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# Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups

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**Abstract**

Let $G$ be a closed subgroup of the semi-direct product of the $n$th Morava stabilizer group $S_n$ with the Galois group of the field extension $\mathbb{F}_{p^n}/\mathbb{F}_p$. We construct a "homotopy fixed point spectrum" $E_n^{hG}$ whose homotopy fixed point spectral sequence involves the continuous cohomology of $G$. These spectra have the expected functorial properties and agree with the Hopkins-Miller fixed point spectra when $G$ is finite.


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*Keywords:* Commutative $S$-algebra; Continuous homotopy fixed point spectra; Morava stabilizer group; Adams spectral sequence

## 1. Introduction

If a (discrete) group $G$ acts on a spectrum $Z$, one can form the homotopy fixed point spectrum, often denoted $Z^{hG}$. It is given by the $G$ fixed points of the function spectrum $F(EG,Z)$, where $EG$ is a contractible free $G$-space. There is then, for each spectrum $X$, a conditionally convergent spectral sequence

$$H^*(G,Z^*X) \Rightarrow [X,Z^{hG}]^*,$$

obtained from the usual filtration of the bar construction for $EG$. This spectral sequence is called the homotopy fixed point spectral sequence. Of course, the construction of such a homotopy fixed

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point spectrum requires that the group act in an appropriate point-set category and not just up to homotopy.

However, there are situations in stable homotopy theory where group actions only exist in the stable category; that is, up to homotopy. In fact, the most important group action in the whole chromatic approach to stable homotopy theory—the action of the extended Morava stabilizer group $G_n$ on the $p$-local Landweber exact spectrum $E_n$ (see [22,5], and Section 1 for a resumé)—arises in this way. Yet the situation in the case of this action is not hopeless. Indeed, H.R. Miller and the second author have proved that $E_n$ is an $A_\infty$ ring spectrum and that the space of $A_\infty$ ring maps from $E_n$ to itself has weakly contractible path components. Furthermore, the set of path components of this space is in bijective correspondence with the set of homotopy classes of ring spectrum maps from $E_n$ to itself (see [27] for an account of this theory.) Since $G_n$ acts on $E_n$ by maps of ring spectra, it follows that the action can be taken to be one of $A_\infty$ maps, and although this action is still only an action up to homotopy, it is an honest action up to “all higher $A_\infty$ homotopies.” Standard techniques then allow one to replace $E_n$ by an equivalent spectrum on which $G_n$ acts on the nose. Hence, if $G$ is a (finite) subgroup of $G_n$, there is an $A_\infty$ homotopy fixed point spectrum $E_n^{hG}$. These spectra have already had a number of interesting applications in stable homotopy theory (see e.g., [23]). Subsequently, P.G. Goerss and Hopkins [12–15] extended the machinery of Hopkins-Miller to the $E_\infty$ setting, and thus $E_n^{hG}$ is an $E_\infty$ ring spectrum.

Unfortunately, this is still not an entirely satisfactory state of affairs. $G_n$ is a profinite group, so one might hope to define, for $G$ a closed subgroup of $G_n$, a “continuous homotopy fixed point spectrum” denoted—abusively—by $E_n^{hG}$ whose “homotopy fixed point spectral sequence” starts with the *continuous* cohomology of $G$. (This is why we restricted to finite subgroups in the previous paragraph.) Indeed, it is the continuous cohomology of $G_n$ that is important in stable homotopy theory—by Morava’s change of rings theorem, the $K(n)_*$-local $E_n$-Adams spectral sequence (see Appendix A) for $\pi_* L_{K(n)}S^0$ has the form

$$H_c^*(G_n, E_{n*}) \Rightarrow \pi_* L_{K(n)}S^0. \tag{1.1}$$

Here $K(n)$ denotes the $n^{\text{th}}$ Morava $K$-theory, and $L_{K(n)}$ denotes $K(n)_*$-localization. The spectral sequence (1.1) thus suggests that $L_{K(n)}S^0$ should be the $G_n$ homotopy fixed point spectrum of $E_n$ in this continuous sense.

The case $n=1$ provides further evidence for the existence of continuous homotopy fixed point spectra. Here we have that $E_1$ is the $p$-completion of the spectrum corepresenting complex $K$-theory and that $G_1 = \mathbb{Z}_p^\times$, the group of multiplicative units in the $p$-adic integers. The element $a$ in $\mathbb{Z}_p^\times$ corresponds to the Adams operation $\psi^a$. Now the component $\Omega_0^\infty E_1$ of the 0th space of $E_1$ containing the base point is just the $p$-completion of $BU$, and, according to Quillen [24], this space is equivalent to the $p$-completion of $BGL(\bar{\mathbb{F}}_l)^+$ for any prime $l$ not equal to $p$. (As usual, $BGL(R)^+$ is the connected space representing the algebraic $K$-theory of the ring $R$, and $\bar{\mathbb{F}}_l$ is the algebraic closure of the field with $l$ elements.) Under this equivalence, the automorphism of $BGL(\bar{\mathbb{F}}_l)^+$ induced by the frobenius automorphism of $\bar{\mathbb{F}}_l$ corresponds to the Adams operation $\psi^l$ on $BU_p$. More generally, the profinite group $\hat{\mathbb{Z}} = \mathrm{Gal}(\bar{\mathbb{F}}_l/\mathbb{F}_l)$ acts on $BGL(\bar{\mathbb{F}}_l)^+$; if $G = \mathrm{Gal}(\bar{\mathbb{F}}_l/k)$ is any closed subgroup of $\hat{\mathbb{Z}}$, then

$$\pi_i(BGL(k)^+) = [\pi_i(BGL(\bar{\mathbb{F}}_l)^+)]^G$$

for all $i \geqslant 1$ [24; Corollary to Theorem 8]. Since

$$H_c^s(G, \pi_i BGL(\bar{\mathbb{F}}_l)^+) = 0$$

for all $i$ and all $s > 0$—again by the computations of Quillen—this suggests that $BGL(k)^+$ should be regarded as the continuous homotopy $G$ fixed point spectrum of $BGL(\bar{\mathbb{F}}_l)^+$.

In this paper, we construct spectra $E_n^{hG}$ for $G$ a closed subgroup of $G_n$, having the desired properties of continuous homotopy fixed point spectra. Our construction proceeds in two steps. First we construct $E_n^{hU}$ for $U$ an open—and hence closed—subgroup of $G_n$, then we construct $E_n^{hG}$ as an appropriate homotopy colimit of the $E_n^{hU}$'s, for $G \subset U$.

We need to introduce a little more notation in order to state our main results. Let $R_{G_n}^+$ denote the category whose objects are continuous finite left $G_n$-sets together with the left $G_n$-set $G_n$. The morphisms are continuous $G_n$-equivariant maps, and we denote by $r_g : G_n \to G_n$ the map given by right multiplication by $g \in G_n$. Let $\mathscr{E}$ denote the category of commutative $S^0$-algebras in the category of $S^0$-modules of May et al. [11]. (A comparison of this category with the category of $\mathscr{L}$-spectra for $\mathscr{L}$ the linear isometries operad—and hence of the category of $E_\infty$ ring spectra—is carried out in [11, II.4].) Finally, since the natural number $n$ will be fixed throughout this paper, we write $\hat{L}$ for the functor $L_{K(n)}$. Here then is our first main result.

**Theorem 1.** *There is a functor $\mathbf{F} : (R_{G_n}^+)^{\mathrm{opp}} \to \mathscr{E}$ with the following properties.*

(i) *$\mathbf{F}(S)$ is $K(n)_*$-local for each object $S$ in $R_{G_n}^+$.*
(ii) *$\mathbf{F}(G_n) = E_n$ and $\mathbf{F}(r_g) : E_n \to E_n$ is the action of $g \in G_n$ on $E_n$.*
(iii) *There is a natural isomorphism*

$$\pi_* \hat{L}(\mathbf{F}(S) \wedge E_n) \approx \mathrm{Map}(S, \mathrm{Map}_c(G_n, E_{n*}))^{G_n}$$

*of completed $\pi_* \hat{L}(E_n \wedge E_n) = \mathrm{Map}_c(G_n, E_{n*})$-comodules, where the action of $G_n$ on $\mathrm{Map}_c(G_n, E_{n*})$, the set of continuous functions from $G_n$ to $E_{n*}$, is given by*

$$(gf)(g') = f(g^{-1}g')$$

*for $g, g' \in G_n$ and $f \in \mathrm{Map}_c(G_n, E_{n*})$. In particular, $\mathbf{F}(*) \simeq \hat{L}S^0$.*
(iv) *Define $E_n^{hU} = \mathbf{F}(G_n/U)$, $U$ an open subgroup of $G_n$, and let $Z$ be any CW-spectrum. There is a natural strongly convergent spectral sequence*

$$H_c^*(U, E_n^* Z) \Rightarrow (E_n^{hU})^* Z$$

*which agrees with the spectral sequence obtained by mapping $Z$ into a $K(n)_*$-local $E_n$-Adams resolution of $E_n^{hU}$.*

**Remark 1.2.** In what follows—and in the proof of Theorem 1—the necessity of working in a precise point-set category of structured ring spectra will become apparent. However, many of our results, such as (iii) and (iv) of Theorem 1, are statements occurring in the stable category. We shall therefore refer to objects as "$S^0$-module spectra" or "commutative $S^0$-algebras" when we wish to emphasize that we need to work at the point-set level and as "CW-spectra" or "ring spectra" when our work takes place in the stable category. Furthermore, $[X, Y]^i$ will always denote the group of maps of degree $-i$ between $X$ and $Y$ in the stable category.

Despite our precautions, there are still some ambiguities. For example, suppose $X$ and $Y$ are commutative $S^0$-algebras, but we wish to understand the stable homotopy type of $X \wedge Y$. Then $X \wedge Y$ might denote the object which is the CW-approximation to the actual point-set level smash product of $X$ and $Y$. Or, $X \wedge Y$ might denote the derived smash product; that is, the smash product of CW-approximations to $X$ and $Y$. However, if $X$ and $Y$ are cofibrant objects in the closed model category structure on $\mathscr{E}$—called $q$-cofibrant in [11]—then these two recipes give the same stable object (see [11, VII, 6]). Since there is a functor $\mathscr{E} \to \mathscr{E}$ sending an object to a weakly equivalent $q$-cofibrant one—in fact, a cell commutative $S^0$-algebra in the sense of [11]—we may as well assume that $\mathbf{F}(S)$, for example, is always $q$-cofibrant and thus eliminate this ambiguity. We will often use this device of functorially approximating by a cell object, even if we do not always mention it.

**Remark 1.3.** Let $I_n = (p, v_1, \ldots, v_{n-1}) \subset E_{n*}$, and, if $Z$ is any CW-spectrum, let $\{Z_\alpha\}$ be the directed system of finite CW-subspectra. Then

$$E_n^* Z = \lim_{\leftarrow \alpha, k} E_n^* Z_\alpha / I_n^k E_n^* Z_\alpha$$

is a profinite continuous $G_n$-module and we define

$$H_c^*(U, E_n^* Z) = \lim_{\leftarrow \alpha, k} H_c^*(U, E_n^* Z_\alpha / I_n^k E_n^* Z_\alpha).$$

In general, if $G$ is a $p$-analytic profinite group and $M$ is a profinite continuous $\mathbb{Z}_p[[G]]$-module, then $\lim_{\leftarrow \alpha} H_c^*(G, M_\alpha)$ is independent of the presentation $M = \lim_{\leftarrow \alpha} M_\alpha$. In fact, it is for example the cohomology of the usual cochain complex whose $j$-cochains are $\mathrm{Map}_c(G^j, M)$ (cf. Proof of Lemma 4.21). We therefore use this as our definition of $H_c^*(G, M)$; see also [31] for a more general treatment of this object.

For the next step, let

$$G_n = U_0 \supsetneq U_1 \supsetneq U_2 \supsetneq \cdots \supsetneq U_i \supsetneq \cdots \tag{1.4}$$

be a sequence of normal open subgroups of $G_n$ with $\bigcap_i U_i = \{e\}$. For example, using the notation at the beginning of Section 1 and the description of $S_n$ in [7, 2.21], we may take $U_i = V_i \rtimes \mathrm{Gal}$, where $V_i$ is the group of power series $\Sigma_{j \geqslant 0}^{\Gamma_n} b_j x^{p^j}$ with $b_j = 0$ for $0 < j < i$ and $b_0 = 1$ if $i > 0$.

**Definition 1.5.** Fix a sequence $\{U_i\}$ as above. For $G$ a closed subgroup of $G_n$, define

$$E_n^{hG} = \hat{L}\, \mathrm{holim}_{\rightarrow i\ \mathscr{E}}\, E_n^{h(U_i G)},$$

where $\mathrm{holim}_{\rightarrow\ \mathscr{E}}$ denotes the homotopy colimit taken in the topological model category $\mathscr{E}$.

**Remark 1.6.** More precisely, one should functorially replace $\mathrm{holim}_{\rightarrow i\ \mathscr{E}} E_n^{h(U_i G)}$ by a weakly equivalent cell commutative $S^0$-algebra before applying $\hat{L}$. Then the construction $E_n^{hG}$ becomes functorial in $\mathscr{E}$ (see [11, VIII, 2]).

We will prove the following result in Section 6.

**Theorem 2.** *The construction* $E_n^{hG}$ *has the following properties*:

(i) $\pi_*\hat{L}(E_n^{hG}\wedge E_n)$ *is naturally isomorphic to* $\mathrm{Map}_c(G_n, E_{n^*})^G$ *as completed* $\mathrm{Map}_c(G_n, E_{n^*})$-*comodules, where* $G$ *acts on* $\mathrm{Map}_c(G_n, E_{n^*})$ *as in Theorem* 1(iii).

(ii) *Let* $Z$ *be any CW-spectrum. The spectral sequence obtained by mapping* $Z$ *into a* $K(n)_*$-*local* $E_n$-*Adams resolution of* $E_n^{hG}$ *is strongly convergent to* $(E_n^{hG})^*Z$ *and has* $E_2$-*term naturally isomorphic to* $H_c^*(G, E_n^*Z)$.

**Remark 1.7.** (i) Using the first part of this theorem, it's easy to see that $E_n^{hG}$ is canonically independent (up to weak equivalence in $\mathscr{E}$) of the choice of sequence $\{U_i\}$.

(ii) Since $G$ is a closed subgroup of a $p$-analytic profinite group, it is itself $p$-analytic (see [9, 10.7]), and therefore $H_c^*(G, E_n^*Z)$ is defined as in Remark 1.3.

Now by Theorem 1, there are commutative diagrams

$$\begin{tikzcd}
E_n^{h(U_iG)} = \mathbf{F}(G_n/U_iG) \arrow[r] \arrow[d] & \mathbf{F}(G_n) = E_n \\
E_n^{h(U_jG)} = \mathbf{F}(G_n/U_jG) \arrow[ur] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_n^{h(U_iG)} = \mathbf{F}(G_n/U_iG) \arrow[r] \arrow[dr] & \mathbf{F}(G_n) = E_n \arrow[d, "\mathbf{F}(r_g)" left, "g" right] \\
 & \mathbf{F}(G_n) = E_n
\end{tikzcd}$$

for $i \leqslant j$ and $g \in G$, where the unlabeled arrows are induced by the evident projections. There is thus a canonical map $E_n^{hG} \to E_n^G$, the $G$-fixed points of $E_n$. Let us denote by $E_n^{h'G}$ the ordinary homotopy fixed point spectrum for the action of $G$ on $E_n$. Composing the above map with the canonical map $E_n^G \to E_n^{h'G}$ ([4, XI, 3.5]) yields a map $E_n^{hG} \to E_n^{h'G}$. The next result will be proved in Section 6.

**Theorem 3.** *Let* $G$ *be a finite subgroup of* $G_n$. *Then*:

(i) *The map* $E_n^{hG} \to E_n^{h'G}$ *described above is a weak equivalence.*

(ii) *Let* $Z$ *be any CW-spectrum. The homotopy fixed point spectral sequence*

$$H^*(G, E_n^*Z) \Rightarrow (E_n^{h'G})^*Z = (E_n^{hG})^*Z$$

*is naturally isomorphic to the spectral sequence of Theorem* 2(ii).

We also have a result on iterated homotopy fixed point spectra. Indeed, if $K$ is a closed and $U$ is a normal open subgroup of $G_n$, then the opposite of the group $W(K) = N(K)/K$ acts on $G_n/UK$ via $xUK \mapsto xhUK$ for $x \in G_n$, $h \in N(K)$. This yields an action of $W(K)$ on $E_n^{h(UK)}$, and hence, upon passing to the homotopy colimit, on $E_n^{hK}$. In particular, if $F$ is a finite subgroup of $W(K)$, we may form $(E_n^{hK})^{hF}$ in the usual way. We can now state our next result.

**Theorem 4.** *Suppose $G$ is a closed subgroup of $G_n$, $K$ is a closed normal subgroup of $G$, and $F \equiv G/K$ is finite. Then $E_n^{hG}$ is naturally equivalent to $(E_n^{hK})^{hF}$.*

Another sort of consistency result is given by examining the case $n=1$. Since the Galois action on $BGL(\bar{\mathbb{F}}_l)^+$ corresponds to the action of $G_1 = \mathbb{Z}_p^\times$ on $E_1$, where once again $l$ is a prime different from $p$, we would expect a relationship between the continuous homotopy fixed points of $BGL(\bar{\mathbb{F}}_l)^+$ and the continuous homotopy fixed point spectra of $E_1$. This is indeed the case.

Let $R$ be a commutative ring, and let $KR$ be the algebraic $K$-theory spectrum of $R$ (so that $\Omega^\infty KR \simeq \mathbb{Z} \times BGL(R)^+$). Quillen's results deloop (see [20, VIII]); hence $(K\bar{\mathbb{F}}_l)_p$ is equivalent to the connective cover of $E_1$, and the action of $t \in \hat{\mathbb{Z}}$ on $(K\bar{\mathbb{F}}_l)_p$ corresponds to the action of $l^t \in \mathbb{Z}_p^\times$ on $E_1$. Now choose $s$ dividing $p-1$ such that $l^s \equiv 1 \bmod (p)$, and define a continuous group monomorphism $\mathbb{Z}_p \to \mathbb{Z}_p^\times$ by sending $a \in \mathbb{Z}_p$ to $l^{sa}$. (The number $s$ is needed to guarantee that $l^{sa}$ makes sense for all $a \in \mathbb{Z}_p$.) If $G$ is a non-trivial closed subgroup of $\mathbb{Z}_p$, then $G = p^j\mathbb{Z}_p$ for some $j \geqslant 0$. Let us also write $G$ for the corresponding closed subgroup in $\mathbb{Z}_p^\times$. We then have the following result.

**Theorem 5.** *With the notation as above, $E_1^{hG} \simeq L_{K(1)}K(k)$, where $k$ is the field of invariants of the action of $sp^j\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$ on $\bar{\mathbb{F}}_l$.*

Our final result, which will be proven as an application of the machinery developed here, was originally due to the second author and H.R. Miller, who suggested that this is the place where it should logically appear.

**Theorem 6** (Hopkins-Miller). *Suppose $c: G_n \to \mathbb{Z}_p$ is a continuous homomorphism, and let $c$ also denote the composition $G_n \xrightarrow{c} \mathbb{Z}_p \to E_{n*}$. Then $c \in H_c^1(G_n, E_{n*})$ survives to $\pi_*\hat{L}S^0$.*

Let us indicate our strategy for constructing $E_n^{hU}$. We will construct a cosimplicial $\mathscr{E}$ spectrum corresponding to the $K(n)_*$-local $E_n$-Adams resolution of $E_n^{hU}$; then we will define $E_n^{hU}$ to be Tot of this cosimplicial spectrum. To do this, we need only determine the (expected) homotopy type of $\hat{L}(E_n^{hU} \wedge E_n)$ together with the comodule structure map

$$\hat{L}(E_n^{hU} \wedge E_n) = \hat{L}(E_n^{hU} \wedge S^0 \wedge E_n) \to \hat{L}(E_n^{hU} \wedge E_n \wedge E_n).$$

Now we might expect a spectral sequence

$$H_c^*(U, \pi_*\hat{L}(E_n \wedge E_n)) \Rightarrow \pi_*\hat{L}(E_n^{hU} \wedge E_n),$$

since there is such a spectral sequence if $U$ is replaced by a finite subgroup of $G_n$ (see Theorem 5.3). But $\pi_*\hat{L}(E_n \wedge E_n) = \mathrm{Map}_c(G_n, E_{n*})$, and the action of $U$ is given as in Theorem 1(iii) (see Section 2). Since $\mathrm{Map}_c(G_n, E_{n*})$ is $U$-acyclic (cf. proof of Lemma 4.20), we should have

$$\pi_*\hat{L}(E_n^{hU} \wedge E_n) = \mathrm{Map}_c(G_n, E_{n*})^U.$$

The comodule structure map is also determined. But, as $E_{n*}$-modules,

$$\mathrm{Map}_c(G_n, E_{n*})^U = \prod_{i=1}^m E_{n*},$$

where $m$ is the cardinality of $G_n/U$. We are thus led to take $X_{G_n/U} = \prod_{i=1}^{m} E_n$ as a model for $\hat{L}(E_n^{hU} \wedge E_n)$. We can also define the corresponding comodule structure map $X_{G_n/U} \to \hat{L}(X_{G_n/U} \wedge E_n)$. (Note that this map is *not*—except in a few very special cases—the product of the maps $E_n = \hat{L}(S^0 \wedge E_n) \to \hat{L}(E_n \wedge E_n)$.) With this construction, we obtain a cosimplicial object, but only in the stable category. However, the technology of Hopkins-Miller as expanded by Goerss-Hopkins is again available to allow us to conclude that the requisite diagrams in fact commute up to all higher homotopies in $\mathscr{E}$. The original cosimplicial object can now be replaced by an equivalent cosimplicial object in $\mathscr{E}$, and Tot can be formed.

The contents of this paper are as follows. In Section 2, we recall the relevant parts of Morava's theory, and in Section 3, we discuss the formation of homotopy inverse limits for certain diagrams commutative up to all higher homotopies. We construct the cosimplicial Adams resolution in Section 4 and identify the resultant spectral sequence as having the form of a (continuous) homotopy fixed point spectral sequence. In Section 5 we compute $\pi_*\hat{L}(\mathbf{F}(S) \wedge E_n)$; this allows us to identify the preceding spectral sequence as a $K(n)_*$-local $E_n$-Adams spectral sequence and completes the proof of Theorem 1. We prove Theorems 2 and 3 in Section 6, Theorem 4 in Section 7, and, finally, Theorems 5 and 6 in Section 8. An appendix summarizes the properties of $K(n)_*$-localizations and $K(n)_*$-local $E_n$-Adams spectral sequences that we need. In particular, we prove the strong convergence of these spectral sequences.

## 2. Resumé of Morava's theory

Let $p$ be a fixed prime, let $n \geqslant 1$, and let $E_n$ denote the spectrum with coefficient ring $E_{n*} = W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$ obtained via the Landweber exact functor theorem for $BP$. $W\mathbb{F}_{p^n}$ denotes the ring of Witt vectors with coefficients in the field $\mathbb{F}_{p^n}$ of $p^n$ elements, and the map $BP_* \xrightarrow{r} E_{n*}$—which also provides $E_n$ with the structure of $BP$-algebra in the stable category—is given by

$$r(v_i) = \begin{cases} u_i u^{1-p^i} & i < n, \\ u^{1-p^n} & i = n, \\ 0 & i > n. \end{cases}$$

Now let $S_n$ denote the $n$th Morava stabilizer group; i.e., the automorphism group of the height $n$ Honda formal group law $\Gamma_n$ over $\mathbb{F}_{p^n}$. Let $\mathrm{Gal} \equiv \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, and let $G_n = S_n \rtimes \mathrm{Gal}$. The Lubin-Tate theory of lifts of formal group laws provides an action of $S_n$ on $E_{n*}$ (see for example [7]), and Gal acts on $E_{n*}$ via its action on $W\mathbb{F}_{p^n}$. If $H$ is a subgroup of Gal, let us write $E_n^H$ for the Landweber exact spectrum with coefficient ring $W(\mathbb{F}_{p^n}^H)[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$, where $\mathbb{F}_{p^n}^H$ is the subfield of $\mathbb{F}_{p^n}$ fixed by the automorphism group $H$.

We first identify the completed Hopf algebroid $\pi_*\hat{L}(E_n \wedge E_n)$ with the split completed Hopf algebroid $(E_{n*}, \mathrm{Map}_c(G_n, \mathbb{Z}_p)\hat{\otimes}_{\mathbb{Z}_p} E_{n*})$; this piece of Morava's theory is crucial to all of our subsequent work. We start by observing that $\mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}}$ is a completed Hopf algebra over $\mathbb{Z}_p$; the diagonal map is given by

$$\mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \to \mathrm{Map}_c(S_n \times S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \xleftarrow{\approx} \mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes} \mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}},$$

where the first map is induced by the multiplication on $S_n$ and the second is an isomorphism by [5, AII.3]. There is also a map

$$\eta_L : E_{n^*}^{\mathrm{Gal}} \to \mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes}_{\mathbb{Z}_p} E_{n^*}^{\mathrm{Gal}} \xrightarrow{\approx} \mathrm{Map}_c(S_n, E_{n^*})^{\mathrm{Gal}}$$

which is given by $\eta_L(x)(g) = g^{-1}x$ for $x \in E_{n^*}^{\mathrm{Gal}}$ and $g \in S_n$. With these structure maps, we obtain a split completed Hopf algebroid $(E_{n^*}^{\mathrm{Gal}}, \mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes}_{\mathbb{Z}_p} E_{n^*}^{\mathrm{Gal}})$. A main result of Morava's theory is the following identification.

**Theorem 2.1.** $(E_{n^*}^{\mathrm{Gal}}, \pi_* \hat{L}(E_n^{\mathrm{Gal}} \wedge E_n^{\mathrm{Gal}}))$ *is isomorphic to* $(E_{n^*}^{\mathrm{Gal}}, \mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes}_{\mathbb{Z}_p} E_{n^*}^{\mathrm{Gal}})$ *as completed Hopf algebroids.*

**Proof.** We showed in [5, Section 4] that $(E_{n^*}^{\mathrm{Gal}}, \mathrm{Map}_c(S_n, W\mathbb{F}_{p^n})^{\mathrm{Gal}} \hat{\otimes} E_{n^*}^{\mathrm{Gal}})$ is isomorphic to a completed Hopf algebroid denoted $(E_{n^*}^{\mathrm{Gal}}, E_{n^*}^{\mathrm{Gal}} \hat{\otimes}_U US \hat{\otimes}_U E_{n^*}^{\mathrm{Gal}})$. But we observed in [6, 3.4] that this Hopf algebroid is isomorphic to $(E_{n^*}^{\mathrm{Gal}}, E_{n^*}^{\mathrm{Gal}} \hat{\otimes}_{BP_*} BP_*BP \hat{\otimes}_{BP_*} E_{n^*}^{\mathrm{Gal}})$, where the completed tensor product here denotes $I_n$-adic completion. Since $\pi_* \hat{L}(E_n^{\mathrm{Gal}} \wedge E_n^{\mathrm{Gal}})$ is the $I_n$-adic completion of $E_{n^*}^{\mathrm{Gal}} E_n^{\mathrm{Gal}} = E_{n^*}^{\mathrm{Gal}} \otimes_{BP_*} BP_*BP \otimes_{BP_*} E_{n^*}^{\mathrm{Gal}}$ (see [16]), the proof is complete. □

To derive the structure of the completed Hopf algebroid $\pi_* \hat{L}(E_n \wedge E_n)$ from Theorem 2.1, we first observe that

$$\begin{aligned} \pi_* \hat{L}(E_n \wedge E_n^{\mathrm{Gal}}) &= \pi_* \hat{L}(E_n^{\mathrm{Gal}} \wedge E_n^{\mathrm{Gal}}) \otimes_{\mathbb{Z}_p} W\mathbb{F}_{p^n} \\ &= \mathrm{Map}_c(S_n, E_{n^*})^{\mathrm{Gal}} \otimes_{\mathbb{Z}_p} W\mathbb{F}_{p^n} \\ &= \mathrm{Map}_c(S_n, E_{n^*}), \end{aligned}$$

where the last equality follows by [5, 5.4]. Then

$$\begin{aligned} \pi_* \hat{L}(E_n \wedge E_n) &= \pi_* \hat{L}(E_n \wedge E_n^{\mathrm{Gal}}) \otimes_{\mathbb{Z}_p} W\mathbb{F}_{p^n} \\ &= \mathrm{Map}_c(S_n, E_{n^*}) \otimes_{\mathbb{Z}_p} W\mathbb{F}_{p^n} \underset{\approx}{\xrightarrow{\tau}} \mathrm{Map}_c(G_n, E_{n^*}), \end{aligned}$$

where

$$\tau(f \otimes w)(s, \sigma) = w \cdot (\sigma^{-1}(f(s)))$$

for $f \in \mathrm{Map}_c(S_n, E_{n^*})$, $w \in W\mathbb{F}_{p^n}$, $s \in S_n$, and $\sigma \in \mathrm{Gal}$. Now consider the split completed Hopf algebroid $(E_{n^*}, \mathrm{Map}_c(G_n, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} E_{n^*})$; once again $\eta_L : E_{n^*} \to \mathrm{Map}_c(G_n, E_{n^*})$ is given by $\eta_L(x)(g) = g^{-1}x$ for $x \in E_{n^*}$ and $g \in G_n$. Observe that this is *not* a Hopf algebroid over $W\mathbb{F}_{p^n}$, since $\eta_L$ and $\eta_R$ are different when restricted to $W\mathbb{F}_{p^n}$. Upon chasing down the identifications, the next result follows from Theorem 2.1.

**Proposition 2.2.** $(E_{n^*}, \pi_* \hat{L}(E_n \wedge E_n))$ *is isomorphic to* $(E_{n^*}, \mathrm{Map}_c(G_n, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} E_{n^*})$ *as completed Hopf algebroids.*

This isomorphism is used to define the action of $G_n$ on $E_n$. Indeed, recall that if $M$ is a completed left $\mathrm{Map}_c(G_n, E_{n^*})$-comodule, then $M$ is a $G_n$-module with the action given by

$$g(m) = \psi(m)(g^{-1}),$$

where

$$\psi : M \to \mathrm{Map}_c(G_n, E_{n*})\hat{\otimes}_{E_{n*}} M \xrightarrow{\approx} \mathrm{Map}_c(G_n, M)$$

is the comodule structure map. In particular, if $X$ is a finite CW-spectrum, $E_{n*}X$ is naturally a $G_n$-module, and the pairing $E_{n*}X \otimes_{E_{n*}} E_{n*}Y \to E_{n*}(X \wedge Y)$ is $G_n$-equivariant, where the left side is given the diagonal action. Since $E_{n*}$ is profinite in each degree, it follows that there exists, in the stable category, a unique action of $G_n$ on $E_n$ by ring spectrum maps inducing the above action on $E_{n*}X$.

With this action, it is immediate that the isomorphism of Proposition 2.2 is given by sending $x \in \pi_*\hat{L}(E_n \wedge E_n)$ to $h_x \in \mathrm{Map}_c(G_n, E_{n*})$, where $h_x(g)$ is given by the composition

$$S \xrightarrow{x} \hat{L}(E_n \wedge E_n) \xrightarrow{\hat{L}(g^{-1} \wedge E_n)} \hat{L}(E_n \wedge E_n) \xrightarrow{\mu} E_n. \tag{2.3}$$

(An identification of this sort first appears in the literature in [30].) Here suspensions have been omitted from the notation and $\mu$ is the ring spectrum multiplication map. Moreover, it is easy to check using (2.3) that the action of $G_n$ on the left factor of $\pi_*\hat{L}(E_n \wedge E_n)$ corresponds to the action of $G_n$ on $\mathrm{Map}_c(G_n, E_{n*})$ described in Theorem 1. For later purposes, we will also need to know the formula for the action of $G_n$ on the right factor of $\pi_*\hat{L}(E_n \wedge E_n)$. If we write $g^R x$ for this action of $g \in G_n$ on $x \in \pi_*\hat{L}(E_n \wedge E_n)$, then it is again easy to see using (2.3) that $g^R x$ corresponds to the map sending $g' \in G_n$ to $g h_x(g'g) \in E_{n*}$.

**Remark 2.4.** In [5], we used Theorem 2.1 to define a natural $W\mathbb{F}_{p^n}$-linear action of $S_n$ on $E_{n*}X$. There is also the evident action of Gal on $E_{n*}X = W\mathbb{F}_{p^n} \otimes_{\mathbb{Z}_p} E_{n*}^{\mathrm{Gal}}X$, and these actions piece together to give a natural action of $G_n$ on $E_{n*}X$, whence an action of $G_n$ on $E_n$ in the stable category. This action is the same as the action defined above.

The identifications of Proposition 2.2 and (2.3) can be generalized to iterated smash products of $E_n$ and beyond. Indeed, if $X$ is a finite spectrum,

$$\pi_*\hat{L}(E_n^{(j+1)} \wedge X) = E_{n*}E_n \hat{\otimes}_{E_{n*}} E_{n*}E_n \hat{\otimes}_{E_{n*}} \cdots \hat{\otimes}_{E_{n*}} E_{n*}E_n \hat{\otimes}_{E_{n*}} E_{n*}X,$$

where $E_{n*}$ acts on the right of each factor $E_{n*}E_n$ and on the left of the factor $E_{n*}X$. But

$$\begin{aligned} E_{n*}E_n \hat{\otimes}_{E_{n*}} \cdots \hat{\otimes}_{E_{n*}} E_{n*}E_n &= (\mathrm{Map}_c(G_n, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{n*}) \hat{\otimes}_{E_{n*}} \cdots \hat{\otimes}_{E_{n*}} (\mathrm{Map}_c(G_n, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E_{n*}) \\ &= \mathrm{Map}_c(G_n, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} \cdots \hat{\otimes}_{\mathbb{Z}_p} \mathrm{Map}_c(G_n, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} E_{n*} \\ &= \mathrm{Map}_c(G_n^j, E_{n*}), \end{aligned}$$

and thus

$$\begin{aligned} E_{n*}E_n \hat{\otimes}_{E_{n*}} \cdots \hat{\otimes}_{E_{n*}} E_{n*}E_n \hat{\otimes}_{E_{n*}} E_{n*}X &= \mathrm{Map}_c(G_n^j, E_{n*}) \hat{\otimes}_{E_{n*}} E_{n*}X \\ &= \mathrm{Map}_c(G_n^j, E_{n*}X). \end{aligned}$$

This isomorphism sends $x \in \pi_*\hat{L}(E_n^{(j+1)} \wedge X)$ to $h_x \in \mathrm{Map}_c(G_n^j, E_{n*}X)$, where $h_x(g_1, \ldots, g_j)$ is given by the composition

$$S \xrightarrow{x} \hat{L}(E_n^{(j+1)} \wedge X) \xrightarrow{\hat{L}(g_1^{-1} \wedge \cdots \wedge g_j^{-1} \wedge E_n \wedge X)} \hat{L}(E_n^{(j+1)} \wedge X) \xrightarrow{\hat{L}(\mu \wedge X)} E_n \wedge X. \tag{2.5}$$

More generally, if $Z$ is any spectrum, there is a natural transformation

$$\tau_j : [Z, \hat{L}(E_n^{(j+1)})]^* \to \mathrm{Map}_c(G_n^j, E_n^* Z) \tag{2.6}$$

such that $\tau_j(x)(g_1, \ldots, g_j)$ is the composite

$$Z \xrightarrow{x} \hat{L}(E_n^{(j+1)}) \xrightarrow{\hat{L}(g_1^{-1} \wedge \cdots \wedge g_j^{-1} \wedge E_n)} \hat{L}(E_n^{(j+1)}) \xrightarrow{\mu} E_n. \tag{2.7}$$

(To show that $\tau_j$ does in fact have its image in $\mathrm{Map}_c(G_n^j, E_{n*}Z)$, it suffices, by the definition of the topology on $E_{n*}Z$, to show that this is the case when $Z$ is finite. But when $Z$ is finite, $\tau_j$ is just the isomorphism described in (2.5) with $X$ the Spanier-Whitehead dual of $Z$.) Since $\mathrm{Map}_c(G_n^j, ?)$ is exact on the category of profinite groups (see [29, I, Theorem 3]), it follows that $\tau_j$ is a natural transformation of cohomology theories satisfying the product axiom. But $\tau_j$ is an isomorphism with $Z = S^0$; therefore $\tau_j$ is an isomorphism for any CW-spectrum $Z$.

## 3. Homotopy inverse limits

Let $h\mathscr{E}$ denote the homotopy category of commutative $S^0$-algebras. That is, two maps $f, g : X \to Y$ in $\mathscr{E}$ are homotopic if $f$ and $g$ lie in the same path component of $\mathscr{E}(X, Y)$, the topological space of $S^0$-algebra maps between $X$ and $Y$. Alternatively, $\mathscr{E}$ is a tensored category (over the category of unbased topological spaces), and $f$ and $g$ are homotopic if there exists a map $h : X \otimes I \to Y$ restricting to $f$ and $g$ on the ends of the cylinder (see [11, VII, 2]).

Now let $J$ be a small category and suppose $X : J \to h\mathscr{E}$ is a functor. In some cases, $X$ can be replaced by a homotopy equivalent *strict* diagram, and then its homotopy inverse limit can be formed.

**Definition 3.1.** Let $X : J \to h\mathscr{E}$ be as above. $X$ is said to be an $h_\infty$-diagram if for each $\alpha : j_1 \to j_2$ in $J$, $\mathscr{E}(Xj_1, Xj_2)_{X\alpha}$, the path component of $\mathscr{E}(Xj_1, Xj_2)$ containing $X\alpha$, is weakly contractible.

The main result of this section is due to Dwyer, Kan, and Smith [10]. We feel that the reader will appreciate an account of the proof.

**Theorem 3.2.** *Let $X : J \to h\mathscr{E}$ be an $h_\infty$-diagram. Then there exists a functor $\bar{X} : J \to \mathscr{E}$ and a natural transformation $\bar{X} \to X$ in $h\mathscr{E}$ such that $\bar{X}(j) \to X(j)$ is a weak equivalence for each $j \in \mathrm{Ob}\, J$.*

The proof makes use of a modification of the cosimplicial replacement of a diagram (cf. [4, XI, 5]). We first recall some notation.

If $Z$ is an unbased topological space and $Y$ is a commutative $S^0$-algebra, let $F(Z, Y)$ denote the cotensor product of $Z$ and $Y$ in $\mathscr{E}$; its underlying $S^0$-module is just the function $S^0$-module of maps from $\Sigma^\infty Z_+$ to $Y$ ([11, VII, 2]). We shall also use the notation $F(\ ,\ )$ to denote function spectra in the stable category.

With $Z$ as above, let $\Gamma Z$ denote the geometric realization of the singular simplicial set of $Z$. This is of course a functorial cofibrant replacement of $Z$; it also has the property that $\Gamma V \times \Gamma W$ is naturally homeomorphic to $\Gamma(V \times W)$ and that this homeomorphism commutes with the projections onto $\Gamma V$ and $\Gamma W$. Hence a pairing $V \times W \to Z$ induces a pairing $\Gamma V \times \Gamma W \to \Gamma Z$.

**Construction 3.3.** Let $X : J \to h\mathscr{E}$ be an $h_\infty$-diagram. Define a cosimplicial commutative $S^0$-algebra $\prod_h^* X$ by

$$\textstyle\prod_h^0 X = \prod_{j\in J} Xj,$$

$$\textstyle\prod_h^n X = \prod_{J_n} F(\Gamma\mathscr{E}X\alpha, Xj_0),$$

where $J_n$ is the set of diagrams

$$\alpha : j_0 \xleftarrow{\alpha_1} j_1 \xleftarrow{\alpha_2} j_2 \leftarrow \cdots \leftarrow j_{n-1} \xleftarrow{\alpha_n} j_n$$

in $J$, and

$$\Gamma\mathscr{E}X\alpha = \Gamma\mathscr{E}(Xj_1, Xj_0)_{X\alpha_1} \times \cdots \times \Gamma\mathscr{E}(Xj_n, Xj_{n-1})_{X\alpha_n}.$$

If $0 < i < n+1$, the coface $d^i$ is defined via the commutative diagram

$$\begin{tikzcd}
\prod_h^n X \arrow[r, "d^i"] \arrow[d, "\pi_{\alpha'}"] & \prod_h^{n+1} X \arrow[d, "\pi_\alpha"] \\
F(\Gamma\mathscr{E}X\alpha', Xj_0) \arrow[r, "d^i_\alpha"] & F(\Gamma\mathscr{E}X\alpha, Xj_0)
\end{tikzcd} \tag{3.4}$$

where $\pi_\alpha$ denotes the projection onto the factor indexed by

$$\alpha : j_0 \xleftarrow{\alpha_1} j_1 \xleftarrow{\alpha_2} j_2 \leftarrow \cdots \leftarrow j_n \xleftarrow{\alpha_{n+1}} j_{n+1},$$

$\alpha'$ denotes the diagram

$$j_0 \xleftarrow{\alpha_1} j_1 \leftarrow \cdots \leftarrow j_{i-1} \xleftarrow{\alpha_i\alpha_{i+1}} j_{i+1} \xleftarrow{\alpha_{i+2}} j_{i+2} \leftarrow \cdots \leftarrow j_n \xleftarrow{\alpha_{n+1}} j_{n+1},$$

and

$$(d^i_\alpha g)(f_1, \ldots, f_{n+1}) = g(f_1, \ldots, f_i f_{i+1}, \ldots, f_{n+1})$$

for $(f_1, \ldots, f_{n+1}) \in \Gamma\mathscr{E}X\alpha$. Here $f_i f_{i+1}$ denotes the image of $(f_i, f_{i+1})$ under the map

$$\Gamma\mathscr{E}(Xj_i, Xj_{i-1}) \times \Gamma\mathscr{E}(Xj_{i+1}, Xj_i) \to \Gamma\mathscr{E}(Xj_{i+1}, Xj_{i-1})$$

induced by the composition pairing

$$\mathscr{E}(Xj_i, Xj_{i-1}) \times \mathscr{E}(Xj_{i+1}, Xj_i) \to \mathscr{E}(Xj_{i+1}, Xj_{i-1}).$$

If $i = 0$, $d^0$ is defined as in (3.4), although now $\alpha'$ is the diagram

$$j_1 \xleftarrow{\alpha_2} j_2 \leftarrow \cdots \xleftarrow{\alpha_{n+1}} j_{n+1},$$

and $d^0_\alpha : F(\Gamma\mathscr{E}X\alpha', Xj_1) \to F(\Gamma\mathscr{E}X_\alpha, Xj_0)$ is defined by

$$(d^0_\alpha g)(f_1, \ldots, f_{n+1}) = f_1(g(f_2, \ldots, f_{n+1})),$$

where $f_1$ also denotes the image of $f_1 \in \Gamma\mathscr{E}(Xj_1, Xj_0)$ in $\mathscr{E}(Xj_1, Xj_0)$.

Finally, if $i=n+1$, $\alpha'$ is the diagram

$$j_0 \overset{\alpha_1}{\leftarrow} j_1 \leftarrow \cdots \leftarrow j_{n-1} \overset{\alpha_n}{\leftarrow} j_n$$

and

$$(d_\alpha^{n+1}g)(f_1,\ldots,f_{n+1}) = g(f_1,\ldots,f_n).$$

As for the codegeneracies, $s^i$ is defined via the commutative diagram

$$\begin{tikzcd} \prod_h^{n+2} X \arrow[r, "s^i"] \arrow[d, "\pi_{\alpha'}"] & \prod_h^{n+1} X \arrow[d, "\pi_\alpha"] \\ F(\Gamma\mathscr{E}X\alpha', Xj_0) \arrow[r, "s_\alpha^i"] & F(\Gamma\mathscr{E}X\alpha, Xj_0) \end{tikzcd} \tag{3.5}$$

where $\alpha'$ is the diagram

$$j_0 \overset{\alpha_1}{\leftarrow} j_1 \overset{\alpha_2}{\leftarrow} j_2 \leftarrow \cdots \leftarrow j_{i-1} \overset{\alpha_i}{\leftarrow} j_i \overset{id}{\leftarrow} j_i \overset{\alpha_{i+1}}{\leftarrow} j_{i+1} \leftarrow \cdots \overset{\alpha_{n+1}}{\leftarrow} j_{n+1}$$

and

$$(s_\alpha^i g)(f_1,\ldots,f_{n+1}) = g(f_1,\ldots,f_i, id, f_{i+1},\ldots,f_{n+1}).$$

Here $id$ denotes the image of $* = \Gamma(*)$ in $\Gamma\mathscr{E}(X_{j_i}, X_{j_i})$ under the evident map.

Recall that a cosimplicial $S^0$-module $Y$ is *fibrant* if the map $s: Y^{n+1} \to M^nY$,

$$M^nY \equiv \{(y^0,\ldots,y^n) \in Y^n \times \cdots \times Y^n : s^i y^j = s^{j-1} y^i \ \forall\ 0 \leqslant i < j \leqslant n\},$$

given by $s(y) = (s^0(y),\ldots,s^n(y))$ is a $q$-fibration—that is, a fibration in the Quillen closed model sense—for all $n \geqslant -1$. (Properly speaking, $M^nY$ should be defined as an equalizer; however we will continue to use this shorthand when to do otherwise would result in more confusion.)

**Lemma 3.6.** *Let $X$ be an $h_\infty$-diagram. Then $\prod_h^* X$ is fibrant.*

**Proof.** Consider first the map $s: \prod_h^1 X \to M^0 \prod_h^* X = \prod_h^0 X$. The composition

$$\prod\nolimits_h^1 X \overset{s}{\to} \prod\nolimits_h^0 X = \prod\nolimits_j Xj \to Xj$$

is given by

$$\prod\nolimits_h^1 X = \prod\nolimits_{j_0 \overset{\alpha}{\leftarrow} j_1} F(\Gamma\mathscr{E}(Xj_1, Xj_0)_{X\alpha}, Xj_0) \to F(\Gamma\mathscr{E}(Xj,Xj)_{id}, Xj) \to Xj,$$

where the last map is evaluation at $id \in \Gamma\mathscr{E}(Xj,Xj)$. Since $id$ is a vertex of $\Gamma\mathscr{E}(Xj,Xj)$, it follows that this last map is a $q$-fibration and hence so is $s$.

Now suppose $n > 1$. Let $J_{n,j}^k$ be the subset of $J_n$ consisting of those $n$-tuples of maps with $j_0 = j$ and $\alpha_k = id$, and set

$$D_{n,j}^k X = \coprod\nolimits_{\alpha \in J_{n,j}^k} \Gamma\mathscr{E}X\alpha.$$

Then let

$$D_{n,j}X = \bigcup_{1 \leqslant k \leqslant n} D^k_{n,j}X,$$

and observe that the map $s\colon \prod_h^n X \to M^{n-1} \prod_h^* X$ restricts to an isomorphism

$$\prod_j F(D_{n,j}X, Xj) \xrightarrow{\approx} M^{n-1} \prod_h^* X.$$

But the inclusion

$$D_{n,j}X \to \coprod_{J_{n,j}} \Gamma \mathscr{E} X\alpha$$

is the inclusion of a summand, where $J_{n,j}$ is the subset of $J_n$ consisting of those $n$-tuples with $j_0 = j$. $s$ is therefore a $q$-fibration in this case as well. □

Recall that, given a fibrant cosimplicial $S^0$-module $Y$, $\mathrm{Tot}\, Y = F(\Delta[\,*\,], Y)$, the $S^0$-module of (unpointed) cosimplicial maps from $\Delta[\,*\,]$ to $Y$. Here $\Delta[\,*\,]$ is the cosimplicial space which in dimension $n$ is the standard $n$-simplex $\Delta[n]$ with the usual coface and codegeneracy maps. Let $Sk_s\Delta[\,*\,]$ be the cosimplicial space which in dimension $n$ is the $s$-skeleton of $\Delta[n]$, and define $\mathrm{Tot}_s\, Y = F(Sk_s\Delta[\,*\,], Y)$. The map $\mathrm{Tot}_{j+1}\, Y \to \mathrm{Tot}_j\, Y$ is a fibration with fiber $F(\Delta[j+1]/\dot{\Delta}[j+1], Y^{j+1} \cap \ker s_0 \cap \cdots \cap \ker s_j)$. By mapping a CW-spectrum $Z$ into the tower $\{\mathrm{Tot}_j\, Y\}$, we obtain a spectral sequence

$$E_2^{s,t} = \pi^s([Z, Y]^t) \Rightarrow [Z, \mathrm{Tot}\, Y]^{t+s} \quad (\text{cf. [4, X.6]}). \tag{3.7}$$

This spectral sequence is strongly convergent in the sense of [4, IX.5.4] provided $\lim\limits_{\leftarrow j}{}^1 E_r^{s,t} = 0$ for all $s$ and $t$.

In particular, if $X$ is an $h_\infty$-diagram, we obtain a spectral sequence

$$E_2^{s,t} = \lim_{\leftarrow J}{}^s([Z, X]^t) \Rightarrow \left[Z, \mathrm{Tot} \prod\nolimits_h^* X\right]^{t+s}. \tag{3.8}$$

This is proved as in [4, XI.7.1], using the fact that $\Gamma\mathscr{E}X_\alpha$ is contractible for all $\alpha$.

With these constructions in hand, we can now prove the main result of this section.

**Proof of Theorem 3.2.** If $j$ is an object of $J$, let $J \backslash j$ be the category whose objects are morphisms $j \to j'$ in $J$ and whose morphisms are the evident commutative diagrams. The evident functor $\mu_j : J \backslash j \to J$ provides us with an $h_\infty$-diagram $\mu_j^* X$ over $J \backslash j$ and hence a functor $\bar{X} : J \to \mathscr{E}$ defined by $\bar{X}(j) = \mathrm{Tot} \prod_h^* \mu_j^* X$.

Now define the map $\bar{X}(j) \to X(j)$ to be the composition

$$\mathrm{Tot} \prod\nolimits_h^* \mu_j^* X \to F\left(\Delta[0], \prod\nolimits_h^0 \mu_j^* X\right) = \prod\nolimits_h^0 \mu_j^* X \xrightarrow{p} Xj, \tag{3.9}$$

where $p$ is the projection onto the factor indexed by the object $j \xrightarrow{id} j$. To prove that

$$\begin{tikzcd} \overline{X}(j) \arrow[r, "\overline{X}f"] \arrow[d] & \overline{X}(j') \arrow[d] \\ X(j) \arrow[r, "Xf"] & X(j') \end{tikzcd}$$

commutes whenever $f : j \to j'$, we must prove that

$$\begin{tikzcd} \overline{X}(j) \arrow[r] \arrow[dr] & X(j) \arrow[d, "Xf"] \\ & X(j') \end{tikzcd} \tag{3.10}$$

commutes, where the diagonal map is the projection onto $\prod_h^0 \mu_j^* X$ followed by the projection onto the factor indexed by $f : j \to j'$. First observe that the two compositions

$$\bar{X}(j) \to F\left(\Delta[1], \prod\nolimits_h^1 \mu_j^* X\right) \underset{F(d^1, id)}{\overset{F(d^0, id)}{\rightrightarrows}} F\left(\Delta[0], \prod\nolimits_h^1 \mu_j^* X\right) \xrightarrow{p_f} X(j')$$

are homotopic, where the last map is the projection onto the factor indexed by $f : (j \xrightarrow{id} j) \to (j \xrightarrow{f} j')$. But these compositions are the same as

$$\bar{X}(j) \to F\left(\Delta[0], \prod\nolimits_h^0 \mu_j^* X\right) \underset{d^1}{\overset{d^0}{\rightrightarrows}} \prod\nolimits_h^1 \mu_j^* X \xrightarrow{p_f} X(j').$$

Now use the definitions of $d^0$ and $d^1$ to check that these maps give the commutative diagram (3.10).

By (3.8), there is a spectral sequence

$$E_2^{s,t} = \lim_{\leftarrow J \setminus j}{}^s \pi_{-t}\, \mu_j^* X \Rightarrow \pi_{-t-s} \bar{X}(j).$$

But $j \to j$ is an initial object of $J \setminus j$; therefore

$$\lim_{\leftarrow J \setminus j}{}^s \pi_t \mu_j^* X = \begin{cases} 0 & s > 0, \\ \pi_t X j & s = 0. \end{cases}$$

Thus $\pi_* \bar{X}(j) = \pi_* X(j)$, and an unraveling of the identifications shows that the map in (3.9) induces the identity on $\pi_*$. $\square$

## 4. Construction of the functor F

We begin by stating the following extensions of the Hopkins-Miller theory due to Goerss and Hopkins. These are the results needed to show that, for $S \in \mathrm{Ob}\, R_{G_n}^+$, the cosimplicial object $C_S$ we will construct in the stable category lifts to an $h_\infty$-diagram, and hence, by Theorem 3.2, to a cosimplicial object in $\mathscr{E}$. Then $\mathbf{F}(S)$ is defined to be $\mathrm{Tot}\, C_S$.

**Theorem 4.1** (Goerss and Hopkins [13]). *Let $E$ and $F$ be q-cofibrant commutative $S^0$-algebras with $E$ Landweber exact. Suppose that $\pi_* E$ is concentrated in even dimensions with an invertible element* *in degree* 2. *Suppose in addition that* $\pi_0 E$ *is* $\mathfrak{m}$-*adically complete for some ideal* $\mathfrak{m}$, *that* $E_0/\mathfrak{m}$ *is an algebra of characteristic* $p$, *and that the relative frobenius for the homomorphism* $E_0/\mathfrak{m} \to E_0F/\mathfrak{m}E_0F$ *is an isomorphism. Then each path component of* $\mathscr{E}(F,E)$ *is weakly contractible, and the Kronecker pairing*

$$\pi_0\mathscr{E}(F,E) \to \mathrm{Hom}_{E_*\text{-alg}}(E_*F, E_*)$$

*is a bijection.*

**Remark 4.2.** Recall that if $A$ is a commutative $\mathbb{F}_p$-algebra, the frobenius homomorphism $\phi_A : A \to A$ is defined by $\phi_A(x) = x^p$. If $f : A \to B$ is a homomorphism of commutative $\mathbb{F}_p$-algebras, the relative frobenius is defined to be the map given by the dotted arrow in the following diagram, where the square is a push-out in the category of commutative $\mathbb{F}_p$-algebras:

$$\begin{tikzcd}
 & & B \\
B \arrow[r] \arrow[urr, bend left, "\phi_B"] & C \arrow[ur, dotted] & \\
A \arrow[u, "f"] \arrow[r, "\phi_A"'] & A \arrow[u] \arrow[uur, bend right, "f"'] &
\end{tikzcd}$$

**Remark 4.3.** The condition that $E$ be Landweber exact can be weakened. Indeed, $E$ need only satisfy a condition of the sort required by Adams [1, III, 13.3] in his construction of universal coefficient spectral sequences. For example, Property 1.1 of [6] suffices. In particular, any Landweber exact spectrum satisfies this property [6, 1.3].

**Theorem 4.4** (Goerss and Hopkins [13]). *Let* $H$ *be a subgroup of* Gal. *Then* $E_n^H$ *has a unique structure of commutative* $S^0$-*algebra descending to its ordinary ring spectrum structure.*

**Remark 4.5.** $E_n^H$ is in fact the $H$ homotopy fixed point spectrum of $E_n$, so our notation is only slightly abusive.

In order to construct $C_S$, we need to apply Theorem 4.1 to spectra of the form $\hat{L}(X \wedge (E_n)^{(j)})$, where $X$ is a finite product of $E_n$'s. The next result enables us to do this.

**Proposition 4.6.** *Let* $E = \hat{L}(X \wedge E_n^{(j)})$ *and let* $F = \hat{L}(Y \wedge E_n^{(k)})$, *where* $X$ *and* $Y$ *are* (*non-empty*) *products of finitely many copies of* $E_n$ *and* $j, k \geqslant 0$. *Then the pair* $F, E$ *satisfies the conditions of Theorem* 4.1; *moreover, the function*

$$\pi_0\mathscr{E}(F,E) \to \mathrm{Hom}_{\mathrm{alg}}(F_*, E_*)$$

*sending a map* $F \to E$ *to the induced map on homotopy groups is one-to-one.*

**Proof.** First observe that $E$ and $F$ are commutative $S^0$-algebras—the key point here is that localization with respect to a homology theory preserves such objects (see [11, VIII]). An easy reduction also shows that it suffices to consider the case where $X = E_n$ and $Y = E_n$. By Morava's theory, $E_* = \mathrm{Map}_c(G_n^j, E_{n*})$ and the action of $E_n$ on the right factor of $E$ induces "right multiplication" of

$E_{n^*}$ on $\mathrm{Map}_c(G_n^j, E_{n^*})$. It therefore follows that $E$ (resp. $F$) is Landweber exact for an appropriate map $BP \to E$ (resp. $BP \to F$) and hence, by the Landweber exact functor theorem, $E_*F$ is a flat $F_*$-module. In addition, $E_0$ is complete with respect to the ideal $\mathfrak{m} = \mathrm{Map}_c(G_n^j, I_n)$, where we also write $I_n = (p, u_1, \dots, u_{n-1}) \subset (E_n)_0$. Now

$$\pi_* \hat{L}(F \wedge E) = \mathrm{Map}_c(G_n^{j+k+1}, E_{n^*})$$

and, just as before, $\hat{L}(F \wedge E)$ is Landweber exact. Since $F_*E$ is flat over $E_*$, it also follows that $F \wedge E$ is Landweber exact. Thus, if $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ is a finite CW-spectrum whose Brown-Peterson homology is $BP_*/(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})$, we have that

$$\begin{aligned} F_*E/F_*E \cdot (p^{i_0}, \dots, v_{n-1}^{i_{n-1}}) &= \pi_*(F \wedge E \wedge M(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})) \\ &\xrightarrow{\approx} \pi_*(\hat{L}(F \wedge E) \wedge M(p^{i_0}, \dots, v_{n-1}^{i_{n-1}})) \\ &= \pi_* \hat{L}(F \wedge E)/\pi_* \hat{L}(F \wedge E) \cdot (p^{i_0}, \dots, v_{n-1}^{i_{n-1}}). \end{aligned}$$

Therefore

$$\begin{aligned} F_*E/F_*E \cdot I_n &= \pi_* \hat{L}(F \wedge E)/\pi_* \hat{L}(F \wedge E) \cdot I_n \\ &= \mathrm{Map}_c(G_n^{j+k+1}, \mathbb{F}_{p^n}[u, u^{-1}]). \end{aligned}$$

Clearly the frobenius is an isomorphism on $E_0/\mathfrak{m}$, and, from the above equality, it is also an isomorphism on $E_0F/\mathfrak{m}E_0F$. This implies that the relative frobenius for $E_0/\mathfrak{m} \to E_0F/\mathfrak{m}E_0F$ is an isomorphism.

Finally, to prove that $\pi_0 \mathscr{E}(F, E) \to \mathrm{Hom}_{\mathrm{alg}}(F_*, E_*)$ is one-to-one, it suffices to prove that the function

$$\mathrm{Hom}_{E_*-\mathrm{alg}}(E_*F, E_*) \to \mathrm{Hom}_{\mathrm{alg}}(F_*, E_*)$$

given by precomposition with the map

$$F_* = \pi_*(S^0 \wedge F) \to \pi_*(E \wedge F) = E_*F$$

is one-to-one. But this follows easily from the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{E_*-\mathrm{alg}}(E_*F, E_*) \arrow[r] \arrow[d, tail] & \mathrm{Hom}_{\mathrm{alg}}(F_*, E_*) \arrow[dd, tail] \\ \mathrm{Hom}_{(E_*\otimes\mathbb{Q})-\mathrm{alg}}(E_*F \otimes \mathbb{Q}, E_* \otimes \mathbb{Q}) \arrow[d, equal] & \\ \mathrm{Hom}_{(E_*\otimes\mathbb{Q})-\mathrm{alg}}(E_* \otimes F_* \otimes \mathbb{Q}, E_* \otimes \mathbb{Q}) \arrow[r, equal] & \mathrm{Hom}_{\mathrm{alg}}(F_* \otimes \mathbb{Q}, E_* \otimes \mathbb{Q}), \end{tikzcd}$$

and the fact that the vertical maps are monomorphisms since any Landweber exact spectrum is torsion free. $\square$

It will also be convenient to know that, for $E$ and $F$ as above, the canonical map from $\pi_0 \mathscr{E}(F, E)$ to the set of ring spectrum maps from $F$ to $E$ in the stable category is a bijection. The next result enables us to conclude this.

**Lemma 4.7.** *Let $E$ be as in Proposition* 4.6, *and let $F$ be a Landweber exact commutative ring spectrum. Then*

(i) $E_*F^{(j)} = \underbrace{E_*F \otimes_{E_*} E_*F \otimes_{E_*} \cdots \otimes_{E_*} E_*F}_{j\text{ times}}$ *for any* $j \geqslant 1$.

(ii) *The Kronecker pairing*

$$[F^{(j)}, E]^* \to \mathrm{Hom}_{E_*}(E_*F^{(j)}, E_*)$$

*is an isomorphism for all* $j \geqslant 1$.

**Proof.** Since $E$ and $F$ are Landweber exact, $E_*F$ is a flat $E_*$-module. This immediately implies (i). As for (ii), begin by observing that $F_1 \wedge F_2$ is Landweber exact if $F_1$ and $F_2$ are. Hence $F^{(j)}$ satisfies the hypotheses of the lemma, so we may assume $j = 1$ without loss of generality.

There is a universal coefficient spectral sequence

$$\mathrm{Ext}^{**}_{E_*}(E_*F, E_*) \Rightarrow E^*F$$

(see Remark 4.3); it thus suffices to show that $\mathrm{Ext}^i_{E_*}(E_*F, E_*) = 0$ for all $i > 0$. We do this by an argument similar to that in [27, 15.6]. Indeed, $E_*F = E_* \otimes_{E_{n*}} E_{n*}F$ and $E_{n*}F$ is flat over $E_{n*}$; hence

$$\mathrm{Ext}^{**}_{E_*}(E_*F, E_*) = \mathrm{Ext}^{**}_{E_{n*}}(E_{n*}F, E_*).$$

We thus need only show that $\mathrm{Ext}^i_{(E_n)_0}((E_n)_0F, E_0) = 0$ for all $i > 0$, where now $E = \hat{L}(E_n^{(j+1)})$ for some $j \geqslant 0$. Let $\mathfrak{m}$ be as in the proof of Proposition 3.6, and write $(E_n)_0F = M$. Then again by flat base change

$$\mathrm{Ext}^i_{(E_n)_0}(M, E_0/\mathfrak{m}) = \mathrm{Ext}^i_{\mathbb{F}_{p^n}}(M/I_nM, E_0/\mathfrak{m}).$$

This last group is trivial for $i > 0$ since $\mathbb{F}_{p^n}$ is a field. Similarly,

$$\mathrm{Ext}^i_{(E_n)_0}(M, \mathfrak{m}^t/\mathfrak{m}^{t+1}) = 0$$

for $i > 0$; now use the fact that $E_0 = \lim\limits_{\leftarrow t} E_0/\mathfrak{m}^t$ to conclude that $\mathrm{Ext}^i_{(E_n)_0}(M, E_0) = 0$ whenever $i > 0$. □

Now let $\mathbf{\Delta}$ be the category whose objects are the nonnegative integers—a typical object will be denoted $[n]$—and whose morphisms from $[n]$ to $[m]$ are the monotone nondecreasing functions from $\{0, 1, \ldots, n\}$ to $\{0, 1, \ldots, m\}$. Recall the definition of $R^+_{G_n}$ from the Introduction. We will construct an $h_\infty$-diagram

$$\mathbf{C} : (R^+_{G_n})^{\mathrm{opp}} \times \mathbf{\Delta} \to h\mathscr{E}$$

such that, for each $S \in \mathrm{Ob}\, R^+_{G_n}$,

$$C_S = \mathbf{C}(S, \quad) : \mathbf{\Delta} \to h\mathscr{E}$$

is a cosimplicial $K(n)_*$-local $E_n$-Adams resolution for the not yet constructed spectrum $\mathbf{F}(S)$. As described in the Introduction (for the case $S = G_n/U$), we seek a $K(n)_*$-local commutative $S^0$-algebra and right $E_n$-module spectrum $X_S$ together with an $\mathscr{E}$-map $d: X_S \to \hat{L}(X_S \wedge E_n)$ such that

$$\pi_* X_S = \mathrm{Map}_{G_n}(S, \mathrm{Map}_c(G_n, E_{n^*})) \tag{4.8}$$

as right $E_{n^*}$-modules, such that

$$\begin{aligned}\pi_* \hat{L}(X_S \wedge E_n) &\xleftarrow{\approx} \pi_* X_S \hat{\otimes}_{E_{n^*}} \pi_* \hat{L}(E_n \wedge E_n)\\ &= \mathrm{Map}_{G_n}(S, \mathrm{Map}_c(G_n, E_{n^*})) \hat{\otimes}_{E_{n^*}} \mathrm{Map}_c(G_n, E_{n^*})\\ &= \mathrm{Map}_{G_n}(S, \mathrm{Map}_c(G_n \times G_n, E_{n^*}))\end{aligned} \tag{4.9}$$

and such that $\pi_* d$ corresponds to the map induced by the group multiplication $G_n \times G_n \to G_n$.

**Proposition 4.10.** *There exists an $h_\infty$-diagram $\mathbf{X}: (R^+_{G_n})^{\mathrm{opp}} \to h\mathscr{E}$ and a natural transformation $d: \mathbf{X} \to \hat{L}(\mathbf{X} \wedge E_n)$ satisfying the above requirements. Moreover, this construction has the following properties*:

(i) *All maps are maps of right $E_n$-module spectra, where $E_n$ acts on the right factor of $\hat{L}(\mathbf{X} \wedge E_n)$.*
(ii) *If $S$ is finite, $X \equiv \mathbf{X}(S)$ is a product of a finite number of copies of $E_n$.*
(iii) *$X_{G_n} = \hat{L}(E_n \wedge E_n)$ and if $r_g: G_n \to G_n$ is right multiplication by $g \in G_n$, $\mathbf{X}(r_g) = \hat{L}(g \wedge E_n)$.*
(iv) *$d(G_n)$ is given by the composition*

$$\hat{L}(E_n \wedge E_n) = \hat{L}(E_n \wedge S^0 \wedge E_n) \to \hat{L}(E_n \wedge E_n \wedge E_n).$$

**Proof.** If $S$ is finite, then $S$ is a finite disjoint union of sets of the form $G_n/U$ for various open subgroups $U$ of $G_n$. We thus need only define $X_{G_n/U}$; $X_S$ can be defined as an appropriate finite product of the $X_{G_n/U}$'s. Define $X_{G_n/U} = \prod_{G_n/U} E_n$; i.e., $X_{G_n/U}$ is a finite product of copies of $E_n$, one for each element of $G_n/U$. Eqs. (4.8) and (4.9) are certainly satisfied. As for the map $d(G_n/U)$, observe that stable $E_n$-module maps $X_{G_n/U} \to \hat{L}(X_{G_n/U} \wedge E_n)$ are in bijective correspondence with $E_{n^*}$-module maps $\pi_* X_{G_n/U} \to \pi_* \hat{L}(X_{G_n/U} \wedge E_n)$. Furthermore, by the results of Goerss-Hopkins together with Lemma 4.7, any stable ring map $X_{G_n/U} \to \hat{L}(X_{G_n/U} \wedge E_n)$ lifts to an $\mathscr{E}$ map, unique up to $\mathscr{E}$ homotopy. Hence $d(G_n/U)$ can be chosen to induce the requisite map on homotopy groups.

If $f: S_1 \to S_2$ is a $G_n$-map with $S_2$ finite, then there exists a unique map $\mathbf{X}f: X_{S_2} \to X_{S_1}$ of $E_n$-module spectra inducing the map

$$f^*: \mathrm{Map}_{G_n}(S_2, \mathrm{Map}_c(G_n, E_{n^*})) \to \mathrm{Map}_{G_n}(S_1, \mathrm{Map}_c(G_n, E_{n^*}))$$

on stable homotopy groups. As before, $\mathbf{X}f$ has a representative in $\mathscr{E}$ and is unique up to $\mathscr{E}$ homotopy.

Finally, the naturality of $\pi_* \mathbf{X}$ and $\pi_* d$ implies, by Proposition 4.6, that $\mathbf{X}$ is a functor and $d$ is a natural transformation. It also follows from 4.6 that $\mathbf{X}$ is an $h_\infty$-diagram. □

**Construction 4.11.** Let $\eta: S^0 \to E_n$ be the unit map, let $\mu: E_n \wedge E_n \to E_n$ be the multiplication map, and, for each $S \in \mathrm{Ob}\, R^+_{G_n}$, let $s: X_S \wedge E_n \to X_S$ be the module structure map. Define an $h_\infty$-diagram

$$\mathbf{C}: (R^+_{G_n})^{\mathrm{opp}} \times \mathbf{\Delta} \to h\mathscr{E}$$

by

$$\mathbf{C}(S, [j]) \equiv C^j_S = \hat{L}(X_S \wedge E_n^{(j)}).$$

The coface and codegeneracy maps are given by

$$d^0(S) = \hat{L}(d(S) \wedge E_n^{(j)}) : C_S^j \to C_S^{j+1},$$

$$d^i(S) = \hat{L}(X_S \wedge E_n^{(i-1)} \wedge \eta \wedge (E_n)^{(j-i+1)}) : C_S^j \to C_S^{j+1}, \quad 1 \leqslant i \leqslant j+1$$

$$s^0(S) = \hat{L}(s \wedge E_n^{(j)}) : C_S^{j+1} \to C_S^j,$$

$$s^i(S) = \hat{L}(X_S \wedge E_n^{(i-1)} \wedge \mu \wedge E_n^{(j-1)}) : C_S^{j+1} \to C_S^j, \quad 1 \leqslant i \leqslant j.$$

That $\mathbf{C}$ is in fact a functor follows from Proposition 4.6; this proposition along with the Goerss-Hopkins results also implies that $\mathbf{C}$ is an $h_\infty$-diagram. Hence by Theorem 2.2, we may lift $\mathbf{C}$ to a diagram in $\mathscr{E}$. From now on, when we refer to $\mathbf{C}$, we actually mean this lift of $\mathbf{C}$.

**Definition 4.12.** The functor $\mathbf{F} : (R_{G_n}^+)^{\text{opp}} \to \mathscr{E}$ of Theorem 1 is defined by $\mathbf{F}(S) = \text{Tot}(\prod^* C_S)$, where, as before, $\prod^* C_S$ is the cosimplicial replacement of the (cosimplicial) diagram $C_S$. Since each $C_S^j$ is $K(n)_*$-local, so is $\mathbf{F}(S)$.

By (3.8), there is a spectral sequence

$$\lim_{\leftarrow \mathbf{\Delta}}{}^s \, [Z, C_S]^t \Rightarrow [Z, \mathbf{F}(S)]^{t+s} \tag{4.13}$$

for any CW-spectrum $Z$. We next claim that

$$\lim_{\leftarrow \mathbf{\Delta}}{}^s \, [Z, C_S]^t = \pi^s [Z, C_S^*]^t, \tag{4.14}$$

where the last term denotes the $s$th cohomology group of the cosimplicial abelian group $[Z, C_S]^t$. This result is of course well-known; however we provide a quick proof for the convenience of the reader.

**Proposition 4.15.** *Let $Ab^{\mathbf{\Delta}}$ be the category of cosimplicial abelian groups. Then the $\delta$-functors $\lim^*_{\leftarrow \mathbf{\Delta}} : Ab^{\mathbf{\Delta}} \to gr_+ Ab$ and $\pi^* : Ab^{\mathbf{\Delta}} \to gr_+ Ab$ are naturally equivalent, where $gr_+ Ab$ denotes the category of nonnegatively graded abelian groups.*

**Proof.** Since $\lim^0_{\leftarrow \mathbf{\Delta}}$ and $\pi^0$ are naturally equivalent, it suffices to prove that $\pi^s$ is effaceable. We begin this proof by making a few recollections. The forgetful functor $U : Ab^{\mathbf{\Delta}} \to gr_+ Ab$ has a right adjoint $V$—if $D$ is an object of $gr_+ Ab$, set

$$(VD)[n] = \prod_{[n] \to [m]} D^m,$$

where the product is taken over all morphisms $[n] \to [m]$ in $\mathbf{\Delta}$. It thus follows that if $D_m$ is injective for each $m$, then $VD$ is injective in $Ab^{\mathbf{\Delta}}$. Now let $C$ be a cosimplicial abelian group, and consider the monomorphism $C \to VUC$. We will show that $\pi^i(VUC) = 0$ for all $i > 0$. Indeed, first observe that $VD = \prod_n V(D(n))$, where

$$D(n)^m = \begin{cases} D^n & m = n, \\ 0 & \text{otherwise.} \end{cases}$$

It then suffices to show that $\pi^i(V(UC(n))) = 0$ for all $n$ and all $i > 0$. But $V(UC(n))$ is just the simplicial cochain complex of the standard $n$-simplex with coefficients in $C^n$. Therefore its cohomology is trivial in positive degrees. □

In general, if $J$ is a small category and $A$ is a $J$-diagram of abelian groups, then $\lim^*_{\leftarrow J} A$ may be computed as the cohomology of the cochain complex of the cosimplicial group $\prod^* A$, the cosimplicial replacement of $A$ (see [4, XI, 6.2]). Thus the $\delta$-functors $\pi^* C$ and $\pi^*(\prod^* C)$ are naturally equivalent on the category of cosimplicial abelian groups. The next result is proved in Appendix B; it will be needed to prove Proposition A.5.

**Proposition 4.16.** *Let $C$ be a cosimplicial abelian group. There is a natural map $T(C): C \to \prod^* C$ of cochain complexes which induces an isomorphism on $\pi^*$. Moreover, $T$ can be chosen so that the composition*

$$C^0 \xrightarrow{T(C)} \prod_j C^j \xrightarrow{\pi} C^{j_0}$$

*sends $x$ to $d^0 \cdots d^0 x$ for any $x \in C^0$, $j_0 \geqslant 0$.*

We can use spectral sequence (4.13) to verify Theorem 1(ii). This will follow easily from the next result.

**Proposition 4.17.** *Consider the maps $E_n \to \hat{L}(E_n \wedge E_n)$ and $\mathbf{F}(G_n) \to \hat{L}(E_n \wedge E_n)$ given by the maps*

$$E_n = E_n \wedge S^0 \xrightarrow{E_n \wedge \eta} \hat{L}(E_n \wedge E_n)$$

*and*

$$\mathbf{F}(G_n) = \mathrm{Tot} \textstyle\prod^* C_{G_n} \to \prod^0 C_{G_n} \to C^0_{G_n} = \hat{L}(E_n \wedge E_n)$$

*respectively. Let $Z$ be any CW-spectrum. Then there is a unique bijection $E_n^* Z \to \mathbf{F}(G_n)^* Z$ such that the diagram*

$$\begin{tikzcd} E_n^* Z \arrow[rr] \arrow[dr] & & \mathbf{F}(G_n)^* Z \arrow[dl] \\ & \widehat{L}(E_n \wedge E_n)^* Z & \end{tikzcd}$$

*commutes.*

**Proof.** The map $E_n \to \hat{L}(E_n \wedge E_n)$ is an augmentation of the cosimplicial object $C^*_{G_n}$ in the stable category. Furthermore, $C^*_{G_n}$ is chain contractible to $E_n$ in the stable category (cf. proof of Lemma 5.4); hence $\pi^*[Z, C^*_{G_n}]^t = E^t_n Z$ concentrated in degree 0. The desired result now follows from (4.13), (4.14), and an unraveling of the identifications. □

**Proof of Theorem 1(ii).** The preceding proposition provides us with a canonical weak equivalence $E_n \to \mathbf{F}(G_n)$. It also implies that the map is $G_n$-equivariant and is a map of ring spectra (in the stable category). By Lemma 4.7 and Theorem 4.1 this weak equivalence lifts to an $\mathscr{E}$ map. □

Finally, we show that, for $S = G_n/U$, the spectral sequence (4.13) has the form of a homotopy fixed point spectral sequence. The proof requires some preparation.

**Definition 4.18.** Let $M$ be an inverse limit of discrete $G_n$-modules, and let $H$ be a closed subgroup of $G_n$. Define a cochain complex $D_H^*(M)$ by

$$D_H^j(M) = (\mathrm{Map}_c(G_n^{j+1}, M))^H$$

with differential $\delta : D_H^j(M) \to D_H^{j+1}(M)$ given by

$$\begin{aligned}\delta f(g_0, g_1, \ldots, g_{j+1}) = &\sum_{i=0}^{j} (-1)^i f(g_0, \ldots, \hat{g}_{i+1}, \ldots, g_{j+1}) \\ &+ (-1)^{j+1} g_{j+1}^{-1} f(g_0 g_{j+1}^{-1}, \ldots, g_j g_{j+1}^{-1}).\end{aligned}$$

**Warning 4.19.** The action of $H$ on $\mathrm{Map}_c(G_n^{j+1}, M)$ is as elsewhere in this paper; that is, if $f \in \mathrm{Map}_c(G_n^{j+1}, M)$ and $h \in H$, then $(hf)(g_0, \ldots, g_{j+1}) = f(h^{-1}g_0, g_1, \ldots, g_{j+1})$.

**Lemma 4.20.** *The $\delta$-functor $H^*(D_H^*(?))$ is equivalent to $H_c^*(H, ?)$ on the abelian category of discrete $G_n$-modules.*

**Proof.** Since $H^0(D_H^*(M)) = M^H$, it suffices to prove that $H^* D_H^*(?)$ is effaceable. To this end, let $N$ be a discrete $G_n$-module, and consider $\mathrm{Map}_c(G_n, N)$. $\mathrm{Map}_c(G_n, N)$ is a discrete $G_n$-module, and there is a $G_n$-equivariant monomorphism $N \to \mathrm{Map}_c(G_n, N)$ defined by $n \mapsto h_n$, where $h_n(g) = g^{-1}n$. Now

$$D_H^j(\mathrm{Map}_c(G_n, N)) = (\mathrm{Map}_c(G_n^{j+1} \times G_n, N))^H,$$

and there is a contracting homotopy

$$q : D_H^{*+1}(\mathrm{Map}_c(G_n, N)) \to D_H^*(\mathrm{Map}_c(G_n, N))$$

given by

$$(qf)(g_0, \ldots, g_j, t) = (-1)^{j+1} f(g_0 t, \ldots, g_j t, t, 1),$$

proving that $H^i(D_H^*(\mathrm{Map}_c(G_n, N))) = 0$ for all $i > 0$. $\square$

We will identify the cochain complex $[Z, C_{G_n/U}^*]^t$ with $D_U^*(E_n^t Z)$. We must thus understand the cohomology of $D_H^*(?)$ for profinite modules.

**Lemma 4.21.** *Let $H$ be a closed subgroup of $G_n$ and let $M$ be a profinite discrete $G_n$-module, say $M = \lim_{\leftarrow \alpha} M_\alpha$, where $\alpha$ ranges over a directed set $I$. Then*

(i) $\lim_{\leftarrow \alpha}^s D_H^j(M_\alpha) = \begin{cases} D_H^j(M) & s = 0 \\ 0 & s > 0 \end{cases}$

(ii) $H^* D_H^*(M) = \lim_{\leftarrow \alpha} H^* D_H^*(M_\alpha) = \lim_{\leftarrow \alpha} H_c^*(H, M_\alpha)$.

**Proof.** We have

$$\lim_{\leftarrow\alpha}{}^{s} D^j_H(M_\alpha) = \pi^s \prod{}^* D^j_H(\mathbf{M}),$$

where $\mathbf{M}$ is the $I$-diagram $\alpha \mapsto M_\alpha$ and $\prod^* D^j_H(\mathbf{M})$ is the cosimplicial replacement of the $I$-diagram $D^j_H(\mathbf{M})$. Now

$$\prod{}^* D^j_H(\mathbf{M}) = D^j_H\left(\prod{}^* \mathbf{M}\right),$$

and since each $M_\alpha$ is finite, $\prod^q \mathbf{M}$ is profinite. Moreover,

$$D^j_H : \text{profinite } G_n\text{-modules} \to Ab$$

is exact; this again follows from the existence of a continuous (set-theoretic) cross-section of an epimorphism of profinite groups ([29, I, Theorem 3]). Therefore

$$\begin{aligned}\lim_{\leftarrow\alpha}{}^{s} D^j_H(M_\alpha) &= D^j_H\left(\pi^s \prod{}^* \mathbf{M}\right)\\ &= D^j_H\left(\lim_{\leftarrow\alpha}{}^{s} M_\alpha\right) = \begin{cases} D^j_H(M) & s = 0\\ 0 & s > 0\end{cases}\end{aligned}$$

by the vanishing of $\lim_{\leftarrow\alpha}^{s}$, $s > 0$, for directed systems of profinite groups.

As for the second part, consider the double cochain complex $\prod^* D^*_H(\mathbf{M})$. This yields two spectral sequences

$$\lim_{\leftarrow\alpha}{}^{s} H^t_c(H, M_\alpha) \Rightarrow H^{s+t}\left(\prod{}^* D^*_H(\mathbf{M})\right),$$

$$H^s\left(\lim_{\leftarrow\alpha}{}^{t} D^*_H(M_\alpha)\right) \Rightarrow H^{s+t}\left(\prod{}^* D^*_H(\mathbf{M})\right).$$

By (i), the second spectral sequence collapses to give

$$H^*\left(\prod{}^* D^*_H(\mathbf{M})\right) = H^*(D^*_H(M)).$$

On the other hand, $H$ is a closed subgroup of a $p$-analytic profinite group and is therefore itself a $p$-analytic group (see [9, 10.7]). Hence $H$ contains an open normal subgroup $U$ which is a Poincaré pro-$p$-group ([19]; see also [28] for a summary). This implies that $H^t_c(H, M_\alpha)$ is finite for each $\alpha$ and so $\lim_{\leftarrow\alpha}^{s} H^t_c(H, M_\alpha) = 0$ for all $s > 0$. The first spectral sequence then collapses to give

$$H^*\left(\prod{}^* D^*_H(\mathbf{M})\right) = \lim_{\leftarrow\alpha} H^*_c(H, M_\alpha).$$

This completes the proof. □

**Lemma 4.22.** *There is a canonical isomorphism* $[Z, C^*_{G_n/U}]^t \approx D^*_U E^t_n Z$ *of cochain complexes, where* $E^*_n Z$ *is topologized as in Remark* 1.3.

**Proof.** The quotient map $G_n \to G_n/U$ induces a map $C^*_{G_n/U} \to C^*_{G_n}$ and hence a map

$$[Z, C^*_{G_n/U}]^t \to [Z, C^*_{G_n}]^t \tag{4.23}$$

of cochain complexes. Now $X_{G_n} = \hat{L}(E_n \wedge E_n)$ (Proposition 4.10); therefore, the map $\tau_j$ of (2.6) provides an isomorphism

$$[Z, C^j_{G_n}]^t \to \mathrm{Map}_c(G_n^{j+1}, E^t_n Z) = D^j_{\{e\}}(E^t_n Z).$$

Moreover, this map is easily seen to be a cochain map and is $G_n$-equivariant, if the action of $g \in G_n$ on $C^j_{G_n}$ is given by

$$\hat{L}(\mathbf{X}(r_g) \wedge E_n \wedge \cdots \wedge E_n) = \hat{L}(g \wedge E_n \wedge E_n \cdots \wedge E_n).$$

Since right multiplication by $g \in U$ is trivial on $G_n/U$, the map in (4.23) is actually a map

$$[Z, C^*_{G_n/U}]^t \to ([Z, C^*_{G_n}]^t)^U \approx (D^*_{\{e\}}(E^t_n Z))^U = D^*_U(E^t_n Z) \tag{4.24}$$

of cochain complexes. Both $[?, C^j_{G_n/U}]^*$ and $D^j_U(E^*_n(?)) = \mathrm{Map}_c(G_n/U \times G^j_n, E^*_n(?))$ are cohomology theories satisfying the product axiom; it thus suffices to prove that the map in (4.24) is an isomorphism when $Z = S^0$. But, by (4.8) and the fact that $X_{G_n/U}$ is a finite product of $E_n$'s (Prop. 4.10), we have that

$$\begin{aligned}\pi_* C^j_{G_n/U} &= \pi_* X_{G_n/U} \otimes_{E_{n*}} \mathrm{Map}_c(G^j_n, E_{n^*})\\ &= \mathrm{Map}_c(G_n/U, E_{n^*}) \otimes_{E_{n*}} \mathrm{Map}_c(G^j_n, E_{n^*}).\end{aligned}$$

We also have

$$\begin{aligned}\pi_* C^j_{G_n} &= \pi_* X_{G_n} \hat{\otimes}_{E_{n*}} \mathrm{Map}_c(G^j_n, E_{n^*})\\ &= \mathrm{Map}_c(G_n, E_{n^*}) \hat{\otimes}_{E_{n*}} \mathrm{Map}_c(G^j_n, E_{n^*})\end{aligned}$$

and the map $\pi_* X_{G_n/U} \to \pi_* X_{G_n}$ corresponds to the homomorphism $\mathrm{Map}_c(G_n/U, E_{n^*}) \to \mathrm{Map}_c(G_n, E_{n^*})$ induced by the quotient map $G_n \to G_n/U$. From this it is clear that

$$\pi_* C^j_{G_n/U} \xrightarrow{\approx} (\pi_* C^j_{G_n})^U,$$

completing the proof.

Finally, we obtain our desired result.

**Proposition 4.25.** *Let* $S = G_n/U$. *The spectral sequence* (4.13) *has* $E_2$*-term canonically isomorphic to* $H^s_c(U, E^t_n Z)$. □

This is the spectral sequence of Theorèm 1(iv). It is strongly convergent because $H^s_c(U, E^t_n Z)$ is a profinite group for each $s$, $t$, and therefore $\varprojlim_r{}^1 / E^{s,t}_r = 0$. □

## 5. The Morava module of $E_n^{hU}$

In this section we complete the proof of Theorem 1. The key step is the identification of $\hat{L}(\mathbf{F}(S)\wedge E_n)$ with $X_S$—this not only immediately implies Theorem 1(iii) but enables us to identify the spectral sequence (4.13) with the $K(n)_*$-local $E_n$-Adams spectral sequence converging to $[Z,\mathbf{F}(S)]^*$. If $S$ is finite, $C_S=\prod_i C_{G_n/U_i}$ for a finite number of open subgroups $U_i$; thus we may assume from the beginning that $S=G_n/U$. (If $S=G_n$, Theorem 1(iii) is a consequence of 1(ii).)

The techniques involved in our computation of $\hat{L}(\mathbf{F}(S)\wedge E_n)$ will be applied in other contexts in Sections 6 and 7; we therefore proceed in a little more generality.

**Notation 5.1.** Write $C:J\to\mathscr{E}$ for any of the following diagrams:

(i) $J=\mathbf{\Delta}$ and $C=C_{G_n/U}$ for $U$ an open subgroup of $G_n$.
(ii) $J=G$ for $G$ a finite subgroup of $G_n$, and $C:G\to\mathscr{E}$ is the diagram given by the action of $G$ on $E_n$.
(iii) $J=F$, where $F$ and $K$ are as in Theorem 4 of the Introduction, and $C:F\to\mathscr{E}$ is the diagram given by the action of $F$ on (the left factor of) $\hat{L}(E_n^{hK}\wedge E_n^{(j+1)})$, $j\geqslant 0$.

Given a diagram $C:J\to\mathscr{E}$, there is a diagram $\hat{L}(C\wedge E_n):J\to\mathscr{E}$ defined by

$$\hat{L}(C\wedge E_n)(j)=\hat{L}(C(j)\wedge E_n),$$
$$\hat{L}(C\wedge E_n)(f)=\hat{L}(C(f)\wedge E_n)$$

for $j$ an object and $f$ a morphism in $J$. There is also a canonical map

$$\hat{L}\left[\left(\mathrm{Tot}\textstyle\prod^* C\right)\wedge E_n\right]\to\mathrm{Tot}\left(\textstyle\prod^*\hat{L}(C\wedge E_n)\right). \tag{5.2}$$

We will prove the following result.

**Theorem 5.3.** *If $C$ is one of the diagrams in Notation 5.1, then the map (5.2) is a weak equivalence.*

Before proving this theorem, we determine its consequence for $C=C_{G_n/U}$. The left side of (5.2) is $\hat{L}(E_n^{hU}\wedge E_n)$. To identify the right side, we examine the spectral sequence ${}^{II}E_r^{*,*}(Z,C)$ obtained by mapping a CW-spectrum $Z$ into the tower of fibrations $\{\mathrm{Tot}_k(\prod^*\hat{L}(C\wedge E_n))\}$.

**Lemma 5.4.** *Let $C=C_{G_n/U}$. Then*

$${}^{II}E_2^{s,t}(Z,C)=\pi^s[Z,\hat{L}(C^*\wedge E_n)]^t=\begin{cases}0 & s>0,\\ [Z,X_{G_n/U}]^t & s=0.\end{cases}$$

*In particular, the map*

$$\begin{aligned}\mathrm{Tot}\left(\textstyle\prod^*\hat{L}(C\wedge E_n)\right)&\to\textstyle\prod^0\hat{L}(C\wedge E_n)\\ &\to\hat{L}(C^0\wedge E_n)\end{aligned}$$

$$\begin{aligned} &\simeq \hat{L}(X_{G_n/U} \wedge E_n) \\ &\xrightarrow{s} X_{G_n/U} \end{aligned}$$

*is a weak equivalence.*

**Proof.** By Proposition 4.15, ${}^{II}E_2^{s,t}(Z,C) = \pi^s[Z, \hat{L}(C^* \wedge E_n)]^t$. Now let $X^*_{G_n/U}$ be the constant cosimplicial spectrum with $X^j_{G_n/U} = X_{G_n/U}$. There are cosimplicial—in the stable category—maps $X^*_{G_n/U} \to \hat{L}(C \wedge E_n)$ and $\hat{L}(C \wedge E_n) \to X^*_{G_n/U}$ given on 0-simplices by $d: X_{G_n/U} \to \hat{L}(X_{G_n/U} \wedge E_n)$ (see Proposition 4.10) and $s: \hat{L}(X_{G_n/U} \wedge E_n) \to X_{G_n/U}$, respectively. Furthermore, these maps are chain homotopy equivalences; use the chain homotopy

$$h: \hat{L}(C^j \wedge E_n) \to \hat{L}(C^{j-1} \wedge E_n)$$

defined by $h = (-1)^j X_{G_n/U} \wedge (E_n)^{(j-1)} \wedge \mu$. This then implies that $\pi^*[Z, \hat{L}(C^* \wedge E_n)]^t$ is as claimed. The weak homotopy equivalence $\mathrm{Tot}(\prod^* \hat{L}(C \wedge E_n)) \to X_{G_n/U}$ is obtained by tracking down the identifications. □

**Corollary 5.5.** *There is a natural weak equivalence $\hat{L}(E_n^{hU} \wedge E_n) \xrightarrow{\nu} X_{G_n/U}$ of commutative $S^0$-algebras and right $E_n$-module spectra such that the diagram*

$$\begin{tikzcd} \hat{L}(E_n^{hU} \wedge E_n) \arrow[r, "\nu"] \arrow[d, equal] & X_{G_n/U} \arrow[dd, "d"] \\ \hat{L}(E_n^{hU} \wedge S^0 \wedge E_n) \arrow[d] & \\ \hat{L}(E_n^{hU} \wedge E_n \wedge E_n) \arrow[r, "\hat{L}(\nu \wedge E_n)"] & \hat{L}(X_{G_n/U} \wedge E_n) \end{tikzcd}$$

*is homotopy commutative. In particular, Theorem* 1(iii) *holds.*

We now turn to the proof of Theorem 5.3. Let ${}^{I}E_r^{*,*}(Z,C)$ denote the spectral sequence obtained by mapping a CW-spectrum $Z$ into the tower

$$\cdots \to \hat{L}\left(\mathrm{Tot}_{k+1}\left(\prod{}^* C\right) \wedge E_n\right) \to \hat{L}\left(\mathrm{Tot}_k\left(\prod{}^* C\right) \wedge E_n\right) \to \cdots;$$

we have

$${}^{I}E_1^{s,t}(Z,C) = [Z, \hat{L}(F_s(C) \wedge E_n)]^{t+s}, \tag{5.6}$$

where $F_s(C)$ is the fiber of $\mathrm{Tot}_s(\prod^* C) \to \mathrm{Tot}_{s-1}(\prod^* C)$. There is a canonical stable map from the (unraveled exact couple of the) tower $\{\hat{L}(\mathrm{Tot}_k(\prod^* C) \wedge E_n)\}$ to $\{\mathrm{Tot}_k(\prod^* \hat{L}(C \wedge E_n))\}$; on fibers this map is the canonical map

$$\hat{L}(F_s(C) \wedge E_n) \to F_s(\hat{L}(C \wedge E_n)). \tag{5.7}$$

**Lemma 5.8.** *Let* ${}^{I}E_2^{*,*}(S^0, C) \to {}^{II}E_2^{*,*}(S^0, C)$ *be the map of spectral sequences described above. If* $C$ *is one of the diagrams in* 5.1, *then this map is an isomorphism.*

**Proof.** If $C$ is a diagram of the form 5.1(ii) or 5.1(iii) then the map (5.7) is an equivalence and hence the desired result follows immediately.

Now let $C = C_{G_n/U}$ and examine ${}^{I}E_2^{*,*}(S^0, C)$. By (4.14), $H^*([Z, F_*(C)]^{t+*}) = \pi^*[Z, C^*]^t$ for any CW-spectrum $Z$, and therefore Lemma 5.4 implies that

$$H^s\pi_{t-*}(F_*(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) = \pi^s\pi_t(C^* \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))$$

$$= \begin{cases} \pi_t(X_{G_n/U} \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) & s = 0, \\ 0 & s \neq 0. \end{cases}$$

In particular, these cohomology groups are all finite. Here $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ is as in the beginning of Section 4; the multi-index $I = (i_0, \ldots, i_{n-1})$ varies over a cofinal sequence as in [6, Section 4], so that

$$\hat{L}Y = \operatorname*{holim}_{\leftarrow I}(Y \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))$$

for any $E(n)_*$-local spectrum $Y$.

We claim that

$$\pi_* \operatorname*{holim}_{\leftarrow I}(F_k(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) = \lim_{\leftarrow I} \pi_*(F_k(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}));$$

that is

$$\lim_{\leftarrow I}{}^1 \pi_*(F_k(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) = 0. \tag{5.9}$$

Assuming this, it follows from the vanishing of $\lim_{\leftarrow I}^1 H^*\pi_{t-*}(F_*(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))$ that

$$H^s\pi_{t-*}\hat{L}(F_*(C) \wedge E_n) = \lim_{\leftarrow I} H^s\pi_{t-*}(F_*(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))$$

$$= \begin{cases} \pi_t X_{G_n/U} & s = 0 \\ 0 & s \neq 0. \end{cases}$$

Furthermore, one checks easily that the map ${}^{I}E_2^{s,t}(S^0, C) \to {}^{II}E_2^{s,t}(S^0, C)$ is the identity under this identification and the identification of Lemma 5.4.

We now verify the claim. $F_k(C)$ is equivalent to a product of $C^j_{G_n/U}$'s and is therefore Landweber exact. Hence $\pi_*(F_k(C) \wedge E_n)$ is a flat $E_{n^*}$-module, so

$$\pi_*(F_k(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) = \pi_*(F_k(C) \wedge E_n)/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}).$$

In particular, the inverse system $\{\pi_*(F_k(C) \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))\}$ is Mittag-Leffler and therefore 5.9 holds. This completes the proof. $\square$

**Corollary 5.10.** *If $C$ is one of the diagrams in Notation* 5.1, *then*

$$\underset{\leftarrow k}{\operatorname{holim}} \hat{L}\left(\mathrm{Tot}_k\left(\prod\nolimits^* C\right) \wedge E_n\right) \xrightarrow{\simeq} \mathrm{Tot}\left(\prod\nolimits^* \hat{L}(C \wedge E_n)\right).$$

**Proof.** If $C$ is a diagram of the form 5.1(ii) or 5.1(iii), the result follows from the fact that the map (5.7) is an equivalence and hence the towers $\{\hat{L}(\mathrm{Tot}_k(\prod^* C) \wedge E_n)\}$ and $\{\mathrm{Tot}_k(\prod^* \hat{L}(C \wedge E_n))\}$ are equivalent.

If $C = C_{G_n/U}$, the result follows from Lemma 5.8 together with the fact that both spectral sequences are strongly convergent in the sense of [4, IX, 5.4]. □

The proof of Theorem 5.3 will now be completed by showing that

$$\hat{L}\left[\left(\mathrm{Tot}\prod\nolimits^* C\right) \wedge E_n\right] \xrightarrow{\simeq} \underset{\leftarrow k}{\operatorname{holim}} \hat{L}\left(\mathrm{Tot}_k\left(\prod\nolimits^* C\right) \wedge E_n\right).$$

We separate off the following key ingredient.

Let

$$\cdots \to Y_k \to Y_{k-1} \to \cdots \to Y_0 \to *$$

be a tower of fibrations of $S^0$-modules, so that the canonical map $\underset{\leftarrow k}{\lim} Y_k \to \underset{\leftarrow k}{\operatorname{holim}} Y_k$ is a weak equivalence. Define $Y^k$ to be the fiber of $\underset{\leftarrow i}{\lim} Y_i \to Y_k$; there is then an inverse system of fibrations

$$\begin{tikzcd} Y^{k+1} \arrow[r] \arrow[d] & \underset{\leftarrow i}{\lim} Y_i \arrow[r] \arrow[d, equal] & Y_{k+1} \arrow[d] \\ Y^k \arrow[r] & \underset{\leftarrow i}{\lim} Y_i \arrow[r] & Y_k \end{tikzcd}.$$

According to [4, XI, 5.5], the map

$$\underset{\leftarrow k}{\operatorname{holim}} \left(\underset{\leftarrow i}{\lim} Y_i\right) \to \underset{\leftarrow k}{\operatorname{holim}} Y_k$$

is a fibration with fiber $\underset{\leftarrow k}{\operatorname{holim}} Y^k$. But the commutative diagram

$$\begin{tikzcd} \underset{\leftarrow k}{\operatorname{holim}}(\underset{\leftarrow i}{\lim} Y_i) \arrow[r] & \underset{\leftarrow k}{\operatorname{holim}} Y_k \\ \underset{\leftarrow k}{\lim}(\underset{\leftarrow i}{\lim} Y_i) \arrow[r, "="] \arrow[u, "\simeq"'] & \underset{\leftarrow k}{\lim} Y_k \arrow[u, "\simeq"'] \end{tikzcd}$$

shows that this map is a weak equivalence, and thus $\underset{\leftarrow k}{\operatorname{holim}} Y^k$ is stably trivial. In certain cases, we can say a good deal more.

**Lemma 5.11.** *Let $\{Y_k\}$ be as above, and let $E_r^{*,*}(Z)$ denote the spectral sequence obtained by mapping the CW-spectrum $Z$ into this tower. Suppose that there exist natural numbers $r_0$ and $s_0$ such that $E_{r_0}^{s,*}(Z) = 0$ for all spectra $Z$ whenever $s > s_0$. Then given $k$, there exists $q$ such that the map $Y^{k+q} \to Y^k$ is stably trivial.*

**Proof.** Let $F_s$ be the fiber of $Y_s \to Y_{s-1}$. There is a diagram

$$\begin{tikzcd} Y^k \arrow[dr] & & Y^{k+1} \arrow[ll] \arrow[dr] & & Y^{k+2} \arrow[ll] & \cdots \arrow[l] \\ & F_{k+1} \arrow[ur] & & F_{k+2} \arrow[ur] & & \end{tikzcd}$$

of exact triangles; let ${}_kE_r^{**}(Z)$ denote the spectral sequence obtained by mapping $Z$ into this diagram. This spectral sequence is isomorphic to the spectral sequence obtained by mapping $Z$ into the tower $\{Y_i\}_{i \geqslant k}$. Hence, ${}_kE_{r_0}^{s,*}(Z) = 0$ for $s > \max\{r_0 - 2, s_0 - k - 1\}$. Since $\underset{\leftarrow i}{\operatorname{holim}}\, Y^i \simeq *$, ${}_kE_r^{**}(Z)$ is conditionally convergent (in the sense of [2]) to $[Z, Y^k]^*$, and thus the horizontal vanishing line implies that

$$\frac{im([Z, Y^{k+s}]^* \to [Z, Y^k]^*)}{im([Z, Y^{k+s+1}]^* \to [Z, Y^k]^*)} = {}_kE_\infty^{s,*}(Z)$$

and that $\{im([Z, Y^{k+s}]^* \to [Z, Y^k]^*)\}_{s \geqslant 0}$ is a complete Hausdorff filtration of $[Z, Y^k]^*$. It then follows that $im([Z, Y^{k+s}]^* \to [Z, Y^k]^*) = 0$ for $s > \max\{r_0 - 2, s_0 - k - 1\}$. But $Z$ is arbitrary; therefore $Y^{k+s} \to Y^k$ is trivial for these values of $s$, completing the proof. □

**Lemma 5.12.** *Let $\{Y_k\}$ satisfy the hypotheses of Lemma* 5.11. *Then, if $W$ and $F$ are any spectra, there is an equivalence*

$$L_F\left[\left(\underset{\leftarrow k}{\operatorname{holim}}\, Y_k\right) \wedge W\right] \xrightarrow{\simeq} \underset{\leftarrow k}{\operatorname{holim}}\, L_F(Y_k \wedge W).$$

**Remark 5.13.** The above map is of course chosen so that composition with the projection onto $L_F(Y_k \wedge W)$ yields the canonical map

$$L_F\left[\left(\underset{\leftarrow k}{\operatorname{holim}}\, Y_k\right) \wedge W\right] \to L_F(Y_k \wedge W).$$

We will show that $\underset{\leftarrow k}{\lim}{}^1\, [Z, L_F(Y_k \wedge W)]^* = 0$ for any spectrum $Z$, so the equivalence of the lemma is uniquely determined.

**Proof.** We have diagrams (in the stable category)

$$\begin{tikzcd} L_F(Y^{k+1}\wedge W) \arrow[r] \arrow[d] & L_F((\operatorname{holim}_{\leftarrow i} Y_i)\wedge W) \arrow[r] \arrow[d, equal] & L_F(Y_{k+1}\wedge W) \arrow[d] \\ L_F(Y^k\wedge W) \arrow[r] & L_F((\operatorname{holim}_{\leftarrow i} Y_i)\wedge W) \arrow[r] & L_F(Y_k\wedge W) \end{tikzcd} \tag{5.14}$$

of fiber sequences. By the previous lemma,

$$\operatorname{holim}_{\leftarrow k} L_F(Y^k\wedge W)\simeq *;$$

from this should follow the desired result. However, we prefer to avoid trying to argue that "the homotopy inverse limit of the fibers is the fiber of the homotopy limits", and instead proceed less generally. Indeed, a diagram chase using (5.14) together with the previous lemma shows that the system $\{[Z, L_F(Y_k\wedge W)]^*\}$ is Mittag-Leffler for any $Z$ and therefore $\lim^1_{\leftarrow k}[Z, L_F(Y_k\wedge W)]^* = 0$. A similar argument also shows that

$$\left[Z, L_F\left(\left(\operatorname{holim}_{\leftarrow i} Y_i\right)\wedge W\right)\right]^* \xrightarrow{\approx} \lim_{\leftarrow k}[Z, L_F(Y_k\wedge W)]^*.$$

This completes the proof. □

**Proof of Theorem 5.3.** Start with cases (i) and (ii) of Notation 5.1. By virtue of the preceding work, we need only show that

$$\hat{L}\left[\left(\operatorname{Tot}\textstyle\prod^* C\right)\wedge E_n\right]\wedge X \xrightarrow{\simeq} \operatorname{holim}_{\leftarrow k}\hat{L}\left(\operatorname{Tot}_k\left(\textstyle\prod^* C\right)\wedge E_n\right)\wedge X \tag{5.15}$$

for some $p$-local finite spectrum $X$ Bousfield equivalent to $S^0_{(p)}$. Nilpotence technology [8, 4.1] tells us that this is the same as requiring $X$ to have torsion free $\mathbb{Z}_{(p)}$-homology.

We will prove (5.15) by finding a torsion free $X$ such that $E_2^{*,*}(Z\wedge DX, C)$ has a horizontal vanishing line independent of $Z$, where $E_r^{*,*}(Z\wedge DX, C)$ denotes the spectral sequence obtained by mapping $Z\wedge DX$ into $\{\operatorname{Tot}_k\prod^* C\}$. But

$$E_2^{s,t}(Z\wedge DX, C) = H_c^s(K, E_n^t(Z\wedge DX))$$

for $K$ some closed subgroup of $G_n$. ($K = U$ in case (i), and $K = G$ in case (ii).) Moreover,

$$H_c^s(K, E_n^t(Z\wedge DX)) = \lim_{\leftarrow\alpha} H_c^{s,-t}(K, E_{n*}X\otimes_{E_{n*}} E_{n*}DZ_\alpha),$$

where $Z = \operatorname{holim}_{\leftarrow\alpha} Z_\alpha$ is a presentation of $Z$ as a direct limit of finite CW-spectra.

Now Hopkins and Ravenel have shown that there exists a finite spectrum $X$ with free $\mathbb{Z}_{(p)}$-homology—in fact, $X$ can be taken to be $S^0$ if $p-1\nmid n$ and a summand of an iterated smash product of a finite complex projective space if $p-1\mid n$—such that $H_c^{s,*}(K, E_{n*}X/I_nE_{n*}X) = 0$ for $s$ bigger than some $s_0$. (This is proved for $K = G_n$ in [26, 8.3.5–7]; from this follows the result for $K$ closed in $G_n$. Or, one can observe that the proof for $G_n$ applies to closed subgroups as well.) An easy induction then shows that $H_c^{s,*}(K, E_{n*}X)$ and hence $H_c^{s,*}(K, E_{n*}X/I_mE_{n*}X)$ vanish for $s > s_0$ and $m\leqslant n$, where $I_m = (p, v_1, \ldots, v_{m-1})$. Finally, proceed by induction on a Landweber filtration of $E_{n*}DZ_\alpha$ to prove that $H_c^{s,*}(K, E_{n*}X \otimes_{E_{n*}} E_{n*}DZ_\alpha) = 0$ for $s > s_0$. (Note that the cross-section theorem [29, I, Theorem 3] allows us to conclude that $H_c^*(K, ?)$ takes short exact sequences of profinite $K$-modules to long exact sequences.)

This vanishing line allows us to apply Lemma 5.12 to the tower $\{(\mathrm{Tot}_k \prod^* C) \wedge X\}$ to complete the proof of (5.15).

As for case iii, we prove that $E_2^{*,*}(Z, C)$ has a horizontal vanishing line independent of $Z$. Indeed,

$$\begin{aligned} E_2^{s,t}(Z, C) &= H^s(F, [Z, \hat{L}(E_n^{hK} \wedge E_n^{(j+1)}]^t) \\ &= H^s(F, \mathrm{Map}_c(G_n^{j+1}, E_n^t Z)^K) \end{aligned}$$

by Proposition 6.3 and (6.5).

Now if $M$ is any discrete $G_n$-module, there is a spectral sequence

$$H^*(F, H_c^*(K, \mathrm{Map}_c(G_n^{j+1}, M))) \Rightarrow H_c^*(G, \mathrm{Map}_c(G_n^{j+1}, M)).$$

But $\mathrm{Map}_c(G_n^{j+1}, M) = \mathrm{Map}_c(G_n, \mathrm{Map}_c(G_n^j, M))$ is both $K$ and $G$-acyclic (see proof of Lemma 4.20); this implies that $\mathrm{Map}_c(G_n^{j+1}, M)^K$ is $F$-acyclic.

If $M$ is profinite, say $M = \lim_{\leftarrow \alpha} M_\alpha$, then there is a spectral sequence

$$\lim_{\leftarrow \alpha}{}^i H^{s-i}(F, \mathrm{Map}_c(G_n^{j+1}, M_\alpha)^K) \Rightarrow H^s(F, \mathrm{Map}_c(G_n^{j+1}, M)^K).$$

But

$$\begin{aligned} H^*(F, \mathrm{Map}_c(G_n^{j+1}, M_\alpha)^K) &= (\mathrm{Map}_c(G_n^{j+1}, M_\alpha)^K)^F \\ &= \mathrm{Map}_c(G_n^{j+1}, M_\alpha)^G \end{aligned}$$

concentrated in degree 0, and by Lemma 4.21(i), $\lim_{\leftarrow \alpha}{}^i \mathrm{Map}_c(G_n^{j+1}, M_\alpha)^G = 0$ for $i > 0$. Thus $E_2^{s,t}(Z, C) = 0$ for $s > 0$, and the proof concludes as before. □

We conclude this section by proving Theorem 1(iv).

**Proposition 5.16.** *Let $U$ be an open subgroup of $G_n$, and let $S = G_n/U$. The spectral sequence (4.13) is naturally isomorphic to the $K(n)_*$-local $E_n$-Adams spectral sequence converging to $[Z, E_n^{hU}]^*$.*

**Proof.** Consider the cosimplicial $S^0$-module $C_{G_n/U}$. By Corollary 5.5,

$$C_{G_n/U}^j \simeq \hat{L}(E_n^{hU} \wedge E_n^{(j+1)})$$

and thus by Remark A.9

$$* \to E_n^{hU} \to C_{G_n/U}^0 \xrightarrow{\delta} \Sigma^{-1} C_{G_n/U}^1 \xrightarrow{\delta} \Sigma^{-2} C_{G_n/U}^2 \to \cdots$$

is a $K(n)_*$-local $E_n$-resolution of $E_n^{hU}$, where $\delta = \sum (-1)^i d^i$. The desired result now follows from Proposition A.5. □

## 6. Homotopy fixed point spectra for closed subgroups of $G_n$

We begin by recalling the construction of the homotopy direct limit in $\mathscr{E}$ for the case of a direct sequence of commutative $S^0$-algebras.

**Definition 6.1.** Let

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} \cdots$$

be a direct sequence of commutative $S^0$-algebras. Then $\operatorname{holim}_{\to i} {}_{\mathscr{E}} A_i = \lim_{\to j} {}_{\mathscr{E}} \bar{A}_j$, where

$$\bar{A}_j = A_0 \otimes I \textstyle\coprod_{f_0} A_1 \otimes I \coprod_{f_1} \cdots \coprod_{f_{j-2}} A_{j-1} \otimes I \coprod_{f_{j-1}} A_j$$

is a $j$-fold mapping cylinder in $\mathscr{E}$. That is, all limits (including tensor products) are to be taken in $\mathscr{E}$.

The next result is crucial in the homotopical analysis of $E_n^{hG}$.

**Lemma 6.2.** *Let $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$ be a sequence of cellular algebra maps between cell commutative $S^0$-algebras. Then there is a natural weak equivalence $\operatorname{holim}_{\to i} {}_{\mathscr{E}} A_i \simeq \operatorname{holim}_{\to i} A_i$ of spectra, where $\operatorname{holim}_{\to i}$ denotes the ordinary homotopy colimit of $\{A_i\}$ regarded as a sequence of spectra.*

**Proof.** Let $\bar{A}_j$ be as in the previous definition. The evident map $\bar{A}_j \to A_j$ is a homotopy equivalence in $\mathscr{E}$ and hence in the category of spectra. Moreover, $\bar{A}_j$ is a relative cell commutative $S^0$-algebra under $\bar{A}_{j-1}$ for each $j \geqslant 1$. This implies that $\bar{A}_{j-1} \to \bar{A}_j$ is a cofibration of underlying spectra [11, VII, 4.14]. Thus, by [11, VII, 3.10], it follows that

$$\operatorname{holim}_{\to i} {}_{\mathscr{E}} A_i = \operatorname{holim}_{\to j} {}_{\mathscr{E}} \bar{A}_j = \lim_{\to j} \bar{A}_j \simeq \operatorname{holim}_{\to j} \bar{A}_j \simeq \operatorname{holim}_{\to j} A_j. \qquad \square$$

We can now identify the $K(n)_*$-local $E_n$-homology of $E_n^{hG}$ and thus prove Theorem 2(i).

**Proposition 6.3.** $\pi_* \hat{L}(E_n^{hG} \wedge E_n) = \operatorname{Map}_c(G_n, E_{n*})^G$ *as completed right* $\pi_* \hat{L}(E_n \wedge E_n)$*-comodule algebras.*

**Proof.** Just compute:

$$\begin{aligned}
\pi_*(E_n^{hG} \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) &= \lim_{\to j} \pi_*(E_n^{h(U_j G)} \wedge E_n \wedge M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})) \\
&= \lim_{\to j} \operatorname{Map}_c(G_n, E_{n*}/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))^{U_j G} \\
&= \operatorname{Map}_c(G_n, E_{n*}/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))^G,
\end{aligned}$$

since $E_{n*}/(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}})$ is discrete. The desired result follows easily. $\square$

It is now also a simple matter to identify $E_n^{hG}$ with the usual homotopy fixed point spectrum when $G$ is finite. As in the Introduction, we denote this spectrum by $E_n^{h'G}$.

**Proposition 6.4.** *Let $G$ be a finite subgroup of $G_n$. The map $E_n^{hG} \to E_n^{h'G}$ described in the Introduction is a weak equivalence.*

**Proof.** Since $E_n^{hG}$ and $E_n^{h'G}$ are both $K(n)_*$-local, it suffices to prove that the map

$$\pi_*\hat{L}(E_n^{hG} \wedge E_n) \to \pi_*\hat{L}(E_n^{h'G} \wedge E_n)$$

is an isomorphism. But we have a commutative diagram

$$\begin{tikzcd} E_n^{hG} \arrow[rr] \arrow[dr] & & E_n^{h'G}, \arrow[dl] \\ & E_n & \end{tikzcd}$$

and by the preceding proposition, $\pi_*\hat{L}(E_n^{hG} \wedge E_n)$ injects into $\pi_*\hat{L}(E_n \wedge E_n)$ with image $\mathrm{Map}_c(G_n, E_{n^*})^G$. On the other hand, Theorem 5.3 implies that $\hat{L}(E_n^{h'G} \wedge E_n) \xrightarrow{\simeq} [\hat{L}(E_n \wedge E_n)]^{h'G}$, where $G$ acts on the left factor $E_n$. But $\pi_*\hat{L}(E_n \wedge E_n) = \mathrm{Map}_c(G_n, E_{n^*})$ is $G$-acyclic (see proof of Lemma 4.20); therefore $\pi_*\hat{L}(E_n^{h'G} \wedge E_n) = \mathrm{Map}_c(G_n, E_{n^*})^G$ as well. This completes the proof. □

Now let $G$ be a closed subgroup of $G_n$ and form a $K(n)_*$-local $E_n$-Adams resolution of $E_n^{hG}$ as in Remark A.9; that is, set

$$C^j_{G_n/G} = \hat{L}(E_n^{hG} \wedge (E_n)^{(j+1)}).$$

As in the proof of Lemma 4.22, there is a natural transformation

$$[Z, C^*_{G_n/G}]^t \to D^*_G E_n^t Z \tag{6.5}$$

of cochain complexes; the proof of Proposition 6.3 generalizes to show that this map is an equivalence when $Z = S^0$. But both $[?, C^j_{G_n/G}]^*$ and $D^j_G E_n^*(?)$ are cohomology theories satisfying the product axiom and hence the map in (6.5) is an isomorphism for all $Z$. Theorem 2(ii) is thus a consequence of Lemmas 4.20 and 4.21.

**Proposition 6.6.** *Let $G$ be a closed subgroup of $G_n$. The $K(n)_*$-local $E_n$-Adams spectral sequence converging to $[Z, E_n^{hG}]^*$ is strongly convergent and has $E_2$-term naturally isomorphic to $H_c^*(G, E_n^* Z)$.*

**Proof.** The only part which has not been proved above is the strong convergence. But this follows from Proposition A.3. □

We can also prove a "covariant" version of Proposition 6.6, although the next result is probably not the most general result which can be achieved in this direction. Recall that $E(n)$ denotes the Landweber exact spectrum with coefficient ring $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$.

**Proposition 6.7.** *Let $X$ be a CW-spectrum such that, for each $E(n)$-module spectrum $M$, there exists a $k$ with $I_n^k M_* X = 0$. Then the $K(n)_*$-local $E_n$-Adams spectral sequence converging to $\pi_*\hat{L}(X \wedge E_n^{hG})$ has $E_2$-term naturally isomorphic to $H_c^*(G, E_{n^*} X)$.*

**Remark 6.8.** The hypotheses imply that $E_{n*}X$ is a discrete $G_n$-module, so $H_c^*(G, E_{n*}X)$ makes good sense.

The proof of this proposition requires a little preparation. Let $L_n$ denote the $E(n)_*$-localization functor. There is a cofiber sequence

$$\Sigma^{-n}M_nS^0 \to L_nS^0 \to L_{n-1}S^0 \tag{6.9}$$

(see [25, Section 5]) and

$$M_nS^0 = L_n\left(\operatorname*{holim}_{\to I} \Sigma^{-n_I} M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})\right), \tag{6.10}$$

$$n_I = \sum_{r=0}^{n-1} 2i_r(p-1),$$

where $I = (i_0, i_1, \ldots, i_{n-1})$ ranges over a cofinal sequence of multi-indices [6, Section 4].

**Lemma 6.11.** *Let $X$ satisfy the hypotheses of Proposition* 6.7, *and let $M$ be an $E(n)$-module spectrum. Then*

(i) *$M \wedge X$ is $K(n)_*$-local*
(ii) $M \wedge X \wedge \Sigma^{-n}M_nS^0 \xrightarrow{\simeq} M \wedge X \wedge L_nS^0 \simeq M \wedge X$.

**Proof.** (i) $M$ is an $E(n)$-module spectrum and is therefore $E(n)_*$-local. Since $E(n)$ is smashing ([26, 7.5.6]), $M \wedge X$ is $E(n)_*$-local. Hence

$$\hat{L}(M \wedge X) = F(\Sigma^{-n}M_nS^0, M \wedge X),$$

and we must show that

$$M \wedge X = F(L_nS^0, M \wedge X) \to F(\Sigma^{-n}M_nS^0, M \wedge X)$$

is an equivalence; i.e.

$$(M \wedge X)^*L_nS^0 \xrightarrow{\cong} (M \wedge X)^*(\Sigma^{-n}M_nS^0).$$

But $\Sigma^{-n}M_nS^0 \to L_nS^0$ is the composite of the maps $L_n\partial_i$, $0 \leqslant i \leqslant n-1$, in the cofibration sequences

$$N_iS^0 \to L_iN_iS^0 \equiv M_iS^0 \to N_{i+1}S^0 \xrightarrow{\partial_i} \Sigma N_iS^0,$$

where $N_0S^0 = S^0$; thus we need only show that $(M \wedge X)^*M_iS^0 = 0$ for $0 \leqslant i \leqslant n-1$. Consider the universal coefficient spectral sequence converging to $(M \wedge X)^*M_iS^0$ whose $E_2$-term is

$$\mathrm{Ext}_{E(n)_*}(v_i^{-1}E(n)_*/(p^\infty, \ldots, v_{i-1}^\infty), M_*X) = \mathrm{Ext}_{E(n)_*}(E(n)_*M_iS^0, M_*X).$$

Since there exists $k$ with $v_i^kM_*X = 0$, it follows that

$$\mathrm{Ext}_{E(n)_*}(v_i^{-1}E(n)_*/(p^\infty, \ldots, v_{i-1}^\infty), M_*X) = 0,$$

completing the proof.

(ii) This follows from 6.9 and the fact that, since $E(n-1)$ is smashing, we have

$$(L_{n-1}S^0) \wedge M \wedge X \simeq L_{n-1}(M \wedge X) \simeq *. \qquad \square$$

**Proof of Proposition 6.7.** By Lemma 6.11, we have

$$\begin{aligned}
\hat{L}(X \wedge E_n^{hG} \wedge E_n^{(j+1)}) &\simeq X \wedge E_n^{hG} \wedge E_n^{(j+1)} \\
&\simeq X \wedge \Sigma^{-n} M_n S^0 \wedge E_n^{hG} \wedge E_n^{(j+1)} \\
&\simeq \underset{\rightarrow}{\mathrm{holim}}\, X_\alpha \wedge \Sigma^{-n_I'} M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}) \wedge E_n^{hG} \wedge E_n^{(j+1)} \\
&\simeq \underset{\rightarrow}{\mathrm{holim}}\, \hat{L}(X_\alpha \wedge \Sigma^{-n_I'} M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}) \wedge E_n^{hG} \wedge E_n^{(j+1)}),
\end{aligned}$$

where $n_I' = n_I + n$, and the homotopy colimit varies over the finite CW-subspectra $X_\alpha$ of $X$ and the sequence of generalized $V(n-1)$'s of (6.10). But, by Proposition 6.6,

$$\begin{aligned}
H^*\pi_t\hat{L}(X \wedge E_n^{hG} \wedge (E_n)^{(*+1)}) &= \underset{\rightarrow}{\lim}\, H^*\pi_t\hat{L}(X_\alpha \wedge \Sigma^{-n_I'} M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}) \wedge E_n^{hG} \wedge (E_n)^{(*+1)}) \\
&= \underset{\rightarrow}{\lim}\, H_c^*(G, (E_n)_t(X_\alpha \wedge \Sigma^{-n_I'} M(p^{i_0}, \ldots, v_{n-1}^{i_{n-1}}))) \\
&= H_c^*(G, (E_n)_t(X \wedge \Sigma^{-n} M_n S^0)) \\
&= H_c^*(G, (E_n)_t X).
\end{aligned}$$

This completes the proof. $\square$

Finally, if $G$ is a finite subgroup of $G_n$, we identify the homotopy fixed point spectral sequence converging to $[Z, E_n^{h'G}]^*$ with the $K(n)_*$-local $E_n$-Adams spectral sequence. Let us first introduce some notation.

If $G$ acts on (the commutative $S^0$-algebra) $X$, write $\prod_G^* X$ for the cosimplicial replacement of the $G$-diagram defined by the action of $G$ on $X$. Also write $F_s(G, X)$ for the fiber of $\mathrm{Tot}_s(\prod_G^* X) \to \mathrm{Tot}_{s-1}(\prod_G^* X)$. The next result proves Theorem 3(ii).

**Proposition 6.12.** *The sequence*

$$* \to E_n^{h'G} \to F_0(G, E_n) \to F_1(G, E_n) \to \cdots$$

*is a $K(n)_*$-local $E_n$-Adams resolution of $E_n^{h'G}$.*

**Proof.** First observe that $F_s(G, E_n)$ is a product of $E_n$'s and is therefore $E_n$-injective. We thus need only show that

$$0 \to [Z, \hat{L}(E_n^{h'G} \wedge E_n)] \to [Z, \hat{L}(F_0(G, E_n) \wedge E_n)] \to \cdots \tag{6.13}$$

is exact for all CW-spectra $Z$. Since $G$ is finite, the cochain complex $\hat{L}(F_*(G,E_n)\wedge E_n)$ is equivalent to $F_*(G,\hat{L}(E_n\wedge E_n))$. Hence

$$\begin{aligned} H^i[Z,\hat{L}(F_*(G,E_n)\wedge E_n)] &= H^i(G,[Z,\hat{L}(E_n\wedge E_n)]) \\ &= H^i(G,\operatorname{Map}_c(G_n,E_n^*Z)) \\ &= \begin{cases} \operatorname{Map}_c(G_n,E_n^*Z)^G & i=0 \\ 0 & i>0 \end{cases} \end{aligned}$$

by the proof of Lemma 4.20. But by Proposition 6.3,

$$[Z,\hat{L}(E_n^{h'G}\wedge E_n)]^* = \operatorname{Map}_c(G_n,E_n^*Z)^G.$$

Tracking down the identifications completes the proof that sequence (6.13) is exact. □

## 7. Proof of Theorem 4

Let $G$ be a closed subgroup of $G_n$, $K$ a closed normal subgroup of $G$, and suppose $F=G/K$ is finite. Then the canonical map $E_n^{hG}\to E_n^{hK}$ factors through $(E_n^{hK})^F$, the $F$ fixed points of $E_n^{hK}$. The next result proves Theorem 4.

**Proposition 7.1.** *The composition $E_n^{hG}\to(E_n^{hK})^F\to(E_n^{hK})^{hF}$ is a weak equivalence.*

**Proof.** Let $F$ also denote the category with one object $*$ whose automorphism group is $F$, and consider the functor $\mathbf{Y}:\mathbf{\Delta}\times F\to\mathscr{E}$ with

$$\mathbf{Y}([j],*)=\hat{L}(E_n^{hK}\wedge E_n^{(j+1)}).$$

$F$ acts on $E_n^{hK}$, and $\mathbf{Y}$ maps morphisms in $\mathbf{\Delta}$ as in Remark A.9. Write $Y^j\equiv\mathbf{Y}([j],*)$. Since

$$E_n^{hK}\xrightarrow{\simeq}\operatorname*{holim}_{\leftarrow\mathbf{\Delta}} Y^j$$

by Corollary A.8, we have that

$$\begin{aligned} (E_n^{hK})^{hF} &\xrightarrow{\simeq} \operatorname*{holim}_{\leftarrow F}\ \operatorname*{holim}_{\leftarrow\mathbf{\Delta}}\ Y^j \\ &\approx \operatorname*{holim}_{\leftarrow\mathbf{\Delta}}(Y^j)^{hF}. \end{aligned}$$

Of course, there is a canonical augmentation $(E_n^{hK})^{hF}\to(Y^0)^{hF}$ and hence an augmentation $E_n^{hG}\to(E_n^{hK})^{hF}\to(Y^0)^{hF}$. We claim that

$$*\to E_n^{hG}\to(Y^0)^{hF}\to\Sigma^{-1}(Y^1)^{hF}\to\cdots$$

is a $K(n)_*$-local $E_n$-Adams resolution of $E_n^{hG}$. Assuming this, it follows from Corollary A.8 that

$$E_n^{hG}\xrightarrow{\simeq}\operatorname*{holim}_{\leftarrow\mathbf{\Delta}}(Y^j)^{hF}$$

and hence that

$$E_n^{hG} \xrightarrow{\sim} (E_n^{hK})^{hF}.$$

To prove the claim, we must show that

$$0 \to [Z, \hat{L}(E_n^{hG} \wedge E_n)] \to [Z, \hat{L}((Y^0)^{hF} \wedge E_n)] \to [Z, \hat{L}((Y^1)^{hF} \wedge E_n)] \to \cdots$$

is exact for any CW-spectrum $Z$. (Since $(Y^i)^{hF}$ is an $E_n$-module spectrum it is $K(n)_*$-local $E_n$-injective.) Begin by recalling Theorem 5.3 which asserts that

$$\hat{L}((Y^i)^{hF} \wedge E_n) \xrightarrow{\sim} [\hat{L}(Y^i \wedge E_n)]^{hF}$$

for all $i \geqslant 0$. But

$$[Z, \hat{L}(Y^i \wedge E_n)]^* = \mathrm{Map}_c(G_n^{i+2}, E_n^* Z)^K$$

by (6.5), and since $\mathrm{Map}_c(G_n^{i+2}, E_n^* Z)^K$ is $F$-acyclic (see proof of Theorem 5.3), it follows that

$$\begin{aligned}
[Z, \hat{L}((Y^i)^{hF} \wedge E_n)]^* &= \mathrm{Map}_c(G_n^{i+2}, E_n^* Z)^G \\
&= \mathrm{Map}_c(G_n^{i+1}, \mathrm{Map}_c(G_n, E_n^* Z))^G \\
&= D_G^i(\mathrm{Map}_c(G_n, E_n^* Z)).
\end{aligned}$$

This is in fact an isomorphism of cochain complexes, so

$$H^i([Z, \hat{L}((Y^*)^{hF} \wedge E_n)]^t) = \begin{cases} \mathrm{Map}_c(G_n, E_n^t Z)^G & i = 0, \\ 0 & i > 0. \end{cases}$$

Since

$$[Z, \hat{L}(E_n^{hG} \wedge E_n)]^t = \mathrm{Map}_c(G_n, E_n^t Z)^G,$$

the claim is proved. □

## 8. Two applications

We begin with the proof of Theorem 6. Let $c : G_n \to \mathbb{Z}_p$ be a surjective continuous homomorphism, and consider the exact sequence of groups

$$0 \to K \to G_n \xrightarrow{c} \mathbb{Z}_p \to 0.$$

Then $\mathbb{Z}_p$ acts—at least regarded as a discrete group—by $S^0$-algebra maps on $E^{hK}$. We have the following result.

**Proposition 8.1.** *Let $t$ be the topological generator 1 of $\mathbb{Z}_p$. Then there is a fiber sequence (in the stable category)*

$$\hat{L}S^0 \xrightarrow{\hat{L}\eta} E_n^{hK} \xrightarrow{id-t} E_n^{hK} \xrightarrow{\partial} \Sigma \hat{L} S^0$$

*where $\eta : S^0 \to E_n^{hK}$ denotes the unit map.*

**Proof.** Let $X$ be the fiber of $id - t$. Since $t$ is an $S^0$-algebra map, the unit $\eta$ factors to give a commutative diagram

$$\begin{tikzcd} X \arrow[rr] & & E_n^{hK} \\ & S^0 \arrow[ul, dashed, "\eta'"] \arrow[ur, "\eta"'] & \end{tikzcd}.$$

We claim that $\eta'$ is a $K(n)_*$-equivalence, and thus $\hat{L}S^0 \xrightarrow{\sim} X$. To prove this, it suffices to show that

$$\pi_*\hat{L}(\eta' \wedge E_n): E_{n^*} \to \pi_*\hat{L}(X \wedge E_n)$$

is an isomorphism. There is a commutative diagram

$$\begin{tikzcd} \pi_*\widehat{L}(E_n^{hK} \wedge E_n) \arrow[r, "(id-t)_*"] \arrow[d, phantom, "\wr"] & \pi_*\widehat{L}(E_n^{hK} \wedge E_n) \arrow[d, phantom, "\wr"] \\ \mathrm{Map}_c(G_n, E_{n^*})^K \arrow[r, "id-\tau_t"] & \mathrm{Map}_c(G_n, E_{n^*})^K, \end{tikzcd}$$

where $\tau_t(f)(g) = f(t^{-1}g)$ for $f \in \mathrm{Map}_c(G_n, E_{n^*})^K$ and $g \in G_n$. (Here $t$ also denotes any element of $G_n$ whose image under $c$ is $t \in \mathbb{Z}_p$.) Since

$$\mathrm{Map}_c(G_n, E_{n^*})^K = \mathrm{Map}_c(G_n/K, E_{n^*}) = \mathrm{Map}_c(\mathbb{Z}_p, E_{n^*}),$$

a standard argument shows that $id - \tau_t$ is surjective and hence

$$\pi_*\hat{L}(X \wedge E_n) = \ker(id - \tau_t).$$

But $\ker(id - \tau_t)$ just consists of the constant maps from $\mathbb{Z}_p$ to $E_{n^*}$; this implies that $\pi_*\hat{L}(\eta' \wedge E_n)$ is an isomorphism, completing the proof. □

The next result implies Theorem 6.

**Proposition 8.2.** *Let $\partial$ be as in Proposition* 8.1. *Then the composition $\partial \circ \eta : S^0 \to \sum \hat{L}S^0$ is detected by $\pm c \in H_c^1(S_n, E_{n^*})^{\mathrm{Gal}}$ in the $K(n)_*$-local $E_n$-Adams spectral sequence.*

**Remark 8.3.** Let $G$ be a profinite group and $M$ a discrete $G$-module, and consider the short exact sequence

$$0 \to M \xrightarrow{i} \mathrm{Map}_c(G, M) \to \mathrm{Map}_c(G, M)/M \to 0,$$

where $i(m)(g) = g^{-1}m$. Then the coboundary map provides an epimorphism

$$(\mathrm{Map}_c(G, M)/M)^G = H_c^0(G, \mathrm{Map}_c(G, M)/M) \to H_c^1(G, M).$$

If $M$ is a trivial $G$-module, this map is an isomorphism; moreover, $(\mathrm{Map}_c(G, M)/M)^G$ is just the group of continuous group homomorphisms from $G$ to $M$. This gives a canonical identification of $H_c^1(G, M)$ with this group of homomorphisms. In particular, the homomorphism $c: G_n \to \mathbb{Z}_p$ defines an element of $\varprojlim_j H_c^1(G_n, \mathbb{Z}/(p^j)) = H_c^1(G_n, \mathbb{Z}_p)$ and hence an element of $H_c^1(G_n, E_{n^*})$.

**Proof of Proposition 8.2.** Write $f=\hat{L}(\partial\circ\eta)$, and consider the cofiber sequence

$$\cdots\to\Sigma^{-1}\hat{L}S^0\xrightarrow{f}\hat{L}S^0\xrightarrow{i}C(f)\xrightarrow{p}\hat{L}S^0\to\cdots. \tag{8.4}$$

By Proposition A.10, $f$ is detected by $\delta(1)\in H_c^1(G_n,E_{n^*})$, where $\delta$ is the coboundary map $H_c^j(G_n,E_{n^*})\to H_c^{j+1}(G_n,E_{n^*})$ for the short exact sequence

$$0\to E_n^*S^0\xrightarrow{p^*}E_n^*C(f)\xrightarrow{i^*}E_n^*S^0\to 0$$

of $G_n$-modules, and $1\in H^0(G_n,E_{n^*})$ is just the unit in $(E_{n^*})^{G_n}=\mathbb{Z}_p$. Now the sequence (8.4) is self-dual; that is, applying the function spectrum functor $F(?,\hat{L}S^0)$ yields the same sequence. Hence $f$ is detected by $\pm\delta'(1)\in H_c^1(G_n,E_{n^*})$, where $\delta'$ denotes the coboundary map for the short exact sequence

$$0\to E_{n^*}^{\wedge}S^0\xrightarrow{i_*}E_{n^*}^{\wedge}C(f)\xrightarrow{p_*}E_{n^*}^{\wedge}S^0\to 0.$$

(By $E_{n^*}^{\wedge}X$, we here mean $\pi_*\hat{L}(X\wedge E_n)$.)

In addition, we have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{L}S^0 \arrow[r, "i"] \arrow[d, equal] & C(f) \arrow[r, "p"] \arrow[d, dashed] & \widehat{L}S^0 \arrow[d, "-\widehat{L}\eta"] \\
\cdots \arrow[r] & \widehat{L}S^0 \arrow[r] & E_n^{hK} \arrow[r, "id-t"] & E_n^{hK}
\end{tikzcd}$$

of cofibration sequences; this yields the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_{n^*}^{\wedge}S^0 \arrow[r] \arrow[d, equal] & E_{n^*}^{\wedge}C(f) \arrow[r] \arrow[d] & E_{n^*}^{\wedge}S^0 \arrow[r] \arrow[d, "-\eta_*"] & 0 \\
0 \arrow[r] & E_{n^*}^{\wedge}S^0 \arrow[r] & \mathrm{Map}_c(G_n,E_{n^*})^K \arrow[r, "id-\tau_t"] \arrow[d, equal] & \mathrm{Map}_c(G_n,E_{n^*})^K \arrow[r] \arrow[d, equal] & 0 \\
& & \mathrm{Map}_c(\mathbb{Z}_p,E_{n^*}) & \mathrm{Map}_c(\mathbb{Z}_p,E_{n^*}) &
\end{tikzcd} \tag{8.5}$$

This diagram is a diagram of $G_n$-modules; the action of $G_n$ on $\mathrm{Map}_c(\mathbb{Z}_p,E_{n^*})$ is given by

$$(gh)(s)=g(h(s+c(g)))$$

for $g\in G_n$, $h\in\mathrm{Map}_c(\mathbb{Z}_p,E_{n^*})$ and $s\in\mathbb{Z}_p$. This follows by naturality from the discussion preceding Remark 2.4. There is also a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_{n^*} \arrow[r] & \mathrm{Map}_c(\mathbb{Z}_p,E_{n^*}) \arrow[r, "id-\tau_t"] & \mathrm{Map}_c(\mathbb{Z}_p,E_{n^*}) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}_p \arrow[r] \arrow[u] & \mathrm{Map}_c(\mathbb{Z}_p,\mathbb{Z}_p) \arrow[r, "id-\tau_t"] \arrow[u] & \mathrm{Map}_c(\mathbb{Z}_p,\mathbb{Z}_p) \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{8.6}$$

of $G_n$-modules. Then $\delta'(1)$ is the image of $-\delta''(1)\in H_c^1(G_n,\mathbb{Z}_p)$ in $H_c^1(G,E_{n^*})$, where $\delta''$ is the coboundary map associated to the bottom exact sequence and $1\in\mathrm{Map}_c(\mathbb{Z}_p,\mathbb{Z}_p)$ is the constant map with value 1. Finally, use the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_p \arrow[r] \arrow[d, equal] & \mathrm{Map}_c(\mathbb{Z}_p, \mathbb{Z}_p) \arrow[r, "id-\tau_t"] \arrow[d, "\mathrm{Map}_c(c,id)"] & \mathrm{Map}_c(\mathbb{Z}_p, \mathbb{Z}_p) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & \mathrm{Map}_c(G_n, \mathbb{Z}_p) \arrow[r] & \mathrm{Map}_c(G_n, \mathbb{Z}_p)/\mathbb{Z}_p \arrow[r] & 0 \end{tikzcd}$$

and Remark 8.3 to complete the proof. □

We next turn to the proof of Theorem 5.

**Lemma 8.7.** *Let $G$ be as in the statement of Theorem 5; that is, $G$ is the closed subgroup of $\mathbb{Z}_p^\times$ generated by $l^{sp^j}$. Then $E_1^{hG}$ is the fiber of $\mathrm{id} - l^{sp^j} : E_1 \to E_1$.*

**Proof.** Let $F$ denote this fiber. Since the composition

$$E_1^{hG} \to E_1 \overset{\mathrm{id}-l^{sp^j}}{\to} E_1$$

is trivial, there is a commutative diagram

$$\begin{tikzcd} F \arrow[r] & E_1 \\ E_1^{hG} \arrow[u] \arrow[ur] & \end{tikzcd}.$$

To show that $E_1^{hG} \overset{\simeq}{\to} F$, it suffices to show that

$$\begin{tikzcd} \pi_* \widehat{L}(E_1^{hG} \wedge E_1) \arrow[r, "\approx"] \arrow[d, phantom, "\wr"] & \pi_* \widehat{L}(F \wedge E_1) \\ \mathrm{Map}_c(\mathbb{Z}_p^\times / G, E_{1*}) & \end{tikzcd}. \tag{8.8}$$

But there is a commutative diagram

$$\begin{tikzcd} \pi_* \widehat{L}(E_1 \wedge E_1) \arrow[r, "(\mathrm{id}-l^{sp^j})_*"] \arrow[d, phantom, "\wr"] & \pi_* \widehat{L}(E_1 \wedge E_1) \arrow[d, phantom, "\wr"] \\ \mathrm{Map}_c(\mathbb{Z}_p^\times, E_{1*}) \arrow[r, "\mathrm{id}-\tau"] & \mathrm{Map}_c(\mathbb{Z}_p^\times, E_{1*}), \end{tikzcd}$$

where here $\tau(f)(u) = f(l^{-sp^j} u)$ for $f \in \mathrm{Map}_c(\mathbb{Z}_p^\times, E_{1*})$ and $u \in \mathbb{Z}_p^\times$. Clearly

$$\ker(\mathrm{id} - \tau) = \mathrm{Map}_c(\mathbb{Z}_p^\times / G, E_{1*});$$

the map in (8.8) is therefore an isomorphism provided that $\mathrm{id} - \tau$ is surjective. But this follows without difficulty from the facts that $G \approx \mathbb{Z}_p$ and $\mathbb{Z}_p^\times / G$ is finite. □

The next result is well known.

**Lemma 8.9.** *Let $f : X \to Y$ be a map in the stable category such that $\pi_i f$ is an isomorphism for all $i$ sufficiently large. Then $L_{K(1)} f : L_{K(1)} X \to L_{K(1)} Y$ is an equivalence.*

**Proof.** Let $M(p^j)$ denote the mod $(p^j)$ Moore spectrum, with $v_1$ self-map $v$. Then

$$f \wedge v^{-1}M(p^j): X \wedge v^{-1}M(p^j) \to Y \wedge v^{-1}M(p^j)$$

is an equivalence, since $\pi_i(f \wedge M(p^j))$ is an isomorphism for all $i$ sufficiently large. But, if $Z$ is any spectrum,

$$L_1 Z \wedge M(p^j) = L_1(Z \wedge M(p^j)) = Z \wedge L_1 M(p^j) = Z \wedge v^{-1}M(p^j)$$

by the telescope conjecture for $n = 1$ (see [26, 7.5.5]), and

$$L_{K(1)}X = \operatorname*{holim}_{\leftarrow j} L_1 X \wedge M(p^j).$$

Hence $L_{K(1)}f$ is an equivalence as desired. □

**Proof of Theorem 5.** Since $K(k)$ is fixed by $\mathrm{Gal}(\bar{\mathbb{F}}_l/k) = \hat{\mathbb{Z}}/sp^j\hat{\mathbb{Z}}$, there is a commutative diagram

$$\begin{tikzcd} K(k) \arrow[r] \arrow[d] & F = E_1^{hG} \arrow[d, shift right] \arrow[d, shift left] \\ K(\bar{\mathbb{F}}_l) \arrow[r] & E_1 \end{tikzcd}.$$

But Quillen showed [24, Theorem 7] that

$$BGL(k)^+ = \Omega_0^\infty K(k) \to \Omega_0^\infty F$$

is an equivalence; hence

$$\pi_i K(k) \xrightarrow{\approx} \pi_i E_1^{hG}$$

for all $i \geqslant 1$. The result now follows from Lemma 8.9. □

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## Appendix A. The $K(n)_*$-local $E_n$-Adams spectral sequence

We begin with some generalities on the Adams-type spectral sequences that we will be considering.

Let $E$ be a commutative ring spectrum (in the stable category), and let $F$ be any spectrum. Then one can construct the $E$-Adams spectral sequence in the $F_*$-local category. In more detail, we follow Miller [21] and define an injective class (see [18]) in this category by declaring an $F_*$-local spectrum $X$ to be $E$-injective if it is a retract of $L_F(Y \wedge E)$ for some spectrum $Y$. A sequence $X' \to X \to X''$ is then $E$-exact if $[X', I] \leftarrow [X, I] \leftarrow [X'', I]$ is exact for every $E$-injective $I$. Given $X$, one may construct an $E$-exact sequence

$$* \to X \to I^0 \to I^1 \to \cdots \tag{A.1}$$

such that $I^s$ is $E$-injective for all $s$. One may then construct a diagram

$$\begin{tikzcd} X = X^0 \arrow[dr, "j"'] & & X^1 \arrow[ll, "i"'] \arrow[dr, "j"'] & & X^2 \arrow[ll, "i"'] \arrow[dr] & \arrow[l] \\ & I^0 \arrow[ur, "k"'] & & I^1 \arrow[ur, "k"'] & & \cdots \end{tikzcd} \tag{A.2}$$

of exact triangles; observe that the map

$$L_F(j \wedge E) : L_F(X^i \wedge E) \to L_F(I^i \wedge E)$$

is a split monomorphism. Conversely, a diagram of exact triangles as in (A.2) with each $L_F(j \wedge E)$ split monic yields an $E$-exact sequence (A.1). Such a diagram (with each $I^j$ $E$-injective) is called an $F_*$-local $E$-Adams resolution of $X$ and is functorial up to chain homotopy.

By mapping a spectrum $Z$ into an $F_*$-local $E$-Adams resolution of $X$, we obtain a spectral sequence, called the $F_*$-local $E$-Adams spectral sequence. The work of Bousfield [3] comes into play in dealing with the convergence question. We define the $F_*$-local $E$-nilpotent spectra to be the smallest class $\mathscr{C}$ of ($F_*$-local) spectra such that

(i) $L_F E \in \mathscr{C}$,
(ii) $L_F(N \wedge X) \in \mathscr{C}$ whenever $N \in \mathscr{C}$,
(iii) $\mathscr{C}$ is closed under retracts and cofibrations.

If $X$ is $F_*$-local $E$-nilpotent, then the proof of [3, Theorem 6.10] applies to show that the $F_*$-local $E$-Adams spectral sequence converges conditionally and strongly to $[Z, X]^*$ for $Z$ any CW-spectrum.

We now specialize to the case $E = E_n$ and $F = K(n)$. Here we have the following result.

**Proposition A.3.** *If $X$ is $K(n)_*$-local, then $X$ is $K(n)_*$-local $E_n$-nilpotent.*

**Proof.** Consider the Landweber exact spectrum $E'(n)$ with coefficient ring $E'(n)_* = \mathbb{Z}_{(p)}[u_1, \ldots, u_{n-1}][u, u^{-1}]$. Since $E'(n)$ is equivalent to a wedge of suspensions of $E(n)$ and $S^0$ is $E(n)$-prenilpotent ([17, Theorem 5.3]), it follows that $S^0$ is $E'(n)$-prenilpotent. But $X$ is $E'(n)_*$-local; therefore $X$ is $E'(n)$-nilpotent. Since $\hat{L}E'(n)$ is a retract of $E_n$, $X$ is also $K(n)_*$-local $E_n$-nilpotent. $\square$

We now examine some further properties of the $K(n)_*$-local $E_n$-Adams spectral sequence which have been used in the text.

Let $X$ be $K(n)_*$-local, and let $C$ be a cosimplicial $S^0$-module with an augmentation $X \to C$ such that

$$* \to X \to C^0 \xrightarrow{\delta} \Sigma^{-1} C^1 \xrightarrow{\delta} \Sigma^{-2} C^2 \to \cdots$$

is a $K(n)_*$-local $E_n$-resolution of $X$. (The suspensions appear so that each map $\delta = \sum (-1)^i d^i$ has degree $-1$.) Consider also the diagram

$$\begin{tikzcd} \operatorname*{holim}_{\leftarrow \Delta} C = \mathrm{Tot}^{-1} \arrow[dr] & & \mathrm{Tot}^0 \arrow[ll] \arrow[dr] & & \mathrm{Tot}^1 \arrow[ll] & \cdots \arrow[l] \\ & F_0 \arrow[ur] & & F_1 \arrow[ur] & & \end{tikzcd} \tag{A.4}$$

of exact triangles, where $\mathrm{Tot}^i$ is the fiber of $\mathrm{Tot}\prod^* C \to \mathrm{Tot}_i \prod^* C$ and $F_i$ is the fiber of $\mathrm{Tot}_i(\prod^* C) \to \mathrm{Tot}_{i-1}(\prod^* C)$ as in Section 5. Since $F_i$ is the product of various $C^j$'s, it is $K(n)_*$-local $E_n$-injective. Therefore, the canonical map $h: X \to \operatorname{holim}_{\leftarrow \Delta} C$ extends to a diagram

$$\begin{tikzcd}
* \arrow[r] & X \arrow[r] \arrow[d, "h"] & C^0 \arrow[r, "\delta"] \arrow[d, "g^0"] & \Sigma^{-1}C^1 \arrow[r, "\delta"] \arrow[d, "h^1"] & \Sigma^{-2}C \arrow[d, "h^2"] & \\
* \arrow[r] & \operatorname{holim}_{\leftarrow \Delta} C \arrow[r] & F_0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & \cdots
\end{tikzcd}$$

of augmented cochain complexes in the stable category, unique up to chain homotopy. This diagram is induced by a map of exact triangles and hence defines a map of spectral sequences.

**Proposition A.5.** *With the notation as above, $\{h^i\}$ induces an isomorphism $\pi^*[Z, C^*] \to H^*[Z, F_*]$ for any spectrum $Z$. Hence the spectral sequence obtained by mapping $Z$ into diagram (A.4) is isomorphic to a $K(n)_*$-local $E_n$-Adams spectral sequence.*

**Proof.** The cochain complex

$$0 \to [Z, F_0]^t \to [Z, F_1]^{t+1} \to [Z, F_2]^{t+2} \to \cdots$$

is the normalized cochain complex of the cosimplicial abelian group $\prod^* [Z, C]^t$. There is then a natural cochain equivalence between these two complexes. Hence by Proposition 4.16, there is a cochain map, natural in $Z$, from $[Z, C^*]$ to $[Z, F_*]$ inducing an isomorphism on cohomology. This map is then induced by a cochain map $\{g^i\}$ from $C^*$ to $F_*$. It now suffices to show that

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "h"] & C^0 \arrow[d, "g^0"] \\
\operatorname{holim}_{\leftarrow \Delta} C \arrow[r] & F_0
\end{tikzcd} \tag{A.6}$$

commutes, for this implies that $\{g^i\}$ is chain homotopic to $\{h^i\}$ and thus induces the same map on cohomology.

To prove the commutativity of (A.6), we need only show that

$$\begin{tikzcd}
\operatorname{holim}_{\leftarrow \Delta} C \arrow[r] \arrow[d, equal] & C^0 \arrow[d, "g^0"] \\
\operatorname{holim}_{\leftarrow \Delta} C \arrow[r] & F_0
\end{tikzcd} \tag{A.7}$$

commutes, where the top map is the canonical map

$$\operatorname*{holim}_{\leftarrow \Delta} C = \mathrm{Tot}\left(\prod\nolimits^* C\right) \to \mathrm{Tot}_0\left(\prod\nolimits^* C\right) = \prod\nolimits_j C^j \to C^0.$$

Now $F_0 = \prod_j C^j$, and by Proposition 4.16, the composition

$$C^0 \xrightarrow{g^0} \prod\nolimits_j C^j \to C^{j_0}$$

is given by $(d_0)^{j_0}$. To prove the commutativity of (A.7), we must therefore prove that

$$\mathrm{Tot}\left(\prod\nolimits^* C\right) \to \prod\nolimits^0 C = \prod\nolimits_j C^j \to C^{j_0}$$

is homotopic to

$$\mathrm{Tot}\left(\prod\nolimits^* C\right) \to \prod\nolimits^0 C = \prod\nolimits_j C^j \to C^0 \xrightarrow{(d_0)^{j_0}} C^{j_0}$$

for each $j_0$. But this follows by a standard argument (cf. proof of Theorem 3.2). □

**Corollary A.8.** *The map* $h : X \to \operatorname*{holim}_{\leftarrow \Delta} C$ *is a weak equivalence.*

**Proof.** By the discussion preceding Lemma 5.11, $\operatorname*{holim}_{\leftarrow i} \mathrm{Tot}^i \simeq *$. Moreover, the spectral sequence $E_r^{**}(S^0, C)$ obtained by applying $\pi_*(?)$ to (A.4) is isomorphic to the $K(n)_*$-local $E_n$-Adams spectral sequence converging strongly to $\pi_* X$. It therefore follows that $E_r^{**}(S^0, C)$ is strongly convergent and that $\pi_* h$ is an isomorphism. □

**Remark A.9.** Given a $K(n)_*$-local $S^0$-module $X$ there is a canonical choice of cosimplicial resolution $C$ as above. Namely, define $C^j = \hat{L}(X \wedge E_n^{(j+1)})$ with the coface and codegeneracy maps defined as in Construction 4.11, where $X_S$ is replaced by $\hat{L}(X \wedge E_n)$.

We conclude with a "geometric boundary theorem" which was used in Section 8.

**Proposition A.10.** *Let $G$ be a closed subgroup of $G_n$, and let*

$$\cdots \to \Sigma^{-1} Z \xrightarrow{\partial} X \xrightarrow{f} Y \xrightarrow{g} Z \to \cdots$$

*be a cofibration sequence with $E_n^* \partial = 0$. Suppose $x \in [X, E_n^{hG}]$ is detected by $u \in H_c^*(G, E_n^t X)$ in the $K(n)_*$-local $E_n$-Adams spectral sequence. Then $x \circ \partial$ is detected by $\delta(u) \in H_c^{s+1}(G, E_n^t Z)$ up to higher filtration, where $\delta$ denotes the coboundary map in $H_c^*(G, ?)$ associated to the short exact sequence*

$$0 \to E_n^* Z \to E_n^* Y \to E_n^* X \to 0.$$

**Remark A.11.** The functor $D_G^*(?)$ of Definition 4.18 is exact on the category of profinite $G_n$-modules (see the proof of Lemma 4.21), so that the coboundary map $\delta$ of the proposition can be defined.

**Proof of A.10.** Let $C$ be the cosimplicial resolution of $E_n^{hG}$ of A.9, and write

$$\begin{tikzcd}
E_n^{hG} = W^0 \arrow[dr, "j"'] & & W^1 \arrow[ll, "i"'] \arrow[dr, "j"'] & & W^2 \arrow[ll, "i"'] \arrow[dr, "j"'] & & \cdots \arrow[ll, "i"'] \\
& C^0 \arrow[ur, "k"'] & & \Sigma^{-1} C^1 \arrow[ur, "k"'] & & \Sigma^{-2} C^2 &
\end{tikzcd}$$

for the associated diagram of exact triangles. Recall also from Lemma 4.22 that $[?, C^*]^t$ is naturally isomorphic to $D^*_G(E^t_n(?))$.

Now $x$ lifts to a map $\bar{x}: X \to W^s$ such that the composition

$$X \xrightarrow{\bar{x}} W^s \xrightarrow{j} \Sigma^{-s} C^s$$

is a representative of $u$ in $D^s_G(E^*_n X)$. The composition $j \circ \bar{x} \circ \partial$ is trivial since $E^*_n \partial = 0$. We may therefore construct a diagram

$$\begin{tikzcd}
\Sigma^{-1} Z \arrow[r, "\partial"] \arrow[d, dashed, "\bar{z}"] & X \arrow[r, "f"] \arrow[d, "\bar{x}"] & Y \arrow[r, "g"] \arrow[d, dashed] & Z \arrow[d, dashed, "\bar{z}"] \\
W^{s+1} \arrow[r, "i"] & W^s \arrow[r, "j"] & \Sigma^{-s} C^s \arrow[r, "k"] & \Sigma W^{s+1}
\end{tikzcd}$$

of cofibration sequences. It now follows easily that $j \circ \bar{z} \in [Z, C^{s+1}]^* = D^{s+1}(E^*_n Z)$ is a representative of $\delta(u)$. But $j \circ \bar{z}$ represents $x \circ \partial$ up to higher filtration as well, completing the proof. □

## Appendix B. Proof of Proposition 4.16

The proof of this proposition requires some preparation.

**Definition B.1.** If $k \geqslant 0$, let $D^*_k$ be the cosimplicial abelian group with

$$D^n_k = \bigoplus_{[k] \to [n]} \mathbb{Z},$$

the sum ranging over all morphisms $[k] \to [n]$ in $\mathbf{\Delta}$. Let $\imath_k \in D^k_k$ denote the element 1 in the summand corresponding to the identity $[k] \to [k]$.

$D^*_k$ has a convenient universal property: If $C$ is a cosimplicial abelian group and $x \in C^k$, there exists a unique map $\tau_x : D^*_k \to C$ of cosimplicial abelian groups with $\tau_x(\imath_k) = x$. We can also put the $D^*_k$'s together.

**Definition B.2.** Let $D^*_*$ be the simplicial cosimplicial abelian group whose cosimplicial group of $k$-simplices is $D^*_k$. If $[m] \to [k]$ is a morphism in $\mathbf{\Delta}$, the map $D^*_k \to D^*_m$ is defined by sending the summand indexed by $[k] \to [n]$ to the summand indexed by $[m] \to [k] \to [n]$ via the identity.

**Lemma B.3.** *There exists a sequence of maps $T_k : D^*_k \to \prod^* D^*_k$ of cosimplicial abelian groups such that*:

(i) *The composition*

$$D^0_0 \xrightarrow{T_0} \prod_j D^j_0 \xrightarrow{\pi} D^{j_0}_0$$

*sends $\imath_0$ to $d^0 \cdots d^0 \imath_0$ for all $j_0 \geqslant 0$.*

(ii) *The diagram*

$$\begin{array}{ccc} D^*_{k+1} & \xrightarrow{\partial} & D^*_k \\ \downarrow{\scriptstyle T_{k+1}} & & \downarrow{\scriptstyle T_k} \\ \prod^* D^*_{k+1} & \xrightarrow{\partial} & \prod^* D^*_k \end{array}$$

*commutes for all* $k \geqslant 0$, *where* $D^*_*$ *and* $\prod^* D^*_*$ *are here regarded as chain complexes of cosimplicial abelian groups and* $\partial$ *denotes their respective boundary maps.*

**Proof.** We construct $T_k$ by induction on $k$. There is a unique cosimplicial map $T_0$ satisfying (i). To construct $T_{k+1}$, it suffices to prove that $T_k(\partial\imath_{k+1})$ is a boundary in the chain complex $\prod^{k+1} D^*_*$; we may then define $T_{k+1}(\imath_{k+1}) = c$, where $\partial c = T_k(\imath_{k+1})$.

We claim that $H_i(\prod^{k+1} D^*_*) = 0$ for all $i > 0$. Indeed, for fixed $i$, $H_i(D^*_*)$ is a cosimplicial group and $H_i(\prod^{k+1} D^*_*) = \prod^{k+1} H_i(D^*_*)$. But the chain complex $D^j_*$ is just the simplicial chain complex of the standard $j$-simplex; therefore

$$H_i(D^j_*) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $0 = T_{k-1}(\partial\partial\imath_{k+1}) = \partial T_k(\partial\imath_{k+1})$ by the inductive hypothesis, it now follows that $T_k(\partial\imath_{k+1})$ is a boundary if $k > 0$. If $k = 0$, use the fact that the maps

$$D^j_0 \xrightarrow{T^j_0} \prod^j D^*_0 \xrightarrow{\pi} D^i_0$$

are augmentation preserving to conclude that $T_1(\partial\imath_1)$ is a boundary as well. This completes the induction and the proof. □

**Proof of Proposition 4.16.** For $x \in C^k$, define $T(C)x$ to be the image of $\imath_k$ under the maps

$$D^k_k \xrightarrow{T_k} \prod^k D^*_k \xrightarrow{\prod^* \tau_x} \prod^k C.$$

Since $\partial : D^*_{k+1} \to D^*_k$ maps $\imath_{k+1}$ to $\sum_{i=0}^{k+1}(-1)^i d^i \imath_k$, it follows that $\tau_{\delta x} = \tau_x \circ \partial$. By B.3(ii), we then have that $T(C)(\delta x)$ is given by the image of $\imath_{k+1}$ under the composition

$$D^{k+1}_{k+1} \xrightarrow{\partial} D^{k+1}_k \xrightarrow{T_k} \prod^{k+1} D^*_k \xrightarrow{\prod^* \tau_x} \prod^{k+1} C.$$

But $(\prod^* \tau_x) \circ T_k$ is a cosimplicial map; therefore

$$\begin{aligned} \left(\left(\prod{}^*\tau_x\right) \circ T_k \circ \partial\right)(\imath_{k+1}) &= \left(\left(\prod{}^*\tau_x\right) \circ T_k\right)\left(\sum_{i=0}^{k+1}(-1)^i d^i \imath_k\right) \\ &= \sum_{i=0}^{k+1} d^i \left(\left(\prod{}^*\tau_x\right) \circ T_k\right)(\imath_k) \\ &= \delta T(C)(x). \end{aligned}$$

Thus $T(C)$ is a cochain map.

Now it is also clear from the definition of $T$ that $T^0$ is as required and hence that $T$ is an isomorphism on $\pi^0$. This implies that $T$ is an isomorphism on $\pi^*$, completing the proof. □

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