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Small ring spectra

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Abstract

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We determine conditions under which the cofibre of a self-map of a ring spectrum is again a ring spectrum. Sufficiently large iterates of v_n self-maps will satisfy this condition.

The main result of this paper gives conditions under which the cofibre of a self-map of a ring spectrum is again a ring spectrum. In particular, sufficiently large iterates of v_n self-maps satisfy this condition. By a ring spectrum, we mean a spectrum X together with maps $\mu : X \wedge X \to X$ and $\eta : S^0 \to X$ such that the composition

$$X = S^0 \wedge X \xrightarrow{\eta \wedge X} X \wedge X \xrightarrow{\mu} X$$

is the identity (in the stable category). Neither associativity nor commutativity is assumed; it is also not even assumed that η is a two-sided unit. We can then prove the following theorem:

Theorem 1. *Let X be a ring spectrum and let $f : \Sigma^{|f|}X \to X$ with $|f|$ even. Suppose that:*

(i) *The map $f \wedge X : \Sigma^{|f|}X \wedge X \to X \wedge X$ is in the center of the ring $[X \wedge X, X \wedge X]_*$.*

(ii) *The diagram*

$$\begin{array}{ccc} \Sigma^{2|f|}X \wedge X & \xrightarrow{X \wedge f^2} & X \wedge X \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ \Sigma^{2|f|}X & \xrightarrow{f^2} & X \end{array}$$

commutes, where $f^2 = f \circ f$.

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Then $C(f^2)$, the cofibre of the map $f^2 : \Sigma^{2|f|}X \to X$, has the structure of a ring spectrum so that the inclusion map $X \to C(f^2)$ is a map of ring spectra.

Now suppose that X is a p-local finite ring spectrum with $K(n-1)_*X = 0$ but $K(n)_*X \neq 0$. As usual $K(i)$ denotes the ith Morava K-theory. Recall that a v_n self-map is a map $g : \Sigma^{|g|}X \to X$ which induces an isomorphism on $K(n)_*X$ and a nilpotent homomorphism on $K(i)_*X$ for $i \neq n$. Then, by the essential uniqueness, naturality, and centrality of v_n self-maps [3, Section 3], it follows that if g is any v_n self-map and $n > 0$, there exists a natural number N such that g^N satisfies conditions (i) and (ii). This implies the next result.

Theorem 2. *Let X be a p-local finite ring spectrum, and let g be a v_n self-map $(n > 0)$. Then there exists a natural number N such that, for each $m > 0$, $C(g^{mN})$ has the structure of a ring spectrum so that the inclusion $X \to C(g^{mN})$ is a map of ring spectra.* □

Working before the nilpotence theorem, Oka obtained some results on ring spectra structures on certain specific finite complexes X with $K(n)_*X \neq 0$ and n small [4]. Of course, in general, one cannot expect such specific results from nilpotence technology. Nevertheless, this type of result is useful in some contexts. For example, in [2], it was sufficient to use the general existence of v_2 self-maps without knowing that any specific power of multiplication by v_2 could be realized. Furthermore, we expect that Theorem 2 will be a technical tool needed to explicitly present the Brown–Comenetz dual I_n of L_nS^0 as a direct limit of finite spectra (cf. [2, 1.5]). (In the absence of the telescope conjecture, this presentation will be in the $E(N)_*$-local homotopy category, where N may be arbitrary.)

Finally, we remark that Theorem 2 may be folklore to certain BP-theorists.

The proof of Theorem 1 requires three lemmas, the last two of which will be proved later. First, we introduce some notation. Given a self map $g : \Sigma^{|g|}X \to X$, there is a cofibration sequence

$$\cdots \to \Sigma^{|g|}X \xrightarrow{g} X \xrightarrow{\iota} C(g) \xrightarrow{\partial} \Sigma^{|g|+1}X \xrightarrow{-g} \Sigma X \to \cdots .$$

Lemma 3. *If f is a self-map of X and $f \wedge X$ is in the center of the ring $[X \wedge X, X \wedge X]_*$, then $f \wedge X = X \wedge f$.*

Proof. Use the fact that $f \wedge X$ commutes with the commutativity automorphism $\tau : X \wedge X \to X \wedge X$. □

Lemma 4. *Let X be any spectrum and suppose that f is a self-map of even degree such that $f \wedge X$ is central. Then there exists a map $h : \Sigma^{2|f|+1}X \wedge X \to X \wedge X$ such that the diagram*

$$\begin{array}{ccc}
\Sigma^{|f|}X \wedge C(f) & \xrightarrow{f \wedge C(f)} & X \wedge C(f) \\
\downarrow{\scriptstyle X \wedge \partial} & & \uparrow{\scriptstyle X \wedge \iota} \\
\Sigma^{|f|}X \wedge \Sigma^{|f|+1}X & \xrightarrow{h} & X \wedge X
\end{array}$$

commutes.

Lemma 5. *Let f and X be as in Lemma* 4. *Then*

$$f^2 \wedge C(f^2) : \Sigma^{2|f|}X \wedge C(f^2) \to X \wedge C(f^2)$$

is trivial.

Proof of Theorem 1. First note that hypothesis (ii) implies the existence of a map $m : X \wedge C(f^2) \to C(f^2)$ such that the diagram

$$\begin{array}{ccccccc}
\Sigma^{2|f|}S^0 \wedge X & \xrightarrow{S^0 \wedge f^2} & S^0 \wedge X & \longrightarrow & S^0 \wedge C(f^2) & \longrightarrow & \Sigma^{2|f|+1}S^0 \wedge X \\
\downarrow{\scriptstyle \eta \wedge X} & & \downarrow{\scriptstyle \eta \wedge X} & & \downarrow{\scriptstyle \eta \wedge C(f^2)} & & \downarrow{\scriptstyle \eta \wedge X} \\
\Sigma^{2|f|}X \wedge X & \xrightarrow{X \wedge f^2} & X \wedge X & \longrightarrow & X \wedge C(f^2) & \longrightarrow & \Sigma^{2|f|+1}X \wedge X \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mu} \\
\Sigma^{2|f|}X & \xrightarrow{f^2} & X & \longrightarrow & C(f^2) & \longrightarrow & \Sigma^{2|f|+1}X
\end{array}$$

commutes, where the rows are cofibration sequences. Now the fact that $\mu \circ (\eta \wedge X) = \mathrm{id}_X$ does not of course imply that $m \circ (\eta \wedge C(f^2))$ is the identity—it does, however, imply that $m \circ (\eta \wedge C(f^2))$ is an automorphism of $C(f^2)$. It is then easy to see that by replacing m with $[m \circ (\eta \wedge C(f^2))]^{-1} \circ m$, we can arrange things so that the above diagram commutes and so that $m \circ (\eta \wedge C(f^2))$ is the identity.

Next, Lemma 5 implies that there exists a retraction $r : C(f^2) \wedge C(f^2) \to X \wedge C(f^2)$. Define $\mu' : C(f^2) \wedge C(f^2) \to C(f^2)$ by $\mu' = m \circ r$ and η' by $\eta' = \iota \circ \eta$. Then one easily checks that these maps give $C(f^2)$ the structure of a ring spectrum and that $\iota : X \to C(f^2)$ is a ring spectrum map. □

Proof of Lemma 4. Begin by observing that, since $X \wedge f = f \wedge X$, the composite

$$\Sigma^{|f|}X \wedge X \xrightarrow{f \wedge X} X \wedge X \to X \wedge C(f)$$

is trivial. There then exists a map

$$g : \Sigma^{|f|}X \wedge \Sigma^{|f|+1}X \to X \wedge C(f)$$

such that the diagram

$$\begin{array}{ccc}
\Sigma^{|f|}X \wedge X & \xrightarrow{f \wedge X} & X \wedge X \\
\big\downarrow{\scriptstyle X \wedge \iota} & & \big\downarrow \\
\Sigma^{|f|}C(X \wedge f) = \Sigma^{|f|}X \wedge C(f) & \xrightarrow{f \wedge C(f)} & X \wedge C(f) \\
\big\downarrow{\scriptstyle X \wedge \partial} & \nearrow{\scriptstyle g} & \\
\Sigma^{|f|}X \wedge \Sigma^{|f|+1}X & &
\end{array}$$

commutes. To complete the proof, we must show that $(X \wedge \partial) \circ g : \Sigma^{2|f|+1}X \wedge X \rightarrow \Sigma^{|f|+1}X \wedge X$ is trivial. For this, it will be convenient to describe g at the point-set level.

Identify $\Sigma^{|f|}X \wedge \Sigma^{|f|+1}X$ with $\Sigma^{|f|}[C(X \wedge f) \cup_{X \wedge X} C(X \wedge X)]$. The cone coordinates are parameterized by $[0, 1]$ with 0 the cone point. Now let $H : \Sigma^{|f|}X \wedge X \wedge I_+ \rightarrow X \wedge X$ be a homotopy with $H_0 = f \wedge X$ and $H_1 = X \wedge f$. Finally, write

$$X \wedge C(f) = C(X \wedge f) = X \wedge X \cup_{X \wedge \Sigma^{|f|}X} C(X \wedge \Sigma^{|f|}X)$$

as usual. Then define $g \mid \Sigma^{|f|}C(X \wedge f)$ to be $f \wedge C(f)$ and define

$$(g \mid \Sigma^{|f|}C(X \wedge X))(x_1 \wedge x_2 \wedge s) = \begin{cases} H(x_1 \wedge x_2, 2 - 2s) & s \geq 1/2\,, \\ x_1 \wedge x_2 \wedge 2s & s \leq 1/2\,. \end{cases}$$

Next, consider the cofibration sequence

$$\begin{aligned} X \wedge X &\xrightarrow{j} C(X \wedge f) \cup_{X \wedge X} C(X \wedge X) \\ &\xrightarrow{\bar{\pi}} (X \wedge \Sigma^{|f|+1}X) \vee \Sigma(X \wedge X) \xrightarrow{\bar{\partial}} \Sigma(X \wedge X) \end{aligned}$$

where j includes $X \wedge X$ onto the base of $C(X \wedge X)$. It is easy to see that

$$(X \wedge \partial) \circ g = k \circ \bar{\pi}$$

(up to homotopy), where

$$\begin{aligned} k \mid \Sigma^{|f|}X \wedge \Sigma^{|f|+1}X &= f \wedge \Sigma^{|f|+1}X\,, \\ k \mid \Sigma^{|f|+1}X \wedge X &= \mathrm{id}\,. \end{aligned}$$

We claim, however, that $\bar{\partial}$ is just $-k$. This implies that $(X \wedge \partial) \circ g$ is trivial, completing the proof.

To prove the claim, note that $\bar{\partial} \mid X \wedge \Sigma^{|f|+1}X$ is just $\partial_1 = -(X \wedge f) = -(f \wedge X)$ in the cofibration sequence

$$\begin{aligned} X \wedge \Sigma^{|f|}X &\xrightarrow{X \wedge f} X \wedge X \rightarrow C(X \wedge f) \\ &\longrightarrow X \wedge \Sigma^{|f|+1}X \xrightarrow{\partial_1} \Sigma(X \wedge X) \end{aligned}$$

and that $\bar{\partial} \mid \Sigma(X \wedge X)$ is just $\partial_2 = -\mathrm{id}$ in the cofibration sequence

$$X \wedge X \xrightarrow{\mathrm{id}} X \wedge X \rightarrow C(X \wedge X) \rightarrow \Sigma(X \wedge X) \xrightarrow{\partial_2} \Sigma(X \wedge X) . \qquad \square$$

Remark. One uses the assumption that $|f|$ is even to get $\partial_1 = -k \mid \Sigma^{|f|}X \wedge \Sigma^{|f|+1}X$.

With Lemma 4 proven, the proof of the last remaining lemma is straightforward.

Proof of Lemma 5. Apply Verdier's axiom [1, Part III, 6.8] to the commutative triangle

$$\begin{array}{ccc} \Sigma^{2|f|}X & \xrightarrow{f^2} & X \\ \downarrow{\scriptstyle f} & \nearrow{\scriptstyle f} & \\ \Sigma^{|f|}X & & \end{array}$$

to obtain a cofibration sequence

$$\Sigma^{|f|}C(f) \xrightarrow{\iota} C(f^2) \xrightarrow{\pi} C(f) \xrightarrow{\delta} \Sigma^{|f|+1}C(f) .$$

Now note that the composition

$$\Sigma^{|f|}X \wedge C(f^2) \xrightarrow{f \wedge C(f^2)} X \wedge C(f^2) \xrightarrow{X \wedge \pi} X \wedge C(f)$$

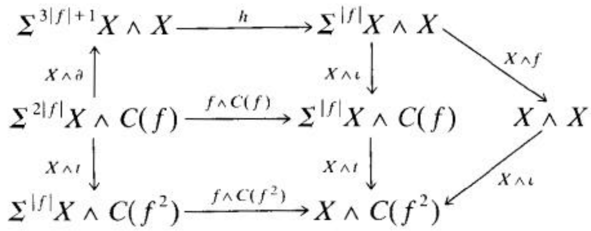
is trivial. This follows from the commutative diagram

$$\begin{array}{ccccc} & \swarrow{\scriptstyle X \wedge \partial} & \Sigma^{|f|}X \wedge C(f^2) & \xrightarrow{f \wedge C(f^2)} & X \wedge C(f^2) \\ & & \downarrow{\scriptstyle X \wedge \pi} & & \downarrow{\scriptstyle X \wedge \pi} \\ \Sigma^{3|f|+1}X \wedge X & & \Sigma^{|f|}X \wedge C(f) & \xrightarrow{f \wedge C(f)} & X \wedge C(f) \\ & & \downarrow{\scriptstyle X \wedge \partial} & & \uparrow{\scriptstyle X \wedge \iota} \\ & \searrow{\scriptstyle X \wedge f} & \Sigma^{2|f|+1}X \wedge X & \xrightarrow{h} & X \wedge X \end{array}$$

and the fact that $h \circ (X \wedge f) = (X \wedge f) \circ h$. There is therefore a map

$$q : \Sigma^{|f|}X \wedge C(f^2) \rightarrow \Sigma^{|f|}X \wedge C(f)$$

with $(X \wedge \iota) \circ q = f \wedge C(f^2)$. But $(f \wedge C(f^2)) \circ (X \wedge \iota)$ is trivial, again because of the commutative diagram



and the fact that $X \wedge f$ is central.

Thus

$$f^2 \wedge C(f^2) = (f \wedge C(f^2)) \circ (f \wedge C(f^2))$$
$$= (f \wedge C(f^2)) \circ (X \wedge t) \circ q = 0\,,$$

completing the proof. □

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