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The spectrum $(P \wedge bo)_{-\infty}$

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1. *Introduction*

There are spectra P_{-k} constructed from stunted real projective spaces as in [**1**] such that $H^*(P_{-k})$ is the span in $\mathbb{Z}/2[x, x^{-1}]$ of those x^i with $i \geq -k$. (All cohomology groups have $\mathbb{Z}/2$-coefficients unless specified otherwise.) Using collapsing maps, these form an inverse system

$$\ldots \to P_{-k-1} \to P_{-k} \to \ldots \to P_0, \tag{1·1}$$

which is similar to those of Lin ([**15**], p. 451). It is a corollary of Lin's work that there is an equivalence of spectra

$$\operatorname{holim}(P_{-k}) \approx \hat{S}^{-1},$$

where holim is the homotopy inverse limit ([**3**], ch. 5) and $\hat{S}^{-1}$ the 2-adic completion of a sphere spectrum. One may denote by $P^{\infty}_{-\infty}$ this holim (P_{-k}), although one must constantly keep in mind that $H^*(P^{\infty}_{-\infty}) \neq \mathbb{Z}/2[x, x^{-1}]$, but rather

$$H^i(P^{\infty}_{-\infty}) = \begin{cases} \mathbb{Z}/2 & i = -1 \\ 0 & i \neq -1. \end{cases}$$

If E is a spectrum, we may apply $E\wedge$ to the inverse system (1·1), and let $(P \wedge E)_{-\infty}$ denote holim $(P_{-k} \wedge E)$. As we shall see, this can be quite different from $P^{\infty}_{-\infty} \wedge E$.

Let bo denote the spectrum for connective ko-theory localized at 2. The spectra $P_k \wedge bo$ have had a variety of applications [**19**, **8**, **9**, **10**, and **18**] and satisfy the periodicity $\Sigma^4 P_k \wedge bo \simeq P_{k+4} \wedge bo$ ([**7**]). For $n \in \mathbb{Z}$, the homotopy groups are [**11**, **19** and **9**]

$$\pi_{4n+k}(P_{4n+1} \wedge bo) \approx \begin{Bmatrix} \hat{\mathbb{Z}}/(2^{(k+3)/2}) & k \equiv 3(8), & k > 0 \\ \mathbb{Z}/(2^{(k+1)/2}) & k \equiv 7(8), & k > 0 \\ \mathbb{Z}/2 & k \equiv 1, 2(8), & k > 0 \\ 0 & \text{otherwise.} & \end{Bmatrix} \tag{1·2}$$

If bo is applied to (1·1), then the homomorphisms $\pi_i(P_{4n-3} \wedge bo) \to \pi_i(P_{4n+1} \wedge bo)$ are surjective when $i \equiv 3(4)$ and 0 otherwise. Then (1·2) implies

$$\operatorname{invlim}(\pi_i(P_{-k} \wedge bo)) \approx \begin{Bmatrix} \hat{\mathbb{Z}}_2 & i \equiv 3(4) \\ 0 & i \not\equiv 3(4), \end{Bmatrix} \tag{1·3}$$

where $\hat{\mathbb{Z}}_2 = \operatorname{invlim}(\mathbb{Z}/2^n)$ is the 2-adic integers. This suggests the following theorem, our main result. Let $\hat{H}$ denote the Eilenberg–MacLane spectrum satisfying

$$\pi_i(\hat{H}) = \begin{cases} \hat{\mathbb{Z}}_2 & i = 0 \\ 0 & i \neq 0. \end{cases}$$

THEOREM 1·4. *There is an equivalence of spectra* $(P \wedge bo)_{-\infty} \approx \bigvee_{i \in \mathbb{Z}} \Sigma^{4i-1} \hat{H}$.

As an immediate corollary of 1·4 we have (using [**19**], 1·6):

COROLLARY 1·5. *There is an equivalence of spectra* $(P \wedge bu)_{-\infty} \approx \bigvee_{i \in \mathbb{Z}} \Sigma^{2i-1}\hat{H}$. This corollary is the case $n = 1$ of the following conjecture.

Conjecture 1·6. Let $BP\langle n\rangle$ denote the spectra associated to the prime 2 which were constructed in [**12**]. There is an equivalence of spectra

$$(P \wedge BP\langle n\rangle)_{-\infty} \approx \bigvee_{i \in \mathbb{Z}} \Sigma^{2i-1} B\hat{P}\langle n-1\rangle,$$

where $\hat{E}$ denotes the 2-adic compeltion of the spectrum E.

The proof of 1·4 occupies Section 2. In Section 3, Theorem 1·4 is applied to construct 2-adic characteristic classes for Spin-bundles.

THEOREM 1·7. *There are elements* $Q_i \in H^{4i}(\mathrm{BSpin}; \hat{\mathbb{Z}}_2)$ *such that*

(i) *the* mod-2 *reduction* $\rho_2 Q_i$ *is the Wu class* v_{4i};

(ii) *there is a map* τ *such that the composite*

$$\mathrm{BSpin} \xrightarrow{\langle Q_i\rangle} \bigvee_{i \geqslant 0} \Sigma^{4i}\hat{H} \xrightarrow{\tau} \Sigma P_1 \wedge bo \to \Sigma P_n \wedge bo$$

is the orientation constructed in [**7**].

The applicability and limitations of this result will be discussed in Section 3. In Section 4, 1·4 is compared with recent work of Jones and Wegmann [**13**].

2. *Proof of Theorem* 1·4

Definition 2·1. Let H denote the Eilenberg–MacLane spectrum for $\mathbb{Z}_{(2)}$, the subring of the rationals with odd denominators. Let $H_{-k} = \bigvee_{j \geqslant -k} \Sigma^{4j-1} H$ and $i = 0 \vee 1$: $H_{-k} \to H_{-k-1} = \Sigma^{-4k-5} H \vee H_{-k}$. Let $\hat{H}_{-k} = \bigvee_{j \geqslant -k} \Sigma^{4j-1}\hat{H}$. Let $\hat{H}_{-\infty} = \bigvee_{j \in \mathbb{Z}} \Sigma^{4j-1}\hat{H} =$ dirlim H_{-k}, where the maps in the direct system are the inclusion $H_{-k} \to H_{-k-1}$.

Let c denote the collapsing map $P_k \to P_{k+l}$ for stunted projective spaces. Most of the work in the proof of 1·4 is incorporated in

THEOREM 2·2. *There are maps* j_k *for all* $k \geqslant 0$, *surjective in* $\pi_{4*-1}(\ )$, *such that the diagrams*

$$\begin{array}{ccc} H_{-k} & \xrightarrow{\ 1\ } & H_{-k} \\ \downarrow{\scriptstyle i} & & \\ H_{-k-1} & & \downarrow{\scriptstyle j_k} \\ \downarrow{\scriptstyle j_{k+1}} & & \\ P_{-4k-3} \wedge bo & \xrightarrow{\ c\ } & P_{-4k+1} \wedge bo \end{array} \qquad \text{commute.}$$

Proof that 2·2 *implies* 1·4

We apply the 2-completion functor to the diagram of 2·2. Since $\pi_j(P_{-4k+1} \wedge bo)$ is finite, $(P_{-4k+1} \wedge bo)^{\wedge}_2 \approx P_{-4k+1} \wedge bo$, so that we obtain maps $\hat{j}_k: H_{-k} \to P_{-4k+1} \wedge bo$. If $l \geqslant k$, let $j_{l,k}$ denote the composite

$$\hat{H}_{-l} \xrightarrow{\hat{j}_l} P_{-4l+1} \wedge bo \xrightarrow{c} P_{-4k+1} \wedge bo.$$

Then 2·2 implies commutativity of

$$\begin{array}{ccccc} \hat{H}_{-l} & \xrightarrow{\ i\ } & \hat{H}_{-l-1} & \longrightarrow & \dots \\ & {\scriptstyle j_{l,k}}\searrow & & \swarrow{\scriptstyle j_{l+1,k}} & \\ & & P_{-4k+1} \wedge bo & & \end{array}$$

inducing a map

$$\hat{H}_{-\infty} \xrightarrow{q_k} P_{-4k+1} \wedge bo.$$

Because each map in the direct system $\hat{H}_{-l} \to \hat{H}_{-l-1} \to \dots$ is an inclusion of a wedge summand, $\lim^1_l [\hat{H}_{-l}, Y] = 0$ for any Y, so that q_k is unique. Thus commutativity of

$$\begin{array}{ccccc} & & \hat{H}_{-\infty} & & \\ & {\scriptstyle q_{k+1}} \swarrow & & \searrow {\scriptstyle q_k} & \\ \dots \longrightarrow & P_{-4k-3} \wedge bo & \xrightarrow{c} & P_{-4k+1} \wedge bo & \end{array} \tag{2·3}$$

is clear because there is a corresponding factorization of the inverse systems. The homomorphism $\pi_*(\hat{H}_{-\infty}) \to \operatorname{invlim}_k \pi_*(P_{-4k+1} \wedge bo)$ is an isomorphism by (1·3) and the surjectivity of $\pi_{4*-1}(j_k)$ given in 2·2. There is an exact sequence

$$0 \to \lim_k{}^1 \pi_{i+1}(P_{-4k+1} \wedge bo) \to \pi_i(P \wedge bo)_{-\infty} \to \operatorname{invlim} \pi_i(P_{-4k+1} \wedge bo) \to 0,$$

and the $\lim^1$-term is 0 because $\pi_{i+1}(P_{-4k+1} \wedge bo)$ is finite. Thus the map $\hat{H}_{-\infty} \to (P \wedge bo)_{-\infty}$ induced by (2·3) induces an isomorphism of homotopy groups and hence is an equivalence of spectra. ▌

In proving 2·2, the following elementary construction and proposition will be useful.

Definition 2·4. If $f: X \to Y \wedge bo$ is any map, let $\bar{f}: X \wedge bo \to Y \wedge bo$ denote the composite $(X \wedge \mu_{bo}) \circ (f \wedge bo)$.

PROPOSITION 2·5. $\bar{f}_*: \pi_*(X \wedge bo) \to \pi_*(Y \wedge bo)$ *is a homomorphism of* $\pi_* bo$*-modules.*

Let Y_k denote the cofibre of a generator of $\pi_{-4k-1}(P^{-4k}_{-4k-3}) \approx \mathbb{Z}/8$. Thus

$$H^*(Y_k) \approx \langle y, Sq^1 y, Sq^2 Sq^1 y : |y| = -4k-3 \rangle$$

and there is a cofibration

$$Y_k \xrightarrow{b_k} S^{-4k} \xrightarrow{\alpha_k} \Sigma P^{-4k}_{-4k-3}.$$

Let $\iota: S^0 \to bo$ denote the unit.

The following result plays a key role in the proof of 2·2.

LEMMA 2·6. *For* $k \geq 0$ *there are maps* g_k *and* f_k *such that*

$$\begin{array}{ccccc} P_{-4k-3} & \xrightarrow{c_k} & P_{-4k+1} & \xrightarrow{a_k} & \Sigma P^{-4k}_{-4k-3} \\ \downarrow {\scriptstyle f_k} & & \downarrow {\scriptstyle g_k} & \square_k & \downarrow {\scriptstyle 1 \wedge \iota} \\ Y_k \wedge bo & \xrightarrow{b_k \wedge bo} & S^{-4k} \wedge bo & \xrightarrow{\alpha_k \wedge bo} & \Sigma P^{-4k}_{-4k-3} \wedge bo \end{array}$$

is a commutative diagram of cofibrations, and maps $h_k: Y_k \to S^{-4k-4} \wedge bo$ *such that* $g_{k+1} = \bar{h}_k \circ f_k$.

Proof. The induction is begun by constructing g_0 so that $\square_0$ commutes. We will need:

LEMMA 2·7. $[P_1, \Sigma P^0_{-3} \wedge bo] \approx \mathbb{Z}/8$, *with a filtration* 1 *generator. Filtration, here and elsewhere, refers to the precise filtration in the Adams spectral sequence.*

Proof. The groups $[P^m_1, \Sigma P^0_{-3} \wedge bo]$ are finite, so that the $\lim^1$-terms vanish and $[P_1, \Sigma P^0_{-3} \wedge bo] \approx \operatorname{invlim} [P^m_1, \Sigma P^0_{-3} \wedge bo]$. The lemma follows from

$$[P^{8n+4}_1, \Sigma P^0_3 \wedge bo] \approx \pi_{-2}(P^{-2}_{-8n-5} \wedge P^0_{-3} \wedge bo) \approx \mathbb{Z}/8$$

on a filtration 1 generator g_n satisfying $i*g_{n+1} = g_n$. The last isomorphism is given above filtration 0 by ([5]; ch. 3). There are no filtration 0 classes because the action of Sq^1 shows that there are no nontrivial homomorphisms $H^*P_1 \to H^*\Sigma P^0_{-3}$. ▌

Let $\lambda: P_1 \to S^0$ be a map such that Sq^n is nonzero on the bottom class of the cofibre for all $n \geq 2$ [17, 14]. Then $\alpha_0\lambda \wedge \iota$ and $a_0 \wedge \iota$ are both filtration 1 elements of the group calculated in 2·7. ⟦Sq^2 is nonzero in the mapping cone of each.⟧ Hence for an appropriate generator u of $\mathbb{Z}/8$, $u\alpha_0\lambda \wedge \iota = a_0 \wedge \iota$. Let $g_0 = u\lambda \wedge \iota$. Then $\square_0$ is satisfied.

Now suppose we have constructed g_k satisfying $\square_k$. Let f_k be the induced map of fibres.

LEMMA 2·8. *The function* $[Y_k, S^{-4k-4} \wedge bo] \xrightarrow{\psi} [P_{-4k-3}, \Sigma P^{-4k-4}_{-4k-7} \wedge bo]$ *defined by* $\psi(h) = (\alpha_{k+1} \wedge bo) \circ \bar{h} \circ f_k$ *is surjective.*

Choose $h_k \in \psi^{-1}(a_{k+1} \wedge \iota)$ and let $g_{k+1} = \bar{h}_k \circ f_k$. Then g_{k+1} satisfies $\square_{k+1}$, completing the inductive proof of 2·6. ▌

Proof of 2·2. Applying $^-$ to the vertical maps in the diagram in 2·6 shows that fibre $(\bar{f}_k) \to$ fibre$(\bar{g}_k)$ is an equivalence, which we use to identify the two. There is a commutative diagram

$$\begin{array}{ccc}
\text{fibre }(\bar{f}_k) & \xrightarrow{=} & \text{fibre }(\bar{g}_k) \\
\downarrow{\scriptstyle \bar{h}'_k} & & \\
\text{fibre }(\bar{g}_{k+1} = \bar{h}_k\bar{f}_k) & & \downarrow \\
\downarrow & & \\
P_{-4k-3} \wedge bo & \xrightarrow{c_k \wedge bo} & P_{-4k+1} \wedge bo
\end{array}$$

which we will show is the diagram of 2·2.

Fibre$(\bar{g}_0) = H_0$ by ([19], 4·5). Suppose we have shown fibre$(\bar{g}_k) = H_{-k}$. We use the following commutative diagram, in which all rows and columns are cofibrations.

$$\begin{array}{ccccccc}
 & & & & \Sigma^{-1}\text{ cof }(\bar{h}_k) & \longrightarrow & Y_k \wedge bo \\
 & & & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle p_k} \\
\text{fibre }(\bar{f}_k) & \xrightarrow{\bar{h}'_k} & \text{fibre }(\bar{h}_k\bar{f}_k) & \longrightarrow & \Sigma^{-1}\text{ cof }(\bar{h}_k) & \xrightarrow{r_k} & \Sigma\text{ fibre }(\bar{f}_k) \\
\downarrow{\scriptstyle i_k} & & \downarrow & & \downarrow & & \downarrow \\
P_{-4k-3} \wedge bo & \xrightarrow{=} & P_{-4k-3} \wedge bo & \longrightarrow & \star & \longrightarrow & \Sigma P_{-4k-3} \wedge bo \\
\downarrow{\scriptstyle \bar{f}_k} & & \downarrow{\scriptstyle \bar{h}_k\bar{f}_k} & & \downarrow & & \downarrow{\scriptstyle \Sigma\bar{f}_k} \\
Y_k \wedge bo & \xrightarrow{\bar{h}_k} & S^{-4k-4} \wedge bo, & \longrightarrow & \text{cof }(\bar{h}_k) & \longrightarrow & \Sigma Y_k \wedge bo
\end{array}$$

LEMMA 2·9. $\text{cof}(\bar{h}_k) = \Sigma^{-4k-4}H$.

$H^*(\Sigma\bar{f}_k; \mathbb{Z}_{(2)})$ is injective, hence $H^*(p_k; \mathbb{Z}_{(2)}) = 0$, and therefore $H^*(r_k; \mathbb{Z}_{(2)}) = 0$. Since fibre$(\bar{f}_k) = H_{-k}$, this implies that $r_k = 0$, and hence the cofibration

$$\begin{array}{ccccc}
\text{fibre }(\bar{f}_k) & \longrightarrow & \text{fibre }(\bar{h}_k\bar{f}_k) & \longrightarrow & \Sigma^{-1}\text{ cof }(\bar{h}_k) \\
\| & & \| & & \| \\
H_{-k} & & \text{fibre }(\bar{g}_{k+1}) & & \Sigma^{-4k-5}H
\end{array}$$

splits, implying fibre $(\bar{g}_{k+1}) = H_{-k-1}$ and thus extending the induction. Surjectivity of $\pi_{-4k-1}(i_k)$ follows from $\pi_{-4k-1}(Y_k \wedge bo) = 0$, and the remaining surjectivity of π_{4*-1} is carried along by the induction. ∎

In the proofs of 2·8 and 2·9 which follow, we abbreviate Ext $(M, \mathbb{Z}/2)$ as Ext (M), and Ext (H^*X) as Ex(X). A_i denotes the subalgebra of the mod 2 Steenrod algebra generated by $\{Sq^n : n \leqslant 2^i\}$. We use charts of $\mathrm{Ext}^{s,t}(\ )$ similar to those of [8] and [9], with co-ordinates $(t-s, s)$. We also use freely the change-of-rings theorem ([5], 3·1).

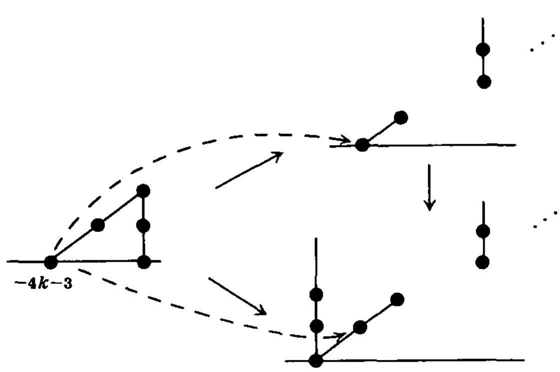
Proof of Lemma 2·9. There is an exact sequence

$$\to \mathrm{Ext}^{s,t}_{A_1}(\Sigma^{-4k-5}\mathbb{Z}/2) \to \mathrm{Ext}^{s,t}_{A_1}(\Sigma^{-4k-5}A_1//A_0) \to \mathrm{Ex}^{s,t}_{A_1}(Y_k) \to \mathrm{Ext}^{s+1,t}_{A_1}(\Sigma^{-4k-5}\mathbb{Z}/2) \to$$
$$\mathrm{Ext}^{s,t}_{A_1}(\Sigma^{-4k-5}A_1//A_0) \approx \mathrm{Ext}^{s,t}_{A_0}(\Sigma^{-5}\mathbb{Z}/2),$$

so that $\mathrm{Ex}_A(Y_k \wedge bo) \approx \mathrm{Ex}_{A_1}(Y_k)$ is given by the chart obtained from that of Ext_{A_1} $(\Sigma^{-4k-4}\mathbb{Z}/2)$ by eliminating the initial tower and decreasing filtration by 1. Applying $\mathrm{Ex}_A(\ )$ to

$$\begin{array}{ccc} & & Y_k \wedge bo \\ & \nearrow & \\ P_{-4k-3} \wedge bo & & \downarrow \bar{h}_k \\ & \searrow & \\ & & S^{-4k-4} \wedge bo \end{array}$$

we obtain



and hence $\mathrm{Ex}_A(\bar{h}_k)$ is nontrivial on the bottom class. Since $\pi_*(\bar{h}_k)$ is $\pi_* bo$-linear by 2·5, it is injective. Thus

$$\pi_i(\mathrm{cof}\,(\bar{h}_k)) \approx \begin{cases} \mathbb{Z}_{(2)} & i = -4k-4 \\ 0 & \text{otherwise.} \end{cases}$$ ∎

Proof of Lemma 2·8. It follows from the definitions that ψ equals the composite

$$[Y_k, S^{-4k-k} \wedge bo] \xrightarrow{\psi_1} [Y_k, \Sigma P^{-4k-4}_{-4k-7} \wedge bo] \xrightarrow{\psi_2} [P_{-4k-3}, \Sigma P^{-4k-4}_{-4k-7} \wedge bo],$$

with $\psi_1(h) = (\alpha_{k+1} \wedge bo) \circ h$ and $\psi_2(l) = \bar{l} \circ f_k$. We show ψ_1 and ψ_2 surjective.

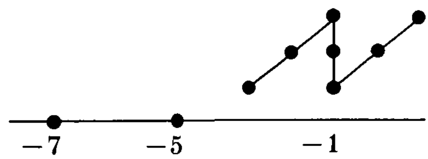
ψ_1 fits into an exact sequence whose next term is $[Y_k, \Sigma Y_{k+1} \wedge bo]$. But $\Sigma Y_{k+1} = \Sigma^{-3} Y_k$ so $[Y_k, \Sigma Y_{k+1} \wedge bo] = \pi_3(DY_k \wedge Y_k \wedge bo)$, where D denotes Spanier–Whitehead dual. If $\theta: DY_k \wedge Y_k \to S^0$ is a duality map, then coker $(H^*(\theta)) \approx \Sigma^{-3}A_1$. Thus $\mathrm{Ex}_{A_1}(DY_k \wedge Y_k) \approx \mathrm{Ex}_A(bo \vee \Sigma^{-3}H\mathbb{Z}/2)$, which is 0 in $t-s = 3$.

To show ψ_2 surjective, we begin by showing both groups are $\mathbb{Z}/8$ on filtration 1 generators. For the target group, this is the same calculation as 2·7.

$[Y_k, \Sigma P_{-4k-7}^{-4k-4} \wedge bo] \approx \pi_{-1}(DY_k \wedge P_{-4k-7}^{-4k-4} \wedge bo)$ can be calculated by using the exact sequence of A_1-modules

$$0 \to \Sigma^{4k+5}\mathbb{Z}/2 \to \Sigma^{4k} A_1//A_0 \to H^*DY_k \to 0$$

to see that $\mathrm{Ex}_{A_1}(DY_k \wedge P_{-4k-7}^{-4k-4})$ is given by the chart



To see that ψ_2 sends one generator to another, note that these can be characterized as maps nontrivial on the bottom cell of Y_k and P_{-4k-3}, respectively. The restriction of f_k to the bottom cell is the (filtration 0) generator of $\pi_{-4k-3}(Y_k \wedge bo)$. If l is nontrivial on the bottom cell, it is clear from the definition that $\bar{l}f_k$ is nontrivial on the bottom cell.

3. *Characteristic classes*

We first expand upon the discussion in [6] that the orientations of [7] factor through $P_{-\infty} \wedge bo$ or $P_{-\infty} \wedge MO\langle\rho\rangle$.

Let ρ be a positive integer congruent to 0, 1, 2, or 4 (mod 8), and let a_ρ denote the order of the cyclic 2-group $\widetilde{KO}(RP^{\rho-1})$. Let $B_N = BO_N\langle\rho\rangle$ denote the classifying space for N-plane bundles trivial on the $(\rho-1)$-skeleton, and $M = MO\langle\rho\rangle$ the associated stable Thom spectrum. Assume $N \equiv 0(a_\rho)$.

The primary M-obstruction for finding k sections on B_N-bundles was defined in [7] to be the map

$$B_N \xrightarrow{\tilde{g}_{N,k}} \Sigma P_{N-k} \wedge M$$

defined by viewing the composite

$$B_N \times P^{k-1} \xrightarrow{\gamma_N \otimes \xi} B_N \to \Sigma^N M$$

as a stable map so that we can consider its restriction to $B_N \wedge P^{k-1}$, dualizing to obtain

$$B_N \xrightarrow{f_k} \Sigma P_{-k}^{-2} \wedge \Sigma^N M,$$

and then following by the composite

$$\Sigma P_{-k}^{-2} \wedge \Sigma^N M \underset{\simeq}{\xrightarrow{e_k^{-1}}} \Sigma P_{N-k}^{N-2} \wedge M \xrightarrow{i} \Sigma P_{N-k} \wedge M,$$

where e_k is the equivalence of [20].

THEOREM 3·1. *For all positive N and L with $N \equiv 0(a_\rho)$, there are maps*

$$B_N \xrightarrow{\tilde{g}_{N,L}} \Sigma P_{N-L} \wedge M$$

such that (i) $c \circ \tilde{g}_{N,L+1} = \tilde{g}_{N,L}$ *and* (ii) *if* $N \geq L$ *then* $\tilde{g}_{N,L} = g_{N,L}$. *Thus there are factorizations*

$$\begin{array}{ccc} & \nearrow^{g_N} & \Sigma(P \wedge M)_{-\infty} \\ B_N & & \downarrow \\ & \searrow_{g_{N,k}} & \Sigma P_{N-k} \wedge M \end{array}$$

Proof. The maps $\tilde{g}_{N,L}$ are constructed as the composites $ie_L^{-1}f_L$, where f_L, e_L, and i are defined similarly to the maps f_k, e_k, and i above. Now (i) follows easily from the observation that e_L may be written as the suspension of the composite

$$P_{N-L}^{N-2} \wedge M \xrightarrow{\Delta} (P_{-L} \wedge P_N)^{(N-2)} \wedge M \xrightarrow{T(N\xi)} (P_{-L} \wedge \Sigma^N M)^{(N-2)} \wedge M$$
$$\hookrightarrow P_{-L}^{-2} \wedge \Sigma^N M \wedge M \xrightarrow{\mu} P_{-L}^{-2} \wedge \Sigma^N M. \blacksquare$$

A similar result is obtained when M is replaced by bo and $N \equiv 0(4)$, using the map $MO\langle 4\rangle = \mathrm{MSpin} \to bo$ of [2]. Compatibility of the maps g_N with respect to increasing N is not clear; however, for any particular bundle one can choose any sufficiently large N. The characteristic class $\langle Q_i\rangle$ of 1·7 is the composite

$$\mathrm{BSpin}_N \xrightarrow{g_N} \Sigma(P \wedge bo)_{-\infty} \approx \bigvee_{i \in \mathbb{Z}} \Sigma^{4i}\hat{H}.$$

The map τ of 1·7 (ii) is the 2-completion of the map

$$\bigvee_{i \geq 0} \Sigma^{4i}H \simeq (S^0 \cup_{u\lambda} CP_1) \wedge bo \to \Sigma P_1 \wedge bo,$$

used in the proof of 2·2. To prove 1·7 (ii) we use the commutative diagram

$$\begin{array}{ccccc}
\bigvee_{i\geq 0} \Sigma^{4i}\hat{H} & \xhookrightarrow{i} \bigvee_{i\in\mathbb{Z}} \Sigma^{4i}H & \xrightarrow{\simeq} & \Sigma(P \wedge bo)_{-\infty} & \\
\downarrow & \searrow^{\tau} & & \downarrow c_\infty & \nwarrow \\
(S^0 \cup CP_1) \wedge bo \simeq \bigvee_{i\geq 0} \Sigma^{4i}\hat{H} & \xrightarrow{\quad i \quad} & & \Sigma P_1 \wedge bo & \xleftarrow{g_{N,N-1}} \mathrm{BSpin}_N. \\
 & & & \downarrow & \swarrow \\
 & & & \Sigma P_m \wedge bo &
\end{array} \qquad (3·2)$$

$g_{N,N-1}$ factors through c_∞ by 3·1. Using the equivalence of 1·4, we obtain a map $\mathrm{BSpin}_N \to \bigvee_{i\in\mathbb{Z}} \Sigma^{4i}\hat{H}$, which factors through $\bigvee_{i\geq 0} \Sigma^{4i}\hat{H}$ because

$$H^*(\mathrm{BSpin}_N; \pi_*(\mathrm{cof}(i))) = 0. \blacksquare$$

Because of 1·7 (ii), the maps $\mathrm{BSpin}_N \to \Sigma P_m \wedge bo$ factor through $\mathrm{BSpin}_N/\mathrm{BSpin}_m$. Thus if

$$X \xrightarrow{\theta} \mathrm{BSpin}_N \xrightarrow{\langle Q_i\rangle} \bigvee_{i\geq 0} \Sigma^{4i}\hat{H} \xrightarrow{\tau_m} \Sigma P_m \wedge bo$$

is nontrivial, then $gd(\theta) > m$.

Despite the fact that Q_i is not canonical, depending upon the choice of the equivalence in 1·4 and perhaps upon N, its mod 2 reduction is, and is given by 1·7 (i). To prove this, we recall from [19] that

$$\bigvee_{i\geq 0} \Sigma^{4i}H \xrightarrow{\tau} \Sigma P_1 \wedge bo$$

satisfies $\tau^*(\sigma a_{4i-1} \otimes 1) = e_{4i}$ with

$$e_{4i} = \iota_{4i} + \sum_{j=0}^{i-1} \chi Sq^{4(i-j)} e_{4j}.$$

The map

$$\mathrm{BSpin} \xrightarrow{g} \Sigma P_1 \wedge bo$$

of [7] satisfies $g^*(\sigma a_{4i-1} \otimes 1) = w_{4i}$. Diagram 3·2 shows $g = \tau \circ Q$. Thus

$$w_{4i} = Q^* e_{4i} = Q^*\iota_{4i} + \sum_{j=0}^{i-1} \chi Sq^{4(i-j)} Q^* e_{4j} = \rho(Q_i) + \sum_{j=0}^{i-1} \chi Sq^{4(i-j)} w_{4j}. \qquad (3·3)$$

The Adem relations and Wu relations imply $\chi Sq^m w_{4i-m} = 0$ if $m \not\equiv 0(4)$. Thus 3·3 becomes

$$\rho(Q_i) = \sum_{m=0}^{4i} \chi Sq^m w_{4i-m} = v_{4i} \quad \text{(see [21]).} \blacksquare$$

In order for the Q_i to be useful in obstruction theory, we need to know more than just their mod 2 reduction, but we have not been able to choose them in a controllable fashion. It is tempting to conjecture that Q might be the multiplicative characteristic class

$$A(\alpha) = \Pi \frac{X_i}{\sinh X_i}$$

(where $\Pi(1+X_i^2)$ is the Pontryagin class $p(\alpha)$). This appeared in the recent work of Crabb ([4], 2·4) and satisfies $\rho A = v$.

Even knowing this, the application to obstruction theory would be quite complicated. For example, if Q is any multiplicative characteristic class reducing to v, then if H_n is the Hopf bundle over quaternionic projective space QP^n,

$$Q(4H_3) = 1 + 4a_1 X + 2a_2 X^2 + 8b_3 X^3,$$

with a_1 and a_2 odd and $b_3 \in \mathbb{Z}_{(2)}$. In [10] we showed $gd(4H_3) > 9$, essentially because $\binom{4}{3} \not\equiv 0 \bmod \pi_{12}(\Sigma P_9 \wedge bo) \approx \mathbb{Z}/8$. From the new perspective, the bundle is classified by the composite

$$QP^3 \xrightarrow{4H} \mathrm{BSpin}_{12} \to \hat{H} \vee \Sigma^4\hat{H} \vee \Sigma^8\hat{H} \vee \Sigma^{12}\hat{H} \to P_9 \wedge bo.$$

The group $[QP^3, \Sigma P_9 bo]$ is $\mathbb{Z}/8$, generated by

$$QP^3 \xrightarrow{c} S^{12} \xrightarrow{g} P_9 \wedge bo.$$

By (3·4), the map

$$QP^3 \xrightarrow{Q_1, Q_2, Q_3} \Sigma^4\hat{H} \vee \Sigma^8\hat{H} \vee \Sigma^{12}\hat{H}$$

has $Q_2 = 2 \cdot \text{odd}$, and the attaching map 2ν of the 12-cell in QP^3 causes this to contribute $2^2 \cdot \text{gen}$ to $[QP^3, \Sigma P_9 \wedge bo]$.

4. *Relationship with the work of Jones and Wegmann*

An easy consequence of Lin's theorem [15] is that for any finite spectrum E there is an equivalence $S^{-1}\hat{E} \to (P \wedge E)_{-\infty}$. Our 1·4 implies that this is not true for $E = bo$.

If E is any spectrum, Jones and Wegmann [13] constructed an inverse system of spectra $P_{-k}E = \Sigma^k D_2(\Sigma^{-k}E)$. Let $P_{-\infty}E = \mathrm{holim}\,(P_{-k}E)$. They showed that if E is a suspension spectrum there are compatible maps $P_{-k} \wedge E \to P_{-k}E$, inducing f_E: $(P \wedge E)_{-\infty} \to P_{-\infty}E$, such that if E is finite and h is a connected (co)homology theory, then $h^*(f_E)$ and $\hat{h}_*(f_E)$ are isomorphisms.

In our preprint we argued from 1·4 that no such map could exist for $E = bo$, but a better argument utilizes the recent result of Wegmann's thesis, that for any spectrum of finite type (e.g. bo) there is an equivalence $g_E: S^{-1}\hat{E} \to P_{-\infty}E$. It is clear that g_{bo} could not factor through $\bigvee \Sigma^{4i-1}\hat{H}$.

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