p,0}(X)$ are Bredon cohomology and homology with coefficients in $\mathbb{Z}$, and in general one has $H^{p,0}(X) = H^p_{sing}(X/\mathbb{Z}_2; \mathbb{Z})$ (but the analog for homology is not quite true).

When $q > 0$ we can now write

$$H^{p-q,-q}(pt) \cong \tilde{H}^{p,0}(S^{q,q}) \cong \tilde{H}^p_{sing}(S^{q,q}/\mathbb{Z}_2; \mathbb{Z}) \cong \tilde{H}^p_{sing}(\Sigma \mathbb{R}P^{q-1}).$$

(For the last isomorphism recall that $S^{q,q}$ is the suspension of the sphere inside $\mathbb{R}^{q,q}$, which is the $(q-1)$-sphere with antipodal action). So when $q > 0$ the groups $H^{*,-q}(pt)$ are the reduced cohomology of $\mathbb{R}P^{q-1}$, with a suitable shifting. See the picture in Theorem 2.8.

When $q > 0$ we can also write

$$H^{p+q,q}(pt) \cong \tilde{H}^{p+q,q}(S^{0,0}) \cong \tilde{H}_{-p-q,-q}(S^{0,0}; \mathbb{Z}) \cong \tilde{H}_{-p,0}(S^{q,q}).$$

The second isomorphism uses equivariant Spanier-Whitehead duality. Now, there is a cofiber sequence $S(\mathbb{R}^{q,q}) \hookrightarrow D(\mathbb{R}^{q,q}) \to S^{q,q}$, where the first two terms are the sphere and disk in $\mathbb{R}^{q,q}$. The induced long exact sequence for $H_{*,0}$ shows that $\tilde{H}_{a,0}(S^{q,q}) \cong \tilde{H}_{a-1,0}(S(\mathbb{R}^{q,q}))$ when $a \neq 0, 1$, and that there is an exact sequence

$$0 \to \tilde{H}_{1,0}(S^{q,q}) \to H_{0,0}(S(\mathbb{R}^{q,q})) \to \mathbb{Z} \to \tilde{H}_{0,0}(S^{q,q}) \to 0. \tag{B.1}$$

The space $S(\mathbb{R}^{q,q})$ has free action, and so $\tilde{H}_{a-1,0}(S(\mathbb{R}^{q,q})) \cong \tilde{H}^{sing}_{a-1}(S(\mathbb{R}^{q,q})/\mathbb{Z}_2) = \tilde{H}^{sing}_{a-1}(\mathbb{R}P^{q-1})$. Hence $H^{p+q,q}(pt) \cong \tilde{H}^{sing}_{-p-1}(\mathbb{R}P^{q-1})$ when $p \neq 0, 1$.

The center map in (B.1) may be seen to coincide with the map $H_{0,0}(\mathbb{Z}/2) \to H_{0,0}(pt)$ induced by $\mathbb{Z}/2 \to pt$. This is the same as the transfer map i_* in the Mackey functor $\mathbb{Z}$, which is the $\times 2$ map $\mathbb{Z} \to \mathbb{Z}$. So $H^{q-1,q}(pt) \cong \tilde{H}_{1,0}(S^{q,q}) = 0$ and $H^{q,q}(pt) \cong \tilde{H}_{0,0}(S^{q,q}) \cong \mathbb{Z}/2$. We have now seen that when $q > 0$ the groups $H^{*,q}(pt)$ are the reduced singular homology groups of $\mathbb{R}P^{q-1}$, read downwards from the group in degree $(q-1, q)$. Again, the reader is referred to the picture that goes with Theorem 2.8.

At this point we have computed the additive groups $H^{*,*}(pt)$ and $H^{*,*}(\mathbb{Z}/2)$, so we turn our attention to the maps between them. Consider the cofiber sequence $\mathbb{Z}/2_+ \to S^{0,0} \to S^{1,1}$ and the induced long exact sequence

$$\cdots \leftarrow H^{0,2n-1}(pt) \leftarrow H^{0,2n}(\mathbb{Z}/2) \xleftarrow{i^*} H^{0,2n}(pt) \leftarrow H^{-1,2n-1}(pt) \leftarrow \cdots$$

When $n \geq 0$ then $H^{0,2n-1}(pt) = 0 = H^{-1,2n-1}(pt)$, and so i^* is an isomorphism. When $n < 0$ we know enough to conclude that coker $i^* \cong \mathbb{Z}/2$, so i^* is multiplication by 2. A $\mathbb{Z}/2$-Mackey functor with both groups equal to $\mathbb{Z}$ is completely determined by its restriction map i^*, so we can deduce that $\underline{H}^{0,2n} = \underline{\mathbb{Z}}$ when $n \geq 0$ and $\underline{H}^{0,-2n} = \underline{\mathbb{Z}}^{op}$ when $n < 0$.

The map $i^*\colon H^{*,*}(pt) \to H^{*,*}(\mathbb{Z}/2)$ is a map of rings, and we know the target is $\mathbb{Z}[u, u^{-1}]$. This allows us to determine the subring $H^{0,2*}(pt)$. Also, the commutativity of the usual diagram

$$\begin{array}{ccc} AG(S^V) \wedge AG(S^W) & \longrightarrow & AG(S^V \wedge S^W) \\ \downarrow t & & \downarrow AG(t) \\ AG(S^W) \wedge AG(S^V) & \longrightarrow & AG(S^W \wedge S^V), \end{array}$$

shows that $H^{*,*}(X)$ is graded-commutative in a certain sense, for any X. For $X = pt$ we know that the groups $H^{*,*}(pt)$ are either $\mathbb{Z}/2$'s or else located in even degrees, so the ring is commutative on-the-nose.

It is not hard to see that $S^{1,1} \to AG(S^{1,1}) \simeq K(\mathbb{Z}(1), 1)$ is a weak equivalence (we know the homotopy groups of the target and its fixed set, so this can be checked directly). Let y denote the composite $S^{0,0} \hookrightarrow S^{1,1} \to K(\mathbb{Z}(1), 1)$. The cofiber sequence $S^{0,0} \hookrightarrow S^{1,1} \to \mathbb{Z}/2_+ \wedge S^{1,0}$ gives us a long exact sequence on $H^{*,*}$ in which one of the maps is multiplication by y. Analysis of this long exact sequence lets us determine all the multiplication-by-y's shown in the diagram in Theorem 2.8.

At this point we have determined almost all of the ring structure on $H^{*,*}(pt)$. If θ_n denotes the class in $H^{0,-2n-1}(pt) \cong \mathbb{Z}/2$ and x the generator of $H^{0,2}(pt) \cong \mathbb{Z}$, we have only to show that $x \cdot \theta_{n+1} = \theta_n$. Let E be the spectrum defined by the cofiber sequence $\Sigma^{0,-2} H\mathbb{Z} \to H\mathbb{Z} \to E$, where the first map denotes multiplication by x. Using what we have already proven, one computes that $E^{n,0}(pt) = 0$ if $n \neq 0$, $E^{0,0}(pt) = \mathbb{Z}/2$, and $E^{n,0}(\mathbb{Z}/2) = 0$ for all n. So E is the Eilenberg-MacLane cohomology theory for the Mackey functor $\underline{E}^{0,0}$, and the nature of this Mackey functor lets us conclude that $E^{n,0}(X) \cong H^n_{sing}(X^{\mathbb{Z}_2};\mathbb{Z}/2)$. So when $n > 0$ we have $E^{0,-n}(pt) \cong \tilde{E}^{n,0}(S^{n,n}) \cong \tilde{H}^n_{sing}(S^0) = 0$. It follows that multiplication by x gives an isomorphism $H^{0,-n-2}(pt) \to H^{0,-n}(pt)$ when $n \geq 2$. This completes the analysis of the ring structure on $H^{*,*}(pt)$—all products can be deduced from the ones we've computed together with commutativity and degree considerations.

B.2. $H\mathbb{Z}_{et}$ **computations.** Recall that the spaces in the Ω-spectrum for $H\underline{\mathbb{Z}}_{et}$ are $K(\mathbb{Z},V)^{E\mathbb{Z}/2}$. From this it follows that $H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}}_{et}$ is a nonequivariant equivalence, and so $H^{*,*}(\mathbb{Z}/2) \to H^{*,*}_{et}(\mathbb{Z}/2)$ is an isomorphism of rings.

Remark 2.13 gives the homotopy type of $K(\mathbb{Z}(n),2n)^{h\mathbb{Z}/2}$, and from this we immediately compute the groups $H^{p,q}_{et}(pt)$ where $p,q \geq 0$ and $p \leq 2q$. The point is that

$$H^{p,q}_{et}(pt) = H^{2q-(2q-p),q}_{et}(pt) = \tilde{H}^{2q,q}_{et}(S^{2q-p,0}) = [S^{2q-p,0}, K(\mathbb{Z}(q),2q)^{h\mathbb{Z}/2}].$$

Using the cofiber sequence $S^{0,0} \hookrightarrow S^{1,1} \to \mathbb{Z}/2_+ \wedge S^{1,0}$ now lets us deduce $H^{p,q}_{et}(pt)$ in the two ranges $(p \leq q)$ and $(p \geq 1)$. In a moment we will show that for all $n > 0$ one has $H^{0,-2n}_{et}(pt) = \mathbb{Z}$, $H^{0,-2n+1}_{et}(pt) = 0$, and the restriction map $H^{0,-2n}_{et}(pt) \to H^{0,-2n}_{et}(\mathbb{Z}/2)$ is an isomorphism. Using these facts, this same cofiber sequence will show that $H^{p,q}_{et}(pt)$ vanishes when both $p < 0$ and $q < 0$.

The above cofiber sequence induces an exact sequence

$$0 = H^{1,-n+1}_{et}(\mathbb{Z}/2) \leftarrow H^{1,-n+1}_{et} \xleftarrow{\cdot y} H^{0,-n}_{et} \leftarrow H^{0,-n+1}_{et}(\mathbb{Z}/2) \xleftarrow{i^*} H^{0,-n+1}_{et}.$$

When $n = 1$ we already know i^* is the identity, and $H^{1,0}_{et} = 0$; so $H^{0,-1}_{et} = 0$. When $n = 2$ we find the exact sequence

$$0 \leftarrow \mathbb{Z}/2 \leftarrow H^{0,-2}_{et} \leftarrow \mathbb{Z} \leftarrow 0,$$

so $H^{0,-2}_{et}$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. But we also know that

$$\begin{aligned} H^{0,-2}_{et} \cong \tilde{H}^{2,0}_{et}(S^{2,2}) \cong \tilde{H}^{2,0}(E\mathbb{Z}/2_+ \wedge S^{2,2}) &\cong \tilde{H}^2_{sing}(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}_2} S^{2,2}) \\ &\cong \tilde{H}^{2,0}_{sing}((S^1_a)_+ \wedge_{\mathbb{Z}_2} S^{2,2}), \end{aligned}$$

where S^1_a denotes the circle with antipodal action, and $S^1_a \hookrightarrow E\mathbb{Z}/2$ is the obvious inclusion. The cofiber sequence $(\mathbb{Z}/2)_+ \hookrightarrow (S^1_a)_+ \to \mathbb{Z}/2_+ \wedge S^{1,0}$ gives a diagram of spaces

$$\begin{array}{ccccccc} \mathbb{Z}/2_+ \wedge_{\mathbb{Z}_2} S^{2,2} & \longrightarrow & (S^1_a)_+ \wedge_{\mathbb{Z}_2} S^{2,2} & \longrightarrow & (\mathbb{Z}/2_+ \wedge S^{1,0}) \wedge_{\mathbb{Z}/2} S^{2,2} & = & S^3 \\ \downarrow & \swarrow & & & & & \\ S^{2,2}/\mathbb{Z}_2 & & & & & & \end{array}$$

where the top row is a cofiber sequence and the two maps to the bottom row squash S^1_a (and $\mathbb{Z}/2$) to a point. Applying H^2_{sing} to this diagram now gives

$$\begin{array}{ccccccc} H^3(S^3) & \longleftarrow & \mathbb{Z} & \longleftarrow & ? & \longleftarrow & 0 \\ & & {\scriptstyle 2}\uparrow & \nearrow & & & \\ & & \mathbb{Z} & & & & \end{array}$$

where ? denotes $H^2_{sing}((S^1_a)_+ \wedge_{\mathbb{Z}_2} S^{2,2})$. We have so far determined that this group is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, and so the only possibility is $\mathbb{Z}$. So we have learned that $H^{0,-2}_{et} = \mathbb{Z}$, and the map $? \to \mathbb{Z}$ in the diagram must be an isomorphism. The diagram now shows that $H^{0,-2} \to H_{et}^{0,-2}$ is multiplication by 2, because this is the map $\mathbb{Z} \to ?$. In the square

$$\begin{array}{ccc} H^{0,-2} & \longrightarrow & H_{et}^{0,-2} \\ \downarrow & & \downarrow \\ H^{0,-2}(\mathbb{Z}/2) & \longrightarrow & H_{et}^{0,-2}(\mathbb{Z}/2) \end{array}$$

we know all the groups are $\mathbb{Z}$, the top and left maps are multiplication by 2, and the bottom map is an isomorphism; so the right vertical map is also an isomorphism.

Using an induction, the above arguments actually show that $H_{et}^{0,-2n+1} = 0$, $H_{et}^{0,-2n} = \mathbb{Z}$, and $H_{et}^{0,-2n} \to H_{et}^{0,-2n}$ is an isomorphism, for all $n \geq 1$. This completes our determination of the groups $H_{et}^{*,*}$, and of the Mackey functors $\underline{H}_{et}^{0,2n}$.

The ring structure on $H_{et}^{0,*}(pt)$ can now be determined by comparing with the known structure of the rings $H^{0,*}(pt)$ and $H^{0,*}(\mathbb{Z}/2)$ (the latter is also $H_{et}^{0,*}(\mathbb{Z}/2)$). The multiplication-by-y's are deduced from the long exact sequences induced by $S^{0,0} \hookrightarrow S^{1,1} \to \mathbb{Z}/2_+ \wedge S^{1,1}$, just as for $H^{*,*}$. This completes the proof of Proposition 7.4.

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