VANISHING LINES IN CHROMATIC HOMOTOPY THEORY

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Abstract. We show that at the prime 2, for any height \( h \) and \( G \subset \mathbb{G}_h \) a finite subgroup of the Morava stabilizer group, the \( RO(G) \)-graded homotopy fixed point spectral sequence for \( E_h \) has a strong horizontal vanishing line of filtration \( N_{h,G} \), a specific number depending on \( h \) and \( G \). It is a consequence of the nilpotence theorem that such homotopy fixed point spectral sequences all admit strong horizontal vanishing lines at some finite filtration. Here, we establish specific bounds for them. Our bounds are sharp for all the known computations of \( E_n^p \) when \( G \) is cyclic.

Our proof is by investigating the effect of the norm functor on the Hill–Hopkins–Ravenel differentials in the Tate spectral sequence. As a consequence, we will also show that the \( RO(G) \)-graded slice spectral sequence of \( (N_{G_2}^G \mathbb{G}_h)^{-1}BP^G \) has a horizontal vanishing line at the same filtration \( N_{h,G} \). As an immediate application, we establish a bound for the orientation order \( \Theta(h, G) \), the smallest number such that the \( \Theta(h, G) \)-fold direct sum of any real vector bundle is \( E_n^G \)-orientable.

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1. Introduction

1.1. Motivation and main theorem. Chromatic homotopy theory originated with Quillen’s remarkable observation of the relationship between the homotopy groups of the complex cobordism spectrum and the Lazard ring \( \mathbb{L} \) [Qui69]. Later on, the work of Miller–Ravenel–Wilson on periodic phenomena in the stable homotopy groups of spheres [MRW77] and Ravenel’s nilpotence conjectures led to what is now called the chromatic point of view. It is a powerful tool that studies periodic phenomena in the stable homotopy category by analyzing the algebraic geometry of smooth one-parameter formal groups. The moduli stack of formal groups has a stratification by height, which corresponds to localization with respect to the Lubin–Tate theories \( E_h \), \( h \geq 0 \). This stratification organizes the search for large scale phenomena in stable homotopy theory.
More precisely, the chromatic convergence theorem of Hopkins and Ravenel \cite{Rav92} exhibits the $p$-local sphere spectrum $S^0_{(p)}$ as the homotopy inverse limit of the chromatic tower

$$\cdots \rightarrow L_{E_0} S^0 \rightarrow \cdots \rightarrow L_{E_1} S^0 \rightarrow L_{E_0} S^0.$$ 

Each stage of the tower, $L_{E_0} S^0$, is the Bousfield localization of the sphere spectrum with respect to the height-$h$ Lubin–Tate theory $E_h$. These localizations can be inductively computed via the chromatic fracture square, which is the homotopy pullback square

$$\begin{array}{ccc}
L_{E_0} S^0 & \rightarrow & L_{K(h)} S^0 \\
\downarrow & & \downarrow \\
L_{E_{n-1}} S^0 & \rightarrow & L_{E_{n-1}} L_{K(h)} S^0.
\end{array}$$

Devinatz and Hopkins \cite{DH03} proved that $L_{K(h)} S^0 \simeq E_h^{hG_h}$, where $G_h$ is the Morava stabilizer group. Here, Goerss–Hopkins–Miller showed that the continuous action of $G_h$ on $\pi_* E_h$ can be refined to a unique $E_\infty$-action of $G_h$ on $E_h$ \cite{Rez98} \cite{GH04}. Furthermore, the $K(h)$-local $E_h$-based Adams spectral sequence of $L_{K(h)} S^0$ can be identified with the $G_h$-homotopy fixed point spectral sequence of $E_h$:

$$E_2^{s,t} = H^s(G_h, \pi_t E_h) \Longrightarrow \pi_{t-s} L_{K(h)} S^0.$$

Henn \cite{Hen07} proposed that the $K(h)$-local sphere $L_{K(h)} S^0$ can be built up from the spectra of the form $E_h^{hG}$, where $G$ is a finite subgroup of $G_h$. This has been explicitly realized at heights 1 and 2 \cite{Hen07} \cite{GHMR05} \cite{Bea15} \cite{BG18}.

From this point of view, the spectra $E_h^{hG}$ are the building blocks of the $p$-local stable homotopy category. In particular, their homotopy groups $\pi_\ast E_h^{hG}$ detects important families of elements in the stable homotopy groups of spheres \cite{HHR16} \cite{LSWX19} \cite{BMQ20}. The computation of these homotopy groups is a central topic in chromatic homotopy theory.

In this paper, we focus on the prime $p = 2$. Hewett classified all the finite subgroups of $G_h$ \cite{Hew95} (see also \cite{Buj12}). If $h = 2^m n$ where $n$ is odd, then when $m \neq 2$, the maximal finite 2-subgroups of $G_h$ are isomorphic to $C_2^m$, the cyclic group of order $2^m$; when $m = 2$, $h$ is of the form $4k - 2$, and the maximal finite 2-subgroups are isomorphic to $Q_8$, the quaternion group.

Due to the difficulty of describing the explicit action of the stabilizer group $G_h$ on $E_h$, computations at the prime 2 have been limited to heights 1 and 2 until the recent computational breakthroughs of Hill–Hopkins–Ravenel \cite{HHR16} (norms of Real bordism and the slice spectral sequence) and Hahn–Shi \cite{HS20} (Real orientation). This has led to the computation of $E_h^{hC_2}$ for all heights \cite{HS20}, and $E_4^{hC_2}$ \cite{HHR16} at height 4.

For $h > 0$ and $H$ a finite 2-subgroup of $G_h$, let $N_{h,H}$ be the positive integer defined as follows:

1. When $(h, H) = (2^{m-1} n, C_2^m)$, $N_{h,H} = 2^{h+m} - 2^m + 1$.
2. When $(h, H) = (4k - 2, Q_8)$, $N_{h,H} = 2^{h+3} - 7$.

Based on the classification of the finite 2-subgroups of $G_h$, this accounts for all possible pairs $(h, H)$.

The main result of this paper is the following:

**Theorem.** At any height $h$ and $G \subset G_h$ a finite subgroup, let $H$ be a Sylow 2-subgroup of $G$. There is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of $E_h$.

Recall that having a strong horizontal vanishing line of filtration $N_{h,H}$ means that the spectral sequence collapses after the $E_{N_{h,H}}$-page, and there are no elements of filtration greater than or equal to $N_{h,H}$ that survives to the $E_\infty$-page.
Our motivation for proving the main theorem is as follows: classically, it is a consequence of the nilpotence theorem that the homotopy fixed point spectral sequences of the Lubin–Tate theories all admit strong horizontal vanishing lines at some finite filtration (see [DH04, Section 5] and [BGH17 Section 2.3]). While theoretically useful, this fact does not help very much when doing concrete computations. Since we don’t know where exactly the vanishing line occurs, we can’t use its mere existence to prove any differentials.

The recent computations of Hill–Shi–Wang–Xu demonstrate that having a precise strong horizontal vanishing line is extremely useful for equivariant computations of Lubin–Tate theories. In [HSWX18], the authors first analyzed the slice spectral sequence of $BP^{(C_4)}\langle 1 \rangle$ (a connective model of $E_2$ equipped with a $C_4$-action). They proved that there is a horizontal vanishing line of filtration 16, and every class on or above this line must die on or before the $E_{13}$-page [HSWX18 Theorem 3.17]. Using this fact, they gave a much shorter proof of all the slice differentials in [HHR17].

For the next case, $BP^{(C_4)}\langle 2 \rangle$ (a connective model of $E_4$ equipped with a $C_4$-action), a similar situation occurred: there is a horizontal vanishing line of filtration 96, and every class on or above this line must vanish on or before the $E_{61}$-page. This theorem is called the Vanishing Theorem [HSWX18 Theorem 9.2], and it is the key to producing all of the higher slice differentials.

The strong vanishing lines proven in our main theorem will render future computations of Lubin–Tate theories and norms of Real bordism theories much more tractable.

1.2. Main results. We will now give a more detailed summary of our results and describe the contents of this paper.

In Section 2 we recall some basic facts of our spectral sequences of interest. More specifically, the classical Tate diagram induces a Tate diagram of spectral sequences (see Section 2 for the definitions of these spectral sequences)

\[
\begin{array}{ccccccc}
\text{HOSS}(X) & \longrightarrow & \text{SliceSS}(X) & \longrightarrow & \text{LSliceSS}(X) \\
\text{HOSS}(X) & \longrightarrow & \text{HFPSS}(X) & \longrightarrow & \text{TateSS}(X).
\end{array}
\]

The interaction between these spectral sequences will be crucial for proving our main theorem.

We will also recall the spectrum $BP^{(G)}$, its slice filtration in the case when $G$ is a cyclic group of order $2^m$, and some special classes on the $E_2$-page of its slice spectral sequence. We prove all the differentials in the $C_2$-slice spectral sequence of $i_{C_2}^* BP^{(G)}$ when $G = C_{2^m}$ and $Q_8$ (Theorem 2.3). While not stated elsewhere, this is a straightforward consequence of [HHR16 Theorem 9.9].

In Section 3 we compare the slice spectral sequence, the homotopy fixed point spectral sequence, and the Tate spectral sequence by analyzing the maps

\[
\text{SliceSS}(X) \longrightarrow \text{HFPSS}(X) \longrightarrow \text{TateSS}(X)
\]

extracted from the Tate diagram of spectral sequences above. Works of Ullman [Ull13] and Bökstedt–Madsen [BM94] show that as maps between integer-graded spectral sequences, both maps induce isomorphisms in a certain range. For our purposes, we extend their isomorphism regions to the $RO(G)$-graded pages.

**Theorem** (Definition 3.1 and Theorem 3.3). For $V \in RO(G)$, let

\[
\tau(V) := \min_{\{e\} \subseteq H \subseteq G} |H| \cdot \dim V^H.
\]
The map from the slice spectral sequence to the homotopy fixed point spectral sequence induces an isomorphism on the $E_2$-page in the region defined by the inequality
\[ \tau(V - s - 1) > |V|. \]
Furthermore, this map induces a one-to-one correspondence between the differentials in this isomorphism region.

The proof of Theorem 3.3 is by application of the main result in Hill–Yarñall [HY18] about the relationship between the slice connectivity of an equivariant spectrum and the connectivity of its geometric fixed points.

The classical analysis about the map from the homotopy fixed point spectral sequence to the Tate spectral sequence almost generalizes immediately to give an $RO(G)$-graded isomorphism region.

**Theorem** (Theorem 3.6). The map from the $RO(G)$-graded homotopy fixed point spectral sequence to the $RO(G)$-graded Tate spectral sequence induces an isomorphism on the $E_2$-page for classes in filtrations $s > 0$, and a surjection for classes in filtration $s = 0$. Furthermore, there is a one-to-one correspondence between differentials whose source is of nonnegative filtration.

In Section 4, we give a brief summary of the norm structure in equivariant spectral sequences. The presence of the norm structure in our spectral sequences allows us to deduce the fate of certain classes in the $G$-equivariant spectral sequence from information in the $H$-equivariant spectral sequence, where $H \subset G$ is a subgroup (Proposition 4.1).

In Section 5, we analyze the Tate spectral sequence of $E_h$.

**Theorem** (Theorem 5.1). For any height $h$ and $G \subset G_h$ a finite subgroup, let $H$ be a Sylow 2-subgroup of $G$. All the classes in the $RO(G)$-graded Tate spectral sequence of $E_h$ vanish after the $E_{N+H}^{h,H}$-page.

At any prime $p$, Mathew and Meier have shown that the map $E_h^{h,G} \to E_h$ is a faithful $G$-Galois extension whenever $G \subset G_h$ is a finite subgroup [MM15, Example 6.2]. This implies that the Tate spectrum $E_h^{h,G}$ is contractible [Rog08, Proposition 6.3.3]. As a consequence, all the classes in the Tate spectral sequence of $E_h$ must eventually vanish. Theorem 5.1 provides a concrete bound for the vanishing page number when $p = 2$.

The proof of Theorem 5.1 requires us to analyze the $G$-equivariant orientation from $BP^{h(G)}$ to $E_h$, as given by [HS20]. This orientation map factors through $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{h(G)}$:

\[ BP^{h(G)} \xrightarrow{} E_h \xrightarrow{} (N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{h(G)} \]

This induces a map of the corresponding Tate spectral sequences:

\[ G\text{-TateSS}((N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{h(G)}) \to G\text{-TateSS}(E_h). \]

Equipped with the results discussed in the previous sections, we first transport the differentials from the $C_2$-slice spectral sequence of $i_{C_2}^{*}(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{h(G)}$ to the $C_2$-Tate spectral sequence of $i_{C_2}^{*}(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{h(G)}$ by using the one-to-one correspondences proven in Section 3. Then, we use the norm structure to deduce that the unit class in the $RO(G)$-graded Tate spectral sequence of $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{h(G)}$ must die on or before the $E_{N+H}^{h,H}$-page. Naturality and multiplicative structure would then imply that the unit class in the $RO(G)$-graded Tate spectral sequence of $E_h$ must also die on or before the $E_{N+H}^{h,H}$-page, and hence all the other classes must vanish after this page as well.
Our proof of the Tate vanishing for $E_h$ works for any $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{[G]}$-module to give the following vanishing theorem.

**Theorem** (see Remark 5.4). Let $M$ be a $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{[G]}$-module, and $H$ a Sylow 2-subgroup of $G$. All the classes in the $RO(G)$-graded Tate spectral sequence of $M$ vanish after the $E_{N_{h,H}}$-page.

In Section 6 we turn our attention to the homotopy fixed point spectral sequence of $E_h$ and prove our main theorem.

**Theorem** (Theorem 6.1). At any height $h$ and $G \subset G_h$ a finite subgroup, let $H$ be a Sylow 2-subgroup of $G$. There is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of $E_h$.

The proof of Theorem 6.1 is by using the vanishing theorem (Theorem 5.1) in the Tate spectral sequence, combined with the comparison theorem (Theorem 3.6) between the homotopy fixed point spectral sequence and the Tate spectral sequence.

Similarly, the same strong horizontal vanishing line exists in the homotopy fixed point spectral sequence for $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{[G]}$.

**Theorem** (Corollary 6.3). For any $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{[G]}$-module $M$, there is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of $M$.

**Corollary** (Corollary 6.4). For all $K(h)$-local finite spectra $Z$, the homotopy fixed point spectral sequence

$$H^*(G, E_t Z) \Rightarrow \pi_{t-s}(BP^G \wedge Z)$$

has a strong horizontal vanishing line at $N_{h,H}$.

In all the current known computations, our strong horizontal vanishing lines are sharp in the case when $H = C_{2^n}$. More precisely, When $H = C_2$, the strong horizontal vanishing line in the homotopy fixed point spectral sequence of $E_{hC_2}$ is at filtration exactly $2^h + 1 - 1$. When $H = C_4$, the strong horizontal vanishing lines in the homotopy fixed point spectral sequences of $E_{hC_4}^2$ and $E_{hC_4}^3$ are of filtrations exactly 13 and 61.

**Conjecture.** The strong horizontal vanishing line in Theorem 6.1 is sharp when the Sylow 2-subgroup $H$ is cyclic.

On the other hand, Bauer’s computation of $tmf$ with the elliptic spectral sequence [Bau08] implies that in the $Q_8$-homotopy fixed point spectral sequence of $E_2$, there is a strong horizontal vanishing line of filtration 23. This is smaller than the bound given by our theorem, which is 25.

**Question.** When $h = 4k - 2$, what is the sharpest bound for the strong horizontal vanishing line in the $RO(Q_8)$-homotopy fixed point spectral sequence of $E_h$?

**Question.** What is the sharpest bound for the strong horizontal vanishing line for the homotopy fixed point spectral sequence of $E_{hQ_8}^G$?

In Section 7 we prove the existence of horizontal vanishing lines in the slice spectral sequence.

**Theorem** (Theorem 7.1). When $G = C_{2^n}$ or $Q_8$, the $RO(G)$-graded slice spectral sequence of any $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{[G]}$-module $M$ admits a horizontal vanishing line of filtration $N_{h,G}$.

In particular, Theorem 7.1 implies that there will be a horizontal vanishing line of filtration 121 in the $C_8$-slice spectral sequence of $\Omega\Sigma$, the detection spectrum of Hill–Hopkins–Ravenel that detects all the Kervaire invariant elements [HHR16].

It is interesting to note that when $G = Q_8$, even though there is no knowledge of the slice filtration of $BP^{[Q_8]}$, yet, Theorem 7.1 still applies to show that the slice spectral sequences of $(N_{C_2}^G \tilde{\psi}_h)^{-1}BP^{[Q_8]}$-modules all have strong horizontal vanishing lines of filtration $N_{h,Q_8}$. 
Question. What is the slice filtration of $BP(\mathbb{Q}_h)$?

Nonequivariantly, Hovey and Sadofsky showed that after $L_{K(n)}$-localization, $v_n^{-1}BP$ splits as a wedge of suspensions of $v_n^{-1}BP(n)$ [HS92, Theorem B]. Their argument generalizes to the $C_2$-equivariant setting to imply a splitting of $v_n^{-1}BP_k(n)$ as a wedge of $(\ast p_2)$-suspensions of $v_n^{-1}BP_k(\langle n \rangle)$ [LLQ19].

When $G = C_{2m}$, since the quotient $(N^G_C \langle \bar{h} \rangle)^{-1}BP(\mathbb{Q}_{2m})\langle n \rangle$ is a $(N^G_C \langle \bar{h} \rangle)^{-1}BP(\mathbb{Q}_{2m})\langle n \rangle$-module, Theorem 7.1 applies to show that there is a horizontal vanishing line of filtration $N_h,C_{2m}$ in its slice spectral sequence. The fact that $(N^G_C \langle \bar{h} \rangle)^{-1}BP(\mathbb{Q}_{2m})\langle n \rangle$ and $(N^G_C \langle \bar{h} \rangle)^{-1}BP(\mathbb{Q}_{2m})\langle n \rangle$ have the same horizontal vanishing line suggests that a similar splitting should occur $C_{2m}$-equivariantly.

Question. After $L_{K(h)}$-localization, does $(N^G_C \langle \bar{h} \rangle)^{-1}BP(\mathbb{Q}_{2m})\langle n \rangle$ split as a wedge of $RO(C_{2m})$-graded suspensions of $(N^G_C \langle \bar{h} \rangle)^{-1}BP(\mathbb{Q}_{2m})\langle n \rangle$?

Without inverting $N^G_C \bar{h}$, there is no horizontal vanishing line in the $RO(C_{2m})$-graded slice spectral sequence of the non-localized quotient $BP(\mathbb{Q}_{2m})\langle n \rangle$, as we have elements of arbitrarily high filtrations on the $E_{\infty}$-page. For example, the tower $a^{-1}_{k+1} \mathbb{Q}$ is an infinite tower containing classes of arbitrarily high filtrations that survive to the $E_{\infty}$-page. Interestingly, computations of $BP_h(n)$, $BP(\mathbb{Q}_2)(1)$, and $BP(\mathbb{Q}_2)(2)$ in [HK01, HHR17, HSX18] suggest that horizontal vanishing lines still exist in the integer-graded slice spectral sequence of the non-localized quotients.

Conjecture. There is a horizontal vanishing line of filtration $N_h,C_{2m}$ in the integer-graded slice spectral sequence of $BP(\mathbb{Q}_{2m})\langle n \rangle$.

Finally, in Section 8, we give an application of our main theorem to study $E^G_h$-orientations of real vector bundles. For $h \geq 0$ and $G \subseteq G_h$ a closed subgroup, let $\Theta(h,G)$ be the smallest number $d$ such that the $d$-fold direct sum of any real vector bundle is $E^G_h$-orientable. At the prime $p = 2$ and when $G = C_2$, Kitchloo and Wilson [KW15] have studied $E^{hG}_2$-orientations. At all primes and when $G = C_p$, Bhattacharya and Chatham [BC21] have studied $E^{hG}_p$-orientations.

We prove the following theorem, which uses the strong vanishing lines proven in Theorem 8.1 to give an upper bound for $\Theta(h,G)$ at the prime 2 for all heights $h \geq 0$ and $G \subseteq G_h$ a finite group.

Theorem. (Theorem 8.4) For any height $h \geq 0$ and $G \subseteq G_h$ a finite subgroup, let $d = 2 \cdot |G| \cdot |H|^\frac{1}{p-1}$, where $H$ is a 2-Sylow subgroup of $G$. Then the $d$-fold direct sum of any real vector bundle is $E^G_h$-orientable.

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2. Preliminaries

In this section, we will discuss the spectral sequences that are of interest to us. We will also collect certain facts about these spectral sequences that we will need in the later sections.

Let $X$ be a $G$-spectrum, and let $P^\bullet X$ be the slice tower of $X$. The Tate diagram

$$
\begin{align*}
E^G_+ \wedge X & \xrightarrow{\sim} X \xrightarrow{} \tilde{E}G \wedge X \\
E^G_+ \wedge F(EG_+, X) & \xrightarrow{} F(EG_+, X) \xrightarrow{} \tilde{E}G \wedge F(EG_+, X)
\end{align*}
$$
induces a diagram of towers:

\[
\begin{array}{c}
EG_+ \wedge P^\ast X \\
\downarrow \simeq \\
\downarrow \\
EG_+ \wedge F(EG_+, P^\ast X) \\
\downarrow \\
F(EG_+, P^\ast X) \\
\downarrow \\
\bar{E}G \wedge P^\ast X.
\end{array}
\]

This diagram of towers further induces a Tate diagram of spectral sequences

\[
\begin{array}{c}
\text{HOSS}(X) \\
\downarrow \\
\downarrow \\
\text{SliceSS}(X) \\
\downarrow \\
\downarrow \\
\text{LSliceSS}(X). \\
\end{array}
\]

(2.1)

\[
\begin{array}{c}
\text{HOSS}(X) \\
\downarrow \\
\downarrow \\
\text{HFPSS}(X) \\
\downarrow \\
\downarrow \\
\text{TateSS}(X). \\
\end{array}
\]

All the spectral sequences in (2.1) are RO(\mathcal{G})-graded spectral sequences. We pause to briefly discuss notations:

1. The spectral sequence associated with the tower \( EG_+ \wedge P^\ast X \) is the homotopy orbit spectral sequence (HOSS) of \( X \). It is a third and fourth quadrant spectral sequence, and it converges to \( _{\mathcal{G}}E_\ast G \wedge X \). In the integer-graded page at the \( G/G \)-level, the spectral sequence converges to \( \pi_\ast G \wedge X = \pi_\ast X_{hG} \).

2. The spectral sequence associated with the tower \( P^\ast X \) is the slice spectral sequence (SliceSS) of \( X \). It is a first and third quadrant spectral sequence, and it converges to \( \pi_\ast X \). In the integer-graded page at the \( G/G \)-level, the spectral sequence converges to \( \pi_\ast G X = \pi_\ast X^G \).

3. Following the treatment of a forthcoming paper by Meier–Shi–Zeng, the spectral sequence associated with the tower \( \bar{E}G \wedge P^\ast X \) is called the localized slice spectral sequence of \( X \) and is denoted by LSliceSS(X). It converges to \( \pi_\ast G \wedge X \).

4. The spectral sequence associated with the tower \( F(E\bar{G}+, P^\ast X) \) is the homotopy fixed point spectral sequence (HFPSS) of \( X \). It is a first and second quadrant spectral sequence, and it converges to \( \pi_\ast F(E\bar{G}+, X) \). In the integer-graded page at the \( G/G \)-level, the spectral sequence converges to \( \pi_\ast G F(E\bar{G}+, X) = \pi_\ast X^{hG} \).

5. The spectral sequence associated with the tower \( \bar{E}G \wedge F(EG_+, P^\ast X) \) is the Tate spectral sequence (TateSS) of \( X \). It has classes in all four quadrants, and it converges to \( \pi_\ast \bar{E}G \wedge F(EG_+, X) \). In the integer-graded page at the \( G/G \)-level, the spectral sequence converges to \( \pi_\ast G \bar{E}G \wedge F(E\bar{G}+, X) = \pi_\ast X_{hG} \).

Let \( \rho_2 \) denote the regular \( C_2 \)-representation. In \textbf{BHSZ21}, it is shown that there are generators

\[ \bar{t}_i \in \pi_{(2^i-1)\rho_2} C_2 \mathbb{Z} \]

such that

\[ \pi_{\rho_2}^{C_2} BP(\mathbb{Z}^{C_2}) \cong \mathbb{Z}_{(2)}[C_m \cdot \bar{t}_1, C_m \cdot \bar{t}_2, \ldots] \]

For a precise definitions of these generators, see formula (1.3) in \textbf{BHSZ21} (also see \textbf{HHR16} Section 5) for analogous generators in \( \pi_{\rho_2}^{C_2} MU(\mathbb{Z}^{C_2}) \). In particular, for \( BP_2 \), the generators \( \bar{t}_i \) can be taken to be the \( \bar{v}_i \)'s, which are the \( C_2 \)-equivariant lifts of the classical \( v_i \)-generators in \( \pi_{\rho_2} BP \).

Similar to the treatment of \( MU(\mathbb{Z}^{C_2}) \) in \textbf{HHR16}, we can build an equivariant refinement

\[ S\mathbb{Z}[C_m \cdot \bar{t}_1, C_m \cdot \bar{t}_2, \ldots] \longrightarrow BP(\mathbb{Z}^{C_2}) \]
from which we can apply the Slice Theorem [HHR16 Theorem 6.1] to show that the slice associated graded of $BP^{G_2}$ is the graded spectrum

$$H\bar{\mathbb{Z}}[C_{2m} \cdot \bar{t}_1, C_{2m} \cdot \bar{t}_2, \ldots].$$

Here, the degree of a summand corresponding to a monomial in the $\bar{t}_i$-generators and their conjugates is the underlying degree.

As a consequence, the slice spectral sequence for the $RO(C_{2m})$-graded homotopy groups of $BP^{G_2}$ has $E_2$-term the $RO(C_{2m})$-graded homotopy of $H\bar{\mathbb{Z}}[C_{2m} \cdot \bar{t}_1, C_{2m} \cdot \bar{t}_2, \ldots]$. To compute this, note that $S^0[C_{2m} \cdot \bar{t}_1, C_{2m} \cdot \bar{t}_2, \ldots]$ can be decomposed into a wedge sum of slice cells of the form

$$C_{2m} + \wedge H_p S^{m_0/2^p},$$

where $p$ ranges over a set of representatives for the orbits of monomials in the $\gamma^i\bar{t}_i$-generators, and $H_p \subset C_{2m}$ is the stabilizer of $p \pmod{2}$. Therefore, it suffices to compute the equivariant homology groups of the representations spheres $S^{m_0/2^p}$ with coefficients in the constant Mackey functor $\bar{\mathbb{Z}}$.

We recall some distinguished elements in the $RO(G)$-graded homotopy groups that we will need in order to name the relevant classes on the $E_2$-page of the slice spectral sequence (see [HHR16 Section 3.4] and [HSWX18 Section 2.2]).

**Definition 2.1.** Let $V$ be a $G$-representation. We will use $a_V : S^0 \to S^V$ to denote its Euler class. This is an element in $\pi_{C_V}^G S^0$. We will also denote its Hurewicz image in $\pi_{C_V}^G H\bar{\mathbb{Z}}$ by $a_V$.

If the representation $V$ has nontrivial fixed points (i.e. $V^G \neq \{0\}$), then $a_V = 0$. Moreover, for any two $G$-representations $V$ and $W$, we have the relation $a_{V \oplus W} = a_V a_W$ in $\pi_{C_V}^G H\bar{\mathbb{Z}}(S^0)$.

**Definition 2.2.** Let $V$ be an orientable $G$-representation. Then a choice of orientation for $V$ gives an isomorphism $H^*_V(S^V; \mathbb{Z}) \cong \mathbb{Z}$. In particular, the restriction map

$$H^*_V(S^V; \mathbb{Z}) \to H^*_V(S^{|V|}; \mathbb{Z})$$

is an isomorphism. Let $u_V \in H^*_V(S^V; \mathbb{Z})$ be the generator that maps to 1 under this restriction isomorphism. The class $u_V$ is called the orientation class of $V$.

The orientation class $u_V$ is stable in $V$. More precisely, if 1 is the trivial representation, then $u_{V \oplus 1} = u_V$. Moreover, if $V$ and $W$ are two orientable $G$-representations, then $V \oplus W$ is also orientable, and $u_{V \oplus W} = u_V u_W$.

The Euler class $a_V$ and the orientation class $u_V$ behave well with respect to the Hill–Hopkins–Ravenel norm functor. More precisely, for $H \subset G$ a subgroup and $V$ a $H$-representation, we have the equalities

\begin{align*}
(2.2) & \quad N^G_H(a_V) = a_{\text{Ind } V} \\
(2.3) & \quad u_{|V| \text{Ind } H} N^G_H(u_V) = u_{\text{Ind } V}
\end{align*}

where $\text{Ind } V = \text{Ind } H \bar{\mathbb{Z}}$ is the induced representation.

When $G = C_{2m}$, let $\lambda_i$, $1 \leq i \leq m$ denote the 2-dimensional real $C_{2m}$-representation corresponding to rotation by $(\frac{2\pi}{2^i})$. In particular, when $i = 1$, the representation $\lambda_1$ corresponds to rotation by $\pi$ and thus equals to $2\sigma$, where $\sigma$ is the real sign representation of $C_{2m}$. When localized at 2, the representations that will be relevant to us are $1, \sigma, \lambda_2, \lambda_3, \ldots, \lambda_m$.

When $G = Q_8$, $RO(Q_8) = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \bar{\mathbb{H}}\}$. The representations $\sigma_i$, $\sigma_j$, and $\sigma_k$ are one-dimensional representations whose kernels are $\langle i \rangle$, $\langle j \rangle$, and $\langle k \rangle$, respectively. The representation $\bar{\mathbb{H}}$ is a four-dimensional irreducible representation, obtained by the action of $Q_8$ on the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ by left multiplication.
For $h \geq 1$, let $\bar{v}_h \in \pi^{G}_{C_{2}^{h+1}-1}\mathcal{P}B^{(G)}$ denote the images of $v_h$-generators under the map

$$BP_R \to i_{C_{2}}^{*}BP^{(G)},$$

which is inclusion into the first factor. The following theorem describes all the differentials in the slice spectral sequence of $i_{C_{2}}^{*}BP^{(G)}$.

**Theorem 2.3.** Let $G = C_{2^{m}}$ or $Q_{8}$. In the $C_{2}$-slice spectral sequence of $i_{C_{2}}^{*}BP^{(G)}$, the differentials are generated under multiplicative structures by the differentials

$$d_{2h+1-1}(u_{2^{h+1}}^{a_{h}^{h+1}}) = \bar{v}_{h}a_{2^{h+1}}^{a_{h}^{h+1}}, \ h \geq 1.$$

**Proof.** When $G = C_{2}$, the claim is immediate from the Slice Differential Theorem of Hill–Hopkins–Ravenel [HHR16, Theorem 9.9]. When $G$ is larger than $C_{2}$, the unit map $BP_R \to i_{C_{2}}^{*}BP^{(G)}$ induces a map

$$(2.4) \quad \text{SliceSS}(BP_R) \to \text{SliceSS}(i_{C_{2}}^{*}BP^{(G)})$$

of slice spectral sequences. For degree reasons, the only possible differentials in SliceSS($i_{C_{2}}^{*}BP^{(G)}$) are of lengths $2^{h+1} - 1$, $h \geq 1$.

We will proceed by using induction on $h$. For the base case, when $h = 1$, we have the $d_{3}$-differential

$$d_{3}(u_{2^{h+1}}) = \bar{v}_{1}a_{2^{h+1}}^{3}$$

in SliceSS($BP_R$). Under the map (2.4), the source is mapped to $u_{2^{h+1}}$ and the target is mapped to $\bar{v}_{1}a_{2^{h+1}}^{3}$. By naturality, $\bar{v}_{1}a_{2^{h+1}}^{3}$ must be killed by a differential of length at most 3. Since the lowest possible differential length is 3 by degree reasons, the $d_{3}$-differential

$$d_{3}(u_{2^{h+1}}) = \bar{v}_{1}a_{2^{h+1}}^{3}$$

must occur in SliceSS($i_{C_{2}}^{*}BP^{(G)}$)). Multiplying this differential by permanent cycles determines the rest of the $d_{3}$-differentials. These are all the $d_{3}$-differentials because after these, there are no more room for other $d_{3}$-differentials.

Suppose now that the induction hypothesis holds for all $1 \leq k \leq h - 1$. For degree reasons, after the $d_{2h-1}$-differentials, the next possible differential is of length $2^{h+1}$. In SliceSS($BP_R$), consider the differential

$$d_{2h+1-1}(u_{2^{h+1}}^{a_{h}^{h+1}}) = \bar{v}_{h}a_{2^{h+1}}^{a_{h}^{h+1}}.$$

The map (2.4) sends both the source and the target of this differential to nonzero classes of the same name in SliceSS($i_{C_{2}}^{*}BP^{(G)}$). By naturality, the image of the target, $\bar{v}_{h}a_{2^{h+1}}^{a_{h}^{h+1}}$, must be killed by a differential of length at most $2^{h+1} - 1$. By degree reasons, it is impossible for this class to be killed by a differential of length smaller than $2^{h+1} - 1$. It follows that the differential

$$d_{2h+1-1}(u_{2^{h+1}}^{a_{h}^{h+1}}) = \bar{v}_{h}a_{2^{h+1}}^{a_{h}^{h+1}}$$

exists in SliceSS($i_{C_{2}}^{*}BP^{(G)}$). The rest of the $d_{2h+1-1}$-differentials are determined by multiplying this differential with permanent cycles. After these differentials, there is no room for other $d_{2h+1-1}$-differentials by degree reasons. This completes the induction step. \qed

3. **Comparison of spectral sequences**

In [Ull13] and [BM94], it is shown that the maps (1) and (2) in (2.4) induce isomorphisms in a certain range in the integer-graded page. For our purposes, we will extend their integral-graded isomorphism ranges to $RO(G)$-graded isomorphism ranges.

**Definition 3.1.** For $V \in RO(G)$, let

$$\tau(V) := \min_{\{c\}_G H \subset G} |H| \cdot \dim V^{H}.$$
Lemma 3.2. For $V \in RO(G)$, the spectrum $S^V \wedge \tilde{E}G$ is of slice $\geq \tau(V)$.

Proof. By [HY18] Theorem 2.5, $S^V \wedge \tilde{E}G$ is of slice $\geq n$ if and only if the geometric fixed points $\Phi^H(S^V \wedge \tilde{E}G) \in \tau_{\geq n/|H|}$ for all $H \subset G$. For $\tilde{E}G$, its underlying space is contractible and its $H$-fixed points is $S^0$ whenever $H$ is a nontrivial subgroup of $G$. Since $\Phi^H S^V = S^V \wedge \tilde{E}G$, $S^V \wedge \tilde{E}G$ is of slice $\geq \tau(V)$. \hfill \Box

Theorem 3.3. The map from the RO(G)-graded slice spectral sequence to the RO(G)-graded homotopy fixed point spectral sequence

$$
\begin{align*}
\pi_{V-s}^G p_{[V]}^X \to & \pi_{V-s}^G F(EG_+, p_{[V]}^X) \\
\pi_{V-s}^G & \to \pi_{V-s}^G F(EG_+, X)
\end{align*}
$$

induces an isomorphism on the $E_2$-page in the region defined by the inequality

$$\tau(V - s - 1) > |V|.$$ 

Furthermore, the map induces a one-to-one correspondence between the differentials in this isomorphism region.

Proof. Applying the functor $F(-, p_{[V]}^X)$ to the cofiber sequence

$$EG_+ \to S^0 \to \tilde{E}G$$

produces the cofiber sequence

$$F(\tilde{E}G, p_{[V]}^X) \to p_{[V]}^X \to F(EG_+, p_{[V]}^X).$$

The long exact sequence in homotopy groups imply that the map

$$\pi_{V-s}^G p_{[V]}^X \to \pi_{V-s}^G F(EG_+, p_{[V]}^X)$$

is an isomorphism when both $\pi_{V-s}^G F(\tilde{E}G, p_{[V]}^X)$ and $\pi_{V-s-1}^G F(\tilde{E}G, p_{[V]}^X)$ are trivial. Since $\pi_{V-s}^G F(\tilde{E}G, p_{[V]}^X) = F(S^* \wedge \tilde{E}G, P_{[V]}^X)$ and $p_{[V]}^X$ is a $|V|$-slice, it suffices to find pairs $(V, s)$ such that $S^{V-s-1} \wedge \tilde{E}G$ is of slice greater than $|V|$. By Lemma 3.2 this is equivalent to $(V, s)$ satisfying the inequality $\tau(V - s - 1) > |V|$.

We will now use induction on $r$ to show that the map of spectral sequences induces a one-to-one correspondence between all the $d_r$-differentials whose source and target are both in the isomorphism region. The base case of the induction, when $r = 1$, is trivial.

For the induction step, suppose that the map induces a one-to-one correspondence between all the $d_r$-differentials in the isomorphism region for all $r' < r$. Let $d_r(x) = y$ be a $d_r$-differential in SliceSS($X$) such that both $x$ and $y$ are in the isomorphism region. By naturality, $y'$ (the image of $y$) must be killed by a differential of length at most $r$ in HFPSS($X$). If the length of this differential is $r$, then the source $x'$ (the image of $x$) and we are done. If the length of this differential is smaller than $r$, then the induction hypothesis implies that the same differential must appear in SliceSS($X$). This would mean that $y$ is killed by a differential of length smaller than $r$, which is a contradiction. Therefore all the $d_r$-differentials in SliceSS($X$) that are in the isomorphism region appear in HFPSS($X$).

On the other hand, let $d_r(x') = y'$ be a $d_r$-differential in HFPSS($X$) such that both $x'$ and $y'$ are in the isomorphism region. Let $x$ be the pre-image of $x'$. By naturality, $x$ must support a differential of length at most $r$. If this differential is of length exactly $r$, then naturality implies that the target must be $y$, the unique preimage of $y'$. If the length is smaller than $r$, then by
the induction hypothesis, \( x' \) must support a differential of length smaller than \( r \) as well. This is a contradiction. Therefore all the \( d_r \)-differentials in \( \text{HFPSS}(X) \) that are in the isomorphism region appear in \( \text{SliceSS}(X) \). This completes the induction step. \( \square \)

**Remark 3.4.** In the integer-graded page, let \( V = t \in \mathbb{Z} \). Let \( m(G) \) be the order of the smallest nontrivial subgroup of \( G \). When \( t - s \geq 1 \), \( \tau(t + V' - s - 1) = m(G)(t - s - 1) \), and the isomorphism region in Theorem 3.3 is defined by the inequality

\[
m(G)(t - s - 1) > t.
\]

This recovers Theorem I.9.4 in [GOH].

**Example 3.5.** When \( G = C_{2^m} \), all the nontrivial \( C_{2^m} \)-representations are rotations, and they have no \( H \)-fixed points when \( H \subset C_{2^m} \) is a nontrivial subgroup. Therefore, if \( V' \in \text{RO}(G) \) is an element consisting only of nontrivial representations, then \( \tau(t + V' - s - 1) = 2(t - s - 1) \) when \( t - s > 1 \).

In the \((s + V')\)-graded page, the isomorphism region in Theorem 3.3 contains pairs \((t + V', s)\) that satisfies the inequality

\[
2(t - s - 1) > t + |V'|,
\]

or equivalently

\[
s < (t - s) - 2 - |V'|.
\]

In particular, the inequality shows that in any of the \( \text{RO}(C_{2^m}) \)-graded page, the isomorphism region is always bounded above by a line of slope 1.

**Theorem 3.6.** The map from the \( \text{RO}(G) \)-graded homotopy fixed point spectral sequence to the \( \text{RO}(G) \)-graded Tate spectral sequence induces an isomorphism on the \( E_2 \)-page for classes in filtrations \( s > 0 \), and a surjection for classes in filtration \( s = 0 \). Furthermore, there is a one-to-one correspondence between differentials whose source is of nonnegative filtration.

**Proof.** The \( E_2 \)-page of the Tate spectral sequence of \( X \) is

\[
E_2^{s,V} = \hat{H}^*(G, \pi_0(S^{-V} \wedge X)) \Longrightarrow \pi_0^G \hat{E}G \wedge F(EG_+, X),
\]

and the \( E_2 \)-page of the homotopy fixed point spectral sequence is

\[
E_2^{s,V} = H^*(G, \pi_0(S^{-V} \wedge X)) \Longrightarrow \pi_0^G F(EG_+, X).
\]

By the definition of Tate cohomology, \( \hat{H}^* = H^* \) when \( s > 0 \). Furthermore, the map \( H^0 \rightarrow \hat{H}^0 \) is a surjection whose kernel is the image of the norm map. This proves the claim about the \( E_2 \)-page. The proof for the one-to-one correspondence of differentials is exactly the same as the proof in Theorem 3.3. \( \square \)

We end this section by discussing the invertibility of certain Euler classes in the Tate spectral sequence. Recall that if \( V \) is a \( G \)-representation such that the fixed point set \( V^H \) is empty whenever \( H \subset G \) is nontrivial, then \( S(\infty V) \) is a geometric model for \( EG \), and \( S^\infty V \) is a geometric model for \( \hat{E}G \). Therefore, for any \( G \)-spectrum \( X \),

\[
\hat{E}G \wedge X \simeq S^\infty V \wedge X = a_0^{-1}X.
\]

Specialized to the case when \( G = C_{2^m} \) and \( Q_8 \), we see that \( \hat{E}C_{2^m} \simeq S^{\infty \lambda_m} \) and \( \hat{E}Q_8 \simeq S^{\infty \mathbb{Z}} \). Moreover, if \( X \) is a \( G \)-spectrum, then the Tate spectral sequence of \( X \) is the spectral sequence associated to the tower \( \hat{E}G \wedge F(EG_+, P^X X) \). This implies that the class \( a_{\lambda_m} \) is invertible in all the \( C_{2^m} \)-Tate spectral sequences, and the class \( a_{\mathbb{Z}} \) is invertible in all the \( Q_8 \)-Tate spectral sequences.
4. The norm structure

In this section, we give a brief summary of results for the norm structure in equivariant spectral sequences. For more detailed discussions, see [Ull13, Chapter I.5], [HHHR17, Section 4], and [MSZ20, Section 3.4].

Consider a tower
\[ \cdots \to P^{i+1} \to P^i \to P^{i-1} \to \cdots \]
of $G$-spectra and let $E^*_{i,j}$ be the associated spectral sequence. Set $P_n = \text{fib}(P^m \to P^{n-1})$ and $P_n = P^{n\infty}$. The towers that will be of relevant to us in this paper are the towers for the slice spectral sequence, the homotopy fixed point spectral sequence, and the Tate spectral sequence.

Let $H \subset G$ be a subgroup. Suppose we have maps $N^G_H P_n \to P_{|G/H|n}$ and $N^G_H P_n \to E^{G/H|n}_{d}$ that are (up to homotopy) compatible with the maps $P_n \to P_{n-1}$.

The following proposition (MSZ20, Proposition 3.7) is a restatement of [Ull13, Proposition 3.3].

Proposition 4.1. Let $x \in E_2(G/H)$ be an element representing zero in $E_{r+1}(G/H)$. Then $N^G_H(x)$ represents zero in $E_{|G/H|(r-1)+2}(G/H)$.

In other words, Proposition 4.1 states that if $x \in E^*_{i,j}(G/H)$ is killed by a $d_r$-differential, then $N^G_H(x) \in E^*_{i,j}(G/H|s|)$ must be killed by a differential of length at most $|G/H|(r-1) + 1$.

Let $\sigma_2$ be the sign representation of $C_2$. As an immediate consequence of Equations (2.2) and (2.3), we have the following proposition.

Proposition 4.2. The following equalities hold:

- $N^C_{C_2^m}(a) = a^{2^{m-2}}_m$
- $N^C_{C_2^m}(u) = \frac{u^{2^{m-1}}}{\prod_{i=2}^{2^m} u^{2^{m-1}i}}$
- $N^G_H(a) = a_H$
- $N^G_H(u) = \frac{u^{2^m}}{u^{2\sigma_2} u^{2\sigma_2} u^{2\sigma_2}}$

Proof. The equalities follow from (2.2), (2.3), and the following facts about induced representations:

- $\text{Ind}^C_{C_2^m}(1) = 1 + \sigma + \sum_{i=2}^{m-1} 2^{i-2}\lambda_i$
- $\text{Ind}^C_{C_2^m}(\sigma_2) = 2^{m-1}\lambda_m$
- $\text{Ind}^C_{C_2^m}(\sigma) = 1 + \sigma + \sigma_j + \sigma_k$
- $\text{Ind}^C_{C_2}(\sigma) = H$. 

\[ \square \]
Theorem 4.3.

1) The class \( N_{C_2}^{2m}(\bar{v}_h)a_{2m}^{2^{m-2}(2^{k+1}-1)} \) in the \( C_2 \)-slice spectral sequence of \( BP(\mathbb{G}_m) \) dies on or before the \( E_{2^{k+m+1}-2} \)-page.

2) The class \( N_{C_2}^{Q_8}(\bar{v}_h)a_{2^{k+1}-1}^{2^{k+1}-1} \) in the \( Q_8 \)-slice spectral sequence of \( BP(\mathbb{Q}_8) \) dies on or before the \( E_{2^{k+1}+1} \)-page.

Proof. By Theorem 2.3, we have the differential
\[
d_{2k+1,k}(u_{2\sigma_2}^{2k-1}) = \bar{v}_h\sigma_2^{2k+1-1}
\]
in the \( C_2 \)-slice spectral sequence of \( i_{C_2}^*BP(\mathbb{G}_m) \) and \( i_{C_2}^*BP(\mathbb{Q}_8) \). Our claims follow by applying Proposition 4.1 and the equations in Proposition 4.2 to \( (H, G, x, r) = (C_2, C_2, \bar{v}_h\sigma_2^{2k+1-1}, 2^{k+1} - 1) \) and \( (C_2, Q_8, \bar{v}_h\sigma_2^{2k+1-1}, 2^{k+1} - 1) \). □

5. Vanishing in the Tate spectral sequence

By the work of Hahn and Shi [HS20], the Lubin–Tate theory \( E_h \) admits an equivariant orientation. More specifically, for \( G \subset \mathbb{G}_h \) a finite subgroup, there is a \( G \)-equivariant map from \( BP(\mathbb{G}) \) to \( E_h \). Furthermore, this \( G \)-equivariant map factors through \( (N_{C_2}^G\bar{v}_h)^{-1}BP(\mathbb{G}) \):

\[
\begin{array}{ccc}
BP(\mathbb{G}) & \longrightarrow & E_h \\
\downarrow & & \\
(N_{C_2}^G\bar{v}_h)^{-1}BP(\mathbb{G}) & \longrightarrow & 
\end{array}
\]

This equivariant orientation induces the following diagram of spectral sequences:

\[
\begin{array}{ccc}
\text{SliceSS}(BP(\mathbb{G})) & \longrightarrow & \text{HFPSS}(BP(\mathbb{G})) \\
\downarrow & & \downarrow \\
\text{HFPSS}(E_h) & \longrightarrow & \text{TateSS}(BP(\mathbb{G})).
\end{array}
\]

For \( h > 0 \) and \( H \) a finite 2-subgroup of \( \mathbb{G}_h \), let \( N_{h,H} \) be a positive integer defined as follows:

1) When \( (h, H) = (2^{m-1}n, C_{2m}) \), \( N_{h,H} = 2^{m+1}n - 2^m + 1 \).

2) When \( (h, H) = (4k - 2, Q_8) \), \( N_{h,H} = 2^{k+3} - 7 \).

Based on the classification of 2-subgroups of \( \mathbb{G}_h \), this accounts for all possible pairs \( (h, H) \).

Theorem 5.1. For any height \( h \) and \( G \subset \mathbb{G}_h \) a finite subgroup, let \( H \) be a Sylow 2-subgroup of \( G \). All the classes in the RO(\( G \))-graded Tate spectral sequence of \( E_h \) vanish after the \( E_{N_{h,H}} \)-page.

In order to prove Theorem 5.1, we will first prove the following lemmas:

Lemma 5.2. Let \( G \) be a finite group and \( H \subset G \) a p-Sylow subgroup. For a p-local \( G \)-spectrum \( X \), if the RO(\( H \))-graded Tate spectral sequence of \( X \) vanish after the \( E_r \)-page, then the RO(\( G \))-graded homotopy fixed point spectral sequence of \( X \) will also vanish after the \( E_r \)-page.

Proof. The restriction and transfer maps induce the following maps of spectral sequences:
\[
G\text{-TateSS}(X) \xrightarrow{\text{res}} H\text{-TateSS}(X) \xrightarrow{\text{tr}} G\text{-TateSS}(X).
\]
The composition map \( \text{tr} \circ \text{res} \) is the degree-\( |G/H| \) map. Since \( |G/H| \) is coprime to \( p \) and \( X \) is \( p \)-local, the composition \( \text{tr} \circ \text{res} \) is an isomorphism. This exhibits the RO(\( G \))-graded Tate spectral sequence as a retract of the RO(\( H \))-graded Tate spectral sequence.

Consider a nonzero class \( x \) on the \( E_2 \)-page of the RO(\( G \))-graded Tate spectral sequence of \( X \). Its image, \( y = \text{res}(x) \), must be nonzero. This is because if \( y = 0 \), then \( 0 = \text{tr}(y) = \text{tr} \circ \text{res}(x) = x \).
which contradicts our assumption. Since $y$ vanishes after the $E_r$-page, it either supports a differential or gets killed by a differential of length at most $r$. If $y$ supports a $d_r$-differential, where $r' \leq r$, then by naturality of the restriction map, $x$ must also support a differential of length at most $r'$. On the other hand, if $y$ is killed by a $d_r$-differential, then by by naturality of the transfer map, $x$ must also be killed by a differential of length at most $r'$. It follows that $x$ must also vanish after the $E_r$-page.

\[ \square \]

**Lemma 5.3.**

1. At height $h = 2^{m-1}n$, the unit class in the $RO(C_{2^m})$-graded Tate spectral sequence of $(N_{C_{2^m}} C_{2^m} \bar{v}_h)^{-1} BP \langle (G') \rangle$ must be killed on or before the $(2^{h+m} - 2^m + 1)$-page.

2. At height $h = 4k - 2$, the unit class in the $RO(Q_8)$-graded Tate spectral sequence of $(N_{C_{2^8}} C_{2^8} \bar{v}_h)^{-1} BP \langle (G') \rangle$ must be killed on or before the $(2^{h+3} - 7)$-page.

**Proof.** For $G = C_{2^m}$ and $Q_8$, consider the map from the $C_2$-slice spectral sequence of $i_{C_2}^* BP \langle (G') \rangle$ to the $C_2$-Tate spectral sequence of $i_{C_2}^* BP \langle (G') \rangle$. Theorem 5.1 combined with the isomorphisms in Theorem 5.3 and Theorem 5.6 shows that we have the differential

$$d_{2h+1-1}(u_{2^{h-1}}) = \bar{v}_h a_{2^{h-1}}^{-1}$$

in the $C_2$-Tate spectral sequence of $i_{C_2}^* BP \langle (G') \rangle$. Since $a_{2^{h-1}}$ is invertible by our discussion in Section 3 after inverting $\bar{v}_h$, we have the differential

$$d_{2h+1-1}(u_{2^{h-1}}^{-1} a_{2^{h-1}}^{-1} a_{2^{h-1}}^{-1} = 1$$

in the $C_2$-Tate spectral sequence of $i_{C_2}^* (N_{C_{2^m}} C_{2^m} \bar{v}_h)^{-1} BP \langle (G') \rangle$. Our claims now follow by applying Proposition 4.1 to $(H, G, x, r) = (C_2, C_{2^m}, 1, 2^{h+1} - 1)$ and $(C_2, Q_8, 1, 2^{h+1} - 1)$. \[ \square \]

**Remark 5.4.** If $M$ is a $(N_{C_{2^m}} C_{2^m} \bar{v}_h)^{-1} BP \langle (G') \rangle$-module, its Tate spectral sequence will also be a module over the Tate spectral sequence of $(N_{C_{2^m}} C_{2^m} \bar{v}_h)^{-1} BP \langle (G') \rangle$. The same proof as the one used in Theorem 5.1 will apply to show the same vanishing results in the Tate spectral sequence of $M$.

6. **Horizontal vanishing lines in the homotopy fixed point spectral sequence**

The vanishing of the Tate spectral sequence (Theorem 6.1) leads to the existence of strong horizontal vanishing lines in the homotopy fixed point spectral sequences of Lubin–Tate theories.

**Theorem 6.1.** At any height $h$ and $G \subset G_h$ a finite subgroup, let $H$ be a Sylow 2-subgroup of $G$. There is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of $E_h$.

We first prove the following lemma, which reduces the general case to the case when $G$ is a finite 2-subgroup of $G_h$.

**Lemma 6.2.** Let $G$ be a finite group and $H \subset G$ a $p$-Sylow subgroup. For a $p$-local $G$-spectrum $X$, if the $RO(H)$-graded homotopy fixed point spectral sequence of $X$ has a vanishing line $L_H$, then the $RO(G)$-graded homotopy fixed point spectral sequence of $X$ will also have $L_H$ as a vanishing line.
Proof. The restriction and transfer maps induce the following maps of spectral sequences:

$$G\text{-HFPSS}(X) \xrightarrow{\text{res}} H\text{-HFPSS}(X) \xrightarrow{\text{tr}} G\text{-HFPSS}(X).$$

The composition map $\text{tr} \circ \text{res}$ is the degree-$|G/H|$ map. Since $|G/H|$ is coprime to $p$ and $X$ is $p$-local, the composition $\text{tr} \circ \text{res}$ is an isomorphism. The $RO(G)$-graded homotopy fixed point spectral sequence is a retract of the $RO(H)$-graded homotopy fixed point spectral sequence. It follows that the vanishing line in the middle spectral sequence, $H\text{-HFPSS}(X)$, will force the same vanishing line in $G\text{-HFPSS}(X)$. \hfill \square

Proof of Theorem 6.7. By Lemma 6.2 we just have to prove the claim for all $(h, H)$. Consider the map

$$H\text{-HFPSS}(E_h) \rightarrow H\text{-TateSS}(E_h).$$

By Theorem [3.6] this map induces an isomorphism of classes above filtration 0 and a one-to-one correspondence of differentials whose sources are in non-negative filtrations.

By Theorem 5.1 all the classes in the Tate spectral sequence vanish after the $N_{h,H}$-page. In particular, this implies that the longest differential is of length at most $N_{h,H}$, and any class of filtration at least $N_{h,H}$ must die from a differential whose source and target both have non-negative filtrations. Combined with the isomorphism in Theorem [3.6] this implies that the homotopy fixed point spectral sequence collapses after the $N_{h,H}$-page, and there is a strong horizontal vanishing line of filtration $N_{h,H}$. \hfill \square

Corollary 6.3. For any $(N_{C_2}^G, v_h)^{-1}BP^{(G)}$-module $M$, there is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of $M$.

Proof. By Remark 6.4 the proof is the same as the proof of Theorem 6.1. \hfill \square

Corollary 6.4. For all $K(h)$-local finite spectra $Z$, the homotopy fixed point spectral sequence

$$H^*(G, E_4Z) \Rightarrow \pi_{-s}(E_{h}^{hG} \wedge Z)$$

has a strong horizontal vanishing line of filtration $N_{h,H}$.

Remark 6.5. The existence of concrete strong horizontal vanishing lines (as given by Theorem 6.1) is very useful for equivariant computations. In [HSWX18], a crucial observation is that there is a horizontal vanishing line of filtration 16 in the slice spectral sequence of $BP^{(C_4)}\{1\}$ (hence in $\text{HFPSS}(E_2)$) and there is a horizontal vanishing line of filtration 96 in the slice spectral sequence of $BP^{(C_4)}\{2\}$ (hence in $\text{HFPSS}(E_4)$). The authors of that paper called this the Vanishing Theorem (see [HSWX18, Theorem 3.17, Theorem 9.2]). Using the Vanishing Theorem, Hill–Shi–Wang–Xu gave a much easier recomputation of the slice spectral sequence of $BP^{(C_4)}\{1\}$ (first computed in [HHR17]) and completely computed all the differentials in the slice spectral sequence of $BP^{(C_4)}\{2\}$.

In a forthcoming paper, we will apply Theorem 6.1 to give an alternative computation of $E_2^{K(2)}$ (the 2-primary $K(2)$-local TMF) by using the equivariant structures present in the homotopy fixed point spectral sequence. Furthermore, we will also use Theorem 6.1 to prove new $RO(G)$-graded periodicities in the homotopy fixed point spectral sequences of $E_h$ (for example, $E_4$ has a periodicity of $36 - 16\lambda - 4\sigma$).

Example 6.6. When $G = C_2$ and at all heights $h$, there is a $d_{2h+1,1}$-differential in the $C_2$-homotopy fixed point spectral sequence of $E_h$, and there is a nonzero class $v_2^2a_{2h+1}^{2h+1-2}$ in bidegree $(2h+1 - 2, 2h+1 - 2)$. Therefore the vanishing line in Theorem 6.1 is sharp for $E_{h}^{hC_2}$. 


Example 6.7. The computations in \cite{HHR17} implies that in the $RO(C_4)$-homotopy fixed point spectral sequence of $E_2$, there exists a $d_{13}$-differential
\[ d_{13}(N^2_2(\bar{t})^5 u_{4\lambda}u_{3\sigma}a_\sigma) = N^2_2(\bar{t})^8 u_{8\sigma}a_{6\lambda} \]
where we let $\lambda = \lambda_2$ and $N^2_2(-) = N^2_{C_2}(-)$ for convenience. Moreover, the class $N^2_2(\bar{t})^{10} u_{4\lambda}u_{10\sigma}a_{6\lambda}$ in bidegree $(28, 12)$ (representing $\kappa^2$) that survives to the $E_\infty$-page. Therefore, our vanishing line is sharp for $E_2^{hC_4}$.

Example 6.8. The computations in \cite{HSWX18} implies that in the $RO(C_4)$-homotopy fixed point spectral sequence of $E_4$, there is a $d_{61}$-differential
\[ d_{61}(N^2_2(\bar{t})^{11} u_{16\lambda}u_{32\sigma}a_{17\lambda}a_{\sigma}) = N^2_2(\bar{t})^{16} u_{48\sigma}a_{48\lambda} \]
Moreover, the class $N^2_2(\bar{t})^{24} N^2_2(\bar{t})^{4} u_{44\lambda}u_{74\sigma}a_{303}$ in bidegree $(236, 60)$ survives to the $E_\infty$-page. Therefore, our vanishing line is sharp for $E_4^{hC_4}$.

**Conjecture 6.9.** The vanishing line given by Theorem 6.1 is sharp when the Sylow 2-subgroup $H$ is cyclic.

Example 6.10. Consider the $RO(Q_8)$-homotopy fixed point spectral sequence of $E_2$. Theorem 6.1 implies that there is a strong horizontal vanishing line of filtration 25. However, the actual vanishing line is of filtration 23. More specifically, by Bauer’s computation \cite{Bau08}, there is a $d_{23}$-differential
\[ d_{23}(\eta \Delta^5) = \Delta^6 k^6 \]
(the notations are adopted from \cite{Ben17} where $\Delta$ and $k$ are generators on $E_2$-page). This implies that in the Tate spectral sequence, there is a $d_{23}$-differential
\[ d_{23}(\eta \Delta^{-1} k^{-6}) = 1. \]
By the same argument as the one given in the proof of Theorem 6.1 and Bauer’s computation, the sharpest vanishing line in the homotopy fixed point spectral sequence is of filtration 23.

**Question 6.11.** When $h = 4k - 2$, what is the sharpest bound for the strong horizontal vanishing line in the $RO(Q_8)$-homotopy fixed point spectral sequence of $E_h$?

### 7. Horizontal vanishing lines in the slice spectral sequence

We will now prove explicit horizontal vanishing lines for the slice spectral sequences of $(N^G_{C_2} \bar{v}_h)^{-1}BP^{(G)}$-modules. Recall from Section 5 that when $(h, G) = (2^{m-1} n, C_{2m})$, $N_h, C_{2m} = 2^{h+m} - 2^m + 1$, and when $(h, G) = (4k - 2, Q_8)$, $N_h, Q_8 = 2^{h+3} - 7$.

**Theorem 7.1.** When $G = C_{2m}$ or $Q_8$, the $RO(G)$-graded slice spectral sequence of any $(N^G_{C_2} \bar{v}_h)^{-1}BP^{(G)}$-module $M$ admits a horizontal vanishing line of filtration $N_h, G$.

**Lemma 7.2.** When $G = C_{2m}$ or $Q_8$, any $(N^G_{C_2} \bar{v}_h)^{-1}BP^{(G)}$-module is cofree.

**Proof.** By \cite{HHR16} Corollary 10.6, we need to show that $\Phi^H(N^G_{C_2} \bar{v}_h)^{-1}BP^{(G)}$ is contractible for all non-trivial $H \subset G$. To do so, it suffices to check that $\Phi^H(N^G_{C_2} \bar{v}_h) = 0$ for all nontrivial $H \subset G$. Recall that $\bar{v}_h \in \pi^{C_2}_{(2^h - 1)_{p_2}} BP^{(G)}$ is defined to be the composition
\[ S^{(2^h - 1)_{p_2}} \xrightarrow{\bar{v}_h} BP \xrightarrow{\bar{C}_2} BP^{(G)}. \]
The claim now follows from the fact that for the class $\bar{v}_h \in \pi^{C_2}_{(2^h - 1)_{p_2}} BP$, $\Phi^{C_2}(\bar{v}_h) = 0$ and therefore $\Phi^H(N^H_{C_2} \bar{v}_h) = \Phi^{C_2}(\bar{v}_h) = 0$ for all nontrivial $H \subset G$. \(\square\)
Proof of Theorem 7.1. Since the spectrum $M$ is cofree by Lemma [7.2], both the slice spectral sequence and the homotopy fixed point spectral sequence converge to the same homotopy groups:

$$\text{SliceSS}(M) \longrightarrow \text{HFPSS}(M)$$

$$\pi_*^G M \longrightarrow \pi_*^G F(EG_+, M).$$

Consider a class $x$ on the $E_2$-page of the slice spectral sequence. We claim that if the filtration of $x$ is at least $N_{h,G}$, then $x$ cannot survive to the $E_{\infty}$-page. This is because if $x$ survives to represent an element in $\pi_*^G M$, then there must be a class $y$ on the $E_2$-page of the homotopy fixed point spectral sequence that also survives to represent the same element in

$$\pi_*^G F(EG_+, M) = \pi_*^G M.$$

Moreover, the filtration of $y$ must be at least the filtration of $x$, which is $\geq N_{h,G}$. This is a contradiction because by Corollary 6.3 there is a strong horizontal vanishing line of filtration $N_{h,G}$ in the homotopy fixed point spectral sequence. \qed

8. $E_{hG}^A$-orientation of real vector bundles

For $h \geq 0$ and $G \subseteq \mathbb{G}_h$ a closed subgroup, let $\Theta(h, G)$ be the smallest number $d$ such that the $d$-fold direct sum of any real vector bundle is $E_{hG}^A$-orientable (defined in Definition 8.1). At the prime $p = 2$ and when $G = C_2$, Kitchloo and Wilson [KW15] have studied $E_{hG}^{BC_2}$-orientations. At all primes and when $G = C_p$, Bhattacharya and Chatham [BC21] have studied $E_{k(p-1)}^{hG_p}$-orientations.

In this section, we will use the strong vanishing lines proven in Theorem 6.1 to give an upper bound for $\Theta(h, G)$ at the prime 2 for all heights $h \geq 0$ and $G \subseteq \mathbb{G}_h$ a finite group.

Definition 8.1. Let $E$ be a multiplicative cohomology theory with multiplication $\mu_E : E \wedge E \to E$, and $\xi$ a virtual $k$-dimensional real vector bundle over a space $X$. Denote the Thom spectrum of $\xi$ by $M\xi$. An $E$-orientation for $\xi$ is a Thom class $u : M\xi \to \Sigma^k E$ such that for any map $f : Y \to X$, the pull-back $u_{f^*} : Mf^*(\xi) \to \Sigma^k E$ induces an equivalence

$$F(\Sigma^k Y_+, E) \xrightarrow{\sim} F(Mf^*(\xi), E),$$

where (8.1) is defined by sending a map $g : \Sigma^k Y_+ \to E$ to the composition

$$Mf^*(\xi) = S^0 \wedge Mf^*(\xi) \xrightarrow{\iota \wedge \text{id}} E \wedge Mf^*(\xi) \xrightarrow{\text{id} \wedge \Delta} E \wedge Mf^*(\xi) \wedge Y_+ \xrightarrow{\text{id} \wedge u_{f^*} \wedge \text{id}} E \wedge \Sigma^k E \wedge Y_+ \xrightarrow{\mu_E \wedge \text{id}} E \wedge \Sigma^k Y_+ \xrightarrow{\mu_E \wedge g} E \wedge E \xrightarrow{\mu_E} E.$$

Here, $\Delta : Mf^*(\xi) \to Mf^*(\xi) \wedge Y_+$ is the Thom diagonal map.

Remark 8.2. If $\xi$ is $E$-oriented, then the equivalence (8.1) induces a Thom isomorphism

$$E^{*-k}(Y_+) \xrightarrow{\sim} E^*(Mf^*(\xi))$$

for any map $f$. In particular, when $f$ is the identity map, there is a Thom isomorphism

$$E^{*-k}(X_+) \xrightarrow{\sim} E^*(M\xi).$$

Note that it follows immediately from Definition 8.1 that for any $E$-oriented bundle $\xi$, its pull back bundle $f^*(\xi)$ is also $E$-oriented. Our definition also recovers the classical definition of
orientations. More precisely, if we take $Y$ to be a point, then the Thom space of the pull back is $S^k$, and the restriction of the Thom class $u$ under the map

$$E^k(Th(\xi)) \rightarrow E^k(S^k)$$

is a unit.

For $X$ a non-equivariant spectrum, we can treat it as a $G$-spectrum equipped with the trivial $G$-action. We have the equivalence

$$F(E_{G+}, F(X, E_h))^G \simeq F(X, F(EG_{G+}, E_h))^G \simeq F(X, F(EG_{G+}, E_h))^G = F(X, E^h_{G+}).$$

This equivalence allows us to use the homotopy fixed point spectral sequence to compute $(E^h_{G+})^*(X)$. The $E_2$-page of this homotopy fixed point spectral sequence is

$$E_2^{s,t} = H^s(G; E^t_hX) \Rightarrow (E^h_{G+})^{s+t}(X).$$

Let $\gamma$ be the universal bundle on $BO$ (it is of virtual dimension zero). Following Kitchloo–Wilson [KWi15], we will denote the Thom spectrum of $E^h_{G+}$, we have the equivalence

$$MO(\delta) \rightarrow MO[2n].$$

Let $\Pi MO := \bigvee_{k \geq 0} MO[2k]$. Lemma 8.3. The homotopy fixed point spectral sequence for $(E^h_{G+})^*(\Pi MO)$ is a multiplicative spectral sequence whose multiplication on the $E_2$-page is commutative.

**Proof.** By [HR20], in order to ensure that the homotopy fixed point spectral sequence has a multiplicative structure, it suffices to construct a $G$-equivariant map

$$F(\Pi MO, E_h) \wedge F(\Pi MO, E_h) \rightarrow F(\Pi MO, E_h).$$

Consider the following map

$$BO \xrightarrow{\Delta} BO \times BO \xrightarrow{[2i, 2j]} BO.$$ 

It induces a map of the corresponding Thom spectra

$$MO[2i + 2j] \rightarrow MO[2i] \wedge MO[2j].$$

If we fix $n$ and combine all such maps for the pairs $(i, j)$ such that $2i + 2j = 2n$, we get a map

$$MO[2n] \rightarrow \bigvee_{2i + 2j = 2n} MO[2i] \wedge MO[2j].$$

Taking the wedge sum of all such maps for all $n \geq 0$ gives the map

$$\Delta: \Pi MO = \bigvee_{n \geq 0} MO[2n] \rightarrow \bigvee_{n \geq 0} \left( \bigvee_{2i + 2j = 2n} MO[2i] \wedge MO[2j] \right) = \Pi MO \wedge \Pi MO.$$ 

We can then define the $G$-equivariant multiplication on $F(\Pi MO, E_h)$ as the following composition:

$$F(\Pi MO, E_h) \wedge F(\Pi MO, E_h) \rightarrow F(\Pi MO \wedge \Pi MO, E_h \wedge E_h) \xrightarrow{(\mu E_h)_{s,t}} F(\Pi MO \wedge \Pi MO, E_h)$$

$$\xrightarrow{\Delta_{s,t}} F(\Pi MO, E_h)$$

To show that the multiplication on the $E_2$-page is commutative, note that since $k\gamma \otimes C = 2k\gamma$ as real vector bundles, $2k\gamma$ is $E_h$-oriented. For each $k \geq 0$, we have a Thom isomorphism

$$E^*_{h}(MO[2k]) \cong E^*_{h}(BO_{+}) \cdot u_{2k}.$$ 

We will show that the $E_2$-page is equal to $E^*_{h}(BO_{+})[u_2]$, equipped with the obvious multiplicative structure (which is commutative).
Consider the composition map
\[ MO[2k] \xrightarrow{k} MO[2] \wedge \cdots \wedge MO[2] \xrightarrow{u_2 \wedge \cdots \wedge u_2} E_h \wedge \cdots \wedge E_h \xrightarrow{\mu} E_h. \]
This composition is \( u_2^k \) by the definition of our multiplicative structure on \( E_h^* (\text{II}MO) \). We claim that \( u_2^k \) is a Thom class for \( MO[2k] \).

Note that by iteratively applying adjunction and the Thom isomorphism, we have
\[ F(MO[2] \wedge \cdots \wedge MO[2], E_h) = F(MO[2] \wedge \cdots \wedge MO[2], F(BO_+, E_h)) = F(MO[2] \wedge \cdots \wedge MO[2] \wedge BO_+, E_h) = \cdots = F(BO_+ \wedge \cdots \wedge BO_+, E_h), \]
and this is given by the Thom class \( u_2 \wedge \cdots \wedge u_2 \). Pulling back this Thom class via the diagonal map \( BO_+ \to BO_+ \wedge \cdots \wedge BO_+ \) gives \( u_2^k \), and it induces the Thom isomorphism
\[ E_h^*(MO[2k]) \cong E_h^*(BO_+) \cdot u_2^k. \]
It follows that we can express the induced multiplication on \( E_2^*(G, E_h^*(\text{II}MO)) \) by using the multiplication on \( \bigoplus_{k \geq 0} E^*(BO_+) \cdot u_2^k \) via the Thom isomorphisms. \( \square \)

**Theorem 8.4.** For any height \( h \geq 0 \) and \( G \subseteq \mathbb{G}_h \) a finite subgroup, let \( d = 2 \cdot |G| \cdot |H|^{\frac{h}{|\text{Syl}_h G| - 1}} \), where \( H \) is a 2-Sylow subgroup of \( G \). Then the \( d \)-fold direct sum of any real vector bundle is \( E_h^{G\text{-orientable}} \).

**Proof.** It suffices to show that for the universal bundle \( \gamma \) on \( BO_+ \), its \( d \)-fold direct sum \( d\gamma \) is \( E_h^{G\text{-orientable}} \). Let \( u_2 : MO[2] \to E_h \) be a Thom class for \( 2\gamma \), as considered in Lemma \( \text{S3} \). For any element \( g \in G \), let \( gu_2 : MO[2] \to E_h \) denote the composition
\[ MO[2] \xrightarrow{u_2} E_h \xrightarrow{g} E_h. \]
Define the map \( u_G : MO[2 \cdot |G|] \to E_h \) to be the composition
\[ u_G : MO[2 \cdot |G|] \xrightarrow{[G]} MO[2] \wedge \cdots \wedge MO[2] \xrightarrow{g_1 u_2 \wedge \cdots \wedge g_{|G|} u_2} E_h \wedge \cdots \wedge E_h \xrightarrow{\mu} E_h, \]
where \( g_1, g_2, \ldots, g_{|G|} \) are all the elements of the group \( G \). It is clear from the definition that \( u_G \) is an element in \( H^0(G, E_h^*(MO[2 \cdot |G|])) \). By a similar argument as the one used in the proof of Lemma \( \text{S3} \), the class \( u_G^k \in \tilde{H}^0(G, E_h^*(MO[2 \cdot |G| \cdot k])) \) is a Thom class for \( MO[2 \cdot |G| \cdot k] \) in the sense that there is a Thom isomorphism
\[ E_h^*(BO_+) \xrightarrow{u_G^k} E_h^*(MO[2 \cdot |G| \cdot k]). \]
If the class \( u_G^k \) is a permanent cycle in the homotopy fixed point spectral sequence for \((E_h^{G\text{-orientable}})^*(MO[2 \cdot |G| \cdot k])\), then by naturality, the map of spectral sequences
\[ H^*(G, E_h^*(BO_+)) \xrightarrow{u_G^k} H^*(G, E_h^*(MO[2 \cdot |G| \cdot k])) \]
induces the Thom isomorphism
\[ (E_h^{G\text{-orientable}})^*(BO_+) \xrightarrow{u_G^k} (E_h^{G\text{-orientable}})^*(MO[2 \cdot |G| \cdot k]) \]
induces an isomorphism
\[(E^G_h)^*(BO_+ \cdot u_G^k) \cong (E^G_h)^*(MO[2 \cdot |G| \cdot k])\]
on the $E_\infty$-page. Moreover, for any map $f : Y \to BO$, the pull back of the class $u_G^k \cdot f^*(u_G^k) \in H^0(G, E^G_h(M f^*(2 \cdot |G| \cdot k\gamma)))$, is also going to be a permanent cycle by naturality:
\[H^*(G, E^G_h(MO[2 \cdot |G| \cdot k])) \to H^*(G, E^G_h(M f^*(2 \cdot |G| \cdot k\gamma)))\]
\[
\begin{array}{c}
H^*(G, E^G_h(MO[2 \cdot |G| \cdot k])) \\
\downarrow
\end{array}
\begin{array}{c}
\rightarrow \ \\
E^G_h(M f^*(2 \cdot |G| \cdot k\gamma))
\end{array}
\]

Therefore, it will induce a Thom isomorphism on the $E_\infty$-page of the homotopy fixed point spectral sequence for $(E^G_h)^*(M f^*(2 \cdot |G| \cdot k\gamma))$ as well.

It remains to find a $k$ such that $u_G^k$ is a permanent cycle. Since there is a splitting map
\[E^G_h(MO[2 \cdot |G| \cdot k]) \to E^G_h(ILMO) \to E^G_h(MO[2 \cdot |G| \cdot k]),\]

the homotopy fixed point spectral sequence for $(E^G_h)^*(MO[2 \cdot |G| \cdot k])$ is a retract of the homotopy fixed point spectral sequence for $(E^G_h)^*(ILMO)$. In particular, the class $u_G^k$ is a permanent cycle in the homotopy fixed point spectral sequence for $(E^G_h)^*(MO[2 \cdot |G| \cdot k])$ if and only if it is a permanent cycle in the homotopy fixed point spectral sequence for $(E^G_h)^*(ILMO)$.

By Lemma 8.3, the homotopy fixed point spectral sequence for $(E^G_h)^*(ILMO)$ has a multiplicative structure that is commutative on the $E_2$-page. Furthermore, all the classes on the $E_2$-page are $H$-torsion, and there can only be differentials of odd length by degree reasons. Since this spectral sequence is a module over the spectral sequence for $(E^G_h)^*(S^0)$, it has a strong horizontal vanishing line of filtration $N_{h,H}$. It follows that when $k = |H|^{N_{h,H}}$, the class $u_G^k$ must be a permanent cycle. This finishes the proof of the theorem. □

**Remark 8.5.** Theorem 8.3 implies that $\Theta(h, G) \leq 2 \cdot |G| \cdot |H|^{|N_{h,H} - 1|/2}$. We would like to remark that our bound is by no means optimal, as we haven’t done any explicit computations with the homotopy fixed point spectral sequence. In [KW15] Theorem 1.4, Kitchloo and Wilson proved via explicit computation of $(E^G_{hC^2})^*(BO(q))$ that the $2^{b+1}$-fold direct sum of any real vector bundle is $E^G_{hC^2}$-orientable. Our bound in this case is $\Theta(h, C^2) \leq 2^{2^{b+1}}$.

The only facts we used to obtain our bounds are that there is a strong horizontal vanishing line of filtration $N_{h,H}$, and that the $E_2$-page is $H$-torsion. With more computational knowledge of the homotopy fixed point spectral sequence of $(E^G_h)^*(ILMO)$, there will be more data to obtain a much better upper bound for $\Theta(h, G)$.

**References**


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