COMPUTATIONS OF HEIGHT 2 HIGHER REAL \( K \)-THEORY SPECTRA AT PRIME 2

ZHIPENG DUAN, HANA JIA KONG, GUCHUAN LI, YUNZE LU, AND GUOZHEN WANG

Abstract. We completely compute the \( G \)-homotopy fixed point spectral sequences at prime 2 for the height 2 Lubin–Tate theory \( E_2 \), in the case of finite subgroups \( G \) of the Morava stabilizer group for \( G = Q_8, SD_{16}, G_{24}, \) and \( G_{48} \). Our computation uses recently developed equivariant techniques since Hill–Hopkins–Ravenel. We also compute the \((\ast - \sigma_i)\)-graded \( Q_8 \)- and \( SD_{16} \)-homotopy fixed point spectral sequences where \( \sigma_i \) is a non-trivial one dimensional \( Q_8 \)-representation.

Contents

1. Introduction and main results 1
2. Preliminaries 4
3. \( E_2 \)-page of the \( Q_8 \)-HFPSS(\( E_2 \)) 12
4. Computation of the integer-graded \( Q_8 \)-HFPSS(\( E_2 \)) 16
5. The \((\ast - \sigma_i)\)-graded computation 27
6. Charts and Tables 35
Appendix A. Group Cohomology 43
References 46

1. Introduction and main results

1.1. Motivation and main results. Chromatic homotopy theory studies large scale phenomena in the stable homotopy category using the algebraic geometry of smooth 1-parameter formal groups [Qui69, Mor85]. The moduli stack of formal groups has a stratification by heights, which in the stable homotopy category corresponds to localizations with respect to the Lubin–Tate theories \( E_n \) for heights \( n \geq 0 \).

We fix a prime \( p \). Let \( \Gamma_n \) be the \( p \)-typical height \( n \) Honda formal group law over \( \mathbb{F}_p \), and let \( S_n \) be the automorphism group of \( \Gamma_n \) (extended to \( \mathbb{F}_{p^n} \)). Let \( G_n = S_n \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \) be the (extended) Morava stabilizer group. Goerss–Hopkins–Miller showed that the continuous action of \( G_n \) on \( \pi_* E_n \) can be refined to a unique \( E_{\infty} \)-action of \( G_n \) on \( E_n \) [Rez98, GH04, Lur18].

At a prime \( p \), theoretically one can assemble the information of \( E_n \) with the \( G_n \) action for all heights \( n \geq 0 \) to recover the \( p \)-local sphere. More precisely, the chromatic convergence theorem due to Hopkins and Ravenel [Rav92] exhibits the \( p \)-local sphere spectrum \( S^0_{(p)} \) as the homotopy inverse limit of the \( E_n \)-local spheres (in the sense of Bousfield [Bou79])

\[
\cdots \rightarrow L_{E_n} S^0 \rightarrow \cdots \rightarrow L_{E_1} S^0 \rightarrow L_{E_0} S^0.
\]
Furthermore, these localizations can be built inductively via the following homotopy pullback square (the chromatic fracture square)

\[
\begin{array}{ccc}
L_{E_n}S^0 & \longrightarrow & L_{K(n)}S^0 \\
\downarrow & & \downarrow \\
L_{E_{n-1}}S^0 & \longrightarrow & L_{E_{n-1}K(n)}S^0
\end{array}
\]

where \(L_{K(n)}\) denotes the localization functor with respect to \(K(n)\), the \(n\)th Morava K-theory. From this point of view, the \(K(n)\)-local sphere \(L_{K(n)}S^0\) is the building block of the \(p\)-local stable homotopy category. Devinnatz and Hopkins showed that \(L_{K(n)}S^0\) is equivalent to the homotopy fixed point \(E^{hG_n}_{\infty}\) [DH04].

Let \(G\) denote a finite subgroup of the Morava stabilizer group. From the finite resolution point of view, the spectrum \(E^{hG}_n\) is the building block of the \(K(n)\)-local stable homotopy category [Hen07, GHMR05]. In particular, its homotopy groups \(\pi_*E^{hG}_n\) detect important families of classes in the stable homotopy groups of spheres [HHR16, LSWX19, BMQ20]. Therefore, the computation of \(E^{hG}_n\) is a central topic in chromatic homotopy theory and extremely challenging in general.

From now on, we focus on the prime \(p = 2\). Hewett classified all the finite subgroups of \(S_n\) [Hew95] (see also [Buj12]). If \(n = 2^{m-1}\ell\) where \(\ell\) is odd, then when \(m \neq 2\), the maximal finite 2-subgroups of \(G_n\) are isomorphic to \(C_{2^m}\), the cyclic group of order \(2^m\); when \(m = 2\), \(n\) is of the form \(4k + 2\), and the maximal finite 2-subgroups are isomorphic to \(Q_8\), the quaternion group.

There are breakthroughs of computations of \(E^{hG}_n\) when \(G\) is cyclic due to the recent development of the equivariant methods [HHR17, HSWX18, BBHS20, HS20]. These computations are done by a new tool called the slice spectral sequence. The slice spectral sequence computations of the norm of real cobordism theories induce computations of \(E^{hG}_n\) at prime 2 for the case \(G = C_{2^m}\). As far as the authors are aware, there are no such computations for the case \(G = Q_8\) due to the lack of the slice information.

At height 2, the group \(Q_8\) first appears as a subgroup of the (small) Morava stabilizer group \(S_2\). Maximal finite subgroups of \(S_2\) are isomorphic to \(G_{24} = Q_8 \rtimes C_3\). Similarly, in the (extended) Morava stabilizer group \(G_2\), there are subgroups isomorphic to \(SD_{16}\) and \(G_{48}\). Homotopy fixed points of \(E_2\) with respect to the above subgroups appear in the finite resolution of \(E^{hG}_2\): the \(K(2)\)-local sphere at prime 2, as building blocks [Bea15, BG18]. Moreover, they also appear in the interplay between chromatic layer 2 and the theory of elliptic curve (see for example [Hop02, DFHH14, HM14, BO16, HL16]). Important examples like \(tmf\) are related to computations of \(E^{hG}_{2\text{G}_{48}}\).

In this paper, we use equivariant methods and a new method, which we called “the vanishing line method”, to compute the \(G\)-homotopy fixed point spectral sequence (\(G\)-HFPPSS) of the height 2 Lubin–Tate theory \(E_2\) at the prime 2 for \(G = Q_8, SD_{16}, G_{24}\) and \(G_{48}\).

Let \(\sigma_i\) (resp. \(\sigma_j, \sigma_k\)) be the one dimensional non-trivial \(Q_8\) representation that \(i \in Q_8\) (resp. \(j, k \in Q_8\)) acts trivially. We compute the \(i\)-graded as well as \((i - \sigma_j)\)-graded \(G\)-HFPPSS for \(E_2\). By symmetry, this gives the \((i - \sigma_j)\)-graded and \((i - \sigma_k)\)-graded \(G\)-HFPPSS for \(E_2\).

**Theorem A.**

1. The integer-graded \(Q_8\)-HFPPSS for \(E_2\) has differentials as listed in Table 8 (also see Figs. 5 to 8). The \(E_{\infty}\)-page with all 2 extensions is presented in Fig. 9.

Furthermore, we have

\[SD_{16}\text{-HFPPSS}(E_2) \otimes_{\mathbb{Z}_2} \mathcal{W}(\mathbb{F}_4) = Q_8\text{-HFPPSS}(E_2),\]

where the tensor products happen on \(E_r\) and \(d_r\) for every \(2 \leq r \leq \infty\).
Theorem B. The integer-graded $G_{24}$-HFPSS for $E_2$ is a subobject of the integer-graded $Q_8$-HFPSS for $E_2$ which consists of classes with $D^m$ where $3 | m$, and the differentials are exactly the same. The $E_{\infty}$-page with all 2 extensions is presented as in Fig. 15. Furthermore, we have

$$SD^{16}_{10}-HFPSS(E_2) \otimes_{\mathbb{Z}_2} \mathbb{W}(\mathbb{F}_4) = Q_8-HFPSS(E_2),$$

where the tensor products happen on $E_r$ and $d_r$ for every $2 \leq r \leq \infty$.

In Theorem B, we only compute the integer-graded part because $\sigma_i$ cannot be lifted to a $G_{24}$-representation.

Theorem A gives the complete computation of the integer-graded $G_{48}$-HFPSS for $E_2$. Though the result is known to experts and can be deduced from the $tmf$ computation [Bau08], as far as the authors are aware, it is not written down in literature before. The $(\ast - \sigma_i)$-graded computation in Theorem A is new. Moreover, our methods for $Q_8$-HFPSS computations are independent of previous computations and can potentially work for higher heights. The first method is the recently developed equivariant method which uses the restriction, transfer and norm structures of the spectral sequence to deduce differentials and hidden extensions. More precisely, we deduce differentials and hidden extensions in the $Q_8$-HFPSS for $E_2$ from differentials in the $C_4$-HFPSS for $E_2$ (computed in [HHR17, BBHS20]) via restrictions, transfers and norms. For example the restriction functor from $Q_8$ to $C_4$ implies a hidden 2 extension from a class at $(54,2)$ to a class at $(54,10)$ in the $Q_8$-HFPSS for $E_2$ (See Lemma 4.23) which is crucial to deduce the $d_{15}$-differential proved in Proposition 4.25. This exempts us from using the Toda bracket shuffling method as in [Bau08, Proposition 8.5 (3)]. $RO(G)$-gradings have been proved to be helpful in computations [HHR17, BBHS20]. For example, for groups $H \subset G$, the norm map on the $E_2$-page of a $G$-HFPSS is only defined after extending to $RO(G)$-gradings [Ull13, HHR17, MSZ20]. Norm maps allow us to pull back and push forward known differentials for new differential information. In our computation, $(\ast - \sigma_i)$-graded $G$-HFPSS for $E_2$ gives an alternative proof of a $d_0$-differential by the norm map (See Proposition 4.43).

We also introduce a new method: “the vanishing line method”. The vanishing result [DLS22] shows that all permanent cycles in filtration at least 25 must be hit, which forces differentials to happen in many cases. For example, in Proposition 4.14 the vanishing line method forces three differentials, including the longest $d_{27}$-differentials, just from the $E_{27}$-page information.

Along the way, we prove properties that help the computation and work for general heights. In particular, we improve the vanishing result in [DLS22, Theorem 6.1] for the $Q_8$ case to which is sharp for all known cases.

Theorem C (Theorem 4.8). The $Q_8$-HFPSS for $E_{4k+2}$ admits a strong vanishing line of filtration $2^{4k+5} - 9$.

Recall that having a strong horizontal vanishing line of filtration $f$ means that the spectral sequence collapses after the $E_f$-page, and any element of filtration greater than or equal to $f$ supports a differential or is hit.

Equivariant methods and the vanishing line method work for general heights. However, the computation of the $E_2$-page of HFPSS($E_{4k+2}$) is not known due to the lack of the explicit $Q_8$-action on $E_{4k+2}$ for $k > 1$.

**Question.** How to describe explicitly the $Q_8$-action on the Lubin-Tate theory $E_{4k+2}$ for $k > 1$?
1.2. **Summary of the contents.** This paper is organized as follows. Section 2 provides a necessary background for the computational tools for the $RO(G)$-graded homotopy fixed point spectral sequence, and the input for the computation of the $Q_8$-HFPSS for $E_2$. In particular, we review the norm structure in $RO(G)$-graded homotopy fixed point spectral sequences (Theorem 2.8) and the interplay between the homotopy fixed point spectral sequences and the Tate spectral sequences in general (Lemma 2.1). We briefly review the $Q_8$-action on $\pi_*(E_2)$ (Eq. (2.3)) and the computation of $RO(C_4)$-graded Mackey-functor-valued $C_4$-HFPSS for $E_2$ (Section 2.4). We take these as the input for the $Q_8$-HFPSS for $E_2$. In Section 3 we compute the $E_2$-page of the integer-graded and $(\ast - \sigma_i)$-graded $Q_8$-HFPSS($E_2$) by Bockstein spectral sequences. In Section 4, we derive all differentials in the integer-graded $Q_8$-HFPSS for $E_2$ via equivariant methods and the vanishing line method (Theorem 4.8). In Section 4.1, we prove the properties of the $Q_8$-HFPSS for $E_2$ that we need for our computation. The vanishing line (Theorem 4.8) works for general heights and is of its own interests. In Section 4.2, we give a complete computation of all differentials in the logical order. The vanishing line method gives some difficult differentials (for example Proposition 4.14). In Section 4.3, we solve all 2 extensions. In Section 4.4, we present alternative proofs for those differentials that can be proved by more than one way.

In Section 5, we also apply equivariant methods and the vanishing line method to compute the $(\ast - \sigma_i)$-graded $Q_8$-HFPSS for $E_2$. In particular, this computation gives an alternative proof of a $d_9$-differential in the integer-graded part. In Section 6, we list figures that present our computation. In Appendix A, we explain algebraic computations of the $Q_8$ group cohomology. In addition, we explain how the Hurewicz image of $E_2^{0C_4}$ helps to compute the restriction map from $Q_8$-HFPSS to $C_4$-HFPSS.

1.3. **Acknowledgements.** The authors would like to thank Agnès Beaudry, Mark Behrens, Paul Goerss, Bert Guillou, XiaoLin Danny Shi, Zhouli Xu, and Mingcong Zeng for helpful conversations. The second author was supported by the National Science Foundation under Grant No. DMS-1926686. The third author is grateful to Max Planck Institute for Mathematics in Bonn for its hospitality and financial support. The fifth author is partially supported by the Shanghai Rising–Star Program under Agreement No. 20QA1401600 and Shanghai Pilot Program for Basic Research–FuDan University 21TQ1400100(21TQ002).

2. **Preliminaries**

2.1. **$RO(G)$-graded homotopy fixed point spectral sequences and Tate spectral sequences.**

Let $X$ be a $G$-spectrum, and let $P^\ast X$ be the slice tower of $X$. We have a diagram of towers:

$$
\begin{array}{ccc}
EG_+ \wedge P^\ast X & \rightarrow & P^\ast X \\
\downarrow \sim & & \downarrow \\
EG_+ \wedge F(EG_+ , P^\ast X) & \rightarrow & F(EG_+ , P^\ast X)
\end{array}
$$

This diagram of towers further induces a Tate diagram of spectral sequences

$$
\begin{array}{cccc}
\text{HOSS}(X) & \rightarrow & \text{SliceSS}(X) & \rightarrow & \text{LSliceSS}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\text{HOSS}(X) & \rightarrow & \text{HFPSS}(X) & \rightarrow & \text{TateSS}(X)
\end{array}
$$

(2.1)
We briefly explain the above notations as follows. We use \( \ast \) to denote an integer and \( \star \) to denote an \( RO(G) \)-grading. We denote the underlying homotopy group \( \pi_0^G(X \wedge S^{-\ast}) \) as a \( G \)-module by \( \pi_\ast^G(X) \).

- The spectral sequence \( \text{HOSS}(X) \) of the tower \( EG_+ \wedge P^\ast X \) is the \( RO(G) \)-graded homotopy fixed point spectral sequence of \( X \) with the \( E_2 \)-page as
  \[
  H_*(G, \pi_\ast^G(X))
  \]
  which converges to \( \pi_{\ast,+}^G(EG_+ \wedge X) \).
- The spectral sequence \( \text{SliceSS}(X) \) of the tower \( P^\ast X \) is the slice spectral sequence of \( X \) with the \( E_2 \)-page as
  \[
  \mathbb{F}_{\ast,+}(P^\ast | \ast| X)
  \]
  which converges to \( \mathbb{F}_{\ast,+}(X) \). Where \( P^\ast X \) is the fiber of \( P^\ast X \to P^{\ast-1}X \) and \( |\ast| \) is the underlying dimension of \( \ast \).
- The spectral sequence \( \text{HFPSS}(X) \) of the tower \( F(EG_+, P^\ast X) \) is the \( RO(G) \)-graded homotopy fixed point spectral sequence of \( X \) with the \( E_2 \)-page as
  \[
  H^*(G, \pi_\ast^G(X))
  \]
  which converges to \( \mathbb{F}^*_{\ast,+}(F(EG_+, X)) \).
- The spectral sequence \( \hat{E}G \wedge P^\ast X \) of the tower is the \( RO(G) \)-graded localized slice spectral sequence. It follows the treatment of a forthcoming paper by Meier-Shi-Zeng.
- The spectral sequence \( \text{TateSS}(X) \) of the tower \( \hat{E}G \wedge F(EG_+, P^\ast X) \) is the \( RO(G) \)-graded Tate spectral sequence of \( X \) with the \( E_2 \)-page as
  \[
  \hat{H}^*(G, \pi_\ast^G(X))
  \]
  which converges to \( \mathbb{F}^*_{\ast,+}(\hat{E}G \wedge F(EG_+, X)) \).

The following result is first proven in [Ull13] for the integer gradings and extended to the \( RO(G) \)-gradings in [DLS22, Theorem 3.3]. It shows that the natural map from \( \text{SliceSS}(X) \) to \( \text{HFPSS}(X) \) is an isomorphism in a certain range.

**Lemma 2.1** ([Ull13], [DLS22]). The map from the \( RO(G) \)-graded slice spectral sequence to the \( RO(G) \)-graded homotopy fixed point spectral sequence

\[
\begin{array}{ccc}
\pi^{\mathcal{G}}_{V,-s} P^{[V]} \to X & \rightarrow & \pi^{\mathcal{G}}_{V,-s} F(EG_+, P^{[V]} X) \\
\downarrow & & \downarrow \\
\pi^{\mathcal{G}}_{V,-s} X & \rightarrow & \pi^{\mathcal{G}}_{V,-s} F(EG_+, X)
\end{array}
\]

induces an isomorphism on the \( E_2 \)-page in the region defined by the inequality

\[
\tau(V - s - 1) > |V|, \quad \tau(V) := \min_{\{G \subseteq H \subseteq G\}} |H| \cdot \dim V^H.
\]

Furthermore, the map induces a one-to-one correspondence between the differentials in this isomorphism region.

We recall two kinds of distinguished classes in the \( RO(G) \)-graded homotopy groups that are useful for naming the relevant classes on the \( E_2 \)-page of the slice spectral sequence (see [HHR16, Section 3.4] and [HSWX18, Section 2.2]) and the homotopy fixed point spectral sequence.

**Definition 2.2.** Let \( V \) be a \( G \)-representation. We denote the inclusion of the fixed points \( S^0 \to S^V \) by \( a_V \). This is a class in \( \pi_0^G S^0 \). Moreover, for a ring spectrum \( X \) with \( G \)-action, we abuse notation to denote the image of \( a_V \) by \( a_V \) under the map \( S^0 \to X \). We will also denote...
the class on the $E_2$-page of the $G$-HFPSS($S^0$) or the $G$-HFPSS($X$) that detects the image of $a_V$ by $a_V$.

By construction, we have the following property.

**Proposition 2.3.** With the above notation, the class $a_V$ on the $E_2$-page of the $G$-HFPSS($X$) is a permanent cycle.

If the representation $V$ has non-trivial fixed points (i.e. $V^G \neq \{0\}$), then $a_V = 0$. Moreover, for any two $G$-representations $V$ and $W$, we have the relation $a_{V \oplus W} = a_V a_W$ in $\pi^G_{V-W}(S^0)$. Moreover, $a_V$-class is always a torsion class, according to [HHR17, Lemma 3.6]

$$|G/G_V|a_V = 0$$

where $G_V$ is the isotropy subgroup of $V$.

For an orientable $G$-representation $V$, a choice of orientation for $V$ gives an isomorphism $H^G_{|V|}(S^V; \mathbb{Z}) \cong \mathbb{Z}$. In particular, the restriction map

$$H^G_{|V|}(S^V, \mathbb{Z}) \longrightarrow H_{|V|}(S^{|V|}, \mathbb{Z})$$

is an isomorphism.

**Definition 2.4.** Let $V$ be an orientable $G$-representation. We define the orientation class of $V$ $u_V \in H^G_{|V|}(S^V; \mathbb{Z})$ to be the generator that maps to 1 under the above restriction isomorphism (2.2).

The orientation class $u_V$ is stable in $V$ in the sense that if 1 is the trivial representation, then $u_{V \oplus 1} = u_V$. Moreover, if $V$ and $W$ are two orientable $G$-representations, then $V \oplus W$ is also orientable with the direct sum orientation, and $u_{V \oplus W} = u_V u_W$.

Norms of $a_V$ classes and $u_V$ classes are given as follows.

**Proposition 2.5.** ([HHR16, Lemma 3.13]) Let $H \subset G$ be a subgroup and $V$ is a $G$-representation

$$N^G_H(a_V) = a_{\text{Ind} V};$$

$$u_{\text{Ind} |V|} N^G_H(u_V) = u_{\text{Ind} V}$$

where $\text{Ind}$ means $\text{Ind}_{|V|}^G$.

Given a $G$-oriented representation $V$ and a $G$-equivariant commutative ring spectrum $X$, by [HHR16, Corollary 4.54] and the unit map $S^0 \to X$, Hill–Hopkins–Ravenel defines the $u_V$ classes on the $E_2$-page of the slice spectral sequence for $X$ via the following map on 0-th slices

$$H^G_{|V|} = P^0_0 S^0 \to P^0_0 X.$$

With Lemma 2.1, we can define $u_V$ classes in the $RO(G)$-graded HFPSS for $X$.

The computation of the TateSS and the HFPSS are closely related. In any $RO(G)$-graded page the natural map from HFPSS($X$) to TateSS($X$) is isomorphic in positive filtration ([DLS22, Theorem 3.6], see also [BM94, Lemma 2.12]).

**Lemma 2.6.** The map from the $RO(G)$-graded homotopy fixed point spectral sequence to the $RO(G)$-graded Tate spectral sequence induces an isomorphism on the $E_2$-page for classes in filtration $s > 0$, and a surjection for classes in filtration $s = 0$. Furthermore, there is a one-to-one correspondence between differentials whose source is in non-negative filtrations.

One advantage of considering Tate spectral sequences is that they are whole plane spectral sequences with more invertible classes. This feature makes the calculations more accessible.

If $V$ is a $G$-representation such that its fixed point set $V^H$ is trivial for any non-trivial subgroup $H$ of $G$, then $S^\infty V$ is a geometric model for $E \hat{G}$. If $X$ is a $G$-spectrum, we have

$$E \hat{G} \wedge X \simeq S^\infty V \wedge X = a_V^{-1} X$$
This implies that for such representation $V$, the class $a_V$ is invertible in the Tate spectral sequence.

**Method 2.7.** When it is multiplicative, the TateSS is extremely useful for proving permanent cycles in the HFPSS. Assume the TateSS of a $G$-spectrum $X$ is multiplicative. Then we can find permanent cycles in the $G$-HFPSS for $X$ as follows. Assume that we find a differential $d_r(a) = b$ in the HFPSS, then there is a corresponding differential $d_r(a') = b'$ in the TateSS by Lemma 2.6. We can move this differential by some $r$-cycle $c'$ in the TateSS such that $d_r(c'a') = c'b'$ is a differential with the source $c'a'$ in a negative filtration and the target $c'b'$ in a non-negative filtration. (One can choose $c' = a_V^{-k}$ for proper integer $k$ where $a_V$ is an invertible class as above.) Then $c'b'$ is a permanent cycle in the TateSS and hence the corresponding class of $c'b'$ in the HFPSS is also a permanent cycle by Lemma 2.6. This method allows us to identify permanent cycles at $E_r$-page for $r < \infty$.

Now we focus on $G = Q_8$ and its subgroups. We will use the following notations for representations of $C_2, C_4$ and $Q_8$.

- When $G = C_2$, $RO(C_2) = \mathbb{Z}\{1, \sigma_2\}$ where $\sigma_2$ is the sign representation.
- When $G = C_4$, $RO(C_4) = \mathbb{Z}\{1, \sigma, \lambda\}$. The representation $\sigma$ is the sign representation and $\lambda$ is the 2-dimensional representation by rotating the plane $\mathbb{R}^2$ by degree $\frac{\pi}{4}$.
- When $G = Q_8$, $RO(Q_8) = \mathbb{Z}\{1, \sigma_i, \sigma_j, \sigma_k, \mathbb{H}\}$. The representations $\sigma_i, \sigma_j, \sigma_k$ are one-dimensional representations where kernels are $C_4(i), C_4(j)$ and $C_4(k)$ i.e., the three $C_4$ subgroups generated by $i, j$ and $k$, respectively. The representation $\mathbb{H}$ is a four-dimensional irreducible representation, obtained by the action of $Q_8$ on the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k$ by left multiplication.

By the above discussion, $S_{\infty, \mathbb{H}}$ is a model of $\tilde{EQ}_8$. Therefore, the class $a_{\mathbb{H}}$ is invertible in any $Q_8$-Tate spectral sequence.

### 2.2. Norm differentials and strong vanishing lines in spectral sequences.

The Hill-Hopkins-Ravenel norm structure holds in nice equivariant spectral sequences. Let $H \subset G$ be a subgroup. Consider the following diagram of $G$-spectra

$$
\cdots \rightarrow P^{n+1} \rightarrow P^n \rightarrow P^{n-1} \rightarrow \cdots
$$

Recall that $P^m$ denotes the fiber of $P^m \to P^{m-1}$ and $P_n = P^\infty_n$.

We denote the spectral sequence associated to this tower by $\{E^n, d_r\}$, where $n$ denotes the filtration and the second grading denotes the $RO(G)$-graded stem. We say the spectral sequence has a norm structure if there are two types of maps $N^G_H P_n \to P_{[G/H]}$ and $N^G_H P_n \to P_{[G/H]}^G$ such that the following two diagrams commute up to homotopy.

\[
\begin{array}{ccc}
N^G_H P_n & \longrightarrow & P_{[G/H]} \\
\downarrow & & \downarrow \\
N^G_H P_{n-1} & \longrightarrow & P_{[G/H]|(n-1)}
\end{array}
\]

\[
\begin{array}{ccc}
N^G_H P_n & \longrightarrow & P_{[G/H]|n} \\
\downarrow & & \downarrow \\
P_{[G/H]} & \longrightarrow & P_{[G/H]|(n-1)}
\end{array}
\]

The norm structure induces a map between towers

\[
\begin{array}{ccc}
\cdots & \longrightarrow & N^G_H P_n \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{[G/H]} \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{[G/H]|-1} \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{(n-1)|G/H|+1} \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & P_{(n-1)|G/H|}
\end{array}
\]

which induces a map from the $E_2$-page of the $H$-level spectral sequence $H-E_2^\bullet$ to the $E_2$-page of the $G$-level spectral sequence $G-E_2^\bullet$ as follows.
Then in the spectral sequence for \( Y \) norm structures, if we have a differential spectral sequence of \( E^2 \).

**Theorem 2.8.** ([Ull13, Proposition I.5.17][HHR17, Theorem 4.7]) In a spectral sequence with norm structures, if we have a differential \( d_r(x) = y \) in the spectral sequence of a \( H \)-spectrum \( X \). Then in the spectral sequence for \( Y = N_H^G(X) \) there is a predicted differential

\[
d_{[G/H][r-1]+1}(a_pN_H^G(x)) = N_H^G(y)
\]

where \( \rho = \text{Ind}_H^G(1) \) and \( \bar{\rho} \) is the reduced representation of \( \rho \).

In [DLS22] the authors use the norm structures to show that every class in \( G \)-TateSS\((E_n) \) is hit before a specific page depending on \( n \) and \( G \).

**Theorem 2.9.** ([DLS22, Theorem 5.1]) At the prime 2, for any height \( n \) and any \( G \subset \mathbb{G}_n \) a finite subgroup, let \( H \) be a Sylow 2-subgroup of \( G \). All the classes in the RO\((G)\)-graded Tate spectral sequence of \( E_n \) vanish after the \( E_{n,H} \)-page. Here \( N_{n,H} \) is a positive integer defined as follows:

- when \( (n,H) = (2^{m-1}l, C_{2^{2m}}) \), \( N_{n,H} = 2^{n+m} - 2^m + 1 \);
- when \( (n,H) = (4k + 2, Q_8) \), \( N_{n,H} = 2^{n+3} - 7 \).

The isomorphism range of the natural map \( G \)-HFPSS\((E_n) \) → \( G \)-TateSS\((E_n) \) implies there is a strong horizontal vanishing line in \( E_{\infty} \)-page of \( G \)-HFPSS\((E_n) \).

**Theorem 2.10.** ([DLS22, Theorem 6.1]) At the prime 2, for any height \( n \) and any \( G \subset \mathbb{G}_n \) a finite subgroup, let \( H \) be a Sylow 2-subgroup of \( G \). There is a strong horizontal vanishing line of filtration \( N_{n,H} \) in the RO\((G)\)-graded homotopy fixed point spectral sequence of \( E_n \).

It turns out that the existence of such horizontal vanishing lines is extremely helpful for determining higher differentials in homotopy fixed point spectral sequences. In particular, for our computation in \( Q_8 \)-HFPSS\((E_2) \), the vanishing line gives an independent proof of several higher differentials in the integer-gradings. Moreover, this vanishing line plays a crucial role in the computation of \((\ast - \sigma_i)\)-graded \( Q_8 \)-HFPSS for \( E_2 \).

### 2.3. Lubin-Tate Theory \( E_2 \) with \( G_2 \)-action.

We fix a pair \( (\mathbb{F}_{p^n}, \Gamma_n) \) where \( \Gamma_n \) is the height \( n \) Honda formal group law over \( \mathbb{F}_p \) extended to \( \mathbb{F}_{p^n} \). Then Lubin-Tate [LT65] shows that there is a universal deformation \( F_n \) defined over a complete local ring

\[
\mathcal{W}(\mathbb{F}_{p^n})[u_1, \ldots, u_{n-1}][u^{-1}]
\]

where \( \mathcal{W}(\mathbb{F}_{p^n}) \) is the \( p \)-typical Witt vector of \( \mathbb{F}_{p^n} \) and \( |u_i| = 0, |u^{-1}| = 2 \). The Landweber exactness theorem shows that this ring can be realized by a complex oriented ring spectrum \( E_n \).

Let \( S_n \) be the automorphism group of \( \Gamma_n \), namely the small \( n \)-th Morava stabilizer group. Let \( G_n = S_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \) be the automorphism group of \( (\mathbb{F}_{p^n}, \Gamma_n) \), namely the (extended) \( n \)-th Morava stabilizer group. By universality, \( \pi_*, E_n \) admits a \( G_n \)-action. The Goerss–Hopkins–Miller theorem [Rez98, GH04, Lur18] lifts this action uniquely to an \( E_\infty \)-action on \( E_n \).

We are interested in computing \( \pi_* E_\infty^G \) for \( G \) a finite subgroup of \( G_n \) via \( G \)-homotopy fixed point spectral sequences. For these computations, the action of the Galois group \( \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \) will not change the differential pattern. More precisely, we review the following result.
Lemma 2.11. ([BG18, Lemma 1.32][BGH22, Lemma 2.2.6, Lemma 2.2.7]) Let $F \subset \mathbb{G}_n$ be a closed subgroup and let $F_0 = F \cap S_n$. Suppose the canonical map

$$F/F_0 \to \mathbb{G}_n/S_n \cong \text{Gal}(\mathbb{F}_p/\mathbb{F}_p)$$

is an isomorphism. Then there is a commutative diagram of homotopy fixed point spectral sequences

$$\begin{array}{ccc}
\mathcal{W}(\mathbb{F}_p) \otimes_{\mathbb{Z}_p} H^*(F, \pi_* E_n) & \xrightarrow{\cong} & \mathcal{W}(\mathbb{F}_p) \otimes_{\mathbb{Z}_p} \pi_*(E^h_{F}) \\
\xrightarrow{H^*(F_0, \pi_* E_n)} & & \xrightarrow{\pi_*(E^h_{F_0})} \\
\end{array}$$

In this paper, we will focus on the case $p = 2$ and $n = 2$. The Galois group $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ is isomorphic to $C_2$ and we write $\mathcal{W}$ for the Witt vector $\mathcal{W}(\mathbb{F}_4)$. There are finite subgroups $Q_8$ and $G_{24} \cong Q_8 \rtimes C_3$ in the small Morava stabilizer group $S_2$ and $SD_8 = Q_8 \rtimes \text{Gal}$ and $G_{24} \cong G_{24} \rtimes \text{Gal}$ in the extended Morava stabilizer group $G_2$. The subgroups $Q_8, G_{24}$ are unique up to conjugacy in $S_2$ [Buj2] (see also [BGH22, Remark 2.4.5]). Therefore, there is no ambiguity in the notation $\pi_* E_2^{h_{Q_8}}$ or $\pi_* E_2^{h_{G_{24}}}$.

The subgroup $Q_8$ and complex orientation coordinates can be chosen specifically from the theory of elliptic curves at prime 2 so that the action has explicit formulas as follows (See [Bea17, Section 2] for more details).

We recall the action of $G_{24}$ on $\pi_* E_2$ [Bea17, Lemma A.1]. The coefficient ring is a complete local ring $\pi_* E_2 = \mathcal{W}[u_1][u_1^{-1}]$ with a maximal ideal $I = (2, u_1)$. Denote $u_1 u^{-1}$ by $v_1$, the generator of the quaternion group $Q_8$ by $i, j, k$ and the generator of $C_3$ by $\omega$. We regard the third root of unit $\zeta$ as a class in the Witt vector $\mathcal{W}$. The $G_{24}$-actions on $u^{-1}$ and $v_1$ are

$$
\begin{align*}
\omega_* (v_1) &= v_1, & \omega_* (u^{-1}) &= \zeta^2 u^{-1}, \\
i_* (v_1) &= \frac{v_1 - u^{-1}}{\zeta^2 - \zeta}, & i_* (v_1) &= \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta}, \\
j_* (v_1) &= \frac{\zeta v_1 - u^{-1}}{\zeta^2 - \zeta}, & j_* (v_1) &= \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}, \\
k_* (v_1) &= \frac{\zeta^2 v_1 - u^{-1}}{\zeta^2 - \zeta}, & k_* (v_1) &= \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}.
\end{align*}
$$

We define $D$ to be $\prod_{g \in Q_8/C_2} g_*(u^{-1})$ which is $Q_8$-invariant. Then $(E_2)_*$ could be expressed as

$$\pi_* E_2 \cong (\mathcal{W}[v_1, u^{-1}][D^{-1}])_{\gamma}$$

which is more convenient for the $Q_8$-cohomology computation.

Lemma 2.12. There is an isomorphism

$$H^*(Q_8, \pi_* E_2) \cong (H^*(Q_8, \mathcal{W}[v_1, u^{-1}][D^{-1}]))_{\gamma}.$$

Proof. Because $D$ is $Q_8$-invariant, we have

$$H^*(Q_8, \mathcal{W}[v_1, u^{-1}][D^{-1}]) \cong H^*(Q_8, \mathcal{W}[v_1, u^{-1}][D^{-1}]).$$

Note that $\mathcal{W}[v_1, u^{-1}][D^{-1}]$ is finitely generated as a $\mathcal{W}$-algebra. Therefore, the completion is an exact functor [AM16, Theorem 10.12] [HS99, Theorem A.1] and we have

$$H^*(Q_8, \pi_* E_2) \cong (H^*(Q_8, \mathcal{W}[v_1, u^{-1}][D^{-1}]))_{\gamma}.$$
2.4. Mackey functor $C_4$-homotopy fixed point spectral sequence for $E_2$. In this subsection, we recall some results on the Mackey-functor-valued $C_4$-HFPSS for $E_2$ in [BBHS20]. See also the slice spectral sequence computation of the truncated $C_4$-normed Real Brown–Petersen spectrum $BP^{((C_4))(1)}$ [HHR17][HSWX18].

**Proposition 2.13.** ([BBHS20, Proposition 5.6]) There is an isomorphism
\[ H^*(C_2, \pi_4 E_2) \cong W[\mu] \langle [\eta^2], [\eta'], a_{\lambda}, a_{\sigma}, [\delta^0_1, u_{2\sigma}] \rangle / (2a_{\sigma}), \]
where the $(\star, \star, \star)$-degree of the classes is given by $|\mu| = (0,0)$, $|\eta^2| = (p_2,0)$, $|a_{\lambda}| = (-\sigma,1)$, and $|u_{2\sigma}| = (2 - 2\sigma,0)$.

We partially rewrite the names of classes on the $E_2$-page of $C_4$-HFPSS($E_2$) in [BBHS20, Proposition 5.10] with slice names. For slice names, see [HHR17, HSWX18] for details. One advantage of using slice names is that it is better to organize differentials by the slice differential theorem [HHR16, Theorem 9.9].

**Proposition 2.14.** ([BBHS20, Proposition 5.10]) There is an isomorphism
\[ H^*(C_4, \pi_4 E_2) \cong W[\mu] \langle T_2, [\eta^2], [\eta'], a_{\lambda}, a_{\sigma}, [\delta^0_1, u_{2\sigma}] \rangle / \sim \]
where $\mu = tr_{C_4}(\mu_0)$, $T_2 = s_1^2 u_{2\sigma}$, $\eta = s_1 a_{\sigma} = tr_{C_4}(r_1 a_{\sigma})$ and $\eta' = s_1 u_{\sigma} a_{\sigma} = tr_{C_4}(r_1 u_{\sigma} a_{\sigma})$. Although $\sigma$ is not an oriented $C_4$-representation, we apply $u_{\sigma}$ here indicating that $\eta'$ is transferred from $r_1 a_{\sigma}$ from integer-graded part in $C_2$-level to $(1 - \sigma)$-page in $C_4$-level. And the relation $\sim$ is the ideal generated by the following relation
\[ 2\eta = 2\eta' = 2a_{\sigma} = 4a_{\lambda} = 0, \quad T_2^2 = \Delta_1((\mu - 2)^2 + 4), \]
\[ \eta^2 u_{2\sigma} = \eta'^2 = T_2 u_{2}\lambda u_{2\sigma}, \quad T_2 \eta' = \delta_1 \mu \eta a_{\lambda} u_{2\sigma}, \]
\[ u_{\lambda} a_{\sigma} = 2a_{\lambda} u_{2\sigma}, \quad \eta a_{\sigma} = \eta' a_{\sigma} = T_2 a_{\sigma} = 0. \]

Here $\Delta_1 = \delta^0_1 u_{2\lambda} u_{2\sigma}$ at $(8,0)$ is an invertible class in $\pi_4 E_2^{hC_4}$.

**Remark 2.15.** Proposition 2.13 and Proposition 2.14 give a full description of the Mackey functor $H^*(C_4, \pi_4 E_2)$ by the Frobenius relation [BBHS20, Remark 5.17] and the multiplicative property of restriction.

**Remark 2.16.** A warning is that one needs to be careful about the isomorphism range (See Lemma 2.1) to translate between the slice spectral sequence and the homotopy spectral sequence. For example, in the $C_4$-SliceSS($BP^{((C_4))(1)}$), the class $u_{2\sigma}$ supports a non-trivial $d_5$-differential [HSWX18, Theorem 3.4], while in the corresponding $C_4$-HFPSS($E_2$), the class $u_{2\sigma}$ actually supports a non-trivial $d_5$-differential [BBHS20, Remark 5.23].

The computation of the Mackey-functor-valued $C_4$-homotopy fixed point spectral sequence for $E_2$ is explained in detail in [BBHS20, Section 5] and presented by [BBHS20, Figure 5.8] and [BBHS20, Figure 5.14].

The $RO(G)$-graded Mackey functor computation is useful even if one only cares about the computation of the integer-graded part $\pi_4 E_2^{hG}$. The following discussion of hidden extensions is a good example. We can use exotic operations (exotic transfers, exotic restrictions, and so on) in Mackey-functor-valued spectral sequences to deduce differentials and hidden extensions inside the spectral sequences. For more detailed definitions and properties of such phenomena, one could refer to [MSZ20, Section 3.3].

In [HHR17, Lemma 4.2], the authors introduce a useful trick to determine exotic restrictions and transfers on the $E_{\infty}$-page of Mackey-functor-valued $G$-HFPSS.
Lemma 2.17. ([HHR17, Lemma 4.2]) Let $G$ be a cyclic 2-group and $G'$ be its index 2 subgroup then in $\pi_\bullet(F(EG_+,X))$ we have

- $\ker(\res^{G'}_G) = \im(a_\sigma)$
- $\im(\tr^{G'}_G) = \ker(a_\sigma)$

where $\sigma$ is the sign representation of $G$.

The following hidden 2 extension in stem 22 is a good example showing that equivariant structures provide extra integer-graded information (see a similar 2 extension in stem 2 in [MSZ20, Remark 5.15]). In [HHR17, Figure 15] and [BBHS20, Figure 5.6], they drew all exotic restrictions and transfers in the $E_\infty$-page of the Mackey functor valued $C_4$-HFSS($E_2$). The 2 extension follows from an exotic transfer and an exotic restriction in 22 stem. We spell out the details in Lemma 2.18.

Lemma 2.18. In the Mackey-functor-valued $C_4$-HFSS for $E_2$, there is an exotic restriction in stem 22 from $\delta^1_8 u_4 a_{2\sigma}$ to $\delta^1_8 u_4 a_{2\sigma}$ and there is an exotic transfer in stem 22 from $\delta^1_8 u_4 a_{2\sigma}$ to $\delta^1_8 u_4 a_{2\sigma}$. As a consequence, there is a hidden 2 extension from $\delta^1_8 u_4 a_{2\sigma}$ to $\delta^1_8 u_4 a_{2\sigma}$.

Proof. According to the computations in [HHR17][BBHS20], in stem 22 there are only three classes survives: $\delta^1_8 u_4 a_{2\sigma}$ and $\delta^1_8 u_4 a_{2\sigma}$ in $C_4$-level and $\delta^1_8 u_4 a_{2\sigma}$ in $C_2$-level. We first claim the class $\delta^1_8 u_4 a_{2\sigma}$ is not in the image of multiplication by $a_{\sigma}$. If there is some $x$ such that $a_{\sigma}x$ is $\delta^1_8 u_4 a_{2\sigma}$, then $x$ is detected by classes at $(22+\sigma,1)$ or $(22+\sigma,0)$. There is only one class at $(22+\sigma,1)$ which is $\delta^1_8 u_4 a_{2\sigma}$ on $E_2$-page. According to [HSWX18, Theorem 3.11], this class supports a $d_{13}$-differential

$$d_{13}(\delta^1_8 u_4 a_{2\sigma}) = \delta^1_8 u_4 a_{2\sigma}d_{13}(u_4 a_{2\sigma}) = \delta^1_8 u_4 a_{2\sigma}a_{\gamma_3}$$

And moreover there is no non-trivial class at $(22+\sigma,0)$. Therefore, in homotopy level there is no class such that its multiplication by $a_{\sigma}$ hits the class $\delta^1_8 u_4 a_{2\sigma}$. Then according to Lemma 2.17, this class must be a non-trivial restriction in homotopy level, and the desired exotic restriction follows from degree reasons.

On the other hand by the gold relation $u_4 a_{2\sigma} = 2a_{2\sigma} a_{\sigma}$ and $2a_{2\sigma} = 0$ we know on $E_2$-page

$$\delta^1_8 u_4 a_{2\sigma} a_{2\sigma} a_{\sigma} = 0$$

Moreover, according to the computation on $* - \sigma$-page of $C_4$-HFSS($E_2$) [BBHS20], there is no hidden $a_{\sigma}$-extension from $\delta^1_8 u_4 a_{2\sigma}$ by degree reasons. Since we have $\im(\tr^{G'}_G) = \ker(a_\sigma)$, the class $\delta^1_8 u_4 a_{2\sigma}$ must be a transfer of a class from $C_2$-level. Then the desired exotic transfer follows from degree reasons. \qed

Remark 2.19. For degree reasons, the class $\delta^1_8 u_4 a_{2\sigma}$ cannot be in the image of the transfer from $C_2$. However, by the gold relation, the product of this class and $a_{\sigma}$ is zero on the $E_2$-page. Therefore, this class must have a hidden $a_{\sigma}$-extension.

Remark 2.20. The hidden 2 extension in Lemma 2.18 will play a crucial rule in deducing several higher differentials in $Q_8$-HFSS($E_2$) (see Lemma 4.23, Proposition 4.25). A similar 2 extension can also be seen in the homotopy groups of tmf in stem 54. The proof of this hidden 2 extension in [Bau08, Proposition 8.5 (3)] uses shuffling arguments of 4-fold Toda brackets. In our $Q_8$-HFSS($E_2$) computation, the corresponding hidden 2 extension follows directly from the $C_4$-computation by restriction (see Lemma 4.23).
2.5. \( RO(G) \)-graded periodicity. When computing HFPSS, another advantage of expanding to \( RO(G) \)-gradings is having more periodicities. These periodicities have their own theoretic importance. They can also move integer-graded calculations to certain \( RO(G) \)-gradings where the calculations might be simpler. In either the slice spectral sequence for \( BP^{(\mathbb{C}^4)^{(1)}} \) [HSWX18] or the \( C_4 \)-homotopy fixed point spectral sequence for \( E_2 \) [HHR17][BBHS20], we have the following periodicities in the \( RO(G) \)-gradings.

**Lemma 2.21.** The following permanent cycles in \( C_4 \)-HFPSS(\(E_2\)) [HHR17][BBHS20] are periodic classes.

- The class \( \bar{d}_1 \) gives \((1 + \sigma + \lambda)\)-periodicity.
- The class \( u_{8\lambda} \) gives \((16 - 8\lambda)\)-periodicity.
- The class \( u_{4\sigma} \) gives \((4 - 4\sigma)\)-periodicity.
- The class \( u_{4\lambda u_{2\sigma}} \) gives \((10 - 4\lambda - 2\sigma)\)-periodicity.

Since the norm functor is symmetric monoidal, we can apply it to the above three invertible permanent cycles, which gives some \( RO(Q_8) \)-periodicities in \( Q_8 \)-HFPSS(\(E_2\)). The quaternion group \( Q_8 \) has three \( C_4 \) subgroups \( C_4(i) \), \( C_4(j) \) and \( C_4(k) \) generated by \( i, j \) and \( k \) respectively. For each \( C_4 \) copy we have the associated \( C_4 \)-periodicities and their norms give \( RO(Q_8) \)-periodicities as follows.

**Corollary 2.22.** We have the following \( RO(Q_8) \)-periodicities in \( Q_8 \)-HFPSS(\(E_2\)).

\[
\begin{align*}
N_{C_4}^{Q_8}(\bar{d}_1) & : 1 + \sigma_i + \sigma_j + \sigma_k + \mathbb{H} \\
N_{C_4}^{Q_8}(u_{4\sigma}) & : 4 + 4\sigma_i - 4\sigma_j - 4\sigma_k \\
& + 4 + 4\sigma_j - 4\sigma_i - 4\sigma_k \\
& + 4 + 4\sigma_k - 4\sigma_i - 4\sigma_j \\
N_{C_4}^{Q_8}(u_{4\lambda u_{2\sigma}}) & : 10 + 10\sigma_i - 2\sigma_j - 2\sigma_k - 4\mathbb{H} \\
& + 10 + 10\sigma_j - 2\sigma_i - 2\sigma_k - 4\mathbb{H} \\
& + 10 + 10\sigma_k - 2\sigma_j - 2\sigma_i - 4\mathbb{H} \\
N_{C_4}^{Q_8}(u_{8\lambda}) & : 16 + 16\sigma_i - 8\mathbb{H} \\
& + 16 + 16\sigma_j - 8\mathbb{H} \\
& + 16 + 16\sigma_k - 8\mathbb{H}
\end{align*}
\]

**Corollary 2.23.** There are periodicities of \( 4 - 4\sigma_i, 4 - 4\sigma_j \) and \( 4 - 4\sigma_k \) in \( Q_8 \)-HFPSS(\(E_2\)).

**Proof.** It suffices to show that \( 4 - 4\sigma_i \) is a periodicity. This periodicity is given by the following product:

\[
N_{C_4(i)}^{Q_8}(u_{4\lambda u_{2\sigma}})N_{C_4(k)}^{Q_8}(u_{4\lambda u_{2\sigma}})N_{C_4(\bar{d})}^{Q_8}(u_{8\lambda})^{-1}N_{C_4(i)}^{Q_8}(u_{2\sigma})^2N_{C_4(j)}^{Q_8}(u_{2\sigma})^{-1}N_{C_4(k)}^{Q_8}(u_{2\sigma})^{-1}.
\]

\[\square\]

3. \( E_2 \)-page of the \( Q_8 \)-HFPSS(\(E_2\))

In this section, we recollect the computation of the \( E_2 \)-page of the integer-graded \( Q_8 \)-HFPSS for \( E_2 \) by the 2-Bockstein spectral sequence (2-BSS) from [Bea17, Bau08]. Then we compute the \( E_2 \)-page of the \((\ast - \sigma_i)\)-graded part by the same method. By Lemma 2.12 we can compute \( H^*(Q_8, \pi_*, E_2) \), the \( E_2 \)-page of the \( Q_8 \)-HFPSS for \( E_2 \), by first computing \( H^*(Q_8, \mathbb{W}[v_1, u^{-1}]) \).
3.1. 2-BSS, integer-graded. The integer-graded 2-Bockstein spectral sequence for $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$ is

$$H^*(Q_8, \mathbb{F}[v_1, u^{-1}])[h_0] \Longrightarrow H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$$

where $h_0$ detects 2. The computation of the $E_1$-page, $H^*(Q_8, \mathbb{F}[v_1, u^{-1}])$, is from [Bau08, Appendix A]. We follow the notation in [Bea17], except that we use $h_1$ for $\eta$ and $h_2$ for $\nu$. The differentials of this 2-BSS are essentially from [Bau08, Section 7] and we list them in Table 1.

**Proposition 3.1.** The bigradings of generators of $H^*(Q_8, \mathbb{F}[v_1, u^{-1}])$ are:

- $|v_1| = (2, 0), \ |D| = (8, 0), \ |k| = (-4, 4), \ |h_1| = (1, 1), \ |h_2| = (3, 1), \ |x| = (-1, 1), \ |y| = (-1, 1), \ |D^{-1}h_2^2y| = (-3, 3)$.

The relation ($\sim$) is generated by:

1. in filtration 1:
   $$v_1 h_2, \ v_1^2 x, \ v_1 y;$$

2. in filtration 2:
   $$h_1 h_2, \ h_2 x - v_1 h_1 x, \ h_1 y - v_1 x^2, \ xy, \ Dy^2 - h_2^2;$$

3. in filtration 3:
   $$h_1^2 Dx - h_2^2, \ Dx^3 - h_2^2 y, \ D(D^{-1}h_2^2y) - h_2^2 y;$$

4. in filtration 4:
   $$h_1^3 - v_1^4 k.$$

**Proof.** Note that the composition

$$H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}]) \xrightarrow{res} H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}]) \xrightarrow{\iota} H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}])$$

is multiplication by $|G_{24}/Q_8| = 3$ which is a unit in the coefficient $\mathbb{F}_4[v_1, u^{-1}]$. This implies that $H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}])$ is just 3 copies of $H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}])$. The result follows from the computation of the cohomology $H^*(G_{24}, \mathbb{F}_4[v_1, u^{-1}])$ in [Bea17, Thm. A.20].

The differentials in the integer-graded 2-BSS for the cohomology $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$ are essentially from [Bau08, Section 7] which are determined by the ones in Table 1 and the multiplicative structure.

<table>
<thead>
<tr>
<th>$s, f$</th>
<th>$x$</th>
<th>$r$</th>
<th>$d_r(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(4k + 2, 0)$</td>
<td>$v_1^{2k+1}$</td>
<td>1</td>
<td>$2v_1^2 h_1$</td>
</tr>
<tr>
<td>$(7, 1)$</td>
<td>$Dx$</td>
<td>1</td>
<td>$2h_1^2$</td>
</tr>
<tr>
<td>$(-1, 1)$</td>
<td>$x$</td>
<td>1</td>
<td>$2y^2$</td>
</tr>
<tr>
<td>$(-1, 1)$</td>
<td>$y$</td>
<td>1</td>
<td>$2x^2$</td>
</tr>
<tr>
<td>$(4, 0)$</td>
<td>$v_1^2$</td>
<td>2</td>
<td>$4h_2$</td>
</tr>
<tr>
<td>$(5, 3)$</td>
<td>$yh_2^2$</td>
<td>3</td>
<td>$8kD$</td>
</tr>
</tbody>
</table>

The 2-Bockstein computation gives the following result (see also [Bau08, Section 7]).

**Theorem 3.2.** Table 2 and Table 3 present $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$, the $E_\infty$-page of the integer-graded 2-Bockstein spectral sequence (also see Fig. 2 and Fig. 3).

**Remark 3.3.** We note that in $H^*(Q_8, \mathbb{W}[v_1, u^{-1}])$, there is an exotic $h_2$-extension

$$h_2 \cdot x^2 h_2 = 4kD$$
by [Bau08, Equation (7.13)] which is useful in later computations.

Table 2. $E_{\infty}$-page, multiplicative generators, integer-graded

<table>
<thead>
<tr>
<th>$(s, f)$</th>
<th>$x$</th>
<th>2-torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-4, 1)$</td>
<td>$k$</td>
<td>$\mathbb{Z}/8$</td>
</tr>
<tr>
<td>$(-2, 2)$</td>
<td>$x^2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$(0, 2)$</td>
<td>$xh_2$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$(1, 1)$</td>
<td>$h_1$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$(3, 1)$</td>
<td>$h_2$</td>
<td>$\mathbb{Z}/4$</td>
</tr>
<tr>
<td>$(5, 1)$</td>
<td>$v_1^2 h_1$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$(8, 0)$</td>
<td>$D$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$(8, 0)$</td>
<td>$v_1^4$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

Table 3. $E_{\infty}$-page, relation generators, integer-graded

<table>
<thead>
<tr>
<th>$f$</th>
<th>relation generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_1^4 h_2$</td>
</tr>
<tr>
<td>2</td>
<td>$h_1 h_2, v_1^4 h_1, h_2, D y^2 - h_2, xh_1 \cdot v_1^4, x^2 \cdot v_1^4, y^2 \cdot v_1^4$</td>
</tr>
<tr>
<td>3</td>
<td>$xh_1 \cdot h_2, xh_1 \cdot v_1^4 h_1, x^2 \cdot v_1^4 h_1, y^2 h_1, y^2 \cdot v_1^4 h_1, D \cdot xh_1 \cdot h_1 - h_2^3$</td>
</tr>
<tr>
<td>4</td>
<td>$h_1^4 - v_1^4 k, (xh_1)^2, (x^2)^2, (y^2)^2, h_2, x^2 \cdot xh_1, y^2 \cdot xh_1, x^2 \cdot y^2, xh_1 \cdot h_1^2, x^2 \cdot h_1^2, y^2 \cdot h_1^2, h_2^2, x^2 - 4kD, y^2 \cdot h_2, xh_1 \cdot h_2^2$</td>
</tr>
</tbody>
</table>

We refer readers to §6 for charts of the $E_1$-page and the $E_{\infty}$-page.

3.2. 2-BSS, $(\ast - \sigma_i)$-graded.

We discuss the RO$(G)$-graded case and restrict it to the $(\ast - \sigma_i)$-graded case. A variation of Lemma 2.12 still holds in this case. Thus we can compute $H^*(Q_8, \pi_{\ast - \sigma_i} E_2)$ by first computing the $(\ast - \sigma_i)$-graded 2-BSS, and then inverting $D$ and taking the completion. Note that after modulo 2, the representation $\sigma_i$ is oriented and the orientation class $u_{\sigma_i}$ gives an isomorphism between $\pi_{*} E_2/2$ and $\pi_{* + 1 - \sigma_i} E_2/2$ as $Q_8$-modules. Therefore, the $E_1$-page of the $(\ast - \sigma_i)$-graded 2-BSS is abstractly isomorphic to that of the integer-graded part. We denote the $E_1$-page by

$$H^*(Q_8, \mathbb{F}_4[v_1, u^{-1}])\{u_{\sigma_i}\}$$

where $u_{\sigma_i}$ denote a generator of the class at $(1 - \sigma_i, 0)$.

Proposition 3.4. In the 2-BSS, there is a differential

$$d_1(u_{\sigma_i}) = 2xu_{\sigma_i} + 2yu_{\sigma_i}.$$ 

Proof. The group cohomology computation shows that $H^1(Q_8, \pi_{1 - \sigma_i} (E_2))$ is 2-torsion according to Proposition A.7. Hence in the 2-BSS, there must be a $d_1$-differential hit the bigrading $(-\sigma_i, 1)$. Then $u_{\sigma_i}$ in the 2-BSS must support a non-trivial $d_1$-differential by degree reasons. Assume that $d_1(u_{\sigma_i}) = 2axu_{\sigma_i} + 2byu_{\sigma_i}$ where $a, b \in \mathbb{F}_4$. By the Leibniz rule, we have $d_1(v_1 u_{\sigma_i}) = 2h_1 u_{\sigma_i} + 2axv_1 u_{\sigma_i}$. Since $h_1$ is a permanent cycle, the Leibniz rule implies that $h_1 u_{\sigma_i}$ also supports a non-trivial $d_1$-differential. Therefore, the $d_1$-target of $v_1 u_{\sigma_i}$ cannot be $2h_1 u_{\sigma_i}$. We deduce that $a \neq 0$.

Similarly, by considering $d_1(h_2^2 u_{\sigma_i})$ and $d_1(y u_{\sigma_i})$, we deduce that $b \neq 0$. 

$\square$
The remaining \((\ast \cdot \sigma_i)\)-graded 2-BSS \(d_1\)-differentials can be determined by the Leibniz rule and the differential on \(u_{\sigma_i}\), in Proposition 3.4.

**Proposition 3.5.** There is a 2-BSS differential
\[
d_2(xr^2u_{\sigma_i}) = 4kr^2u_{\sigma_i}.
\]

**Proof.** By Example A.5, the class at \((1 - \sigma_i, 4)\) is 4-torsion in the \(E_\infty\)-page. This forces the desired \(d_2\)-differential.

We list non-trivial differentials on classes of the form \{multiplicative generators\}\(u_{\sigma_i}\) in the table below.

<table>
<thead>
<tr>
<th>((s, f))</th>
<th>(x)</th>
<th>(d_2(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1 - \sigma_i, 0))</td>
<td>(u_{\sigma_i})</td>
<td>(2ru_{\sigma_i} + 2gu_{\sigma_i})</td>
</tr>
<tr>
<td>((-\sigma_i, 1))</td>
<td>(xu_{\sigma_i})</td>
<td>(2x^2u_{\sigma_i} + 2y^2u_{\sigma_i})</td>
</tr>
<tr>
<td>((3 - \sigma_i, 0))</td>
<td>(v_1 u_{\sigma_i})</td>
<td>(2ru_{\sigma_i} + 2xv_1 u_{\sigma_i})</td>
</tr>
<tr>
<td>((4 - \sigma_i, 1))</td>
<td>(h_2 u_{\sigma_i})</td>
<td>(2rxh_2 u_{\sigma_i} + 2yh_2 u_{\sigma_i})</td>
</tr>
<tr>
<td>((2 - \sigma_i, 3))</td>
<td>(x^2 y u_{\sigma_i})</td>
<td>(4kr^2 u_{\sigma_i})</td>
</tr>
</tbody>
</table>

**Theorem 3.6.** Table 5 and Table 6 present \(H^*(\mathbb{Q}_8, \mathbb{W}[v_1, w^{-1}])\), the \(E_\infty\)-page of the \((\ast \cdot \sigma_i)\)-graded 2-Bockstein spectral sequence.

**Proof.** The result follows from the 2-BSS computation.

<table>
<thead>
<tr>
<th>((s, f))</th>
<th>(x)</th>
<th>2-torsion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-2, 2))</td>
<td>(x^2 + y^2) (u_{\sigma_i})</td>
<td>(\mathbb{Z}/2)</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>(x + y) (u_{\sigma_i})</td>
<td>(\mathbb{Z}/2)</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>(h_1 + x v_1) (u_{\sigma_i})</td>
<td>(\mathbb{Z}/2)</td>
</tr>
<tr>
<td>((0, 2))</td>
<td>(v_1^2 u_{\sigma_i})</td>
<td>(\mathbb{Z})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(f)</th>
<th>relation generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>({h_1 + x v_1} u_{\sigma_i} \cdot v_1^4 - v_1^4 u_{\sigma_i} \cdot v_1^4 h_1, v_1^4 u_{\sigma_i} \cdot h_2, {x + y} u_{\sigma_i} \cdot v_1^4)</td>
</tr>
<tr>
<td>2</td>
<td>(h_1^2 \cdot x - h_1 + x v_1) (u_{\sigma_i} \cdot x h_1, {x + y} u_{\sigma_i} \cdot v_1^2 h_1, {x^2 + y^2} u_{\sigma_i} \cdot v_1^2)</td>
</tr>
<tr>
<td>3</td>
<td>({h_1 + x v_1} u_{\sigma_i} \cdot x - {x^2 + y^2} u_{\sigma_i} \cdot h_2, {h_1 + x v_1} u_{\sigma_i} \cdot y^2, {x^2 + y^2} u_{\sigma_i} \cdot v_1^2 h_1)</td>
</tr>
<tr>
<td>4</td>
<td>({x^2 + y^2} u_{\sigma_i} \cdot h_1^2, {x^2 + y^2} u_{\sigma_i} \cdot h_2^2, {x^2 + y^2} u_{\sigma_i} \cdot x h_1, {x^2 + y^2} u_{\sigma_i} \cdot x h_1)</td>
</tr>
</tbody>
</table>

We refer the readers to §6 for charts of the \(E_1\)-page and the \(E_\infty\)-page.

By Lemma 2.12, in both the integer-graded and the \((\ast \cdot \sigma_i)\)-graded case, the \(E_2\)-page of \(\mathbb{Q}_8\)-HFPSS(\(E_2\)) follows from Theorem 3.2 and Theorem 3.6.
Remark 3.7. The $E_2$-page of TateSS($E_2$) follows by further inverting the class $k$ from that of $Q_8$-HFPSS($E_2$), and then replacing the 0-line with the cokernel of the norm map.

4. Computation of the integer-graded $Q_8$-HFPSS($E_2$)

In this section, we derive all differentials in the integer-graded $Q_8$-HFPSS for $E_2$ via the following two methods.

1. Equivariant methods: apply the restrictions, transfers, and norms to deduce differentials in the $Q_8$-HFPSS for $E_2$ from the $C_4$-HFPSS for $E_2$;

2. The vanishing line method: use the fact that the $Q_8$-HFPSS for $E_2$ admits a strong vanishing line of filtration 23 (Theorem 4.8, for general cases, see [DLS22, Theorem 6.1]) to force differentials.

We also solve all hidden 2 extensions via equivariant methods and investigation of the Tate spectral sequence.

We will rename several classes on the $E_2$-page of the $Q_8$-HFPSS for $E_2$ as follows. The advantage is that these names are compatible with the $tmf$ computation and the Hurewicz images in $E_{hQ}^2$ (see [Bau08], also compare to [Isa18]). For example, we rename the class $kD^3$ by $g$, which is compatible with [Bau08] and suggests that this class detects the Hurewicz image of $\bar{\kappa}$ (see 4.9). Table 7.

<table>
<thead>
<tr>
<th>Classes</th>
<th>Bauer's notation</th>
<th>Bigrading</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Dxh_1$</td>
<td>$c$</td>
<td>(8, 2)</td>
</tr>
<tr>
<td>$D^2x^2$</td>
<td>$d$</td>
<td>(14, 2)</td>
</tr>
<tr>
<td>$kD^3$</td>
<td>$g$</td>
<td>(20, 4)</td>
</tr>
</tbody>
</table>

When we talk about the restriction map from $Q_8$ to $C_4$, the subgroup $C_4$ usually indicates the subgroup $C_4(\langle i \rangle)$ generated by $i$ if there is no further specification. Some of the arguments in the proofs of this section are easier to see when accompanied by charts in §6.

4.1. General properties of the $Q_8$-HFPSS for $E_{4k+2}$. It is a result of Shi–Wang–Xu, using the Slice Differential Theorem and the norm functor of Hill–Hopkins–Ravenel [HHR16], that the homotopy fixed point spectrum $E_{4k+2}^{hQ_8}$ is $2^{4k+6}$-periodic.

The periodicity of $E_{2}^{hQ_8}$ is known by computation to be 64 classically. Here we give a proof that $E_{2}^{hQ_8}$ is 64-periodic before compute it using $Q_8$-HFPSS.

Proposition 4.1. The homotopy groups of the spectrum $E_2^{hQ_8}$ is 64-periodic and the periodicity class can be given by the class $D^8$.

Proof. The product $$N_{C_4}^{Q_8}(\bar{1})^8N_{C_4(\langle i \rangle)}^{Q_8}(u_{4\tau})^2N_{C_4(\langle j \rangle)}^{Q_8}(u_{8\lambda})N_{C_4(\langle j \rangle)}^{Q_8}(u_{4\eta})^4N_{C_4(\langle k \rangle)}^{Q_8}(u_{4\sigma})^4$$ gives the 64 periodicity of $E_{2}^{hQ_8}$. This product is in bigrading $(64, 0)$ and is invertible. On the other hand, the generator $D^8$ of $\pi_{64}(E_2)$ is $Q_8$-invariant and invertible. Therefore, this periodicity class is $D^8$ up to a unit. $\square$

From now on we can simply view $D^8$ as a periodicity class of $E_2^{hQ_8}$. In the following property, we show that the $Q_8$-HFPSS for $E_2$ can split into three parts such that there are no differentials across different parts.
Note that the universal space $EG_{24}$ can be viewed as a model for $EQ_8$. The transfer and the restriction of the genuine spectrum $F(EG_{24}, E_2)$ give a sequence $E_2^{hG_{24}} \xrightarrow{res} E_2^{hQ_8} \xrightarrow{tr} E_2^{hG_{24}}$, which is compatible with the filtration of the HFPSS.

**Proposition 4.2.** The composition

$$E_2^{hG_{24}} \xrightarrow{res} E_2^{hQ_8} \xrightarrow{tr} E_2^{hG_{24}}$$

is an equivalence. In particular, the $G_{24}$-HFPSS for $E_2$ splits as a summand of the $Q_8$-HFPSS for $E_2$.

**Proof.** The composition $tr \circ res$ is multiplication by $|G_{24}|/|Q_8| = 3$. All spectra are 2-local and 3 is coprime to 2 so this composition is an equivalence. □

We identify the $E_2$-page of $Q_8$-HFPSS($E_2$) as a free module over the $E_2$-page of $G_{24}$-HFPSS($E_2$) generated by $\{1, D, D^2\}$.

**Corollary 4.3.** Let $a, b$ be two classes on the $E_2$-page of $G_{24}$-HFPSS($E_2$). View $a, b$ as classes in $Q_8$-HFPSS($E_2$) and consider classes $aD^{k_a}, bD^{k_b}$ where $k_a, k_b \in \{0, 1, 2\}$. Then there is a differential $d_\sigma a D^{k_a} = b D^{k_b}$ in the $Q_8$-HFPSS($E_2$) iff there is a differential $d_\sigma a = b$ in the $G_{24}$-HFPSS($E_2$) and $k_a = k_b$.

**Proof.** When $k_a = 0$, this follows from Proposition 4.2. For $k_a = 1$, note that the $Q_8$-HFPSS for $E_2$ is $D^k$-periodic by Proposition 4.1. The two differentials

1. $d_\sigma a D = b D^{k_b}$
2. $d_\sigma (a D^9) = b D^{k_b+8}$

imply each other. We observe that the class $D^9$ is a class in $G_{24}$-HFPSS($E_2$). Then by the case $k_a = 0$, the differential (2) happens in $G_{24}$-HFPSS($E_2$). This implies the desired result. The case $k_a = 2$ is similar. □

As a consequence, the computation of the $Q_8$-HFPSS for $E_2$ splits into three copies with the same differential patterns and there are no differentials across different copies. In particular, the $G_{24}$-HFPSS for $E_2$ is 192-periodic.

**Remark 4.4.** A similar statement holds for general heights $4k + 2$. A maximal finite subgroup in $S_{4k+2}$ is $Q_8 \times C_3(2^{2k+1}+1)$ (Hew95) (Buj12, Section 4.3). The computation of the $Q_8$-HFPSS for $E_{4k+2}$ also splits into copies of the computation of the $Q_8 \times C_3(2^{2k+1}+1)$-HFPSS for $E_{4k+2}$.

**Remark 4.5.** The $G_{24}$-HFPSS for $E_2$ computation is essentially the same as the 2-local $tmf$ computation (Bau08). However, our computation only relies on the $C_4$ computation of $E_2$ and hence is an independent computation of the classical $tmf$ computations.

In Theorem 4.8, we will improve the horizontal vanishing line result of the $Q_8$-HFPSS for $E_{4k+2}$ in Theorem 2.10. In the case of the $Q_8$-HFPSS for $E_2$, the improved vanishing line of filtration 23 turns out to be sharp by computation. We start with the following fact.

**Proposition 4.6.** Let $H\mathbb{Z}$ be the Eilenberg-Mac Lane spectrum with trivial $Q_8$-action. Then on the $E_2$-page of $Q_8$-HFPSS($H\mathbb{Z}$), the product $a_\sigma a_\sigma a_\sigma$ is trivial.

**Proof.** We prove a stronger statement that the whole group $H^3(Q_8, \pi_{3-\sigma_i, \sigma_j, \sigma_k}(H\mathbb{Z}))$, where the class $a_\sigma a_\sigma a_\sigma$ lies in, is trivial. According to Proposition A.7, the group $H^3(Q_8, \mathbb{Z})$ is trivial. We observe that the homotopy group $\pi_{3-\sigma_i, \sigma_j, \sigma_k}(H\mathbb{Z})$ as a $Q_8$-module is a copy of $\mathbb{Z}$ with trivial $Q_8$-action ($\sigma_i \otimes \sigma_j \otimes \sigma_k$ is a trivial $Q_8$-representation). Then we have

$$H^3(Q_8, \pi_{3-\sigma_i, \sigma_j, \sigma_k}(H\mathbb{Z})) = (\pi_{3-\sigma_i, \sigma_j, \sigma_k}(H\mathbb{Z}))^{Q_8} \cong \mathbb{Z}.$$
Similarly we also have
\[ H^0(Q_8, \pi_{-3+\sigma_1+\sigma_3+\sigma_2}(H\mathbb{Z})) = (\pi_{-3+\sigma_1+\sigma_3+\sigma_2}(H\mathbb{Z}))^{Q_8} \cong \mathbb{Z}. \]
Let \( u \) be a generator of \( H^0(Q_8, \pi_{-3-\sigma_1-\sigma_3}(H\mathbb{Z})) \). Then the class \( u \) is invertible on the \( E_2 \)-page of HFPSS for \( H\mathbb{Z} \) by the following paring
\[ \pi_{3-\sigma_1-\sigma_3}(H\mathbb{Z}) \otimes \pi_{-3+\sigma_1+\sigma_3+\sigma_2}(H\mathbb{Z}) \cong \mathbb{Z}. \]
Therefore, the class \( u \) induces an isomorphism \( H^3(Q_8, \pi_{3-\sigma_1-\sigma_3}(H\mathbb{Z})) \cong H^3(Q_8, \mathbb{Z}) \), the latter of which is trivial. \( \square \)

**Remark 4.7.** We thanked Guillou for confirming and explaining Proposition 4.6. This proposition also follows from Guillou and Slone’s computation of quaternionic Eilenberg–Mac Lane spectra [GS22].

**Theorem 4.8.** The RO\((Q_8)\)-graded \( Q_8 \)-TateSS for \( E_{4k+2} \) vanishes after \( E_{2k+5-9} \)-page. And the RO\((Q_8)\)-graded \( Q_8 \)-HFPSS for \( E_{4k+2} \) admits a strong vanishing line of filtration \( 2^{4k+5} - 9 \).

**Proof.** Denote the height \( 4k+2 \) by \( h \). We briefly review the proof of the vanishing line of filtration \( 2^{h+3} - 7 \) in [DLS22, Theorem 6.1] and explain the filtration improvement by \( 2 \). By Theorem 2.8, in the \( Q_8 \)-TateSS\((E_h)\), there is a predicted differential
\[ (4.1) \quad d_{2h+3-7}(N_{C_2,Q_8}^{Q_8}((h-1)^{u_{\sigma_2}+1}a_{\sigma_2}^{1-2^{h+1}})(a_0) = 1. \]
By naturality, the unit \( 1 \) has to be hit by a differential \( d_r \) with \( r \leq 2^{h+3} - 7 \). Note that since \( 1 \) is hit, the spectral sequence vanishes at \( E_r \)-page.
The ring map \( \mathbb{Z} \to \pi_*(E_h) \) induces a map between \( E_2 \)-pages of the \( Q_8 \)-HFPSS for \( H\mathbb{Z} \) and \( E_h \). Then the naturality forces the source of Eq. (4.1) is trivial since \( a_0 = a_{\sigma_2}a_{\sigma_1}a_{\sigma_2} = 0 \) by Proposition 4.6. For degree reasons, we conclude \( r \leq 2^{h+3} - 9 \). So every class in the \( Q_8 \)-TateSS\((E_h)\) will disappear on or before the \( E_{2k+3-9} \)-page. Finally by Lemma 2.6 there is a strong vanishing line of filtration \( 2^{4k+5} - 9 \). \( \square \)

**Lemma 4.9.** In the \( Q_8 \)-HFPSS for \( E_2 \), the class \( h_1, h_2, g \) are permanent cycles.

**Proof.** Consider the following maps
\[ S^0 \xrightarrow{\text{unit}} E_2^{hQ_8} \xrightarrow{r_{c_2}} E_2^{hC_2}. \]
By [LSWX19, Theorem 1.8], the class \( \bar{c} \in \pi_{20}S^0 \) maps to a non-trivial class in \( E_2^{hC_2} \) in filtration 4 in the \( C_2 \)-HFPSS for \( E_2 \). Thus the image of \( \bar{c} \) in \( \pi_*(E_2^{hQ_8}) \) is non-trivial. For degree reasons, it is detected by the class \( g \) in \( Q_8 \)-HFPSS\((E_2)\). The proofs for \( h_1, h_2 \) are similar. \( \square \)

We only use the Hurewicz image of \( E_2^{hC_2} \) as the input. This has been systematically studied in [LSWX19]. Our method does not assume the knowledge of the Hurewicz image of \( E_2^{hC_1} \).

**4.2. Differentials in the integer-graded pages.** We suggest readers refer to the charts while reading the proofs in this section.

All statements about differentials in this subsection are differentials in integer-graded \( Q_8 \)-HFPSS\((E_2)\) if there is no specification.

**Proposition 4.10.** The class \( v_0^i \) in \((12,0)\) supports a \( d_3 \)-differential
\[ d_3(v_0^i) = v_1^ih_1^3. \]
As for the classes $2D TateSS$ for $E E$

The result follows by naturality.

\[ \text{Proof.} \]

By construction, we have $\text{res}_{C_4}^Q(v_1^6) = T_2^3$, $\text{res}_{C_4}^Q(h_1) = \eta$. In $C_4$-HFPS$SS(E_2)$, \cite[Proposition 5.21]{BBHS20} implies that we have

\[ d_3(T_2^3) = T_2^2 \eta^3. \]

The result follows by naturality. \hfill \square

\[ \text{Corollary 4.11.} \] The class $v_1^2 h_1$ at $(5, 1)$ supports a $d_3$-differential

\[ d_3(v_1^2 h_1) = h_1^4. \]

\[ \text{Proof.} \]

By Proposition 4.10, we have $d_3(v_1^6 h_1) = v_1^4 h_1^4$. Note that $v_1^4$ is a 3-cycle. This forces the desired $d_3$-differential. \hfill \square

Proposition 4.10 produces a family of $d_3$-differentials by the Leibniz rule:

\[ d_3(D^m g^s v_1^{4l+2} h_1^n) = D^m g^s v_1^{4l+3}, \quad \text{and} \quad d_3(D^m g^s v_1^{4l} h_1^n) = D^m g^s h_1^{n+3} \]

for any $(m, s, l, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

For degree reasons (and the following proposition), these are all the non-trivial $d_3$-differentials.

\[ \text{Proposition 4.12.} \] The following classes survive to the $E_\infty$-page.

\[ 2D^m v_1^{4l+2}, D^m v_1^{4l}, D^m v_1^{4l} h_1, D^m v_1^{4l} h_1^2, \quad (m, l) \in \mathbb{Z} \times \mathbb{Z}_{\geq 1}. \]

\[ \text{Proof.} \]

The classes $D^m v_1^{4l}, D^m v_1^{4l} h_1, D^m v_1^{4l} h_1^2$ cannot be hit by degree reasons. They are permanent cycles by Lemma 2.6 and the $Q_s$-TateSS($E_2$) $d_3$-differentials

\[ d_3(D^{m+3} g^{−1} v_1^{4l−2} h_1^n) = D^m v_1^{4l} h_1^{n+3}, \quad m, l, n \in \mathbb{Z}, \quad l \neq 0. \]

As for the classes $2D^m v_1^{4l+2}$, we consider the additive norm map

\[ H_0(Q_s,(E_2)_s) \xrightarrow{N} H^0(Q_s,(E_2)_s) \]

where $N(x) = \sum_{g \in Q_s} g(x)$. By the $Q_s$-action formulas (Eq. (2.3)), we have

\[ N(v_1^{2l+1} (u−1)^{2l+1}) = \sum_{g \in Q_s} g(v_1^{2l+1} (u−1)^{2l+1}) \]

\[ = 2v_1^{2l+1} (u−1)^{2l+1} + 2 \binom{v_1 + 2u−1}{\zeta^2 - \zeta}^{2l+1} \binom{v_1 - u−1}{\zeta^2 - \zeta}^{2l+1} \]

\[ + 2 \binom{v_1 + 2\zeta^2 u−1}{\zeta^2 - \zeta}^{2l+1} \binom{\zeta v_1 - u−1}{\zeta^2 - \zeta}^{2l+1} + 2 \binom{v_1 + 2\zeta u−1}{\zeta^2 - \zeta}^{2l+1} \binom{\zeta^2 v_1 - u−1}{\zeta^2 - \zeta}^{2l+1}. \]

The leading term of the above formula on the $E_\infty$-page of the 2-BSS for $H^*(Q_s,\mathbb{W}[v_1,u−1])$ is $2v_1^{2l+2}$ for $l \geq 1$. Then we have

\[ N(D^m v_1^{2l+1} (u−1)^{2l+1}) = D^m \sum_{g \in Q_s} g(v_1^{2l+1} (u−1)^{2l+1}) = 2D^m v_1^{4l+2} \]

since $D$ is $Q_s$-invariant. As the additive norm map is the $d_1$-differential on $E_1$-page of the $Q_s$-TateSS for $E_2$, we have the classes $2D^m v_1^{4l+2}$ are permanent cycles who survive to the $E_\infty$-page by Lemma 2.6. \hfill \square

\[ \text{Remark 4.13.} \] All the classes supporting or receiving non-trivial $d_3$-differentials and all classes in Proposition 4.12 are sometimes referred to as the $bo$-pattern. They match the pattern of (many copies of) $\pi_*KO$, the homotopy groups of the real $K$-theory. See \cite[Definition 2.1]{BG18} for more details.
The following result is the first example of the strong vanishing line method (Theorem 4.8). The method gives differentials of three lengths (including the longest \( d_{23} \)-differential) all at once (see Fig. 1).

**Proposition 4.14.** There are differentials

1. \( d_5(D^{-13}gh^3dh_2) = 4D^{-16}gh^7 \);
2. \( d_{13}(D^{-7}gh^4ch_1) = 2D^{-16}gh^7 \);
3. \( d_{23}(D^{-1}gh_1) = D^{-16}gh^7 \).

**Proof.** We observe that in filtration 28 \( \geq 23 \), by multiplying \( 2 \)-extension from 2, \( 2 \)-HFPSS splits into three parts. On the \( E_2 \)-page, these three parts are modules over the \( E_2 \)-page of \( G_{24} \)-HFPSS and all differentials do not cross different copies. In Fig. 1, we highlight the relevant copy. By inspection, we obtain the desired \( d_5 \), \( d_{13} \) and \( d_{23} \)-differentials. \( \square \)

**Corollary 4.15.** The class \( D \) at \((8,0)\) supports a \( d_5 \)-differential

\[ d_5(D) = D^{-2}gh_2. \]

**Proof.** Note that \( D^8 \) is an invertible permanent cycle (Proposition 4.1), and \( g^5 \) is a permanent cycle (Lemma 4.9). By Proposition 4.14(1) and the Leibniz rule, there is a \( d_5 \)-differential

\[ d_5(D^3dh_2) = 4g^2. \]

The relation \( dh_2^2 = 4g \) (see Remark 3.3 under 2BSS names) forces the following \( d_5 \)-differential

\[ d_5(D^3d) = gdh_2. \]

With Eq. (4.3), it suffices to show \( D^2d \) is a 5-cycle. In fact, the only possible \( d_5 \) target of \( D^2d \) supports a differential

\[ d_5(D^{-1}gdh_2) = 4D^{-4}g^3, \]

by multiplying \( D^{-4}gh_2 \) with Eq. (4.3). Note that \( D^{-4} \) is a 5-cycle since \( D \) is a 3-cycle. \( \square \)

All the remaining \( d_5 \)-differentials follow from the Leibniz rule. There are no more \( d_5 \)-differentials by degree reasons and Corollary 4.3.

We also get a \( d_9 \)-differential from the \( d_{13} \)-differential in Proposition 4.14(2).

**Corollary 4.16.** The class \( Dc \) at \((16,2)\) supports a \( d_9 \)-differential

\[ d_9(Dc) = D^{-5}gh^4dh_1. \]

**Proof.** We observe that in \( Q_8 \)-HFPSS the \( h_1 \)-extension from \( Dc \) to \( Dch_1 \). We proof by contradiction. Suppose that \( Dc \) does not support the claimed \( d_9 \)-differential. Then for degree reasons, \( Dc \) becomes a 13-cycle. However, this contradicts Proposition 4.14 since \( Dch_1 \) supports a non-trivial \( d_{13} \)-differential. \( \square \)

**Proposition 4.17.** The classes \( 4D \) and \( 2D^2 \) at \((16,0)\) support the following \( d_7 \)-differentials

1. \( d_7(4D) = D^{-2}gh_1^3 \);
2. \( d_7(2D^2) = D^{-1}gh_1^3 \).

**Proof.** By Corollary 4.15 and the hidden 2 extension from \( 2h_2 \) to \( h_1^3 \) (see [Tod62]), \( D^{-2}gh_1^3 \) has to be hit by a differential. For degree reasons and Corollary 4.3, the only possible source is either \( 4D \). The second \( d_7 \)-differential follows similarly from \( d_5(D^2) = 2D^{-1}gh_2 \). \( \square \)
The $d_7$-differential on $D^4$ (which we prove in Proposition 4.28) turns out to be a hard one, as it does not follow from primary relations like the Leibniz rule or (hidden) extensions. We will first prove several $d_9,d_{11}$-differentials, and then the $d_7$-differential follows from the vanishing line method.

**Proposition 4.18.** The class $D^5ch_1$ at $(49,3)$ and the class $D^5c$ at $(48,2)$ supports the following differentials.

1. $d_{13}(D^5ch_1) = 2D^{-4}g^4$;
2. $d_9(D^5c) = D^{-1}g^2dh_1$.

**Proof.** By a similar argument as in Corollary 4.16, it is enough to show the (1). We first observe that the class $2D^{-4}g^4$ is in the image of the transfer map from $C_4$-HFPSS($E_2$) since

$$\text{tr} \circ \text{res}(D^{-4}g^4) = [Q_8 : C_4]D^{-4}g^4 = 2D^{-4}g^4.$$
According to [BBHS20, Proposition 5.28], the class \( \text{res}(D^{-4}g^4) \) receives a \( d_{13} \)-differential in \( C_4 \)-HFPSS(\( E_2 \)). The naturality forces that \( 2D^{-4}g^4 \) dies on or before the \( E_{13} \)-page in \( Q_8 \)-HFPSS(\( E_2 \)). The only possibility is the desired \( d_{13} \)-differential by Corollary 4.3 and degree reasons.

**Remark 4.19.** Since \( C_4 \)-HFPSS(\( E_2 \)) is 32-periodic with the periodicity class \( \Delta_1^4 = \delta^4_{15}u_8\lambda u_8\sigma \) [HHR17][BBHS20], the same argument in the proof of Proposition 4.18 gives an alternative proof of Proposition 4.14(2) and Corollary 4.16.

**Lemma 4.20.** The class \( D^3h_1 \) is a permanent cycle.

**Proof.** By Corollary 4.3, it suffices to show \( D^3h_1 \) is a permanent cycle in \( G_{24} \)-HFPSS(\( E_2 \)). For degree reasons, \( D^3h_1 \) can only possibly hit \( D^{-3}g \) or \( 2D^{-12}g^6 \) in \( G_{24} \)-HFPSS(\( E_2 \)). Because \( D^{-8}g \), \( g \) are permanent cycles, Proposition 4.18 implies

\[
d_{13}(D^{-3}g^2ch_1) = 2D^{-12}g^6 \quad \text{and} \quad d_9(D^{-3}g) = D^{-9}g^3dh_1.
\]

Therefore, the class \( D^3h_1 \) has to be a permanent cycle.

**Remark 4.21.** It turns out that \( D^3h_1 \) is hit by a \( d_{23} \)-differential in the Tate spectral sequence by Corollary 4.22.

**Corollary 4.22.** There are non-trivial \( d_{23} \)-differentials

1. \( d_{23}(D^2h_2^2) = D^{-13}g^9h_1 \);
2. \( d_{23}(D^5h_1^3) = D^{-10}g^6h_1^2 \).

**Proof.** The claimed \( d_{23} \)-differentials follow from Proposition 4.14(3) and Lemma 4.20.

We write \( m \doteq n \) if \( m = ln \) for some \( l \in \mathbb{W}(F_4)^\times \).

**Lemma 4.23.** There is a hidden 2 extension from \( D^6h_2^2 \) to \( g^2d \).

**Proof.** According to Lemma 2.18, there is a hidden 2 extension in stem 54 from \( \Delta_1^4\delta^4_{15}u_{4\lambda}u_{4\sigma}a_{2\sigma} \) to \( \Delta_1^4\delta^4_{15}u_{4\lambda}u_{4\sigma}a_{2\sigma} \) in the \( C_4 \)-HFPSS(\( E_2 \)) since it is \( \Delta_1^4 \)-periodic. Note that the restriction of \( D \) to the \( E_2 \)-page of the \( C_4 \)-HFPSS(\( E_2 \)) is invertible then it equals \( \Delta_1 \) up to a unit, i.e., \( \text{res}^Q_{C_4}(D) = \Delta_1 \). In Appendix A we show that the restriction of the classes \( h_2, d \) and \( g \) are non-trivial. Then in stem 54 of \( Q_8 \)-HFPSS(\( E_2 \)), we have the following two restrictions

\[
\text{res}^Q_{C_4}(D^6h_2^2) = \Delta_1^4\delta^4_{15}u_{4\lambda}u_{4\sigma}a_{2\sigma},
\]

\[
\text{res}^Q_{C_4}(g^2d) = \Delta_1^4\delta^4_{15}u_{4\lambda}u_{4\sigma}a_{4\lambda}a_{2\sigma}.
\]

Note that in \( G_{24} \)-HFPSS(\( E_2 \)), there are no other classes between these two filtrations. Then the naturality forces a hidden 2 extension from \( D^6h_2^2 \) to \( g^2d \) in \( G_{24} \)-HFPSS(\( E_2 \)). This hidden 2 extension also happens in \( Q_8 \)-HFPSS(\( E_2 \)) by Corollary 4.3.

As the \( C_4 \)-HFPSS for \( E_2 \) is 32-periodic, a similar proof gives the following hidden 2 extension in stem 22 in the \( Q_8 \)-HFPSS for \( E_2 \).

**Corollary 4.24.** There is a hidden 2 extension from \( D^2h_2^2 \) to \( D^{-4}g^2d \).

**Proposition 4.25.** The classes \( 2Dh_2 \) at \((11, 1)\) and \( 2D^5h_2 \) at \((43, 1)\) support \( d_{13} \)-differentials

1. \( d_{13}(2Dh_2) = D^{-8}g^3d \);
2. \( d_{13}(2D^5h_2) = D^{-4}g^9d \).
Proof. (1) By Lemma 4.23 and the $E_{\infty}$-page class $g$, there is a hidden 2 extension from $D^{-2}gh^2_2$ to $D^{-8}g^3d$ in stem 10 of the $Q_8$-HFPSS for $E_2$. By Corollary 4.15, we have
$$d_5(Dh_2) = D^{-2}gh^2_2.$$ Then the hidden 2 extension forces the desired differential.

(2) It follows similarly from the hidden 2 extension from $D^2g^2h^2_2$ to $D^{-4}g^3d$ by Corollary 4.24.

**Remark 4.26.** In Bauer’s computation for $tmf$ [Bau08], the hidden 2 extension in Lemma 4.23 is proved using four-fold Toda brackets. In our approach, the hidden 2 extension follows from the restriction and the $C_4$-HFPSS hidden 2 extension, which again is forced by the exotic restrictions and transfers in Lemma 2.18.

**Lemma 4.27.** The class $Dh_3^2$ is a permanent cycle.

**Proof.** The class $Dh_3^2$ is a cycle. By Corollary 4.3 and degree reasons, $Dh_3^2$ can only possibly hit $D^{-5}g^3d$ and $D^{-14}g^6h^2_1$, of which the former is hit by a $d_1$-differential by Proposition 4.27 and the latter supports a $d_{23}$-differential by Corollary 4.22 ($D^{-14}g^6h^2_1 = D^{-10}g^6D^2h^2_1$). The result thereby follows.

**Proposition 4.28.** The class $D^4$ at (32, 0) supports a $d_7$-differential
$$d_7(D^4) = Dh^3_1.$$  

**Proof.** Note that $g$ and $D^{-8}$ are permanent cycles. Then by Lemma 4.27 the class $D^{-15}h^3_1g^6$ at (3, 27) is also a permanent cycle. This class has to be hit by a differential via the vanishing line method (Theorem 4.8). By Corollary 4.3, the potential source is either $D^{-3}gc$ or $D^{-12}g^5$. The former supports a $d_9$ by Proposition 4.18. Therefore, the only possibility is the $d_7$-differential $d_7(D^{-12}g^5) = D^{-15}h^3_1g^6$. Since $D^{-8}g^5$ is a permanent cycle, the result follows.

All $d_7$-differentials follow from Proposition 4.17, Proposition 4.28 and the Leibniz rule.

Before proving the next two $d_9$-differentials in Corollary 4.32, we need to first prove a permanent cycle in Lemma 4.29 and two $d_{11}$-differentials in Proposition 4.30.

**Lemma 4.29.** The class $D^3dh_1$ is a permanent cycle.

**Proof.** By Corollary 4.16 in the $Q_8$-TateSS for $E_2$, we have a $d_9$-differential
$$d_9(D^5g^{-2}c) = D^3dh_1.$$ Then $D^3dh_1$ is a permanent cycle in the $Q_8$-TateSS. By Lemma 2.6 it is also a permanent cycle in $Q_8$-HFPSS($E_2$).

**Proposition 4.30.** The classes $D^2d$ at (30, 2) and $D^6d$ at (62, 2) support $d_{11}$-differentials

1. $d_{11}(D^2d) = D^{-4}g^3h_1$;
2. $d_{11}(D^6d) = g^1h_1$.

**Proof.** According to Proposition A.7, the restriction of the class $d$ from $Q_8$-HFPSS($E_2$) to $C_4$-HFPSS($E_2$) is non-trivial, and supports a non-trivial $d_{13}$-differential by [BBHS20, Proposition 5.28]. This implies the class $D^2d$ supports a non-trivial differential with a length at most 13. The desired differential in (1) follows by degree reasons. The proof for (2) is similar since $C_4$-HFPSS($E_2$) is 32-periodic.

**Corollary 4.31.** The classes $D^2dh_1$ at (31, 3) and $D^6dh_1$ at (63, 3) support $d_{11}$-differentials
(1) \(d_{11}(D^2dh_1) = D^{-4}g^3h_1^2\);
(2) \(d_{11}(D^5dh_1) = g^3h_1^2\).

**Corollary 4.32.** The classes \(Dh_1\) at \((9,1)\) and \(D^5h_1\) at \((41,1)\) support \(d_9\)-differentials

(1) \(d_9(Dh_1) = D^{-5}g^2c;\)
(2) \(d_9(D^5h_1) = D^{-1}g^2c.\)

**Proof.** By Corollary 4.3 and degree reasons, the class \(Dh_1\) either supports a non-trivial \(d_9\)-differential or is an 11-cycle. We show it is the first case.

If \(Dh_1\) were a 11-cycle then by Proposition 4.30 and the Leibniz rule, there would be a \(d_{11}\)-differential
\[d_{11}(D^5dh_1) = D^{-3}g^3h_1^2,\]
This contradicts Lemma 4.29. Therefore, we have the desired \(d_9\)-differential in (1). The proof for (2) is similar. \(\square\)

**Proposition 4.33.** The class \(D^{-1}h_1\) is a 13-cycle.

**Proof.** Since \(D^9\) is the periodic class, it suffices to prove that \(D^7h_1\) is a 13-cycle. The \(D^7h_1\) is a 7-cycle from our computation of \(E_9\)-page. It cannot support a \(d_9\)-differential since the possible target \(g^2Dc\) supports a differential by Corollary 4.16. Then for degree reasons, \(D^7h_1\) is a 13-cycle. \(\square\)

**Corollary 4.34.** The classes \(D^2c\) at \((24,2)\) and \(D^6c\) at \((56,2)\) support \(d_9\)-differentials

(1) \(d_9(D^2c) = D^{-4}g^2dh_1;\)
(2) \(d_9(D^6c) = g^2dh_1.\)

**Proof.** Suppose \(D^2c\) doesn’t support a non-trivial \(d_9\)-differential. Then for degree reasons, it is a 13-cycle. However, since \(D^{-1}h_1\) is also a 13-cycle, the Leibniz rule show that \(Dh_1c\) is also a 13-cycle. This contradicts Proposition 4.14 and proves the \(d_9\)-differential in (1). The \(d_9\)-differential in (2) follows similarly by Proposition 4.18. \(\square\)

**Corollary 4.35.** The classes \(Ddh_1\) at \((23,3)\) and \(D^5dh_1\) at \((55,3)\) support \(d_{11}\)-differentials

(1) \(d_{11}(Ddh_1) = D^{-5}g^3h_1^2;\)
(2) \(d_{11}(D^5dh_1) = D^{-1}g^3h_1^2.\)

**Proof.** According to Proposition 4.33, the class \(D^{-1}h_1\) is a 13-cycle. Then these two \(d_{11}\)-differentials follow by Proposition 4.30 and the Leibniz rule. \(\square\)

**Lemma 4.36.** The class \(d\) is a permanent cycle.

**Proof.** Proposition 4.25 shows \(d\) is hit by a \(d_{11}\)-differential from \(2D^9g^{-3}h_2\) in \(Q_8\)-TateSS\((E_2)\). By Lemma 2.6 \(d\) is a permanent cycle. \(\square\)

**Remark 4.37.** In fact, the class \(d\) is in the image of the Hurewicz map \(S^0 \to E_2^{hQ_8}\). This follows from the Hurewicz image of \(E_2^{hC_4}\) [HSWX18, Figure 12] (see Proposition A.8).

**Proposition 4.38.** The classes \(D^2h_1\) at \((17,1)\) and \(D^6h_1\) at \((49,1)\) support \(d_9\)-differentials

(1) \(d_9(D^2h_1) = g^2D^{-4}c;\)
(2) \(d_9(D^6h_1) = g^2c.\)

**Proof.** We prove by contradiction. Assume \(D^2h_1\) does not support the desired differential. Then it is a 11-cycle by degree reasons. The Leibniz rule forces the class \(Dh_1\) to support a non-trivial \(d_{11}\)-differential but this contradicts Lemma 4.36. The proof of (2) is similar. \(\square\)
Table 8 lists the differentials we have computed so far. They generate differentials via the Leibniz rule. By inspection, these are all non-trivial differentials since the remaining classes are permanent cycles by Method 2.7.

4.3. Extension problem. Now we solve all the 2-extensions on the $E_{\infty}$-page.

**Theorem 4.39.** All the hidden 2 extensions in the integer-graded $G_{24}$-HFSS($E_2$) are displayed in Fig. 10 by gray vertical lines.

**Proof.** Since the $G_{24}$-HFSS for $E_2$ is 192-periodic, it suffices to consider the stem range from 0 to 192. We divide these 2 extensions into three types by their proofs. The first type follows from the fact that in homotopy groups of spheres $4\nu = \eta^3$ and $h_1$ detects $\eta$, $h_2$ detects $\nu$ (Lemma 4.9). This type of hidden 2 extensions happens in stem 3, 27, 51, 99, 123 and 147 in the period from 0 to 192.

The second type consists of the 2 extensions in stem 54 and 150. The proof of the first is in Lemma 4.23, and proof of the second is similar using the 32-periodicity of $C_2$-HFSS($E_2$) and Lemma 2.18.

The third type consists of three hidden 2 extensions in the first period. The first one is in stem 110 from $D^{12}d$ to $D^6g^3h_1^2$. The other two in stem 130 and 150 (from filtration 10 to 22) follow from the first one by multiplying $g$ and $g^2$ respectively.

In $G_{24}$-TateSS($E_2$) we have the following two differentials by Proposition 4.25 and Corollary 4.22.

$$d_{13}(2D^{21}g^{-3}h_2) = D^{12}d,$$

$$d_{23}(D^{21}g^{-3}h_1^3) = D^6g^3h_1^2.$$

Now consider the cofibration

$$(E_2)_{hG_{24}} \rightarrow E_2^{hG_{24}} \rightarrow E_2^{[G_{24}]}$$

In the negative filtrations in $G_{24}$-TateSS($E_2$), there is a hidden 2 extension from $2D^{21}g^{-3}h_2$ to $D^{21}g^{-3}h_1^2$, then this hidden 2 extension under the additive norm map gives a 2 extension relation in $\pi_*(E_2)^{hG_{24}}$ from an element detected by $D^{12}d$ to some element detected by $D^6g^3h_1^2$. This forces a hidden 2 extension from $D^{12}d$ to $D^6g^3h_1^2$ in $G_{24}$-HFSS($E_2$).

We claim there are no further 2 extensions in $G_{24}$-HFSS($E_2$). By degree reasons, the other possible hidden 2 extensions either have sources that are $h_1$ divisible or have targets that support $h_1$ extensions. Therefore, the hidden 2 extensions cannot happen in these cases. \[\square\]

**Corollary 4.40.** All the hidden 2 extensions in the integer-graded $Q_{18}$-HFSS($E_2$) are displayed in Fig. 9 by gray vertical lines.

**Proof.** This follows from Theorem 4.39 and Proposition 4.2. \[\square\]

Our result of 2 extensions via the equivariant and the Tate methods matches the tmf computation in [Bau08]. In [Bau08], because his arguments for proving differentials rely on (hidden) $\eta$ and $\nu$ extensions, almost all these hidden extension are also computed (there are another $\nu$ extension from $D^{15}h_1^2$ at $122, 2$ and its $\bar{\kappa}$ multiples [Isa09, Lemma 5.3]). Here our new methods only use hidden 2 extensions and the $h_1$, $h_2$ multiplications on the $E_2$-page. Therefore, we do not need to work out hidden $\eta$ and $\nu$ extensions and in our figures we only draw $h_1$, $h_2$ multiplications.

4.4. Differentials: alternative methods. In this subsection, we revisit several differentials in the integer-graded part via different approaches.

**Proposition 4.41.** The class $D$ at $(8, 0)$ supports a $d_5$-differential

$$d_5(D) = D^{-2}gh_2.$$
Proof. The restriction of $D$ to the $C_4$-HFPSS for $E_2$ is $\Delta_1$, which supports a non-trivial $d_5$-differential according to [BBHS20, Proposition 5.24]. By naturality, $D$ must support a non-trivial differential with length $\leq 5$. Then by Corollary 4.3 and degree reasons, it has to be $d_5(D) = D^{-2}gh_2$. \hfill \square

Moreover, given all $d_5, d_7$-differentials, then the vanishing line forces the $d_{11}$-differential in Proposition 4.30.

**Proposition 4.42.** The class $D^6d$ at (62, 2) supports a $d_{11}$-differential

$$d_{11}(D^6d) = g^3h_1.$$  

Proof. It is enough to prove the $d_{11}$-differential

$$d_{11}(D^6g^5dh_1) = g^8h_1^2$$

since $g$ is invertible in the $Q_8$-TateSS for $E_2$. The target $g^8h_1^2$ is a permanent cycle in filtration $34 \geq 23$. By Theorem 4.8 and Theorem 2.9 it has to be hit by a differential. Since $D^6g^5dh_1$ is a 7-cycle, the only possibility is the desired $d_{11}$-differential. \hfill \square

We here present another proof of the $d_9$-differential in Proposition 4.38 which combines the partial calculations in $(\ast-\sigma)$-gradings by the norm method (see Proposition 5.16).

**Proposition 4.43.** The class $D^2h_1$ at (17, 1) supports a $d_9$-differential

$$d_9(D^2h_1) = D^{-4}g^2c.$$  

Proof. Suppose the claimed $d_9$-differential doesn’t happen, then $D^2h_1$ is a 9-cycle. According to Lemma 5.6, the class $(x+y)D^4u_\sigma$, is a 9-cycle. Then the Leibniz rule implies that $(x+y)D^6h_1u_\sigma$, is also a 9-cycle. This contradicts the fact that $(x+y)D^6h_1u_\sigma$, supports a non-trivial $d_9$-differential in Proposition 5.16. \hfill \square

4.5. **Summary of differentials.** We summarize differentials in Table 8. All differentials follow from this list by the Leibniz rule.

<table>
<thead>
<tr>
<th>top rule $(s, f)$</th>
<th>$x$</th>
<th>$r$</th>
<th>$d_\nu(x)$</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>(12, 0)</td>
<td>$v_i^6$</td>
<td>3</td>
<td>$v_i^4h_1^3$</td>
<td>Proposition 4.10 (restriction)</td>
</tr>
<tr>
<td>(8, 0)</td>
<td>$D$</td>
<td>5</td>
<td>$D^{-2}gh_2$</td>
<td>Corollary 4.15 (vanishing line) or Proposition 4.41 (restriction)</td>
</tr>
<tr>
<td>(8, 0)</td>
<td>$4D$</td>
<td>7</td>
<td>$D^{-2}gh_1^3$</td>
<td>Proposition 4.17 ($8\nu = \eta^3$)</td>
</tr>
<tr>
<td>(16, 0)</td>
<td>$2D^2$</td>
<td>7</td>
<td>$D^{-1}gh_1^3$</td>
<td>Proposition 4.17</td>
</tr>
<tr>
<td>(32, 0)</td>
<td>$D^4$</td>
<td>7</td>
<td>$Dgh_1^3$</td>
<td>Proposition 4.28 (vanishing line)</td>
</tr>
<tr>
<td>(9, 1)</td>
<td>$Dh_1$</td>
<td>9</td>
<td>$D^{-5}g^2c$</td>
<td>Corollary 4.32</td>
</tr>
<tr>
<td>(41, 1)</td>
<td>$D^5h_1$</td>
<td>9</td>
<td>$D^{-1}g^2c$</td>
<td>Corollary 4.32</td>
</tr>
<tr>
<td>(16, 2)</td>
<td>$Dc$</td>
<td>9</td>
<td>$D^{-5}g^2dh_1$</td>
<td>Corollary 4.16</td>
</tr>
<tr>
<td>(48, 2)</td>
<td>$D^5c$</td>
<td>9</td>
<td>$D^{-1}g^2dh_1$</td>
<td>Proposition 4.18</td>
</tr>
<tr>
<td>(17, 1)</td>
<td>$D^2h_1$</td>
<td>9</td>
<td>$D^{-4}g^2c$</td>
<td>Proposition 4.38</td>
</tr>
<tr>
<td>(49, 1)</td>
<td>$D^6h_1$</td>
<td>9</td>
<td>$g^2c$</td>
<td>Proposition 4.38</td>
</tr>
<tr>
<td>(24, 2)</td>
<td>$D^2c$</td>
<td>9</td>
<td>$D^{-4}g^2dh_1$</td>
<td>Corollary 4.34</td>
</tr>
<tr>
<td>(56, 2)</td>
<td>$D^6c$</td>
<td>9</td>
<td>$g^2dh_1$</td>
<td>Corollary 4.34</td>
</tr>
<tr>
<td>(30, 2)</td>
<td>$D^2d$</td>
<td>11</td>
<td>$D^{-4}g^3h_1$</td>
<td>Proposition 4.30 (restriction)</td>
</tr>
<tr>
<td>(62, 2)</td>
<td>$D^6d$</td>
<td>11</td>
<td>$g^3h_1$</td>
<td>Proposition 4.30 (restriction)</td>
</tr>
</tbody>
</table>
5. The \((\ast - \sigma_i)\)-Graded Computation

In this section, we compute the \((\ast - \sigma_i)\)-graded \(Q_8\)-HFPSS for \(E_2\). We adapt the following convention: a class at \((n - \sigma, m)\) will be denoted as in degree \((n - 1, m)\). Since the \(Q_8\)-representation \(\sigma_i\) cannot be lifted to \(G_{24}\), in this section, we only consider the groups \(Q_8\) and \(SD_{16}\). We name classes by their names in the 2-BSS in Table 5, and also use 2-BSS names for the integer-graded classes as it makes the multiplication relation clearer.

**Proposition 5.1.** The class \(v_1^2u_{\sigma_i}\) at \((4, 0)\) supports a \(d_3\)-differential

\[
d_3(v_1^2u_{\sigma_i}) = h_1^3u_{\sigma_i}.
\]

**Proof.** We consider the restriction map from \((\ast - \sigma_i)\)-graded \(Q_8\)-HFPSS\((E_2)\) to the integer-graded \(C_4\)-HFPSS\((E_2)\). Note that the \(C_4\)-invariant element \(T_2 \in H^0(C_4, v_1E_2)\) equals \(v_1^2\) modulo 2. This implies \(\text{res}_{C_4}^{-}(v_1^2u_{\sigma_i}) = T_2\). Recall that in the \(C_4\)-HFPSS for \(E_2\), the class \(T_2\) supports a non-trivial \(d_3\)-differential [\([BBHS20, Proposition 5.21]\)]. The class \(v_1^2u_{\sigma_i}\) must support a non-trivial differential of length \(\leq 3\). By degree reasons, we have

\[
d_3(v_1^2u_{\sigma_i}) = h_1^3u_{\sigma_i}.
\]

Since the \((\ast - \sigma_i)\)-graded part is a module over the integer-graded part, this \(d_3\)-differential implies a family of \(d_3\)-differentials as follows:

\[
d_3(k^4D^m_1v_1^{4l+2}h_1^n u_{\sigma_i}) = k^4D^m_1v_1^{4l+3} u_{\sigma_i},
\]

where \(k, m, l, n \in \mathbb{Z}\) and \(n \geq 0\). By taking out these \(d_3\)-differentials, an argument similar to the proof in Proposition 4.12 shows that the following classes are permanent cycles

\[
2D^m_1v_1^{4l-2}, D^m_1v_1^{4l}, D^m_1v_1^{4l}h_1, D^m_1v_1^{4l}h_1^2
\]

where \(l \geq 1\) and \(m \geq 0\). All the classes above either support non-trivial \(d_3\)-differentials or are permanent cycles. Similar to the \(bo\)-pattern in the integer graded part, we do not need to consider this part in later computations of higher differentials.

However, this is not the only kind of \(d_3\)-differentials in \((\ast - \sigma_i)\)-graded part. In order to derive the second kind of \(d_3\)-differentials, we first need to show the \(d_3\)-differential pattern and several other facts.

**Lemma 5.2.** The class \(\{x + y\}u_{\sigma_i}\) is a permanent cycle.
Proof. For degree reasons, this class is $a_{\sigma}$, on the $E_2$-page defined in Definition 2.2. By Proposition 2.3, this class is a permanent cycle.

Corollary 5.3. The class $\{x+y\} D u_{\sigma}$, at (7,1) supports a $d_5$-differential
\[ d_5(\{x+y\} D u_{\sigma}) = k\{yh_2 + xh_1v_1\} D u_{\sigma}. \]

Proof. Since the $(* - \sigma)$-graded part is a module over the integer-graded part, the claimed differential follows from Lemma 5.2, Corollary 5.15 and the Leibniz rule.

Corollary 5.3 generates the first kind of $d_5$-differentials via the Leibniz rule.

Lemma 5.4. The class $\{x^2 + y^2\} D u_{\sigma}$ is a permanent cycle.

Proof. According to Proposition 4.38, there is a $d_5$-differential $d_5(D^5h_1) = k^2 D^7 x h_1$. Then the Leibniz rule implies that the class $k^2 x^2 h_1 D^7 u_{\sigma} = k^2 D^7 x h_1 \cdot \{x+y\} u_{\sigma}$ is hit by a differential of length $\leq 9$. For degree reasons, it is hit by either a $d_5$-differential or a $d_7$-differential. In either case, the degree reasons force $k^2 \{x^2 + y^2\} D^7 u_{\sigma}$ to be hit on or before the $E_9$-page. Then the class $\{x^2 + y^2\} D u_{\sigma}$ must be a permanent cycle; otherwise the class $k^3 \{x^2 + y^2\} D^7 u_{\sigma}$ would support a non-trivial differential since
\[ k^2 \{x^2 + y^2\} D^7 u_{\sigma} = \{x^2 + y^2\} D u_{\sigma} \cdot k^2 D^6 \]
where $k^2 D^6 = g^2$ is a permanent cycle that survives to $E_{\infty}$-page in the integer-graded part.

Corollary 5.5. The class $\{x^2 + y^2\} u_{\sigma}$, at $(-2,2)$ supports a $d_5$-differential
\[ d_5(\{x^2 + y^2\} u_{\sigma}) = k\{x+y\} h^2_7 u_{\sigma}. \]

All $d_5$-differentials in $(* - \sigma)$-graded part follows from Corollary 5.3 and Corollary 5.5 by the Leibniz rule.

Lemma 5.6. The class $\{x+y\} D^4 u_{\sigma}$ is a 11-cycle.

Proof. According to [BBHS20, Remark 5.23], the class $u_{\sigma}$ is a 5-cycle in the $C_4$-HFPSS for $E_2$. Therefore, Theorem 2.8 implies that $N_{C_4}^Q(u_{\sigma}) a_{\sigma}$ is a 9-cycle. Because the norm functor is symmetric monoidal [HHR16] and $u_{\sigma}$ is an invertible class on the $E_2$-page of the $C_4$-HFPSS for $E_2$, the class $N_{C_4}^Q(u_{\sigma})$ in the $Q_4$-HFPSS for $E_2$ is also invertible on the $E_2$-page. Hence $N_{C_4}^Q(u_{\sigma}) \cdot a_{\sigma}$ is non-trivial on the $E_2$-page. By multiplying $N_{C_4}^Q(u_{\sigma}) \cdot a_{\sigma}$, with the periodicity classes in Corollary 2.22, we get a non-trivial class at $(31,1)$. For degree reasons, this class must be $\{x+y\} D^4 u_{\sigma}$ (up to a unit). This implies that $\{x+y\} D^4 u_{\sigma}$ is also a 9-cycle. For degree reasons, $\{x+y\} D^4 u_{\sigma}$ is a 11-cycle.

Remark 5.7. We will show in Proposition 5.20 that the above class supports a non-trivial $d_{13}$-differential.

Corollary 5.8. The classes $x^3 u_{\sigma}$, at $(-3,3)$ and $x^3 D^4 u_{\sigma}$, at $(29,3)$ support $d_{11}$-differentials
\[ (1) \quad d_{11}(x^3 u_{\sigma}) = k^3 \{x+y\} D h_1 u_{\sigma}; \]
\[ (2) \quad d_{11}(x^3 D^4 u_{\sigma}) = k^4 \{x+y\} D^3 h_1 u_{\sigma}. \]

Proof. According to Proposition 4.30, there is a $d_{11}$-differential in the integer-gradings
\[ d_{11}(x^2) = k^3 D h_1. \]

Note that $\{x+y\} u_{\sigma}$ and $\{x+y\} D^4 u_{\sigma}$ are both 11-cycles. By the Leibniz rule, we have
\[ d_{11}(x^3 u_{\sigma}) = \{x+y\} u_{\sigma} d_{11}(x^2) = k^3 \{x+y\} D h_1 u_{\sigma}. \]

The proof of the second $d_{11}$-differential is similar.

□
Proposition 5.9. The class $\{h_1 + xv_1\}u_{\sigma_i}$ at $(1, 1)$ supports a $d_3$-differential

$$d_3(\{h_1 + xv_1\}u_{\sigma_i}) = 2kv^2_iu_{\sigma_i}.$$

Proof. We argue by contradiction. Suppose this differential doesn’t happen. By Corollary 5.8 the class $k^2\{x + y\}D^2h_1u_{\sigma_i}$ must be hit by a differential of length $\leq 11$. Because $kD^3 = g$ is a permanent cycle, the class $k^2\{x + y\}D^2u_{\sigma_i}$, has to be hit by a differential of length $\leq 11$ in the $Q_8$-TateSS for $E_2$. So is the class $k^2\{x + y\}D^2h_1^2u_{\sigma_i}$. For degree reasons, the class $k^2\{x + y\}D^2h_1^2u_{\sigma_i}$ has to be hit by the following $d_9$-differential

$$d_9(\{h_1^2 + xh_1v_1\}Du_{\sigma_i}) = k^2\{x + y\}D^2h_1^2u_{\sigma_i}.$$

By Lemma 2.6 this $d_9$-differential also happens in the $Q_8$-HFPSS for $E_2$. This forces the following $d_9$-differential

$$d_9(\{h_1 + xv_1\}Du_{\sigma_i}) = k^2\{x + y\}D^2h_1u_{\sigma_i}.$$

Since $D^{-1}h_1$ is a 9-cycle according to Proposition 4.33, there is a non-trivial $d_9$-differential

$$d_9(\{h_1^2 + xh_1v_1\}u_{\sigma_i}) = k^2\{x + y\}Dh_1^2u_{\sigma_i}.$$

By the assumption that $\{h_1 + xv_1\}u_{\sigma_i}$ is a 3-cycle and degree reasons, this class survives to the $E_9$-page. Then the above $d_9$-differential forces

$$d_9(\{h_1 + xv_1\}u_{\sigma_i}) = k^2\{x + y\}Dh_1u_{\sigma_i}.$$

Recall that $Dh_1$ supports a non-trivial $d_9$-differential by Proposition 4.33. Then by the Leibniz rule, we have

$$d_9(\{h_1^2 + xh_1v_1\}Du_{\sigma_i}) = d_9(Dh_1) \cdot \{h_1 + xv_1\}u_{\sigma_i} + Dh_1 \cdot d_9(\{h_1 + xv_1\}u_{\sigma_i}) = 0.$$

This contradicts Eq. (5.1).

Therefore, the claimed $d_9$-differential must happen. \hfill \Box

Remark 5.10. Proposition 5.9 shows that $2kv^2_iu_{\sigma_i}$ is hit by a $d_3$-differential. Recall that the class $kv^2_iu_{\sigma_i}$ itself supports a non-trivial $d_9$-differential by Proposition 5.1.

By the above discussion and by inspection, all $d_3$-differentials in the $(\ast - \sigma_i)$-graded part follows from Proposition 5.1, Proposition 5.9 and the Leibniz rule.

Proposition 5.11. The classes $\{h_1^2 + xh_1v_1\}Du_{\sigma_i}$ at (10, 2) and $\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}$ at (42, 2) support $d_9$-differentials

1. $d_9(\{h_1^2 + xh_1v_1\}Du_{\sigma_i}) = k^2\{x + y\}h_1D^2u_{\sigma_i}$;
2. $d_9(\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}) = k^2\{x + y\}h_1^2D^3u_{\sigma_i}$.

Proof. Because $kD^3 = g$ is an invertible permanent cycle in $Q_8$-TateSS($E_2$), the $d_{11}$-differential in Corollary 5.8

$$d_{11}(x^3D^4u_{\sigma_i}) = k^3\{x + y\}h_1D^5u_{\sigma_i},$$

implies that in the $(\ast - \sigma_i)$-graded $Q_8$-TateSS($E_2$) we have

$$d_{11}(k^{-1}x^3Du_{\sigma_i}) = (kD^3)^{-1}d_{11}(x^3D^4u_{\sigma_i}) = k^2\{x + y\}h_1D^2u_{\sigma_i}.$$

Since $k^2\{x + y\}h_1D^2u_{\sigma_i}$ is hit by a $d_{11}$-differential in $Q_8$-TateSS($E_2$), its $h_1$ extension, $k^2\{x + y\}h_1^2D^2u_{\sigma_i}$, has to be hit or on before the $E_{11}$-page. For degree reasons, this class $k^2\{x + y\}D^2h_1^2u_{\sigma_i}$ must be hit by the claimed $d_9$-differential in $Q_8$-TateSS($E_2$). By Lemma 2.6, the first claimed $d_9$-differential also happens in $Q_8$-HFPSS($E_2$). The second $d_9$-differential follows similarly. \hfill \Box

We have the following $d_9$-differentials by the Leibniz rule and integer-graded $d_9$-differentials.

Proposition 5.12. We have the following $d_9$-differentials

1. $d_9(\{x + y\}h_1Du_{\sigma_i}) = k^2x^2h_1D^2u_{\sigma_i}$.
According to Corollary 4.32, in the integer-graded $Q$- which contradicts Proposition 5.12. □

We prove the first differential, and the proofs of the rest three differentials are similar. According to Corollary 4.32, in the integer-graded $Q$-HFPSS($E_2$) we have 

$$d_9(Dh_1) = k^2xh_1.$$ 

Note that the class $\{x + y\} u_{\sigma_i}$ is a permanent cycle by Lemma 5.2. Then the Leibniz rule implies 

$$d_9(\{x + y\} h_1 D u_{\sigma_i} = \{x + y\} u_{\sigma_i} d_9(Dh_1) = k^2x^2h_1D^2u_{\sigma_i}. \square$$ 

**Corollary 5.13.** The classes $\{x + y\} D^2u_{\sigma_i}$ at (15,1) and $\{x + y\} D^6u_{\sigma_i}$ at (47,1) support $d_9$-differentials

$$(1) \quad d_9(\{x + y\} D^2u_{\sigma_i} = k^2(x^2 + y^2)D^3u_{\sigma_i};$$

$$(2) \quad d_9(\{x + y\} D^6u_{\sigma_i} = k^2(x^2 + y^2)D^7u_{\sigma_i}.$$ 

**Proof.** By Proposition 4.33, the class $D^{-1}h_1$ is a 9-cycle. These two $d_9$-differentials hold since otherwise the classes $(\{x + y\} Dh_1 u_{\sigma_i}$ and $\{x + y\} D^5h_1 u_{\sigma_i}$ would be 9-cycles by the Leibniz rule, which contradicts Proposition 5.12. □

In order to derive the last type of $d_9$-differential, we first need to show the following $d_{17}$-differential in the $(* - \sigma_i)$-graded part.

**Proposition 5.14.** The class $\{h_1^2 + xh_1v_1\} u_{\sigma_i}$ at (2,2) supports a $d_{17}$-differential 

$$d_{17}\{h_1^2 + xh_1v_1\} u_{\sigma_i} = k^4\{x + y\} h_1^2 D^2u_{\sigma_i}.$$ 

**Proof.** Consider the class $k^6\{h_1^2 + xh_1v_1\} D^{10}u_{\sigma_i}$ in filtration 26. By Theorem 4.8 this class cannot survive to the $E_{17}$-page.

After the $E_5$-page, all the potential sources that could support a differential hitting the class $k^6\{h_1^2 + xh_1v_1\} D^{10}u_{\sigma_i}$ are $k^3x^2h_1 D^9u_{\sigma_i}$, $k^3\{x + y\} D^9u_{\sigma_i}$ and $k^2x^2h_1 D^8u_{\sigma_i}$. We run out all three possibilities one by one. The class $k^3x^2h_1 D^9u_{\sigma_i}$ is hit by the following $d_9$-differential in Proposition 5.12

$$d_9(kxh_1 D^9u_{\sigma_i}) = k^2D^3d_9(\{x + y\} h_1 D^6u_{\sigma_i}) = k^3x^2h_1 D^9u_{\sigma_i}.$$ 

The class $k^3\{x + y\} D^9u_{\sigma_i}$ is a permanent cycle since $\{x + y\} u_{\sigma_i}$ is a permanent cycle by Lemma 5.2. The class $k^3x^2h_1 D^8$ is also a permanent cycle since it is hit by a known $d_9$-differential in the $Q_8$-TateSS for $E_2$ according to Proposition 5.12

$$d_9(k^{-1}\{x + y\} D^{10}h_1 u_{\sigma_i}) = kx^2h_1 D^8u_{\sigma_i}.$$ 

Therefore, the class $k^6\{h_1^2 + xh_1v_1\} D^{10}u_{\sigma_i}$ must support a non-trivial differential. Since $kD^3 = g$ is an invertible permanent cycle in TateSS, the class $\{h_1^2 + xh_1v_1\} u_{\sigma_i} = D^3(kD^3)^{-6}k^6\{h_1^2 + xh_1v_1\} D^{10}u_{\sigma_i}$ also has to support a non-trivial differential. Then for degree reasons, since $k^8\{h_1^2 + xh_1v_1\} D^{10}u_{\sigma_i}$ is 5-cycle, the only potential targets are $k^4xh_1^2 D^2u_{\sigma_i}$ and $k^2x^3 D^4u_{\sigma_i}$. However, the class $k^5x^3 D^3u_{\sigma_i}$ supports the following $d_{11}$-differential by Corollary 5.8:

$$d_{11}(k^5x^3 D^3u_{\sigma_i}) = (kD^3)^5D^{-16}d_{11}(x^3 D^4u_{\sigma_i}) = k^8 D^4\{x + y\} h_1 u_{\sigma_i}.$$ 

Therefore, the class $\{h_1^2 + xh_1v_1\} u_{\sigma_i}$ supports the desired $d_{17}$-differentials

$$d_{17}\{h_1^2 + xh_1v_1\} u_{\sigma_i} = k^4\{x + y\} h_1^2 D^2u_{\sigma_i}.$$ 

□
It turns out that this is the only $d_{17}$-differential in one period of the $(\ast - \sigma_i)$-graded part of $Q_8$-HFPSS($E_2$).

**Proposition 5.15.** The classes $\{x^2 + y^2\}D^3u_{\sigma_i}$, at $(22, 2)$ and $\{x^2 + y^2\}D^7u_{\sigma_i}$, at $(54, 2)$ support $d_9$-differentials

1. $d_9(\{x^2 + y^2\}D^3u_{\sigma_i}) = k^2x^3D^4u_{\sigma_i};$
2. $d_9(\{x^2 + y^2\}D^7u_{\sigma_i}) = k^2x^3D^4u_{\sigma_i};$

**Proof.** According to Proposition A.7, the restriction of $\{x^2 + y^2\}u_{\sigma_i}$ to the integer-graded $C_4$-HFPSS for $E_2$ is non-trivial. It implies the following restriction by degree reasons

$$\text{res}_D(\{x^2 + y^2\}D^3u_{\sigma_i}) = \delta_1^0u_{\sigma_1}u_{\sigma_2}u_{\sigma_3}.$$  

We now prove that the class $\{x^2 + y^2\}D^3u_{\sigma_i}$ supports a non-trivial differential by contradiction. Suppose that the class $\{x^2 + y^2\}D^3u_{\sigma_i}$ is a permanent cycle that survives to the $E_{\infty}$-page. Note that its $C_4$-restriction $\delta_{1}^0u_{\sigma_1}u_{\sigma_2}u_{\sigma_3}$ has a hidden 2 extension in $\pi_*(E_2^{C_4})$ by Lemma 2.18. Then $\{x^2+y^2\}D^3u_{\sigma_i}$ also has a hidden 2 extension in the $E_{\infty}$-page. However, since hidden extensions and natural maps between spectral sequences will not decrease filtration, the potential target of the hidden 2 extension from the class $\{x^2+y^2\}D^3u_{\sigma_i}$ can only be $k^2\{x^2+y^2\}D^4u_{\sigma_i}, k\{y_1h_2 + xh_1v_1\}D^3u_{\sigma_i}$ and $k\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}$ by degree reasons. However, the first class $k^2\{x^2+y^2\}D^3u_{\sigma_i}$ supports a non-trivial $d_5$-differential by Corollary 5.5

$$d_5(k^2\{x^2+y^2\}D^3u_{\sigma_i}) = k^3xh_1^2D^4u_{\sigma_i}.$$  

The second class $k\{y_1h_2 + xh_1v_1\}D^3u_{\sigma_i}$ is hit by a $d_5$-differential by Corollary 5.3

$$d_5(k\{x+y\}D^3u_{\sigma_i}) = k\{y_1h_2 + xh_1v_1\}D^3u_{\sigma_i}.$$  

The third class $k\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}$ supports a $d_{17}$-differential by Proposition 5.14

$$d_{17}(k\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}) = k^5xh_1^2D^5u_{\sigma_i}.$$  

Therefore, all the potential targets of the hidden 2 extension from the class $\{x^2+y^2\}D^3u_{\sigma_i}$ will not survive to the $E_{\infty}$-page. This is a contradiction. Hence the class $\{x^2+y^2\}D^3u_{\sigma_i}$ must support a non-trivial differential.

The $E_{5}$-page, the only two potential targets are $k^2x^3D^4u_{\sigma_i}$ and $k^3\{x+y\}h_1^2D^4u_{\sigma_i}$ by degree reasons. However, the class $k^3\{x+y\}h_1^2D^3u_{\sigma_i}$ is hit by the following $d_{17}$-differential by Proposition 5.14 and the Leibniz rule

$$d_{17}(k\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}) = kD^3d_{17}(\{h_1^2 + xh_1v_1\}D^3u_{\sigma_i}) = k^5\{x+y\}h_1^2D^5u_{\sigma_i}.$$  

Then the first desired $d_9$-differential follows. The proof of the second $d_9$-differential in the statement is similar since the $C_4$-HFPSS for $E_2$ is 32-periodic.

Actually, we can apply the norm method to get a $d_9$-differential directly (after the calculation of $E_3$-page) which is independent of the $d_9$ information in the integer-graded part.

**Proposition 5.16.** There is a normed $d_9$-differential in $(\ast - \sigma_i)$-page

$$d_9(\{x+y\}D^6u_{\sigma_i}) = k^2\{x^2+y^2\}D^7u_{\sigma_i}.$$  

**Proof.** According to [HHR17, Theorem 11.13], the class $u_{\sigma_2}$ supports a non-trivial $d_5$-differential in $C_4$-HFPSS($E_2$)

$$d_5(u_{\sigma_2}) = \bar{\delta}_1u_{\sigma_1}u_{\sigma_2}u_{\sigma_3}.$$  

Then Theorem 2.8 implies there is a predicted $d_9$-differential in $Q_8$-HFPSS($E_2$)

$$d_9(N_{Q_8}^{C_4}(u_{\sigma_2})) = N_{Q_8}^{C_4}(\bar{\delta}_1)N_{Q_8}^{C_4}(u_{\sigma_2})u_{\sigma_3}u_{\sigma_4}.$$
We claim the target of this predicted $d_9$-differential is non-trivial on the $E_2$-page. It suffices to show that the class $a_{\sigma_j, \alpha_j}$ is non-trivial since $N^{Q_8}_{C_4}(u_{\lambda})a_{2\mathbb{H}}$ is invertible in TateSS($E_2$). We observe that

$$\text{res}^Q_{C_4}(a_{\sigma_j, \alpha_j}) = a_{2\sigma_j}$$

where $a_{2\sigma}$ is non-trivial in $C_4$-HFPPSS($E_2$). This implies that $a_{\sigma_j, \alpha_j}$ is also non-trivial. Therefore, the non-trivial class on the $E_2$-page $N^{Q_8}_{C_4}(1)N^{Q_8}_{C_4}(u_{\lambda})a_{2\mathbb{H}}a_{\sigma_j}a_{2\sigma_j}$ must be hit on or before the $E_8$-page. By multiplying this class with the periodicity classes in Corollary 4.22, we get a non-trivial class on the $E_8$-page. By multiplying this class with the periodicity classes in Corollary 2.22, we get a non-trivial class on the $E_9$-page (up to a unit) by degree reasons. Therefore, the class $k^2\{x^2 + y^2\}D^7u_{\sigma_j}$ has to be hit on or before the $E_9$-page too. For degree reasons, the desired $d_9$-differential happens in the $(\ast - \sigma_j)$-graded part.

All $d_9$-differentials follow from Proposition 5.11, Proposition 5.12, Corollary 5.13, Proposition 5.15 and the Leibniz rule.

**Proposition 5.17.** The classes \(\{x + y\}h_1^2D^2u_{\sigma_j}\) at (17, 3) and \(\{x + y\}h_1D^7u_{\sigma_j}\) at (56, 2) support $d_{23}$-differentials

\[
(1) \quad d_{23}\{x + y\}h_1^2D^2u_{\sigma_j} = k^6\{x + y\}h_1D^4u_{\sigma_j}; \\
(2) \quad d_{23}\{x + y\}h_1D^7u_{\sigma_j} = k^6\{x + y\}D^5u_{\sigma_j}.
\]

**Proof.** By Corollary 4.22 we have the following two $d_{23}$-differentials in the integer-graded part

\[
d_{23}\{x + y\}h_1^2D^2u_{\sigma_j} = k^6h_1D^5, \quad d_{23}\{x + y\}D^7h_1 = k^6D^{10}.
\]

Note that the class $\{x + y\}h_{\bar{\sigma}_j, \alpha_j}$ is a permanent cycle by Lemma 5.2. Then the desired two differentials follow from these $d_{23}$-differentials and the Leibniz rule.

All $d_9$-differentials follow from Proposition 5.17 and the Leibniz rule.

**Lemma 5.18.** The classes \(\{h_1^2 + xh_1v_1\}D^7u_{\sigma_j}\) and \(\{h_1^2 + xh_1v_1\}D^4u_{\sigma_j}\) are both permanent cycles.

**Proof.** After the $E_8$-page, the potential targets of \(\{h_1^2 + xh_1v_1\}D^7u_{\sigma_j}\) are the classes $k^2\{x + y\}h_1^2D^5u_{\sigma_j}$ and $k^3xh_1D^6u_{\sigma_j}$, since lengths of differentials in the $RO(Q_8)$-graded $Q_8$-HFPPSS($E_2$) are less than or equal to 23 by Theorem 4.8. However, the class $k^2\{x + y\}h_1^2D^5u_{\sigma_j}$ supports a non-trivial $d_{23}$-differential by Proposition 5.17 and the class $k^3xh_1D^6u_{\sigma_j}$ supports a non-trivial $d_{11}$-differential by Corollary 5.8. By similar reasons, the class $\{h_1^2 + xh_1v_1\}D^4u_{\sigma_j}$ is also a permanent cycle.

**Proposition 5.19.** There are four non-trivial $d_{11}$-differentials

\[
(1) \quad d_{11}(xh_1D^2u_{\sigma_j}) = k^3\{h_1^2 + xh_1v_1\}D^3u_{\sigma_j}; \\
(2) \quad d_{11}(xh_1D^3u_{\sigma_j}) = k^3\{h_1^2 + xh_1v_1\}D^4u_{\sigma_j}; \\
(3) \quad d_{11}(xh_1D^6u_{\sigma_j}) = k^3\{h_1^2 + xh_1v_1\}D^7u_{\sigma_j}; \\
(4) \quad d_{11}(xh_1D^8u_{\sigma_j}) = k^3\{h_1^2 + xh_1v_1\}D^9u_{\sigma_j}.
\]

**Proof.** We prove the last $d_{11}$-differential, and the proofs for the other differentials are similar. The class \(\{h_1^2 + xh_1v_1\}D^7u_{\sigma_j}\) is a permanent cycle by Lemma 5.18. Then $k^6\{h_1^2 + xh_1v_1\}D^9u_{\sigma_j} = D^{-16}\{h_1^2 + xh_1v_1\}D^7u_{\sigma_j}(kD^3)^6$ is also a permanent cycle. Since the filtration of this class is 26 which is greater than 23, Theorem 4.8 implies that the class $k^3\{h_1^2 + xh_1v_1\}D^7u_{\sigma_j}$ must be hit by a differential. For degree reasons, we have the $d_{11}$-differentials

\[
d_{11}(k^3xh_1D^8u_{\sigma_j}) = k^6\{h_1^2 + xh_1v_1\}D^9u_{\sigma_j}.
\]

Note that the class $(k^3D)^{-1} = (k^3D^{-1})^{-1}$ is an invertible permanent cycle in $Q_8$-TateSS($E_2$). Then the Leibniz rule implies the desired $d_{11}$-differential in $Q_8$-TateSS($E_2$). By Lemma 2.6 this $d_{11}$-differential also happens in $Q_8$-HFPPSS($E_2$).
All $d_{13}$-differentials follow from Corollary 5.8, Proposition 5.19 and the Leibniz rule.

**Proposition 5.20.** The class $\{x + y\}D^4u_\sigma$, at $(31, 1)$ supports a $d_{13}$-differential

$$d_{13}(\{x + y\}D^4u_\sigma) = k^3\{h_1^2 + xh_1v_1\}D^3u_\sigma.$$

**Proof.** We first claim the class $k^3\{h_1^2 + xh_1v_1\}D^5u_\sigma$, is a permanent cycle. In the $Q_8$-TateSS for $E_2$, by multiplying it with $k^{-3}D^{-9} \cdot D^3$, we obtain $\{h_1^2 + xh_1v_1\}D^4u_\sigma$, which is a permanent cycle by Lemma 5.18. So $k^3\{h_1^2 + xh_1v_1\}D^5u_\sigma$ is also a permanent cycle in the $Q_8$-HFPSS for $E_2$.

Next we consider the class $k^3\{h_1^2 + xh_1v_1\}D^6u_\sigma = k^3\{h_1^2 + xh_1v_1\}D^5u_\sigma \cdot k^3D^9 \cdot D^{-8}$ above the vanishing line. By Theorem 4.8 it must be hit by a differential since it is a permanent cycle. Then for degree reasons, the only two possible sources are $\{x + y\}D^4u_\sigma$, and $x^2h_1D^4u_\sigma$. Note that the class $x^2h_1D^4$ is a permanent cycle since it is hit by a $d_9$-differential in $Q_8$-TateSS($E_2$)

$$d_9(k^{-2}xh_1D^3u_\sigma) = x^2h_1D^4u_\sigma.$$

Therefore, the claimed $d_{13}$-differential must happen. \[\square\]

This $d_{13}$-differential can also be deduced via the norm method.

**Second proof of Proposition 5.20.** According to [HHR17, Theorem 11.13][HSWX18, Corollary 3.14], there is a $d_7$-differential in the $C_4$-HFPSS for $E_2$

$$d_7(u_{4\lambda}) = \delta_1\eta'\eta_2\alpha_{3\lambda}.$$

Then Theorem 2.8 shows that there is a predicted $d_{13}$-differential

$$d_{13}(N_{Q_8}^{Q_8}(u_{4\lambda})a_\sigma) = N_{Q_8}^{Q_8}(\delta_1)N_{Q_8}^{Q_8}(\eta')N_{Q_8}^{Q_8}(u_{2\lambda})a_{3\lambda}.$$

According to [Sch11, Proposition 10.4 (viii)], $\text{res}_{Q_8}^{Q_8}N_{Q_8}^{Q_8}(\eta') = \eta'^2$ is non-trivial. Then $N_{Q_8}^{Q_8}(\eta')$ is non-trivial on the $E_2$-page and so is the class $N_{Q_8}^{Q_8}(\delta_1)N_{Q_8}^{Q_8}(\eta')N_{Q_8}^{Q_8}(u_{2\lambda})a_{3\lambda}$. By multiplying the non-trivial class $N_{Q_8}^{Q_8}(\delta_1)N_{Q_8}^{Q_8}(\eta')N_{Q_8}^{Q_8}(u_{2\lambda})a_{3\lambda}$ with the periodicity classes in Corollary 2.22, we get a non-trivial class at $(30, 14)$ on the $E_2$-page, which has to be the class $k^3\{h_1^2 + xh_1v_1\}D^5u_\sigma$ by degree reasons. Therefore, the class $k^3\{h_1^2 + xh_1v_1\}D^5u_\sigma$ must be hit on or before the $E_{13}$-page. For degree reasons, the desired $d_{13}$-differential follows. \[\square\]

All $d_{13}$-differentials follow from Proposition 5.20 and the Leibniz rule.

Table 9 lists the differentials we have computed so far. They generate differentials via the Leibniz rule. By inspection, these are all non-trivial differentials since the remaining classes are permanent cycles by Method 2.7.

**Theorem 5.21.** There are no hidden 2 extensions on the $E_{\infty}$-page of $(\ast - \sigma_\ast)$-graded $Q_8$-HFPSS($E_2$).

**Proof.** Since $a_\sigma = \{x + y\}u_\sigma$, already lives in the homotopy group $\pi_{-\sigma_\ast}(E_2^{hQ_8})$ and it is 2-torsion. Then in $(\ast - \sigma_\ast)$-graded $Q_8$-HFPSS($E_2$) all $a_\sigma$, multiplies that survive to $E_\infty$-page can not support hidden 2 extensions. Then for degree reasons, the only possible 2 extensions are from $\{x^2 + y^2\}D^{4k+1}u_\sigma$ to $k\{h_1^2 + xh_1v_1\}D^{4k+1}u_\sigma$, where $k \in \mathbb{Z}$. We now show there is actually no hidden 2 extension on the class $\{x^2 + y^2\}Du_\sigma$; and the rest are similar. We observe that in $(\ast - \sigma_\ast)$-graded $Q_8$-TateSS($E_2$), there is a differential

$$d_9(k^{-2}(x + y)u_\sigma) = \{x^2 + y^2\}Du_\sigma.$$

By sparseness of the $Q_8$-TateSS for $E_2$, the above differential implies that under the homotopy group map induced by the additive norm map

$$N: (E_2)_hQ_8 \rightarrow E_2^{hQ_8},$$
the elements detected by \( k^{-2}(x + y)u_{\sigma_i} \) maps to elements detected by \( \{x^2 + y^2\}Du_{\sigma_i} \). However, in \((\ast - \sigma_i)\)-graded \( Q_8\)-HOSS\( (E_2) \)

\[2k^{-2}(x + y)u_{\sigma_i} = 0,\]

then it forces

\[2\{x^2 + y^2\}Du_{\sigma_i} = 0.\]

\( \square \)

The result is presented in Fig. 15. We only draw \( h_1, h_2 \) extensions from the \( E_2 \)-page.

5.1. **Summary of differentials.** Differentials in \((\ast - \sigma)\)-graded part are given by Table 9. All differentials follow from this list by multiplying permanent cycles and the Leibniz rule.

<table>
<thead>
<tr>
<th>((s, f))</th>
<th>(x)</th>
<th>(r)</th>
<th>(d_r(x))</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1) (h_1 + x v_1)u_{\sigma_i}</td>
<td>2</td>
<td>2kv_1^2u_{\sigma_i}</td>
<td>Proposition 5.9</td>
<td></td>
</tr>
<tr>
<td>(4, 0) (v_1^2u_{\sigma_i})</td>
<td>3</td>
<td>(h_1^3u_{\sigma_i})</td>
<td>Proposition 5.1 (restriction)</td>
<td></td>
</tr>
<tr>
<td>(7, 1) (x + y)Du_{\sigma_i}</td>
<td>5</td>
<td>(k{yh_2 + xh_1v_1}Du_{\sigma_i})</td>
<td>Corollary 5.3 (module structure)</td>
<td></td>
</tr>
<tr>
<td>(14, 2) (x^2 + y^2D^2u_{\sigma_i})</td>
<td>5</td>
<td>(kxh_1^2D^2u_{\sigma_i})</td>
<td>Corollary 5.5 (module structure)</td>
<td></td>
</tr>
<tr>
<td>(10, 2) (h_1^2 + xh_1v_1)Du_{\sigma_i}</td>
<td>9</td>
<td>(k^2{x + y}h_1^2D^2u_{\sigma_i})</td>
<td>Proposition 5.11</td>
<td></td>
</tr>
<tr>
<td>(42, 2) (h_1^4 + xh_1v_1)D^3u_{\sigma_i}</td>
<td>9</td>
<td>(k^2{x + y}h_1^2D^3u_{\sigma_i})</td>
<td>Proposition 5.11</td>
<td></td>
</tr>
<tr>
<td>(8, 2) (x + y)h_1Du_{\sigma_i}</td>
<td>9</td>
<td>(k^2xh_1^2D^2u_{\sigma_i})</td>
<td>Proposition 5.12 (module structure)</td>
<td></td>
</tr>
<tr>
<td>(40, 2) (x + y)h_1D^3u_{\sigma_i}</td>
<td>9</td>
<td>(k^2x^2h_1D^3u_{\sigma_i})</td>
<td>Proposition 5.12</td>
<td></td>
</tr>
<tr>
<td>(15, 1) (x + y)D^2u_{\sigma_i}</td>
<td>9</td>
<td>(k^2{x^2 + y^2}D^3u_{\sigma_i})</td>
<td>Corollary 5.13</td>
<td></td>
</tr>
<tr>
<td>(47, 1) (x + y)D^6u_{\sigma_i}</td>
<td>9</td>
<td>(k^2x^2 + y^2)D^7u_{\sigma_i})</td>
<td>Corollary 5.13</td>
<td></td>
</tr>
<tr>
<td>(22, 2) (x^2 + y^2D^4u_{\sigma_i})</td>
<td>9</td>
<td>(k^2x^3D^4u_{\sigma_i})</td>
<td>Proposition 5.15 (hidden 2 extension)</td>
<td></td>
</tr>
<tr>
<td>(54, 2) (x^2 + y^2D^7u_{\sigma_i})</td>
<td>9</td>
<td>(k^2x^3D^8u_{\sigma_i})</td>
<td>Proposition 5.15</td>
<td></td>
</tr>
<tr>
<td>(15, 3) (x^2h_1D^3u_{\sigma_i})</td>
<td>11</td>
<td>(k^3{h_1^2 + xh_1v_1}D^3u_{\sigma_i})</td>
<td>Proposition 5.19 (vanishing line)</td>
<td></td>
</tr>
<tr>
<td>(47, 3) (x^2h_1D^4u_{\sigma_i})</td>
<td>11</td>
<td>(k^3{h_1^2 + xh_1v_1}D^4u_{\sigma_i})</td>
<td>Proposition 5.19</td>
<td></td>
</tr>
<tr>
<td>(23, 3) (x^2h_1D^3u_{\sigma_i})</td>
<td>11</td>
<td>(k^3{h_1^2 + xh_1v_1}D^4u_{\sigma_i})</td>
<td>Proposition 5.19</td>
<td></td>
</tr>
<tr>
<td>(55, 3) (x^2h_1D^7u_{\sigma_i})</td>
<td>11</td>
<td>(k^3{h_1^2 + xh_1v_1}D^8u_{\sigma_i})</td>
<td>Proposition 5.19</td>
<td></td>
</tr>
<tr>
<td>(29, 3) (x^3D^4u_{\sigma_i})</td>
<td>11</td>
<td>(k^3xh_1D^5u_{\sigma_i})</td>
<td>Corollary 5.8 (module structure)</td>
<td></td>
</tr>
<tr>
<td>(61, 3) (x^3D^8u_{\sigma_i})</td>
<td>11</td>
<td>(k^3xh_1D^9u_{\sigma_i})</td>
<td>Corollary 5.8</td>
<td></td>
</tr>
<tr>
<td>(31, 1) (x + y)D^4u_{\sigma_i})</td>
<td>13</td>
<td>(k^3{h_1^2 + xh_1v_1}D^5u_{\sigma_i})</td>
<td>Proposition 5.20 (vanishing line or norm differential)</td>
<td></td>
</tr>
<tr>
<td>(2, 2) (h_1^2 + xh_1v_1)u_{\sigma_i}</td>
<td>17</td>
<td>(k^4{x + y}h_1^2D^2u_{\sigma_i})</td>
<td>Proposition 5.14 (vanishing line)</td>
<td></td>
</tr>
<tr>
<td>(17, 3) (x + y)h_1^2D^5u_{\sigma_i})</td>
<td>23</td>
<td>(k^5{x + y}h_1D^5u_{\sigma_i})</td>
<td>Proposition 5.17 (module structure)</td>
<td></td>
</tr>
<tr>
<td>(56, 2) (x + y)h_1D^{10}u_{\sigma_i})</td>
<td>23</td>
<td>(k^5{x + y}D^{10}u_{\sigma_i})</td>
<td>Proposition 5.17</td>
<td></td>
</tr>
</tbody>
</table>
6. Charts and Tables

6.1. Keys for the charts. In all charts, a gray line denotes a multiplication. See the following table for the keys.

<table>
<thead>
<tr>
<th>line</th>
<th>meanings</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertical</td>
<td>2 multiplication</td>
</tr>
<tr>
<td>slope 1</td>
<td>$h_1$ multiplication</td>
</tr>
<tr>
<td>slope 1/3</td>
<td>$h_2$ multiplication</td>
</tr>
<tr>
<td>dashed (only in 2BSS)</td>
<td>hidden extension</td>
</tr>
</tbody>
</table>

The colored lines denote the differentials. We use different colors to distinguish different lengths.

<table>
<thead>
<tr>
<th>class</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>dot</td>
<td>$k$</td>
</tr>
<tr>
<td>blue dot</td>
<td>$k\lbrack j \rbrack$</td>
</tr>
<tr>
<td>red dot</td>
<td>$k\lbrack j \rbrack { j }$</td>
</tr>
<tr>
<td>square</td>
<td>$\mathbb{W}(k)$</td>
</tr>
</tbody>
</table>

Here $k$ is $\mathbb{F}_2$ for $G = SD_{16}$ or $G_{48}$, and is $\mathbb{F}_4$ for $G = Q_8$ or $G_{24}$; $j$ is $v_1^{12}D^{-3}$ for $G_{24}$ or $G_{48}$, and $v_1^4D^{-1}$ otherwise.

Remark 6.1. We elaborate more on boxes and dots connected by vertical lines in the same bidegree. Such pattern is a 2-adic presentation of a class. Namely, the bottom dot is generated by the generator and represents a 2-torsion copy, the dot or box just above is generated by twice the generator, and so on.

For example, on the $E_{\infty}$-page of the integral degrees (Fig. 9), in bigrading $(32, 0)$ the bottom red dot represents the class $\mathbb{W}/2[v_1^4D^{-1} \{ v_1^4D^3 \}]$ and the blue box above represents $\mathbb{W}[v_1^4D^{-1} \{ 2D^4 \}]$; Note that there is a 2 extension. Thus the class at $(32, 0)$ is $\mathbb{W}[v_1^4D^{-1} \{ v_1^4D^3 \}] \oplus \mathbb{W}\{2D^4\}$.

Such presentations help to demonstrate where the differentials or extensions come from. For example, in Fig. 5 in bigrading $(12, 0)$, only the generator $v_1^4$ supports a non-trivial $d_3$-differential and $2v_1^4$ survives. This convention is due to Dan Isaksen.

Remark 6.2. We comment on the extensions between dots of different colors. For example, in the bidegree $(24, 0)$ and $(25, 1)$ in Fig. 9, there is an $h_1$ multiplication connecting a red and a blue dot. The red dot represents the class $\mathbb{W}/2[\{ v_1^4D^{-1} \} \{ v_1^4D^3 \}]$ and the blue dot represents the class $\mathbb{W}/2[\{ v_1^4D^{-1} \} \{ h_1D^3 \}]$. The $h_1$ multiplication happens whenever it is indicated by the class names. Note that the class $\mathbb{W}/2\{ h_1D^3 \}$ is not $h_1$-divisible in this case since the source is missing.

6.1.1. 2-BSS.

Fig. 2 - Fig. 4 are charts for the 2-Bockstein spectral sequences. All three charts have $(8, 0)$ periodicity by multiplying $D$ and $(-4, 4)$ periodicity by multiplying $k$ (except the $v_1$ local classes in low filtration). We only depict part of the spectral sequence here, which contains a full periodic range.

In Fig. 2, a blue line indicates the multiplication by $x$, while an orange line indicates the multiplication by $y$. 

Recall the \((\ast - \sigma_i)\)-graded part and the integer-graded part have isomorphic \(E_1\)-pages. When interpret the chart as the \((\ast - \sigma_i)\)-graded part, the name of a class at \((s, f)\) is its label multiplied by \(u_{\sigma_i}\), and its degree is \((s + 1 - \sigma_i, f)\). For example, the class 1 at \((0, 0)\), when interpreted as an \((\ast - \sigma_i)\)-graded part class, denotes \(u_{\sigma_i}\) at \((1 - \sigma_i, 0)\) in the 2BSS.

Fig. 3 and Fig. 4 show the \(E_\infty\)-page of 2BSS, for the integer-graded part and \((\ast - \sigma_i)\)-graded part respectively.

6.1.2. \(HFPSS\).

Fig. 5–Fig. 9 depict the integer degree calculation of the integer-graded \(G\)-HFPSS\(E_2\) for \(G = Q_8\) or \(SD_{16}\), and Fig. 11–Fig. 15 depict the \((\ast - \sigma_i)\)-graded calculation. Both \(E_2\)-pages are \((8, 0)\) periodic by multiplying \(D\), and other pages are \((64, 0)\) periodic by multiplying \(D^8\). All charts are \((20, 4)\) periodic by multiplying \(kD^3\) (except the \(v_1\) local classes in low filtration). The differentials are denoted by the colored lines with their length classified by the color. When the target or the source of the differential is out of range, we replace the line with an arrow. There are horizontal vanishing lines in filtration 23 on \(E_\infty\)-pages.

![Figure 2. The \(E_1\)-page of the integer/(\(\ast - \sigma_i\))-graded 2BSS.](image2)

![Figure 3. The \(E_\infty\)-page of the integer-graded 2BSS.](image3)
COMPUTATIONS OF HEIGHT 2 HIGHER REAL $k$-THEORY SPECTRA AT PRIME 2

**Figure 4.** The $E_{\infty}$-page of the $(\ast - \sigma)$-graded 2BSS.

**Figure 5.** The $E_3$-page of the integer-graded $Q_6$-HFPSS($E_2$). The red lines are $d_3$-differentials.
Figure 6. The $E_5$-page of the integer-graded $Q_8$-HFPSS($E_2$). The green lines are $d_5$-differentials. The blue lines are $d_7$-differentials.

Figure 7. The $E_9$-page of the integer-graded $Q_8$-HFPSS($E_2$). The purple lines are $d_9$-differentials.
Figure 8. The $E_{11}$-page of the integer-graded $Q_8$-HFSS($E_2$). The brown lines are $d_{11}$-differentials. The magenta lines are $d_{13}$-differentials. The green lines are $d_{13}$-differentials.

Figure 9. The $E_\infty$-page of the integer-graded $Q_8/SD_{16}$-HFSS($E_2$).
Figure 10. The $E_{\infty}$-page of the integer-graded $G_{24}/G_{44}$-HFPSS($E_2$).
Figure 11. The $E_2$-page of the $(s - s)$-graded $Q^*_s$-HFPSS($E_2$). The red lines are $d_3$-differentials.

Figure 12. The $E_2$-page of the $(s - s)$-graded $Q^*_s$-HFPSS($E_2$). The green lines are $d_3$-differentials.
Figure 13. The $E_0$-page of the $(\ast - \sigma_1)$-graded $Q_8$-HFPSS($E_2$). The purple lines are $d_9$-differentials.

Figure 14. The $E_{11}$-page of the $(\ast - \sigma_1)$-graded $Q_8$-HFPSS($E_2$). The brown lines are $d_{11}$-differentials. The magenta lines are $d_{13}$-differentials. The blue lines are $d_{17}$-differentials. The green lines are $d_{13}$-differentials.
APPENDIX A. GROUP COHOMOLOGY

In this appendix, we collect and present examples of computations of group cohomology. There are two main applications: one is to calculate it as the input for the $E_2$-page of the integer- and $(s-\sigma)$-graded homotopy fixed points spectral sequences for $E_2$, the other is to utilize restrictions, transfers, and norm maps for proofs of differentials. All the rests needed for our computation of the $Q_8$-HFPSS for $E_2$ are listed in Proposition A.7.

Let $Q_8$ be presented as

$$Q_8 = \langle i, j \mid i^4 = j^2 = iji^{-1} \rangle$$

with its real representation ring $RO(Q_8) = \mathbb{Z}[1, \sigma_1, \sigma_2, \sigma_3, \sigma_4]$.

To calculate $H^*(Q_8; A)$ we will use the following 4-periodic free $\mathbb{Z}[Q_8]$-resolution:

$$0 \leftarrow \mathbb{Z} \xleftarrow{\sum} X_0 \xleftarrow{d_0} X_1 \xleftarrow{d_1} X_2 \xleftarrow{d_2} ...$$

where $X_0 = \mathbb{Z}[Q_8]\{a_0\}$, $\nabla(a_0) = 1$, and for $k \geq 0$,

$$X_{4k+1} = \mathbb{Z}[Q_8]\{b_{k,1}, b_{k,2}\}, \quad d(b_{k,1}) = (i-1)a_k,$$

$$X_{4k+2} = \mathbb{Z}[Q_8]\{c_{k,1}, c_{k,2}\}, \quad d(c_{k,1}) = (1+i)b_{k,1} - (1+j)b_{k,2},$$

$$X_{4k+3} = \mathbb{Z}[Q_8]\{e_k\}, \quad d(e_k) = (i-1)c_{k,1} - (ij-1)c_{k,2},$$

$$X_{4k+4} = \mathbb{Z}[Q_8]\{a_{k+1}\}, \quad d(a_{k+1}) = \sum_{g \in Q_8} g \cdot e_k.$$

Suppose that $A$ is a $Q_8$-module, then $H^*(Q_8; A)$ is the cohomology of the cochain complex

$$A \xrightarrow{d_0} A \oplus A \xrightarrow{d_1} A \oplus A \xrightarrow{d_2} A \xrightarrow{d_3} A \rightarrow ...$$

where the differentials (by abuse of notation) are given by the following matrices

$$d_{4k} = \begin{pmatrix} i-1 & j-1 \\ j & i \\ \end{pmatrix}, \quad d_{4k+1} = \begin{pmatrix} 1+i & -1-j \\ 1+i & -1+i \\ \end{pmatrix}, \quad d_{4k+2} = \begin{pmatrix} i-1 & i-j \\ i-1 & i-j \\ \end{pmatrix},$$

and $d_{4k+3} = \sum_{g \in Q_8} g$. 

---

Figure 15. The $E_\infty$-page of the $(s-\sigma)$-graded $Q_8/SD_{16}$-HFPSS$(E_2)$. 

---
We record here the group cohomology of $Q_8$ with trivial $\mathbb{Z}$ coefficients
\[
H^{4k+2}(Q_8, \mathbb{Z}) = \mathbb{Z}/2 \otimes \mathbb{Z}/2,
\]
\[
H^{4k+4}(Q_8, \mathbb{Z}) = \mathbb{Z}/8,
\]
\[
H^{2r+1}(Q_8, \mathbb{Z}) = 0,
\]
where $k \geq 0, q \geq 0$, and the generator of $H^4(Q_8, \mathbb{Z})$ gives the 4-periodicity.

In addition to the integer-graded $Q_8$-HFPSS for $E_2$, we also compute the $(s - \sigma_i)$-graded part. For this purpose, we study the structure of $\pi_*E_2 \otimes \sigma_i$ as a $Q_8$-module, which is given by the following analog of [HM17, Lemma 4.6]:

**Lemma A.1.** Let $E$ be a $Q_8$-spectrum. Then
\[
\pi_*^e(E \wedge S^{1-\sigma_i}) \cong \pi_*^e E \otimes \sigma_i
\]
as $Q_8$-modules.

Recall that we define $v_1 = u_1 u^{-1}$ and its $Q_8$ action is given in Eq. (2.3). By Lemma 2.12, we may first compute $H^*(Q_8, \mathbb{W}[u^{-1}, v_1])$, and then invert $D$ and complete at $I = (2, u_1)$.

**Remark A.2.** If we define $s = i_*(u^{-1})$ and denote $u^{-1}$ by $t$, then the actions of $Q_8$ on $s, t$ are given by
\[
i_*(s) = -t, \quad i_*(t) = s
\]
\[
j_*(s) = -\zeta^2 s + \zeta t, \quad j_*(t) = \zeta s + \zeta^2 t
\]
\[
k_*(s) = \zeta s + \zeta^2 t, \quad k_*(t) = \zeta^2 s - \zeta t
\]
For computational purposes, it is equivalent to replace generators $u^{-1}, v_1$ by $s, t$, and the form of the action turns out to be more compact.

We first calculate the 0-th cohomology ring. Behrens and Ormsby [BO16] have determined the $C_4(i)$-invariants:

**Proposition A.3.** Let $b_2 = s^2 + t^2$, $b_4 = s^4 - st^3$ and $\delta = s^2 t^2$, then
\[
H^0(C_4, \mathbb{W}[u^{-1}, v_1]) = \mathbb{W}[b_2, b_4, \delta]/(b_2^2 - b_4^2 \delta + 4 \delta^2).
\]
The $j$-actions on $b_2, b_4, \delta$ are the following:
\[
j_*(b_2) = -b_2,
\]
\[
j_*(b_4) = -(2\zeta + 1)b_2^2 + 7b_4 + 8(2\zeta + 1)\delta,
\]
\[
j_*(\delta) = b_2^2 + 2(2\zeta + 1)b_4 - 7\delta.
\]

**Proposition A.4.** We have the 0-th cohomology ring
\[
H^0(Q_8, \mathbb{W}[u^{-1}, v_1]) = \mathbb{W}[s_1, s_2, s_3]/(s_1^2 = 4(2\zeta + 1)s_2^2 s_2 + 16s_1 s_2^2)
\]
where $s_1 = b_2^2$, $s_2 = b_4 + (2\zeta + 1)\delta$, and $s_3 = b_2^2 + 2(2\zeta + 1)b_4 b_2 - 8b_2^2 \delta$.

**Proof.** Since $\pi_*E_2$ is 16-periodic, it suffices to compute the $j$-invariants of $H^0(C_4, \mathbb{W}[u^{-1}, v_1])$ in low degrees. The result follows by direct computation. \hfill $\Box$

In the main computations, we sometimes need to rely on explicit group cohomology results. The following is an example.

**Example A.5.** The calculation of $H^4(Q_8, \pi_*E_2 \otimes \sigma_i) \cong \mathbb{W}/4$.

The cochain complex at degree 4 looks like
\[
\mathbb{W}\{s^2, st, t^2\} \xrightarrow{d_3} \mathbb{W}\{s^2, st, t^2\} \xrightarrow{d_4} \mathbb{W}\{s^2, st, t^2\}^2
\]
By Lemma A.1, the actions are
\[
i_*(s^2) = t^2, \quad i_*(st) = -st, \quad i_*(t^2) = s^2
\]
\[
j_*(s) = -\zeta s^2 + 2st - \zeta^2 t^2, \quad j_*(t) = \zeta s + \zeta^2 t, \quad j_*(t^2) = \zeta^2 s^2 + 2st + \zeta t^2.
\]
Therefore, \(\ker d_4 = \ker (i - 1) = \mathbb{W}\{s^2 + t^2\}\).

Meanwhile, since we have
\[
d_3(s^2) = 4(s^2 + t^2),
\]
\[
d_3(st) = 0,
\]
\[
d_3(t^2) = 4(s^2 + t^2),
\]
we conclude that \(H^4(Q_8, \pi_4 E_2 \otimes \sigma_i) \cong \mathbb{W}/4\).

We also calculate a couple of restriction maps in group cohomology. In the case of the integer-

graded part, most calculations are easy. By Proposition A.8 we deduce that the generators 
\(\eta, \nu, c, d, g\) have to restrict non-trivially to their \(C_4\)-counterparts, which lie in the Hurewicz image. 
For the \((* - \alpha)\)-graded part, some chain level calculations seem to be inevitable.

Example A.6. In the integer-graded part, calculate \(\text{res}_{C_4(i)}^Q D^{-2}d \neq 0\). This is used in the proof of Proposition 4.30.

The class \(D^{-2}d\) lies in bigrading \((-2, 2)\). We are looking at the degree 0 part of \(\mathbb{W}[u^{-1}, v_1]\).

The generator of \(H^2(Q_8, \mathbb{W}[1])\) is given by the cochain
\[
\alpha : \mathbb{Z}[Q_8]\{c_{0,1}, c_{0,2}\} \to \mathbb{W}\{1\},
\]
\[
c_{0,1} \mapsto 1, \quad c_{0,2} \mapsto 0.
\]

Restricting to \(C_4(i)\), we rewrite \(X_2 = \mathbb{Z}[Q_8]\{c_{0,1}, c_{0,2}\}\) as \(\mathbb{Z}[C_4(i)]\{c_{0,1}, jc_{0,1}, c_{0,2}, jc_{0,2}\}\), and similarly for \(X_1\). Then \(\alpha\) restricts to the cochain
\[
\alpha : \mathbb{Z}[Q_8]\{c_{0,1}, jc_{0,1}, c_{0,2}, jc_{0,2}\} \to \mathbb{W}\{1\},
\]
\[
c_{0,1}, jc_{0,1} \mapsto 1, \quad c_{0,2}, jc_{0,2} \mapsto 0.
\]

Now we check the image of \(d_1\). Let \(\beta_1, \beta_2, \beta_3, \beta_4\) be the dual basis of \(b_{0,1}, jb_{0,1}, b_{0,2}, jb_{0,2}\) in \(\text{Hom}_{C_4(i)}(X_1, \mathbb{W}\{1\})\). The image of \(\beta_1\) is calculated by evaluating \(\beta_1 \circ d_1\) at the \(C_4(i)\)-basis of \(X_2\). As an example, we have
\[
(\beta_1 \circ d_1)(c_{1,0}) = \beta_1((1 + i)b_{0,1} - b_{0,2} - jb_{0,2}) = 2.
\]

Similarly, we verify that the restriction of \(\alpha\) does not lie in the coboundary; hence the restriction

is non-trivial.

Sometimes the restriction to \(C_4(i)\) is trivial, but it becomes non-trivial when restricted to \(C_4(j)\) or \(C_4(k)\). By similar calculations we have \(\text{res}_{C_4(j)}^Q (x + y)u_{\sigma_j} = 0\), while \(\text{res}_{C_4(k)}^Q (x + y)u_{\sigma_k} \neq 0\).

Finally, we present the collection of calculated results.

Proposition A.7. Summary of calculated group cohomology

- \(H^3(Q_8, \mathbb{Z}) = 0\).
- \(H^4(Q_8, \pi_4 E_2 \otimes \sigma_i) = \mathbb{W}/4\).
- \(H^3(Q_8, \pi_4 E_2 \otimes \sigma_i) = \mathbb{W}/2\).
- \(H^2(Q_8, \pi_4 E_2 \otimes \sigma_i) = \mathbb{W}/2 \oplus \mathbb{W}/2\).
- \(H^1(Q_8, \pi_6 E_2 \otimes \sigma_i) = \mathbb{W}/2\).

Summary of calculated restrictions

- \(\text{res}_{C_4(i)}^Q h_1 \neq 0\).
- \(\text{res}_{C_4(k)}^Q h_2 \neq 0\).
• \( \text{res}^{Q_8}_1 d \neq 0 \).
• \( \text{res}^{Q_8}_2 g \neq 0 \).
• \( \text{res}^{Q_8}_3 \{x^2 + y^2\} u_\sigma \neq 0 \).

In fact, the restriction map from \( H^*(Q_8, \pi_*, E_2) \) to \( H^*(C_4, \pi_*, E_2) \) is determined by the Hurewicz image of \( E_2^{hC_4} \). The direct algebraic computation we gave above could potentially adapt to computations at higher heights.

We recall the known result of the Hurewicz image result of \( E_2^{hC_4} \). We follow names introduced in Proposition 2.14.

**Proposition A.8.** (see [HSWX18, Figure 12]) The following classes on the \( E_\infty \)-page of the \( C_4 \)-HFPSS for \( E_2 \) detects images of the Hurewicz map: \( S^0 \rightarrow E_2^{hC_4} \):

- \( s_1 a_{\sigma_2} \) at \((1,1)\) detects the image of \( \eta \in \pi_1 S^0 \),
- \( \delta_1 u_\lambda a_\sigma \) at \((3,1)\) detects the image of \( \nu \in \pi_3 S^0 \),
- \( \delta_1^4 u_{4\sigma} u_{4\lambda} \) at \((8,8)\) detects the image of \( \epsilon \in \pi_8 S^0 \),
- \( \delta_1^4 u_{4\lambda} u_{2} a_{2\sigma} \) at \((14,2)\) detects the image of \( \kappa \in \pi_{14} S^0 \),
- \( \delta_1^4 u_{4\lambda} u_{60} a_{2\lambda} \) at \((20,4)\) detects the image of \( \bar{\kappa} \in \pi_{20} S^0 \).

The unit map \( S^0 \rightarrow E_2^{hC_4} \) factors as:

\[
S^0 \xrightarrow{\text{unit}} E_2^{hQ_8} \xrightarrow{\text{res}} E_2^{hC_4}.
\]

There is a map of spectral sequences from the Adams–Novikov spectral sequence of the sphere to the \( C_4 \)-HFPSS for \( E_2 \), and it factors through the \( Q_8 \)-HFPSS for \( E_2 \). By comparing the Adams–Novikov spectral sequence of the sphere (e.g., see [Rav78, Table 2]) and the \( C_4 \)-HFPSS for \( E_2 \), we see that the classes detecting \( \eta, \nu, g, d \) do not jump filtrations under this map. Hence in the \( Q_8 \)-HFPSS for \( E_2 \), these classes are detected by classes \( h_1, h_2, d, g \), and the \( C_4 \)-restriction of these classes are non-trivial as follows.

**Proposition A.9.** The restriction map from the \( E_2 \)-page of the \( Q_8 \)-HFPSS for \( E_2 \) to the \( E_2 \)-page of the \( C_4 \)-HFPSS for \( E_2 \) is determined by the following and the multiplicative structure.

\[
\begin{aligned}
\text{res}^{Q_8}_{C_4}(h_1) &= \eta, \\
\text{res}^{Q_8}_{C_4}(c) &= 0, \\
\text{res}^{Q_8}_{C_4}(g) &= \delta_1^4 u_{4\lambda} u_{60} a_{2\lambda}.
\end{aligned}
\]

The element \( \epsilon \in \pi_8 S^0 \) is detected by a class at filtration 2 in the Adams–Novikov spectral sequence of the sphere. However, the image of \( \epsilon \) in \( \pi_8 E_2^{hC_4} \) is detected by \( \delta_1^4 u_{4\sigma} a_{4\lambda} \) at filtration 8 in the \( C_4 \)-HFPSS for \( E_2 \). There is a jump of filtration by 6. By degree reasons, in \( Q_8 \)-HFPSS(\( E_2 \)), the image of \( \epsilon \) could be potentially detected by a class of filtration \( 2 \leq f \leq 8 \). By the fact that the unit map \( S^0 \rightarrow E_2^{hQ_8} \) further factors through \( S^0 \xrightarrow{\text{unit}} E_2^{hG_2} \), the image of \( \epsilon \) is detected by the class \( c \) at \((8,2)\) (up to a unit) in \( Q_8 \)-HFPSS(\( E_2 \)). Therefore, there is an exotic restriction in \( \text{HFPSS} \) from \( Q_8 \) to \( C_4 \) that maps the class \( c \) to the class \( \delta_1^4 u_{4\sigma} a_{4\lambda} \).

**References**


---

**School of Mathematical Sciences, Nanjing Normal University**  
*Email address: zhipeng@njnu.edu.cn*

**School of Mathematics, Institute for Advanced Study**  
*Email address: hana.jia.kong@gmail.com*

**Department of Mathematics, University of Michigan**  
*Email address: guchuan@umich.edu*

**Department of Mathematics, University of California San Diego**  
*Email address: yu1248@ucsd.edu*

**Shanghai Center for Mathematical Sciences**  
*Email address: wangguozhen@fudan.edu.cn*