K-Theory **15:** 201–252, 1998. 201

Spaces over a Category and Assembly Maps in Isomorphism Conjectures in K- and L-Theory

JAMES F. DAVIS⋆[1] and WOLFGANG LÜCK[2]
[1]*Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.*
e-mail: jfdavis@indiana.edu
[2] *Institut für Mathematik, Westfaelische Wilhelms-Universtitaet, 48149 Muenster, Germany*
e-mail: lueck@math.uni-muenster.de

(Received: April 1997)

Abstract. We give a unified approach to the Isomorphism Conjecture of Farrell and Jones on the algebraic K- and L-theory of integral group rings and to the Baum–Connes Conjecture on the topological K-theory of reduced C^*-algebras of groups. The approach is through spectra over the orbit category of a discrete group G. We give several points of view on the assembly map for a family of subgroups and characterize such assembly maps by a universal property generalizing the results of Weiss and Williams to the equivariant setting. The main tools are spaces and spectra over a category and their associated generalized homology and cohomology theories, and homotopy limits.

Mathematics Subject Classification (1991). 57-XX.

Key words: Algebraic K- and L-theory, Baum–Connes Conjecture, assembly maps, spaces and spectra over a category.

0. Introduction

Glen Bredon [5] introduced the *orbit category* $\mathrm{Or}(G)$ of a group G. Objects are homogeneous spaces G/H, considered as left G-sets, and morphisms are G-maps. This is a useful construct for organizing the study of fixed sets and quotients of G-actions. If G acts on a set X, there is the contravariant fixed point functor $\mathrm{Or}(G) \longrightarrow$ SETS given by $G/H \mapsto X^H = \mathrm{map}_G(G/H, X)$ and the covariant quotient space functor $\mathrm{Or}(G) \longrightarrow$ SETS given by $G/H \mapsto X/H = X \times_G G/H$. Bredon used the orbit category to define equivariant cohomology theory and to develop equivariant obstruction theory.

Examples of covariant functors from the orbit category of a discrete group G to Abelian groups are given by algebraic K-theory $K_i(\mathbb{Z}H)$, algebraic L-theory $L_i(\mathbb{Z}H)$, and the K-theory $K_i^{\mathrm{top}}(C_r^*(H))$ of the reduced C^*-algebra of H. In Section 2, we express each of these as the composite of a functor $\mathrm{Or}(G) \longrightarrow$ SPECTRA with the ith homotopy group. We use these functors to give a clean formulation

⋆Supported by the Alexander von Humboldt-Stiftung and the National Science Foundation. James F. Davis wishes to thank the Johannes Gutenberg-Universität in Mainz for its hospitality when this work was initiated.

of some of the main conjectures of high-dimensional topology: the Isomorphism Conjecture of Farrell–Jones [15] (which implies the Borel/Novikov Conjecture) and the Baum–Connes Conjecture in the case of discrete groups.

Our motivation was in part to obtain such a formulation and in part to set the stage for explicit computations based on isomorphism conjectures. We give computations of K- and L-groups of group rings in a separate paper [8]. Our formulation is used by Kimberly Pearson [27] to show that the Whitehead group $\mathrm{Wh}(G)$ and the reduced K_0-group $\tilde{K}_0(\mathbb{Z}G)$ vanish for two-dimensional crystallographic groups. We also hope our formulations will prove useful in the further study of isomorphism conjectures and in the related study of manifolds admitting metrics of positive scalar curvature.

Sections 1, 3, 4 and 7 contain foundational background, independent of assembly maps and algebraic K-theory. Section 2 is devoted to K-theory, and Sections 5 and 6 to assembly maps. More precisely, in Section 1 we discuss the adjointness of mapping spaces and tensor (or balanced) products over a category, as well as the notions of spaces and spectra over a category. In Section 2, we define our three main examples of $\mathrm{Or}(G)$-spectra: $\mathbf{K}^{\mathrm{alg}}$, $\mathbf{L}$, and $\mathbf{K}^{\mathrm{top}}$. (These are all nonconnective spectra; they have homotopy groups in negative dimensions.) They are all defined by first assigning to an object G/H, the transformation groupoid $\overline{G/H}$, whose objects are elements of G/H, and whose morphisms are given by multiplication by a group element, and then assigning a spectrum to a groupoid. In the $\mathbf{K}^{\mathrm{top}}$-case there is an intermediate step of considering the C^*-category of a groupoid and a spectrum of a C^*-category, derived from Bott periodicity.

In Section 3 we discuss free CW-complexes over a category $\mathcal{C}$, the universal free CW-complex $E\mathcal{C}$ over a category $\mathcal{C}$, and homotopy (co)-limits $E\mathcal{C} \otimes_{\mathcal{C}} X$ of a $\mathcal{C}$-space X. The ideas here are well-known to the experts (see, e.g., [11]), but our approach, relying on homological methods and avoiding simplicial methods, may appeal to an algebraist. By approximating a $\mathcal{C}$-space X by a free $\mathcal{C}$-CW-complex, we define in Section 4 homology $H_*^{\mathcal{C}}(X; \mathbf{E})$ and cohomology $H_{\mathcal{C}}^*(X; \mathbf{E})$ of a space X with coefficients in a $\mathcal{C}$-spectrum $\mathbf{E}$. We give an Atiyah–Hirzebruch type spectral sequence for these theories.

With regard to the assembly maps arising in the Isomorphism Conjectures, we give three points of view in Section 5. Let $\mathcal{F}$ be a family of subgroups of G, closed under taking subgroups and conjugation. Let $\mathbf{E}\colon \mathrm{Or}(G) \longrightarrow$ SPECTRA be a covariant functor. We define a functor

$$\mathbf{E}_{\%}\colon G\text{-SPACES} \longrightarrow \text{SPECTRA}$$

by setting $\mathbf{E}_{\%}(X) = (G/H \longrightarrow X^H)_+ \otimes_{\mathrm{Or}(G)} \mathbf{E}$. Then $\pi_*(\mathbf{E}_{\%}(X))$ is an equivariant homology theory in the sense of Bredon [5]. Let $E(G, \mathcal{F})$ be the classifying space for a family of subgroups of G, i.e. it is a G-CW-complex so that $E(G, \mathcal{F})^H$ is contractible for subgroups H in $\mathcal{F}$ and is empty for H not in $\mathcal{F}$. The map

$$\pi_*\mathbf{E}_{\%}(E(G, \mathcal{F})) \longrightarrow \pi_*\mathbf{E}_{\%}(G/G)$$

given by applying $\mathbf{E}_{\%}$ to the constant map and then taking homotopy groups is called the $(\mathbf{E}, \mathcal{F}, G)$-*assembly map.* We say the $(\mathbf{E}, \mathcal{F}, G)$-*isomorphism conjecture* holds if the $(\mathbf{E}, \mathcal{F}, G)$-assembly map is an isomorphism. When $\mathcal{F} = \mathcal{VC}$, the family of virtual cyclic subgroups of G, (i.e. $H \in \mathcal{VC}$ if and only if H has a cyclic subgroup of finite index), the isomorphism conjectures of Farrell–Jones [15] for algebraic K- and L-theory are equivalent to the $(\mathbf{K}^{\mathrm{alg}}, \mathcal{VC}, G)$- and $(\mathbf{L}, \mathcal{VC}, G)$-isomorphism conjectures, where $\mathbf{K}^{\mathrm{alg}}$ and $\mathbf{L}$ $(= \mathbf{L}^{\langle -\infty \rangle})$ are $\mathrm{Or}(G)$-spectra associated to algebraic K- and L-theories. When $\mathcal{F} = \mathcal{FIN}$, the family of finite subgroups of G, and $\mathbf{K}^{\mathrm{top}}$ is the $\mathrm{Or}(G)$-spectra associated with the K-theory of C^*-algebras, then the $(\mathbf{K}^{\mathrm{top}}, \mathcal{FIN}, G)$-Isomorphism Conjecture is equivalent to the Baum–Connes Conjecture for the discrete group G (see Section 5). When $\mathcal{F} = 1$, the family consisting only of the trivial subgroup of G, then the $(\mathbf{K}^{\mathrm{alg}}, 1, G)$, $(\mathbf{L}, 1, G)$, and $(\mathbf{K}^{\mathrm{top}}, 1, G)$-assembly maps can be identified with maps $H_*(BG; \mathbf{K}^{\mathrm{alg}}(\mathbb{Z})) \longrightarrow K_*(\mathbb{Z}G)$, $H_*(BG; \mathbf{L}(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}G)$, and $H_*(BG; \mathbf{K}^{\mathrm{top}}(\mathbb{C})) \longrightarrow K_*^{\mathrm{top}}(C_r^* G)$.

We give three variant ways of expressing the $(\mathbf{E}, \mathcal{F}, G)$-assembly map: by approximating $\mathbf{E}$ by $\mathbf{E}_{\%}$ as above, in terms of homotopy colimits, and in terms of a generalized homology theory over a category. The first definition is the quickest and leads to an axiomatic characterization; the last two are well-suited for computations.

Let $\mathrm{Or}(G, \mathcal{F})$ be the restricted orbit category, where the objects are G/H with $H \in \mathcal{F}$. The $(\mathbf{E}, \mathcal{F}, G)$-assembly map is equivalent to the map

$$\pi_*(\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}) \longrightarrow \pi_*(\operatorname*{hocolim}_{\mathrm{Or}(G)} \mathbf{E})$$

induced by the inclusion of the restricted orbit category in the full orbit category. Since $E(G, \mathcal{F})$ is only defined up to G-homotopy type, it is natural for us to define homotopy limits and colimits as a homotopy type, rather than a fixed space or spectra; we take this approach in Section 3.

Given a family $\mathcal{F}$ of subgroups of G, define the $\mathrm{Or}(G)$-space $\{*\}_{\mathcal{F}}$ to be the functor which sends G/H to a point if H is in $\mathcal{F}$ and to the empty set otherwise. Let $\{*\}$ be the trivial $\mathrm{Or}(G)$-space, which sends G/H to a point for all H. The third point of view is to identify the $(\mathbf{E}, \mathcal{F}, G)$-assembly map with the map

$$H_*^{\mathrm{Or}(G)}(\{*\}_{\mathcal{F}}; \mathbf{E}) \longrightarrow H_*^{\mathrm{Or}(G)}(\{*\}; \mathbf{E})$$

induced by the inclusion map of $\mathrm{Or}(G)$-spaces, $\{*\}_{\mathcal{F}} \longrightarrow \{*\}$.

Section 6 gives a characterization of assembly maps, generalizing that of Weiss–Williams [42] to the equivariant setting. Associated to a homotopy invariant functor

$$\mathbf{E}\colon G\text{-SPACES} \longrightarrow \text{SPECTRA},$$

we define a new functor

$$\mathbf{E}^{\%}\colon G\text{-SPACES} \longrightarrow \text{SPECTRA},$$

and a natural transformation

$$\mathbf{A}\colon \mathbf{E}^{\%} \longrightarrow \mathbf{E},$$

where $\mathbf{A}(G/H)$ is a homotopy equivalence for all orbits G/H. Here $\mathbf{E}^{\%}$ is the 'best approximation' of $\mathbf{E}$ by an excisive functor, in particular $\pi_*(\mathbf{E}^{\%}(X))$ is an equivariant homology theory. When $\mathbf{E}(X) = \mathbf{K}^{\mathrm{alg}}(\Pi(EG \times_G X))$ where Π is the fundamental groupoid, then the map $\pi_*(\mathbf{A}(E(G, \mathcal{F})))$ is equivalent to the $(\mathbf{K}^{\mathrm{alg}}, \mathcal{F}, G)$-assembly map. (We define $\mathbf{K}^{\mathrm{alg}}$ of a groupoid in Section 2.) An analogous statement holds for L-theory and for the topological K-theory of C^*-algebras. This gives a fourth point of view on assembly maps.

In Section 7 we make explicit the correspondence between G-spaces and $\mathrm{Or}(G)$-spaces which has been implicit throughout the paper.

We thank Erik Pedersen for warning us about two pitfalls related to the spectra of algebraic K- and L-theory and Stephan Stolz for discussions on the material of Section 2.

1. Spaces and Spectra over a Category

This section gives basic definitions and examples of spaces and spectra over a small (discrete) category $\mathcal{C}$ and discusses the adjointness of the tensor product and mapping space. Our main example for $\mathcal{C}$ is due to Bredon [5]:

DEFINITION 1.1. Let G be a group and $\mathcal{F}$ be a *family of subgroups*, i.e. a non-empty set of subgroups of G closed under taking conjugates and subgroups. The *orbit category* $\mathrm{Or}(G)$ has as objects homogeneous G-spaces G/H and as morphisms G-maps. The *orbit category* $\mathrm{Or}(G, \mathcal{F})$ *with respect to* $\mathcal{F}$ is the full subcategory of $\mathrm{Or}(G)$ consisting of those objects G/H for which H belongs to $\mathcal{F}$. □

Examples of families are $\mathcal{F} = \{H \subset G \mid X^H \neq \emptyset\}$ for a G-space X, the finite subgroups of G, and the virtually cyclic subgroups of G. Notice that the automorphism group of an object G/H can be identified with the Weyl group $W(H) = N(H)/H$. Furthermore, if H is finite, then any endomorphism of G/H is invertible, but not in general [23, Lemma 1.31 on page 22]. We will always work in the category of compactly generated spaces (see [37] and [43, I.4]).

DEFINITION 1.2. A *covariant* (*contravariant*) $\mathcal{C}$*-space* X *over the category* $\mathcal{C}$ is a covariant (contravariant) functor

$$X: \mathcal{C} \longrightarrow \mathrm{SPACES}$$

from $\mathcal{C}$ into the category of compactly generated spaces. A map between $\mathcal{C}$-spaces is a natural transformation of such functors. Given $\mathcal{C}$-spaces X and Y, denote by $\hom_{\mathcal{C}}(X, Y)$ the space of maps of $\mathcal{C}$-spaces from X to Y with the subspace topology coming from the obvious inclusion into $\prod_{c \in \mathrm{Ob}(\mathcal{C})} \mathrm{map}(X(c), Y(c))$. □

Likewise we can define a $\mathcal{C}$-set and an $R\mathcal{C}$-module. For a ring R, a $R\mathcal{C}$-module is a functor M from $\mathcal{C}$ to the category of R-modules. For two $R\mathcal{C}$-modules M and N of the same variance, $\hom_{R\mathcal{C}}(M, N)$ is the Abelian group of natural transformations from M to N. We can form kernels and cokernels, so the category of $R\mathcal{C}$-modules is an Abelian category, and thus one can use homological algebra to study $R\mathcal{C}$-modules (see [23]).

Let G be a group. Let 1 be the family consisting of precisely one element, namely the trivial group. Then $\mathrm{Or}(G, 1)$ is a category with a single object, and G can be identified with the set of morphisms by sending $g \in G$ to the automorphism $G/1 \longrightarrow G/1$ which maps g' to $g'g^{-1}$. A covariant (contravariant) $\mathrm{Or}(G, 1)$-space is the same as a left (right) G-space. Maps of $\mathrm{Or}(G, 1)$-spaces correspond to G-maps. For a different example of an orbit category, let $\mathbb{Z}_p$ be the cyclic group of order p for a prime number p. A contravariant $\mathrm{Or}(\mathbb{Z}_p)$-space Y is specified by a $\mathbb{Z}_p$-space $Y(\mathbb{Z}_p/\{1\})$, a space $Y(\mathbb{Z}_p/\mathbb{Z}_p)$, and a map $Y(\mathbb{Z}_p/\mathbb{Z}_p) \longrightarrow Y(\mathbb{Z}_p/\{1\})^{\mathbb{Z}_p}$.

EXAMPLE 1.3. Let Y be a left G-space and $\mathcal{F}$ be a family of subgroups. Define the associated *contravariant* $\mathrm{Or}(G, \mathcal{F})$*-space* $\mathrm{map}_G(-, Y)$ by

$$\mathrm{map}_G(-, Y)\colon \mathrm{Or}(G, \mathcal{F}) \longrightarrow \mathrm{SPACES} \quad G/H \mapsto \mathrm{map}_G(G/H, Y) = Y^H. \quad \square$$

The tensor product of a contravariant $\mathcal{C}$-space with a covariant $\mathcal{C}$ space yields a topological space.

DEFINITION 1.4. Let X be a contravariant and Y be a covariant $\mathcal{C}$-space. Define their *tensor product* to be the space

$$X \otimes_{\mathcal{C}} Y = \coprod_{c \in \mathrm{Ob}(\mathcal{C})} X(c) \times Y(c)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi\colon c \longrightarrow d$ in $\mathcal{C}$ and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. $\square$

The tensor product and the hom space are called the coend and end constructions in category theory [24, pages 219 and 222]. A lot of well-known constructions are special cases of it.

Recall that the category of covariant (contravariant) $\mathrm{Or}(G, 1)$-spaces is the category of left (right) G-spaces. The balanced product $X \times_G Y$ of a right G-space X and of a left G-space Y can be identified with the tensor product $X \otimes_{\mathrm{Or}(G,1)} Y$. The mapping space $\mathrm{map}_G(X, Y)$ of two left (right) G-spaces X and Y can be identified with $\mathrm{hom}_{\mathrm{Or}(G,1)}(X, Y)$.

The main property of the tensor product is the following.

LEMMA 1.5. *Let X be a contravariant $\mathcal{C}$-space, Y be a covariant $\mathcal{C}$-space and Z be a space. Denote by* $\mathrm{map}(Y, Z)$ *the obvious contravariant $\mathcal{C}$-space whose value at an object c is the mapping space* $\mathrm{map}(Y(c), Z)$. *Then there is a homeomorphism natural in X, Y and Z*

$$T = T(X, Y, Z)\colon \mathrm{map}(X \otimes_{\mathcal{C}} Y, Z) \longrightarrow \mathrm{hom}_{\mathcal{C}}(X, \mathrm{map}(Y, Z))$$

Proof. We only indicate the definition of T. Given a map $g\colon X \otimes_{\mathcal{C}} Y \longrightarrow Z$, we have to specify for each object c in $\mathcal{C}$ a map $T(g)(c)\colon X(c) \longrightarrow \mathrm{map}(Y(c), Z)$. This is the same as specifying a map $X(c) \times Y(c) \longrightarrow Z$ which is defined to be the composition of g with the obvious map from $X(c) \times Y(c)$ to $X \otimes_{\mathcal{C}} Y$. □

In particular, Lemma 1.5 says that for a fixed covariant $\mathcal{C}$-space Y the functor $-\otimes_{\mathcal{C}} Y$ from the category of contravariant $\mathcal{C}$-spaces to the category of spaces and the functor $\mathrm{map}(Y, -)$ from the category of spaces to the category of contravariant $\mathcal{C}$-spaces are adjoint. Similarly if N is a covariant $R\mathcal{C}$-module, then there is adjoint to $\hom_{R\mathcal{C}}(N, -)$, namely the tensor product of $R\mathcal{C}$-modules $- \otimes_{R\mathcal{C}} N$ (see [10, p. 79], [23, p. 166]). Many properties of these products can be proven via the adjoint property, rather than referring back to the definition. These products are reminiscent of the analogous situation of a right R-module X, a left R-module Y and an Abelian group Z, the tensor product $X \otimes_R Y$, the R-module $\hom_{\mathbb{Z}}(Y, Z)$. Here there is a natural adjoint isomorphism

$$\hom_{\mathbb{Z}}(X \otimes_R Y, Z) \longrightarrow \hom_R(X, \hom_{\mathbb{Z}}(Y, Z)).$$

LEMMA 1.6. *Let X be a space and let Y and Z be covariant (contravariant) $\mathcal{C}$-spaces. Let $X \times Y$ be the obvious covariant (contravariant) $\mathcal{C}$-space. There is a homeomorphism, natural in X, Y, and Z*

$$T(X, Y, Z)\colon \hom_{\mathcal{C}}(X \times Y, Z) \longrightarrow \mathrm{map}(X, \hom_{\mathcal{C}}(Y, Z)).$$ □

EXAMPLE 1.7. Let $\mathbf{\Delta}$ be the category of finite-ordered sets, i.e. for each nonnegative integer p we have an object $[p] = \{0, 1, \ldots, p\}$ and morphisms are monotone increasing functions. A *simplicial space* $X.$ is by definition a contravariant $\mathbf{\Delta}$-space and a *cosimplicial space* is a covariant $\mathbf{\Delta}$-space. A *simplicial set* is a contravariant $\mathbf{\Delta}$-set. It can be considered as a simplicial space by using the discrete topology. Define a covariant $\mathbf{\Delta}$-space $\Delta.$ by assigning to $[p]$ the standard p-simplex and to a monotone function the obvious simplicial map. Given a topological space Y, the *associated simplicial set* $S.Y$ is given by $\mathrm{map}(\Delta., Y)_d$. (The subscript d indicates that we equip this mapping space with the discrete topology, in contrast to the usual convention.) The *geometric realization* $|X.|$ of a simplicial space $X.$ is the space $X. \otimes_{\mathbf{\Delta}} \Delta.$. The geometric realization of a simplicial set has the structure of a CW-complex where each nondegenerate p-simplex corresponds to a p-cell.

We get from Lemma 1.5 that these two functors are adjoint, i.e. for a simplicial space $X.$ and a space Y there is a natural homeomorphism

$$T(X., Y)\colon \mathrm{map}(|X.|, Y) \longrightarrow \hom_{\mathbf{\Delta}}(X., S.Y).$$

In particular, we get for a space Y the natural map given by the adjoint of the identity on $S.Y$

$$t(Y)\colon |S.Y| \longrightarrow Y$$

which is known to be a weak homotopy equivalence. Hence $t(Y)$ is a functorial construction of a CW-approximation of the space Y. For more information about simplicial spaces and sets we refer, for instance, to [4, 7, 22, 25]. □

Next we introduce spectra over a category $\mathcal{C}$. Let SPACES$_+$ be the category of pointed spaces. Recall that objects are compactly generated spaces X with base points for which the inclusion of the base point is a cofibration and morphisms are pointed maps. We define the category SPECTRA of spectra as follows. A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n): E(n) \wedge S^1 \longrightarrow E(n+1)$. A (*strong*) *map* of spectra (sometimes also called function in the literature) $\mathbf{f}: \mathbf{E} \longrightarrow \mathbf{E}'$ is a sequence of maps $f(n): E(n) \longrightarrow E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ \left(f(n) \wedge \mathrm{id}_{S^1}\right)$ for all $n \in \mathbb{Z}$. This should not be confused with the notion of map of spectra in the stable category (see [1, III.2]). Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) = \mathrm{colim}_{k \to \infty} \pi_{i+k}(E(k))$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism and the homomorphism induced by the structure map. A *weak homotopy equivalence* of spectra is a map $\mathbf{f}: \mathbf{E} \longrightarrow \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups. A spectrum $\mathbf{E}$ is called Ω-*spectrum* if for each structure map, its adjoint $E(n) \longrightarrow \Omega E(n+1) = \mathrm{map}(S^1, E(n+1))$ is a weak homotopy equivalence of spaces. We denote by Ω-$SPECTRA$ the corresponding full subcategory of *SPECTRA*.

A *pointed* $\mathcal{C}$-*space*, resp. a $\mathcal{C}$-*spectrum*, resp. $\mathcal{C}$-Ω-*spectrum*, is a functor from $\mathcal{C}$ to SPACES$_+$, resp. SPECTRA, resp. Ω-SPECTRA. We have introduced tensor product of $\mathcal{C}$-spaces in Definitions 1.4 and mapping spaces of $\mathcal{C}$-spaces in Definition 1.2. These notions extend to pointed spaces, one simply has to replace disjoint unions $\coprod$ and Cartesian products $\prod$ by wedge products $\vee$ and smash products $\wedge$ and mapping spaces by pointed mapping spaces. All the adjunction properties remain true. Any $\mathcal{C}$-space X determines a pointed $\mathcal{C}$-space $X_+ = X \coprod \{*\}$ by adjoining a base point. Here $\{*\}$ denotes a $\mathcal{C}$-space which assigns to any object a single point. It is called the *trivial* $\mathcal{C}$-*space*.

A $\mathcal{C}$-spectrum $\mathbf{E}$ can also be thought of as a sequence $\{E(n) \mid n \in \mathbb{Z}\}$ of pointed $\mathcal{C}$-spaces and the structure maps as maps of pointed $\mathcal{C}$-spaces. With this interpretation it is obvious what the *tensor product spectrum* $X \otimes_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed $\mathcal{C}$-space and a covariant $\mathcal{C}$-spectrum means. The canonical associativity homeomorphisms

$$(X \otimes_{\mathcal{C}} E(n)) \wedge S^1 \longrightarrow X \otimes_{\mathcal{C}} (E(n) \wedge S^1)$$

are used in order to define the structure maps. It is given on representatives by sending $(x \otimes_{\mathcal{C}} e) \wedge z$ to $x \otimes_{\mathcal{C}} (e \wedge z)$. More abstractly, it is induced by the following composition of natural bijections coming from various adjunctions where Z is a pointed space

$$\operatorname{map}\Big((X \otimes_{\mathcal{C}} E(n)) \wedge S^1, Z\Big) \longrightarrow \operatorname{map}\Big(X \otimes_{\mathcal{C}} E(n), \operatorname{map}(S^1, Z)\Big) \longrightarrow$$
$$\hom_{\mathcal{C}}\Big(X, \operatorname{map}\Big(E(n), \operatorname{map}(S^1, Z)\Big)\Big) \longrightarrow \hom_{\mathcal{C}}\Big(X, \operatorname{map}(E(n) \wedge S^1, Z)\Big)$$
$$\longrightarrow \operatorname{map}(X \otimes_{\mathcal{C}} (E(n) \wedge S^1), Z).$$

Similarly one defines the *mapping space spectrum* $\hom_{\mathcal{C}}(X, \mathbf{E})$ of a pointed $\mathcal{C}$-space X and a $\mathcal{C}$-spectrum $\mathbf{E}$ using the canonical map of pointed spaces (which is not a homeomorphism in general)

$$\hom_{\mathcal{C}}(X, E(n)) \wedge S^1 \longrightarrow \hom_{\mathcal{C}}(X, E(n) \wedge S^1).$$

This map assigns to $\phi \wedge z$ the map of $\mathcal{C}$-spaces from X to $E(n) \wedge S^1$ which sends $x \in X(c)$ to $\phi(c)(x) \wedge z \in E(n)(c) \wedge S^1$ for $c \in \operatorname{Ob}(\mathcal{C})$.

A *homotopy of maps of spectra* $f_k\colon \mathbf{E} \longrightarrow \mathbf{F}$ is a map of spectra $h\colon [0,1]_+ \wedge \mathbf{E} \longrightarrow \mathbf{F}$ whose composition with the inclusion $i_k\colon \mathbf{E} \longrightarrow [0,1]_+ \wedge \mathbf{E}, \quad e \mapsto k \wedge e$ is f_k for $k = 0, 1$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *$\mathcal{C}$-$\mathcal{D}$-space* is a covariant $\mathcal{C} \times \mathcal{D}^{\mathrm{op}}$-space where $\mathcal{D}^{\mathrm{op}}$ is the opposite of $\mathcal{D}$ which has the same objects as $\mathcal{D}$ and is obtained by reversing the direction of all arrows in $\mathcal{D}$. This is the analogue of a R-S-bimodule for two rings R and S. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a covariant functor. We get a $\mathcal{D}$-$\mathcal{C}$-space $\operatorname{mor}_{\mathcal{D}}(F(?), ??)$ where we use the discrete topology on the set of morphisms. Here ? is the variable in $\mathcal{C}$ and ?? is the variable in $\mathcal{D}$. Analogously one defines a $\mathcal{C}$-$\mathcal{D}$-space $\operatorname{mor}_{\mathcal{D}}(??, F(?))$.

DEFINITION 1.8. Given a covariant (resp. contravariant) $\mathcal{C}$-space X, define the *induction of X with F* to be the covariant (resp. contravariant) $\mathcal{D}$-space

$$F_*X = \operatorname{mor}_{\mathcal{D}}(F(?), ??) \otimes_{\mathcal{C}} X$$

respectively

$$F_*X = X \otimes_{\mathcal{C}} \operatorname{mor}_{\mathcal{D}}(??, F(?))$$

and the *coinduction of X with F* to be the covariant (resp. contravariant) $\mathcal{D}$-space

$$F_!X = \hom_{\mathcal{C}}(\operatorname{mor}_{\mathcal{D}}(??, F(?)), X)$$

respectively

$$F_!X = \hom_{\mathcal{C}}(\operatorname{mor}_{\mathcal{D}}(F(?), ??), X).$$

Given a covariant (contravariant) $\mathcal{D}$-space Y, define *the restriction of Y with F* to be the covariant (contravariant) $\mathcal{C}$-space $F^*Y = Y \circ F$. □

There are corresponding definitions for $\mathcal{C}$-sets and $R\mathcal{C}$-modules (see [10, p. 80], [23, p. 166] for induction of modules). For example, if M is a covariant $R\mathcal{C}$-module, then $F_*M = R\operatorname{mor}_{\mathcal{D}}(F(?), ??) \otimes_{R\mathcal{C}} M$, where for a set S the notation RS is the free R-module generated by the set S. The key properties of (co)-induction and restriction are the following adjoint properties.

LEMMA 1.9. *There are natural adjunction homeomorphisms*

$$\begin{aligned}
&\hom_{\mathcal{D}}(F_*X, Y) \longrightarrow \hom_{\mathcal{C}}(X, F^*Y);\\
&\hom_{\mathcal{C}}(F^*X, Y) \longrightarrow \hom_{\mathcal{D}}(X, F_!Y);\\
&F_*X \otimes_{\mathcal{D}} Y \longrightarrow X \otimes_{\mathcal{C}} F^*Y;\\
&Y \otimes_{\mathcal{D}} F_*X \longrightarrow F^*Y \otimes_{\mathcal{C}} X;
\end{aligned}$$

for a $\mathcal{C}$-space X and $\mathcal{D}$-space Y of the required variance.

Proof. Notice for a covariant $\mathcal{D}$-space Y that there are natural homeomorphisms of covariant $\mathcal{C}$-spaces

$$\mathrm{mor}_{\mathcal{D}}(??, F(?)) \otimes_{\mathcal{D}} Y \longrightarrow F^*Y \longrightarrow \hom_{\mathcal{D}}\left(\mathrm{mor}_{\mathcal{D}}(F(?), ??), Y\right)$$

and analogously for contravariant Y. Now the claim follows from the adjointness of tensor product and hom and the associativity of tensor product. □

2. *K*- and *L*-Theory Spectra over the Orbit Category

In this section we construct the main examples of spectra over the orbit category

$$\begin{aligned}
&\mathbf{K}^{\mathrm{alg}}\colon \mathrm{Or}(G) \longrightarrow \Omega\text{-SPECTRA},\\
&\mathbf{L}\colon \mathrm{Or}(G) \longrightarrow \Omega\text{-SPECTRA},\\
&\mathbf{K}^{\mathrm{top}}\colon \mathrm{Or}(G) \longrightarrow \Omega\text{-SPECTRA}.
\end{aligned}$$

These functors are necessary for the statements of the various Isomorphism Conjectures. First we outline what we would naively like to do, explain why this does not work and then give the details of the correct construction.

The three functors defined over the orbit category will be related to the more classical functors

$$\begin{aligned}
&\mathbf{K}^{\mathrm{alg}}\colon \mathrm{RINGS} \longrightarrow \Omega\text{-SPECTRA},\\
&\mathbf{L}\colon \mathrm{RINGS}^{\mathrm{inv}} \longrightarrow \Omega\text{-SPECTRA},\\
&\mathbf{K}^{\mathrm{top}}\colon C^*\text{-ALGEBRAS} \longrightarrow \Omega\text{-SPECTRA},
\end{aligned}$$

where $\mathrm{RINGS}^{\mathrm{inv}}$ is the category of rings with involution. The classical functors were defined by Gersten [17] and Wagoner [39] for algebraic K-theory, by Quinn–Ranicki [33] for algebraic L-theory, and by using Bott periodicity for C^*-algebras (see [40] for a discussion of Bott periodicity for C^*-algebras and also the end of this section for a functorial approach). The homotopy groups of these spectra give the algebraic K-groups of Quillen–Bass, the surgery obstruction L-groups of Wall, and the topological K-groups of C^*-algebras. These are all nonconnective spectra; the homotopy groups in negative dimensions are nontrivial. In L-theory our notation is an abbreviation for $\mathbf{L} = \mathbf{L}^{\langle \mathbf{j} \rangle}$ for $j \in \mathbb{Z} \amalg \{-\infty\}$, $j \leqslant 2$, where the superscript refers to the K-theory allowed. We would like our functors defined on the orbit category to have the property that the spectra $\mathbf{K}^{\mathrm{alg}}(G/H)$, $\mathbf{L}(G/H)$ and $\mathbf{K}^{\mathrm{top}}(G/H)$ have the weak homotopy type of the spectra $\mathbf{K}^{\mathrm{alg}}(\mathbb{Z}H)$, $\mathbf{L}(\mathbb{Z}H)$ and $\mathbf{K}^{\mathrm{top}}(C_r^*H)$, respectively, where $\mathbb{Z}H$ is the integral group ring and C_r^*H is the reduced C^*-algebra of H (see [29] for a definition). We would also like our functor to be correct on morphisms. Notice that a morphism from G/H to G/K is given by right multiplication $r_g\colon G/H \longrightarrow G/K, g'H \mapsto g'gK$ provided $g \in G$ satisfies $g^{-1}Hg \subset K$. The induced homomorphism $c_g\colon H \longrightarrow K, h \mapsto g^{-1}hg$ gives a map of rings (with involution) from $\mathbb{Z}H$ to $\mathbb{Z}K$, and, at least if the index of $c_g(H)$ in K is finite, a map on reduced C^*-algebras. We would like the functors applied to the morphism r_g in the orbit category to match up with the 'classical' functors on rings, rings with involution, and C^*-algebras.

The naive approach is define $\mathbf{K}^{\mathrm{alg}}(G/H)$, $\mathbf{L}(G/H)$ and $\mathbf{K}^{\mathrm{top}}(G/H)$ as the spectra $\mathbf{K}^{\mathrm{alg}}(\mathbb{Z}H)$, $\mathbf{L}(\mathbb{Z}H)$ and $\mathbf{K}^{\mathrm{top}}(C_r^*H)$, respectively. This definition works fine for objects, but fails for morphisms. The problem is that g in c_g is not unique, because for any $k \in K$, clearly g and gk define the same morphism in the orbit category. Hence this definition makes sense only if $c_k\colon K \longrightarrow K$ induces the *identity* on the various spectra associated to K. This is actually true on the level of homotopy groups, but not on the level of the spectra themselves. However, it is important to construct these functors for spectra and not only for homotopy groups of spectra in order to deal with assembly maps and the various Isomorphism Conjectures. Thus we must thicken up the spectra. The problems with constructing the functor $\mathbf{K}^{\mathrm{top}}\colon C^*$-ALGEBRAS $\longrightarrow \Omega$-SPECTRA are particularly involved. P. Baum and J. Block, and P. Baum and G. Comezana have approaches to this construction, quite different from ours.

The general strategy for a solution of this problem is the following. Let GROUPOIDS be the category of (discrete) groupoids with functors of groupoids as morphisms. (A groupoid is a small category, all of whose morphisms are isomorphisms.) Let GROUPOIDS$^{\mathrm{inj}}$ be the subcategory consisting of those functors $F\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ which are faithful, i.e. for any two objects x, y in $\mathcal{G}_0$ the induced map $\mathrm{mor}_{\mathcal{G}_0}(x, y) \longrightarrow \mathrm{mor}_{\mathcal{G}_1}(F(x), F(y))$ is injective. In the first step one defines a covariant functor

$$GR\colon \mathrm{Or}(G) \longrightarrow \mathrm{GROUPOIDS}^{\mathrm{inj}}$$

from the orbit category to the category of groupoids as follows. A left G-set S defines a groupoid $\bar{S}$ where $\mathrm{Ob}(\bar{S}) = S$ and for $s, t \in S$, $\mathrm{mor}(s, t) = \{g \in G \mid gs = t\}$. The composition law is given by group multiplication. Obviously a map of left G-sets defines a functor of the associated groupoids. The category $\overline{G/H}$ is equivalent to the category $\mathrm{Or}(H, 1) = H$ and hence $\overline{G/H}$ can serve as a substitute for the subgroup H.

Next one extends the definition of the algebraic K- and L-theory spectra of the integral group ring of a group and the topological K-theory spectrum of the reduced C^*-algebra of a group to the category of groupoids. The composition of this extension with the functor GR above yields covariant functors from the orbit category to the category of spectra. We will see that their value at each object G/H is homotopy equivalent to the corresponding spectrum associated to H. The main effort is now to construct these extensions to the category of groupoids, which will be denoted in the same way as the three functors we want to construct:

$$\mathbf{K}^{\mathrm{alg}}\colon \mathrm{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA},$$
$$\mathbf{L}\colon \mathrm{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA},$$
$$\mathbf{K}^{\mathrm{top}}\colon \mathrm{GROUPOIDS}^{\mathrm{inj}} \longrightarrow \Omega\text{-SPECTRA}.$$

For this purpose we must introduce some additional structures on categories. Recall that a category $\mathcal{C}$ is *small* if the objects in $\mathcal{C}$ form a set and for any two objects x and y the morphisms from x to y form a set. In the sequel all categories are assumed to be small. We will recall and introduce additional structures on $\mathcal{C}$.

Let R be a commutative ring with unit. We call $\mathcal{C}$ a *R-category* if for any two objects x and y the set $\mathrm{mor}_{\mathcal{C}}(x, y)$ of morphisms from x to y carries the structure of a R-module such that composition induces a R-bilinear map $\mathrm{mor}_{\mathcal{C}}(x, y) \times \mathrm{mor}_{\mathcal{C}}(y, z) \longrightarrow \mathrm{mor}_{\mathcal{C}}(x, z)$ for all objects x, y and z in $\mathcal{C}$.

Suppose that R comes with an involution of rings $R \longrightarrow R$, $r \mapsto \bar{r}$. A *R-category with involution* is a R-category $\mathcal{C}$ with a collection of maps

$$*_{x,y}\colon \mathrm{mor}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{mor}_{\mathcal{C}}(y, x) \qquad x, y, \in \mathrm{Ob}(\mathcal{C})$$

such that the following conditions are satisfied:

1. $*_{x,y}(\lambda \cdot f + \mu \cdot g) = \bar{\lambda} \cdot *_{x,y}(f) + \bar{\mu} \cdot *_{x,y}(g)$ for all $\lambda, \mu \in R$, objects $x, y \in \mathrm{Ob}(\mathcal{C})$, and morphisms $f, g\colon x \longrightarrow y$;
2. $*_{x,y} \circ *_{y,x} = \mathrm{id}$ for all objects $x, y \in \mathrm{Ob}(\mathcal{C})$;
3. $*_{x,z}(g \circ f) = *_{x,y}(f) \circ *_{y,z}(g)$ for all $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $f\colon x \longrightarrow y$ and $g\colon y \longrightarrow z$.

In the sequel we abbreviate $*_{x,y}(f)$ by f^*. In this notation the conditions above become $(\lambda f + \mu g)^* = \bar{\lambda} f^* + \bar{\mu} g^*$, $(f^*)^* = f$ and $(g \circ f)^* = f^* \circ g^*$.

We call a R-category (with involution) an *additive R-category* (*with involution*) if it possesses a sum $\oplus$ and the obvious compatibility conditions with the R-module structures (and the involution) on the morphisms are fulfilled.

The notion of a C^*-category was defined by Ghez–Lima–Roberts [18] and we give the definition below in our language. Equip the complex numbers with the involution of rings given by complex conjugation. A *C^*-category* $\mathcal{C}$ is a $\mathbb{C}$-category with involution such that for each two objects $x, y \in \mathrm{Ob}(\mathcal{C})$ there is a norm $\| \ \|_{x,y}$ on each complex vector space $\mathrm{mor}_{\mathcal{C}}(x, y)$ such that the following conditions are satisfied:

1. $(\mathrm{mor}_{\mathcal{C}}(x, y), \| \ \|_{x,y})$ is a Banach space for all objects $x, y \in \mathrm{Ob}(\mathcal{C})$;
2. $\|g \circ f\|_{x,z} \leqslant \|g\|_{y,z} \cdot \|f\|_{x,y}$ for all $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $f\colon x \longrightarrow y$ and $g\colon y \longrightarrow z$;

3. $\|f^* \circ f\|_{x,x} = \|f\|^2_{x,y}$ for all $x, y \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $f: x \longrightarrow y$;
4. For every $f \in \mathrm{mor}_{\mathcal{C}}(x, y)$, there is a $g \in \mathrm{mor}_{\mathcal{C}}(x, x)$ so that $f^* \circ f = g^* \circ g$.

In the sequel we abbreviate $\|f\|_{x,y}$ by $\|f\|$ and we will consider a C^*-category as a topological category by equipping the set of objects with the discrete topology and the set $\mathrm{mor}_{\mathcal{C}}(x, y)$ with the topology which is induced by the norm.

EXAMPLE 2.1. Let $\mathcal{C}$ be a category with precisely one object x. Then the structure of a R-category on $\mathcal{C}$ gives $\mathrm{mor}_{\mathcal{C}}(x, x)$ the structure of a central R-algebra with unit id_x. The additional structure of an involution is given by a map $*: \mathrm{mor}_{\mathcal{C}}(x, x) \longrightarrow \mathrm{mor}_{\mathcal{C}}(x, x)$ satisfying:

$$* (\lambda \cdot f + \mu \cdot g) = \bar{\lambda} \cdot *(f) + \bar{\mu} \cdot *(g),$$
$$* \circ * = \mathrm{id} \quad \text{and} \quad * (g \circ f) = *(f) \circ *(g).$$

The structure of a C^*-category on $\mathcal{C}$ is the same as the structure of a C^*-algebra on the set $\mathrm{mor}_{\mathcal{C}}(x, x)$ with id_x as unit. The structure of a topological category on $\mathcal{C}$ is the structure of a topological space on $\mathrm{mor}_{\mathcal{C}}(x, x)$ such that composition defines a continuous map. □

Next we construct from a category (for example, a groupoid) other categories with the structures described above. Given a category $\mathcal{C}$, the *associated R-category $R\mathcal{C}$* has the same objects as $\mathcal{C}$ and its morphism set $\mathrm{mor}_{R\mathcal{C}}(x, y)$ from x to y is given by the free R-module $R\,\mathrm{mor}_{\mathcal{C}}(x, y)$ generated by the set $\mathrm{mor}_{\mathcal{C}}(x, y)$. The composition is induced by the composition in $\mathcal{C}$ in the obvious way. Notice that the functor $\mathcal{C} \mapsto R\mathcal{C}$ is the left adjoint of the forgetful functor from the category of R-categories to the category of small categories.

Let $\mathcal{G}$ be a groupoid and R a commutative ring with unit and involution. Then $R\mathcal{G}$ inherits the structure of a R-category with involution by defining

$$\left(\sum_{i=1}^{r} \lambda_i f_i\right)^* := \sum_{i=1}^{r} \bar{\lambda} f_i^{-1}.$$

Next we explain how the category with involution $\mathbb{C}\mathcal{G}$ can be completed to a C^*-category $C_r^*\mathcal{G}$. It will have the same objects as $\mathcal{G}$. Consider two objects $x, y \in \mathrm{Ob}(\mathcal{G})$. If $\mathrm{mor}_{\mathcal{G}}(x, y)$ is empty, put $\mathrm{mor}_{C_r^*\mathcal{G}}(x, y) = 0$. Suppose that $\mathrm{mor}_{\mathcal{G}}(x, y)$ is nonempty. Choose some object $z \in \mathrm{Ob}(\mathcal{G})$ such that $\mathrm{mor}_{\mathcal{G}}(z, x)$ is nonempty, for instance one could choose $z = x$. For a set S let $l^2(S)$ be the Hilbert space with S as Hilbert basis. Define a $\mathbb{C}$-linear map

$$i_{x,y;z}: \mathbb{C}\mathrm{mor}_{\mathcal{G}}(x, y) \longrightarrow \mathcal{B}(l^2(\mathrm{mor}_{\mathcal{G}}(z, x)), l^2(\mathrm{mor}_{\mathcal{G}}(z, y)))$$

by sending $f \in \mathrm{mor}_{\mathcal{G}}(x, y)$ to the bounded operator from $l^2(\mathrm{mor}_{\mathcal{G}}(z, x))$ to $l^2(\mathrm{mor}_{\mathcal{G}}(z, y))$ given by composition with f. On the target of $i_{x,y;z}$ we have the operator norm $\|\ \ \|$. Define:

$$\|u\|_{x,y} := \|i_{x,y;z}(u)\| \quad \text{for } u \in \mathrm{mor}_{\mathbb{C}\mathcal{G}}(x, y) = \mathbb{C}\,\mathrm{mor}_{\mathcal{G}}(x, y).$$

One easily checks that this norm $\| \ \|_{x,y}$ is independent of the choice of z. The Banach space of morphisms in $C_r^*\mathcal{G}$ from x to y is the completion of $\mathrm{mor}_{\mathbb{C}\mathcal{G}}(x, y)$ with respect to the norm $\| \ \|_{x,y}$. We will denote the induced norm on the completion $\mathrm{mor}_{C_r^*\mathcal{G}}(x, y)$ again by $\| \ \|_{x,y}$ and sometimes abbreviate by $\| \ \|$. One easily checks that $*_{x,y}\colon \mathrm{mor}_{\mathbb{C}\mathcal{G}}(x, y) \longrightarrow \mathrm{mor}_{\mathbb{C}\mathcal{G}}(y, x)$ is an isometry since it is compatible with applying the maps $i_{x,y;z}$ and $i_{y,x;z}$ and taking adjoints of operators. Therefore it induces an isometry denoted in the same way

$$*_{x,y}\colon \mathrm{mor}_{C_r^*\mathcal{G}}(x, y) \longrightarrow \mathrm{mor}_{C_r^*\mathcal{G}}(y, x).$$

Composition defines a $\mathbb{C}$-bilinear map $\mathrm{mor}_{\mathbb{C}\mathcal{G}}(x, y) \times \mathrm{mor}_{\mathbb{C}\mathcal{G}}(y, z) \longrightarrow \mathrm{mor}_{\mathbb{C}\mathcal{G}}(x, z)$ which satisfies $\|g \circ f\|_{x,z} \leqslant \|g\|_{y,z} \cdot \|f\|_{x,y}$. Hence it induces a map on the completions

$$\mathrm{mor}_{C_r^*\mathcal{G}}(x, y) \times \mathrm{mor}_{C_r^*\mathcal{G}}(y, z) \longrightarrow \mathrm{mor}_{C_r^*\mathcal{G}}(x, z)$$

with the same inequality for the norms. This is the composition in $C_r^*\mathcal{G}$. One easily verifies that $C_r^*\mathcal{G}$ satisfies all the axioms of a C^*-category.

EXAMPLE 2.2. Let G be a group. It defines a groupoid $\mathcal{G}$ with one object and G as its automorphism group. Then $R\mathcal{G}$ is just the group ring RG and $C_r^*\mathcal{G}$ is just the reduced group C^*-algebra C_r^*G under the identifications of Example 2.1. □

The assignment of a C^*-category $C_r^*\mathcal{G}$ to a groupoid $\mathcal{G}$ gives a functor

$$C_r^*\colon \mathrm{GROUPOIDS}^{\mathrm{inj}} \longrightarrow C^*\text{-CATEGORIES},$$

where C^*-CATEGORIES is the category of small C^*-categories. The inj-condition that a functor $F\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ is faithful is used to guarantee that the map $\mathrm{mor}_{\mathbb{C}\mathcal{G}_0}(x, y) \longrightarrow \mathrm{mor}_{\mathbb{C}\mathcal{G}_1}(F(x), F(y))$ extends to $\mathrm{mor}_{C_r^*\mathcal{G}_0}(x, y) \longrightarrow \mathrm{mor}_{C_r^*\mathcal{G}_1}(F(x), F(y))$, for all $x, y \in \mathrm{Ob}(\mathcal{G}_0)$.

Remark 2.3. We make a few remarks on functoriality (or lack thereof) of C^*-algebras, which motivate our use of C^*-categories. First note that the assignment of a C^*-algebra C_r^*H to a group H cannot be extended to a functor from the category of groups to the category of C^*-algebras. For instance, the reduced C^*-algebra $C_r^*(\mathbb{Z}*\mathbb{Z})$ of the free group on two letters is simple [31] and hence admits no C^*-homomorphism to the reduced C^*-algebra $\mathbb{C}$ of the trivial group.

There is a notion of the C^*-algebra of a groupoid, but it is poorly behaved with respect to functoriality. To a discrete groupoid $\mathcal{G}$, one can associate the complex groupoid ring $\mathbb{C}\mathcal{G}$, which as a $\mathbb{C}$-vector space has a basis consisting of the morphisms in the groupoid. The product of two basis elements is the composite if defined and is zero otherwise. The completion of $\mathbb{C}\mathcal{G}$ in $\mathcal{B}(l^2(\mathcal{G}), l^2(\mathcal{G}))$ in the operator norm is called the reduced C^*-algebra of the groupoid and which we denote $C_r^*\mathcal{G}$-alg. If $\mathcal{G}$ is connected (any two objects are isomorphic), and H is the automorphism group of an object, then it can be shown (via Morita theory) that the spectra $\mathbf{K}^{\mathrm{top}}(C_r^*\mathcal{G}\text{-alg})$

and $\mathbf{K}^{\text{top}}(C_r^* H)$ have the same weak homotopy type. The second naive approach to the construction of a functor

$$\mathbf{K}^{\text{top}} : \mathrm{Or}(G) \longrightarrow \Omega\text{-SPECTRA}$$

is to define $\mathbf{K}^{\text{top}}(G/H)$ to be $\mathbf{K}^{\text{top}}(C_r^* \overline{G/H}\text{-alg})$. While this approach is basically correct for algebraic K- and L-theory, it fails for C^*-algebras because the C^*-algebra of a groupoid does not define a functor from the category GROUPOIDS$^{\text{inj}}$ to C^*-ALGEBRAS. Indeed, consider the groupoid $\mathcal{G}[n]$ with n objects and precisely one morphism between two objects. Notice that the obvious functor from $\mathcal{G}[n]$ to $\mathcal{G}[1]$ has an obvious right inverse. Hence it would induce a surjective C^*-homomorphism between the associated C^*-algebras but this is impossible for $n \geqslant 2$ as the associated C^*-algebra of $\mathcal{G}[n]$ is the simple algebra $M_n(\mathbb{C})$. Another counterexample comes from a morphism in the orbit category. Let G be any infinite group and consider the map of groupoids $\overline{G/1} \longrightarrow \overline{G/G}$ where G acts on $G/1$ effectively and transitively by left multiplication and G acts trivially on G/G. An easy computation with the operator norm shows that this map of groupoids does not extend to a map of the reduced C^*-algebras of the groupoids. We take the trouble to discuss this because mistakes have been made in the literature on this point and to motivate our definition of the functor C_r^*: GROUPOIDS$^{\text{inj}} \longrightarrow C^*$-CATEGORIES. Below we will define the $\mathbf{K}^{\text{top}}$-functor from C^*-CATEGORIES to SPECTRA. Note that after applying homotopy groups, one gets maps on the K-theory of reduced C^*-algebras of the groupoids, independent of Morita theory and without maps on the C^*-algebras themselves. □

We recall some basic constructions we will need later.

Let $\mathcal{C}$ be a R-category. We define a new R-category $\mathcal{C}_\oplus$, called the *symmetric monoidal R-category associated to $\mathcal{C}$* with an associative and commutative sum $\oplus$ as follows. The objects in $\mathcal{C}_\oplus$ are n-tuples $\underline{x} = (x_1, x_2, \ldots, x_n)$ consisting of objects $x_i \in \mathrm{Ob}(\mathcal{C})$ for $n = 0, 1, 2, \ldots$. We will think of the empty set as 0-tuple which we denote by 0. The R-module of morphisms from $\underline{x} = (x_1, \ldots, x_m)$ to $\underline{y} = (y_1, \ldots, y_n)$ is given by

$$\mathrm{mor}_{\mathcal{C}_\oplus}(\underline{x}, \underline{y}) := \oplus_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n} \mathrm{mor}_{\mathcal{C}}(x_i, y_j).$$

Given a morphism $f : \underline{x} \longrightarrow \underline{y}$, we denote by $f_{i,j} : x_i \longrightarrow y_j$ the component which belongs to $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. If x or y is the empty tuple, then $\mathrm{mor}_{\mathcal{C}_\oplus}(x, y)$ is defined to be the trivial R-module. The composition of $f : \underline{x} \longrightarrow \underline{y}$ and $g : \underline{y} \longrightarrow \underline{z}$ for objects $\underline{x} = (x_1, \ldots, x_m)$, $\underline{y} = (y_1, \ldots, y_n)$ and $\underline{z} = (z_1, \ldots, z_p)$ is defined by

$$(g \circ f)_{i,k} = \sum_{j=1}^{n} g_{j,k} \circ f_{i,j}.$$

The sum on $\mathcal{C}_\oplus$ is defined on objects by sticking the tuples together, i.e. for $\underline{x} = (x_1, \ldots, x_m)$ and $\underline{y} = (y_1, \ldots, y_n)$ define

$$\underline{x} \oplus \underline{y} := (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

The definition of the sum of two morphisms is now obvious. Notice that this sum is (strictly) associative, i.e. $(\underline{x} \oplus \underline{y}) \oplus \underline{z}$ and $\underline{x} \oplus (\underline{y} \oplus \underline{z})$ are the same objects and analogously for morphisms. Moreover, there is a natural isomorphism

$$\underline{x} \oplus \underline{y} \longrightarrow \underline{y} \oplus \underline{x}$$

and all obvious compatibility conditions hold. The zero object is given by the empty tuple 0. These data define the structure of a symmetric monoidal R-category on $\mathcal{C}_\oplus$. Notice that the functor $\mathcal{C} \mapsto \mathcal{C}_\oplus$ is the left adjoint of the forgetful functor from symmetric monoidal R-categories to R-categories.

Given a category $\mathcal{C}$, define its *idempotent completion* $\mathcal{P}(\mathcal{C})$ to be the following category. An object in $\mathcal{P}(\mathcal{C})$ is an endomorphism $p\colon x \longrightarrow x$ in $\mathcal{C}$ which is an idempotent, i.e. $p \circ p = p$. A morphism in $\mathcal{P}(\mathcal{C})$ from $p\colon x \longrightarrow x$ to $q\colon y \longrightarrow y$ is a morphism $f\colon x \longrightarrow y$ in $\mathcal{C}$ satisfying $q \circ f \circ p = f$. The identity on the object $p\colon x \longrightarrow x$ in $\mathcal{P}(\mathcal{C})$ is given by the morphism $p\colon x \longrightarrow x$ in $\mathcal{C}$. If $\mathcal{C}$ has the structure of a R-category or of a symmetric monoidal R-category, then $\mathcal{P}(\mathcal{C})$ inherits such a structure in the obvious way.

For a category $\mathcal{C}$, let $\mathrm{Iso}(\mathcal{C})$ be the subcategory of $\mathcal{C}$ with the same objects as $\mathcal{C}$, but whose morphisms are the isomorphisms of $\mathcal{C}$. If $\mathcal{C}$ is a symmetric monoidal R-category, then so is $\mathrm{Iso}(\mathcal{C})$.

Let $\mathcal{C}$ be a symmetric monoidal R-category, all of whose morphisms are isomorphisms. Then its *group completion* is the following symmetric monoidal R-category $\mathcal{C}\hat{}$. An object in $\mathcal{C}$ is a pair (x, y) of objects in $\mathcal{C}$. A morphism in $\mathcal{C}\hat{}$ from (x, y) to $(\bar{x}, \bar{y})$ is given by equivalence classes of triples (z, f, g) consisting of an object z in $\mathcal{C}$ and isomorphisms $f\colon x \oplus z \longrightarrow \bar{x}$ and $g\colon y \oplus z \longrightarrow \bar{y}$. We call two such triples (z, f, g) and (z', f', g') equivalent if there is an isomorphism $h\colon z \longrightarrow z'$ which satisfies $f' \circ (\mathrm{id}_x \oplus h) = f$ and $g' \circ (\mathrm{id}_y \oplus h) = g$. The sum on $\mathcal{C}\hat{}$ is given by

$$(x, y) \oplus (\bar{x}, \bar{y}) := (x \oplus \bar{x}, y \oplus \bar{y}).$$

If $\mathcal{C}$ is a C^*-category, then $\mathcal{C}_\oplus$ and $\mathcal{P}(\mathcal{C})$ inherit the structure of a C^*-category where one should modify the definition of $\mathcal{P}(\mathcal{C})$ by requiring that each object $p\colon x \longrightarrow x$ is a self-adjoint idempotent, i.e. $p \circ p = p$ and $p^* = p$. Moreover, $\mathcal{C}_\oplus$, $\mathcal{P}(\mathcal{C}_\oplus)$ and $(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$ inherit the structure of topological categories where the set of objects always gets the discrete topology.

Next we can construct the desired functors from GROUPOIDS and GROUPOIDS$^{\mathrm{inj}}$ to Ω-SPECTRA. The covariant functor *nonconnective algebraic K-theory spectrum of a groupoid with coefficients in R*

$$\mathbf{K}^{\mathrm{alg}}\colon \text{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA}$$

assigns to a groupoid $\mathcal{G}$ the nonconnective K-theory spectrum of the small additive category $(\mathrm{Iso}(\mathcal{P}(R\mathcal{G}_\oplus)))\hat{}$. (See [28] for the construction of the nonconnective K-theory spectrum of a small additive category.)

Next we define the covariant functor *periodic algebraic L-theory spectrum of a groupoid with coefficients in R*

$$\mathbf{L} = \mathbf{L}^{\mathbf{h}} \colon \text{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA}$$

where we assume that R is a commutative ring with unit and involution. Then $R\mathcal{G}$ and hence $R\mathcal{G}_{\oplus}$ inherit an involution. We apply the construction of the periodic algebraic L-theory spectrum in [33, Example 13.6 on page 139]. If one uses the idempotent completion, one gets the projective version

$$\mathbf{L}^{\mathbf{p}} \colon \text{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA}.$$

Taking the Whitehead torsion into account yields the simple version

$$\mathbf{L}^{\mathbf{s}} \colon \text{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA}.$$

More generally, one obtains for $j \in \mathbb{Z} \amalg \{-\infty\},\ j \leqslant 2$,

$$\mathbf{L}^{\langle \mathbf{j} \rangle} \colon \text{GROUPOIDS} \longrightarrow \Omega\text{-SPECTRA},$$

where $\mathbf{L}^{\langle \mathbf{j} \rangle}$ is $\mathbf{L}^{\mathbf{s}}$, $\mathbf{L}^{\mathbf{h}}$, $\mathbf{L}^{\mathbf{p}}$ for $j = 2, 1, 0$.

Next we construct the covariant functor *nonconnective topological K-theory spectrum*

$$\mathbf{K}^{\mathrm{top}} \colon \text{GROUPOIDS}^{\mathrm{inj}} \longrightarrow \Omega\text{-SPECTRA}.$$

We do this by composing the functor

$$C_r^* \colon \text{GROUPOIDS}^{\mathrm{inj}} \longrightarrow C^*\text{-CATEGORIES},$$

with the functor

$$\mathbf{K}^{\mathrm{top}} \colon C^*\text{-CATEGORIES} \longrightarrow \Omega\text{-SPECTRA},$$

which we are about to construct. Let $\mathbb{C}$ denote both the complex numbers and the obvious C^*-category with precisely one object denoted by $\underline{1}$. We have introduced the category $\mathbb{C}_{\oplus}$ before. We denote by $\underline{n}$ the n-fold sum of the object $\underline{1}$. In this notation $\mathbb{C}_{\oplus}$ has as objects $\{\underline{n} \mid n = 0, 1, 2, \ldots\}$, the sum is $\underline{m} \oplus \underline{n} = \underline{m+n}$ for $m, n = 0, 1, 2, \ldots$ and the Banach space of morphisms from $\underline{m}$ to $\underline{n}$ is just given by the (n, m)-matrices with complex entries. Let $\mathcal{C}$ be any $\mathbb{C}$-category. We define a functor

$$\otimes \colon \mathbb{C}_{\oplus} \times \mathcal{C}_{\oplus} \longrightarrow \mathcal{C}_{\oplus}$$

as follows. We assign to an object $\underline{n} \in \mathbb{C}_{\oplus}$ and an object $\underline{x} \in \mathcal{C}_{\oplus}$ the object $\underline{n} \otimes \underline{x}$ which is the n-fold direct sum $\oplus_{i=1}^{n} \underline{x}$. Let $f \colon \underline{m} \longrightarrow \underline{n}$ be a morphism in $\mathbb{C}_{\oplus}$ and $g \colon \underline{x} \longrightarrow \underline{y}$ be a morphism in $\mathcal{C}_{\oplus}$. Define $f \otimes g \colon \underline{m} \otimes \underline{x} \longrightarrow \underline{n} \otimes \underline{y}$, to be the morphism whose component from the ith copy of $\underline{x}$ in $\underline{m} \otimes \underline{x}$ to the jth copy of $\underline{y}$ in $\underline{n} \otimes \underline{y}$ is $f_{i,j} \cdot g$, where $f_{i,j} \in \mathbb{C}$ is the component of f from the ith coordinate of $\underline{m}$ to the jth coordinate of $\underline{n}$. One easily checks that $f \otimes g$ is a functor. For objects $\underline{m}$

and $\underline{n}$ in $\mathbb{C}_\oplus$ and an object $\underline{x}$ in $\mathcal{C}_\oplus$ we have $(\underline{m} \oplus \underline{n}) \otimes \underline{x} = (\underline{m} \otimes \underline{x}) \oplus (\underline{n} \otimes \underline{x})$. For an object $\underline{n}$ in $\mathbb{C}_\oplus$ and objects $\underline{x}$ and $\underline{y}$ in $\mathcal{C}_\oplus$ we have a natural isomorphism $\underline{n} \otimes (\underline{x} \oplus \underline{y}) \cong (\underline{n} \otimes \underline{x}) \oplus (\underline{n} \otimes \underline{y})$. Obviously this functor sends the subcategories $\{\underline{0}\} \times \mathcal{C}_\oplus$ and $\mathbb{C}_\oplus \times \{0\}$ to $\{0\}$, where $\{\underline{0}\}$ and $\{0\}$ denote the obvious subcategories with one object.

Let $\mathcal{C}$ be any C^*-category. Then the construction above applies to $\mathcal{P}(\mathcal{C}_\oplus)$. It extends to a functor

$$\otimes\colon (\mathrm{Iso}(\mathbb{C}_\oplus))\hat{} \times (\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{} \longrightarrow (\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$$

in the obvious way. Notice that $(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$ inherits from $\mathcal{C}$ the structure of a topological category for which the set of objects is discrete. With respect to these topological structures the functor above is a functor of topological categories. Given a topological category $\mathcal{D}$, let $B\mathcal{D}$ be its classifying space [34] (whose construction takes the topology into account). Given topological categories $\mathcal{D}$ and $\mathcal{D}'$, the projections induce a homeomorphism

$$B(\mathcal{D} \times \mathcal{D}') \longrightarrow B\mathcal{D} \times B\mathcal{D}'.$$

Hence the functor above induces a map

$$B(\mathrm{Iso}(\mathbb{C}_\oplus))\hat{} \times B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{} \longrightarrow B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$$

for any C^*-category $\mathcal{C}$. Since it sends $B(\mathrm{Iso}(\mathbb{C}_\oplus))\hat{} \vee B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$ to the base point $B\{0\} \subset B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$, we obtain a map, natural in $\mathcal{C}$,

$$\mu\colon B(\mathrm{Iso}(\mathbb{C}_\oplus))\hat{} \wedge B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{} \longrightarrow B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}.$$

The category $\mathrm{Iso}(\mathbb{C}_\oplus)$ can be identified with the disjoint union $\coprod_{n \geqslant 0} GL(n, \mathbb{C})$. Let $GL(\mathbb{C}) = \mathrm{colim}_{n\to\infty} GL(n, \mathbb{C})$. Let $\mathbb{Z} \times GL(\mathbb{C})$ be the symmetric monoidal category whose objects (and monoidal sum) are given by the integers, and so that $\mathrm{mor}_{\mathbb{Z}\times GL(\mathbb{C})}(m, n)$ is empty if $m \neq n$ and is $GL(\mathbb{C})$ if $m = n$. There is an obvious functor $\mathrm{Iso}(\mathbb{C}_\oplus) \longrightarrow \mathbb{Z} \times GL(\mathbb{C})$. Using Quillen's group completion theorem [19, pp. 220–221], it follows that $B\mathrm{Iso}(\mathbb{C}_\oplus)\hat{}$ has the homotopy type of $\mathbb{Z} \times BGL(\mathbb{C})$. Let $b\colon S^2 \longrightarrow B\mathrm{Iso}(\mathbb{C}_\oplus)\hat{}$ be a fixed representative of the Bott element in $\pi_2(B\mathrm{Iso}(\mathbb{C}_\oplus)\hat{}) = K^{-2}(\{pt.\})$. Then b and μ yield a map, natural in $\mathcal{C}$,

$$S^2 \wedge B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{} \longrightarrow B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}.$$

Its adjoint is also natural in $\mathcal{C}$ and denoted by

$$\beta\colon B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{} \longrightarrow \Omega^2 B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}.$$

Define the *nonconnective topological K-theory spectrum* $\mathbf{K}^{\mathrm{top}}(\mathcal{C})$ of the C^*-category $\mathcal{C}$ by the space $B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$ in even dimensions, by the space $\Omega B(\mathrm{Iso}(\mathcal{P}(\mathcal{C}_\oplus)))\hat{}$ in odd dimensions and by the structure maps which are the identity in odd dimensions and β in even dimensions. (Another construction is suggested by [14, Remark

VIII.4.4. on page 186].) We claim that the proof of Bott periodicity for C^*-algebras carries over to C^*-categories. Hence $\mathbf{K}^{\mathrm{top}}(\mathcal{C})$ is a Ω-spectrum. We will only be interested in the case where $\mathcal{C}$ is $C_r^*\mathcal{G}$ for a connected groupoid and in this case the claim follows from Bott periodicity for the reduced group C^*-algebra of the automorphism group of an object in $\mathcal{G}$ and Lemma 2.4.

We make some remarks about the constructions of the spectra of groupoids above and give some basic properties.

There are obvious equivalences of additive categories from $R\mathcal{G}_\oplus$, resp. $\mathcal{P}(R\mathcal{G}_\oplus)$, to the category of finitely generated free $R\mathcal{G}$-modules, resp. finitely generated projective $R\mathcal{G}$-modules, as defined in [23, Section 9]. Notice that these module categories are not small, in contrast to $R\mathcal{G}_\oplus$ and $\mathcal{P}(R\mathcal{G}_\oplus)$. A functor $F\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ induces a functor from the category of finitely generated free, resp. projective, $R\mathcal{G}_0$-modules to the corresponding category over $\mathcal{G}_1$ by induction. However, if we have a second functor $G\colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, then the functor induced on the module categories by $G \circ F$ and the composition of the functors induced by F and G on the module categories are not the same, they agree only up to natural equivalence. In order to avoid this technical problem, we prefer the small category $R\mathcal{G}_\oplus$ and its idempotent completion since there the composition of the functors induced by F and G is the same as the functor induced by $G \circ F$, so that we get honest functors from GROUPOIDS to Ω-SPECTRA.

As mentioned earlier, the functors $\mathbf{K}^{\mathrm{alg}}$, $\mathbf{L}^{\langle \mathbf{j} \rangle}$, and $\mathbf{K}^{\mathrm{top}}$ defined on the orbit category are given by the composition of the groupoid-valued functor GR and the spectra-valued functors defined above. The automorphism group of the object eH in $\overline{G/H}$ for the identity element $e \in G$ is just the subgroup H. Hence the next lemma proves what we have already claimed before, namely, that the spectra we assign to $\overline{G/H}$ are homotopy equivalent to the spectra associated to H. In particular, we get for all $n \in \mathbb{Z}$ and $j \in \mathbb{Z} \amalg \{-\infty\}$, $j \leqslant 2$,

$$\pi_n(\mathbf{K}^{\mathrm{alg}}(G/H)) \cong K_n^{\mathrm{alg}}(\mathbb{Z}H)$$
$$\pi_n(\mathbf{L}^{\langle \mathbf{j} \rangle}(G/H)) \cong L_n^{\langle j \rangle}(\mathbb{Z}H)$$
$$\pi_n(\mathbf{K}^{\mathrm{top}}(G/H)) \cong K_n(C^*H)$$

LEMMA 2.4.

(1) *If $F_i\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ for $i = 0, 1$ are functors of groupoids and $T\colon F_0 \longrightarrow F_1$ is a natural transformation between them, then the induced maps of spectra*

$$\mathbf{K}^{\mathrm{alg}}(F_i)\colon \mathbf{K}^{\mathrm{alg}}(\mathcal{G}_0) \longrightarrow \mathbf{K}^{\mathrm{alg}}(\mathcal{G}_1)$$

are homotopy equivalent and analogously for $\mathbf{L}^{\langle \mathbf{j} \rangle}$ and $\mathbf{K}^{\mathrm{top}}$;

(2) *Let $\mathcal{G}$ be a groupoid. Suppose that $\mathcal{G}$ is connected, i.e. there is a morphism between any two objects. For an object $x \in \mathrm{Ob}(\mathcal{G})$, let $\mathcal{G}_x$ be the full subgroupoid with precisely one object, namely x. Then the inclusion $i_x\colon \mathcal{G}_x \longrightarrow \mathcal{G}$ induces a homotopy equivalence*

$$\mathbf{K}^{\mathrm{alg}}(i_x)\colon \mathbf{K}^{\mathrm{alg}}(\mathcal{G}_x) \longrightarrow \mathbf{K}^{\mathrm{alg}}(\mathcal{G})$$

and $\mathbf{K}^{\mathrm{alg}}(\mathcal{G}_x)$ *is isomorphic to the spectrum* $\mathbf{K}^{\mathrm{alg}}$ *associated to the group ring* $R\ \mathrm{aut}_{\mathcal{G}}(x)$. *The analogous statements hold for* $\mathbf{L}^{\langle \mathbf{j} \rangle}$ *and* $\mathbf{K}^{\mathrm{top}}$.

Proof. Obviously (2) follows from (1). We indicate the proof of (1) in the case of $\mathbf{K}^{\mathrm{top}}$, the other cases are analogous if one inspects the definitions in [28, 33]. One easily checks that a natural transformation between F_0 to F_1 induces a natural transformation between the induced functors from $(\mathrm{Iso}(\mathcal{P}(C_r^*\mathcal{G}_{0_\oplus})))\hat{}$ to $(\mathrm{Iso}(\mathcal{P}(C_r^*\mathcal{G}_{1_\oplus})))\hat{}$. Let [1] be the category having two objects, namely 0 and 1, and three morphisms, namely the identities on 0 and 1 and one morphism from 0 to 1. Then the natural transformation above can be viewed as a functor of topological categories from $(\mathrm{Iso}(\mathcal{P}(C_r^*\mathcal{G}_{0_\oplus})))\hat{}\times [1]$ to $(\mathrm{Iso}(\mathcal{P}(C_r^*\mathcal{G}_{1_\oplus})))\hat{}$. Since the classifying space of a product is the product of the classifying spaces and the classifying space of [1] is $[0,1]$, we obtain a map

$$h\colon B(\mathrm{Iso}(\mathcal{P}(C_r^*\mathcal{G}_{0_\oplus})))\hat{}\times [0,1] \longrightarrow B(\mathrm{Iso}(\mathcal{P}(C_r^*\mathcal{G}_{1_\oplus})))\hat{}.$$

One easily checks that this induces the desired homotopy of maps of spectra. □

3. *CW*-Approximations and Homotopy Limits

In this section we give the basic definitions and properties of spaces and CW-complexes over a small category $\mathcal{C}$. We show that the Whitehead Theorem and CW-approximations carry over from spaces to $\mathcal{C}$-spaces. We emphasize the parallels between a category and a group, thinking of a group as a category with a single object, all of whose morphisms are invertible. We define $E\mathcal{C}$, the universal free contractible $\mathcal{C}$-space, and use this to define the homotopy colimit $E\mathcal{C} \otimes_{\mathcal{C}} X$, the analogue of the Borel construction $EG \times_G X$.

Consider the set $\mathrm{Ob}(\mathcal{C})$ as a small category in the trivial way, i.e. the set of objects is $\mathrm{Ob}(\mathcal{C})$ itself and the only morphisms are the identity morphisms. A map of two $\mathrm{Ob}(\mathcal{C})$-spaces is a collection of maps $\{f(c)\colon X(c) \longrightarrow Y(c) \mid c \in \mathrm{Ob}(\mathcal{C})\}$. There is a forgetful functor

$$F\colon \mathcal{C}\text{-SPACES} \longrightarrow \mathrm{Ob}(\mathcal{C})\text{-SPACES}.$$

Define a functor

$$B\colon \mathrm{Ob}(\mathcal{C})\text{-SPACES} \longrightarrow \mathcal{C}\text{-SPACES}$$

by sending a contravariant $\mathrm{Ob}(\mathcal{C})$-space $X(-)$ to $\coprod_{c\in \mathrm{Ob}(\mathcal{C})} \mathrm{mor}_{\mathcal{C}}(-,c) \times X(c)$. In the covariant case one uses $\mathrm{mor}_{\mathcal{C}}(c,-)$.

LEMMA 3.1. *The functor B is the left adjoint of F.*

Proof. This means that there is a natural bijection

$$T(X, Y)\colon \hom_{\mathcal{C}}(B(X), Y) \longrightarrow \hom_{\mathrm{Ob}(\mathcal{C})}(X, F(Y))$$

for all $\mathrm{Ob}(\mathcal{C})$-spaces X and for all $\mathcal{C}$-spaces Y. Actually $T(X, Y)$ will even be a homeomorphism. For $f\colon B(X) = \coprod_{c\in\mathrm{Ob}(\mathcal{C})} \mathrm{mor}_{\mathcal{C}}(-, c) \times X(c) \longrightarrow Y(-)$ define $T(X, Y)f$ by restricting f to $X(-) = \{\mathrm{id}_-\} \times X(-)$. The inverse $T(X, Y)^{-1}$ assigns to a map g of $\mathrm{Ob}(\mathcal{C})$-spaces the following transformation:

$$B(X) = \coprod_{c\in\mathrm{Ob}(\mathcal{C})} \mathrm{mor}_{\mathcal{C}}(-, c) \times X(c) \longrightarrow Y(-),$$
$$(\phi, x) \mapsto Y(\phi) \circ g(c)(x). \qquad \square$$

Let R be a ring. There is also an adjoint to the forgetful functor from $R\mathcal{C}$-MOD to $\mathrm{Ob}(\mathcal{C})$-SETS. It is defined as $B(X(-)) = \oplus_{c\in\mathrm{Ob}(\mathcal{C})} R(\mathrm{mor}_{\mathcal{C}}(-, c) \times X(c))$. A free $R\mathcal{C}$-module is a module isomorphic to one in the image of B. Notice the analogy between Lemma 3.1 and the adjoint pair consisting of the forgetful functor from R-modules to sets and the functor assigning to a set S the free R-module RS generated by S.

We have already mentioned that the category of $\mathrm{Or}(G, 1)$-spaces is the category of G-spaces and the category $\mathrm{Ob}(\mathrm{Or}(G, 1))$-spaces is the category of spaces. Under this identification the forgetful functor F just forgets the G-action and B sends a space Z to the G-space $G \times Z$ where G acts in the obvious way.

Notice that the notions of coproduct, product, pushout, pullback, colimit, and limit exist in the category of $\mathcal{C}$-spaces. They are constructed by applying these notions in the category SPACES objectwise. For instance, the pushout of a diagram of $\mathcal{C}$-spaces $X_1 \longleftarrow X_0 \longrightarrow X_2$ is defined as the functor $X\colon \mathcal{C} \longrightarrow$ SPACES whose value at an object c in $\mathcal{C}$ is the pushout of the diagram of spaces $X_1(c) \longleftarrow X_0(c) \longrightarrow X_2(c)$. We mention that sometimes in the literature the terms direct limit and inverse limit are used instead of colimit and limit. We will always use the names colimit and limit.

Given a $\mathcal{C}$-space X and a space Y, we obtain the $\mathcal{C}$-space $X \times Y$ by assigning to an object c the space $X(c) \times Y$. Taking $Y = [0, 1]$, it is now clear what a *homotopy of maps of $\mathcal{C}$-spaces* means. A map $f\colon X \longrightarrow Y$ of $\mathcal{C}$-spaces is a *cofibration (fibration) of $\mathcal{C}$-spaces* if it has the homotopy extension property (homotopy lifting property) for all $\mathcal{C}$-spaces. If f is a (co)-fibration of $\mathcal{C}$-spaces, its evaluation $f(c)\colon X(c) \longrightarrow Y(c)$ is a (co)-fibration of $\mathrm{aut}(c)$-spaces for all objects c in $\mathcal{C}$. The proof of this fact is a simple abstract manipulation of the homotopy lifting (extension) property and various adjunctions. Notice that the converse is not true.

Next we extend the notion of a CW-complex for spaces to $\mathcal{C}$-spaces. We will see that the notion of a free $\mathcal{C}$-CW-complex is very similar to the the notion of an ordinary CW-complex and that standard results and their proofs for CW-complexes generalize in a straightforward manner to the case of free $\mathcal{C}$-CW-complexes. This leads to easy proofs of known and new results whose strategy is very close to classical ideas and patterns.

DEFINITION 3.2. A *contravariant free $\mathcal{C}$-CW-complex* X is a contravariant $\mathcal{C}$-space X together with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset X = \bigcup_{n \geqslant 0} X_n$$

such that $X = \operatorname{colim}_{n\to\infty} X_n$ and for any $n \geqslant 0$ the *n-skeleton* X_n is obtained from the $(n-1)$-skeleton X_{n-1} by attaching free contravariant $\mathcal{C}$-n-cells, i.e. there exists a pushout of $\mathcal{C}$-spaces of the form

$$\begin{array}{ccc} \coprod_{i\in I_n} \operatorname{mor}_{\mathcal{C}}(-,c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} \operatorname{mor}_{\mathcal{C}}(-,c_i) \times D^{n} & \longrightarrow & X_n \end{array}$$

where the vertical maps are inclusions, I_n is an index set, and the c_i are objects of $\mathcal{C}$. We refer to the inclusion functor $\operatorname{mor}_{\mathcal{C}}(-,c_i) \times \operatorname{int} D^n \longrightarrow X$ as a free $\mathcal{C}$-n-cell based at c_i. A free $\mathcal{C}$-CW-complex has *dimension* $\leqslant n$ if $X = X_n$. The definition of a *covariant free $\mathcal{C}$-CW-complex* is analogous. □

Note that the trivial contravariant (covariant) $\mathcal{C}$-space which sends every object to a point is not in general a free $\mathcal{C}$-CW-complex unless $\mathcal{C}$ has a final (initial) object.

The more general notion of a $\mathcal{C}$-CW-complex was defined by Dror Farjoun [11, 1.16 and 2.1] (see also [30]). We shall deal almost exclusively with *free* $\mathcal{C}$-CW-complexes. For a free $\mathcal{C}$-CW-complex X, the *cellular chain complex* $C_*(X)(-)$, $c \mapsto C_*(X(c))$ is a $\mathcal{C}$-chain complex of free $\mathbb{Z}\mathcal{C}$-modules. Notice that a free $\mathcal{C}$-CW-complex X defines a functor from $\mathcal{C}$ to CW-COMPLEXES, but not any functor from $\mathcal{C}$ to CW-COMPLEXES is a free $\mathcal{C}$-CW-complex.

If Y is a G-CW-complex, then $\operatorname{map}_G(-,Y)$ (which sends $G/H \mapsto Y^H$) is an example of a free $\operatorname{Or}(G)$-CW-complex. A G-cell of Y of orbit type G/H corresponds to a $\operatorname{Or}(G)$-cell of $\operatorname{map}_G(-,Y)$ based at G/H. Recall that the category of $\operatorname{Or}(G,1)$-spaces coincides with the category of G-spaces. Under this identification a *free* $\operatorname{Or}(G,1)$-CW-complex is the same as a *free G-CW*-complex.

Recall that a map $f\colon X \longrightarrow Y$ of spaces is *n-connected* for $n \geqslant 0$ if and only if for all points x in X the induced map $\pi_k(f,x)\colon \pi_k(X,x) \longrightarrow \pi_k(Y,f(x))$ is bijective for all $k < n$ and surjective for $k = n$. It is a *weak homotopy equivalence* if it is n-connected for all $n \geqslant 0$.

DEFINITION 3.3. A map $f\colon X \longrightarrow Y$ of $\mathcal{C}$-spaces is *n-connected* (a *weak homotopy equivalence*) if for all objects c the map of spaces $f(c)\colon X(c) \longrightarrow Y(c)$ is n-connected (a weak homotopy equivalence). □

The constant map $EG \longrightarrow \{*\}$ is a weak homotopy equivalence, but not a homotopy equivalence of $\operatorname{Or}(G,1)$-spaces.

The following result is well-known for ordinary CW-complexes [43, IV. Theorem 7.16 and 7.17 on page 182]. See also [11, Proposition 2.9] and [30, Theorem 3.4].

THEOREM 3.4. *Let* $f: Y \longrightarrow Z$ *be a map of* $\mathcal{C}$*-spaces and* X *be a* $\mathcal{C}$*-space. The map on homotopy classes of maps between* $\mathcal{C}$*-spaces induced by composition with* f *is denoted by* $f_*: [X, Y]^{\mathcal{C}} \longrightarrow [X, Z]^{\mathcal{C}}$.

1. *Then* f *is* n*-connected if and only if* f_* *is bijective for any free* $\mathcal{C}$*-CW-complex* X *with* $\dim(X) < n$ *and surjective for any free* $\mathcal{C}$*-CW-complex* X *with* $\dim(X) \leqslant n$.
2. *Then* f *is a weak homotopy equivalence if and only if* f_* *is bijective for any free* $\mathcal{C}$*-CW-complex* X.

Proof. We only give the proof of the second assertion in the special case where Z is the trivial $\mathcal{C}$-space, i.e. $Z(c) = \{*\}$ for all objects c in $\mathcal{C}$. Then it is easy to figure out the full proof following the classical proof in [43, IV. Theorem 7.16 and 7.17 on page 182].

We begin with the 'if' statement. Suppose that $[X, Y]^{\mathcal{C}}$ consists of one element for each free $\mathcal{C}$-CW-complex X. We then choose $X = \mathrm{mor}_{\mathcal{C}}(-, c) \times S^k$, for a fixed $c \in \mathrm{Ob}(\mathcal{C})$. From Lemma 3.1 we obtain a natural homeomorphism

$$\hom_{\mathcal{C}}(\mathrm{mor}_{\mathcal{C}}(-, c) \times S^k, Y) \longrightarrow \mathrm{map}(S^k, Y(c))$$

and thus a natural bijection

$$[\mathrm{mor}_{\mathcal{C}}(-, c) \times S^k, Y]^{\mathcal{C}} \longrightarrow [S^k, Y(c)].$$

Hence for all objects c in $\mathcal{C}$ any map from S^k to $Y(c)$ is nullhomotopic. This implies that f is a weak homotopy equivalence.

Next we prove the 'only if' statement. Suppose that f is a weak homotopy equivalence. We must show for any free $\mathcal{C}$-CW-complex X that any map of $\mathcal{C}$-spaces $g: X \longrightarrow Y$ is nullhomotopic, or in other words, extends to the cone on X. The cone on X is obtained from X by attaching $\mathcal{C}$-cells. Therefore it suffices to show that any map of $\mathcal{C}$-spaces $\mathrm{mor}_{\mathcal{C}}(-, c) \times S^{n-1} \longrightarrow Y$ can be extended to a map $\mathrm{mor}_{\mathcal{C}}(-, c) \times D^n \longrightarrow Y$. Such a problem reduces to extending a map from S^{n-1} to $Y(c)$ to D^n. This can be done as $Y(c)$ has the weak homotopy type of a point by assumption. □

COROLLARY 3.5. *A weak homotopy equivalence between free* $\mathcal{C}$*-CW-complexes is a homotopy equivalence.*

Proof. Let $f: Y \longrightarrow X$ be a weak homotopy equivalence between free $\mathcal{C}$-CW-complexes. By Theorem 3.4, there is a $g: X \longrightarrow Y$ so that $f_*[g] = [f \circ g] = [\mathrm{id}_X]$. Thus g is a weak homotopy equivalence. To show that g is the homotopy inverse of f, we need only show that g has a right homotopy inverse, but this follows by Theorem 3.4 again. □

DEFINITION 3.6. Let (X, A) be a pair of $\mathcal{C}$-spaces. A $\mathcal{C}$-CW*-approximation*

$$(u, v): (X', A') \longrightarrow (X, A)$$

consists of a free $\mathcal{C}$-CW-pair (X', A') together with a map of pairs (u, v) of $\mathcal{C}$-spaces such that both u and v are weak homotopy equivalences of $\mathcal{C}$-spaces. A *$\mathcal{C}$-CW-approximation of a space X* is a $\mathcal{C}$-CW-approximation of the pair $(X, \emptyset)$. □

This is a categorical generalization of the notion of a CW-approximation for a topological space X (see [43, V.3]). By taking (f, g) to be the identity in Theorem 3.7 below we see that $\mathcal{C}$-CW-approximations exist and are unique up to homotopy.

THEOREM 3.7. *Let (X, A) be a pair of $\mathcal{C}$-spaces.*

(1) (*Existence*) *There exists a $\mathcal{C}$-CW-approximation of (X, A);*

(2) (*Uniqueness*) *Given a map of pairs $(f, g)\colon (X, A) \longrightarrow (Y, B)$ of $\mathcal{C}$-spaces and given $\mathcal{C}$-CW-approximations $(u, v)\colon (X', A') \longrightarrow (X, A)$ and $(a, b)\colon (Y', B') \longrightarrow (Y, B)$, then there exists a map of pairs (f', g') : $(X', A') \longrightarrow (Y', B')$ so that the diagram*

$$\begin{array}{ccc} (X', A') & \xrightarrow{(u,v)} & (X, A) \\ {\scriptstyle (f', g')}\downarrow & & \downarrow{\scriptstyle (f, g)} \\ (Y', B') & \xrightarrow{(a,b)} & (Y, B) \end{array}$$

commutes up to homotopy. Furthermore, the map (f', g') is unique up to homotopy.

Proof. Existence of a $\mathcal{C}$-CW-approximation is an inductive construction done by attaching n-cells to obtain a n-connected map and finally taking a colimit. Uniqueness follows from the relative versions of Theorem 3.4 and Corollary 3.5. □

DEFINITION 3.8. Let $E\mathcal{C}$ denote any free $\mathcal{C}$-CW-complex so that $E\mathcal{C}(c)$ is contractible for all objects c. □

Since $E\mathcal{C}$ is a $\mathcal{C}$-CW-approximation of the trivial $\mathcal{C}$-space, $E\mathcal{C}$ exists and is unique up to homotopy type. Note there is a contravariant $E\mathcal{C}$ and a covariant $E\mathcal{C}$. They are not closely related, but one can identify the contravariant $E\mathcal{C}$ with the covariant $E\mathcal{C}^{\mathrm{op}}$. There are functorial constructions of $\mathcal{C}$-CW-approximations and hence for $E\mathcal{C}$, which we describe at the end of this section. However, often it is useful to have smaller and more flexible models.

If $\mathcal{C} = \mathrm{Or}(G, 1)$, then $E\mathcal{C}$ can be identified with EG, a contractible free G-CW-complex. If $\mathcal{C}$ has a final object, then we may take the contravariant $E\mathcal{C}$ to be the trivial $\mathcal{C}$-space, which is a single $\mathcal{C}$-0-cell based at the final object. Similarly, if $\mathcal{C}$ has an initial object, the trivial $\mathcal{C}$-space is a covariant $E\mathcal{C}$. If G is a *crystallographic group*, i.e. a discrete subgroup of the isometries of $\mathbb{R}^n$ so that $\mathbb{R}^n/G$ is compact, then $(G/H \longmapsto (\mathbb{R}^n)^H)$ is a contravariant $E\mathrm{Or}(G, \mathcal{FIN})$, where $\mathcal{FIN}$ is the family of finite subgroups. More generally, if $E(G, \mathcal{F})$ is classifying space for a family of subgroups of a discrete group G, then $(G/H \longmapsto E(G, \mathcal{F})^H)$ is a model for $E\mathrm{Or}(G, \mathcal{F})$. This example is expanded on in Section 7.

EXAMPLE 3.9. Let ω be the category whose objects are the nonnegative integers and whose morphisms are given by the arrows below, their composites, and the identity maps.

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

Then we may take a contravariant $E\omega$ to be $(E\omega)(i) = [i, \infty)$, whose zero skeleton is obtained by intersecting each space with the integers. For each nonnegative integer i, there is $\mathcal{C}$-0-cell and a $\mathcal{C}$-1-cell based at i. We may take the covariant $E\omega$ to be the trivial $\mathcal{C}$-space. □

DEFINITION 3.10. The *classifying space of a category* $\mathcal{C}$ is the space $B\mathcal{C} = E\mathcal{C} \otimes_{\mathcal{C}} \{*\}$, where $\{*\}$ is the trivial $\mathcal{C}$-space and $E\mathcal{C}$ is a contravariant $\mathcal{C}$-CW-approximation of the trivial $\mathcal{C}$-space. □

The classifying space $B\mathcal{C}$ is a CW-complex defined only up to homotopy type. We will recall its functorial definition later in this section.

THEOREM 3.11. *Let $f: Y \longrightarrow Z$ be a weak homotopy equivalence of covariant $\mathcal{C}$-spaces. Then for any contravariant free $\mathcal{C}$-CW-complex X the induced map*

$$\mathrm{id}_X \otimes_{\mathcal{C}} f: X \otimes_{\mathcal{C}} Y \longrightarrow X \otimes_{\mathcal{C}} Z$$

is a weak homotopy equivalence. A similar statement holds for weak homotopy equivalences of contravariant $\mathcal{C}$-spaces.

Let X be a covariant (contravariant) free $\mathcal{C}$-CW-complex and $f: Y \longrightarrow Z$ be a weak homotopy equivalence of covariant (contravariant) $\mathcal{C}$-spaces. Then the induced map

$$\hom_{\mathcal{C}}(\mathrm{id}, f) : \hom_{\mathcal{C}}(X, Y) \longrightarrow \hom_{\mathcal{C}}(X, Z)$$

is a weak homotopy equivalence.

Proof. We will prove the claim by induction over the skeletons and the cells in X. We only consider the case $\mathrm{id}_X \otimes_{\mathcal{C}} f$. The functor $- \otimes_{\mathcal{C}} Y$ is compatible with colimits, using the standard trick from category theory that a functor with a right adjoint commutes with arbitrary colimits (see [24, Chapter V, Section 5]). Hence the pushout specifying how X_n is obtained from X_{n-1} by attaching cells remains a pushout after applying $- \otimes_{\mathcal{C}} Y$. Moreover, the left vertical arrow in this pushout is a cofibration and $\mathrm{id}_{X_n} \otimes_{\mathcal{C}} f$ is the pushout of three weak homotopy equivalences. Hence it is itself a weak homotopy equivalence by excision theorem of Blakers–Massey [43, VII.7]. Analogously one argues to show that the colimit of the maps $\mathrm{id}_{X_n} \otimes_{\mathcal{C}} f$ is $\mathrm{id}_X \otimes_{\mathcal{C}} f$ and each inclusion $X_n \otimes_{\mathcal{C}} Y \longrightarrow X_{n+1} \otimes_{\mathcal{C}} Y$ is a cofibration. This implies that $\mathrm{id}_X \otimes_{\mathcal{C}} f$ is a weak homotopy equivalence. The proof of the assertion for hom is similar. □

Next we give some definitions, which are in close analogy with group cohomology and homological algebra.

DEFINITION 3.12. Let M be a covariant $\mathbb{Z}\mathcal{C}$-module, X a covariant $\mathcal{C}$-space, and $\mathbf{E}$ a covariant $\mathcal{C}$-spectrum. Define the *colimit* and the *limit* of M over $\mathcal{C}$ to be the Abelian groups

$$\operatorname*{colim}_{\mathcal{C}} M = \mathbb{Z} \otimes_{\mathbb{Z}\mathcal{C}} M \quad \text{and} \quad \lim_{\mathcal{C}} M = \hom_{\mathbb{Z}\mathcal{C}}(\mathbb{Z}, M).$$

Define the *colimit of X over $\mathcal{C}$* and the *limit of X over $\mathcal{C}$* to be the topological spaces

$$\operatorname*{colim}_{\mathcal{C}} X = \{*\} \otimes_{\mathcal{C}} X \quad \text{and} \quad \lim_{\mathcal{C}} X = \hom_{\mathcal{C}}(\{*\}, X).$$

Define the *colimit of* $\mathbf{E}$ *over* $\mathcal{C}$ and the *limit of* $\mathbf{E}$ *over* $\mathcal{C}$ to be the spectra

$$\operatorname*{colim}_{\mathcal{C}} \mathbf{E} = \{*\} \otimes_{\mathcal{C}} \mathbf{E} \quad \text{and} \quad \lim_{\mathcal{C}} \mathbf{E} = \hom_{\mathcal{C}}(\{*\}, \mathbf{E}). \qquad \square$$

The above definitions are standard and the universal properties follow from the adjunctions in Lemmas 1.5 and 1.6. Here $\mathbb{Z}$ represents the trivial $\mathbb{Z}\mathcal{C}$-module, with $\mathbb{Z}(c) \equiv \mathbb{Z}$ and $\{*\}$ is the trivial $\mathcal{C}$-space. It is also convenient to define colimits and limits of contravariant functors over $\mathcal{C}$, by applying the above definitions to the functors considered as covariant functors on $\mathcal{C}^{\mathrm{op}}$. We next discuss the higher derived functors of the above limits.

DEFINITION 3.13. If M is a covariant $\mathbb{Z}\mathcal{C}$-module, define

$$H_i(\mathcal{C}; M) = H_i(C_*(E\mathcal{C}) \otimes_{\mathbb{Z}\mathcal{C}} M) \quad \text{and}$$
$$H^i(\mathcal{C}; M) = H^i(\mathrm{Hom}_{\mathbb{Z}\mathcal{C}}(C_*(E\mathcal{C}), M)).$$

If X is a covariant $\mathcal{C}$-space, define the *homotopy colimit* and the *homotopy limit* of X over $\mathcal{C}$ as

$$\operatorname*{hocolim}_{\mathcal{C}} X = E\mathcal{C} \otimes_{\mathcal{C}} X \quad \text{and} \quad \operatorname*{holim}_{\mathcal{C}} X = \hom_{\mathcal{C}}(E\mathcal{C}, X).$$

If $\mathbf{E}$ is a covariant $\mathcal{C}$-spectrum, define the *homotopy colimit* and the *homotopy limit* of $\mathbf{E}$ over $\mathcal{C}$ as

$$\operatorname*{hocolim}_{\mathcal{C}} \mathbf{E} = E\mathcal{C} \otimes_{\mathcal{C}} \mathbf{E} \quad \text{and} \quad \operatorname*{holim}_{\mathcal{C}} \mathbf{E} = \hom_{\mathcal{C}}(E\mathcal{C}, \mathbf{E}). \qquad \square$$

One must be careful about the variances on $E\mathcal{C}$ in the above definitions. In the left-hand appearances of $E\mathcal{C}$ we are taking the contravariant version, while on the right we want the covariant version. In the definition of the higher limits H^i and colimits H_i, the $\mathbb{Z}\mathcal{C}$-chain complex $C_*(E\mathcal{C})$ can be replaced by any projective $\mathbb{Z}\mathcal{C}$-resolution of $\mathbb{Z}$. As above we define homology, cohomology, hocolimits, and holimits of contravariant functors by considering them as functors defined on the opposite category. For properties of H_i and H^i, see, for example, [23] and for properties of homotopy limits see for instance [4], [12, §9] and [21]. One obtains functorial definitions if one uses the functorial construction $E^{\mathrm{bar}}\mathcal{C}$ for $E\mathcal{C}$. Since $E\mathcal{C}$ maps to $\{*\}$, there are maps $\operatorname{hocolim}_{\mathcal{C}} X \longrightarrow \operatorname{colim}_{\mathcal{C}} X$ and $\lim_{\mathcal{C}} X \longrightarrow \operatorname{holim}_{\mathcal{C}} X$. They are not, in general, weak homotopy equivalences, although the first map is if X is a free $\mathcal{C}$-CW-complex. The basic property of homotopy limits is that if $X \longrightarrow Y$ is a weak homotopy equivalence, then so are the induced maps $\mathrm{hocolim}_{\mathcal{C}} X \longrightarrow \mathrm{hocolim}_{\mathcal{C}} Y$ and $\mathrm{holim}_{\mathcal{C}} Y \longrightarrow \mathrm{holim}_{\mathcal{C}} X$; this follows from Theorem 3.11.

EXAMPLE 3.14. Let ω be the category from Example 3.9. Let M and N be covariant and contravariant $\mathbb{Z}\mathcal{C}$-modules, respectively. Then it is easy to see that $H_i(\omega; M)$ is $\mathrm{colim}_{j\to\infty} M(j)$ for $i = 0$ and zero for $i > 0$, that $H^i(\omega; M)$ is $M(0)$ for $i = 0$ and zero for $i > 0$, that $H_i(\omega; N)$ is $N(0)$ for $i = 0$ and zero for $i > 0$, and that $H^i(\omega; N)$ is $\lim_{j\to\infty} N(j)$ for $i = 0$, Milnor's $\lim^1_{j\to\infty} N(j)$ for $i = 1$, and zero for $i > 1$.

Let X and Y be covariant and contravariant $\mathcal{C}$-spaces, respectively. Then with the $E\omega$'s from Example 3.9 $\mathrm{hocolim}_{\omega}\ X$ is the infinite mapping telescope of

$$X(0) \longrightarrow X(1) \longrightarrow X(2) \longrightarrow X(3) \longrightarrow \cdots.$$

Clearly $\mathrm{holim}_{\omega}\ X = X(0)$ and $\mathrm{hocolim}_{\omega}\ Y = Y(0)$. Now $\mathrm{holim}_{\omega}\ Y$ is a bit bigger, it is the subspace of

$$\mathrm{map}([0,\infty), Y(0)) \times \mathrm{map}([1,\infty), Y(1)) \times \mathrm{map}([2,\infty), Y(2)) \times \cdots,$$

consisting of all tuples $(\gamma_0, \gamma_1, \gamma_2, \ldots)$ so that the composite of $[i,\infty) \xrightarrow{\gamma_i} Y(i) \longrightarrow Y(i-1)$ equals γ_{i-1} restricted to $[i,\infty)$. □

DEFINITION 3.15. Let X be a $\mathcal{C}$-space and M a $\mathbb{Z}\mathcal{C}$-module. Let $X' \longrightarrow X$ be a $\mathcal{C}$-CW-approximation. If X is contravariant and M is covariant, define

$$H_p^{\mathcal{C}}(X; M) = H_p(C_*(X') \otimes_{\mathbb{Z}\mathcal{C}} M),$$

where $C_*(X')$ is the cellular chain complex of X'. There is a similar definition if X is covariant and M is contravariant. If X and M have the same variance, define

$$H^p_{\mathcal{C}}(X; M) = H^p(\hom_{\mathbb{Z}\mathcal{C}}(C_*(X'), M)).$$ □

When $\mathcal{C} = \mathrm{Or}(G, 1)$, $H_p^{\mathcal{C}}(X; M)$ is Borel equivariant homology $H_p^G(X; M) = H_p(EG \times_G X; M)$. When $\mathcal{C} = \mathrm{Or}(G)$ and X is the the fixed point functor $G/H \mapsto Z^H$ of a G-CW-complex Z, then $H_p^{\mathcal{C}}(X; M)$ is Bredon equivariant homology of Z with coefficients in M.

Remark 3.16. One of the original motivations for Bredon's introduction of the orbit category was equivariant obstruction theory, and it is clear that all the ingredients are in place for the development of obstruction theory for the study of $\mathcal{C}$-maps between a free $\mathcal{C}$-CW-space and a $\mathcal{C}$-space, but we leave the task of finding the precise formulation to a reader motivated by specific applications. Local coefficient systems are particularly subtle, see [26]. □

Next we recall functorial constructions of classifying spaces and $\mathcal{C}$-CW-approximations (see for instance [4,21,34]). We will need some of the details later in Section 6. View the ordered set $[p] = \{0, 1, 2, \ldots, p\}$ as a category, namely, objects are the elements and there is precisely one morphism from i to j if $i \leqslant j$ and no morphism otherwise. Continuing with the terminology from Example 1.7, we get a covariant functor

$$[\ ]\colon \mathbf{\Delta} \longrightarrow \text{CATEGORIES}$$

from the category of finite ordered sets into the category of small categories. The *nerve* of a category $\mathcal{C}$ is the simplicial set

$$N.\mathcal{C}\colon \mathbf{\Delta} \longrightarrow \text{SETS}, \quad [p] \mapsto \text{func}([p], \mathcal{C}).$$

More explicitly, $N_p\mathcal{C}$ consists of diagrams in $\mathcal{C}$ of the form

$$c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} c_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{p-1}} c_p.$$

The *bar resolution model* $B^{\text{bar}}\mathcal{C}$ for the classifying space of a category $\mathcal{C}$ is the geometric realization $|N.\mathcal{C}|$ of its nerve where we regard a simplicial set as a simplicial space by using the discrete topology. It has the nice properties (see [34]) that it is functorial, that $B^{\text{bar}}(\mathcal{C} \times \mathcal{D}) = B^{\text{bar}}\mathcal{C} \times B^{\text{bar}}\mathcal{D}$, that $B^{\text{bar}}\mathcal{C} = B^{\text{bar}}(\mathcal{C}^{\text{op}})$, and that a natural transformation from a functor F_0 to a functor F_1 induces a homotopy between the maps $B^{\text{bar}}F_0$ and $B^{\text{bar}}F_1$ on the bar resolution models. In particular, an equivalence of categories gives a homotopy equivalence on the bar resolution models of the classifying spaces. From Example 1.7 we get that $B^{\text{bar}}\mathcal{C}$ comes with a canonical CW-complex structure such that there is a bijective correspondence between the set of sequences of composable morphisms

$$c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} c_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{p-1}} c_p$$

where no morphism is the identity and the set of p-cells. Any functor induces a cellular map. We will justify the term 'model of the classifying space' shortly.

Given two objects ? and ?? in $\mathcal{C}$, define the category $?\downarrow\mathcal{C}\downarrow??$ as follows. An object is a diagram $? \xrightarrow{\alpha} c \xrightarrow{\beta} ??$ in $\mathcal{C}$. A morphism from $? \xrightarrow{\alpha} c \xrightarrow{\beta} ??$ to $? \xrightarrow{\alpha'} c' \xrightarrow{\beta'} ??$ is a commutative diagram in $\mathcal{C}$ of the shape

$$\begin{array}{ccccc} ? & \xrightarrow{\alpha} & c & \xrightarrow{\beta} & ?? \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \text{id}} \\ ? & \xrightarrow{\alpha'} & c' & \xrightarrow{\beta'} & ?? \end{array}$$

Let $\overline{\text{mor}_{\mathcal{C}}(?, ??)}$ be the category whose set of objects is $\text{mor}_{\mathcal{C}}(?, ??)$ and whose only morphisms are the identity morphism of objects. Consider the functor

$$\text{pr}\colon ?\downarrow\mathcal{C}\downarrow?? \longrightarrow \overline{\text{mor}_{\mathcal{C}}(?, ??)} \quad (? \xrightarrow{\alpha} c \xrightarrow{\beta} ??) \mapsto (\beta \circ \alpha\colon ? \longrightarrow ??)\,.$$

LEMMA 3.17. *The map of contravariant $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-spaces*

$$B^{\mathrm{bar}}\mathrm{pr}\colon B^{\mathrm{bar}}\, ?\downarrow\mathcal{C}\downarrow ?? \longrightarrow B^{\mathrm{bar}}\overline{\mathrm{mor}_{\mathcal{C}}(?, ??)} = \mathrm{mor}_{\mathcal{C}}(?, ??)$$

is a $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-$CW$-approximation.

Proof. First we verify that $B^{\mathrm{bar}}\mathrm{pr}$ is a weak homotopy equivalence. Fix objects c, c' of $\mathcal{C}$. Define functors

$$j\colon \overline{\mathrm{mor}_{\mathcal{C}}(c, c')} \longrightarrow c\downarrow\mathcal{C}\downarrow c' \qquad \left(c \xrightarrow{\alpha} c'\right) \mapsto \left(c \xrightarrow{\mathrm{id}} c \xrightarrow{\alpha} c'\right).$$

$$\mathrm{pr}(c, c')\colon c\downarrow\mathcal{C}\downarrow c' \longrightarrow \overline{\mathrm{mor}_{\mathcal{C}}(c, c')} \quad (c \xrightarrow{\alpha} d \xrightarrow{\beta} c') \mapsto (\beta \circ \alpha\colon c \longrightarrow c').$$

These give homotopy equivalences after applying B^{bar}, since $\mathrm{pr}(c, c') \circ j$ is the identity and there is a natural transformation $S\colon j \circ \mathrm{pr}(c, c') \longrightarrow \mathrm{id}$ defined by assigning to an object $c \xrightarrow{\alpha} d \xrightarrow{\beta} c'$ in $c\downarrow\mathcal{C}\downarrow c'$ the morphism in $c\downarrow\mathcal{C}\downarrow c'$

$$\begin{array}{ccccc} c & \xrightarrow{\mathrm{id}} & c & \xrightarrow{\beta\circ\alpha} & c' \\ {\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\mathrm{id}}\downarrow \\ c & \xrightarrow{\alpha} & d & \xrightarrow{\beta} & c' \end{array}$$

We next show that $B^{\mathrm{bar}}\, ?\downarrow\mathcal{C}\downarrow ??$ is a free $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-$CW$-complex. The canonical skeletal filtration on the classifying space of a category induces a filtration on $B^{\mathrm{bar}}\, ?\downarrow\mathcal{C}\downarrow ??$ such that

$$B^{\mathrm{bar}}\, ?\downarrow\mathcal{C}\downarrow ?? = \mathrm{colim}_{p\to\infty} B^{\mathrm{bar}}_p\, ?\downarrow\mathcal{C}\downarrow ??\,.$$

Moreover, there is a pushout of contravariant $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$-spaces

$$\begin{array}{ccc} (n.d.N_p\, ?\downarrow\mathcal{C}\downarrow ??) \times S^{p-1} & \longrightarrow & B^{\mathrm{bar}}_{p-1}\, ?\downarrow\mathcal{C}\downarrow ?? \\ \downarrow & & \downarrow \\ (n.d.N_p\, ?\downarrow\mathcal{C}\downarrow ??) \times D^{p} & \longrightarrow & B^{\mathrm{bar}}_{p}\, ?\downarrow\mathcal{C}\downarrow ?? \end{array}$$

where $n.d.N_p\, ?\downarrow\mathcal{C}\downarrow ??$ is the set of nondegenerate p-simplices of the nerve of $?\downarrow\mathcal{C}\downarrow ??$. This set can be identified with the disjoint union of the $\mathcal{C}$-$\mathcal{C}$-sets $\mathrm{mor}_{\mathcal{C}}(?, c_0) \times \mathrm{mor}_{\mathcal{C}}(c_p, ??)$ where the disjoint union runs over the sequences

$$c_0 \xrightarrow{\phi_0} c_1 \xrightarrow{\phi_1} c_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{p-1}} c_p$$

where no morphism ϕ_i is the identity. Such sequences thus give the indexing set for the p-cells. □

From Example 1.7 we get that for any $\mathcal{C}$-space X, there is a weak homotopy equivalence of $\mathcal{C}$-spaces

$$t\colon |S.X| \longrightarrow X.$$

such that $|S.X|$ is functor from $\mathcal{C}$ to CW-COMPLEXES. Notice that this does not mean that $|S.X|$ itself is a free $\mathcal{C}$-CW-complex.

DEFINITION 3.18. Let X be a contravariant $\mathcal{C}$-space. The tensor product taking over the variable ?? yields contravariant $\mathcal{C}$-spaces $X \otimes_{\mathcal{C}} B^{\text{bar}}\, ?\downarrow\mathcal{C}\downarrow ??$ and $X \otimes_{\mathcal{C}} \text{mor}_{\mathcal{C}}(?, ??)$. Define a map of contravariant $\mathcal{C}$-spaces

$$p_X\colon X \otimes_{\mathcal{C}} B^{\text{bar}}\, ?\downarrow\mathcal{C}\downarrow ?? \xrightarrow{\text{id}\otimes_{\mathcal{C}} B^{\text{bar}}\text{pr}} X \otimes_{\mathcal{C}} \text{mor}_{\mathcal{C}}(?, ??) \xrightarrow{\cong} X$$

where the second map is the canonical isomorphism given by $x \otimes \phi \mapsto X(\phi)(x)$. Define a map of contravariant $\mathcal{C}$-spaces

$$a_X\colon |S.X| \otimes_{\mathcal{C}} B^{\text{bar}}\, ?\downarrow\mathcal{C}\downarrow ?? \xrightarrow{t\otimes_{\mathcal{C}}\text{id}} X \otimes_{\mathcal{C}} B^{\text{bar}}\, ?\downarrow\mathcal{C}\downarrow ?? \xrightarrow{p_X} X. \qquad \square$$

LEMMA 3.19. *Let X be a contravariant $\mathcal{C}$-space. Then:*

(1) *p_X is a weak homotopy equivalence of contravariant $\mathcal{C}$-spaces, i.e. $p_X(c)$ is a weak equivalence of spaces for all objects c in $\mathcal{C}$.*

(2) *Suppose that X is a contravariant functor from $\mathcal{C}$ to CW-COMPLEXES, i.e. for each object c in $\mathcal{C}$ there is a CW-structure on $X(c)$ and each morphism $f\colon c \longrightarrow c'$ in $\mathcal{C}$ induces a cellular map $X(f)\colon X(c') \longrightarrow X(c)$. Suppose Y is a contravariant free $\mathcal{D} \times \mathcal{C}^{\text{op}}$-$CW$-complex. Then the contravariant $\mathcal{D}$-space $X \otimes_{\mathcal{C}} Y$ inherits the structure of a free $\mathcal{D}$-CW-complex.*

(3) *The map $a_X\colon |S.X| \otimes_{\mathcal{C}} B^{\text{bar}}\, ?\downarrow\mathcal{C}\downarrow ?? \longrightarrow X$ is a $\mathcal{C}$-CW-approximation.*

Proof. (1) Fix an object c in $\mathcal{C}$. Then

$$B\text{pr}(c, ??)\colon B^{\text{bar}}\, c\downarrow\mathcal{C}\downarrow ?? \longrightarrow \text{mor}_{\mathcal{C}}(c, ??)$$

is a weak homotopy equivalence of free $\mathcal{C}$-CW-complexes, hence is a $\mathcal{C}$-homotopy equivalence. Thus $p_X(c)$ is a homotopy equivalence.

(2) We will only indicate what the skeleta and cells are. The p-skeleton of $X \otimes_{\mathcal{C}} Y$ is $\cup_{i+j=p} X_i \otimes_{\mathcal{C}} Y_j$. A free $\mathcal{D}\times\mathcal{C}^{\text{op}}$-$j$-cell of Y based at (d, c) together with a i-cell of $X(c)$ gives rise to a free $\mathcal{D}$-$i+j$-cell based at d. More precisely, if $\Phi\colon D^i \longrightarrow X(c)$ is a characteristic map for a i-cell of $X(c)$ and if $\Psi\colon \text{mor}_{\mathcal{D}}(?, d) \times \text{mor}_{\mathcal{C}}(c, ??) \times D^j \longrightarrow Y$ is a characteristic map for a free $\mathcal{D} \times \mathcal{C}^{\text{op}}$-$j$-cell of Y based at (d, c), then the characteristic map

$$\text{mor}_{\mathcal{D}}(?, d) \times D^i \times D^j \longrightarrow X \otimes_{\mathcal{C}} Y$$

is given by

$$(f, a, b) \mapsto [\Phi(a), \Psi(f, \text{id}_c, b)].$$

(3) Follows from Lemma 3.17, (1), (2) above, and Theorem 3.11. □

If one takes $X = \{*\}$ in the construction above, one obtains the contravariant *bar $\mathcal{C}$-CW-approximation* of $\{*\}$

$$E^{\text{bar}}\mathcal{C} \ := \ \{*\} \otimes_{\mathcal{C}} B^{\text{bar}} ?\downarrow\mathcal{C}\downarrow ??\,.$$

More explicitly it is given as follows. For an object ? in $\mathcal{C}$ let $?\downarrow\mathcal{C}$ be the *category of objects under* ?. An object in $?\downarrow\mathcal{C}$ is a morphism $\phi\colon ? \longrightarrow c$ in $\mathcal{C}$ with ? as source. A morphism in $?\downarrow\mathcal{C}$ from $\phi_0\colon ? \longrightarrow c_0$ to $\phi_1\colon ? \longrightarrow c_1$ is given by a morphism $h\colon c_0 \longrightarrow c_1$ in $\mathcal{C}$ satisfying $\phi_1 = h \circ \phi_0$. A morphism $\psi\colon c \longrightarrow d$ in $\mathcal{C}$ defines a functor $\psi \downarrow \mathcal{C}\colon d \downarrow \mathcal{C} \longrightarrow c \downarrow \mathcal{C}$ by composition with ψ from the right. Then

$$E^{\text{bar}}\mathcal{C}\colon \mathcal{C}\colon \longrightarrow \text{SPACES}, \quad c \mapsto B^{\text{bar}}\, c\downarrow\mathcal{C}\,.$$

One easily checks that $E^{\text{bar}}\mathcal{C} \otimes_{\mathcal{C}} \{*\} = B^{\text{bar}}\mathcal{C}$ and thereby justifies our notation.

4. (Co-)Homology Associated to Spectra over a Category

In this section we introduce the homology and cohomology theories associated to a spectrum over a category. We then explain a kind of Atiyah–Hirzebruch type spectral sequence.

DEFINITION 4.1. Let (X, A) be a pair of pointed $\mathcal{C}$-spaces. Denote the reduced cone of the pointed space A by $\text{cone}(A)$. For a $\mathcal{C}$-spectrum $\mathbf{E}$ of the opposite variance as (X, A) define

$$\mathbf{E}_p^{\mathcal{C}}(X, A) = \pi_p(X \cup_A \text{cone}(A) \otimes_{\mathcal{C}} \mathbf{E}).$$

For a $\mathcal{C}$-spectrum $\mathbf{E}$ of the same variance as (X, A), define

$$\mathbf{E}^p_{\mathcal{C}}(X, A) = \pi_{-p}(\text{hom}_{\mathcal{C}}(X \cup_A \text{cone}(A), \mathbf{E})).$$

If A is just a point, we omit A from the notation. □

If $\mathcal{C}$ is the trivial category consisting of precisely one object and one morphism, then the homology and cohomology as defined in Definition 4.1 reduces to the classical definition of the reduced homology and cohomology of a pair with coefficients in a spectrum. This is obvious for homology whereas for cohomology one uses the natural bijection induced by the adjunction

$$\pi_{p+k}(\text{map}(X, E(k))) \ \longrightarrow [X \wedge S^{p+k}, E(k)].$$

Notice that writing homology and cohomology in terms of tensor product and mapping space spectra is analogous to the definition of the homology and cohomology of a chain complex C_* with coefficients in a module M as the homology of $C_* \otimes M$, respectively, $\text{Hom}(C_*, M)$.

LEMMA 4.2. *The homology and cohomology groups defined in Definition 4.1 are generalized reduced homology and cohomology theories for pointed $\mathcal{C}$-spaces.*

Proof. The proof is exactly as in the case of spaces, i.e. where $\mathcal{C}$ is the trivial category. For instance, let us check the long cohomology sequence of a pair (X, A) of pointed $\mathcal{C}$-spaces. The following diagram is a pushout

$$\begin{array}{ccc} A & \xrightarrow{i} & X \cup_A (A \wedge [0,1]_+) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \{*\} & \xrightarrow{j} & X \cup_A \operatorname{cone} A \end{array}$$

where i is the cofibration given by the inclusion and p and q are the projections. The functor $\hom_{\mathcal{C}}(-, Y)$ for a fixed pointed covariant $\mathcal{C}$-space Y has a left adjoint, namely $- \otimes_{\mathcal{C}} Y$. Hence the following diagram is a pullback and $\hom_{\mathcal{C}}(i, \mathrm{id}_{E(n)})$ is a fibration for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc} \hom_{\mathcal{C}}(X \cup_A \operatorname{cone}(A), E(n)) & \xrightarrow{\hom_{\mathcal{C}}(q, \mathrm{id}_{E(n)})} & \hom_{\mathcal{C}}(X \cup_A (A \wedge [0,1]_+), E(n)) \\ {\scriptstyle \hom_{\mathcal{C}}(j, \mathrm{id}_{E(n)})}\downarrow & & \downarrow{\scriptstyle \hom_{\mathcal{C}}(i, \mathrm{id}_{E(n)})} \\ \hom_{\mathcal{C}}(\{*\}, E(n))V & \xrightarrow{\hom_{\mathcal{C}}(p, \mathrm{id}_{E(n)})} & \hom_{\mathcal{C}}(A, E(n)) \end{array}$$

Hence we get for $n \in \mathbb{Z}$ fibrations of pointed spaces

$$\hom_{\mathcal{C}}(X \cup_A \operatorname{cone}(A), E(n)) \xrightarrow{\hom_{\mathcal{C}}(q, \mathrm{id}_{E(n)})} \hom_{\mathcal{C}}(X \cup_A (A \wedge [0,1]_+), E(n)) \xrightarrow{\hom_{\mathcal{C}}(i, \mathrm{id}_{E(n)})} \hom_{\mathcal{C}}(A, E(n)).$$

They are compatible with the structure maps. Now the colimit over their long homotopy sequences yields the desired long cohomology sequence of the pair since the canonical projection from $X \cup_A (A \wedge [0,1]_+)$ to X is a homotopy equivalence of pointed $\mathcal{C}$-spaces.

The suspension isomorphism is induced by the following identifications:

$$\begin{aligned} &\pi_{p+k}(\hom_{\mathcal{C}}(X \wedge S^1, E(k))) = \pi_{p+k}(\operatorname{map}(S^1, \hom_{\mathcal{C}}(X, E(k)))) \\ &\quad = \pi_{p+k}(\Omega \hom_{\mathcal{C}}(X, E(k))) = \pi_{p+k+1}(\hom_{\mathcal{C}}(X, E(k))). \end{aligned}$$ □

Recall that a weak homotopy equivalence of $\mathcal{C}$-spaces is a $\mathcal{C}$-map $X \longrightarrow Y$ so that $X(c) \longrightarrow Y(c)$ is a weak homotopy equivalence for all objects $c \in \mathrm{Ob}(\mathcal{C})$. The WHE-axiom says that a weak homotopy equivalence $f\colon X \longrightarrow Y$ of pointed spaces induces isomorphisms on homology, resp. cohomology. This is not necessarily satisfied for $\mathbf{E}_p^{\mathcal{C}}$ and $\mathbf{E}_{\mathcal{C}}^p$ as the following example shows. Let G be a group and $\mathcal{C} = \mathrm{Or}(G, 1)$. Recall that a contravariant pointed $\mathrm{Or}(G, 1)$-space is a space with a base point preserving right G-action. Let $\mathbf{E}$ be the ordinary Eilenberg–MacLane spectrum with $\pi_0(\mathbf{E}) = \mathbb{Z}$, considered as a covariant $\mathrm{Or}(G, 1)$-spectrum by the trivial G-action. The projection $p\colon EG_+ \longrightarrow \{*\}_+$ is a weak homotopy equivalence of pointed $\mathrm{Or}(G, 1)$-spaces. We get

$$\mathbf{E}_q^{\mathrm{Or}(G,1)}(EG_+) = H_q(BG) \quad \text{and} \quad \mathbf{E}_q^{\mathrm{Or}(G,1)}(\{*\}_+) = H_q(\{*\}),$$

where H_* is ordinary homology. Obviously these two groups do not coincide in general.

Our goal is to get unreduced homology and cohomology theories for (unpointed) $\mathcal{C}$-spaces which satisfy both the disjoint union axiom and the WHE-axiom. To be more precise, a homology theory means that homotopic maps of pairs of $\mathcal{C}$-spaces induce the same maps on the homology groups, that there are long exact sequences of pairs (X, A), and for a pushout of $\mathcal{C}$-spaces

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

the map $(j_2, i_1)\colon (X_2, X_0) \longrightarrow (X, X_1)$ induces an isomorphism on homology provided that $i_2\colon X_0 \longrightarrow X_2$ is a cofibration of $\mathcal{C}$-spaces. If the homology theory satisfies the WHE-axiom, it suffices to require that for each object c the map $i_2(c)\colon X_0(c) \longrightarrow X_2(c)$ is a cofibration of spaces. The disjoint union axiom says that for an arbitrary disjoint union the obvious map from the direct sum of the homology groups of the various summands to the homology of the disjoint union is an isomorphism. (For cohomology the direct sum has to be substituted by the direct product and the map goes the other way round.) For this purpose we need $\mathcal{C}$-CW-approximations (Definition 3.6) in order to generalize the usual procedure for spaces to $\mathcal{C}$-spaces (cf. [38, 7.68]).

DEFINITION 4.3. Let (X, A) be a pair of $\mathcal{C}$-spaces. Let $(u, v)\colon (X', A') \longrightarrow (X, A)$ be a $\mathcal{C}$-CW-approximation. For a $\mathcal{C}$-spectrum $\mathbf{E}$ of the opposite variance as (X, A), define the *homology of* (X, A) *with coefficients in* $\mathbf{E}$ by

$$H_p^{\mathcal{C}}(X, A; \mathbf{E}) = \mathbf{E}_p^{\mathcal{C}}(X'_+, A'_+)$$

and

$$H_p^{\mathcal{C}}(X; \mathbf{E}) = H_p^{\mathcal{C}}(X, \emptyset; \mathbf{E}).$$

Given a $\mathcal{C}$-spectrum $\mathbf{E}$ of the same variance as (X, A), define the *cohomology of* (X, A) *with coefficients in* $\mathbf{E}$ by

$$H^p_{\mathcal{C}}(X, A; \mathbf{E}) = \mathbf{E}^p_{\mathcal{C}}(X'_+, A'_+)$$

and

$$H^p_{\mathcal{C}}(X; \mathbf{E}) = H^p_{\mathcal{C}}(X, \emptyset; \mathbf{E}). \qquad \square$$

The above homology and cohomology are well-defined by the existence and uniqueness of $\mathcal{C}$-CW-approximations. Furthermore, by Theorem 3.4, given a map of pairs of $\mathcal{C}$-spaces $(X, A) \longrightarrow (Y, B)$, there is an induced map of their $\mathcal{C}$-CW-approximations which is uniquely up to homotopy determined by the property that the following diagram commutes up to homotopy

$$\begin{array}{ccc} (X', A') & \longrightarrow & (X, A) \\ \downarrow & & \downarrow \\ (Y', B') & \longrightarrow & (Y, B) \end{array}$$

Thus for a map of $\mathcal{C}$-pairs, there are corresponding maps of homology and cohomology groups. We always have natural maps

$$H_p^{\mathcal{C}}(X, A; \mathbf{E}) \longrightarrow \mathbf{E}_p^{\mathcal{C}}(X, A)$$

and

$$\mathbf{E}_{\mathcal{C}}^p(X, A) \longrightarrow H_{\mathcal{C}}^p(X, A; \mathbf{E}).$$

They are isomorphisms if (X, A) is a free $\mathcal{C}$-CW-pair, but not in general.

LEMMA 4.4. *$H_p^{\mathcal{C}}(X, A; \mathbf{E})$ and $H_{\mathcal{C}}^p(X, A; \mathbf{E})$ are unreduced homology and cohomology theories on pairs of $\mathcal{C}$-spaces which satisfy the WHE-axiom. The homology theory satisfies the disjoint union axiom. The cohomology theory satisfies the disjoint union axiom provided that $\mathbf{E}$ is a $\mathcal{C}$-Ω-spectrum.*

Proof. The first claim follows from Lemma 4.2 and Theorem 3.4.

The homology theory satisfies the disjoint union axiom for finite disjoint unions. We get the disjoint union axiom for arbitrary coproducts, if we show that the homology theory commutes with arbitrary colimits. This follows from the fact that the functor $- \otimes_{\mathcal{C}} E(k)$ has a right adjoint and commutes therefore with arbitrary colimits and that two colimits of systems of Abelian groups commute.

To check the disjoint union axiom for the cohomology theory, it suffices to do this for a disjoint union $\coprod_{i \in I} X_i$ of free $\mathcal{C}$-CW-complexes. We conclude from Theorem 3.11 for any free $\mathcal{C}$-CW-complex Y that $\hom_{\mathcal{C}}(Y, \mathbf{E})$ is a Ω-spectrum since $\mathbf{E}$ is a $\mathcal{C}$-Ω-spectrum and hence

$$\pi_p(\hom_{\mathcal{C}}(Y, \mathbf{E})) = \pi_{p+k}(\hom_{\mathcal{C}}(Y, E(k))),$$

provided $p + k \geqslant 0$. Now the claim follows from the adjunction homeomorphism

$$\hom_{\mathcal{C}}\left(\left(\coprod_{i \in I} X_i\right)_+, E(k)\right) \xrightarrow{\cong} \prod_{i \in I} \hom_{\mathcal{C}}((X_i)_+, E(k)). \qquad \square$$

Notice that without the condition that $\mathbf{E}$ is a $\mathcal{C}$-Ω-spectrum the associated cohomology theory does not have to satisfy the disjoint union axiom.

LEMMA 4.5. *Let X be a $\mathcal{C}$-space with a filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

such that $X = \mathrm{colim}_{n\to\infty} X_n$. Let $\mathbf{E}$ be a $\mathcal{C}$-spectrum with the opposite, respectively the same, variance as X.

(1) *The natural map*

$$\mathrm{colim}_{n\to\infty} H_p^{\mathcal{C}}(X_n; \mathbf{E}) \longrightarrow H_p^{\mathcal{C}}(X; \mathbf{E})$$

is an isomorphism for $p \in \mathbb{Z}$.

(2) *Let $\mathbf{E}$ be a $\mathcal{C}$-Ω-spectrum. There is a natural exact sequence*

$$0 \to \lim^1_{n\to\infty} H^{p-1}_{\mathcal{C}}(X_n; \mathbf{E}) \to H^p_{\mathcal{C}}(X; \mathbf{E}) \to \lim_{n\to\infty} H^p_{\mathcal{C}}(X_n; \mathbf{E}) \to 0$$

for all $p \in \mathbb{Z}$.

Proof. The proof is exactly as in the case where $\mathcal{C}$ is the trivial category which is due to Milnor and can be found for instance in [38, 7.53,7.66,7.73] or [43, Theorem XIII.1.1 on page 604 and Theorem XIII.1.3 on page 605]. □

Lemmas 4.4 and 4.5 imply

LEMMA 4.6. *Let $\mathbf{E}$ and $\mathbf{F}$ be $\mathcal{C}$-spectra and $\mathbf{f}\colon \mathbf{E} \longrightarrow \mathbf{F}$ be a (strong) map of $\mathcal{C}$-spectra. It induces a natural transformation*

$$\mathbf{f}_*\colon H_*^{\mathcal{C}}(X; \mathbf{E}) \longrightarrow H_*^{\mathcal{C}}(X; \mathbf{F}).$$

If $\mathbf{f}$ is a weak equivalence, then $\mathbf{f}_*$ is an isomorphism. The analogous statement holds for cohomology provided that $\mathbf{E}$ and $\mathbf{F}$ are $\mathcal{C}$-Ω-spectra. □

Any cohomology theory on the category of CW-complexes satisfying the disjoint union axiom can be represented by a Ω-spectrum. This is a consequence of Brown's representation theorem and proven for instance in [38, Chapter 9]. The proof goes through with some obvious modifications also in the case of free $\mathcal{C}$-CW-complexes. This does not contradict the remark in [11, 5.8] since in our setting we allow for free $\mathcal{C}$-CW-complexes only cells of the type $\mathrm{mor}(-, c)$ and the objects of $\mathcal{C}$ form a set by assumption whereas in [11] all homotopy types of orbits can occur and these homotopy types do not form a set.

Finally, we remark that a filtration of X gives a spectral sequence.

THEOREM 4.7. *Let X be a contravariant $\mathcal{C}$-space with a filtration*

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

such that $X = \mathrm{colim}_{n\to\infty} X_n$.

(1) *Let $\mathbf{E}$ be a covariant $\mathcal{C}$-spectrum $\mathbf{E}$. Then there is a spectral (homology) sequence $E^r_{p,q}, d^r_{p,q}\colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ whose E_1-term is given by*

$$E^1_{p,q} = H^{\mathcal{C}}_{p+q}(X_p, X_{p-1}; \mathbf{E})$$

and the first differential is the composition

$$d^1_{p,q}: E^1_{p,q} = H^{\mathcal{C}}_{p+q}(X_p, X_{p-1}, \mathbf{E}) \longrightarrow H^{\mathcal{C}}_{p+q-1}(X_{p-1}, \mathbf{E}) \longrightarrow H^{\mathcal{C}}_{p+q-1}(X_{p-1}, X_{p-2}; \mathbf{E}) = E^1_{p-1,q}$$

where the first map is the boundary operator of the pair (X_p, X_{p-1}) *and the second induced by the inclusion. The* E^∞*-term is given by*

$$E^\infty_{p,q} = \operatorname{colim}_{r\to\infty} E^r_{p,q}.$$

This spectral sequence converges to $H^{\mathcal{C}}_{p+q}(X; \mathbf{E})$*, i.e. there is an ascending filtration* $F_{p,m-p}H^{\mathcal{C}}_m(X, \mathbf{E})$ *of* $H^{\mathcal{C}}_m(X, \mathbf{E})$ *such that*

$$F_{p,q}H^{\mathcal{C}}_{p+q}(X, \mathbf{E})/F_{p-1,q+1}H^{\mathcal{C}}_{p+q}(X, \mathbf{E}) \cong E^\infty_{p,q}.$$

(2) *Let* $\mathbf{E}$ *be a contravariant* $\mathcal{C}$*-*Ω*-spectrum. Then there is a spectral (cohomology) sequence* $E^{p,q}_r$, $d^{p,q}_r: E^r_{p,q} \longrightarrow E^r_{p+r,q-r+1}$ *whose* E^1*-term is given by*

$$E^{p,q}_1 = H^{p+q}_{\mathcal{C}}(X_p, X_{p-1}; \mathbf{E})$$

and the first differential is the composition

$$d^1_{p,q}: E^1_{p,q} = H^{p+q}_{\mathcal{C}}(X_p, X_{p-1}, \mathbf{E}) \longrightarrow H^{p+q}_{\mathcal{C}}(X_p, \mathbf{E}) \longrightarrow H^{p+q+1}_{\mathcal{C}}(X_{p+1}, X_p; \mathbf{E}) = E^1_{p+1,q}$$

where the first map is induced by the inclusion and the second is the boundary operator of the pair (X_{p+1}, X_p)*. The* E^∞*-term is given by*

$$E^\infty_{p,q} = \lim_{r\to\infty} E^r_{p,q}.$$

There is a descending filtration $F^{p,m-p}\lim_{n\to\infty} H^m_{\mathcal{C}}(X_n; \mathbf{E})$ *of* $\lim_{n\to\infty} H^m_{\mathcal{C}}(X_n; \mathbf{E})$ *such that there is an exact sequence*

$$0 \longrightarrow F^{p,q}\lim_{n\to\infty} H^{p+q}_{\mathcal{C}}(X_n; \mathbf{E})/F^{p+1,q-1}\lim_{n\to\infty} H^{p+q}_{\mathcal{C}}(X_n; \mathbf{E}) \longrightarrow E^{p,q}_\infty \longrightarrow \lim^1_{m\to\infty} H^{p+q}_{\mathcal{C}}(X_{p+m}, X_p; \mathbf{E}) \longrightarrow \lim^1_{m\to\infty} H^{p+q}_{\mathcal{C}}(X_{p+m}, X_{p-1}; \mathbf{E}).$$

If one of the following conditions is satisfied

(a) *The filtration is finite, i.e. there is* $n \geqslant -1$ *such that* $X = X_n$*;*
(b) *The inclusion of* X_p *into* X_{p+1} *is* p*-connected for* $p \in \mathbb{Z}$ *and there is* $m \in \mathbb{Z}$ *such that* $\pi_q(\mathbf{E}(C))$ *vanishes for all objects* $c \in \mathrm{Ob}(\mathcal{C})$ *and* $q > m$*;*

then the spectral sequence converges to $H^{p+q}_{\mathcal{C}}(X; \mathbf{E})$*, i.e. there is a descending filtration* $F^{p,m-p}H^m_{\mathcal{C}}(X, \mathbf{E})$ *of* $H^m_{\mathcal{C}}(X, \mathbf{E})$ *such that*

$$F^{p,q}H^{p+q}_{\mathcal{C}}(X; \mathbf{E})/F^{p+1,q-1}H^{p+q}_{\mathcal{C}}(X; \mathbf{E}) \cong E^{p,q}_\infty.$$

Proof. Again this is a variation of the case where $\mathcal{C}$ is the trivial category (see [38, 7.75,15.6 and Remark 3 on page 352]) or [43, Theorem XIII.3.2. on page 614 and Theorem XIII.3.6. on page 616]. □

Suppose in Theorem 4.7 that X is a free $\mathcal{C}$-CW-complex and X_n its n-skeleton. Then the E^2-term, respectively E_2-term, of the spectral sequence in Theorem 4.7 can be identified with

$$E^2_{p,q} = H^{\mathcal{C}}_p(X; H^{\mathcal{C}}_q(\{*\}; \mathbf{E})) = H^{\mathcal{C}}_p(X; \pi_q(\mathbf{E})),$$

respectively

$$E_2^{p,q} = H^p_{\mathcal{C}}(X; H^q_{\mathcal{C}}(\{*\}; \mathbf{E})) = H^p_{\mathcal{C}}(X; \pi_{-q}(\mathbf{E})).$$

One gets the same spectral sequence as in Theorem 4.7 if one takes a dual point of view. Namely, one does not filter X by its skeleta, but uses a Postnikov decomposition of **E**. The Atiyah–Hirzebruch spectral sequence [38, 15.7] is a special case of Theorem 4.7. Quinn's spectral sequence [32, Theorem 8.7] coincides with Theorem 4.7 when the stratified system of fibrations is given by a group action.

Taking $X = E\mathcal{C}$, filtering by skeleta, and identifying the E^2 and E^∞-terms, one gets the *homotopy colimit spectral sequence*

$$H_p(\mathcal{C}; \pi_q(\mathbf{E})) \Longrightarrow \pi_{p+q}(\operatorname{hocolim}_{\mathcal{C}} \mathbf{E})$$

and the *homotopy limit spectral sequence*

$$H^p(\mathcal{C}; \pi_{-q}(\mathbf{E})) \Longrightarrow \pi_{-p-q}(\operatorname{holim}_{\mathcal{C}} \mathbf{E})$$

analogous to those of Bousfield–Kan [4, XII, 5.7 on page 339 and XI, 7.1 on page 309].

5. Assembly Maps and Isomorphism Conjectures

In this section we give three equivalent definitions of assembly maps, each of which corresponds to a certain point of view. Then we explain the Isomorphism Conjectures for the three $\mathrm{Or}(G)$-spectra introduced in Section 2. We will define assembly maps given the following data: a (discrete) group G, a non–empty family of subgroups $\mathcal{F}$, closed under inclusion and conjugation, and a covariant $\mathrm{Or}(G)$-spectrum **E**.

5.1. Assembly by extension from homogeneous spaces to G-spaces

Let **E** be a covariant $\mathrm{Or}(G)$-spectrum. We define an extension of **E** to the category of G-spaces by

$$\mathbf{E}_{\%}\colon G\text{-SPACES} \longrightarrow \text{SPECTRA} \quad X \mapsto \operatorname{map}_G(-, X)_+ \otimes_{\mathrm{Or}(G)} \mathbf{E}.$$

Recall that

$$\mathrm{map}_G(-, X)_+ \otimes_{\mathrm{Or}(G)} \mathbf{E} = \coprod_{H \subset G} X_+^H \wedge \mathbf{E}(G/H)/\sim,$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for $x \in X_+^K$, $y \in \mathbf{E}(G/H)$ and $\phi\colon G/H \longrightarrow G/K$. This construction is functorial in $\mathbf{E}$, i.e. a map of $\mathrm{Or}(G)$-spectra $\mathbf{T}\colon \mathbf{E} \longrightarrow \mathbf{F}$ induces a map of G-SPACES-spectra $\mathbf{T}_\%\colon \mathbf{E}_\% \longrightarrow \mathbf{F}_\%$.

Let $E(G, \mathcal{F})$ be a classifying space of G with respect to a family $\mathcal{F}$ (see [5] or [10]), i.e. a G-CW-complex such that the H-fixed point set is contractible if $H \in \mathcal{F}$ and empty otherwise. Such classifying spaces were introduced by tom Dieck [9, 10] and are unique up to G-homotopy type. We will give another point of view on these spaces in Section 7. The projection induces a map

$$\mathbf{E}_\%(\mathrm{pr})\colon \mathbf{E}_\%(E(G, \mathcal{F})) \longrightarrow \mathbf{E}_\%(G/G) = \mathbf{E}(G/G)$$

which is called *assembly map*. The map $\pi_*(\mathbf{E}_\%(\mathrm{pr}))$ is the $(\mathbf{E}, \mathcal{F}, G)$-assembly map referred to in the introduction. □

5.2. ASSEMBLY AS HOMOTOPY COLIMIT

We first discuss the behavior of homotopy limits under change of category. Consider a covariant functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$. We introduced F_*X in Definition 1.8. Since $E\mathcal{C}$ is a free $\mathcal{C}$-CW-complex, we can apply Theorem 3.4 to the weak homotopy equivalence of $\mathcal{C}$-spaces $F^*E\mathcal{D} \longrightarrow \{*\}$, and get a $\mathcal{C}$-map $E\mathcal{C} \longrightarrow F^*E\mathcal{D}$, which is unique up to homotopy. It induces a map of $\mathcal{D}$-spaces $f\colon F_*E\mathcal{C} \longrightarrow E\mathcal{D}$ by Lemma 1.9. Let X be a covariant $\mathcal{D}$-space. Then the *assembly map*

$$F_*\colon \operatorname*{hocolim}_{\mathcal{C}} F^*X \longrightarrow \operatorname*{hocolim}_{\mathcal{D}} X$$

is given by the composition

$$E\mathcal{C} \otimes_{\mathcal{C}} F^*X \xrightarrow{g} F_*E\mathcal{C} \otimes_{\mathcal{D}} X \xrightarrow{f \otimes_{\mathcal{D}} \mathrm{id}} E\mathcal{D} \otimes_{\mathcal{D}} X$$

where the map g is the homeomorphism from Lemma 1.9. This assembly map is unique up to homotopy. There is also an assembly map if the covariant $\mathcal{D}$-space X is replaced by a covariant $\mathcal{D}$-spectrum $\mathbf{E}$. If one uses the functorial models $E^{\mathrm{bar}}\mathcal{C}$ and $E^{\mathrm{bar}}\mathcal{D}$, there is a functorial construction of the map $E^{\mathrm{bar}}\mathcal{C} \longrightarrow F^*E^{\mathrm{bar}}\mathcal{D}$ and hence of the assembly map.

Let

$$I\colon \mathrm{Or}(G, \mathcal{F}) \longrightarrow \mathrm{Or}(G)$$

be the inclusion functor. Define the *assembly map*

$$I_*\colon \operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} I^*\mathbf{E} \longrightarrow \operatorname*{hocolim}_{\mathrm{Or}(G)} \mathbf{E} = \mathbf{E}(G/G),$$

where the homotopy colimit over the orbit category of $\mathbf{E}$ is $\mathbf{E}(G/G)$ because the orbit category has the terminal object G/G. This assembly map can be identified with the assembly map defined earlier by taking $E\mathrm{Or}(G) = \{*\}$ and $E\mathrm{Or}(G, \mathcal{F}) = \mathrm{map}_G(-, E(G, \mathcal{F}))$. The $(\mathbf{E}, \mathcal{F}, G)$-assembly map is obtained by applying homotopy groups. □

5.3. ASSEMBLY FROM THE HOMOLOGICAL POINT OF VIEW

Let $\{*\}_{\mathcal{F}}$ be the $\mathrm{Or}(G)$-space defined by setting $\{*\}_{\mathcal{F}}(G/H)$ to be a point if $H \in \mathcal{F}$ and to be empty otherwise. Let inc: $\{*\}_{\mathcal{F}} \longrightarrow \{*\}$ be the inclusion map of $\mathrm{Or}(G)$-spaces. It follows from definitions that the $(\mathbf{E}, \mathcal{F}, G)$-assembly map can be identified with the map

$$H_i^{\mathrm{Or}(G)}(\mathrm{inc})\colon H_i^{\mathrm{Or}(G)}(\{*\}_{\mathcal{F}}; \mathbf{E}) \longrightarrow H_i^{\mathrm{Or}(G)}(\{*\}; \mathbf{E}) = \pi_i(\mathbf{E}(G/G)). \qquad \square$$

DEFINITION 5.1. The $(\mathbf{E}, \mathcal{F}, G)$-*Isomorphism Conjecture* for a discrete group G, a family of subgroups $\mathcal{F}$, and a covariant $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ is that the $(\mathbf{E}, \mathcal{F}, G)$-assembly map is an isomorphism. For an integer i, the $(\mathbf{E}, \mathcal{F}, G, i)$-Isomorphism Conjecture is that the $(\mathbf{E}, \mathcal{F}, G)$-assembly map is an isomorphism in dimension i.

Of course for an arbitrary $(\mathbf{E}, \mathcal{F}, G)$, the Isomorphism Conjecture need not be valid. However, the Isomorphism Conjecture is always true (and therefore pointless!) when $\mathcal{F}$ is the family of all subgroups. The main problem is given G and $\mathbf{E}$ to find a *small* family $\mathcal{F}$ for which the Isomorphism Conjecture is true. The proper $\mathcal{F}$ to choose for the functors $\mathbf{K}$, $\mathbf{L}^{\langle \mathbf{j} \rangle}$, and $\mathbf{K}^{\mathrm{top}}$ will be discussed later in this section.

The main point of the validity of the $(\mathbf{E}, \mathcal{F}, G)$-Isomorphism Conjecture is that it allows the computation of $\pi_*(\mathbf{E}(G/G))$ from $\pi_*(\mathbf{E}(G/H))$ for $H \in \mathcal{F}$ and the structure of the restricted orbit category $\mathrm{Or}(G, \mathcal{F})$. Here are two examples which were historically important in algebraic K-theory.

EXAMPLE 5.2. Let G be an amalgamated free product of H_1 and H_2 along a subgroup K. Let $\mathcal{F}$ be the smallest family (closed under subgroups and conjugation) containing H_1 and H_2. The $E(G, \mathcal{F})$ can be taken to be a tree, where the isotropy group of an edge is conjugate to K and the isotropy group of a vertex is conjugate to H_1 or H_2. The $(\mathbf{E}, \mathcal{F}, G)$-Isomorphism Conjecture and the material in Section 4, give a long exact Mayer–Vietoris exact sequence

$$\cdots \longrightarrow \pi_i(\mathbf{E}(G/K)) \longrightarrow \pi_i(\mathbf{E}(G/H_1)) \oplus \pi_i(\mathbf{E}(G/H_2)) \longrightarrow \pi_i(\mathbf{E}(G/G)) \longrightarrow \cdots \qquad \square$$

EXAMPLE 5.3. Let G be a semidirect product given by the action of an infinite cyclic group on a group K. Let $\mathcal{F}$ be the family of all subgroups of K. Then $E(G, \mathcal{F})$ can be taken to be a $\mathbb{R}$, with the isotropy group K at every point. The $(\mathbf{E}, \mathcal{F}, G)$-Isomorphism Conjecture and the material in Section 4, give a long exact Wang exact sequence

$$\cdots \longrightarrow \pi_i(\mathbf{E}(G/K)) \longrightarrow \pi_i(\mathbf{E}(G/G)) \longrightarrow \pi_{i-1}(\mathbf{E}(G/K)) \longrightarrow \cdots \qquad \square$$

The following observation both motivates Isomorphism Conjectures and can be helpful in computation of $\mathcal{H}_*(BG)$ for a generalized homology theory $\mathcal{H}$ and a discrete group G.

LEMMA 5.4. *Let* $\mathbf{S}$ *be a fixed spectrum and* G *be a discrete group. Define an* $\mathrm{Or}(G)$*-spectrum* $\mathbf{E}$ *by* $\mathbf{E}(G/H) = (EG \times_G G/H)_+ \wedge \mathbf{S}$. *For any family* $\mathcal{F}$ *of subgroups of* G*, the* $(\mathbf{E}, \mathcal{F}, G)$*-Isomorphism Conjecture is valid.*

Proof. Let $\nabla\colon \mathrm{Or}(G) \longrightarrow \mathrm{SPACES}$ be the covariant functor $\nabla(G/H) = G/H$. Note that the $\mathrm{Or}(G)$-space ∇ has a left G-action defined by left multiplication of an element g on G/H. We have

$$\begin{aligned}
\mathbf{E}_{\%}(E(G,\mathcal{F})) &= E(G,\mathcal{F})^H \otimes_{\mathrm{Or}(G)} ((EG \times_G G/H)_+ \wedge \mathbf{S}) \\
&= (EG \times_G (E(G,\mathcal{F})^H \otimes_{\mathrm{Or}(G)} \nabla))_+ \wedge \mathbf{S} \\
&= (EG \times_G E(G,\mathcal{F}))_+ \wedge \mathbf{S} \\
&\xrightarrow{A} (EG \times_G G/G)_+ \wedge \mathbf{S} \\
&= \mathbf{E}_{\%}(G/G).
\end{aligned}$$

The first, second, and fourth equalities are clear. The third equality holds since one can identify any left G-space X with the left G-space $X^H \otimes_{\mathrm{Or}(G)} \nabla$ by Theorem 7.4 (1). The map A is the assembly map $\mathbf{E}_{\%}(\mathrm{pr})$. Since $\{e\} \in \mathcal{F}$, we see $E(G,\mathcal{F}) = E(G,\mathcal{F})^{\{e\}}$ is contractible, and hence $EG \times_G E(G,\mathcal{F}) \longrightarrow EG \times_G G/G$ is a homotopy equivalence. The Atiyah–Hirzebruch spectral sequence then shows A is a weak homotopy equivalence. □

Given a contravariant functor $\mathbf{E}\colon \mathrm{Or}(G) \longrightarrow \Omega$-SPECTRA, there is a dual assembly map obtained by reversing arrows and replacing $\otimes_{\mathrm{Or}(G)}$ by $\hom_{\mathrm{Or}(G)}$, hocolimits by holimits, and homology by cohomology. The analogue of the last lemma remains valid.

Now we consider the covariant $\mathrm{Or}(G)$-spectra of Section 2. When $\mathbf{E}$ equals the algebraic K-theory spectra $\mathbf{K}^{\mathrm{alg}}$ or the algebraic L-theory spectra $\mathbf{L}^{\langle -\infty \rangle}$ of Section 2 and $\mathcal{F}$ is the family $\mathcal{VC}$ of virtually cyclic subgroups of G, then the Isomorphism Conjecture is the one of Farrell–Jones [15]. An element of $\mathcal{VC}$ is a subgroup of G which in turn has a cyclic subgroup of finite index. Farrell and Jones use Quinn's version of the assembly map which can be identified with the one presented here by the characterization given in Section 6 and the fact that the source of Quinn's assembly map is a homology theory on the category of G-$\mathcal{VC}$-CW-complexes [32, Proposition 8.4 on page 421]. The Isomorphism Conjecture computes the algebraic K-groups, resp. $L^{\langle -\infty \rangle}$-groups, of the integral group ring of G in terms of the corresponding groups for all virtually cyclic subgroups of G. Here $L^{\langle -\infty \rangle} = \lim_{j \to -\infty} L^{\langle j \rangle}$. For the integers $\mathbb{Z}$, $L^{\langle j \rangle}(\mathbb{Z})$ is independent of j, and conjecturely $L^{\langle j \rangle}(\mathbb{Z}G)$ is independent of j for any torsion-free group G. The Isomorphism Conjecture for $\mathbf{K}^{\mathrm{alg}}$ has been proven rationally for discrete cocompact subgroups of virtually connected Lie groups by Farrell and Jones [15]. The $(\mathbf{K}^{\mathrm{alg}}, \mathcal{VC}, G, i)$-Isomorphism Conjecture for such groups with $i < 2$ also follows from [15]. The Isomorphism Conjecture for $\mathbf{L}^{\langle \mathbf{j} \rangle}$ has been proven for crystallographic groups if one inverts 2 by Yamasaki [44]. Notice that after inverting 2 the spectrum $\mathbf{L}^{\langle \mathbf{j} \rangle}$ is independent of j. The Isomorphism Conjecture for $\mathbf{K}^{\mathrm{alg}}$ and $\mathbf{L}^{\langle -\infty \rangle}$ together imply the Novikov Conjecture and (for dimensions greater than 4) the Borel Conjecture. The Borel Conjecture says that two aspherical closed manifolds with isomorphic fundamental groups are homeomorphic and any homotopy equivalence between them is homotopic to a homeomorphism. A survey on these conjectures is given in [16]. Related issues are discussed in [41, Chapter 14].

When $\mathbf{E}$ equals the topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ defined in Section 2 and $\mathcal{F}$ is the family $\mathcal{FIN}$ of finite subgroups of G, then the Isomorphism Conjecture is the Baum–Connes Conjecture [3, Conjecture 3.15 on page 254]. The identification is not obvious and will explained at the end of Section 6. For information about the Baum–Connes map we refer to [3] or [20].

EXAMPLE 5.5. Let $\mathbf{E}$ be a covariant $\mathrm{Or}(G)$-spectrum and $\mathcal{F} = 1$ the trivial family. The domain of the $(\mathbf{E}, 1, G)$-assembly map is $\mathbf{E}_{\%}(E(G,1)) = EG_+ \wedge_G \mathbf{E}(G/1)$. Now suppose there is a functor $\mathbf{J}\colon \mathrm{GROUPOIDS}^{\mathrm{inj}} \longrightarrow \mathrm{SPECTRA}$ so that $\mathbf{E}(G/H) = \mathbf{J}(\overline{G/H})$. Then the morphism of groupoids $\overline{G/1} \longrightarrow \overline{1/1}$ gives a map of spectra $\mathbf{E}(G/1) \longrightarrow \mathbf{E}(1/1)$ which is G-equivariant, where $\mathbf{E}(G/1)$ is given the $G = \mathrm{aut}_{\mathrm{Or}(G)}(G/1)$-action and $\mathbf{E}(1/1)$ is given the trivial G-action. Now suppose $\mathbf{J}$ has the additional property that given functors of groupoids $F_i\colon \mathcal{G}_0 \longrightarrow \mathcal{G}_1$ for $i = 0, 1$ and a natural transformation $T\colon F_0 \longrightarrow F_1$, then the maps of spectra $\mathbf{J}(F_0)$ and $\mathbf{J}(F_1)$ are homotopic. (See Lemma 2 to see that these hypotheses are valid where $\mathbf{E}$ is $\mathbf{K}^{\mathrm{alg}}$, $\mathbf{L}^{\langle \mathbf{j} \rangle}$, or $\mathbf{K}^{\mathrm{top}}$.) Since $\overline{G/1} \longrightarrow \overline{1/1}$ is a natural equivalence of groupoids, the map $\mathbf{E}(G/1) \longrightarrow \mathbf{E}(1/1)$ is a homotopy equivalence, which is in addition a G-map. It follows that

$$\mathbf{E}_{\%}(E(G/1)) = EG_+ \wedge_G \mathbf{E}(G/1) \longrightarrow BG_+ \wedge \mathbf{E}(1/1)$$

is a weak homotopy equivalence.

Thus the $(\mathbf{E}, 1, G)$-assembly map for the three $\mathrm{Or}(G)$-spectra of Section 2 can be identified with the 'classical' assembly maps

$$A\colon H_i(BG; \mathbf{K}^{\mathrm{alg}}(\mathbb{Z})) \longrightarrow K_i(\mathbb{Z}G),$$

$$A\colon H_i(BG; \mathbf{L}^{\langle -\infty \rangle}(\mathbb{Z})) \longrightarrow L_i^{\langle -\infty \rangle}(\mathbb{Z}G),$$

$$A\colon H_i(BG; \mathbf{K}^{\mathrm{top}}(\mathbb{C})) \longrightarrow \mathbf{K}_i^{\mathrm{top}}(C_r^*(G)).$$

The last map has an interpretation in terms of taking the index of elliptic operators. The Novikov Conjecture is equivalent to the conjecture that the middle map is rationally injective and is implied by the conjecture that the last map is rationally injective, which is in turn implied by the Baum–Connes conjecture.

It is easy to check that there are finite groups G for which none of the three assembly maps above is an isomorphism. However, it is conjectured that when G is torsion-free, that all three maps are isomorphisms. Indeed, the $(\mathbf{K}^{\mathrm{alg}}, \mathcal{VC}, G)$, $(\mathbf{L}^{\langle -\infty \rangle}, \mathcal{VC}, G)$, and $(\mathbf{K}^{\mathrm{top}}, \mathcal{FIN}, G)$ Isomorphism Conjectures applied to a torsion free group G are equivalent to the conjectures that the maps labeled A are isomorphisms. This is obvious in the $(\mathbf{K}^{\mathrm{top}}, \mathcal{FIN}, G)$-case, and is shown by Farrell-Jones [15, 1.6.1 and Remark A.11] in the other two cases. □

6. Characterization of Assembly Maps

In this section we characterize assembly maps by a universal property. This is useful for identifying different constructions of assembly maps (for example, assembly maps arising controlled topology, or from geometric techniques) and generalizes work of Weiss and Williams [42] from the case of a trivial group to the case of a general discrete group G.

We first review the nonequivariant version. Let $\mathbf{E}$ be a spectrum. Then one can define a functor

$$\mathbf{E}_{\%}\colon \text{SPACES} \longrightarrow \text{SPECTRA} \quad X \mapsto X_+ \wedge \mathbf{E}.$$

When this functor is restricted to the category of CW-complexes, it is excisive, in particular $\pi_*(\mathbf{E}_{\%}(-))$ is a generalized homology theory.

Now suppose

$$\mathbf{F}\colon \text{SPACES} \longrightarrow \text{SPECTRA}$$

is a (weakly) homotopy invariant functor, i.e. it takes (weak) homotopy equivalences to (weak) homotopy equivalences*. Then Weiss–Williams [42] construct a functor

$$\mathbf{F}^{\%}\colon CW\text{-COMPLEXES} \longrightarrow \text{SPECTRA}$$

and natural transformations

$$\mathbf{A}_{\mathbf{F}}\colon \mathbf{F}^{\%} \longrightarrow \mathbf{F}; \qquad \mathbf{B}_{\mathbf{F}}\colon \mathbf{F}^{\%} \longrightarrow (\mathbf{F}\mid_{\{*\}})_{\%};$$

which induce a (weak) homotopy equivalence of spectra $\mathbf{A}_{\mathbf{F}}(\{*\})$ and (weak) homotopy equivalences of spectra $\mathbf{B}_{\mathbf{F}}(X)$ for all CW-complexes X. Thus $\mathbf{F}^{\%}$ is a (weakly) excisive approximation for $\mathbf{F}$. The map $\mathbf{A}_{\mathbf{F}}(X)$ should be thought of as an assembly map, and when $X = BG$ and $\mathbf{F} = \mathbf{K}^{\mathrm{alg}}(\Pi(X)$, applying homotopy groups gives the classical K-theory assembly map

$$H_*(X; \mathbf{K}^{\mathrm{alg}}(\mathbb{Z})) \longrightarrow K_*^{\mathrm{alg}}(\mathbb{Z}\pi_1 X).$$

We now proceed to give the equivariant version of the above. We associate to a covariant $\mathrm{Or}(G, \mathcal{F})$-spectrum $\mathbf{E}$ an extension

$$\mathbf{E}_{\%}^{\mathcal{F}}\colon G\text{-SPACES} \longrightarrow \text{SPECTRA} \quad X \mapsto \mathrm{map}_G(-, X)_+ \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}.$$

*The example to be kept in mind is $\mathbf{F}(X) = \mathbf{K}^{\mathrm{alg}}(\Pi(X))$, the algebraic K-spectrum of the fundamental groupoid. This functor is homotopy invariant, but is neither excisive as a functor of X, nor continuous as a functor of topological categories.

Notice that this construction depends on $\mathcal{F}$. If $\mathbf{E}$ is a $\mathrm{Or}(G)$-spectrum, we have introduced $\mathbf{E}_{\%}$ already in Section 5. There is a natural transformation $\mathbf{S}\colon (\mathbf{E}\mid_{\mathrm{Or}(G,\mathcal{F})})^{\mathcal{F}}_{\%} \longrightarrow \mathbf{E}_{\%}$ of G-SPACES-spectra. A *G-$\mathcal{F}$-space* (*G-$\mathcal{F}$-CW-complex*) is a G-space (G-CW-complex) such that the isotropy group G_x of each point $x \in X$ is contained in the family $\mathcal{F}$. The map $\mathbf{S}(X)$ is an isomorphism if X is a G-$\mathcal{F}$-CW-complex but not in general. For instance for $X = G/G$ and $\mathcal{F}$ the trivial family 1 we get $(\mathbf{E}\mid_{\mathrm{Or}(G,\mathcal{F})})^{\mathcal{F}}_{\%}(G/G) = \mathbf{E}(G/1)/G$ and $\mathbf{E}_{\%}(G/G) = \mathbf{E}(G/G)$. We will omit the superscript $\mathcal{F}$ in $\mathbf{E}^{\mathcal{F}}_{\%}$ when it is clear from the context. Notice that this construction is functorial in $\mathbf{E}$, i.e. a map of $\mathrm{Or}(G,\mathcal{F})$-spectra $\mathbf{T}\colon \mathbf{E} \longrightarrow \mathbf{F}$ induces a map of G-SPACES-spectra $\mathbf{T}_{\%}\colon \mathbf{E}_{\%} \longrightarrow \mathbf{F}_{\%}$. Recall that a map (isomorphism) of spectra $\mathbf{f}\colon \mathbf{E} \longrightarrow \mathbf{F}$ is a collection of maps (homeomorphisms) $f(n)\colon \mathbf{E}(n) \longrightarrow \mathbf{F}(n)$ which are compatible with the structure maps. An isomorphism of $\mathcal{C}$-spectra is a map of $\mathcal{C}$-spectra whose evaluation at each object is an isomorphism of spectra.

LEMMA 6.1. *Let* $\mathbf{E}$ *be a covariant* $\mathrm{Or}(G,\mathcal{F})$*-spectrum. Then:*

(1) *The canonical map* $\mathbf{E}_{\%}(X) \cup_{\mathbf{E}_{\%}(f)} \mathbf{E}_{\%}(Y) \longrightarrow \mathbf{E}_{\%}(X \cup_f Y)$ *is an isomorphism, where* $f\colon A \longrightarrow Y$ *is a G-map and A is a closed, G-invariant subset of X.*
(2) *The canonical map* $\mathrm{colim}_{n\to\infty}\mathbf{E}_{\%}(X_n) \longrightarrow \mathbf{E}_{\%}(\mathrm{colim}_{n\to\infty} X_n)$ *is an isomorphism, where* $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$ *is a sequence of G-cofibrations.*
(3) *The canonical map* $Z_+ \wedge \mathbf{E}_{\%}(X) \longrightarrow \mathbf{E}_{\%}(Z \times X)$ *is an isomorphism, where Z is a space and X is a G-space.*
(4) *The canonical map* $\mathbf{E}_{\%}(G/H) \longrightarrow \mathbf{E}(G/H)$ *is an isomorphism for all* $H \in \mathcal{F}$.

Proof. It can be checked directly that the H-fixed point set functor $\mathrm{map}_G(G/H, -)$ commutes with attaching a G-space to a G-space along a G-map and with colimits of G-cofibrations indexed by the nonnegative integers. Parts (1) and (2) follow from the fact that $- \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}$ commutes with colimits, since it has an right adjoint by Lemma 1.5. Parts (3) and (4) follow from the definition of $\mathbf{E}_{\%}$. □

LEMMA 6.2. *Let* $\mathbf{E}$ *be a covariant* $\mathrm{Or}(G,\mathcal{F})$*-spectrum. Then the extension* $\mathbf{E} \mapsto \mathbf{E}_{\%}$ *is uniquely determined on the category of G-$\mathcal{F}$-CW-complexes up to isomorphism of G-$\mathcal{F}$-CW-*COMPLEXES*-spectra by the properties of Lemma 6.1.*

Proof. Let $\mathbf{E} \mapsto \mathbf{E}_{\$}$ be another such extension. There is a (*a priori* not necessarily continuous) set-theoretic natural transformation

$$\mathbf{T}(X)\colon \mathbf{E}_{\%}(X) = X_+ \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E} \longrightarrow \mathbf{E}_{\$}(X)$$

which sends an element represented by $(x\colon G/H \longrightarrow X, e) \in \mathrm{map}_G(G/H, X) \times \mathbf{E}(G/H)$ to $\mathbf{E}_{\$}(x)(e)$. Since any G-$\mathcal{F}$-CW-complex is constructed from orbits G/H with $H \in \mathcal{F}$ via products with disks, attaching a G-space to a G-space along a G-map, and colimits over the nonnegative integers, $\mathbf{T}(X)$ is continuous and is an isomorphism for all G-$\mathcal{F}$-CW-complexes X. □

Lemma 6.2 is a characterization of $\mathbf{E} \mapsto \mathbf{E}_{\%}$ up to isomorphism. Next we give a homotopy theoretic characterization.

A covariant functor $\mathbf{E}\colon G$-$\mathcal{F}$-CW-COMPLEXES $\longrightarrow$ SPECTRA is called (*weakly*) *$\mathcal{F}$-homotopy invariant* if it sends G-homotopy equivalences to (weak) homotopy equivalences of spectra. The functor $\mathbf{E}$ is (*weakly*) *$\mathcal{F}$-excisive* if it has the following four properties. First, it is (weakly) $\mathcal{F}$-homotopy invariant. Second, $\mathbf{E}(\emptyset)$ is contractible. Third, it respects homotopy pushouts up to (weak) homotopy equivalence, i.e. if the G-$\mathcal{F}$-CW-complex X is the union of G-CW-subcomplexes X_1 and X_2 with intersection X_0, then the canonical map from the homotopy pushout of $\mathbf{E}(X_2) \longrightarrow \mathbf{E}(X_0) \longleftarrow \mathbf{E}(X_2)$, which is obtained by gluing the mapping cylinders together along $\mathbf{E}(X_0)$, to $\mathbf{E}(X)$ is a (weak) homotopy equivalence of spectra. Finally, $\mathbf{E}$ respects countable disjoint unions up to (weak) homotopy, i.e. the natural map $\vee_{i\in I}\mathbf{E}(X_i) \longrightarrow \mathbf{E}(\coprod_{i\in I} X_i)$ is a (weak) homotopy equivalence for all countable index sets I. The last condition implies that the natural map from the homotopy colimit of the system $\mathbf{E}(X_n)$ coming from the skeletal filtration of a G-$\mathcal{F}$-CW-complex X, i.e. the infinite mapping telescope, to $\mathbf{E}(X)$ is a (weak) homotopy equivalence of spectra. Notice that $\mathbf{E}$ is weakly $\mathcal{F}$-excisive if and only if $\pi_q(\mathbf{E}(X))$ defines a homology theory on the category of G-$\mathcal{F}$-CW-complexes, satisfying the disjoint union axiom for countable disjoint unions.

THEOREM 6.3.

(1) *Suppose* $\mathbf{E}\colon \mathrm{Or}(G,\mathcal{F}) \longrightarrow$ SPECTRA *is a covariant functor. Then* $\mathbf{E}_{\%}$ *is $\mathcal{F}$-excisive.*
(2) *Let* $\mathbf{T}\colon \mathbf{E} \longrightarrow \mathbf{F}$ *be a transformation of* (*weakly*) *$\mathcal{F}$-excisive functors* $\mathbf{E}$ *and* $\mathbf{F}$ *from* G-$\mathcal{F}$-CW-COMPLEXES *to* SPECTRA *so that* $\mathbf{T}(G/H)$ *is a* (*weak*) *homotopy equivalence of spectra for all* $H \in \mathcal{F}$. *Then* $\mathbf{T}(X)$ *is a (weak) homotopy equivalence of spectra for all* G-$\mathcal{F}$-CW-*complexes* X.
(3) *For any* (*weakly*) *$\mathcal{F}$-homotopy invariant functor* $\mathbf{E}$ *from* G-$\mathcal{F}$-CW-COMPLEXES *to* SPECTRA, *there is a (weakly) $\mathcal{F}$-excisive functor* $\mathbf{E}^{\%}$ *from* G-$\mathcal{F}$-CW-COMPLEXES *to* SPECTRA *and there are natural transformations*
$$\mathbf{A_E}\colon \mathbf{E}^{\%} \longrightarrow \mathbf{E}; \qquad \mathbf{B_E}\colon \mathbf{E}^{\%} \longrightarrow (\mathbf{E}\mid_{\mathrm{Or}(G,\mathcal{F})})_{\%};$$
which induce (*weak*) *homotopy equivalences of spectra* $\mathbf{A_E}(G/H)$ *for all* $H \in \mathcal{F}$ *and (weak) homotopy equivalences of spectra* $\mathbf{B_E}(X)$ *for all* G-$\mathcal{F}$-CW-*complexes* X. *Given a family* $\mathcal{F}' \subset \mathcal{F}$, $\mathbf{E}$ *is* (*weakly*) *$\mathcal{F}'$-excisive if and only if* $\mathbf{A_E}(X)$ *is a* (*weak*) *homotopy equivalence of spectra for all* G-$\mathcal{F}'$-CW-*complexes* X.

Proof. (1) Follows from Lemma 6.1.
(2) Use the fact that a (weak) homotopy colimit of homotopy equivalences of spectra is again a (weak) homotopy equivalence of spectra.
(3) Define $\mathbf{E}^{\%}(X)$ by the spectrum

$$\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} B^{\mathrm{bar}}\, ?\!\downarrow \mathrm{Or}(G,\mathcal{F}) \\ \times \mathbf{\Delta}\!\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} \mathbf{E}(- \times \Delta.)$$

where $-$, resp. ., runs over $\mathrm{Or}(G)$, resp. $\mathbf{\Delta}$, the subscript d in $\mathrm{map}_G(- \times \Delta.,X)_d$ indicates that we equip this mapping space in contrast to the usual convention with the discrete topology and $B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F}) \times \mathbf{\Delta} \downarrow ??$ was introduced at the end of Section 3. Define the transformation $\mathbf{A_E}(X)\colon \mathbf{E}^{\%}(X) \longrightarrow \mathbf{E}(X)$ by the following diagram

$$\begin{array}{c}
\mathbf{E}(X)\\
\big\uparrow c_1\\
\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} \mathbf{E}(- \times \Delta.)\\
\big\uparrow p_{\mathrm{map}_G(-\times\Delta.,X)_d}\otimes \mathrm{id}\\
\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F}) \times \mathbf{\Delta}\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} \mathbf{E}(- \times \Delta.)
\end{array}$$

where $p_{\mathrm{map}_G(-\times\Delta.,X)_d}$ was introduced in Definition 3.18 and here and in the next diagram c_k refers to the canonical map whose definition is obvious from the context. Define the transformation $\mathbf{B_E}(X)\colon \mathbf{E}^{\%}(X) \longrightarrow (\mathbf{E}\,|_{\mathrm{Or}(G,\mathcal{F})})_{\%}(X)$ by the following diagram

$$\begin{array}{c}
\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F}) \times \mathbf{\Delta}\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} \mathbf{E}(- \times \Delta.)\\
\big\downarrow \mathrm{id}\otimes\mathrm{id}\otimes\mathbf{E}(\mathrm{pr})\\
\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F}) \times \mathbf{\Delta}\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} \mathbf{E}(-)\\
\big\downarrow \mathrm{id}\otimes c_2\otimes\mathrm{id}\ \cong\\
\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ?? \times B^{\text{bar}}\,?\downarrow \mathbf{\Delta}\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})\times\mathbf{\Delta}} \mathbf{E}(-)\\
\big\downarrow c_3\ \cong\\
(\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathbf{\Delta}\downarrow ?? \otimes_{\mathbf{\Delta}} \{*\}) \otimes_{\mathrm{Or}(G,\mathcal{F})} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}(-)\\
\big\downarrow (\mathrm{id}\otimes c_4)\otimes\mathrm{id}\otimes\mathrm{id}\ \cong\\
(\mathrm{map}_G(- \times \Delta.,X)_d \otimes_{\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathbf{\Delta}) \otimes_{\mathrm{Or}(G,\mathcal{F})} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}(-)\\
\big\downarrow (c_5\otimes\mathrm{id})\otimes\mathrm{id}\otimes\mathrm{id}\ \cong\\
(\mathrm{map}(\Delta.,\mathrm{map}_G(-,X))_d \otimes_{\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathbf{\Delta}) \otimes_{\mathrm{Or}(G,\mathcal{F})} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}(-)\\
\big\downarrow (\mathrm{id}\otimes q)\otimes\mathrm{id}\otimes\mathrm{id}\\
(\mathrm{map}(\Delta.,\mathrm{map}_G(-,X))_d \otimes_{\mathbf{\Delta}} \Delta.) \otimes_{\mathrm{Or}(G,\mathcal{F})} B^{\text{bar}}\,?\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ?? \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}(-)\\
\big\downarrow a_{\mathrm{map}_G(-,X)}\otimes\mathrm{id}\\
\mathrm{map}_G(-,X) \otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}(-)
\end{array}$$

where the canonical map $q\colon B^{\text{bar}}\,?\downarrow \mathbf{\Delta} \longrightarrow \Delta_?$ is defined in [4, Example XI.2.6 on page 293] and $a_{\mathrm{map}_G(-,X)}$ was introduced in Definition 3.18.

Next we show that $\mathbf{B_E}(X)$ is a (weak) homotopy equivalence provided that X is a G-$\mathcal{F}$-CW-complex. Since $\mathbf{E}$ is (weakly) $\mathcal{F}$-excisive, the map $\mathbf{E}(\mathrm{pr})\colon \mathbf{E}(G/H \times \Delta_n) \longrightarrow \mathbf{E}(G/H)$ is a (weak) homotopy equivalence for all $H \in \mathcal{F}$. Hence the first map in the diagram above $\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{E}(\mathrm{pr})$ is a weak homotopy equivalence because of Theorem 3.11. The next four maps are all isomorphisms. The map

$$\begin{aligned}
\mathrm{id} \otimes q\colon &\mathrm{map}(\Delta.,\mathrm{map}_G(-,X))_d \otimes_{\mathbf{\Delta}} B^{\text{bar}}\,?\downarrow \mathbf{\Delta}\\
&\longrightarrow \mathrm{map}(\Delta.,\mathrm{map}_G(-,X))_d \otimes_{\mathbf{\Delta}} \Delta.
\end{aligned}$$

is a weak homotopy equivalence of $\mathrm{Or}(G,\mathcal{F})$-spaces [4, XII.3.4 on page 331]. Because of Theorem 3.11 the map

$$\begin{aligned}&(\mathrm{id}\otimes q)\otimes \mathrm{id}\colon (\mathrm{map}(\Delta.,\mathrm{map}_G(-,X))_d \otimes_{\mathbf{\Delta}} B^{\mathrm{bar}}\,?\!\downarrow \mathbf{\Delta})\\ &\quad\otimes_{\mathrm{Or}(G,\mathcal{F})} B^{\mathrm{bar}}\,?\!\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ??\\ &\quad\longrightarrow \big(\mathrm{map}(\Delta.,\mathrm{map}_G(-,X))_d \otimes_{\mathbf{\Delta}} \Delta\big) \otimes_{\mathrm{Or}(G,\mathcal{F})} B^{\mathrm{bar}}\,?\!\downarrow \mathrm{Or}(G,\mathcal{F})\downarrow ??\end{aligned}$$

is a weak $\mathrm{Or}(G,\mathcal{F})$-homotopy equivalence of $\mathrm{Or}(G,\mathcal{F})$-spaces. Since the domain and target are free $\mathrm{Or}(G,\mathcal{F})$-$CW$-complexes by Lemma 3.19, it is a homotopy equivalence of $\mathrm{Or}(G)$-spaces by Corollary 3.5. Hence the map $(\mathrm{id}\otimes q)\otimes \mathrm{id}\otimes \mathrm{id}$ in the diagram above is a homotopy equivalence.

As we assume that X is a G-$\mathcal{F}$-CW-complex $\mathrm{map}_G(-,X)$ is a $\mathrm{Or}(G,\mathcal{F})$-$CW$-complex. Since $a_{\mathrm{map}_G(-,X)}$ is a $\mathrm{Or}(G,\mathcal{F})$-$CW$-approximation by Lemma 3.19 Corollary 3.5 implies that it is a homotopy equivalence of $\mathrm{Or}(G,\mathcal{F})$-$CW$-complexes. Hence the last map in the diagram above $a_{\mathrm{map}_G(-,X)}\otimes \mathrm{id}$ is a homotopy equivalence. This shows that $\mathbf{B_E}(X)$ is a (weak) homotopy equivalence.

In the case $X = G/H$ for $H \in \mathcal{F}$ the composition of the (weak) homotopy equivalence $\mathbf{B_E}(G/H)$ with the canonical isomorphism $\mathrm{map}_G(-,G/H)\otimes_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}(-) \longrightarrow \mathbf{E}(G/H)$ agrees with $\mathbf{A_E}(G/H)$. Hence $\mathbf{A_E}(G/H)$ is a (weak) homotopy equivalence for all G/H with $H\in\mathcal{F}$. This finishes the proof of Theorem 6.3. □

The map $\mathbf{A_E}$ is called an *assembly map for* $\mathbf{E}$.

EXAMPLE 6.4. For a topological space X, the *fundamental groupoid* $\Pi(X)$ is the category whose objects are points in X and whose morphism set $\mathrm{mor}_{\Pi(X)}(x,y)$ is given by equivalence classes of paths from x to y, where the equivalence relation is homotopy rel $\{0,1\}$. A map of spaces gives a map of fundamental groupoids. A homotopy equivalence of spaces gives a natural equivalence of fundamental groupoids. If X is path-connected and $x_0 \in X$, then the inclusion of the fundamental group $\pi_1(X,x_0) \longrightarrow \Pi(X)$ is a natural equivalence of groupoids.

Let $\mathbf{K}^{\mathrm{alg}}$: GROUPOIDS $\longrightarrow$ SPECTRA be the functor from Section 2. By Lemma 2, $\mathbf{K}^{\mathrm{alg}}$ has the property that a natural equivalence of groupoids gives a homotopy equivalence of spectra.

One can define a homotopy invariant functor $\mathbf{E}\colon CW$-COMPLEXES $\longrightarrow$ SPECTRA by $\mathbf{E}(X) = \mathbf{K}^{\mathrm{alg}}(\Pi(X))$. We apply Theorem 6.3 in the case where G is the trivial group (note that for $G = 1$, Theorem 6.3 is due to Weiss–Williams [42]). The map $\mathbf{B_E}$ gives a homotopy equivalence from $\mathbf{E}^{\%}(X)$ to $X_+ \wedge \mathbf{K}^{\mathrm{alg}}(\mathbb{Z})$, where $\mathbf{K}^{\mathrm{alg}}(\mathbb{Z})$ is the algebraic K-spectrum of the ring $\mathbb{Z}$. After one applies the nth homotopy group to the assembly map

$$\mathbf{A_E}\colon \mathbf{E}^{\%}(X) \longrightarrow \mathbf{E}(X)$$

one obtains the algebraic $\mathbf{K}$-theory assembly map

$$A\colon H_n(X;\mathbf{K}^{\mathrm{alg}}(\mathbb{Z})) \longrightarrow K_n^{\mathrm{alg}}(\mathbb{Z}\pi_1 X).$$

Next consider a discrete group G and a family of subgroups $\mathcal{F}$. One can then define an $\mathcal{F}$-homotopy invariant functor

$$\mathbf{E}\colon G\text{-}CW\text{-COMPLEXES} \longrightarrow \text{SPECTRA}$$

by setting $\mathbf{E}(X) = \mathbf{K}^{\mathrm{alg}}(\Pi(EG \times_G X))$. If X is simply-connected, there is a natural equivalence of groupoids

$$G = \mathrm{Or}(G, 1) \longrightarrow \Pi(EG \times_G X).$$

Using this identification, we have a fourth point of view on the $(\mathbf{K}^{\mathrm{alg}}, \mathcal{F}, G)$-assembly map, namely it is

$$\pi_*(\mathbf{A_E}(E(G, \mathcal{F})))\colon \pi_*(\mathbf{E}^{\%}(E(G, \mathcal{F}))) \longrightarrow \pi_*(\mathbf{E}(E(G, \mathcal{F}))).$$

The case of algebraic L-theory is analogous. For a map of spaces $X \longrightarrow Y$, the map of groupoids $\Pi(X) \longrightarrow \Pi(Y)$ need not be a morphism in $\text{GROUPOIDS}^{\mathrm{inj}}$. However, all relevant maps in the definition of $\mathbf{A_E}$ and $\mathbf{B_E}$ have this property, so that the analogous point of view holds also for the topological K-theory of C^*-algebras. □

Next we give for a covariant $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ an equivalent definition of $\mathbf{E}^{\%}$ which is closer to the construction in [42]. Let $\mathrm{simp}_G(X)$ be the category having as morphisms pairs $(G/H \times [n], \sigma)$ which consists of an object $G/H \times [n]$ in $\mathrm{Or}(G, \mathcal{F}) \times \mathbf{\Delta}$ and a G-map $\sigma\colon G/H \times \Delta_n \longrightarrow X$. A morphism from $(G/H \times [n], \sigma)$ to $(G/K \times [m], \tau)$ is a morphism $f \times u\colon G/H \times [n] \longrightarrow G/K \times [m]$ in $\mathrm{Or}(G, \mathcal{F}) \times \mathbf{\Delta}$ such that the composition of the induced map $G/H \times \Delta_n \longrightarrow G/K \times \Delta_m$ with τ is σ. This is the equivariant version of the construction in [35, Appendix A] applied to the simplicial set $S.X$ associated to a space X. We get a covariant functor $\mathbf{E}(- \times \Delta.)$ from $\mathrm{simp}_G(X)$ to SPECTRA by $(G/K \times [m], \sigma) \mapsto \mathbf{E}(G/K \times \Delta_m)$. We briefly indicate how one can identify

$$\mathbf{E}^{\%}(X) = \operatorname*{hocolim}_{\mathrm{simp}_G(X)} \mathbf{E}(- \times \Delta.).$$

Let $P\colon \mathrm{simp}_G(X) \longrightarrow \mathrm{Or}(G) \times \mathbf{\Delta}$ be the obvious forgetful functor. It suffices to construct a natural isomorphism of $\mathrm{Or}(G) \times \mathbf{\Delta}$-spaces

$$B^{\mathrm{bar}}\, ?\!\downarrow \mathrm{simp}_G(X) \ \otimes_{\mathrm{simp}_G(X)} \mathrm{mor}_{\mathrm{Or}(G)\times\mathbf{\Delta}}(??, P(?))$$
$$\longrightarrow \mathrm{map}_G(- \times \Delta., X) \otimes_{\mathrm{Or}(G)\times\mathbf{\Delta}} B^{\mathrm{bar}}\, ??\!\downarrow \mathrm{Or}(G) \times \mathbf{\Delta}\!\downarrow - \times . \quad .$$

It will be implemented by the following natural bijection of simplicial sets for a given object $G/K \times [m]$ in $\mathrm{Or}(G) \times \mathbf{\Delta}$ where p runs over $0, 1, 2, \ldots$

$$N_p\, ?\!\downarrow \mathrm{simp}_G(X) \ \otimes_{\mathrm{simp}_G(X)} \mathrm{mor}_{\mathrm{Or}(G)\times\mathbf{\Delta}}(G/K \times [m], P(?)) \longrightarrow$$
$$\mathrm{map}_G(- \times \Delta., X) \otimes_{\mathrm{Or}(G)\times\mathbf{\Delta}} N_p\, G/K \times [m]\!\downarrow \mathrm{Or}(G) \times \mathbf{\Delta}\!\downarrow - \times . \quad .$$

An element in the source is represented for $? = (G/H \times [n], \sigma)$ by the pair

$$((G/H \times [n], \sigma) \longrightarrow (G/H_0 \times [n_0], \sigma_0) \longrightarrow \cdots \longrightarrow (G/H_p \times [n_p], \sigma_p)) \\ \times (G/K \times [m] \longrightarrow G/H \times [n]).$$

It is sent to the element in the target represented by

$$(\sigma_p : G/H_p \times \Delta_{n_p} \longrightarrow X) \times \\ \times (G/K \times [m] \longrightarrow G/H \times [n] \longrightarrow G/H_0 \times [n_0] \longrightarrow \cdots \longrightarrow G/H_p \times [n_p]).$$

This is indeed a bijection since $G/H_0 \times [n_0] \longrightarrow \cdots \longrightarrow G/H_p \times [n_p]$ and σ_p determine $\sigma_0, \ldots, \sigma_{p-1}$.

Next we explain why Theorem 6.3 characterizes the assembly map in the sense that $\mathbf{A_E} : \mathbf{E}^{\%} \longrightarrow \mathbf{E}$ is the universal approximation from the left by a (weakly) $\mathcal{F}$-excisive functor of a (weakly) $\mathcal{F}$-homotopy invariant functor $\mathbf{E}$ from G-$\mathcal{F}$-CW-COMPLEXES to SPECTRA. The argument is the same as in [42, page 336]. Namely, let $\mathbf{T} : \mathbf{F} \longrightarrow \mathbf{E}$ be a transformation of functors from G-$\mathcal{F}$-CW-COMPLEXES to SPECTRA such that $\mathbf{F}$ is (weakly) $\mathcal{F}$-excisive and $\mathbf{T}(G/H)$ is a (weak) homotopy equivalence for all $H \in \mathcal{F}$. Then for any G-$\mathcal{F}$-CW-complex X the following diagram commutes:

$$\begin{array}{ccc} \mathbf{F}^{\%}(X) & \xrightarrow{\mathbf{A_F}(X)} & \mathbf{F}(X) \\ \mathbf{T}^{\%}(X) \Big\downarrow & & \Big\downarrow \mathbf{T}(X) \\ \mathbf{E}^{\%}(X) & \xrightarrow{\mathbf{A_E}(X)} & \mathbf{E}(X) \end{array}$$

and $\mathbf{A_F}(X)$ and $\mathbf{T}^{\%}(X)$ are (weak) homotopy equivalences. Hence one may say that $\mathbf{T}(X)$ factorizes over $\mathbf{A_E}(\mathbf{X})$.

One may be tempted to define a natural transformation $\mathbf{S} : \mathbf{E}_{\%} \longrightarrow \mathbf{E}$ as indicated in the proof of Lemma 6.2. Then $\mathbf{S}(X)$ is a well-defined bijection of sets but is not necessarily continuous because we do not want to assume that $\mathbf{E}$ is continuous, i.e. that the induced map from $\hom_{\mathcal{C}}(X, Y)$ to $\hom_{\mathcal{C}}(\mathbf{E}(X), \mathbf{E}(Y))$ is continuous for all G-$\mathcal{F}$-CW-complexes X and Y. The construction above uses the (weak) $\mathcal{F}$-homotopy invariance of $\mathbf{E}$ instead.

Finally we say how one can identify the Baum–Connes-map of [3] with the map induced on homotopy groups by the assembly map

$$(\mathbf{K}^{\mathrm{top}})_{\%}(E(G, \mathcal{FIN})) \longrightarrow (\mathbf{K}^{\mathrm{top}})_{\%}(G/G).$$

This problem has been considered by many people including Baum, Bloch, Carlsson, Comezana, Higson, Pedersen, Roe, and Stolz. Our characterization of the assembly map allows the proof of this identification for all possible models of $\mathbf{K}^{\mathrm{top}}$ if it has been done for one model. We shall use the following construction due to Carlsson *et al.* [6].

There is a functor **P** from the category of G-spaces to the category of spectra with G-action and strong maps as morphisms such that the functor

$$\mathbf{Q}\colon G\text{-}CW\text{-COMPLEXES} \longrightarrow \text{SPECTRA}$$

obtained from **P** by taking the G-fixed point set of G-spectra has the following properties.

(1) There are identifications of the homotopy groups of $\mathbf{Q}(E(G,\mathcal{FIN}))$ with the source of the Baum–Connes map and of $\mathbf{Q}(G/G)$ with the target of the Baum–Connes map such that the Baum–Connes map itself agrees with the map induced by the projection $\mathbf{Q}(E(G,\mathcal{FIN})) \longrightarrow \mathbf{Q}(G/G)$.

(2) There is a weak equivalence of $\mathrm{Or}(G)$-spectra
$$\mathbf{U}\colon \mathbf{K}^{\mathrm{top}} \longrightarrow \mathbf{Q}|_{\mathrm{Or}(G)}.$$

(3) **Q** is weakly $\mathcal{FIN}$-excisive.

Then we obtain from Theorem 6.3 for $\mathcal{F}$ the family of all subgroups and $\mathcal{F}' = \mathcal{FIN}$ a commutative diagram of spectra whose vertical maps are all weak homotopy equivalences.

$$\begin{array}{ccc}
(\mathbf{K}^{\mathrm{top}})_{\%}(E(G,\mathcal{FIN})) & \longrightarrow & (\mathbf{K}^{\mathrm{top}})_{\%}(G/G) \\
\downarrow{\scriptstyle \mathbf{U}_{\%}(E(G,\mathcal{FIN}))} & & \downarrow{\scriptstyle \mathbf{U}_{\%}(G/G)} \\
(\mathbf{Q}|_{\mathrm{Or}(G)})_{\%}(E(G,\mathcal{FIN})) & \longrightarrow & (\mathbf{Q}|_{\mathrm{Or}(G)})_{\%}(G/G) \\
\uparrow{\scriptstyle \mathbf{B}_{\mathbf{Q}}(E(G,\mathcal{FIN}))} & & \uparrow{\scriptstyle \mathbf{B}_{\mathbf{Q}}(G/G)} \\
\mathbf{Q}^{\%}(E(G,\mathcal{FIN})) & \longrightarrow & \mathbf{Q}^{\%}(G/G) \\
\downarrow{\scriptstyle \mathbf{A}_{\mathbf{Q}}(E(G,\mathcal{FIN}))} & & \downarrow{\scriptstyle \mathbf{A}_{\mathbf{Q}}(G/G)} \\
\mathbf{Q}(E(G,\mathcal{FIN})) & \longrightarrow & \mathbf{Q}(G/G)
\end{array}$$

Hence the map induced on homotopy groups by the top horizontal arrow is the Baum–Connes map.

7. *G*-Spaces and Or(*G*)-spaces

In this section we discuss the orbit category in more detail, and give a correspondence between G-spaces with isotropy in $\mathcal{F}$ and $\mathrm{Or}(G,\mathcal{F})$-spaces. This in turn will give a correspondence between classifying spaces of G with respect to $\mathcal{F}$ and models of $E\mathrm{Or}(G,\mathcal{F})$ and will thereby give a source of natural examples. As usual, let G be a discrete group and $\mathcal{F}$ a nonempty family of subgroups closed under conjugation and inclusion. A G-space X is a *G-$\mathcal{F}$-space* if the isotropy subgroup of each point in X is contained in $\mathcal{F}$. Let $\mathrm{Or}(G,\mathcal{F})$ be the restricted orbit category whose objects are G/H for $H \in \mathcal{F}$ and whose morphisms are G-maps.

Next we explain how one gets from G-$\mathcal{F}$-spaces to $\mathrm{Or}(G,\mathcal{F})$-spaces and *vice versa*. We will get a correspondence up to homeomorphism, not only up to homotopy (cf. [11, Theorem 3.11], [13, 30]).

DEFINITION 7.1. Given a left G-space Y, define the *associated contravariant* $\mathrm{Or}(G, \mathcal{F})$*-space* $\mathrm{map}_G(-, Y)$ by

$$\mathrm{Or}(G, \mathcal{F}) \longrightarrow \mathrm{SPACES} \quad G/H \mapsto \mathrm{map}_G(G/H, Y) = Y^H.$$

Let ∇ be the covariant $\mathrm{Or}(G, \mathcal{F})$-space given by sending G/H to itself. Given a contravariant $\mathrm{Or}(G, \mathcal{F})$-space X define the *associated left* G*-*$\mathcal{F}$*-space* $\hat{X}$ by

$$\hat{X} = X \otimes_{\mathrm{Or}(G,\mathcal{F})} \nabla.$$

The left action of an element $g \in G$ is given by $\mathrm{id} \otimes_{\mathrm{Or}(G,\mathcal{F})} L_g$ where $L_g \colon G/H \longrightarrow G/H$ is the map of covariant $\mathrm{Or}(G, \mathcal{F})$-spaces given by left multiplication with g.
□

The notation for the functor ∇ is intended to be reminiscent of the cosimplicial space $\Delta.$ from Example 1.7.

LEMMA 7.2. *The functors in Definition 7.1 are adjoint, i.e. for a contravariant* $\mathrm{Or}(G, \mathcal{F})$*-space* X *and a left* G*-space* Y *there is a natural homeomorphism*

$$T(X, Y) \colon \mathrm{map}_G(\hat{X}, Y) \longrightarrow \mathrm{hom}_{\mathrm{Or}(G,\mathcal{F})}(X, \mathrm{map}_G(-, Y)).$$

Proof. If we neglect the G-action on Y, we get from Lemma 1.5 a natural homeomorphism

$$\mathrm{map}(\hat{X}, Y) \longrightarrow \mathrm{hom}_{\mathrm{Or}(G,\mathcal{F})}(X, \mathrm{map}(-, Y)).$$

Using the transformations L_g and the G-action on Y one defines appropriate G-actions on the source and target of this map and checks that this map is G-equivariant. Hence it induces a homeomorphism on the G-fixed point set which is just $T(X, Y)$. Of course one can define for instance $T(X, Y)^{-1}$ explicitly. Given $f \colon X \longrightarrow \mathrm{map}_G(-, Y)$ we define $T(X, Y)^{-1}(f)$ by specifying for each G/H a map $X(G/H) \times G/H \longrightarrow Y$. It sends (x, gH) to the value of $f(G/H)(x)$ at gH. □

LEMMA 7.3. *The map*

$$f \colon X(G/1) \longrightarrow \hat{X} \quad x \mapsto [x, 1]$$

is a G*-homeomorphism.*

Proof. The inverse $f^{-1} \colon \hat{X} \longrightarrow X(G/1)$ assigns to an element represented by (x, gH) the element $X(q_{gH})(x)$ where $q_{gH} \colon G/1 \longrightarrow G/H$ sends g' to $g'gH$. □

Let X be a contravariant $\mathrm{Or}(G, \mathcal{F})$-space. Obviously the projection $\mathrm{pr} \colon G/1 \longrightarrow G/H$ induces a map $X(\mathrm{pr}) : X(G/H) \longrightarrow X(G/1)^H$. Now one easily checks using Lemma 7.3 above the following:

THEOREM 7.4.

(1) *Given a left G-$\mathcal{F}$-space Y, the adjoint of the identity on* $\mathrm{map}_G(-, Y)$ *under the adjunction of Lemma 7.2 is a natural G-homeomorphism*

$$T(Y)\colon \widehat{\mathrm{map}_G(-, Y)} \longrightarrow Y.$$

It is induced by the map

$$\coprod_{H \in \mathcal{F}} \mathrm{map}_G(G/H, Y) \times G/H \longrightarrow Y, \quad (\phi, gH) \mapsto \phi(gH).$$

(2) *Given a contravariant* $\mathrm{Or}(G, \mathcal{F})$*-space X, the adjoint of the identity on $\hat{X}$ under the adjunction of Lemma 7.2 is a natural map of* $\mathrm{Or}(G, \mathcal{F})$*-spaces*

$$S(X)\colon X \longrightarrow \mathrm{map}_G(-, \hat{X}).$$

Given $H \in \mathcal{F}$, the map $S(X)(G/H)$ maps the element $x \in X(G/H)$ to the element in $\mathrm{map}_G(G/H, \hat{X}) = \left(X \otimes_{\mathrm{Or}(G,\mathcal{F})} \nabla\right)^H$ *represented by $(x, eH) \in X(G/H) \times G/H$. It is an isomorphism of* $\mathrm{Or}(G, \mathcal{F})$*-spaces if and only if for each $H \in \mathcal{F}$ the projection* $\mathrm{pr}\colon G/1 \longrightarrow G/H$ *induces a homeomorphism* $X(\mathrm{pr})\colon X(G/H) \longrightarrow X(G/1)^H$*. This condition is satisfied if X is a free* $\mathrm{Or}(G, \mathcal{F})$*-$CW$-complex.*

(3) *If Y is left G-$\mathcal{F}$-CW-complex, then* $\mathrm{map}_G(-, Y)$ *is a free* $\mathrm{Or}(G, \mathcal{F})$*-$CW$-complex. There is a bijective correspondence between the G-cells in Y of type G/H and the* $\mathrm{Or}(G, \mathcal{F})$*-cells in* $\mathrm{map}_G(-, Y)$ *based at the object G/H. The analogous statement holds for a free* $\mathrm{Or}(G, \mathcal{F})$*-$CW$-complex X and $\hat{X}$.* □

The bar resolution is a natural construction, however, it is a 'very big' model. Models with a fewer number of cells can be very convenient for concrete calculations and arise often as follows.

DEFINITION 7.5. Let G be a group and $\mathcal{F}$ be a family of subgroups. A *classifying space $E(G, \mathcal{F})$ of G with respect to $\mathcal{F}$* is a left G-CW-complex such that $E(G, \mathcal{F})^H$ is contractible for $H \in \mathcal{F}$ and empty otherwise. □

The existence of $E(G, \mathcal{F})$ and proofs that for any G-$\mathcal{F}$-CW-complex X there is precisely one G-map up to G-homotopy from X to $E(G, \mathcal{F})$ and thus that two such classifying spaces are G-homotopy equivalent, is given in [9],[10, I.6]. Another construction and proof of the results above come from Theorem 3.4 and the following result which is a direct consequence of Theorem 7.4.

LEMMA 7.6. *Let G be a group and $\mathcal{F}$ be a family of subgroups.*

(1) *If $E(G, \mathcal{F})$ is a classifying space of G with respect to $\mathcal{F}$, then the associated contravariant* $\mathrm{Or}(G, \mathcal{F})$*-space*

$$\mathrm{map}_G(-, E(G, \mathcal{F}))$$

is a model for $E\mathrm{Or}(G, \mathcal{F})$*;*

(2) *Given a model* $E\mathrm{Or}(G, \mathcal{F})$*, then the* G*-space* $E\widehat{\mathrm{Or}(G}, \mathcal{F})$ *is a classifying space of* G *with respect to* $\mathcal{F}$. □

EXAMPLE 7.7. Sometimes geometry yields small examples of classifying spaces and resolutions. We have already mentioned this in the case where G is a crystallographic group. Generalizing this, let G be a discrete subgroup of a Lie group L with a finite number of components. If K is a maximal compact subgroup of L, then L/K is homeomorphic to $\mathbb{R}^n$ and L/K can be taken as a model for $E(G, \mathcal{FIN})$, where $\mathcal{FIN}$ is the family of finite subgroups. Generalizing further, let G be a group of finite virtual cohomological dimension. Then there is finite-dimensional classifying space $E(G, \mathcal{FIN})$ (see [36, Proposition 12]) and hence a finite-dimensional model for $E\mathrm{Or}(G, \mathcal{FIN})$. Many examples of such groups are discussed by Serre in [36]. More examples of nice geometric models for $E(G, \mathcal{FIN})$ can be found in [3, Section 2]. □

References

1. Adams, J. F.: *Stable Homotopy and Generalised Homology*, Chicago Lecture Notes in Math., University of Chicago Press, 1974.
2. Bass, H.: *Algebraic K-theory*, Benjamin, 1968.
3. Baum, P., Connes, A. and Higson, N.: *Classifying Space for Proper Actions and K-theory of Group* C^**-algebras*, in C^**-algebras: 1943–1993* (*San Antonio, TX, 1993*), Contemp. Math. 167, Amer. Math. Soc., Providence, 1994, pp. 240–291.
4. Bousfield, A. K. and Kan, D. M.: *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. 304, Springer, New York, 1972.
5. Bredon, G. E.: *Equivariant Cohomology Theories*, Lecture Notes in Math. 34, Springer, New York, 1967.
6. Carlsson, G., Pedersen, E. K. and Roe, J.: Controlled C^*-theory and the injectivity of the Baum–Connes map, in preparation.
7. Curtis, E. B.: Simplicial homotopy theory, *Adv. Math.* **6** (1971), 107–209.
8. Davis, J. F. and Lück, W.: Computations of K- and L-groups of groups rings based on isomorphism conjectures, in preparation.
9. tom Dieck, T.: Orbittypen und äquivariante Homologie I, *Arch. Math.* **23** (1972), 307–317.
10. tom Dieck, T.: *Transformation Groups*, Studies in Math. 8, de Gruyter, 1987.
11. Dror Farjoun, E.: Homotopy and homology of diagrams of spaces, in *Algebraic Topology* (*Seattle, Wash. 1985*), Lecture Notes in Math. 1286, Springer, New York, 1987, pp. 93–134.
12. Dwyer, W. and Kan, D.: A classification theorem for diagrams of simplicial sets, *Topology* **23** (1984), 139–155.
13. Elmendorf, A. D.: Systems of fixed point sets, *Trans. Amer. Math. Soc.* **227** (1983), 275–284.
14. Elmendorf, A. D., Kriz, I., Mandell, M. A. and May, P.: Rings, modules, and algebras in stable homotopy theory, Math. Surveys Monographs 47, Amer. Math. Soc., Providence, 1997.
15. Farrell, F. T. and Jones, L. E.: Isomorphism conjectures in algebraic K-theory, *J. Amer. Math. Soc.* **6** (1993), 249–298.
16. Ferry, S. C., Ranicki, A. and Rosenberg, J.: A history and survey of the Novikov Conjecture, in Proc. Conf. *Novikov Conjectures, Index Theorems and Rigidity*, I, Oberwolfach 1993, London Math. Soc. Lecture Notes Ser. 226, Cambridge, University Press, 1995, pp. 7–66.
17. Gersten, S. M.: On the spectrum of algebraic K-theory, *Bull. Amer. Math. Soc.* **78** (1972), 216–219 (1972).
18. Ghez, P., Lima, R. and Roberts, J. E.: W^*-Categories, *Pacific J. Math.* **120** (1985), 79–109.
19. Grayson, D.: Higher algebraic K-theory: II [after Daniel Quillen], in *Algebraic K-Theory, Evanston, 1976*, Lecture Notes in Math. 551, Springer, New York, 1976.

20. Higson, N., Pedersen, E. K. and Roe, J.: C^*-algebras and controlled topology, *K-Theory* **11** (1997), 209–239.
21. Hollender, J. and Vogt, R. M.: Modules of topological spaces, applications to homotopy limits and E_∞-structures, *Arch. Math.* **59** (1992), 115–129.
22. Lamotke, K.: *Semisimpliziale algebraische Topologie*, Grundlehren Math. Wiss. 147, Springer, New York, 1968.
23. Lück, W.: *Transformation Groups and Algebraic K-Theory*, Lecture Notes in Math. 1408, Springer, New York, 1989.
24. MacLane, S.: *Categories for the Working Mathematician*, Graduate Texts in Math. 5, Springer, New York, 1971.
25. May, P.: *Simplicial Objects in Algebraic Topology*, van Nostrand, New York, 1967.
26. Moerdijk, I. and Svensson, J.-A.: A Shapiro lemma for diagrams of spaces with applications to equivariant topology, *Compositio Math.* **96** (1995), 249–282.
27. Pearson, K.: Algebraic K-theory of two-dimensional crystallographic groups, to appear in *K-Theory*.
28. Pedersen, E. K. and Weibel, C. A.: *A non-connective delooping of algebraic K-theory*, in *Algebraic and Geometric Topology*, *Proc. Conf. Rutgers Univ., New Brunswick 1983*, Lecture Notes in Math. 1126, Springer, New York, 1985, pp. 166–181.
29. Pedersen, G. K. *C^*-algebras and their Automorphism Groups*, Academic Press, New York, 1979.
30. Piacenza, R. J.; Homotopy theory of diagrams and CW-complexes over a category, *Canad. J. Math.* **43** (1991), 814–824.
31. Powers, R. T.: Simplicity of the C^*-algebra associated with the free group on two generators, *Duke Math. J.* **42** (1975), 151–156.
32. Quinn, F.: Ends of maps II, *Invent. Math.* **68** (1982), 353–424.
33. Ranicki, A.: *Algebraic L-Theory and Topological Manifolds*, Cambridge Tracts in Math. 102, Cambridge University Press, 1992.
34. Segal, G. B.: Classifying spaces and spectral sequences, *Publ. I.H.E.S.* **34** (1968), 105–112.
35. Segal, G. B.: Categories and cohomology theories, *Topology* **13** (1974), 293–312.
36. Serre, J.-P.: Cohomologie des groupes discrets, in *Prospects in Mathematics*, Ann. Math Studies 70, Princeton University Press, 1981, pp. 77–169.
37. Steenrod, N. E.: A convenient category of topological spaces, *Michigan. Math. J.* **14** (1967), 133–152.
38. Switzer, R. M.: *Algebraic Topology–Homotopy and Homology*, Grundlehren Math. Wiss. 212, Springer, New York, 1975.
39. Wagoner, J. B.: Delooping classifying spaces in algebraic K-theory, *Topology* **11** (1972), 349–370.
40. Wegge-Olsen, N. E.: *K-Theory and C^*-Algebras*, Oxford University Press, 1993.
41. Weinberger, S.: *The Topological Classification of Stratified Spaces*, University of Chicago Press, 1994.
42. Weiss, M. and Williams, B: Assembly, in *Proc. Conf. Novikov Conjectures, Index Theorems and Rigidity* II, *Oberwolfach 1993*, Lecture Notes Ser. 227, Cambridge University Press, 1995, pp. 332–352.
43. Whitehead, G. W.: *Elements of Homotopy Theory*, Graduate Texts in Math., Springer, New York, 1978.
44. Yamasaki, M. : L-groups of crystallographic groups, *Invent. Math.* **88** (1987), 571–602.