

Thom Complexes and the Spectrum $\mathrm{tmf}_{(2)}$

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Abstract

Many interesting spectra can be constructed as Thom spectra of easily constructed bundles. Mahowald [2] showed that bu and bo cannot be realized as E_1 Thom spectra. We use related techniques to show that $\mathrm{tmf}_{(2)}$ also cannot be realized as an E_1 Thom spectrum.

1 Introduction

Theorem 1.1. *The spectrum $\mathrm{tmf}_{(2)}$ is not a Thom spectrum of an H -map from a loop space to $\mathrm{BGL}_1(S)$.*

We will prove this by contradiction. We show:

Proposition 1.2. *Suppose that Z is a loop space and $f : Z \rightarrow \mathrm{BGL}_1(S)$ is an H -map such that the Thom spectrum of f is $\mathrm{tmf}_{(2)}$. Then there are spaces X and Y with cell structures as in Section 1 and a map $g : \Sigma^8 X \rightarrow Y$ such that the cohomology of the cofiber C has a cup product $x_9 x_{13} = x_{22}$ where $x_9 \in H^9(C; \mathbb{Z})$, $x_{13} \in H^{13}(C; \mathbb{Z})$, and $x_{22} \in H^{22}(C; \mathbb{Z})$ are generators.*

Proposition 1.3. *Suppose X and Y are spaces with cell structures as in Figure 1, suppose $g : \Sigma^8 X \rightarrow Y$ is any map and C is the cofiber of g . Then there is a space D with $H^*(D; \mathbb{Z}) \cong \mathbb{Z}\{x_9, x_{13}, x_{22}\}$ and a map $D \rightarrow C$ inducing a surjection on cohomology.*

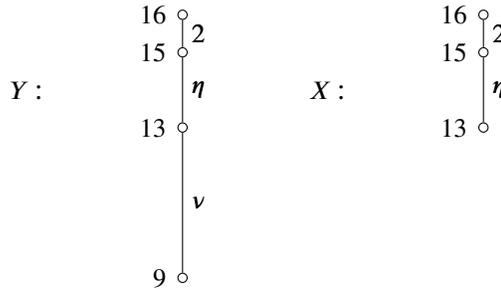


Figure 1: The cell structures of spaces X and Y .

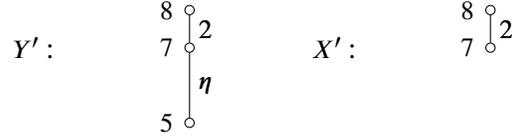


Figure 2: The cell structures of X' and Y' .

If we take the map g to be the map produced in Proposition 1.2, then the resulting space D has a perfect cup pairing $x_9x_{13} = x_{22}$. James has a classification theorem that says what attaching maps and cup product structures are possible on 3-cell CW complexes [1, Theorem 1.2]. Using this we show:

Proposition 1.4. *Suppose D is a space with $H^*(D; \mathbb{Z}) \cong \mathbb{Z}\{x_9, x_{13}, x_{22}\}$ and $Sq^4(\bar{x}_9) = \bar{x}_{13}$ where \bar{x}_i denotes the image of x_i under the reduction map $H^*(D; \mathbb{Z}) \rightarrow H^*(D; \mathbb{Z}/2)$. Then $x_9x_{13} = 2kx_{22}$ for some $k \in \mathbb{Z}$.*

Propositions 1.3 and 1.4 show that the conclusion of Proposition 1.2 is a contradiction, which proves Theorem 1.1.

1.1 Comparison to Mahowald

Our argument is closely based on Mahowald's argument in [2] that bo is not a Thom spectrum. For comparison, we reformulate Mahowald's argument in a parallel form to ours to make the similarities and the differences apparent. Proposition 1.2 is an analogue of:

Proposition 1.5 ([2, Discussion on page 294]). *Suppose that Z is a loop space and $f : Z \rightarrow BGL_1(S)$ is an H -map such that the Thom spectrum of f is bo . Then there are spaces X' and Y' with cell structures as indicated in Section 1.1 and a map $\Sigma^4 X' \rightarrow Y'$ such that the cohomology of the cofiber C' has a cup product $x_5x_7 = x_{13}$ where $x_5 \in H^5(C'; \mathbb{Z})$, $x_7 \in H^7(C'; \mathbb{Z})$, and $x_{12} \in H^{12}(C'; \mathbb{Z})$ are generators.*

Proposition 1.3 is an analogue of:

Proposition 1.6 ([2, Lemma 3 and discussion on page 294]). *Suppose $g : \Sigma^4 X' \rightarrow Y'$ is any map and C' is the cofiber. Then there is a space D' with $H^*(D'; \mathbb{Z}) \cong \mathbb{Z}\{x_5, x_7, x_{12}\}$ and a map $D' \rightarrow C'$ inducing a surjection on cohomology.*

Again taking g to be the map produced in Proposition 1.2, then the resulting space D' has a perfect cup pairing $x_5x_7 = x_{12}$. This means that D' is an S^5 bundle over S^7 . The 7-cell in D' is attached to the 5-cell by an η so D' has no section. Mahowald deduces a contradiction:

Lemma 1.7 ([2, Lemma 4]). *Every 5-sphere bundle over S^7 has a section.*

The proof of Proposition 1.2 is exactly the same as the proof of Proposition 1.5, we merely fill in details. The proof of Proposition 1.3 is completely different from the proof of Proposition 1.6. The analog of Lemma 1.7 in our setting would state that every 9-sphere bundle over S^{13} has a section, but this is false – using [1, Theorem 1.2], it is possible to show that there exists a space D with $H^*(D; \mathbb{Z}) = \mathbb{Z}\{x_9, x_{13}, x_{22}\}$, with $D^{(13)} \simeq C(2v_9)$ and with $x_9x_{13} = x_{22}$. This is an S^9 bundle over S^{13} with no section. We deduce Lemma 1.7 from the following analog of Proposition 1.4:

Proposition 1.8. *Suppose D' is a space with $H^*(D'; \mathbb{Z}) \cong \mathbb{Z}\{x_5, x_7, x_{12}\}$ and $Sq^2(\bar{x}_5) = \bar{x}_7$ where \bar{x}_i denotes the image of x_i under the reduction map $H^*(D'; \mathbb{Z}) \rightarrow H^*(D'; \mathbb{Z}/2)$. Then $x_9 x_{13} = 2k x_{22}$ for some $k \in \mathbb{Z}$.*

Proof that Proposition 1.8 implies Lemma 1.7. A 5-sphere bundle over S^7 is a space D' with a 5-cell, a 7-cell, and a 12-cell, where in $H^*(X)$, $x_5 x_7 = x_{12}$. Such a space has a section if the attaching map $S^6 \rightarrow S^5$ of the 7-cell is null. Proposition 1.8 says that the attaching map cannot be η_5 , so the remaining possibility is that it is null. \square

Mahowald has a different proof of Lemma 1.7.

Acknowledgements

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2 Proof of Proposition 1.2

For a space X let $X^{(d)}$ denote the d -skeleton of X and let $X_{(d)}$ denote the cofiber of the inclusion map $X^{(d-1)} \rightarrow X$.

Proof of Proposition 1.2. Suppose that Z is a loop space and $Z \rightarrow \text{BGL}_1(S)$ is an H -map such that the Thom spectrum is $\text{tmf}_{(2)}$. Then $H_*(Z; \mathbb{F}_2) \cong \mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}, x_{31}, \dots]$ where under the Thom isomorphism $H_*(Z) \cong H_*(\text{tmf}_{(2)}; \mathbb{F}_2) = \mathbb{F}_2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \xi_5, \dots]$, the class x_i maps to the multiplicative generator of $H_*(\text{tmf}_{(2)}; \mathbb{F}_2)$ in the same degree. Because Z is a loop space, the inclusion $Z^{(15)} \rightarrow Z$ of the 15 skeleton of Z extends to a map $f : \Omega \Sigma Z^{(15)} \rightarrow Z$. Let F be the fiber of f and let $\bar{F} = F^{(25)}$. Let $g : \Sigma \bar{F} \rightarrow \Sigma Z^{(15)}$ be the adjoint to inclusion of fiber map $\bar{F} \rightarrow Z^{(15)}$. The Steenrod action on the homology of $\Sigma Z^{(15)}$ shows that it has the cell structure indicated for the space Y , so we can take $Y = \Sigma Z^{(15)}$ and $X = Y_{(13)}$. By Lemmas 2.1 and 2.2, we are finished. \square

Lemma 2.1. *There is a unique space X with cell structure as in Section 1. In the context of the proof of Proposition 1.2, $\Sigma \bar{F} \simeq \Sigma^8 X$.*

Proof. Suppose X_1 and X_2 are two spaces with the cell structure as in Section 1. Since the bottom cell of X is in dimension 13 and the top cell is in dimension $16 < 2 \times 13 - 1$, there is an isomorphism $[X_1, X_2] \rightarrow [\Sigma^{\infty+} X_1, \Sigma^{\infty} X_2]$ so it suffices to show that $\Sigma^{\infty} X$ is uniquely determined by its \mathbb{F}_2 cohomology. The E_2 page of the Adams spectral sequence $\text{Ext}(H_*(X_1; \mathbb{F}_2), H_*(X_2; \mathbb{F}_2))$ is displayed in Figure 3, and the (-1) -stem is empty, so X is uniquely determined by its cohomology.

So to show $\Sigma \bar{F} \simeq \Sigma^8 X$ it suffices to check that that $H^*(\Sigma \bar{F}; \mathbb{F}_2) \cong H^*(\Sigma^8 X; \mathbb{F}_2)$. We are computing the fiber of $f : \Omega \Sigma Z^{(15)} \rightarrow Z$ through dimension 23. The homology $H_*(\Omega \Sigma Z^{(15)}; \mathbb{F}_2)$ is the free associative algebra on $H_*(Z^{(15)}; \mathbb{F}_2)$ and $H_*(f; \mathbb{F}_2)$ is the map $\mathbb{F}_2\langle x_8, x_{12}, x_{14}, x_{15} \rangle \rightarrow \mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}, x_{31}, \dots]$ from the associative algebra to the commutative algebra with kernel generated by commutators, surjective through dimension 30. Thus, f is 20-connective and since Z is 8-connective, and the sequence $0 \rightarrow H_*(F) \rightarrow H_*(\Omega \Sigma Z^{(15)}) \rightarrow H_*(Z) \rightarrow 0$ is exact through dimension 27. Thus, $H_*(\bar{F}) = \mathbb{F}_2\{[x_8, x_{12}], [x_8, x_{14}], [x_8, x_{15}]\}$, where the coaction comes from the coaction on the x_i 's. We conclude that $H_*(\Sigma \bar{F}; \mathbb{F}_2) \cong H_*(\Sigma^8 X; \mathbb{F}_2)$. \square

Lemma 2.2. *In the context of the proof of Proposition 1.2, the space C has cohomology ring as follows: $H^*(C; \mathbb{F}_2) = \mathbb{F}_2\{\alpha_9, \beta_{13}, \gamma_{15}, \delta_{16}, \alpha_9\beta_{13}, \alpha_9\gamma_{15}, \alpha_9\gamma_{16}\}$. The Steenrod action is generated by $Sq^4(\alpha_9) = \beta_{13}$, $Sq^2(\beta_{13}) = \gamma_{15}$ and $Sq^1(\gamma_{15}) = \delta_{16}$.*

Proof. The fiber sequence $F \rightarrow \Omega\Sigma Z^{(15)} \rightarrow Z$ deloops to $F' \rightarrow \Sigma Z^{(15)} \rightarrow BZ$, so $F = \Omega F'$. Let $h: \bar{F} \rightarrow \Omega\Sigma Z^{(15)}$ be the inclusion of the fiber. The map $g: \Sigma\bar{F} \rightarrow \Sigma Z^{(15)}$ is the composite $\Sigma\bar{F} \rightarrow \bar{F}' \rightarrow \Sigma Z^{(15)}$. Since the composition $\bar{F}' \rightarrow \Sigma Z^{(15)} \rightarrow BZ$ is null there are maps in the following diagram:

$$\begin{array}{ccccc}
 \Sigma\bar{F} \simeq \Sigma\Omega\bar{F}' & & & & C \\
 \downarrow & \searrow & & \nearrow & \downarrow \\
 F' & \longrightarrow & \Sigma X & \longrightarrow & C' \\
 & & & \searrow & \downarrow \\
 & & & & BZ
 \end{array}$$

where C' is the cofiber of $F' \rightarrow \Sigma X$. I claim that the maps $C \rightarrow C'$ and $C' \rightarrow BZ$ induce isomorphisms in cohomology in degree ≤ 22 . Since BZ is 9-connective and F' is 21-connective, by [4, Theorem 6.1] the map $C' \rightarrow BZ$ induces an isomorphism in cohomology through degree $21 + 9 - 1 = 29$ and the map $\Sigma\Omega F' \rightarrow F'$ induces an isomorphism in cohomology through degree $21 + 20 - 1 = 40$ so the map $C \rightarrow C'$ induces an isomorphism in cohomology through dimension 41. It remains to compute the cohomology of BZ in this range. The homology of Z is polynomial, so the cohomology is a divided power algebra $H^*(Z) = \Gamma[y_8, y_{12}, y_{14}, y_{15}, y_{31}, \dots]$ so it follows that $H^*(BZ) = \Lambda(\sigma y_8, \sigma y_{12}, \sigma y_{14}, \sigma y_{15}, \sigma y_{31}, \dots)$. \square

3 Extracting the three-cell complex

In this section we prove Proposition 1.3. We show in Lemma 3.2 that any composite $\Sigma^8 X \rightarrow Y \rightarrow X$ is a smash product $\alpha \wedge id_X$ for some $\alpha \in \pi_8(S)$. Let $i: S^{21} \rightarrow \Sigma^8 X$ be the inclusion of the bottom cell. In Lemma 3.4 we show that because the map $\Sigma^8 X \rightarrow Y \rightarrow X$ is a smash product, any composite $S^{21} \rightarrow \Sigma^8 X \rightarrow Y \rightarrow Y_{(15)}$ is null. We deduce that $g \circ i$ factors through the 21 skeleton of the fiber of $Y \rightarrow Y_{(15)}$, which is $C(v_9)$ by Lemma 3.3. From this we deduce Proposition 1.3.

Lemma 3.1. *The map $\pi_8(S) \rightarrow [\Sigma^8 X, X]$ given by $\alpha \mapsto \alpha \wedge id_X$ is injective.*

Proof. First note that since X has its bottom cell in dimension 13 and $\Sigma^8 X$ has its top cell in dimension 23 which is less than or equal to $2 \times 13 - 2$, there is an isomorphism $[\Sigma^8 X, X] \rightarrow [\Sigma^{\infty+8} X, \Sigma^{\infty} X]$. I claim that the further composition $\pi_8(S) \rightarrow [\Sigma^{\infty+8} X, \Sigma^{\infty} X] \rightarrow [\Sigma^{\infty+8} X, \Sigma^{\infty} S^{16}]$ is injective, where the second map is squeezing off to the top cell of X . Let $A = \Sigma^{13} D\Sigma^{\infty} X$, which has the following cell structure:

$$\begin{array}{c}
 3 \circ \\
 \left. \vphantom{3} \right\} \eta \\
 1 \circ \\
 \left. \vphantom{1} \right\} 2 \\
 0 \circ
 \end{array}$$

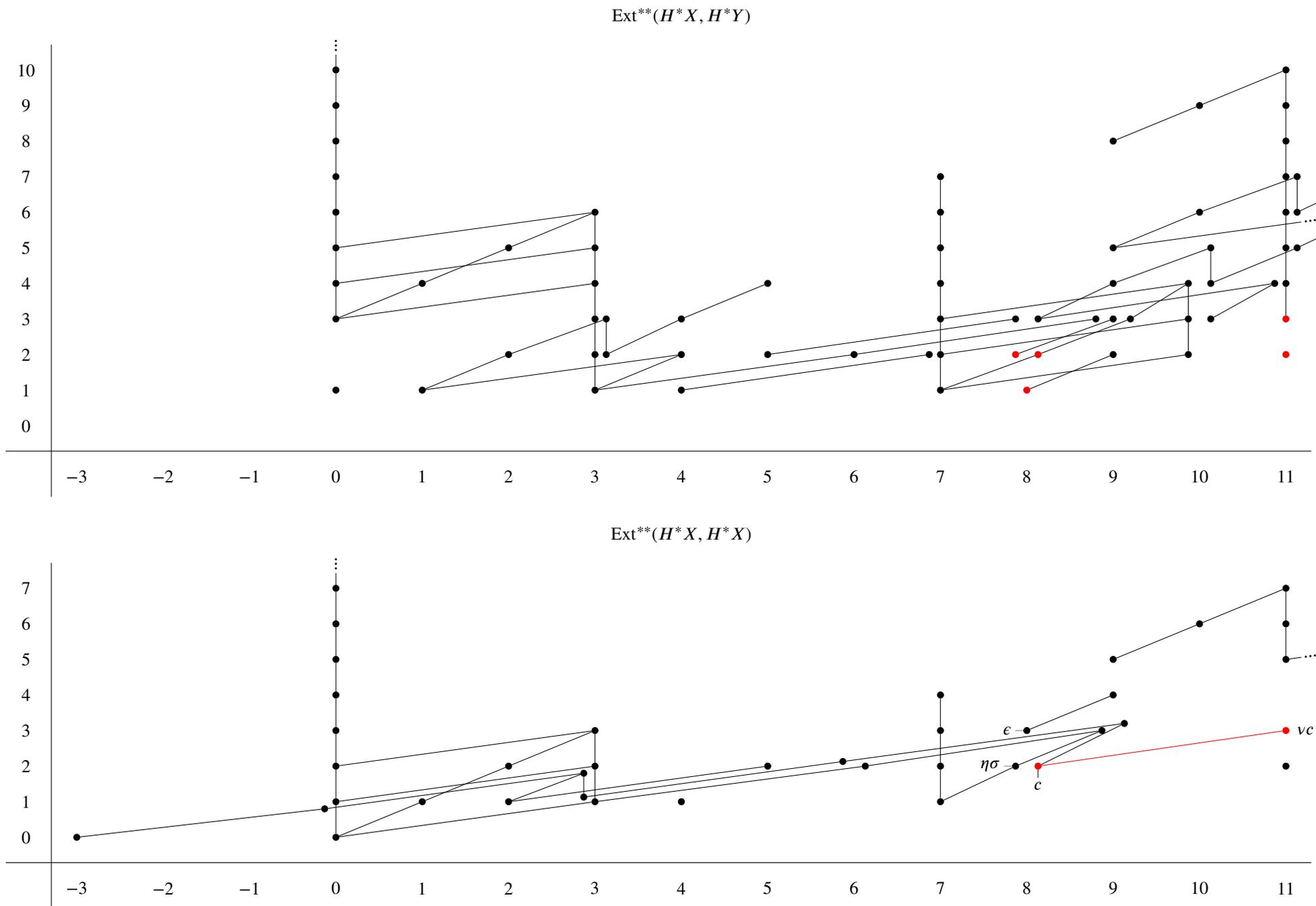


Figure 3: The E_2 pages of the Adams spectral sequences computing $\pi_*F(\Sigma^\infty X, \Sigma^\infty Y)$ (top) and $\pi_*F(\Sigma^\infty X, \Sigma^\infty X)$ (bottom).

The map $\pi_8(S) \rightarrow [\Sigma^{\infty+8}X, \Sigma^{\infty}X] \rightarrow \pi_8(A)$ is induced by the inclusion of the bottom cell of A . Thus, we need to show that no Atiyah Hirzebruch differentials hit $\eta\sigma$ or ϵ on the bottom cell of A . The first differential is multiplication by two, and since $\pi_8(S)$ is all two-torsion, this does not hit anything. The second differential is given by the Toda bracket $\langle -, \eta, 2 \rangle : \ker(\eta : \pi_6(S) \rightarrow \pi_7(S)) \rightarrow \pi_8(S)$. Since $\pi_6(S)$ consists just of v^2 , it suffices to show that the Toda bracket $\langle v^2, \eta, 2 \rangle = 0$. By shuffling, $\langle v^2, \eta, 2 \rangle = \langle v, v\eta, 2 \rangle = 0$ because $v\eta = 0$, and the indeterminacy is the image of v^2 and 2 in π_8 which is trivial. \square

Lemma 3.2. *The image of $\pi_8 F(X, Y) \rightarrow \pi_8 F(X, X)$ is contained in the image of the map $\pi_8 S \rightarrow \pi_8 F(X, X)$ – that is, they are maps of the form $\alpha \wedge id_X$.*

Proof. Refer to Figure 3. By Lemma 3.1, the two classes labeled $\eta\sigma$ and ϵ are the images of $\eta\sigma, \epsilon \in \pi_8(S)$. Note that the 8-stem in $\text{Ext}^{**}(H^*X, H^*X)$ is $\mathbb{Z}/2\{\epsilon, \eta\sigma, c\}$ so that $\pi_8 F(X, X)$ is either $(\mathbb{Z}/2)^2$ or $(\mathbb{Z}/2)^3$ depending on whether or not the class c supports an Adams d_2 hitting 8σ . If it does support such a differential, the map $\pi_8 S \rightarrow \pi_8 F(X, X)$ is surjective and we're done. Otherwise, it suffices to show that c is not in the image of the map $p : \pi_8(\Sigma^{\infty}X, \Sigma^{\infty}Y) \rightarrow \pi_8(\Sigma^{\infty}X, \Sigma^{\infty}X)$. I claim that for all x in the 8-stem of $\text{Ext}^{**}(H^*X, H^*Y)$, vx is detected in filtration at least 4. Since vc is nonzero and in bidegree $(11, 3)$, this implies that c is not in the image of p . To see that vx is detected in filtration at least 4 note that v multiplication on the 8-stem is zero in the associated graded, so multiplication by v raises filtration by at least 2. This implies that vx cannot be the class in $(11, 2)$. Since the class in $(11, 3)$ is 256-torsion, it can't be divisible by v so vx is detected in filtration at least 4 as needed. \square

Lemma 3.3. *Let F be the fiber of $Y \rightarrow Y_{(15)}$. Then $F^{(21)} \simeq C(v_9)$.*

Proof. Since the composite $C(v_9) = Y^{(13)} \rightarrow Y \rightarrow Y_{(15)}$ is null, there is a natural map $C(v_9) \rightarrow F$. We compute the Serre spectral sequence for the cohomology of the fiber sequence $F \rightarrow Y \rightarrow Y_{(15)}$ and see that the map $C(v_9) \rightarrow F$ is an equivalence through degree 22. See Figure 4. \square

Lemma 3.4. *There is a commutative square:*

$$\begin{array}{ccccc} S^{21} & \longrightarrow & C(v_9) & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^8 X & \xrightarrow{g} & Y & \longrightarrow & C \end{array}$$

where rows are cofiber sequences, the map $S^{21} \rightarrow \Sigma^8 X$ is the inclusion of the bottom cell and the map $C(v_9) \rightarrow Y$ is the inclusion of the fiber of $Y \rightarrow Y_{(15)}$. The map $D \rightarrow C$ is an isomorphism in cohomology in degrees 9, 13, and 22.

Proof. By Lemma 3.2, the composite $\Sigma^8 X \rightarrow Y \rightarrow X$ is a smash product $\alpha \wedge id_X$ for some $\alpha \in \pi_8 S$. We get a commutative square:

$$\begin{array}{ccccc} S^{21} & \xrightarrow{\alpha} & S^{13} & \longrightarrow & * \\ \downarrow \Sigma^8 i & & \downarrow i & & \downarrow \\ \Sigma^8 X & \xrightarrow{g} & Y & \longrightarrow & X \\ & \searrow \alpha \wedge id_X & & & \downarrow \\ & & X & \longrightarrow & X_{(15)} \end{array}$$

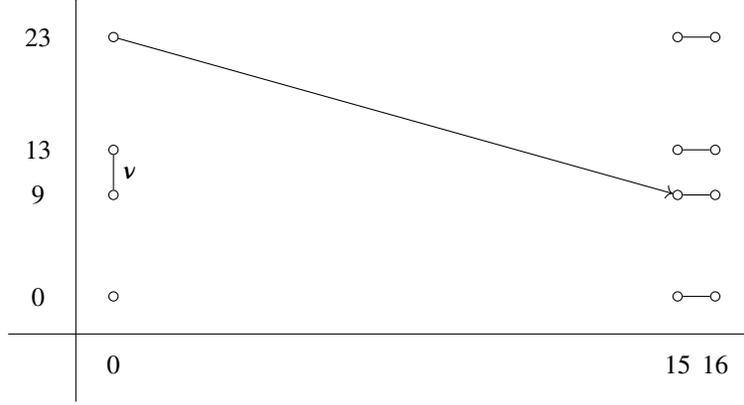


Figure 4: The Serre spectral sequence for $F \rightarrow Y \rightarrow Y_{(15)}$

thus the composite of $g \circ \Sigma^8 i : S^{21} \rightarrow Y$ with the projection $Y \rightarrow Y_{(15)}$ is null and factors through the fiber F of $Y \rightarrow Y_{(15)} = X_{(15)}$. In fact it factors through $F^{(21)} \simeq C(v_9)$. Thus there is a map h making the following diagram commute:

$$\begin{array}{ccc} S^{21} & \xrightarrow{h} & C(v_9) \\ \downarrow & & \downarrow \\ \Sigma^8 X & \xrightarrow{g} & Y \end{array}$$

The map $S^{21} \rightarrow \Sigma^8 X$ is an isomorphism in cohomology through dimension 21, and the map $C(v_9) \rightarrow Y$ is an isomorphism in cohomology through dimension 13 and also in dimension 22, so the map $D \rightarrow C$ is an isomorphism in cohomology in dimensions 9, 13 and 21. \square

Proposition 1.3 is an immediate consequence of Lemma 3.4.

4 Unstable calculations to prove Proposition 1.4

The main ingredient of Proposition 1.4 is the following theorem of James, which tells us which cup product structures on 3-cell complexes exist. Suppose that K is a three cell CW complex with cells in dimension q , n , and $n + q$. For $\alpha \in \pi_{n-1} S^q$ and m an integer, say that K has type (m, α) if the attaching map of the n cell to the q cell is given by α and the integral cohomology $H^*(K; \mathbb{Z}) = \mathbb{Z}\{x_q, y_n, z_{n+q}\}$ has cup product $x_q y_n = m z_{n+q}$.

Theorem 4.1 ([1, Theorem 1.2]). *Let $\alpha \in \pi_{n-1}(S^q)$ where $n - 1 > q \geq 2$. Let $[\alpha, i_q]$ denote the Whitehead product of α and a generator $i_q \in \pi_q(S^q)$. There exists a complex K of type (m, α) if and only if $m[\alpha, i_q]$ is contained in the image of left composition with α : $\alpha_* : \pi_{n+q-2}(S^{n-1}) \rightarrow \pi_{n+q-2}(S^q)$.*

We apply this with $\alpha = v_9$ to show that no three-cell complex of type $(1, v_9)$ exists, which is a reformulation of Proposition 1.4:

Proposition 4.2. *A three cell complex D of type (m, v_9) exists if and only if m is even.*

Lemma 4.3. *The map $v_9 : \pi_{20}(S^{12}) \rightarrow \pi_{20}(S^9)$ is zero.*

Proof. According to [3, Theorem 7.1], $\pi_{20}S^{12} = \mathbb{Z}/2\{\epsilon_{12}, \bar{v}_{12}\}$. By [3, Equations 7.18], $v_6 \circ \epsilon_9 = 2\bar{v}_6 \circ v_{14}$ so suspending this gives $v_9 \circ \epsilon_{12} = 2\bar{v}_9 \circ v_{17}$. According to [3, Theorem 7.1], $2\bar{v}_7 = 0$ so $v_9 \circ \epsilon_{12} = 0$.

By [3, Lemma 6.4], $\eta_{12} \circ \sigma_{13} = \epsilon_{12} + \bar{v}_{12}$. Since $v_6 \circ \eta_9 = 0$, we see that $v_9 \circ \bar{v}_{12} = v_9 \circ (\epsilon_{12} + \bar{v}_{12}) = v_9 \circ \eta_{12} \circ \sigma_{13} = 0$. \square

Lemma 4.4. *The Whitehead product $[v_9, i_9]$ is of order two.*

Proof. First note that $[v_9, i_9] = [i_9, i_9] \circ v_{17}$. By [3, Theorem 7.1], $\pi_{17}S^9 = \mathbb{Z}/2\{\epsilon_9, \bar{v}_9, \sigma_9 \circ \eta_{16}\}$, where under suspension $\bar{v}_{10} = \epsilon_{10} + \sigma_{10} \circ \eta_{17}$. By [3, Theorem 7.4], $\pi_{20}S^9 = \mathbb{Z}/8\{\zeta_9\} \oplus \mathbb{Z}/2\{\bar{v}_9 \circ v_{17}\}$, where the element ζ_9 is stably $P(v)$ and $\bar{v}_9 \circ v_{17}$ is in the kernel of suspension. According to [3, Equation 7.1], $[i_9, i_9] = \bar{v}_9 + \epsilon_9 + \sigma_9 \circ \eta_{16}$ is the nonzero element of the kernel of suspension in $\pi_{17}S^9$. Since $\eta_{16} \circ v_{17} = 0 = \epsilon_9 \circ v_{17}$, we have $[v_9, i_9] = [i_9, i_9] \circ v_{17} = \bar{v}_9 \circ v_{17}$ is the nontrivial element of the kernel of suspension in $\pi_{20}S^9$. \square

Proof of Proposition 4.2. We apply Theorem 4.1 with $q = 9$, $n = 13$, $\alpha = v_9$. By Lemma 4.4, the Whitehead product $[v_9, i_9]$ is a nonzero two-torsion element, but by Lemma 4.3, the image of $v_9 : \pi_{20}S^{12} \rightarrow \pi_{20}S^9$ is zero. A type (m, v_9) complex exists if and only if $m[v_9, i_9] = 0$ which is true when m is even. \square

References

- [1] IM James. Note on cup-products. *Proceedings of the American Mathematical Society*, 8(2):374–383, 1957.
- [2] Mark Mahowald. Thom complexes and the spectra bo and bu . In Haynes R. Miller and Douglas C. Ravenel, editors, *Algebraic Topology*, pages 293–297, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.
- [3] Hiroshi Toda. *Composition Methods in Homotopy Groups of Spheres*, volume 49 of *Annals of Mathematics*. Princeton University Press, 1962.
- [4] George W Whitehead. *Elements of Homotopy Theory*, volume 61 of *Graduate Texts in Mathematics*. Springer Science & Business Media, 1978.