CHROMATIC PICARD GROUPS AT LARGE PRIMES

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Abstract. As a consequence of the algebraicity of chromatic homotopy at large primes, we show that the Hopkins' Picard group of the $K(n)$-local category coincides with the algebraic one when $2p-2 > n^2+n$.

1. Introduction

If $\mathcal{C}$ is a symmetric monoidal ∞-category, then we can consider equivalence classes of invertible objects X, that is, those such that there exists a Y satisfying $X \otimes Y \simeq \mathbb{1}$. This is often a set, rather than a proper class, and it inherits a group multiplication induced from the tensor product. We call the resulting group the Picard group and denote it by $\mathrm{Pic}(\mathcal{C})$.

Following ideas of Hopkins, the study of Picard groups was brought into chromatic homotopy theory [HMS94], [Str92]. In this context, $\mathcal{C}$ is usually taken to be the ∞-category of $E(n)$- or $K(n)$-local spectra at a fixed prime.

As a general rule, one expects the answers to be algebraic when the prime is large compared to the height. To explain what we mean, let us focus on the $E(n)$-local case first. In this context, taking rational homology defines a homomorphism

$$H\mathbb{Q}_* : \mathrm{Pic}(\mathcal{S}p_{E(n)}) \to \mathrm{Pic}(\mathbb{Q}),$$

where by the latter we denote the Picard group of graded rational vector spaces, which is isomorphic to $\mathbb{Z}$. This homomorphism is in fact a split surjection, with splitting $k \mapsto S^k_{E(n)}$.

Then, it is a result of Hovey and Sadofsky that when $2p-2 > n^2+n$, the algebraic comparison map is an isomorphism, so that we have $\mathrm{Pic}(\mathcal{S}p_{E(n)}) \simeq \mathbb{Z}$ [HS99a]. This is in stark contrast with what happens at small primes; for example, we have $\mathrm{Pic}(\mathcal{S}p_{E(1)}) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ at $p=2$, and $\mathrm{Pic}(\mathcal{S}p_{E(2)}) \simeq \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$ at $p=3$ [HS99a], [GHMR14].

To study the $K(n)$-local case, one needs a more subtle algebraic invariant. More precisely, we define the completed E-homology as

$$E^\vee_* X := \pi_* L_{K(n)}(E \wedge X),$$

where E is the Morava E-theory spectrum of height n. When it's finitely generated, $E^\vee_* X$ has a canonical structure of an L-complete comodule over $E^\vee_* E$ [Bak09] [BH16][1.22]. The latter Hopf algebroid can be described explicitly as $E^\vee_* E \simeq \mathrm{map}^c(\mathbb{G}_n, E_*)$, the space of continuous functions on the Morava stabilizer group, with structure maps induced from the action of $\mathbb{G}_n$ [DH04].

If X is $K(n)$-locally invertible, then $E^\vee_* X$ is an invertible $E^\vee_* E$-comodule, which gives a homomorphism $\mathrm{Pic}(\mathcal{S}p_{K(n)}) \to \mathrm{Pic}(E^\vee_* E)$ into the algebraic Picard group, given by isomorphisms classes of invertible comodules.

The algebraic Picard group can be expressed in terms of cohomology of the Morava stabilizer group; to do so, one observes that an invertible $E^\vee_* E$-comodule is the same as an invertible E_*-module equipped with a compatible continuous action of $\mathbb{G}_n$. Since E_0 is a regular local ring, any such module is free of rank one, and so we have a short exact sequence

$$0 \to \mathrm{Pic}^0(E^\vee_* E) \to \mathrm{Pic}(E^\vee_* E) \to \mathbb{Z}/2 \to 0,$$

where $\mathrm{Pic}^0(E^\vee_* E)$ is the subgroup of those invertible modules which are concentrated in even degrees. Since E_* is 2-periodic, any such module is determined by its degree zero part, which yields an isomorphism $\mathrm{Pic}^0(E^\vee_* E) \simeq H^1_c(\mathbb{G}_n, E_0^\times)$ by standard considerations [GHMR14].

Due to a classical argument using the sparsity of the Adams-Novikov spectral sequence, one knows that $\mathrm{Pic}(\mathcal{S}p_{K(n)}) \to \mathrm{Pic}(E^{\vee}_{*}E)$ is injective when $2p-2 > n^2$ and $(p-1) \nmid n$ [HMS94][7.5]. On the other hand, surjectivity was not known except at low heights, where both sides can be computed explicitly.

Our main result gives a range in which the comparison map is in fact an isomorphism.

Theorem 1.1 (2.5)**.** *When $2p-2 > n^2+n$, $\mathrm{Pic}(\mathcal{S}p_{K(n)}) \to \mathrm{Pic}(E^{\vee}_{*}E)$ is an isomorphism.*

The proof of **Theorem 1.1** rests on the recent chromatic algebraicity result of the author which states that if $p > n^2+n+1$, then there exists an equivalence $h\mathcal{S}p_E \simeq h\mathcal{D}(E_*E)$ between the homotopy categories of E-local spectra and differential E_*E-comodules [Pst18].

In fact, to prove the isomorphism between Picard groups, we do not need the equivalence of homotopy categories, but only the weaker statement that any E_*E-comodule can be canonically realized as a homology of a certain E-local spectrum. Thus, **Theorem 1.1** holds in a slightly larger range of primes than chromatic algebraicity.

As was pointed to us by Paul Goerss, an alternative proof could be obtained using the descent spectral sequence for $S^0_{K(n)} \to E$, which is $K(n)$-local pro-Galois extension with Galois group $\mathbb{G}_n$ [Rog05]. Unfortunately, the Morava stabilizer group is profinite rather than finite, and to our knowledge no construction of the needed spectral sequence in this case appears in the literature.

Since it is of potential interest, we present the alternative approach in **Remark 2.6**, but it should be interpreted as conditional on the construction of the descent spectral sequence. The proof given in the main body of the paper is independant from this argument.

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2. Chromatic Picard groups at large primes

We let p denote the prime and n the height, both of which are fixed. By E we denote the Morava E-theory spectrum, this is an even periodic Landweber exact spectrum associated to the Lubin-Tate ring $E_0 \simeq W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]$. In particular, E_0 is a complete regular local ring of dimension n, with maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$.

The completion functor $M \to \varprojlim M/\mathfrak{m}^k M$ on E_*-modules is neither right or left exact, but it has a right exact left derived functor which we denote by L_0 [HS99b][Appendix A]. We say a module M is L-complete if the natural map $M \to L_0 M$ is an isomorphism. If M, N are modules, then we denote their L-complete tensor product by $M \widehat{\otimes}_{E_*} N := L_0(M \otimes_{E_*} N)$.

We let K denote the associated Morava K-theory spectrum; this is the unique E-module with homotopy groups $K_* \simeq E_*/\mathfrak{m}$. The spectrum K is Bousfield equivalent to the classical Morava K-theory spectrum $K(n)$ satisfying $K(n)_* \simeq \mathbb{F}_p[v_n^{\pm 1}]$.

Following [HMS94], [Str92], for a spectrum X we define its completed E-homology as

$$E^{\vee}_{*}X := \pi_* L_K(E \wedge X).$$

One can show that $E^{\vee}_{*}X$ is an L-complete E_*-module, and if it is finitely generated, then it has a structure of a comodule over $E^{\vee}_{*}E$ [HS99b][8.5], [Bak09], [BH16][1.22]. If X is finite, or more generally if E_*X is L-complete, then $E^{\vee}_{*}X \simeq E_*X$ [Hov04][3.2].

We would like to understand the Picard group $\mathrm{Pic}(\mathcal{S}p_K)$ of equivalence classes of invertible K-local spectra. We begin by recalling the following fundamental result.

Theorem 2.1 ([HMS94](1.3))**.** *A spectrum X is K-locally invertible if and only if $E^{\vee}_{*}X$ is free of rank one over E_*; equivalently, is an invertible $E^{\vee}_{*}E$-comodule.*

As a consequence of **Theorem 2.1**, we obtain a homomorphism $E^{\vee}_{*} : \mathrm{Pic}(\mathcal{S}p_K) \to Pic(E^{\vee}_{*}E)$ from the K-local Picard group into the Picard group of $E^{\vee}_{*}E$, given by isomorphisms classes of

$E^\vee_* E$-comodules which are invertible under the tensor product. One can describe $E^\vee_* E$ and the associated Picard group in terms of the Morava stabilizer group, which we now recall.

Since E is even periodic, it is complex orientable and the associated formal group is the universal deformation of the Honda formal group law Γ of height n over $\mathbb{F}_{p^n}$. This endows E_0 with an action of the Morava stabilizer group $\mathbb{G}_n := \mathrm{Aut}(\mathbb{F}_{p^n}, \Gamma)$, which by the Goerss-Hopkins-Miller theorem lifts to an action on E by maps of commutative ring spectra [GH].

The action of $\mathbb{G}_n$ on E induces an isomorphism $E^\vee_* E \simeq \mathrm{map}_c(\mathbb{G}_n, E_*)$, where the latter is the space of continuous functions on the Morava stabilizer group [DH04]. If M is an $E^\vee_* E$-comodule, then this identification endows it with an action of $\mathbb{G}_n$, and if M is finitely generated over E_*, then this action is continuous in the $\mathfrak{m}$-adic topology and any such continuous action determines a comodule structure [BH16][5.4].

We deduce that the data of an invertible $E^\vee_* E$-comodule is the same as that of a an invertible E_*-module equipped with a compatible continuous action of $\mathbb{G}_n$, this allows one to give a homological description of $\mathrm{Pic}(E^\vee_* E)$, as we recalled in the introduction.

Our goal is to prove that the homomorphism $\mathrm{Pic}(\mathcal{S}p_K) \to \mathrm{Pic}(E^\vee_* E)$ is an isomorphism at large primes. We start with injectivity, which is classical, but since the proof is enlightening, and not particularly difficult, we briefly recall the argument.

Proposition 2.2 ([HMS94](7.5))**.** *If $2p-2 \geq n^2$ and $(p-1) \nmid n$, then the comparison map $E^\vee_* : \mathrm{Pic}(\mathcal{S}p_K) \to \mathrm{Pic}(E^\vee_* E)$ is injective.*

Proof. Suppose that $X \in \mathrm{Pic}(\mathcal{S}p_K)$; since $E^\vee_* X$ is free of rank one, in particular finitely generated, we have the K-local E-based Adams spectral sequence of the form

$$\widehat{\mathrm{Ext}}^{s,t}_{E^\vee_* E}(E_*, E^\vee_* X) \Rightarrow \pi_{t-s} X$$

and an isomorphism $\mathrm{Ext}^{s,t}_{E^\vee_* E}(E_*, E^\vee_* X) \simeq H^s_c(\mathbb{G}_n, E^\vee_t X)$ between the E_2-term and the continuous cohomology of the Morava stabilizer group [BH16][3.1, 4.1].

The E_2-term is concentrated in internal degrees divisible by $2p-2$ and if $(p-1) \nmid n$, then it has a horizontal vanishing line at n^2, the homological dimension of the Morava stabilizer group [Hea15][4.2.1]. It follows that under the given assumptions the spectral sequence collapses for degree reasons.

Now, suppose that X is in the kernel of $E^\vee_* : \mathrm{Pic}(\mathcal{S}p_K) \to \mathrm{Pic}(E^\vee_* E)$, so that we have an isomorphism $E^\vee_* X \simeq E^\vee_* S^0_K \simeq E_*$. As observed above, the E-based Adams spectral sequence collapses, and it follows that the chosen isomorphism is necessarily an infinite cycle and so descends to an equivalence $X \simeq S^0_K$. This ends the argument. □

We move on to the surjectivity of the comparison map; this is the heart of the problem. We start with two short, technical lemmas.

Lemma 2.3. *We have $\varinjlim \mathrm{Ext}_{E_*}(E_*/\mathfrak{m}^k, K_*) \simeq K_*$, concentrated in homological degree zero.*

Proof. Since E_* is 2-periodic and the above modules are even graded, it is enough to prove that $\varinjlim \mathrm{Ext}_{E_0}(E_0/\mathfrak{m}^k, K_0) \simeq K_0$, concentrated in homological degree zero. Since E_0 is a regular local ring, local duality implies that $\varinjlim \mathrm{Ext}^i_{E_0}(E_0/\mathfrak{m}^k, K_0) \simeq \mathrm{Ext}^{n-i}_{E_0}(K_0, E_0)^\vee$, where by $(-)^\vee$ we denote the Matlis dual [BH98]. Because $K_0 \simeq E_0/\mathfrak{m}$ is the unique simple E_0-module, it is Matlis self-dual and we deduce that it is enough to show that $\mathrm{Ext}_{E_0}(K_0, E_0) \simeq K_0$, concentrated in homological degree n.

More generally, we claim that $\mathrm{Ext}_{E_0}(E_0/I_k, E_0) \simeq E_0/I_k$, concentrated in homological degree k, where $I_k = (p, u_1, \ldots, u_{k-1})$ for any $0 \leq k \leq n$. This is clear for $k = 0$ and the general case follows by induction from the long exact sequence of Ext-groups associated to

$$0 \to E_0/I_{k-1} \to E_0/I_{k-1} \to E_0/I_k \to 0,$$

which ends the proof. □

Lemma 2.4. *Let $X \simeq \varprojlim X_i$ be a limit diagram of K-local spectra such that X_i and X are K-locally dualizable. Then, for any K-local spectrum Y we have $L_K(Y \wedge X) \simeq \varprojlim L_K(Y \wedge X_i)$.*

Proof. Consider the collection of all K-local spectra Y such that the needed condition holds. Since X and X_i are dualizable, smashing with them preserves K-local limits and we deduce that this collection is closed under limits. Since it also contains S^0_K by assumption, we deduce that it is necessarily all of $\mathcal{S}p_K$ by [HS99b][7.5]. □

The following is the main result of this note.

Theorem 2.5. *Let $2p-2 > n^2+n$. Then, $E^\vee_* : \mathrm{Pic}(\mathcal{S}p_K) \to \mathrm{Pic}(E^\vee_* E)$ is an isomorphism.*

Proof. If $2p-2 > n^2+n$, then $2p-2 \geq n^2$ and $(p-1) \nmid n$ and we've seen in **Proposition 2.2** that under these conditions the homomorphism between Picard groups is injective.

To verify surjectivity, we have to prove that if $M \in \mathrm{Pic}(E^\vee_* E)$, there exists a K-locally invertible spectrum X with $E^\vee_* X \simeq M$. Observe that as an E_*-module, M is necessarily free of rank one [HS99b][A.9] and, without loss of generality, we can assume that it is even graded. Then, for each $k \geq 1$ we have $M/\mathfrak{m}^k M \simeq E_*/\mathfrak{m}^k$ as an E_*-module and so

$$E^\vee_* E \widehat{\otimes}_{E_*} M/\mathfrak{m}^k M \simeq E_* E \widehat{\otimes}_{E_*} M/\mathfrak{m}^k M \simeq E_* E \otimes_{E_*} M/\mathfrak{m}^k M,$$

where the first isomorphism is [HS99b][A.7] and the second follows from the fact that the last term is an $E_*/\mathfrak{m}^k$-module and so is already L-complete. Thus, we deduce that $M/\mathfrak{m}^k M$ is an E_*E-comodule in the usual, non-complete sense.

Under the assumption $2p-2 > n^2+n$, in [Pst18][2.14] we construct the Bousfield splitting functor $\beta : \mathcal{C}omod_{E_*E} \to h\mathcal{S}p_E$ valued in the homotopy category of E-local spectra with the property that $E_*\beta M \simeq M$ for any $M \in \mathcal{C}omod_{E_*E}$.

By construction, we have $E_*\beta(M/\mathfrak{m}^k M) \simeq M/\mathfrak{m}^k M$ and since the latter is L-complete, we deduce from [Hov04][3.2] that $E^\vee_*\beta(M/\mathfrak{m}^k M) \simeq E_*\beta(M/\mathfrak{m}^k M) \simeq M/\mathfrak{m}^k M$. Since these are concentrated in degrees divisible by $2p-2 > n$, the universal coefficient spectral sequence collapses and induces an isomorphism $\mathrm{Ext}_{E_*}(E_*/\mathfrak{m}^k, K_*) \simeq K^*\beta(M/\mathfrak{m}^k M)$ up to regrading.

We let $X_k := L_K\beta(M/\mathfrak{m}^k M)$; by the above, this is a K-local spectrum with $E^\vee_* X_k \simeq M/\mathfrak{m}^k M$ and, up to regrading, $K^* X_k \simeq \mathrm{Ext}_{E_*}(E_*/\mathfrak{m}^k, K_*)$. The first condition implies that $E^\vee_* X_k$ is degreewise finite and so X_k is a finite K-local spectrum of type n by [HS99b][8.5].

We have maps $X_k \to X_{k-1}$ induced from the projections $M/\mathfrak{m}^k \to M/\mathfrak{m}^{k-1}$, well-defined up to homotopy, and we let $X := \varprojlim{}_{\mathrm{holim}} X_k$ denote the corresponding homotopy limit. Here, by the latter we mean that we pick a lift of the tower of X_k to the ∞-category $\mathcal{S}p_K$ and we compute the limit there. It is classical that up to equivalence the homotopy limit does not depend on the choice of that lift, since it can be defined using the triangulated structure alone.

We first show that X is invertible. Since X_k are dualizable, we have $X \simeq D(\varinjlim DX_k)$, where $D := F(-, S^0_K)$ is the K-local Spanier-Whitehead dual and the colimit is the K-local one. Thus, it is enough to show that $\varinjlim DX_k$ is invertible; which we will verify by showing that $K_*(\varinjlim DX_k) \simeq K_*$. We have

$$K_*(\varinjlim DX_k) \simeq \varinjlim K^* X_k \simeq \varinjlim \mathrm{Ext}_{E_*}(E_*/\mathfrak{m}^k, K_*)$$

where we have used the description of K^*X_k given above. Then, the needed statement follows from **Lemma 2.3**, and we deduce that $\varinjlim DX_k$, hence X, is invertible.

Since X_k and X are K-locally dualizable, $L_K(E \wedge X) \simeq \varprojlim L_K(E \wedge X_k)$ by **Lemma 2.4**. After passing to homotopy groups, we obtain the Milnor exact sequence

$$0 \to \varprojlim{}^1 (M/\mathfrak{m}^k M)[-1] \to E^\vee_* X \to \varprojlim M/\mathfrak{m}^k M \to 0$$

and since M is free of rank one, the $\varprojlim^1$-term vanishes. We deduce that the second map must be an isomorphism, which ends the proof since $M \simeq \varprojlim M/\mathfrak{m}^k M$. □

Remark 2.6. The following alternative approach to the proof of **Theorem 2.5**, based on the descent spectral sequence, was pointed to us by Paul Goerss. As explained in the introduction, the needed spectral sequence does not seem to appear in the literature, and what follows should be interpreted as conditional on its existence.

If $\mathcal{C}$ is a presentably symmetric monoidal ∞-category, then the Picard group can be lifted to the Picard space, which we will denote by $\mathcal{P}ic(\mathcal{C})$ [MS16]. The latter is the ∞-groupoid of invertible objects in $\mathcal{C}$; it is an $\mathbb{E}_\infty$-space with multiplication induced from the tensor product.

The Picard group itself can be recovered through the relation $\mathrm{Pic}(\mathcal{C}) = \pi_0\mathcal{P}ic(\mathcal{C})$. The higher homotopy groups of the Picard space are easy to describe, as we have $\pi_t\mathcal{P}ic(\mathcal{C}) \simeq \pi_{t-1}\mathrm{aut}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ for $t > 0$, where $\mathbb{1}$ is the monoidal unit and $\mathrm{aut}_{\mathcal{C}}$ denotes the space of self-equivalences.

By the work of Devinatz and Hopkins, the map $S^0_K \to E$ of commutative ring spectra is a $K(n)$-local pro-Galois extension in the sense of Rognes with Galois group $\mathbb{G}_n$ [DH04], [Rog05]. One then expects that there should exist a descent spectral sequence

$$H^s_c(\mathbb{G}_n, \pi_t\mathcal{P}ic(\mathcal{M}od_E)) \Rightarrow \pi_{t-s}\mathcal{P}ic(\mathcal{S}p_K)$$

with differentials d_r of degree $(r, r-1)$ and where the action of $\mathbb{G}_n$ on $\mathcal{M}od_E$ is induced from that on E. A spectral sequence of this precise from does not appear in the literature, but a construction for finite Galois groups can be found in [MS16], [GL16].

To get hold on the E_2-term, we need to understand the homotopy of the Picard space of $\mathcal{M}od_E$, but this is not difficult. Since E is even periodic and E_0 is regular local, any invertible E-module is free and so $\pi_0\mathcal{P}ic(\mathcal{M}od_E) \simeq \mathbb{Z}/2$ [BR05]. Moreover, because E is the monoidal unit of $\mathcal{M}od_E$, we have $\pi_1\mathcal{P}ic(\mathcal{M}od_E) \simeq E_0^\times$ and $\pi_t\mathcal{P}ic(\mathcal{M}od_E) \simeq E_{t-1}$ for $t \geq 2$.

If $(p-1) \nmid n$, then the Morava stabilizer group is of finite homological dimension n^2 and the E_2-term has a horizontal vanishing line. Furthermore, by standard considerations $H^s(\mathbb{G}_n, E_t)$ vanishes unless t is divisible by $2p-2$ [Hea15][4.2.1].

It follows that if $2p - 2 \geq n^2$ and $(p-1) \nmid n$, then if drawn using the Adams grading, the $-1 \leq t - s \leq 1$ region of the above spectral sequence looks like

$$\begin{array}{ccc} H^1_c(\mathbb{G}_n, \mathbb{Z}/2) & H^1_c(\mathbb{G}_n, E_0^\times) & 0 \\ & & \\ 0 & H^0_c(\mathbb{G}_n, \mathbb{Z}/2) & H^0_c(\mathbb{G}_n, E_0^\times) \\ & \xrightarrow{\quad t-s \quad} & \end{array},$$

with only zeroes above. We deduce that in this range this spectral sequence collapses and yields a short exact sequence

$$0 \to H^1_c(\mathbb{G}_n, E_0^\times) \to \pi_0\mathcal{P}ic(\mathcal{S}p_K) \to \mathbb{Z}/2 \to 0.$$

This means that the topological Picard group $\mathrm{Pic}(\mathcal{S}p_K) \simeq \pi_0\mathcal{P}ic(\mathcal{S}p_K)$ fits into a short exact sequence of the same form as the algebraic one, as explained in the introduction. One can then verify that $\mathrm{Pic}(\mathcal{S}p_K) \to \mathrm{Pic}(E^\vee_*E)$ fits into a map of short exact sequences which is then an isomorphism by the five-lemma, giving a different proof of **Theorem 2.5**.

In fact, the bound obtained in this way is slightly sharper, as one only needs $2p - 2 \geq n^2$, rather than $2p - 2 > n^2 + n$. This comes from the fact that this argument avoids the use of the E-local category, since the homological dimension of E_*E is $n^2 + n$, while the homological dimension of $\mathbb{G}_n$ is just n^2.

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