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Topological hypercovers and $\mathbb{A}^1$-realizations

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Abstract. We show that if U_* is a hypercover of a topological space X then the natural map hocolim $U_* \to X$ is a weak equivalence. This fact is used to construct topological realization functors for the $\mathbb{A}^1$-homotopy theory of schemes over real and complex fields. In an appendix, we also prove a theorem about computing homotopy colimits of spaces that are not cofibrant.

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1 Introduction

Let X be a topological space, and let $\mathcal{U} = \{U_a\}$ be an open cover of X. From this data one may build the **Čech complex** $\check{C}(\mathcal{U})_*$, which is the simplicial space

$$\coprod U_{a_0} \leftleftarrows \coprod U_{a_0 a_1} \; \overset{\longleftarrow}{\leftleftarrows} \; \coprod U_{a_0 a_1 a_2} \cdots$$

Here $U_{a_0 \cdots a_n} = U_{a_0} \cap \cdots \cap U_{a_n}$, and the face maps are obtained by omitting indices – we have chosen not to draw the degeneracies for typographical reasons. Segal [S1] proved that if X has a partition of unity subordinate to $\mathcal{U}$ then the map $|\check{C}(\mathcal{U})_*| \to X$ is a *homotopy* equivalence, where $|-|$ denotes geometric realization. Our first goal in this paper is to generalize this result to the following theorem.

Theorem 1.1. *For every open cover $\mathcal{U}$ of X, the map* hocolim $\check{C}(\mathcal{U})_* \to X$ *is a weak equivalence (where the hocolim is taken over the simplicial indexing category).*

There are two steps in the argument. First, we prove that $|\check{C}(\mathcal{U})_*| \to X$ is a weak equivalence for arbitrary open covers. It is possible to deduce this from Segal's

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result, making use of the fact that weak equivalences are detected by spheres, and spheres always have partitions of unity. But instead of going this route we give a proof that avoids Segal's theorem completely, and is quite elementary.

The second step is to deal with the difference between $|\check{C}(\mathfrak{U})_*|$ and $\operatorname{hocolim} \check{C}(\mathfrak{U})_*$. For any simplicial object W_* in a model category, there are general criteria for when its geometric realization agrees with its homotopy colimit (cf. [H, Th. 18.7.4]); unfortunately these criteria apply only when the objects W_n are all cofibrant, and we are definitely not assuming that the open sets U_a and their intersections are cofibrant. It turns out that this cofibrancy issue *never* matters, which is somewhat of a surprise. Here is one specific result which we will use later:

Theorem 1.2. *Let U_* be a simplicial space that has free degeneracies (see Definition A.4), but do not assume that each U_n is cofibrant. Then the homotopy colimit of U_* is weakly equivalent to the realization of U_*.*

This result is similar to [S2, A.5], but different in that it is about weak equivalences rather than homotopy equivalences. We in fact prove that when taking homotopy colimits for any diagram of topological spaces one doesn't first have to make all the spaces involved cofibrant; the usual formulas are already homotopy-invariant. This is definitely a non-standard fact, but we've banished it to an appendix so it won't distract the reader from the general theme of the paper. On the other hand it is a useful result, and we'd like to call attention to it.

The main goal of this paper is generalizing Theorem 1.1 so that it applies to 'hypercovers', rather than just Čech covers. These are defined in detail in Section 4, but for now we will just give an intuitive definition. An open hypercover of a space X is a simplicial space U_* such that

(1) Each U_n is a disjoint union of open subsets of X,
(2) The spaces appearing in U_0 are an open cover of X,
(3) The spaces in U_1 cover the double intersections of those in level 0,
(4) The spaces in U_2 cover the triple intersections of those in level 1, and so on.

Of course making sense of (4) – especially the 'and so on' part – requires a certain amount of bookkeeping, which is why we are postponing the formal definition. But the essence is that hypercovers are like Čech complexes except that instead of taking the double intersections at level 1 we may refine them further, and we may continue this refining process at each level. Our second main result concerns homotopy colimits of hypercovers.

Theorem 1.3. *If U_* is an open hypercover of a space X, then the natural map* $\operatorname{hocolim} U_* \to X$ *is a weak equivalence.*

This result could almost be considered folklore since everyone immediately agrees it's true, but a proof seems to be missing from the literature. One might consider tackling it by appealing to the Whitehead theorem, proving an isomorphism on fundamental groupoids and homology with local coefficients. This is the approach taken in [F, Prop. 8.1] in the related context of étale hypercovers, but this is messy and obscures in computation the underlying geometric explanation of the theorem. In the case of topological spaces, the isomorphism on fundamental groupoids was the subject of the paper [RT] (although they only dealt with Čech complexes, not hypercovers). The approach we take here, on the other hand, is very elementary. The idea is to reduce to the case of Čech covers in a clever way.

Our interest in these results arose from attempts to understand topological realization functors in the $\mathbb{A}^1$-homotopy theory of schemes [MV]. Given an algebraic variety X defined over $\mathbb{C}$, there is an associated topological space $X(\mathbb{C})$ obtained by giving X the analytic topology. Of course this should extend to a map of 'homotopy theories' from the Morel-Voevodsky category $\mathrm{Spc}(\mathbb{C})$ to the category of topological spaces. In [MV] this extension is only provided at the level of homotopy categories, but we are interested in extending it to the model category level.

The first thing to note is that $\mathrm{Spc}(\mathbb{C})$ must be replaced by a Quillen-equivalent model, denoted $\mathrm{Spc}'(\mathbb{C})$ in section 5. Once this is done, the key fact needed to make things work is precisely Theorem 1.3. This is worked out in detail in Section 5, following the basic program of [D2, Rem. 8.2] or [I]. We also prove that taking analytic spaces for schemes defined over $\mathbb{R}$ induces a Quillen map from $\mathrm{Spc}'(\mathbb{R})$ to $\mathbb{Z}_2$-equivariant topological spaces:

Theorem 1.4. *The functor $X \mapsto X(\mathbb{C})$ extends to left Quillen functors of the form $Spc'(\mathbb{C}) \to \mathcal{T}op$ and $Spc'(\mathbb{R}) \to \mathbb{Z}_2 - \mathcal{T}op$.*

Finally, we give in this paper several interesting corollaries to Theorem 1.3. On the whole these seem too disparate to recount in the introduction, but as an example let us mention two of them. We refer the reader to Sections 3 and 4 for more results like these.

Corollary 1.5. *Let $E \to B$ be any map which is locally split (for example, a covering space), and form the associated Čech complex $\check{C}(E)_*$ given by*

$$\check{C}(E)_n := E_B^{n+1} = E \times_B E \times_B \cdots \times_B E.$$

Then the natural map hocolim $\check{C}(E)_* \to B$ *is a weak equivalence.*

Corollary 1.6. *Let $\mathcal{U}$ be an open cover of a space X with the property that every finite intersection $U_{a_0\cdots a_n}$ is covered by other elements of $\mathcal{U}$. Form the diagram consisting of all the U_a's and all the inclusions between them. Then the homotopy colimit of this diagram is weakly equivalent to X.*

The first corollary is an immediate consequence of Proposition 4.10, and the second is restated and proved as Proposition 4.6(c).

Using open covers to give homotopy decompositions for spaces, or to detect weak equivalences, is of course a classical topic. In addition to [S1] it is worthwhile to mention [Mc1], [Mc2], and [Dk]. Hypercovers were invented by Verdier in [SGA4, Expose V, Sec. 7], where they were used as a way of computing sheaf cohomology in arbitrary Grothendieck topologies. Finally, after writing this paper we learned of the unpublished preprint [Si], which deals with some overlapping issues from a different perspective.

We would like to express our thanks to Bill Dwyer, Michael Mandell, and Jeff Smith for several useful conversations about these results, and to Boris Chorny for pointing out an error in an earlier version. We thank Phil Hirschhorn for helpful conversations about the results in the appendix, in particular for his ideas on removing an annoying T_1 separation condition. The final form of Lemmas A.2 and A.3 is something we owe to him.

1.7 Notation and terminology

We assume that the reader is familiar with homotopy colimits, and in a few places also with the theory of model categories. The original reference for the latter is [Q], but we generally follow [H] in notation and terminology ([Ho] is also a good reference). Regarding homotopy colimits, [H] uses 'hocolim D' to denote the result of applying a certain explicit formula to any diagram D. This has the disadvantage that the resulting object has the correct homotopy type only when the diagram consists entirely of cofibrant objects. We instead adopt the position that 'hocolim D' should *always* denote the correct homotopy-invariant construction: it is obtained by first applying cofibrant-replacement to the objects in the diagram, and only then using the usual explicit formulas. In model-theoretic terms, homotopy colimit is the left derived functor of the ordinary colimit functor, when the category of diagrams is given the projective model structure (see below). Having made the previous point, we now get to say that for topological spaces it isn't really necessary because of Appendix A.

We review one last piece of machinery, used often in the body of the paper. Given a small category I, recall that there is a model structure on the category $s\mathcal{S}et^I$ of diagrams of simplicial sets such that a map is a weak equivalence (resp., fibration) if it is so in every spot of the diagram [H, Sec. 11.6]. We call this the *projective* model structure on $s\mathcal{S}et^I$, and the cofibrant diagrams have the property that the homotopy colimit and ordinary colimit are weakly equivalent.

Finally, some notation: Throughout this paper our open covers $\mathcal{U} = \{U_a\}$ are always indexed by a set A. In particular, we are allowing the possibility that $U_a = U_{a'}$ for different values $a \neq a'$. For every finite set $\sigma = \{a_0, \ldots, a_n\}$ in A, we'll write U_σ or $U_{a_0 \cdots a_n}$ for $U_{a_0} \cap \cdots \cap U_{a_n}$. Also, once and for all we fix our model for Δ^n as the subset of $\mathbb{R}^{n+1}$ consisting of $(n+1)$-tuples $t = (t_0, \ldots, t_n)$ such that $0 \leq t_i \leq 1$ for all i and $\Sigma_{i=0}^n t_i = 1$. The symbol $\mathcal{T}op$ denotes the category of *all* topological spaces – we don't assume any hypotheses like compactly-generated, although the results all work in that context as well. Also, the symbol $s\mathcal{S}et$ denotes the category of simplicial sets.

2 Čech complexes

The purpose of this section is to prove the following:

Theorem 2.1. *For any open cover $\mathcal{U}$ of a topological space X, the natural map $\pi : |\check{C}(\mathcal{U})_*| \to X$ is a weak equivalence.*

We start by recalling the following result and its corollary:

Proposition 2.2 (Gray). *Let $f\colon X \to Y$ be a map of spaces and let U and V form an open cover of Y. Suppose that the induced maps*

$$f^{-1}U \to U, \qquad f^{-1}V \to V, \quad \text{and} \quad f^{-1}(U \cap V) \to U \cap V$$

are all weak equivalences. Then $X \to Y$ is also a weak equivalence.

This is proven (in more generality) in [G, 16.24], using an elegant small-simplices argument. With enough technology it can also be done by a Whitehead-type theorem: it's easy to see that $X \to Y$ is an isomorphism on π_0, a souped-up van Kampen theorem yields the isomorphism on π_1, and for homology with local coefficients one uses the Mayer-Vietoris exact sequence. Gray's argument is much nicer, though.

Corollary 2.3 (May). *Let $f\colon X \to Y$ be a map of spaces and let $\mathfrak{U} = \{U_a\}$ be an open cover of Y. Suppose that $f^{-1}U_\sigma \to U_\sigma$ is a weak equivalence for every finite set σ of indices. Then $X \to Y$ is also a weak equivalence.*

May deduces the generalization by a quick application of Zorn's Lemma [M2, Cor. 1.4]: look at the set of all opens W such that $f^{-1}(W \cap U_\sigma) \to W \cap U_\sigma$ is a weak equivalence for all σ, including $\sigma = \emptyset$. This set has maximal element, and Gray's result shows it must be X. In an earlier paper McCord proved a more general version of this result [Mc1, Th. 6], but the proof is quite a bit more complicated.

Proof of Theorem 2.1. Let $\epsilon\colon E_* \to X$ be an augmented simplicial space and let $p\colon |E_*| \to X$ be the induced map. For any $V \subseteq X$, there is a continuous bijection from the realization of $[n] \mapsto \epsilon_n^{-1}V$ to the space $p^{-1}V$. If V is open in X, one checks that this is actually a homeomorphism.

In our particular example of $\check{C}(\mathfrak{U})_* \to X$, we can now say that given any open set V in X, the space $\pi^{-1}V$ is homeomorphic to the space $|\check{C}(\mathfrak{U}')_*|$, where $\mathfrak{U}'$ is the open cover $\{U_a \cap V\}$ of the space V. We want to consider the maps $\pi^{-1}(U_\sigma) \to U_\sigma$, but in this case the cover $\mathfrak{U}'$ of U_σ actually contains the whole space U_σ as one of its elements. Using Corollary 2.3 we therefore reduce the theorem to the case where the open cover contains the whole space as one of its elements, which is the following lemma. □

Lemma 2.4. *Let $\mathfrak{U}$ be an open cover of X such that $U_b = X$ for some index b. Then the natural map $|\check{C}(\mathfrak{U})_*| \to X$ is a weak equivalence (in fact, a homotopy equivalence).*

Proof. There is a section $\chi\colon X \to |\check{C}(\mathfrak{U})_*|$ obtained from the map $U_b \otimes \Delta^0 \to |\check{C}(\mathfrak{U})_*|$ and the identification $U_b = X$. We only need to show that $\chi\pi$ is homotopic to the identity.

Let $\check{C}(\mathfrak{U})_* \times I$ be the simplicial space obtained by crossing all the levels of $\check{C}(\mathfrak{U})_*$ with the unit interval. Then $|\check{C}(\mathfrak{U})_* \times I|$ is the quotient

$$\left[\coprod_{a_0 \cdots a_n} U_{a_0 \cdots a_n} \times \Delta^n \times I\right] / \sim$$

where the relations are the usual ones, not affecting the I factor at all. Define a map $|\check{C}(\mathfrak{U})_* \times I| \to |\check{C}(\mathfrak{U})_*|$ in the following way. Take an element $(x, t_0, \dots, t_n, s)$ where x belongs to $U_{a_0 \cdots a_n}$ and $(t_0, \dots, t_n)$ belongs to Δ^n, and send it to the element $(x, 1-s, st_0, \dots, st_n)$ in the factor $U_{ba_0 \dots a_n} \otimes \Delta^{n+1}$. This definition respects the various identifications.

Now, there is also an obvious map $f\colon |\check{C}(\mathfrak{U})_* \times I| \to |\check{C}(\mathfrak{U})_*| \times I$ induced by sending (x, t, s) to $((x, t), s)$. We claim that this is a homeomorphism, thereby giving us a homotopy $|\check{C}(\mathfrak{U})_*| \times I \to |\check{C}(\mathfrak{U})_*|$ between $\chi\pi$ and the identity. The reason f is a homeomorphism is just because geometric realization and crossing with I are both left adjoints, and the right adjoints are easily seen to commute. It is important that I and Δ^n are locally compact Hausdorff so that the relevant mapping spaces with compact-open topologies have the correct adjointness properties. □

It's possible to give a 'slick' proof of the above lemma by noticing that there is a contracting homotopy for the simplicial space $\check{C}(\mathfrak{U})_*$. One needs to check that homotopic maps of simplicial spaces give homotopic maps on realizations, but this is essentially the above argument. See the proof of Proposition 4.10 for a broader perspective.

2.5 Connection with Segal's results

To close this section we make the connection between our Theorem 2.1 and the result proven in [S1]. Segal doesn't explicitly deal with Čech complexes, but the objects he deals with turn out to be homeomorphic to them. This connection will be needed later on.

Let A be the indexing set for a cover $\mathfrak{U}$. We have already introduced the Čech complex $\check{C}(\mathfrak{U})_*$, but if A is given an ordering we may also consider the **ordered Čech complex** $\check{C}^o(\mathfrak{U})_*$ which is often easier to work with. This is the simplicial space given by $\check{C}^o(\mathfrak{U})_n = \coprod_{a_0 \cdots a_n} U_{a_0 \cdots a_n}$, where the coproduct ranges over all *ordered* multi-indices in A. That is, we only consider multi-indices for which $a_0 \le a_1 \le \cdots \le a_n$. Note that there is an inclusion of simplicial spaces $\check{C}^o(\mathfrak{U})_* \to \check{C}(\mathfrak{U})_*$.

Proposition 2.6. *The map $\check{C}^o(\mathfrak{U})_* \to \check{C}(\mathfrak{U})_*$ induces a homotopy equivalence $|\check{C}^o(\mathfrak{U})_*| \to |\check{C}(\mathfrak{U})_*|$.*

Proof. For any (not necessarily ordered) multi-index $a_0 \cdots a_n$, there is a canonical reordering $a_{\sigma 0} \cdots a_{\sigma n}$ such that $a_{\sigma 0} \le \cdots \le a_{\sigma n}$. If $a_i = a_j$ for some $i < j$, then always choose $\sigma i < \sigma j$. This allows us to define an inverse map $|\check{C}(\mathfrak{U})_*| \to |\check{C}^o(\mathfrak{U})_*|$. If (x, t) is an element of $U_{a_0 \cdots a_n} \otimes \Delta^n$, then send (x, t) to the element $(x, \sigma t)$ of $U_{a_{\sigma 0} \cdots a_{\sigma n}} \otimes \Delta^n$, where σt is defined by $(\sigma t)_i = t_{\sigma i}$.

One composition is the equal to the identity. It remains to construct a homotopy $H : |\check{C}(\mathfrak{U})_*| \times I \to |\check{C}(\mathfrak{U})_*|$ between the other composition and the identity. As in the proof of Lemma 2.4, we use the space $|\check{C}(\mathfrak{U})_* \times I|$ rather than $|\check{C}(\mathfrak{U})_*| \times I$. We define H as follows: An element (x, t) of $U_{a_0 \cdots a_n} \otimes \Delta^n$ is equivalent in $|\check{C}(\mathfrak{U})_*|$ to the element $(x, t_0, \dots, t_n, 0, \dots, 0)$ of $U_{a_0 \cdots a_n a_{\sigma 0} \cdots a_{\sigma n}} \otimes \Delta^{2n+1}$.

Also, $(x, \sigma t)$ is equivalent in $|\check{C}(\mathfrak{U})_*|$ to the element $(x, 0, \dots, 0, t_{\sigma 0}, \dots, t_{\sigma n})$ of $U_{a_0 \cdots a_n a_{\sigma 0} \cdots a_{\sigma n}} \otimes \Delta^{2n+1}$. Define $H((x,t), s)$ to be the element

$$(x, st_0, \dots, st_n, (1-s)t_{\sigma 0}, \dots, (1-s)t_{\sigma n})$$

of $U_{a_0 \cdots a_n a_{\sigma 0} \cdots a_{\sigma n}} \otimes \Delta^{2n+1}$. □

Proposition 2.7. *Let $\mathfrak{U}$ be an open cover of a space X indexed by a set A. Consider the realization of the simplicial space*

$$[n] \mapsto \coprod_{\sigma_0 \subseteq \cdots \subseteq \sigma_n} U_{\sigma_n},$$

where the coproduct is indexed by chains of nonempty, finite subsets of A. This realization is homeomorphic to the realization $|\check{C}^o(\mathfrak{U})_*|$ of the ordered Čech complex and is homotopy equivalent to $|\check{C}(\mathfrak{U})_*|$.

The realization in the above proposition is the object considered in [S1]. The ordered Čech complex is another construction of the same space, which for us seems somewhat easier to work with. One disadvantage, of course, is that it is not natural: a total ordering on A must be chosen to begin with.

Proof. The second claim follows from the first claim and Proposition 2.6.

For the first claim, it is convenient to use a slightly unusual construction of $|\check{C}^o(\mathfrak{U})_*|$. When forming the geometric realization, instead of forming Cartesian products with Δ^k we instead form products with $\operatorname{sd}\Delta^k$; since they are homeomorphic it doesn't matter which one we use. Given this, the key observation is that we can coordinatize $\operatorname{sd}\Delta^k$ in the following way: assuming that the vertices of Δ^k are labelled by the numbers $0, \dots, k$ in the usual way, a point on $\operatorname{sd}\Delta^k$ is represented uniquely by a chain of proper inclusions $\sigma_0 \subset \cdots \subset \sigma_j$ of subsets of $\{0, \dots, k\}$ together with an element t of Δ^j. Essentially, the chain of subsets determines in which sub-simplex the point lies, and then t gives local coordinates inside that sub-simplex.

Using this coordinate scheme, we can write down maps in both directions between the two realizations

$$\Bigl[\coprod_{\sigma_0 \subseteq \cdots \subseteq \sigma_n} U_{\sigma_n} \times \Delta^n\Bigr]/\sim \qquad \text{and} \qquad \Bigl[\coprod_{a_0 \leq \cdots \leq a_k} U_{a_0 \cdots a_k} \times \operatorname{sd}\Delta^k\Bigr]/\sim .$$

For instance, let's give the map from left to right. Using degeneracy relations, a point p in the left space can be represented by a chain of *proper* inclusions $\sigma_0 \subset \cdots \subset \sigma_n$, a point x of U_{σ_n}, and an element t of Δ^n. Let $a_0, a_1, \dots, a_k$ be the ordered list of elements of σ_n. The chain σ_* together with t defines a point s in $\operatorname{sd}\Delta^k$, and so we map p to the pair (x, s). It is easy to see that this map is well-defined and continuous, and just as easy to write down its inverse. □

In the case that $\{U_a\}$ admits a partition of unity $\{\psi_a\}$ it is fairly easy to see that the map $\pi\colon |\check{C}^o(\mathfrak{U})_*| \to X$ admits a section: First, a point x of X has a neighborhood which intersects the support of ψ_a only for finitely many indices

$a = a_0, \dots, a_n$. The section χ sends x to the point of $|\check{C}^o(\mathcal{U})_*|$ represented by (x, t) in $U_{a_0 \cdots a_n} \otimes \Delta^n$ where $t_i = \psi_{a_i}(x)$. One has to check that χ is continuous (use the local-finiteness of the partition of unity), and that $\chi\pi \simeq id$ via a straight-line homotopy. See Proposition 4.1 of [S1].

3 Passing to homotopy colimits

The results of the previous section all concerned geometric realizations. In this section we translate these into results about various homotopy colimits. In general, there is a 'Reedy cofibrancy' condition on simplicial spaces which guarantees that geometric realization and homotopy colimit agree. Unfortunately our Čech complexes are not Reedy cofibrant, due to the fact that the open sets appearing in them are not necessarily cofibrant spaces. However, Theorem 1.2 shows that in the category of topological spaces this cofibrancy issue is unimportant: homotopy colimits can be computed naively, without first making things cofibrant.

Theorem 3.1. *If $\mathcal{U}$ is an open cover of a space X, then the natural map* hocolim $\check{C}(\mathcal{U})_* \to X$ *is a weak equivalence.*

Proof. By Theorem 1.2, the map hocolim $\check{C}(\mathcal{U})_* \to |\check{C}(\mathcal{U})_*|$ is a weak equivalence because the Čech complex has free degeneracies in the sense of Definition A.4. Theorem 2.1 now gives the result. □

Here are several alternative formulations:

Proposition 3.2. *Let A be an indexing set for the cover $\mathcal{U}$, and let $\mathcal{P}_A$ denote the partially ordered set consisting of all nonempty finite subsets of A. Let Γ denote the functor $\mathcal{P}_A^{op} \to \mathcal{T}op$ which sends σ to U_σ. Then the natural map* hocolim $\Gamma \to X$ *is a weak equivalence.*

Proof. To construct hocolim Γ we can take the realization of the simplicial replacement for Γ (by Theorem A.7 we don't need to first make the spaces cofibrant). That is, we take the realization of the simplicial space

$$[n] \mapsto \coprod_{\sigma_0 \subseteq \cdots \subseteq \sigma_n} U_{\sigma_n},$$

where the coproduct is indexed by chains of nonempty, finite subsets of A. Now Proposition 2.7 tells us that this realization is homotopy equivalent to $|\check{C}(\mathcal{U})_*|$, so Theorem 2.1 finishes the proof. □

Corollary 3.3. *Let $\mathcal{P}_{\mathcal{U}}$ denote the subcategory of $\mathcal{T}op$ whose objects are the open sets U_a belonging to $\mathcal{U}$ together with their finite intersections; the morphisms are the inclusions of open subsets of X. Let Γ denote the inclusion functor $\mathcal{P}_{\mathcal{U}} \to \mathcal{T}op$. Then the natural map* hocolim $\Gamma \to X$ *is a weak equivalence.*

Proof. Consider the obvious functor $F\colon \mathcal{P}_A^{op} \to \mathcal{P}_{\mathcal{U}}$ sending σ to U_σ. We will show that it is homotopy cofinal, so for every object V in $\mathcal{P}_{\mathcal{U}}$ we prove that the undercategory $(V \downarrow F)$ is contractible. It suffices to show that any map $K \to N(V \downarrow F)$ can be extended over the cone on K, as K ranges over all finite simplicial sets. Every n-simplex s in K maps to a chain of open sets $V \to U_{\sigma_0} \to U_{\sigma_1} \to \cdots \to U_{\sigma_n}$ in $(V \downarrow F)$. Since K has only finitely-many non-degenerate simplices, only finitely-many of the U_σ will ever appear. Define μ to be the union of all the σ_i arising from the map $K \to N(V \downarrow F)$. To extend the map over CK, we send the cone on s to the $(n+1)$-simplex corresponding to the chain $V \to U_\mu \to U_{\sigma_0} \to U_{\sigma_1} \to \cdots \to U_{\sigma_n}$. □

The following corollary was shown to us by Bill Dwyer. Let $(\mathcal{T}op \downarrow X)_{\mathcal{U}}$ denote the full subcategory of $(\mathcal{T}op \downarrow X)$ consisting of all maps $Z \to X$ that factor through the space $E = \coprod_a U_a$. Let $\Gamma\colon (\mathcal{T}op \downarrow X)_{\mathcal{U}} \to \mathcal{T}op$ be the canonical functor sending $Z \to X$ to Z. We would like to claim that the homotopy colimit of the diagram Γ is weakly equivalent to X, but $(\mathcal{T}op \downarrow X)_{\mathcal{U}}$ is not a small category. So we choose an infinite cardinal κ larger than the size of E and restrict to the spaces Z that have at most κ elements. As the proof of the following corollary indicates, the weak homotopy type of hocolim Γ is independent of the choice of κ, as long as κ is sufficiently large.

Corollary 3.4. *For the functor* $\Gamma\colon (\mathcal{T}op \downarrow X)_{\mathcal{U}} \to \mathcal{T}op$ *defined above, the natural map* hocolim $\Gamma \to X$ *is a weak equivalence.*

Proof. The nth level of the Čech complex is $E_X^n := E \times_X E \times_X \cdots \times_X E$ (n factors). Let's write $\mathcal{C} = (\mathcal{T}op \downarrow X)_{\mathcal{U}}$, for brevity. So we have the functor $F\colon \Delta^{op} \to \mathcal{C}$ given by $[n] \mapsto E_X^n$. The composition $\Delta^{op} \to \mathcal{C} \to \mathcal{T}op$ is just $\check{C}(\mathcal{U})_*$. Because of Theorem 3.1, it will be enough to show that F is homotopy cofinal.

For this we pick an object $z : Z \to X$ in $\mathcal{C}$ and show that $(z \downarrow F)$ is contractible. This undercategory is isomorphic to the category of simplices of K [H, 15.1.16], where K is the simplicial set sending $[n]$ to $\mathrm{Hom}_{\mathcal{C}}(z, E_X^n)$. But observe that $\mathrm{Hom}_{\mathcal{C}}(z, E_X^n)$ is equal to T^n where $T = \mathrm{Hom}_{\mathcal{C}}(z, E_X)$. So K is the simplicial set $[n] \mapsto T^n$, which is contractible because T is nonempty (using the fact that $z\colon Z \to X$ factors through E). Thus $(z \downarrow F)$ is isomorphic to the category of simplices of a contractible simplicial set, and therefore has a contractible nerve. □

Corollary 3.5 (Small simplices theorem)**.** *Let* $Sing_{\mathcal{U}} X$ *denote the simplicial set whose* n*-simplices are the maps* $\Delta^n \to X$ *that factor through some* U_a*. Then* $Sing_{\mathcal{U}} X \to Sing\, X$ *is a weak equivalence.*

Proof. Let $\mathcal{P}_A$ be the category defined in Proposition 3.2, where A is the indexing set for the cover. Consider the diagram $\Gamma\colon \mathcal{P}_A^{op} \to s\mathcal{S}et$ defined by $\Gamma(\sigma) = Sing\,(U_\sigma)$. By general nonsense $|\mathrm{hocolim}\, \Gamma| \simeq \mathrm{hocolim}\, |\Gamma|$. Also, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{hocolim}_{\mathcal{P}_A^{op}} |Sing\, U_\sigma| \arrow[r] \arrow[d] & |Sing\, X| \arrow[d] \\
\mathrm{hocolim}_{\mathcal{P}_A^{op}} U_\sigma \arrow[r] & X
\end{tikzcd}$$

in which the vertical maps are weak equivalences because the natural map $|Sing\, Y| \to Y$ is a weak equivalence for every space Y. We know from Proposition 3.2 that the bottom horizontal map is a weak equivalence, so the top horizontal map is also a weak equivalence. We conclude that the map $\operatorname{hocolim} \Gamma \to Sing\, X$ is a weak equivalence of simplicial sets. Therefore, we shall compare $\operatorname{hocolim} \Gamma$ and $Sing_{\mathcal{U}} X$.

For the moment, assume that A is finite. In this case $\mathcal{P}_A^{op}$ is a Reedy category [Ho, Def. 5.2.1], where we think of all the maps as increasing degree. Since there are no non-identity downward maps, the fibrations are objectwise in the Reedy model structure on $s\mathcal{S}et^{\mathcal{P}_A^{op}}$ (see [Ho, Th. 5.2.5]). So in this case the Reedy and projective model structures (cf. Section 1.7) are the same. In particular, a Reedy-cofibrant diagram is also projective-cofibrant, which guarantees that the homotopy colimit and the ordinary colimit are weakly equivalent.

The functor Γ may be checked to be Reedy cofibrant: at the spot indexed by $\sigma = \{a_0, \ldots, a_n\}$, the latching object is the subobject of $Sing\, U_\sigma$ consisting of all simplices which are contained in some other U_b. The fact that it is actually a subobject says that the latching map is a cofibration. So we know that $\operatorname{hocolim} \Gamma$ and $\operatorname{colim} \Gamma$ are weakly equivalent. It is easy to check that $\operatorname{colim} \Gamma \cong Sing_{\mathcal{U}} X$. We have shown that if $\mathcal{U}$ is a finite cover, then $Sing_{\mathcal{U}} X$ is weakly equivalent to $Sing\, X$.

Now let A be arbitrarily large. For any finite subcollection $\mathcal{U}'$, let $\cup\mathcal{U}'$ denote the union of the open sets in $\mathcal{U}'$. Then we know the map $Sing_{\mathcal{U}'}(\cup\mathcal{U}') \to Sing\,(\cup\mathcal{U}')$ is a weak equivalence. But $Sing_{\mathcal{U}} X \to Sing\, X$ is the filtered colimit of these maps, where the indexing category is the poset of all finite subcollections $\mathcal{U}'$. This uses that each space Δ^n is compact. Our result now follows from the fact that filtered colimits of simplicial sets preserve weak equivalences. □

4 Hypercovering Theorems

In this section we define hypercovers, and then prove our main result, Theorem 1.3. We go on to deduce various corollaries.

Before giving a rigorous definition of hypercovers, we need to recall a few pieces of machinery related to simplicial objects. For any category $\mathcal{C}$, let $s\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. Likewise, let $s_{\leq n}\mathcal{C}$ denote the category of truncated simplicial objects of dimension n. There is the obvious forgetful functor $\mathrm{sk}_n : s\mathcal{C} \to s_{\leq n}\mathcal{C}$, and if $\mathcal{C}$ has all finite limits then sk_n has a right adjoint called cosk_n; these are the **skeleton** and **coskeleton** functors. If U_* belongs to $s\mathcal{C}$ then we'll often abbreviate $\mathrm{cosk}_n(\mathrm{sk}_n\, U)_*$ as just $\mathrm{cosk}_n\, U_*$. Finally, the ***n*th matching object $M_n U$** is defined to be the nth object of $\mathrm{cosk}_{n-1}\, U_*$. There is a canonical map of simplicial spaces $U_* \to \mathrm{cosk}_{n-1}\, U_*$, and in level n it gives $U_n \to M_n U$. In levels less than n, this map is the identity. We write cosk_n^X for the nth coskeleton functor for $s(\mathcal{C} \downarrow X)$.

These definitions have somewhat easier interpretations when $\mathcal{C}$ is the category of topological spaces. To describe these, note that any simplicial set may be regarded as a simplicial space which is discrete in every dimension, and if U_* and W_* are simplicial spaces then the set of maps from U_* to W_* has a natural topology coming from the compact-open topology on function spaces. Using these observations, one checks that

(i) $U_n \cong \mathrm{Map}(\Delta^n, U_*)$,
(ii) $[\mathrm{cosk}_n U]_k \cong \mathrm{Map}(sk_n \Delta^k, U_*)$, and
(iii) $M_n U \cong \mathrm{Map}(\partial\Delta^n, U_*)$.

The first property is immediate from the Yoneda lemma. The second property follows from the first and the adjunction between sk_n and cosk_n. The third property is a special case of the second.

Finally, say that a map of spaces $Z \to X$ is an **open covering map** if it is isomorphic to a map of the form $\coprod_a U_a \to X$ where $\{U_a\}$ is an open cover of X.

Definition 4.1. *A* ***hypercover*** *of a space X is an augmented simplicial space $U_* \to X$ such that the maps $U_n \to M_n^X U$ are open covering maps for all $n \geq 0$. Here $M_n^X U$ denotes the nth matching object of U_* computed in the category $s(\mathcal{T}op \downarrow X)$ of simplicial spaces over X.*

Note that $M_0^X U \cong X$, so the condition for $n = 0$ says that $U_0 \to X$ is an open covering map. Also $M_1^X U \cong U_0 \times_X U_0$, so when $n = 1$ we are requiring $U_1 \to U_0 \times_X U_0$ to be an open covering map. The reader should be aware that when $n > 1$ the objects $M_n U$ and $M_n^X U$ turn out to be isomorphic, so one can forget about the extra complication of the overcategory.

Using properties (i)–(iii) above, it can be checked that if $U_* \to X$ is a hypercover and $K \to L$ is an inclusion of finite simplicial sets, then the map $\mathrm{Map}(L, U_*) \to \mathrm{Map}(K, U_*)$ is also an open covering map. From this, it follows that $\mathrm{cosk}_n^X U_* \to X$ is a hypercover whenever $U_* \to X$ is a hypercover. Also, each map $U_k \to [\mathrm{cosk}_n^X U]_k$ is an open covering map.

We leave it to the reader to check that in a hypercover each U_n must be a disjoint union of open subsets of X, and that Čech complexes are the hypercovers for which the maps $U_n \to M_n^X U$ are all *isomorphisms*. Generalizing this, a hypercover $U_* \to X$ is called **bounded** if there exists an N such that the maps $U_n \to M_n^X U$ are isomorphisms for all $n > N$. The smallest such N for which this happens is called the dimension of the hypercover. Said intuitively, the bounded hypercovers of dimension N are the hypercovers for which the refinement process stops after the Nth level. A hypercover $U_* \to X$ has dimension at most N if and only if $U_* \cong \mathrm{cosk}_N^X U_*$.

Lemma 4.2. *If $U_* \to X$ is a bounded hypercover, then* hocolim $U_* \to X$ *is a weak equivalence.*

A more detailed version of the following proof, given in the context of an arbitrary Grothendieck topology, appears in [DHI].

Proof. We proceed by induction, starting from the fact that bounded hypercovers of dimension 0 are just Čech complexes and therefore are handled by Theorem 3.1.

Suppose that $U_* \to X$ is a bounded hypercover of dimension $n + 1$. Define V_* to be $\mathrm{cosk}_n U_*$, so V_* is a bounded hypercover of dimension at most n. Therefore, we may assume by induction that hocolim $V_* \to X$ is a weak equivalence. The canonical map $U_* \to V_*$ gives an open covering map $U_{n+1} \to V_{n+1}$, by the very definition of what it means for U_* to be a hypercover (since $V_{n+1} = M_{n+1}U$). In fact, one can check that $U_k \to V_k$ is an open covering map for all k.

Consider the following bisimplicial object, augmented horizontally by V_*:

$$\begin{tikzcd} V_* & U_* \arrow[l] & U_* \times_{V_*} U_* \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd}$$

The kth row is the (augmented) Čech complex for the open covering map $U_k \to V_k$. Note that for $0 \le k \le n$ the kth row is the constant simplicial object with value U_k because $U_k \to V_k$ is the identity. Call this bisimplicial object (without the horizontal augmentation) W_{**}.

Let D_* denote the diagonal of W_{**}. Standard homotopy theory tells us that hocolim D_* may be computed (up to weak equivalence) by first taking the homotopy colimits of the rows of W_{**}, and then taking the homotopy colimits of the resulting simplicial object. But the homotopy colimit of the kth row is just V_k by Theorem 3.1. Since V_* is a bounded hypercover of dimension at most n, we have assumed that hocolim V_* is weakly equivalent to X. So hocolim $D_* \to X$ is a weak equivalence.

We claim that U_* is a retract, over X, of D_*. Note first that one has, in complete generality, a map $U_* \to D_*$; in dimension k it is the unique horizontal degeneracy $W_{0k} \to W_{kk}$ (which is the diagonal map in the Čech complex).

To produce a map $D_* \to U_*$ it is enough to give $\mathrm{sk}_{n+1} D_* \to \mathrm{sk}_{n+1} U_*$, because $U_* = \mathrm{cosk}_{n+1} U_*$. Notice that $\mathrm{sk}_n D_* = \mathrm{sk}_n U_*$. Choosing any face map $[0] \to [n+1]$ gives a map $W_{n+1,n+1} \to W_{0,n+1}$, which is just $D_{n+1} \to U_{n+1}$. This induces a corresponding map $\mathrm{sk}_{n+1} D_* \to sk_{n+1}U_*$ as desired.

It is straightforward to check that $U_* \to D_* \to U_*$ is the identity (because $U_* = \mathrm{cosk}_{n+1} U_*$ one only has to check it on $(n+1)$-skeleta), and all the maps commute with the augmentations down to X. We have already shown that hocolim $D_* \to X$ is a weak equivalence. Since hocolim $U_* \to X$ is a retract of hocolim $D_* \to X$, it must also be a weak equivalence. □

Theorem 4.3. *If $U_* \to X$ is a hypercover then the maps* hocolim $U_* \to |U_*| \to X$ *are both weak equivalences.*

Proof. The fact that hocolim $U_* \to |U_*|$ is a weak equivalence follows from Theorem 1.2: the simplicial object U_* has free degeneracies (Definition A.4).

To show that hocolim $U_* \to X$ is a weak equivalence, note first that we have an isomorphism $\pi_k(\mathrm{hocolim}\, U_*) \to \pi_k(\mathrm{hocolim}\, [\mathrm{cosk}_{k+1} U_*])$. This is true for any map of simplicial spaces $X_* \to Y_*$ which is an isomorphism on $(k+1)$-skeleta – an easy proof is to apply the singular functor everywhere to get into bisimplicial sets, then use the diagonal in place of hocolim. But $\mathrm{cosk}_{k+1} U_*$ is a bounded hypercover, so Lemma 4.2 gives us the isomorphism $\pi_k(\mathrm{hocolim}[\mathrm{cosk}_{k+1} U_*]) \xrightarrow{\cong} \pi_k X$. □

4.4 Complete covers

In this section we replace hypercovers with a related concept which captures the same phenomena. This second approach was suggested to us by Jeff Smith.

Definition 4.5. *An open cover* $\mathcal{U} = \{U_a\}$ *of a space* X *is called* ***complete*** *if for all finite sets* σ *of indices, the intersection* U_σ *is covered by elements of* $\mathcal{U}$. *It is called a* ***Čech cover*** *if every* U_σ *is again an element of the cover.*

Complete covers appear in [DT, Satz 2.2], where they were used in the context of identifying quasi-fibrations. The paper [Mc1] then used them to detect weak equivalences.

We blur the distinction between a cover and the full subcategory that it spans inside the category of open sets of X. Given a cover $\mathcal{U}$, we can construct an associated simplicial space in the following way: For any $n \geq 0$, let P_n denote the category of nonempty subsets of $\{0, \dots, n\}$, where the maps are the inclusions. Note that the assignment $[n] \mapsto P_n$ defines a cosimplicial category in the obvious way. (Application of the nerve functor everywhere gives the cosimplicial space $[n] \to \operatorname{sd} \Delta^n$.)

Define Ω_* to be the simplicial space

$$[n] \mapsto \coprod_{F\colon P_n^{op} \to \mathcal{U}} F(\{0, \dots, n\}),$$

where the coproduct runs over all functors $P_n^{op} \to \mathcal{U}$. The faces and degeneracies are induced by those in P in the expected way.

To give a point in Ω_3, for example, is to give the following data:

(1) A sequence of opens $U_0, \dots, U_3$ in $\mathcal{U}$,
(2) 6 open subsets $U_{01}, U_{02}, \dots, U_{23}$ in $\mathcal{U}$ such that $U_{ij} \subseteq U_i \cap U_j$;
(3) 4 open subsets $U_{012}, \dots, U_{123}$ in $\mathcal{U}$ such that $U_{ijk} \subseteq U_{ij} \cap U_{jk} \cap U_{ik}$;
(4) An open subset U_{0123} in $\mathcal{U}$ which is contained in all the U_{ijk};
(5) A point on U_{0123}.

It is usually helpful to think of these open sets as indexed by the faces of a 3-simplex.

In forming the Čech complex of a cover $\mathcal{U}$ we are throwing in all the finite intersections U_σ into the higher levels of the simplicial object, and these are typically objects which are not in $\mathcal{U}$ itself. The simplicial object Ω_* is in some sense the closest thing we can get to a Čech complex while requiring all the open sets to belong to $\mathcal{U}$.

Proposition 4.6.

(a) If the cover $\mathcal{U}$ *is complete then* Ω_* *is a hypercover of* X.
(b) Regarding $\mathcal{U}$ *as a category, let* $\Gamma\colon \mathcal{U} \to \mathcal{T}op$ *be the obvious inclusion. Then* $\operatorname{hocolim} \Gamma \simeq |\Omega_*|$.
(c) If the cover $\mathcal{U}$ *is complete then the natural map* $\operatorname{hocolim} \Gamma \to X$ *is a weak equivalence.*

Proof. For part (a), consider the full subcategory $\bar{P}_n$ of P_n consisting of all objects except for $\{0, 1, \dots, n\}$. Then the matching space $M_n\Omega$ is equal to

$$\coprod_{\bar{F}\colon \bar{P}_n^{op} \to \mathcal{U}} \Big[\bigcap_{\sigma \in \bar{P}_n} \bar{F}(\sigma)\Big].$$

For example, a point in $M_3\Omega$ is determined by the data in (1)–(3) above, together with a point in $U_{012} \cap U_{013} \cap U_{023} \cap U_{123}$.

Since the cover is complete, for each functor $\bar{F}\colon \bar{P}_n^{op} \to \mathcal{U}$ and each element x of $\bigcap_{\sigma\in\bar{P}_n} \bar{F}(\sigma)$, there exists an extension F of $\bar{F}$ to P_n^{op} such that x belongs to $F(\{0,\dots,n\})$. This shows that $\Omega_n \to M_n\Omega$ is an open covering map, which finishes part (a).

Now we proceed to part (b). To form hocolim Γ we can work in the Strom model structure on $\mathcal{T}op$ (see Appendix A), where we first take the simplicial replacement

$$[n] \mapsto \coprod_{U_0\to\cdots\to U_n} U_0$$

and then form the realization. Here the coproduct is indexed over all functors $\Delta^n \to \mathcal{U}$, where Δ^n denotes the category of n composable maps. Note that Ω_* was formed in almost the same way as the simplicial replacement of Γ, except we indexed the coproduct by functors $P_n^{op} \to \mathcal{U}$. Each P_n is essentially just a subdivision of Δ^n, so it's not surprising that $|\Omega_*|$ is another model of the homotopy colimit.

In somewhat more detail: Let sd$'$ denote the 'opposite' of the usual subdivision functor on $s\mathcal{S}et$, in which the orientations of all the simplices have been changed so that they point away from the barycentres, rather than towards them. (We need this because we are using P_n^{op} rather than P_n.) The functor sd$'$ has a right adjoint Ex$'$. There is a natural 'first vertex map' sd$'$ $K \to K$, inducing $K \to \mathrm{Ex}'\,K$. Given our diagram $\Gamma\colon \mathcal{U} \to \mathcal{T}op$, the realization of the simplicial replacement is isomorphic to the coend $\Gamma \otimes_{\mathcal{U}} B$, where $B\colon \mathcal{U} \to s\mathcal{S}et$ sends U_a to the classifying space $B(U_a \downarrow \mathcal{U})$. Likewise, one checks that the realization of Ω_* is isomorphic to the coend $\Gamma \otimes_{\mathcal{U}} \mathrm{Ex}'\,B$, where $\mathrm{Ex}'\,B$ is the obvious composite functor. The natural map $B \to \mathrm{Ex}'\,B$ is an objectwise weak equivalence. The object B of $s\mathcal{S}et^{\mathcal{U}}$ is cofibrant (see [H, Cor. 14.8.8]), where this diagram category has the projective model structure described in Section 1.7. The exact same arguments show that $\mathrm{Ex}'\,B$ is also cofibrant in this structure. So we have an objectwise weak equivalence between two cofibrant diagrams. The diagram $\Gamma\colon \mathcal{U} \to \mathcal{T}op$ is objectwise cofibrant (since we are working with the Strom model structure on $\mathcal{T}op$), and so by [H, Cor. 18.4.5] it follows that $\Gamma \otimes_{\mathcal{U}} B \to \Gamma \otimes_{\mathcal{U}} \mathrm{Ex}'\,B$ is a weak equivalence.

Finally, part (c) is an immediate consequence of (a), (b), and Theorem 4.3. □

The following corollary was originally proven by McCord [Mc1, Th. 6], but is an easy consequence of our hypercovering theorem. It generalizes May's result from Corollary 2.3, which handled the case of Čech covers. For the proof we will need the following observations: (1) If $U \to X$ is an open covering map and $f\colon Y \to X$ is any map, there is an induced open covering map $Y \times_X U \to Y$. (2) If $U_* \to X$ is a hypercover and $f\colon Y \to X$ is a map of spaces, one gets a hypercover $f^{-1}U_* \to Y$ whose space in level n is $Y \times_X U_n$.

Corollary 4.7. *Let $f\colon X \to Y$ be a map of spaces. Suppose there is a complete cover $\mathcal{U} = \{U_a\}$ of Y such that each $f^{-1}(U_a) \to U_a$ is a weak equivalence. Then f itself is a weak equivalence.*

Proof. From $\mathcal{U}$ form the associated hypercover Ω^Y_* as described in the paragraph preceding Proposition 4.6. Pulling this back to X gives a hypercover $\Omega^X_* := f^{-1}\Omega^Y_*$, as described above (note that this is *not* the hypercover associated to the covering $\{f^{-1}U_a\}$). Now f induces a map $\Omega^X_* \to \Omega^Y_*$ compatible with the augmentations. This map of simplicial spaces is a levelwise weak equivalence, by assumption. Upon taking homotopy colimits we get

$$\begin{tikzcd}
\operatorname{hocolim} \Omega^X_* \arrow[r, "\sim"] \arrow[d, "\sim"'] & \operatorname{hocolim} \Omega^Y_* \arrow[d, "\sim"] \\
X \arrow[r] & Y,
\end{tikzcd}$$

and so we conclude that $X \to Y$ is also a weak equivalence. □

4.8 Generalized hypercovers for topological spaces

Up until now we have only considered open covers, but now we turn to a broader notion. We'll say that a map $p \colon E \to B$ of spaces is a **generalized cover** if it is locally split: that is, every element of B has a neighborhood U such that $p^{-1}(U) \to U$ admits a section. Observe that covering spaces, and in fact fibre bundles in general, are generalized covers.

Definition 4.9. *An augmented simplicial space $U_* \to X$ is a* ***generalized hypercover*** *of X if the maps $U_n \to M^X_n U$ are generalized covers.*

Proposition 4.10. *If U_* is a generalized hypercover of X then* $\operatorname{hocolim} U_* \to X$ *is a weak equivalence.*

Proof. The general results of [DHI] show that this is a formal consequence of Theorem 4.3, because open covers and generalized covers generate the same Grothendieck topology on topological spaces. To keep within the spirit of the present paper, however, we will also give an elementary proof.

One can almost repeat all of the results leading up to Theorem 4.3 for generalized covers verbatim, but there is the slight problem that for a generalized cover $E \to X$ the Čech complex $\check{C}(E)$ need not be Reedy cofibrant, even in the Strom model category. Here $\check{C}(E)_*$ is the simplicial space whose nth level is $E \times_X \cdots \times_X E$ (with $n+1$ factors), and for Reedy cofibrancy one needs to know that the inclusion of the 'fat diagonals' in each $\check{C}(E)_n$ are cofibrations. Although this will be satisfied in almost all real-life situations, we avoid the issue by working only with hocolims.

Using the same arguments as in Lemma 4.2 and Theorem 4.3, one reduces to showing that if $E \to X$ is a generalized cover then $\pi \colon \operatorname{hocolim} \check{C}(E)_* \to X$ is a weak equivalence. Since $E \to X$ is a generalized cover, there is an open cover $\{U_a\}$ of X such that the map has a section over each U_a. By Corollary 2.3, it suffices to prove that $\pi^{-1}(U_\sigma) \to U_\sigma$ is a weak equivalence for every σ. But for any open $U \subseteq X$, $\pi^{-1}(U)$ is homeomorphic to $\operatorname{hocolim} \check{C}(p^{-1}U)_*$ (the argument is the same as for realizations – see the first paragraph of the proof of Theorem 2.1). So at this point we have further reduced to the case where $E \to X$ has a section.

But a section gives a contracting homotopy for the augmented simplicial space $\check{C}(E)_* \to X$, or equivalently a homotopy equivalence between $\check{C}(E)_*$ and the constant simplicial space with X in every dimension. So if Δ^1 denotes the usual simplicial set, regarded as a discrete simplicial space, we have a homotopy $\check{C}(E)_* \times \Delta^1 \to \check{C}(E)_*$. One can check that if Z_* is any simplicial space, then the two inclusions $Z_* \hookrightarrow Z_* \times \Delta^1$ induce weak equivalences after taking hocolims (there are different ways to prove this, one of which is to quote [D1, Thm 6.1]). It follows that a homotopy equivalence between simplicial spaces induces a homotopy equivalence between their hocolims, and so hocolim $\check{C}(E)_* \to X$ is a homotopy equivalence. □

Corollary 1.5 is an immediate consequence of the above proposition.

Example 4.11. Let G be a topological group and consider the usual G-fibre bundle $\xi \colon EG \to BG$. Form the Čech complex $\check{C}(\xi)_*$, which is a generalized hypercover of BG. Using only the fact that EG has a free G-action, one can see that the nth level of $\check{C}(\xi)_*$ is homeomorphic to $G^n \times EG$, and the face and degeneracy maps are the familiar ones of the two-sided bar construction $B(*, G, EG)$. Now using that EG is contractible, we find that $\check{C}(\xi)_*$ is levelwise weakly equivalent to the simplicial space

$$\begin{tikzcd} * & G \arrow[l, shift left] \arrow[l, shift right] & G \times G \cdots . \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

The above proposition tells us that $|\check{C}(\xi)_*| \simeq BG$, and so in this way we recover the usual bar construction for BG.

5 Topological realization functors for $\mathbb{A}^1$-homotopy theory

Let k be a field. Morel and Voevodsky [MV] produced a model category Spc(k) which captures the 'motivic homotopy theory' of smooth schemes over k. Here Spc(k) stands for 'spaces over k'. There are several possible choices for what model category to choose, all Quillen equivalent in the end, but we choose our reference point to be the category of simplicial presheaves on the Nisnevich site of smooth schemes over Spec k, with the model structure provided in [J, Appendix B].

When k comes with an embedding $k \hookrightarrow \mathbb{C}$, then any k-scheme X gives rise to a topological space $X(\mathbb{C})$ consisting of its $\mathbb{C}$-valued points with the analytic topology. A natural expectation is to use this functor to relate Spc(k) to the usual model category $\mathcal{T}op$ of topological spaces. Morel and Voevodsky showed how to extend this functor on the level of homotopy categories (by somewhat awkward methods), but they didn't produce functors at the model category level. In this section we use Proposition 4.10 to produce such functors, with the small provision that we have to replace Spc(k) with a Quillen-equivalent variant. We also address the situation when $k \hookrightarrow \mathbb{R}$, in which case one can construct topological realization functors into $\mathbb{Z}_2$-equivariant spaces.

As in [D2], a Quillen pair $L \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon R$ will be called a *Quillen map* $\mathcal{M} \to \mathcal{N}$.

5.1 The Complex case

Let $\mathcal{T}$ denote either the Nisnevich or étale Grothendieck topology on the category Sm/k of smooth k-schemes. In the terminology of [D2], let $\mathrm{Spc}'(k)_{\mathcal{T}}$ denote the universal model category built from Sm/k subject to the following relations:

(1) $X \amalg Y \xrightarrow{\sim} (X \cup Y)$ (here $\amalg$ denotes the coproduct in our model category, whereas $\cup$ denotes disjoint union of schemes);

(2) $\mathrm{hocolim}\, U_* \xrightarrow{\sim} X$ for any $\mathcal{T}$-hypercover U_* of a smooth scheme X (called 'basal hypercovers' in [DHI]);

(3) $X \times \mathbb{A}^1 \xrightarrow{\sim} X$.

(Relation (1) is morally a special case of (2), but must be included separately for technical reasons – see [DHI]).

The model categories $\mathrm{Spc}(k)_{\mathcal{T}}$ and $\mathrm{Spc}'(k)_{\mathcal{T}}$ have the same underlying category and the same class of weak equivalences, but differ in their notions of cofibration and fibration. They are injective and projective versions of the same homotopy theory.

Theorem 5.2. *There are Quillen maps $Spc'(k)_{et} \to \mathcal{T}op$ and $Spc'(k)_{Nis} \to \mathcal{T}op$ sending a smooth k-scheme X to $X(\mathbb{C})$.*

Proof. By general nonsense from [D2], to give a Quillen map $\mathrm{Spc}'(k)_{\mathcal{T}} \to \mathcal{T}op$ we just need to give a functor $Sm/k \to \mathcal{T}op$ which respects the above relations. The functor we're interested in is $X \mapsto X(\mathbb{C})$, and this clearly preserves relations (1) and (3). In the case of the étale topology, the fact that it preserves relation (2) is just Proposition 4.10; the point is that if $p\colon E \to B$ is an étale cover, then $p(\mathbb{C})\colon E(\mathbb{C}) \to B(\mathbb{C})$ satisfies the hypotheses of the inverse function theorem and hence is locally split.

Since the étale topology is finer than the Nisnevich topology, there is an obvious map $\mathrm{Spc}'(k)_{Nis} \to \mathrm{Spc}'(k)_{et}$ (in essence, there are more relations of type (2) for the étale topology). So one also gets a topological realization map $\mathrm{Spc}'(k)_{Nis} \to \mathcal{T}op$ by composition. □

It is possible to show that the functor $X \mapsto X(\mathbb{C})$ takes elementary distinguished squares [MV, Sec. 3.1, Def. 1.3] to homotopy pushout squares of topological spaces. Together with results of [B], this can be used to give an alternative proof of the above theorem for the Nisnevich topology.

5.3 The Real case

If we have a Real field $k \hookrightarrow \mathbb{R}$, then the space $X(\mathbb{C})$ comes equipped with an action of the group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$. So we might hope to compare $\mathrm{Spc}'(k)$ to a model category of $\mathbb{Z}_2$-equivariant spaces.

Recall that if G is a finite group then there are two notions of weak equivalence for G-spaces, called the **non-equivariant** and **G-equivariant** equivalences. An equivariant map $X \to Y$ is a non-equivariant equivalence if it is a weak equivalence after forgetting the equivariant structure, and it is a G-equivariant equivalence if $X^H \to Y^H$ is a non-equivariant weak equivalence for every subgroup $H \subseteq G$. There are associated G-equivariant and non-equivariant model structures on the category of G-spaces, which we will denote $\mathcal{T}op(G)$ and $\mathcal{T}op(G)_{\text{non}}$.

If $p\colon E \to B$ is an equivariant map which is also a covering space (non-equivariantly), the map $\operatorname{hocolim} \check{C}(E)_* \to B$ is a non-equivariant equivalence but not necessarily a G-equivariant equivalence. For instance, if p is $G \to *$ then the map $\operatorname{hocolim} \check{C}(E)_* \to B$ is equal to $EG \to *$. So when we have a subfield $k \hookrightarrow \mathbb{R}$ the arguments given above show that the functor $X \mapsto X(\mathbb{C})$ induces a Quillen map $\mathrm{Spc}'(k)_{et} \to \mathcal{T}op(\mathbb{Z}_2)_{\text{non}}$, but not a Quillen map $\mathrm{Spc}'(k)_{et} \to \mathcal{T}op(\mathbb{Z}_2)$. However, when we use the Nisnevich topology something special happens.

Lemma 5.4. *If $E \to B$ is a Nisnevich cover of k-schemes, then $E(\mathbb{C})^{\mathbb{Z}_2} \to B(\mathbb{C})^{\mathbb{Z}_2}$ is locally split.*

For a counterexample to this in the case of étale covers, try $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$.

Proof. First note that $X(\mathbb{C})^{\mathbb{Z}_2}$ is homeomorphic to $X(\mathbb{R})$ for any scheme X over k. The map $p(\mathbb{R}) : E(\mathbb{R}) \to B(\mathbb{R})$ is surjective by the defining property of Nisnevich covers; every $\mathbb{R}$-point in B lifts to E.

By definition of étale covers, $p(\mathbb{R})$ satisfies the hypothesis of the inverse function theorem. Since $p(\mathbb{R})$ is surjective, it is locally split. □

Theorem 5.5. *There is a Quillen map $\mathrm{Spc}'(k)_{Nis} \to \mathcal{T}op(\mathbb{Z}_2)$ sending a smooth k-scheme X to $X(\mathbb{C})$.*

Proof. The argument exactly parallels the non-equivariant case in Theorem 5.2, so the only nontrivial part is to show that if $U_* \to X$ is a Nisnevich hypercover then the map $\operatorname{hocolim} U_*(\mathbb{C}) \to X(\mathbb{C})$ is a $\mathbb{Z}_2$-equivariant weak equivalence of $\mathbb{Z}_2$-spaces. The fact that it is a non-equivariant equivalence has already been discussed in Theorem 5.2, because $U_* \to X$ is in particular an étale hypercover. So we must consider what happens when we take $\mathbb{Z}_2$-fixed points.

It is a fact that for any diagram D of G-spaces (G any finite group) and any subgroup H of G, one has $(\operatorname{hocolim} D)^H \simeq (\operatorname{hocolim} D^H)$ (see Remark 5.6 below). So we just need to convince ourselves that $\operatorname{hocolim}\{U_*(\mathbb{C})^{\mathbb{Z}_2}\} \to X(\mathbb{C})^{\mathbb{Z}_2}$ is a non-equivariant weak equivalence. But by the above lemma one sees that $U_*(\mathbb{C})^{\mathbb{Z}_2}$ is a generalized hypercover of $X(\mathbb{C})^{\mathbb{Z}_2}$, and so the result is an instance of Proposition 4.10. □

Remark 5.6. In the above proof we needed the fact that $(\operatorname{hocolim} D)^H$ is weakly equivalent to $\operatorname{hocolim}(D^H)$. This is well-known in equivariant topology, but it's hard to find an actual reference. We give a brief sketch, for which we are grateful to Michael Mandell.

First of all, it clearly suffices to consider the case where all the D_i are cofibrant. This means in particular that they are Hausdorff. We form $\operatorname{hocolim} D$ by first writing down the simplicial replacement of the diagram, and then taking geometric realization. Taking H-fixed points obviously commutes with the simplicial replacement functor, so it suffices to worry about the geometric realization part. But one can check that if X_* is a simplicial space in which all X_n are Hausdorff, then

$|X_*|^H$ is homeomorphic to $|X_*^H|$. To do this, use the skeletal filtration on $|X_*|$ and the fact that $|\mathrm{Sk}_n X_*|$ is obtained from $|\mathrm{Sk}_{n-1} X_*|$ by pushing out along a closed inclusion (this is one of the places where the Hausdorff condition is needed). Check that taking fixed-points commutes with filtered colimits, and for Hausdorff spaces it also commutes with pushouts along closed inclusions.

Appendix A. Homotopy colimits for diagrams of non-cofibrant spaces

Let $\mathcal{T}op$ denote the category of all topological spaces, with its usual model category structure. Given a diagram $D\colon I \to \mathcal{T}op$, the usual instructions for computing the homotopy colimit of D are (1) to apply a cofibrant-replacement functor to every object in the diagram, and (2) to then use an explicit formula like that of Bousfield-Kan [BK, Sec. XII.2]. This is the situation in an arbitrary model category. In this section we show that for the special case of $\mathcal{T}op$, the first step of cofibrant-replacement is actually not needed. What we show is that no matter what formula one uses for computing homotopy colimits – whether it is the Bousfield-Kan formula or your favorite alternative – that formula always gives a homotopy invariant construction in $\mathcal{T}op$, even without the cofibrant-replacement step. This fact seems to be folklore in certain circles, although not well-known in others.

The most useful way to formulate this result seems to be in model category terms, as a comparison between the usual model structure on $\mathcal{T}op$ and the Strom model structure, where everything is cofibrant. See Theorem A.7.

To begin with, we need the following

Lemma A.1. *Let $A \to B$ and $X \to Y$ be weak equivalences. Given a diagram*

$$\begin{tikzcd}
A \times D^n \arrow[d] & A \times S^{n-1} \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
B \times D^n & B \times S^{n-1} \arrow[l] \arrow[r] & Y,
\end{tikzcd}$$

where the maps in the left-hand-square are the obvious ones, the induced map from the pushout of the top row to the pushout of the bottom row is also a weak equivalence.

Note that if A and B are cofibrant then this is an easy consequence of left-properness for $\mathcal{T}op$, but we claim the result in greater generality.

Proof. Let X_A and Y_B be the pushouts of the top and bottom rows respectively, and write $f\colon X_A \to Y_B$ for the map between them. We will produce a suitable cover of these spaces and use Proposition 2.2.

Let U_B be the pushout of

$$B \times (D^n - \{0\}) \longleftarrow B \times S^{n-1} \longrightarrow Y.$$

Write D_ϵ for $\{x \in D^n\colon |x| < \epsilon\}$ (where $0 < \epsilon < 1$), and let $V_B = B \times D_\epsilon$. The spaces U_B and V_B clearly form an open cover of Y_B, and notice that U_B deformation-retracts down to Y. The intersection $U_B \cap V_B$ is equal to $B \times (D_\epsilon - \{0\})$.

The same definitions give us a cover $\{U_A, V_A\}$ of X_A, and it is easy to check that $f^{-1}(U_B) = U_A$ and $f^{-1}(V_B) = V_A$. So the map $f^{-1}(V_B) \to V_B$ is the map $A \times D_\epsilon \to B \times D_\epsilon$, which is a weak equivalence. Similar reasoning shows that $f^{-1}(U_B \cap V_B) \to U_B \cap V_B$ is a weak equivalence. Finally, one argues that $f^{-1}(U_B) \to U_B$ is a weak equivalence because it deformation-retracts down to $X \to Y$. Proposition 2.2 now shows that $X_A \to Y_B$ is a weak equivalence. □

We'll say that an inclusion $Y \hookrightarrow Z$ is **relatively T_1** if given any open set U in Y and any point z of $Z \backslash U$, there is an open set W of Z such that $U \subseteq W$ and $z \notin W$ (compare the similar definition from [Ho, p. 50]). It follows that if E is any finite subset of $Z \backslash U$, one can find an open set $W \subseteq Z$ which contains U and doesn't intersect E. Note that a space X is T_1 precisely if all the inclusions $\{x\} \hookrightarrow X$ are relatively T_1.

Lemma A.2. *Given a pushout diagram of the form*

$$\begin{tikzcd}
A \times S^n \arrow[r] \arrow[d, hook] & Y \arrow[d, hook, dotted] \\
A \times D^{n+1} \arrow[r, dotted] & Z,
\end{tikzcd}$$

the inclusion $Y \hookrightarrow Z$ is relatively T_1.

Proof. Suppose given a point z in Z and an open U in Y. Either z is in Y or else it is represented by a pair (a, t) where t is in the interior of D^{n+1}. The argument works the same for the two cases, and so for convenience we'll assume the latter.

Pull back U to $A \times S^n$ and express it as a union of rectangles $V_i \times W_i$, where V_i is open in A and W_i is open in S^n. Each W_i can be fattened into an open subset W_i' of D^{n+1} with the properties that $W_i' \cap S^n = W_i$ and W_i' does not contain t.

Let M be the union of the $V_i \times W_i'$; it is an open subset of $A \times D^{n+1}$. Let N be the union of the images of M and U in Z. One checks that $N \cap Y = U$, and the pullback of N to $A \times D^{n+1}$ is M. So N is open in Z and N contains U, but N does not contain z. □

The following lemma is well-known for closed inclusions of T_1-spaces (see also [Ho, Prop. 2.4.2]). The usual proof still works in our case.

Lemma A.3. *Suppose that $Y_1 \hookrightarrow Y_2 \hookrightarrow \cdots$ is a sequence of relatively T_1 inclusions and that K is a compact space. Then any map $f\colon K \to \operatorname{colim} Y$ factors through some Y_k.*

Proof. Suppose the map does not factor through any Y_k. By taking a subsequence of Y if necessary, we can find a sequence of points $k_1, k_2, \ldots$ in K with the property that $f(k_i)$ lies in $Y_i \backslash Y_{i-1}$.

Pick an n and set $V_n = Y_n$. Next, choose an open set V_{n+1} in Y_{n+1} which contains V_n but doesn't contain $f(k_{n+1})$. Then pick an open set V_{n+2} in Y_{n+2} which contains V_{n+1} but neither $f(k_{n+1})$ nor $f(k_{n+2})$. Continuing this process gives an infinite sequence of opens, so their colimit W_n is an open subset of $\operatorname{colim} Y$.

As n varies, the open subspaces W_n form a cover of $\operatorname{colim} Y$. But $f(K)$ is a compact subspace of $\operatorname{colim} Y$, and it is not covered by any finite subcover. This is a contradiction. □

We now need some machinery related to simplicial spaces.

Definition A.4. *A simplicial space X_* is said to be* ***split****, or to have* ***free degeneracies****, if there exist subspaces $N_k \hookrightarrow X_k$ such that the canonical map*

$$\coprod_{\sigma} N_\sigma \to X_k$$

is an isomorphism. Here the variable σ ranges over all surjective maps in Δ of the form $[k] \to [n]$, N_σ denotes a copy of N_n, and the map $N_\sigma \to X_k$ is the one induced by $\sigma^*\colon X_n \to X_k$ (see [AM, Def. 8.1]).

The idea is that the spaces N_k represent the 'non-degenerate' part of X_k, sitting inside of X_k as a direct summand. It is an easy exercise to check that if X_* has free degeneracies and all the N_k are cofibrant spaces, then X_* is Reedy cofibrant in $s\mathcal{T}op$.

If X_* is any simplicial space, let $\operatorname{Sk}_n X_*$ be the simplicial space equaling X_* through dimension n and equaling the degenerate subspaces of X_* in larger dimensions. This is slightly different than the n-truncated simplicial space $\operatorname{sk}_n X_*$. There are maps $\operatorname{Sk}_0 X_* \to \operatorname{Sk}_1 X_* \to \cdots$ and the colimit is X_*. It follows that $|X_*|$ is equal to $\operatorname{colim}_n |\operatorname{Sk}_n X_*|$, using that geometric realization is a left adjoint (and this doesn't require any assumptions on X, only hinging upon the fact that the spaces Δ^n are locally compact Hausdorff). An important point is that when X_* has free degeneracies the space $|\operatorname{Sk}_n X_*|$ is obtained from $|\operatorname{Sk}_{n-1} X_*|$ via the pushout diagram

$$\begin{tikzcd}
N_n \times \partial\Delta^n \arrow[r] \arrow[d] & |\operatorname{Sk}_{n-1} X_*| \arrow[d, dotted] \\
N_n \times \Delta^n \arrow[r, dotted] & |\operatorname{Sk}_n X_*|.
\end{tikzcd} \tag{A.1}$$

Proposition A.5. *Let X_* be a simplicial space with free degeneracies. If K is a compact space then any map $K \to |X_*|$ factors through some $|\operatorname{Sk}_n X_*|$.*

Proof. This is a direct application of Lemmas A.2 and A.3, using the skeletal filtration of $|X_*|$ and the pushout square (A.1). □

The following corollary is the crucial ingredient for Theorem A.7. It is very similar to things in the literature, notably [M1, Th. 11.13] and [S2, Lem. A.5]. May's result assumes the spaces are compactly-generated and Hausdorff, and also that the realizations are simply-connected. Segal's result is more similar to ours, and the proofs follow the same pattern, but he works with homotopy equivalences rather than weak equivalences.

Corollary A.6. *If $X_* \to Y_*$ is a map of simplicial spaces with free degeneracies such that $X_n \to Y_n$ is a weak equivalence for each n, then $|X_*| \to |Y_*|$ is also a weak equivalence.*

Proof. For every k and every basepoint $*$ of X_0, there is an isomorphism

$$\operatorname*{colim}_n \pi_k(|\operatorname{Sk}_n X_*|, *) \to \pi_k(|X_*|, *)$$

(and the same statement holds with X_* replaced by Y_*). This follows from Proposition A.5, taking K to be a sphere. Therefore, it suffices to show that $|\operatorname{Sk}_n X_*| \to |\operatorname{Sk}_n Y_*|$ is a weak equivalence. Using induction, this follows from the pushout square (A.1) and Lemma A.1. □

Recall that the Strom model category is a model structure for topological spaces, denoted $\mathcal{T}op^S$, in which the weak equivalences are homotopy equivalences and the cofibrations (resp., fibrations) are the Hurewicz cofibrations (resp., fibrations). Note that all objects are cofibrant in this structure.

Theorem A.7. *Let $D\colon I \to \mathcal{T}op$ be a diagram of spaces. Then the homotopy colimits of D as computed in $\mathcal{T}op$ and $\mathcal{T}op^S$ have the same weak homotopy type.*

Proof. We don't know if the Strom model category is simplicial – the usual definitions for the simplicial action run into adjointness problems because $\mathcal{T}op$ is the category of *all* topological spaces. However, one may check directly that the homotopy colimit of a diagram in $\mathcal{T}op^S$ can still be computed by first taking the simplicial replacement of D and then applying the geometric realization functor, as usual. No cofibrant-replacement is necessary, since all objects are cofibrant.

In $\mathcal{T}op$ we first apply a cofibrant-replacement functor to all the objects in the diagram, and only then do we take simplicial replacement and realize. Simplicial replacements always have free degeneracies (see [D2, Proof of Lem. 2.7]), hence Corollary A.6 applies. □

Remark A.8. Theorem A.7 also holds if one uses the category of compactly-generated, weak Hausdorff spaces with its usual model structure. The same proofs work, with some extra caution that the various colimits are what they're supposed to be.

Finally, we give a short proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem A.7, we can compute the homotopy colimit in the Strom model category. In this model structure U_* is Reedy cofibrant because it has free degeneracies. So the realization already has the correct homotopy type. □

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