THE CATEGORY OF CGWH SPACES

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It is well-known that the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces is a convenient setting for homotopy theory and algebraic topology. In this paper we give an expository account of this category. Most of what we cover is well-known, but we have added some points about the interaction between limits and colimits that we have found useful in applications to be discussed elsewhere. There are several places in the literature where one can read about compactly generated spaces (such as [3]) and other places where one can learn how to modify the theory to include the weak Hausdorff condition (such as [2]). There is also an account of CGWH spaces in the unpublished PhD thesis of Jim McClure. However, we are not aware of an openly available, resonably self-contained exposition, apart from this one. We assume general familiarity with category theory, and the main properties of compact Hausdorff spaces, and little else.

1. Basic definitions

It will be convenient to think of a topology as being specified by the collection of *closed* sets. Thus, a topology on a set X will mean a collection ζ of subsets of X that is closed under finite unions and arbitrary intersections, and contains $\emptyset$ and X. We write $F \subseteq_C X$ and $U \subseteq_O X$ to indicate that F is a closed subset of X, and U is an open subset of X.

Definition 1.1. A subset $Y \subseteq X$ is *k-closed* if $u^{-1}Y$ is closed in K for every compact Hausdorff space K and every continuous map $u\colon K \to X$. We write $k(\zeta)$ for the collection of k-closed sets. It is easy to check that this is a topology on X, and that $\zeta \subseteq k(\zeta)$. We write kX for the set X equipped with the topology $k(\zeta)$. We say that X is *compactly generated (CG)* if $kX = X$.

Definition 1.2. A topological space X is *weakly Hausdorff (WH)* if for every compact Hausdorff K and every continuous map $u\colon K \to X$ the image $u(K)$ is closed in X.

The next two propositions imply that the vast majority of spaces in common use are CGWH.

Proposition 1.3. *A Hausdorff space is weakly Hausdorff.*

Proof. If X is Hausdorff, K is compact Hausdorff and $u\colon K \to X$ is continuous then $u(K)$ is a compact subset of a Hausdorff space and thus is closed. □

Lemma 1.4. *Suppose that X is WH.*

- (a) *Every point is closed (so X is T_1).*
- (b) *If K is compact Hausdorff and $u\colon K \to X$ is continuous then $u(K)$ is compact Hausdorff with respect to the subspace topology.*
- (c) *A subset $Y \subseteq X$ is k-closed iff $Y \cap K$ is closed in K for every subset $K \subseteq X$ that is compact Hausdorff with respect to the subspace topology.*

Proof. Part (a) is trivial. Let $u\colon K \to X$ be as in (b), and put $L = u(K)$, which is closed in X by hypothesis. If $F \subseteq K$ is closed then it is compact Hausdorff so $u(F)$ is also closed in X and thus in K, so $u\colon K \to L$ is a closed map. If $a, b \in K$ and $a \neq b$ then $u^{-1}\{a\}$ and $u^{-1}\{b\}$ are disjoint closed subspaces of the compact Hausdorff space K, so they have disjoint neighbourhoods, say U and V. Put

$$U' = \{x \in X \mid u^{-1}\{x\} \subseteq U\} = \{x \in X \mid u^{-1}\{x\} \cap (K \setminus U) = \emptyset\} = L \setminus u(K \setminus U)$$
$$V' = \{x \in X \mid u^{-1}\{x\} \subseteq V\} = \{x \in X \mid u^{-1}\{x\} \cap (K \setminus V) = \emptyset\} = L \setminus u(K \setminus V).$$

From the first description we see that $a \in U'$ and $b \in V'$ and $U' \cap V' = \emptyset$. We also note that $K \setminus U$ is compact so $u(K \setminus U)$ is closed in L so $U' = L \setminus u(K \setminus U)$ is open in L. Similarly, V' is open, so we have found

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disjoint open neighbourhoods of a and b in L. This shows that L is Hausdorff. This proves (b), and (c) follows easily. □

Definition 1.5. Let X be a space, and Y a subset of X. We say that Y is *sequentially closed* if whenever (y_n) is a sequence in Y converging to a point $x \in X$, we have $x \in Y$; clearly every closed subspace is sequentially closed. We say that X is a *sequential space* if every sequentially closed subset is closed. We also say that X is *first countable* if every point has a countable basis of neighbourhoods. It is easy to see that metric spaces are first countable and that first countable spaces are sequential.

Proposition 1.6. *Every sequential space (and thus every metric space and every first countable space) is compactly generated.*

Proof. Let X be a sequential space, and Y a k-closed subset; it suffices to show that Y is sequentially closed. Suppose that we have a convergent sequence $y_k \to x$ with $y_k \in Y$; we need to show that $x \in Y$. Let K be the one-point compactification of $\mathbb{N}$, and define a map $u\colon K \to X$ by $u(k) = y_k$ and $u(\infty) = x$. This is continuous because the sequence converges. As Y is k-closed, we conclude that $u^{-1}Y$ is closed in K, but $\mathbb{N} \subseteq u^{-1}Y$ and $\mathbb{N}$ is dense in K so $\infty \in u^{-1}Y$ so $x = u(\infty) \in Y$ as required. □

Proposition 1.7. *Every locally compact Hausdorff space is CGWH.*

Proof. Let X be a locally compact Hausdorff space, and Y a k-closed subset. Suppose that $x \in \overline{Y}$; we need to show that $x \in Y$. As X is locally compact, x has a neighbourhood U such that $K = \overline{U}$ is compact. If V is a neighbourhood of x then so is $V \cap K$, and $x \in \overline{Y}$, so $V \cap K \cap Y \neq \emptyset$; this shows that $x \in \overline{K \cap Y}$. On the other hand, as Y is compactly closed and the inclusion $j\colon K \rightarrowtail X$ is continuous we see that $K \cap Y = j^{-1}Y$ is closed in K. Thus $x \in Y$ as required.

This shows that X is CG, and it is WH by Proposition 1.3. □

Recall that a function is continuous if and only if the preimage of any closed set is closed. This makes the following lemma trivial.

Lemma 1.8. *If K is compact Hausdorff then a function $u\colon K \to X$ is continuous with respect to ζ if and only if it is continuous with respect to $k(\zeta)$.* □

Corollary 1.9. *For any space X we have $k^2X = kX$ and thus kX is compactly generated.* □

Corollary 1.10. *Let X be a CG space and Y an arbitrary space. Then a function $f\colon X \to Y$ is continuous if and only if it is continuous when thought of as a function $X \to kY$.*

Proof. Suppose that $f\colon X \to kY$ is continuous. If $Z \subseteq Y$ is closed in the original topology then it is also closed in kY so $f^{-1}Z$ is closed in X; this means that $f\colon X \to Y$ is continuous.

Less trivially, suppose that $f\colon X \to Y$ is continuous, and that $Z \subseteq Y$ is k-closed; we need to check that $f^{-1}Z$ is closed in X. Consider a compact Hausdorff space K and a continuous map $u\colon K \to X$. Then $fu\colon K \to Y$ is continuous and Z is k-closed so $u^{-1}f^{-1}Z = (fu)^{-1}Z$ is closed in K. This means that $f^{-1}Z$ is k-closed in X, and thus closed because X is CG. □

Proposition 1.11. *Let X be a CG space and Y an arbitrary space. Then a function $f\colon X \to Y$ is continuous if and only if $fu\colon K \to Y$ is continuous for all compact Hausdorff K and all continuous maps $u\colon K \to X$.*

Proof. Suppose that fu is continuous for all K and u. Let $F \subseteq Y$ be closed. For any K and u, the continuity of uf means that $f^{-1}(u^{-1}F) = (uf)^{-1}F$ is closed. This shows that $f^{-1}F$ is k-closed and thus closed, which means that f is continuous. The opposite implication is trivial. □

Proposition 1.12. *If ζ and ξ are topologies on X with $\zeta \subseteq \xi$ then $k(\zeta) \subseteq k(\xi)$.*

Proof. Consider a set $F \in k(\zeta)$; we must show that F is k-closed with respect to ξ. Consider a compact Hausdorff space K and a ξ-continuous map $u\colon K \to X$. As $\zeta \subseteq \xi$ we see that u is also ζ-continuous. As F is k-closed with respect to ζ we see that $u^{-1}(F)$ is closed in K. Thus $F \in k(\xi)$. □

2. New spaces for old

Proposition 2.1. *If X is a CG space and E is an equivalence relation on X then the quotient $Y = X/E$ is CG.*

Proof. Write $q\colon X \to Y$ be the quotient map. Let Z be a k-closed subset of Y. By Corollary 1.10, the function q is continuous when thought of as a map $X \to kY$, so $q^{-1}Z$ is closed in X. By the definition of the quotient topology, we conclude that Z is closed in Y. □

Proposition 2.2. *If $\{X_i\}$ is a family of CG spaces then their disjoint union $X = \coprod_i X_i$ is CG.*

Proof. Let $Z \subseteq X$ be k-closed. Then Z has the form $\coprod_i Z_i$, where $Z_i = Z \cap X_i$, and it is sufficient to check that Z_i is closed in X_i. As X_i is CG, it is enough to check that Z_i is k-closed in X_i. Consider a map $u\colon K \to X_i$. Then the composite $v = (K \xrightarrow{u} X_i \rightarrowtail X)$ is continuous and $u^{-1}Z_i = v^{-1}Z$, which is closed because Z is k-closed in X. □

Definition 2.3. Given two spaces X and Y, we shall write $X \times_0 Y$ for the product space equipped with the usual product topology. This need not be CG even if X and Y are. We thus define $X \times Y = k(X \times_0 Y)$. Similarly, given an indexed family of (possibly infinitely many) spaces X_i we write $\prod_{0,i} X_i$ for their product under the usual topology and $\prod_i X_i = k\prod_{0,i} X_i$.

Proposition 2.4. *Let $\{X_i\}$ be a family of CG spaces. Then the projection maps $\pi_i\colon \prod_i X_i \to X_i$ are continuous. Moreover, for any CG space Y, a map $f\colon Y \to \prod_i X_i$ is continuous if and only if each component $f_i = \pi_i \circ f$ is continuous. (This means that $\prod_i X_i$ is the product of the objects X_i in the category of CG spaces.)*

Proof. This is immediate from Corollary 1.10 and standard properties of the ordinary product topology. □

We will often need the following result.

Lemma 2.5 (Tube Lemma)**.** *Suppose that X is compact and $y \in Y$, and that U is an open subset of $X \times_0 Y$ that contains $X \times \{y\}$. Then there is a neighbourhood V of y in Y such that $X \times_0 V \subseteq U$.*

Proof. For each $x \in X$ we have $(x, y) \in U$ and U is open so there is a neighbourhood U_x of x in X and a neighbourhood V_x of y in Y such that $U_x \times V_y \subseteq U$. As X is compact, there is some finite set $x_1, \ldots, x_n$ such that $U_{x_1} \cup \ldots \cup U_{x_n} = X$. Write $V = V_{x_1} \cap \ldots \cap V_{x_n}$. It is easy to see that V is a neighbourhood of y and that $X \times V \subseteq U$. □

Proposition 2.6. *If X is a locally compact Hausdorff space and Y is CG then $X \times_0 Y$ is CG and thus $X \times Y = X \times_0 Y$.*

Proof. Suppose that $Z \subseteq X \times_0 Y$ is k-closed; we need to check that it is closed in the ordinary product topology. Suppose that $(x, y) \notin Z$. The map $i_y\colon x' \mapsto (x', y)$ is continuous. It follows that $i_y^{-1}Z = \{x' \in X \mid (x', y) \in Z\}$ is k-closed in the CG space X and thus closed. By standard properties of locally compact Hausdorff spaces, we can choose an open neighbourhood U of x in X such that $\overline{U}$ is compact and $\overline{U} \cap i_y^{-1}Z = \emptyset$, or equivalently $\overline{U} \times \{y\} \cap Z = \emptyset$. Now write $V = \{y' \in Y \mid \overline{U} \times \{y\} \cap Z = \emptyset$. We claim that this is open. Let K be a compact Hausdorff space, and $u\colon K \to Y$ a continuous map; it will be enough to check that $u^{-1}V$ is open in K. Let Z' be the preimage of Z under $1 \times u\colon \overline{U} \times K \to X \times Y$. As Z is k-closed, we know that Z' is closed in $\overline{U} \times K$ and thus compact, so the projection of Z' on K is compact and thus closed. However, this projection is easily seen to be the complement of $u^{-1}V$, so $u^{-1}V$ is open as required. Thus $U \times V$ is an open neighbourhood of (x, y) in $X \times_0 Y$ which does not meet Z. It follows that Z is closed in $X \times_0 Y$, as required. □

Proposition 2.7. *Suppose that X and Y are both first countable (and thus CG by Proposition 1.6). Then $X \times_0 Y$ is also first countable and therefore CG, so $X \times Y = X \times_0 Y$*

Proof. Given a point $(x, y) \in X \times Y$ we choose a countable basis of neighbourhoods U_i for x in X, and a countable basis of neighbourhoods V_j for $y \in Y$. Then the sets $U_i \times V_j$ give a countable basis of neighbourhoods for $(x, y) \in X \times Y$. □

Definition 2.8. Let X and Y be CG spaces. For any compact Hausdorff K and any map $u\colon K \to X$ and any open set $U \subseteq Y$, we write

$$W(u, K, U) = \{ \text{ maps } f\colon X \to Y \mid fu(K) \subseteq U\}.$$

If K is a compact subspace of X and $u\colon K \to X$ is the inclusion then we write $W(K, U)$ for $W(u, K, u)$. We write $C_0(X, Y)$ for the set of maps $f\colon X \to Y$, equipped with the smallest topology for which the sets $W(u, K, U)$ are open (this is called the *compact-open topology*). We also write $C(X, Y) = kC_0(X, Y)$.

Remark 2.9. If Z is a closed subspace of Y then $C(X, Z)$ is closed in $C(X, Y)$, because $C(X, Z) = \bigcap_x W(\{x\}, Z^c)^c$.

Lemma 2.10. *If $g\colon Y \to Z$ is continuous, then so is the map $g_*\colon C(X, Y) \to C(X, Z)$ defined by $g_*(t) = g \circ t$. If $f\colon W \to X$ is continuous then so is the map $f^*\colon C(X, Y) \to C(W, Y)$ defined by $f^*(t) = t \circ f$.*

Proof. Consider a compact Hausdorff space K, a map $u\colon K \to X$ and an open subset $U \subseteq Z$. We then have

$$(g_*)^{-1}W(u, K, U) = \{t\colon X \to Y \mid gtu(K) \subseteq U\} = W(u, K, g^{-1}U).$$

It follows easily that the preimage of any open subset of $C_0(X, Z)$ is open in $C_0(X, Y)$ and thus in $C(X, Y)$; by Corollary 1.10 we conclude that g_* is continuous as a map $C(X, Y) \to C(X, Z)$.

Now consider a map $v\colon L \to W$ with L compact Hausdorff, and an open subset $V \subseteq Y$. Then

$$(f^*)^{-1}W(v, L, V) = \{t\colon X \to Y \mid tgv(L) \subseteq V\} = W(gv, L, V).$$

By the same logic, we conclude that $f^*\colon C(X, Y) \to C(W, Y)$ is continuous. □

Proposition 2.11. *Define functions $ev_{X,Y}\colon X \times C(X, Y) \to Y$ and $inj_{X,Y}\colon Y \to C(X, X \times Y)$ by*

$$ev(x, f) = f(x)$$
$$inj(y)(x) = (x, y).$$

Then ev and inj are continuous.

Proof. We first consider inj. By Corollary 1.10, it is enough to show that inj is continuous as a map $Y \to C_0(X, X \times Y)$, or equivalently that $\text{inj}^{-1}W(u, K, U)$ is open in Y for every $u\colon K \to X$ and every open set $U \subseteq X \times Y$. As Y is CG, it is equivalent to check that $v^{-1}\text{inj}^{-1}W(u, K, U)$ is open in L for every compact Hausdorff L and every map $v\colon L \to Y$. Note that $u \times v\colon K \times L \to X \times Y$ is continuous, so $(u \times v)^{-1}U$ is open in $K \times L$, so $\{b \in L \mid K \times \{b\} \subseteq (u \times v)^{-1}U\}$ is open in L by the Tube Lemma. It is easy to check that this set is the same as $v^{-1}\text{inj}^{-1}W(u, K, U)$, which completes the proof.

We next consider the evaluation map $\text{ev}\colon X \times C(X, Y) \to Y$. Consider an open set $U \subseteq Y$, a compact Hausdorff space K, and a map $u\colon K \to X \times C(X, Y)$. It will be enough to show that $V = u^{-1}\text{ev}^{-1}U$ is open in K. Let $v\colon K \to X$ and $w\colon K \to C(X, Y)$ be the two components of u, so $V = \{a \in K \mid w(a)(v(a)) \in U\}$. Suppose that $a \in V$. As $w(a) \circ v\colon K \to Y$ is continuous, we can choose a compact neighbourhood L of a in K such that $w(a)(v(L)) \subseteq U$. This means that $w(a) \in W(v, L, U) \subseteq C(X, Y)$. As $w\colon K \to C(X, Y)$ is continous, the set $N = w^{-1}W(v, L, U)$ is a neighbourhood of a in K. If $b \in N \cap L$ then $w(b)(v(b)) \in w(b)(v(L)) \subseteq U$, so $b \in V$. Thus the neighbourhood $N \cap L$ of a is contained in V. This shows that V is open, as required. □

Proposition 2.12. *There is a natural homeomorphism $adj_{X,Y,Z}\colon C(X, C(Y, Z)) \to C(X \times Y, Z)$ given by $adj(f)(x, y) = f(x)(y)$. Thus, the category $\mathcal{U}$ of CGWH spaces is cartesian closed.*

Proof. Write $D(X, Y)$ for the set of all (possibly discontinuous) functions $X \to Y$. It is clear that there is a bijection between functions $f\colon X \to D(Y, Z)$ and functions $g\colon X \times Y \to Z$ defined by $g(x, y) = f(x)(y)$. We first claim that g is continuous if and only if

(1) $f(x)\colon Y \to Z$ is continuous for each $x \in X$, so that f can be considered as a function $X \to C(X, Y)$.
(2) f is continuous when considered as a function $X \to C(X, Y)$.

Indeed, if f satisfies these conditions then g is the composite

$$X \times Y \xrightarrow{f \times 1} C(Y, Z) \times Y \xrightarrow{\text{ev}} Y$$

which is continuous by Proposition 2.11. Conversely, suppose that g is continuous. If $x \in X$ then we have a continuous map $i_x\colon Y \to X \times Y$ defined by $i_x(y) = (x, y)$ and $f(x) = g \circ i_x$ so $f(x)$ is continuous. Moreover, f is the composite

$$X \xrightarrow{\text{inj}} C(Y, X \times Y) \xrightarrow{g_*} C(Y, Z)$$

which is continuous by Lemma 2.10 and Proposition 2.11.

It follows from the above that we have a bijection $\text{adj}\colon C(X, C(Y, Z)) \to C(X \times Y, Z)$, which already means that $\mathcal{U}$ is cartesian closed.

However we still need to show that adj is a homeomorphism. This is true by a purely formal argument, which we now explain. We know that the evaluation map $\text{ev}_{X,C(Y,Z)}\colon X \times C(X, C(Y, Z)) \to C(Y, Z)$ is continuous, as is $\text{ev}_{Y,Z}\colon Y \times C(Y, Z) \to Z$. It follows that the composite

$$\text{ev}_{Y,Z} \circ (1_Y \times \text{ev}_{X,C(Y,Z)})\colon Y \times X \times C(X, C(Y, Z)) \to Z$$

is continuous. It follows that the adjoint map

$$C(X, C(Y, Z)) \to C(X \times Y, Z)$$

is also continuous. However, this last map is just adj itself. Thus, adj is continuous.

Similarly, we know that the evaluation map $X \times Y \times C(X \times Y, Z) \to Z$ is continuous. It follows that the adjoint map $X \times C(X \times Y, Z) \to C(Y, Z)$ is continuous. Applying the same argument again, we see that the adjoint map $C(X \times Y, Z) \to C(X, C(Y, Z))$ is continuous. However, this last map is just the inverse of adj. This proves that adj is a homeomorphism.

We give a second proof which may be found more conceptual. For any space W, we have natural bijections

$$\begin{aligned} C(W, C(X, C(Y, Z))) &\xrightarrow{\text{adj}_{W,X,C(Y,Z)}} C(W \times X, C(Y, Z)) \\ &\xrightarrow{\text{adj}_{W\times X,Y,Z}} C(W \times X \times Y, Z) \\ &\xleftarrow{\text{adj}_{W,X\times Y,Z}} C(W, C(X \times Y, Z)). \end{aligned}$$

This means that $C(X, C(Y, Z))$ and $C(X \times Y, Z)$ represent the same contravariant functor from spaces to sets, and it follows by Yoneda's Lemma that there is a natural homeomorphism

$$C(X, C(Y, Z)) = C(X \times Y, Z).$$

Applying Yoneda's Lemma just comes down to considering the cases $W = C(X, C(Y, Z))$ and $W = C(X \times Y, Z)$, as we did previously. □

Proposition 2.13. *If X is compact Hausdorff and Y is a metric space then $C(X, Y)$ is a metric space, with metric $d(f, g) = \max_{x \in X} d(f(x), g(x))$.*

Proof. Let ζ be the compact-open topology on $C(X, Y)$, so the official topology on $C(X, Y)$ is $k(\zeta)$. Next, observe that the definition of $d(f, g)$ makes sense: the map $x \mapsto d(f(x), g(x))$ is a continuous real-valued function on a compact Hausdorff space, so it has a maximum. It is easy to check that $d(f, g)$ defines a metric; we write ξ for the resulting topology, so $\xi = k(\xi)$ by Proposition 1.6.

Suppose that we have a subbasic open set $W(u, K, U)$ for ζ, and a point $f \in W(u, K, U)$, so that $f\colon X \to Y$ and $fu(K) \subseteq U$. Then $a \mapsto d(fu(a), U^c)$ is a continuous, strictly positive, real-valued function on the compact space K, so it has a lower bound $\epsilon > 0$. It is easy to see that the open ball $B(\epsilon, f) = \{g \mid d(g, f) < \epsilon\}$ is contained in $W(u, K, U)$. It follows that $W(u, K, U)$ is open with respect to ξ, so $\zeta \subseteq \xi$. It follows using Proposition 1.12 that $k(\zeta) \subseteq k(\xi) = \xi$.

Conversely, consider a point $f \in C(X, Y)$ and an open ball $B(\epsilon, f)$ around f. The sets $f^{-1}B(y, \epsilon/3)$ (as y runs over Y) form an open cover of X. We may therefore choose finitely many points $y_1, \ldots, y_n \in Y$ such that the sets $f^{-1}B(y_i, \epsilon/3)$ cover X. We write $K_i = f^{-1}\overline{B}(y_i, \epsilon/3)$ and $U_i = B(y_i, \epsilon/2)$, so that the K_i are compact and cover X, and $f(K_i) \subseteq U_i$. It is clear that the set $N = \bigcap_i W(K_i, U_i)$ is a neighbourhood of f in the compact-open topology. We claim that $N \subseteq B(f, \epsilon)$. Indeed, suppose that $g \in N$ and $x \in X$; we need to show that $d(f(x), g(x)) < \epsilon$. We know that $x \in K_i$ for some i, so both $f(x)$ and $g(x)$ lie in $B(y_i, \epsilon/3)$, and the required inequality follows immediately. This shows that $B(f, \epsilon)$ is a neighbourhood of f in ζ, so $\xi \subseteq \zeta \subseteq k(\zeta) \subseteq k(\xi) = \xi$. required. □

Proposition 2.14. *A CG space X is weakly Hausdorff if and only if the diagonal subspace $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in $X \times X$.*

Proof. Suppose that X is weakly Hausdorff. First, observe that every one-point set $\{x\} \subset X$ is certainly a continuous image of a compact Hausdorff space and thus is closed in X, so X is T_1. Next, consider a compact Hausdorff space K and a map $u = (v,w)\colon K \to X \times X$. It will be enough to show that $u^{-1}\Delta_X = \{a \in K \mid v(a) = w(a)\}$ is closed in K. Suppose that $a \notin u^{-1}\Delta_X$, so $v(a) \neq w(a)$. Then $U = \{b \mid v(b) \neq w(a)\}$ is an open neighbourhood of a (because $\{w(a)\}$ is closed in X). By a standard lemma, there is an open neighbourhood V of a in K such that $\overline{V} \subseteq U$, or equivalently $w(a) \notin v(\overline{V})$. This means that a lies in the set $W = w^{-1}(v(\overline{V})^c)$. The weak Hausdorff condition implies that $v(\overline{V})$ is closed in X and thus W is open in K. We claim that $(V \cap W) \cap u^{-1}\Delta_X = \emptyset$. Indeed, if $b \in V \cap W$ then $v(b) \in v(\overline{V})$ but $w(b) \in v(\overline{V})^c$ by the definition of W, so $v(b) \neq w(b)$, so $u(b) = (v(b), w(b)) \notin \Delta_X$. This shows that $u^{-1}\Delta_X$ is closed in K, as required.

Conversely, suppose that Δ_X is closed in $X \times X$. Let K be a compact Hausdorff space, and $u\colon K \to X$ a map. Given any other compact Hausdorff space L and any map $v\colon L \to X$, we define $M = \{(a,b) \in K \times L \mid u(a) = v(b)\} \subseteq K \times L$. This can also be described as $(u \times v)^{-1}\Delta_X$, so it is closed in $K \times L$ and thus compact. It follows that the projection $\pi_L M$ is compact and thus closed in L. However, it is easy to see that $\pi_L M = v^{-1}(u(K))$. Thus shows that $u(K)$ is k-closed in X, and hence closed. This means that X is weakly Hausdorff. ☐

Corollary 2.15. *If X and Y are CGWH and $f, g\colon X \to Y$ are continuous then $\{x \mid f(x) = g(x)\} = (f,g)^{-1}\Delta_Y$ is closed in X.* ☐

Corollary 2.16. *The product of an arbitrary family of CGWH spaces (with the CG topology) is CGWH.*

Proof. Consider a product $X = \prod_i X_i$ of CGWH spaces with projection maps $\pi_i\colon X \to X_i$. We know that Δ_{X_i} is closed in $X_i \times X_i$ and $\pi_i \times \pi_i\colon X \times X \to X_i \times X_i$ is continuous so $D_i = (\pi_i \times \pi_i)^{-1}\Delta_{X_i}$ is closed in $X \times X$. It is clear that $\Delta_X = \bigcap_i D_i$, so Δ_X is closed in $X \times X$ and X is WH. ☐

The following proposition is a good example of something which is usually true for the ordinary product of general spaces, but requires messy hypotheses. In our compactly generated context, it is true without restriction.

Proposition 2.17. *Let X and Y be CG spaces and E an equivalence relation on X. Let E' be the equivalence relation on $X \times Y$ defined by $(x_0, y_0)E'(x_1, y_1)$ if and only if $x_0 E x_1$ and $y_0 = y_1$. Then the natural bijection $(X \times Y)/E' \to (X/E) \times Y$ is a homeomorphism.*

Proof. Let $q\colon X \to X/E$ and $q'\colon X \times Y \to (X \times Y)/E'$ be the quotient maps. We have a continuous map $q \times 1\colon X \times Y \to (X/E) \times Y$, which evidently respects the equivalence relation E', so we have an induced continuous map $f\colon (X \times Y)/E' \to (X/E) \times Y$. On the other hand, the adjoint of q' is a continuous map $X \to C(Y, (X \times Y)/E')$ respecting E, so we get an induced map $g^{\#}\colon X/E \to C(Y, (X \times Y)/E')$. The adjoint of this is a continuous map $g\colon (X/E) \times Y \to (X \times Y)/E'$. It is easy to check that f and g are just the evident bijections and thus that $fg = 1$ and $gf = 1$. ☐

We can rephrase this in slightly different language as follows.

Definition 2.18. A map $f\colon X \to Y$ is a *quotient map* if it is surjective, and Y is topologised as a quotient of X. Explicitly, we mean that a subset $Z \subseteq Y$ is closed if and only if $q^{-1}Z$ is closed in X. It is clear that a composition of quotient maps is a quotient map.

Remark 2.19. If $f\colon X \to Y$ is surjective and either open or closed, it is easy to see that it is a quotient map.

Proposition 2.20. *If $f\colon W \to X$ and $g\colon Y \to Z$ are quotient maps of CG spaces, then so is $f \times g\colon W \times Y \to X \times Z$.*

Proof. It is immediate from Proposition 2.17 that $f \times 1_Y\colon W \times Y \to X \times Y$ and $1_X \times g\colon X \times Y \to X \times Z$ are quotient maps, and $f \times g = (1_X \times g) \circ (f \times 1_Y)$. ☐

Corollary 2.21. *Let E be an equivalence relation on a CG space X. Then X/E is WH if and only if E is closed in $X \times X$. (Here we are identifying a relation R on X with the set $\{(x, y) \mid xRy\} \subseteq X \times X$.)*

Proof. We know from Proposition 2.14 that X/E is WH if and only if $\Delta_{X/E}$ is closed in $X/E \times X/E$. Let $q\colon X \to X/E$ be the quotient map, so Proposition 2.20 tells us that $q \times q\colon X \times X \to X/E \times X/E$ is a quotient map, so $\Delta_{X/E}$ is closed if and only if $(q \times q)^{-1}\Delta_{X/E}$ is closed in $X \times X$. It is easy to see that $(q \times q)^{-1}\Delta_{X/E} = E$, so we conclude that X/E is WH if and only if E is closed in $X \times X$. □

Proposition 2.22. *Let X be a CG space. Then there is a smallest closed equivalence relation E on X. If we write $hX = X/E$ then h defines a functor from CG spaces to CGWH spaces, which is left adjoint to the inclusion of CGWH spaces in CG spaces. In other words, any map from X to a CGWH space factors uniquely through hX.*

Proof. Let $\mathcal{R}$ be the set of all equivalence relations R on X such that R is closed as a subset of $X \times X$. (There is at least one such relation, namely $R = X \times X$.) It is trivial to check that $E = \bigcap_{R \in \mathcal{R}} R$ is an equivalence relation and is closed in $X \times X$; clearly it is the smallest such. Corollary 2.21 tells us that $hX = X/E$ is CGWH. If Y is a CGWH space and $f\colon X \to Y$ is continuous then it is not hard to see that $R = \{(x, x') \mid f(x) = f(x')\} = (f \times f)^{-1}\Delta_Y$ is a closed equivalence relation on X. It follows that $E \subseteq R$, and thus that f factors through a unique continuous map $hX \to Y$. This implies that h is a functor and is left adjoint to the inclusion of CGWH spaces in CG spaces. □

Corollary 2.23. *The category of CGWH spaces has colimits for small diagrams, obtained by applying the functor h to the colimit as calculated in the category of all spaces.* □

Proposition 2.24. *If X is CG and Y is CGWH then $C(X, Y)$ is CGWH. Thus, the category of CGWH spaces is cartesian closed.*

Proof. By definition, $C(X, Y)$ is CG; we need only check that it is WH, or equivalently that $\Delta_{C(X,Y)}$ is closed in $C(X, Y) \times C(X, Y)$. Define $\mathrm{ev}_x\colon C(X, Y) \to Y$ by $\mathrm{ev}_x(f) = f(x)$. For any open set $U \subseteq Y$ we have $\mathrm{ev}_x^{-1}(U) = W(\{x\}, U)$, which is open in $C(X, Y)$, so ev_x is continuous. Moreover, we have $\Delta_{C(X,Y)} = \bigcap_x (\mathrm{ev}_x \times \mathrm{ev}_x)^{-1}\Delta_Y$, which is closed because Δ_Y is. □

Definition 2.25. Let X be a CGWH space (with topology ζ), and let Y be a subset of X. Let ζ_Y^0 denote the ordinary subspace topology on Y (so $\zeta_Y^0 = \{F \cap Y \mid F \in \zeta\}$) and put $\zeta_Y = k(\zeta_Y^0)$; we call this the CGWH subspace topology. Let $i_Y\colon Y \to X$ be the inclusion map. As $\zeta_Y^0 \subseteq \zeta_Y$ we see that i_Y is continuous with respect to ζ_Y and ζ.

Lemma 2.26. *If Y is open or closed then $\zeta_Y = \zeta_Y^0$.*

Proof. The case where Y is closed is easy, so we will assume instead that Y is open.

Suppose that $F \in \zeta_Y$, and put $V = Y \setminus F$. We must prove that F is closed in the ordinary subspace topology on the open set Y, or equivalently that V is open in X. Consider a compact Hausdorff space K and a continuous map $u\colon K \to X$; as X is compactly generated, it will be enough to show that $u^{-1}V$ is open in K. Suppose that $a \in u^{-1}V$. As Y is open in X, we know that $u^{-1}Y$ is an open neighbourhood of a in K. As K is compact Hausdorff and thus regular, we can choose an open neighbourhood N of a in K such that $\overline{N} \subseteq u^{-1}V$. Now $\overline{N}$ is a compact Hausdorff space, and u restricts to give a continuous map $v\colon \overline{N} \to X$ with image contained in Y. Thus, as $F \in \zeta|_Y$, we see that $v^{-1}F$ is closed in $\overline{N}$, so $u^{-1}V \cap \overline{N} = v^{-1}V$ is open in $\overline{N}$. This means that $u^{-1}V \cap N$ is open in N and thus in K, so $u^{-1}V$ is a neighbourhood of a in K. This proves that $u^{-1}V$ is open in K, as required. □

Definition 2.27. A continuous map $i\colon Y \to X$ of CGWH spaces is an *inclusion* if it is injective, and the resulting map $Y \to i(Y)$ is a homeomorphism if $i(Y)$ is given the CGWH subspace topology inherited from X. Using Lemma 2.26 we see that an inclusion sends closed sets in Y to closed sets in X iff the image $i(Y)$ is closed in X. If so, we say that i is a *closed inclusion*. If Y is just a subset of X and $i(y) = y$ for all Y, we say that i is the *identity inclusion*.

Lemma 2.28. *Let $i\colon Y \to X$ be a continuous injective map of CGWH spaces. Then i is an inclusion if and only if it has the following property: if T is CGWH and $f\colon T \to Y$ is such that $if\colon T \to X$ is continuous, then f is continuous.*

Proof. Let $P(i)$ denote the statement that i has the property mentioned above.

Let Y be a subset of X (with the CGWH subspace topology) and let $i_Y : Y \to X$ be the identity inclusion. Suppose that $f : T \to Y$ is such that $i_Y f$ is continuous. This immediately implies that f is continuous with respect to ζ_Y^0, but T is CGWH so this is equivalent to continuity with respect to ζ_Y (by Corollary 1.10). Thus $P(i_Y)$ holds, and it follows that $P(i)$ holds for any inclusion.

Conversely, suppose we have an injective map $i : Y \to X$ such that $P(i)$ holds. Put $Y' = i(Y)$ and give this the CGWH subspace topology. Let $f : Y \to Y'$ be the bijection induced by i, so $i_{Y'} f = i$ and $i f^{-1} = i_{Y'}$. Note that $P(i_{Y'})$ holds by the previous paragraph, and it follows that f is continuous. We are given that $P(i)$ holds, and it follows that f^{-1} is continuous. Thus f is a homeomorphism, so i is an inclusion. □

Corollary 2.29. *Let $Y \xrightarrow{i} X \xrightarrow{r} Y$ be continuous maps of CGWH spaces such that $ri = 1_Y$. Then i is a closed inclusion and r is a quotient map.*

Proof. Let $f : T \to Y$ be such that if is continuous; then $f = rif$ is continuous as well. The lemma therefore tells us that i is an inclusion. One checks that

$$i(Y) = \{x \in X \mid ir(x) = x\} = (ir, 1_X)^{-1}(\Delta_X),$$

so $i(Y)$ is closed in X, so i is a closed inclusion. Clearly r is surjective. If $F \subseteq Y$ is such that $r^{-1}F$ is closed, then the set $F = (ri)^{-1}(F) = i^{-1}(r^{-1}(F))$ is itself closed. This shows that r is a quotient map. □

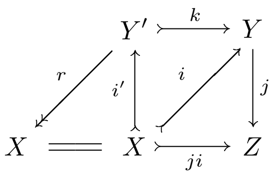
Proposition 2.30. *The category of CGWH spaces has limits for all small diagrams, and they are preserved by the forgetful functor to sets.*

Proof. Suppose we have a diagram $\{X_i\}$ of CGWH spaces. The limit calculated in the category of sets is a certain subset $X \subseteq \prod_i X_i$. We give $\prod_i X_i$ the CGWH product topology, and observe (using Corollary 2.15) that X is then a closed subspace. If we give X the subspace topology, it is easy to check that this makes it the limit in the CGWH category. □

Proposition 2.31. *Let $X \xrightarrow{i} Y \xrightarrow{j} Z$ be continuous maps of CGWH spaces.*

(a) *If i and j are inclusions then so is ji.*
(b) *If i and j are closed inclusions then so is ji.*
(c) *If ji is an inclusion then so is i.*
(d) *If ji is a closed inclusion then so is i.*

Proof. Parts (a) and (b) are easy, and part (c) follows from Lemma 2.28. Thus, in (d) we know that i is an inclusion and we just have to check that it is closed. By assumption, $ji(X)$ is closed in Z, so the set $Y' = j^{-1}(ji(X))$ is closed in Y, and it clearly contains $i(X)$. Let $k : Y' \to Y$ be the inclusion, and let i' be i regarded as a map to Y', so $ki' = i$. Next, if $y \in Y'$ then $j(y) = ji(x)$ for some x, and this x is unique because ji is injective, so we can denote it by $r(y)$. This gives a map $r : Y' \to X$ with $jir = jk$. In particular, jir is continuous and ji is an inclusion so r is continuous. We also have $jiri' = jki' = ji$, and ji is injective, so $ri' = 1_X$. It follows from Corollary 2.29 that i' is a closed inclusion, and k is also a closed inclusion, so $i = ki'$ is a closed inclusion as claimed. The maps considered are indicated in the following diagram:



□

Proposition 2.32. *Let X, Y and Z be CGWH spaces, and let $X \xrightarrow{i} Y$ be an inclusion. Then the map $i \times 1 : X \times Z \to Y \times Z$ is again an inclusion. If i is closed then so is $i \times 1$.*

Proof. Suppose that $(u, v) : W \to X \times Z$ and that the map $(i \times 1) \circ (u, v) = (iu, v) : W \to Y \times Z$ is continuous. This means that iu and v are continuous, and i is an inclusion, so u is also continuous, so (u, v) is continuous. This proves that $i \times 1$ is an inclusion. If i is closed then $i(X)$ is closed in Y so $(i \times 1)(X \times Z) = i(X) \times Z$ is closed in $Y \times Z$, so $i \times 1$ is closed (by the remarks in Definition 2.27). □

Proposition 2.33. *Suppose we have a pullback square as shown, in which i is an inclusion. Then i' is also an inclusion. Moreover, if i is closed then so is i'.*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \underset{i}{\rightarrowtail} & Y. \end{array}$$

Proof. Suppose we have a map $u\colon W \to X'$ such that $i'u$ is continuous. We then see that the map $if'u = fi'u$ is continuous, and i is an inclusion, so $f'u$ is continuous. As $i'u$ and $f'u$ are continuous, the pullback property tells us that u is continuous. This proves that i' is an inclusion. Now suppose that i is closed. Then one checks that $i'(X') = f^{-1}i(X)$, which is closed in Y' because f is continuous and i is closed. It follows that i' is a closed inclusion. □

Lemma 2.34. *If the diagram*

$$\begin{array}{ccc} W & \overset{f}{\rightarrowtail} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[k]{} & Z \end{array}$$

is a pullback of sets and f is a closed inclusion, then it is a pullback of spaces.

Proof. We need to show that W is topologised as a subspace of $X \times Y$, or equivalently that the map $W \xrightarrow{(f,g)} X \times Y$ is a closed inclusion. Let $p\colon X \times Y \to X$ be the projection. Then $p \circ (f, g) = f$ is a closed inclusion, so (f, g) is a closed inclusion by Proposition 2.31(d). □

Proposition 2.35. *Suppose we have a pushout square as shown, in which i is a closed inclusion. Then j is also a closed inclusion, and the square is a pullback. Moreover, the pushout is created in the category of sets.*

$$\begin{array}{ccc} W & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[j]{} & Z \end{array}$$

Proof. We may assume that W is a closed subspace of X and that i is the identity inclusion. We first analyse the pushout of i and f in the category of sets. Put $Z' = (X \setminus W) \amalg Y$ (just considered as a set, for the moment). Let $p\colon X \amalg Y \to Z'$ be given by the identity on the subset $Z' \subseteq X \amalg Y$, and by f on the subset $W \subseteq X \subseteq X \amalg Y$. Let $g'\colon X \to Z'$ and $j'\colon Y \to Z'$ be the restrictions of p. One can check directly that the square

$$\begin{array}{ccc} W & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow[j']{} & Z' \end{array}$$

is a pushout in the category of sets. We give Z' the unique topology for which p is a quotient map. We will need to show that this topology is CGWH, or equivalently that the equivalence relation

$$E = \mathrm{eq}(p) = \{(a, b) \in (X \amalg Y)^2 \mid p(a) = p(b)\}$$

is closed in $X \amalg Y$. For this we put $G = \{(w, y) \in W \times Y \mid f(w) = y\}$, which is closed in $W \times Y$, and thus also in $X \times Y$. Put $G' = \{(y, x) \mid (x, y) \in G\} \subseteq_C Y \times X$ and

$$E_0 = \mathrm{eq}(f) = \{(w, w') =\in W^2 \mid f(w) = f(w')\} \subseteq_C W^2 \subseteq_C X^2$$
$$E_1 = (\Delta_X \cup E_0) \amalg G \amalg G' \amalg \Delta_Y \subseteq_C X^2 \amalg (X \times Y) \amalg (Y \times X) \amalg Y^2 = (X \amalg Y)^2.$$

Then E_1 is visibly closed, and one checks directly that $E = E_1$. Thus Z' is CGWH, so it is also the pushout in the category $\mathcal{U}$, so we can identify Z' with Z. It follows that j is injective and that for any closed set $F \subseteq Y$ we have

$$p^{-1}j(F) = f^{-1}(F) \amalg F \subseteq X \amalg Y,$$

which is closed. Thus j is a closed inclusion. Moreover, we now see that the square is a pullback of sets, so it is a pullback of spaces by Lemma 2.34. $\square$

Proposition 2.36. *Consider a pullback square in $\mathcal{U}$ as shown, in which q is a quotient map. Then p is also a quotient map.*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Proof. First consider the special case where g is a closed inclusion. Proposition 2.33 tells us that f is also a closed inclusion. As the pullback is created in the category of sets, we can check by a small diagram chase that p is surjective and that $q^{-1}g(F) = fp^{-1}(F)$ for all subsets $F \subseteq Y$. Now suppose that $p^{-1}(F)$ is closed. As f is a closed inclusion we deduce that $q^{-1}g(F) = fp^{-1}(F)$ is closed, and as q is a quotient map this means that $g(F)$ is closed. As g is injective we have $F = g^{-1}g(F)$, so F is closed. This proves that p is a quotient map, as claimed.

For the general case, it is formal that the square below is also a pullback:

$$\begin{array}{ccc} W & \xrightarrow{(f,p)} & X \times Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q\times 1} \\ Y & \xrightarrow[(g,1)]{} & Z \times Y \end{array}$$

Here $q \times 1$ is a quotient map by Proposition 2.20, and $(g, 1)$ is a closed inclusion by Corollary 2.29, so p is a quotient map by the special case already considered. $\square$

Proposition 2.37. *Let X, Y and Z be CGWH spaces, and let $X \xrightarrow{i} Y$ be an inclusion. Then the map $i_*\colon C(Z, X) \to C(Z, Y)$ is again an inclusion. Moreover, if i is closed then so is i_*.*

Proof. Consider a map $w\colon W \to C(Z, X)$ such that $i_* \circ w\colon W \to C(Z, Y)$ is continuous. This means that the adjoint map $\text{adj}^{-1}(i_* \circ w)\colon W \times Z \to Y$ is continuous, but this is the same as the composite

$$W \times Z \xrightarrow{\text{adj}^{-1}(w)} X \xrightarrow{i} Y,$$

and i is an inclusion, so $\text{adj}^{-1}(w)$ is continuous, so w is continuous. This proves that i_* is an inclusion. If i is closed then the set $i_*(C(Z, X)) = C(Z, i(X))$ is closed in $C(Z, Y)$ by Remark 2.9, so i_* is a closed inclusion. $\square$

Definition 2.38. Let X be a CGWH space, and let Y be a closed subspace. Define X/Y to make the square on the left below a pushout:

$$\begin{array}{ccc} Y \amalg 0 & \overset{i}{\rightarrowtail} & X \amalg 0 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle q} \\ 1 & \underset{z}{\rightarrowtail} & X/Y \end{array} \qquad\qquad \begin{array}{ccc} Y & \overset{i}{\rightarrowtail} & X \\ \downarrow & & \downarrow \\ 1 & \underset{z}{\rightarrowtail} & X/Y \end{array}$$

By Proposition 2.35, the pushout is created in the category of sets, and the square is also a pullback. If $Y = \emptyset$ then $X/Y = X \amalg \{0\}$; otherwise, one checks that the right hand square is also a pushout and a pullback. We also let $p\colon X \amalg \{0\} \to X/Y$ be the constant map with value z, and note that $\{x \in X \mid p(x) = q(x)\} = Y \amalg \{0\}$.

Remark 2.39. If $F \subseteq X$ is a closed set then $q^{-1}(q(F))$ is either F (if $F \cap Y = \emptyset$) or $F \cup Y$. Either way $q^{-1}(q(F))$ is closed, and q is a quotient map, so $q(F)$ is closed in X/Y. This shows that q is a closed map.

Proposition 2.40. *Suppose we have a CGWH space X, a closed subspace Y, and another CGWH space Z. Then $(X \times Z)/(Y \times Z) = (X/Y) \times Z$.*

Proof. This is essentially Proposition 2.20 applied to the quotient map $X \to X/Y$. □

3. Limits and regularity

From now on we will take a somewhat more categorical viewpoint, and investigate the relationship between various kinds of limits and colimits in the category $\mathcal{U}$ of CGWH spaces. We will also change our terminology slightly: the word "space" will refer to a CGWH space unless we explicitly say otherwise.

3.1. **Regularity.** Let $\mathcal{C}$ be a category with finite limits and colimits. Then any parallel pair of maps $f, g\colon A \to B$ has an equaliser $\mathrm{eq}(f,g) \to A$ and a coequaliser $B \to \mathrm{coeq}(f,g)$. We also write $\mathrm{eq}(f)$ for the equaliser of $f\pi_1, f\pi_2\colon A^2 \to B$ and $\mathrm{coeq}(f)$ for the dual thing. Equivalently, we have pullback and pushout squares as shown.

$$\begin{array}{ccc} \mathrm{eq}(f) & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ A \times A & \xrightarrow[f\times f]{} & B \times B \end{array} \qquad\qquad \begin{array}{ccc} A \amalg A & \xrightarrow{f \amalg f} & B \amalg B \\ {\scriptstyle\nabla}\downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{coeq}(f) \end{array}$$

or

$$\begin{array}{ccc} \mathrm{eq}(f) & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & A \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & \mathrm{coeq}(f) \end{array}$$

In other words, $\mathrm{eq}(f) = A \times_B A$ and $\mathrm{coeq}(f) = B \amalg_A B$.

Recall that a map $f\colon B \to C$ is said to be a *regular epimorphism* if it is the coequaliser of some pair of maps $A \rightrightarrows B$, or equivalently if it is the coequaliser of the obvious pair of maps $\mathrm{eq}(f) \rightrightarrows B$. Dually, f is a *regular monomorphism* if it is the equaliser of some pair of maps $C \rightrightarrows D$.

We say that a category is *regular* if every map can be factored as a regular epimorphism followed by a monomorphism, and pullbacks of regular epimorphisms are regular epimorphisms. *Coregularity* is defined dually, and *biregular* means regular and coregular.

Theorem 3.1.
(a) *A map in $\mathcal{U}$ is a monomorphism iff it is injective, and an epimorphism iff it has dense image.*
(b) *A map in $\mathcal{U}$ is a regular monomorphism iff it is a closed inclusion.*
(c) *A map in $\mathcal{U}$ is a regular epimorphism iff it is a quotient map.*
(d) *A product, coproduct or composite of (regular) monomorphisms is a (regular) monomorphism.*
(e) *A coproduct, finite product or composite of (regular) epimorphisms is a (regular) epimorphism.*
(f) *$\mathcal{U}$ is biregular*

Proof of Theorem 3.1. (a): By definition, $i\colon A \to B$ is a monomorphism iff the induced map of sets $i_*\colon \mathcal{U}(X,A) \to \mathcal{U}(X,B)$ is injective for all X. This clearly holds if f is injective; for the converse, take X to be a single point, so that $\mathcal{U}(X,A)$ is just the underlying set of A.

Now suppose that $r\colon A \to B$ has dense image; we claim that r is an epimorphism. Indeed, suppose we have two maps $g, h\colon B \to X$ with $gr = hr$. Then the set

$$\mathrm{eq}(g,h) = \{b \in B \mid g(b) = h(b)\} = (g \times h)^{-1}\Delta_X$$

is a closed subspace of B containing the image of r. As this image is dense, we have $C = B$ and $g = h$. Thus r is an epimorphism.

Conversely, suppose that $r\colon A \to B$ is an epimorphism. Let A' be the closure of the image of r, and apply Definition 2.38 to A'. This gives a pair of maps $p, q\colon B \to B/A'$ and a point $z \in B/A'$ with $p^{-1}\{z\} = B$ and $q^{-1}\{z\} = A'$. It follows that $pr = qr$, but r is epi, so $p = q$. This means that $A' = q^{-1}\{z\} = p^{-1}\{z\} = B$, so $r(A)$ is dense as claimed.

(b): Any regular monomorphism $i\colon A \to B$ is the equaliser of some parallel pair of arrows $g, h\colon B \to X$. The equaliser was originally constructed by taking the closed set $\{b \in B \mid g(b) = h(b)\} = (g \times h)^{-1}\Delta$ and giving it the subspace topology; so i is a closed inclusion.

Conversely, let $i\colon A \to B$ be the inclusion of a closed subspace. Then i is the equaliser of the maps $p, q\colon B \to B/A$ in Definition 2.38, so it is a regular monomorphism.

(c): Let $r\colon A \to B$ be a regular epimorphism, so r is the coequaliser of some pair of arrows $g, h\colon X \to A$. There exist closed equivalence relations $R \subseteq A^2$ such that the set $S = \{(g(x), h(x)) \mid x \in X\}$ is contained in R (for example, $R = A^2$). Let R be the intersection of all such relations. Clearly R itself is a closed equivalence relation, so we have a quotient map $q\colon A \to B' = A/R$, and $B' \in \mathcal{U}$ by Corollary 2.21. This is easily seen to be a coequaliser of g and h, so it can be identified with r, so r is a quotient map.

Conversely, let $r\colon A \to B$ be a quotient map, so $B = A/R$ for some equivalence relation R, which must be a closed subspace of A^2 because B is WH. It is then clear that r is a coequaliser for the two projections $\pi_0, \pi_1\colon R \to A$, so it is a regular epimorphism.

(d): In any category, it is trivial that a product of (regular) monomorphisms is a (regular) monomorphism. The corresponding facts for coproducts and composites follow from parts (a) and (b) and the explicit construction of coproducts.

(e): The statements about coproducts are again formal, as is the fact that a composite of epimorphisms is epi. It follows easily from (c) that composites of regular epimorphisms are regular epimorphisms. Proposition 2.20 tells us that finite products of quotient maps are quotient maps, and we now know that quotient maps are the same as regular epimorphisms.

(f): Consider a map $f\colon A \to B$. Write $R = \mathrm{eq}(f)$, which is a closed equivalence relation on A. Let C be the closure of the image of f, topologised as a subspace of B. We then have a factorisation of f as a composite

$$A \to A/R \to C \to B.$$

The factorisation $A \to A/R \to B$ displays f as a regular epimorphism followed by a monomorphism. The factorisation $A \to C \to B$ displays f as an epimorphism followed by a regular monomorphism.

Proposition 2.36 now tells us that pullbacks of regular epis are regular epi, so $\mathcal{U}$ is regular. Proposition 2.35 tells us that pushouts of regular monos are regular mono, so $\mathcal{U}$ is coregular. □

3.2. **Filtered colimits.**

Definition 3.2. Recall that a category I is *filtered* if

(a) $I \neq \emptyset$
(b) For any objects $i, j \in I$ there is another object k and maps $i \xrightarrow{u} k \xleftarrow{v} j$ in I.
(c) For any parallel pair of morphisms $u, v\colon i \to j$ in I there is another morphism $w\colon j \to k$ with $wu = wv$.

Lemma 3.3. *Let $\{X_i\}_{i \in I}$ be a filtered diagram of closed inclusions of spaces, with colimit X. Then the underlying set of X is the colimit of the underlying sets of the X_i, and the maps $X_i \to X$ are closed inclusions.*

Proof. We first claim that if $u, v\colon i \to j$ in I then $u_* = v_*\colon X_i \to X_j$. Indeed, if w is as in axion (c) above then certainly $w_*u_* = w_*v_*$, but w_* is assumed to be a closed inclusion, so $u_* = v_*$.

Next suppose we have $i, j, m \in I$ and there exist maps $i \xrightarrow{u} m \xleftarrow{v} j$. We put

$$R_{ij}^m = \{(x, y) \in X_i \times X_j \mid u_*(x) = v_*(y)\},$$

which is closed in $X_i \times X_j$. By the previous paragraph, this is independent of the choice of u and v. In the case $m = i = j$ we can take $u = v = 1_i$ to see that $R_{ii}^i = \Delta_{X_i}$.

If there exists a morphism $f\colon m \to m'$ then, using the fact that $f_*\colon X_m \to X_{m'}$ is injective we see that $R_{ij}^m = R_{ij}^{m'}$. Even if there is no map $m \to m'$ we can certainly choose m'' with maps $m \to m'' \leftarrow m'$ so we still have $R_{ij}^m = R_{ij}^{m'}$. We write R_{ij} for this set. If $(x,y) \in R_{ij}$ and $(y,z) \in R_{jk}$ then, by choosing on object m that admits maps from all of i, j and k, we see that $(x,z) \in R_{ik}$.

Now put $T = \coprod_i X_i$, so $T^2 = \coprod_{i,j} X_i \times X_j$. As R_{ij} is closed in $X_i \times X_j$ we see that the set $R = \coprod_{i,j} R_{ij}$ is closed in T^2. It is also an equivalence relation, so we have a CGWH space $X = T/R$. Let $q \colon T \to X$ be the quotient map, and let $f_i \colon X_i \to X$ be the obvious map. It is now straightforward to check that these form a universal cone, so $\varinjlim_i X_i = X$, and this colimit is created in the category of sets. As $R_{ii} = \Delta_{X_i}$ we see that f_i is injective. Suppose we have a closed set $F \subseteq X_i$; we claim that $f_i(F)$ is closed in X. It will suffice to show that $q^{-1}(f_i(F))$ is closed in T, or that $X_j \cap q^{-1}(f_i(F))$ is closed in X_j for all j. To see this choose an object m and maps $i \xrightarrow{u} m \xleftarrow{v} j$, giving closed inclusions $X_i \xrightarrow{u_*} X_j \xleftarrow{v_*} X_k$. One checks that $X_j \cap q^{-1}(f_i(F)) = v_*^{-1}(u_*(F))$, which is closed in X_j as required. This proves that f_i is a closed inclusion. □

Definition 3.4. We say that a filtered diagram $\{A_i\}$ of closed inclusions is *strongly filtered* if every compact subset of $A = \varinjlim_i A_i$ lies in the image of some A_i.

Remark 3.5. A convergent sequence together with its limit is compact. Using this, one sees that any compact metric space A is the filtered colimit of its countable compact subspaces. This diagram is not strongly filtered unless A is countable.

Lemma 3.6. *A sequence of closed inclusions is strongly filtered. The diagram of finite subcomplexes of a CW complex is strongly filtered. More generally, let $\{A_i\}$ be a directed system of subsets of $A = \varinjlim_i A_i$, and suppose that there are disjoint sets B_j such that each A_i is the union of a finite set of B's. Then the family $\{A_i\}$ is strongly filtered.*

Proof. This argument is well-known, but it seems a little less well-known exactly what one needs to make it work.

The first and second claims follow from the third, by taking $B_i = A_i \setminus A_{i-1}$ in the first case, or taking the B's to be the open cells in the second. So suppose that the A's and B's are as in the third case. Suppose that $C \subseteq A$ is compact. For each j such that $B_j \cap C \neq \emptyset$, choose $b_j \in B_j \cap C$. Let D be the set of these b_j's. For any $E \subseteq D$ and any i, we see that $E \cap A_i$ is finite and thus closed in A_i. It follows that E is closed in A. As this holds for all $E \subseteq D$, we conclude that D is a discrete closed subset of the compact set C, hence D is finite. Thus C is contained in some finite union of B's. As the diagram of A's is directed, we conclude that $C \subseteq A_i$ for some i. □

Lemma 3.7. *Let $\{A_i\}$ be a directed family of closed subsets of B, and write $A = \bigcup_i A_i \subseteq B$. Consider the conditions*

(a) *A is closed in B, and is homeomorphic to $\varinjlim_i A_i$.*

(b) *For any compact set $C \subseteq B$, we have $C \cap A = C \cap A_i$ for some i.*

Then (b) implies (a), and the converse holds if $\{A_i\}$ is strongly filtered.

Proof. (b)⇒(a): Let $C \subseteq B$ be compact. Then $C \cap A = C \cap A_i$ for some i, and this is closed in C because A_i is closed in B. Thus A is compactly closed and thus closed in B. Similarly, suppose that $D \subseteq A$ is such that $D \cap A_i$ is closed for all i. Then for any compact C we can choose i such that $C \cap A = C \cap A_i$, so $D \cap C = D \cap C \cap A_i$, and this is again closed in C; thus D is closed in B. Thus, the subspace topology on A coincides with the colimit topology.

(a)⇒(b): Suppose that $\{A_i\}$ is strongly filtered and (a) holds. Suppose that $C \subseteq B$ is compact. As A is closed, we see that $C \cap A$ is compact. As $A = \varinjlim_i A_i$ is a strongly filtered colimit, we see that $C \cap A \subseteq A_i$ for some i, so $C \cap A = C \cap A_i$. □

Lemma 3.8. *If X is compact, then the functor $C(X, -)$ preserves strongly filtered colimits of closed inclusions.*

Proof. Let $\{A_i\}$ be a strongly filtered diagram with colimit A. As $C(X,-)$ has a left adjoint, it preserves regular monos, so $\{C(X, A_i)\}$ is a diagram of closed inclusions. Any map $X \to A$ has compact image; by the definition of a strongly filtered colimit, it therefore factors through some A_i. This means that the natural map $b\colon \varinjlim_i C(X,A_i) \to C(X,A)$ is a continuous bijection. Now let W be another compact space. Proposition 2.12 gives us a diagram

$$\begin{array}{ccccc}
\varinjlim C(W, C(X,A_i)) & \longrightarrow & C(W, \varinjlim C(X,A_i)) & \xrightarrow{b_*} & C(W, C(X,A)) \\
{\scriptstyle a}\downarrow{\scriptstyle \simeq} & & & & {\scriptstyle \simeq}\uparrow{\scriptstyle d} \\
\varinjlim C(W\times X, A_i) & & \xrightarrow[c]{\qquad\qquad\qquad} & & C(W\times X, A)
\end{array}$$

in which a and d are homeomorphisms. The maps a and d are bijective for formal reasons. The map c is bijective by the previous paragraph, as $W \times X$ is compact. As $b\colon \varinjlim C(X,A_i) \to C(X,A)$ is bijective, the map b_* is mono. Commutativity of the diagram now shows that b is also epi, so it is a bijection. As any space is a colimit of compact spaces, we conclude that $C(W, \varinjlim C(X,A_i)) = C(W, C(X,A))$ for noncompact W's as well. Thus $\varinjlim C(X,A_i)$ is homeomorphic to $C(X,A)$ by Yoneda's lemma.

Now suppose that $K \subseteq C(X,A)$ is compact. Then the image of the compact space $K \times X$ under the evaluation map is a compact subspace of A, hence contained in some A_k; it follows that $K \subseteq C(X,A_k)$. Thus, the diagram $\{C(X,A_i)\}$ is strongly filtered. □

Lemma 3.9. *Consider a diagram of the form*

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{i_0} & A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & \cdots \\
{\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\
B_0 & \xrightarrow[j_0]{} & B_1 & \xrightarrow[j_1]{} & B_2 & \xrightarrow[j_2]{} & \cdots
\end{array}$$

in which all maps are closed inclusions and all squares are pullbacks of sets (and therefore of spaces, by lemma 2.34). Write A_∞ and B_∞ for the evident colimits. Then the diagram

$$\begin{array}{ccc}
A_i & \longrightarrow & A_\infty \\
{\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f_\infty} \\
B_i & \longrightarrow & B_\infty
\end{array}$$

is a pullback square of closed inclusions.

Proof. Using Lemma 3.3, we see that the square is a pullback of sets, that the horizontal maps are closed inclusions, and that the vertical maps are injective. It follows from Lemma 2.34 that it is a pullback of spaces. The left hand vertical is a closed inclusion by assumption. If $C \subseteq A_\infty$ is closed then $f_\infty(C) \cap B_k = f_k(C \cap A_k)$ by the pullback property, and this is clearly closed in B_k. As $B_\infty = \varinjlim_k B_k$, we conclude that $f_\infty(C)$ is closed in B_∞, so f_∞ is a closed inclusion. □

Corollary 3.10. *Let $\{A_{k,l}\}$ be a diagram of closed inclusions indexed by $\mathbb{N}^2$, such that each square*

$$\begin{array}{ccc}
A_{k,l} & \longrightarrow & A_{k+1,l} \\
\downarrow & & \downarrow \\
A_{k,l+1} & \longrightarrow & A_{k+1,l+1}
\end{array}$$

is a pullback. Define $A_{\infty,l}$, $A_{k,\infty}$ and $A_{\infty,\infty}$ to be the obvious colimits. Then the resulting diagram indexed by $(\mathbb{N} \cup \{\infty\})^2$ again consists of pullback squares of closed inclusions.

3.3. **β-epimorphisms.** For any set S we let βS be the space of ultrafilters on S, or equivalently the maximal ideal space of the Banach algebra $C^*(S)$ of bounded functions $S \to \mathbb{R}$. This is called the Stone-Cech compactification of the discrete space S. The basic point is that βS is a compact Hausdorff space containing S as a discrete open subspace, and that any function $u: S \to X$ (where X is compact Hausdorff) extends uniquely to a continuous map $\beta S \to X$. In other words, β is left adjoint to the forgetful functor from compact Hausdorff spaces to sets. (In fact, a theorem of Manes tells us that the category of such spaces is equivalent to the category of algebras for β, considered as a monad in the category of sets.) All this is covered in [1].

Definition 3.11. We say that a map $f: X \to Y$ of CGWH spaces is a β-epimorphism if it satisfies the following equivalent conditions.

(a) For every set S and map $u: \beta S \to Y$ there is a map $v: \beta S \to X$ with $fv = u$.
(b) If $L \subseteq Y$ is compact then there is a compact set $K \subseteq Y$ such that $f(K) = L$.
(c) If $L \subseteq Y$ is compact then there is a compact set $K \subseteq Y$ such that $f(K) \supseteq L$.

Proof of equivalence. Suppose that (a) holds and that we are given a compact set $L \subseteq Y$. We have a counit map $\beta L \twoheadrightarrow L$ which we compose with the inclusion $L \rightarrowtail Y$ to get $u: \beta L \to Y$, which we can lift to a map $v: \beta L \to X$. We then put $K = v(\beta L)$ and observe that this is compact and has $f(K) = L$.

It is trivial that (b) implies (c). On the other hand, given K satisfying (c) we can take $K' = K \cap f^{-1}(L)$ to see that (b) holds as well.

Now suppose that (b) holds and we are given $u: \beta S \to Y$. Put $L = u(\beta S)$; this is a compact Hausdorff subset of Y, so we can choose a compact Hausdorff set $K \subseteq X$ with $f(K) = L$. Now choose a function $v: S \to K$ lifting the composite $S \rightarrowtail \beta S \xrightarrow{u} K$, and extend over βS using the universal property. This gives the required lifting of u to X. □

Proposition 3.12. *If f is a β-epimorphism, then it is a regular epimorphism.*

Proof. We can take L to be a point in (b) to see that f is surjective. Suppose that $Z \subseteq Y$ is such that $f^{-1}Z$ is closed in X; it will be enough to check that Z is closed in Y, for then f is a quotient map and thus a regular epi. As Y is CGWH it is enough to check that $Z \cap L$ is closed when L is compact. We can choose a compact set $K \subseteq X$ with $f(K) = L$ and then $Z \cap L = f(f^{-1}(Z) \cap K)$ but $f^{-1}(Z)$ is closed so $f^{-1}(Z) \cap K$ is compact Hausdorff so $f(f^{-1}(Z) \cap K)$ is closed as required. □

Lemma 3.13. *The class of β-epimorphisms is closed under products, pullbacks and composition.* □

Lemma 3.14. *If X is a metric space and $\emptyset \neq W \subseteq X$ is closed and $f: X \to Y := X/W$ is the projection then f is β-epi.*

Proof. Suppose that $L \subseteq Y$ is compact. Put $L' = L \setminus \{0\}$ and $K' = f^{-1}L'$; one checks that $f: K' \to L'$ is a homeomorphism. Let K be the closure of K' in X and choose $w \in W$, so $f(K \cup \{w\}) = L \cup \{0\}$. It is thus enough to show that K is compact, or equivalently that K' is totally bounded.

Fix $\epsilon > 0$. Let S be a subset of K' with the property that $d(a, b) > \epsilon/2$ when $a, b \in S$ and $a \neq b$. Any Cauchy sequence in S is clearly eventually constant, and it follows that S is a discrete closed subspace of X. Also $S \cap W = \emptyset$ so $f^{-1}f(S) = S$ and f is a quotient map so $f(S)$ is closed so $U := f(S)^c$ is open in Y. For each $s \in S$, choose a small open ball B_s of radius at most $\epsilon/4$ around s such that $B_s \cap W = \emptyset$ and put $V_s = f(B_s)$, which is an open neighbourhood of $f(s)$ in Y. The sets U and V_s cover Y and so some finite subcollection covers L. However $f(s) \in L$ and V_s is the only set in the cover that contains $f(s)$ so each V_s must be in the finite subcover so S must be finite.

It follows from this that we can choose a maximal set S with the stated properties, and it is easy to see that such an S is an ϵ-net for K'. Thus K' is totally bounded, as required. □

Lemma 3.15. *If X is a CW complex and $\emptyset \neq W \subseteq X$ is a subcomplex and $f: X \to Y := X/W$ is the projection then f is β-epi.*

Proof. Suppose that $L \subseteq X/W$ is compact. Then L is contained in a finite subcomplex, so we may assume that it is itself a finite subcomplex. Each open cell lifts to an open cell in X whose closure is a finite complex and thus compact. It follows easily that we can choose a compact subcomplex $K \subseteq X$ with $f(K) = L$, as required. □

Definition 3.16. We say that a surjective map $f\colon X \to Y$ is *proper* if it satisfies the following equivalent conditions.

(a) f is closed and has compact fibres.
(b) The preimage of any compact set is compact.

(It is immediate from (b) that a proper map is β-epi and thus regular epi.)

Proof of equivalence. Suppose that (a) holds and that $L \subseteq Y$ is compact, and put $K = f^{-1}(L)$; we need to show that K is compact. Let $\{F_i\}$ be a collection of nonempty closed subsets of K that is closed under finite intersections. It suffices to show that $\bigcap_i F_i \neq \emptyset$. The sets $f(F_i)$ are closed in the compact space L and any finite intersection of them is nonempty, so $\bigcap_i f(F_i)$ contains some point y say. The sets $F_i \cap f^{-1}\{y\}$ are thus nonempty closed subsets of the compact space $f^{-1}\{y\}$ and are closed under finite intersections, so $\bigcap_i F_i \cap f^{-1}\{y\} \neq \emptyset$ so $\bigcap_i F_i \neq \emptyset$ as required.

Conversely, suppose that (b) holds. It is immediate that f has compact fibres. Suppose that $F \subseteq X$ is closed; we need to show that $f(F)$ is closed. As Y is CGWH, it suffices to show that $f(F) \cap L$ is closed when $L \subseteq Y$ is compact, but then $K = f^{-1}(L)$ is compact in X so $F \cap K$ is compact so $f(F) \cap L = f(F \cap K)$ is closed as required. ☐

Proposition 3.17. *A proper continuous bijection of CGWH spaces is a homeomorphism.*

Proof. Let $f\colon Y \to Z$ be a proper continuous bijection. As remarked above, f is regular epi and clearly also mono and thus an isomorphism. For a more direct argument, we need only show that f^{-1} is continuous or equivalently that f is closed. Let $X \subseteq Y$ be closed, and let $L \subseteq Z$ be compact. It will suffice to show that $f(X) \cap L$ is closed. As f is proper we see that $K := f^{-1}(L)$ is compact so $X \cap K$ is compact, and images of compact sets are always compact so the set $f(X) \cap L = f(X \cap K)$ is compact, and thus closed as required. **check that this does not need strong Hausdorff property**. ☐

Remark 3.18. The obvious map $[0,1) \amalg [0,1] \to [0,1]$ is a split epimorphism (and thus a β-epi and a regular epi) with compact fibres, but is not proper.

4. Pointed spaces

Definition 4.1. Let $\mathcal{T}$ denote the category of CGWH spaces equipped with a specified basepoint $0_X \in X$. The morphisms from X to Y are the continuous maps $f\colon X \to Y$ for which $f(0_X) = 0_Y$. We will use the symbol 0 to denote the constant map $X \to Y$ taking everything to 0_Y, and also for the based space whose only point is the basepoint.

Definition 4.2. If X is an unbased space we write X_+ for $X \amalg \{0\}$, with the new point 0 taken as the basepoint. If Y is a based space we write Y_- for Y regarded as an unbased space. We then have an evident adjunction $\mathcal{T}(X_+, Y) = \mathcal{U}(X, Y_-)$.

Remark 4.3. Given a space $X \in \mathcal{U}$ and a closed subspace Y, we can form $X/Y = (X \amalg \{0\})/E$ as in Definition 2.38, and take the image of 0 as the basepoint. Most often we will do this when X is a based space and Y contains the basepoint. In that case we can regard X/Y as a quotient of X, and then $0_{X/Y}$ is the image of 0_X.

Remark 4.4. If $(X_i)_{i \in I}$ is any diagram in $\mathcal{T}$ then we can form the inverse limit $X = \varprojlim_i X_i$ in $\mathcal{U}$. The points $(0_{X_i})_i$ define a point 0_X of this inverse limit, and we take this as the basepoint in X. This makes X into the inverse limit of the diagram in $\mathcal{T}$. More formally, we can say that $\mathcal{T}$ has all small limits, and they are created by the forgetful functor $\mathcal{T} \to \mathcal{U}$. In particular, the categorical product of based spaces X and Y is just the space $X \times Y$ with the basepoint $0_{X \times Y} = (0_X, 0_Y)$.

Before discussing colimits, we need a small preliminary.

Definition 4.5. Given a category I, we let $\sim$ be the smallest equivalence relation on $\mathrm{obj}(I)$ such that $i \sim j$ whenever there is a morphism from i to j. We write $\pi_0(I)$ for the set of equivalence classes, and we say that I is *connected* if $\pi_0(X)$ is a singleton. For example, the usual indexing categories for coequalisers, pushouts and quotients by group actions are connected, but those for products are not.

Remark 4.6. Let $\mathcal{C}$ be a category with finite products and coproducts, let X be an object of $\mathcal{C}$, and regard it as a constant functor $I \to \mathcal{C}$. Then one checks that $\varinjlim_I X = \coprod_{i \in \pi_0(I)} X$ and $\varprojlim_I X = \prod_{i \in \pi_0(I)} X$. In particular, if I is connected then $\varinjlim_I X = X = \varprojlim_I X$.

Remark 4.7. Let $X : I \to \mathcal{T}$ be a diagram of pointed spaces, and let X' be the colimit in $\mathcal{U}$. Let 0 denote the constant I-diagram whose value is the one-point space, so $\varinjlim_I 0 = \pi_0(I)$ (with the discrete topology). There are evident maps $0 \to X \to 0$ of I-diagrams, giving maps $\pi_0(I) \to X' \to \pi_0(I)$ in $\mathcal{U}$. The composite is the identity on $\pi_0(I)$, showing that $\pi_0(I)$ is embedded as a closed subspace in X'. One checks that X'/Z is the colimit of X in $\mathcal{T}$, showing that $\mathcal{T}$ has all small colimits. Moreover, if I is connected we see that colimits for I-diagrams are created in $\mathcal{U}$. In particular, pushouts and coequalisers are the same whether calculated in $\mathcal{T}$ or in $\mathcal{U}$. However, the categorical coproduct in $\mathcal{T}$ is the wedge product, defined by

$$X \vee Y = (X \amalg Y)/(\{0_X\} \amalg \{0_Y\})$$

Note that there is a canonical map $k : X \vee Y \to X \times Y$ given by $k(x) = (x, 0_Y)$ for $x \in X$ and $k(y) = (0_X, y)$ for $y \in Y$.

Theorem 4.8. *Let $f : A \to B$ be a morphism in $\mathcal{T}$.*

(a) *f is a monomorphism iff it is injective, and an epimorphism iff it has dense image.*
(b) *f is a regular monomorphism iff it is a homeomorphism of A with a closed subset of B (with the usual subspace topology), or in other words a closed inclusion.*
(c) *f is a regular epimorphism iff it is surjective and B has the quotient topology.*
(d) *A coproduct, product or composite of (regular) monomorphisms is a (regular) monomorphism.*
(e) *A coproduct, finite product or composite of (regular) epimorphisms is a (regular) epimorphism.*
(f) *$\mathcal{T}$ is biregular.*

Proof. In part (e), it is categorical nonsense that coproducts preserve coequalisers and so preserve regular epimorphisms. In part (d), we leave aside for the moment the claim that coproducts preserve (regular) monomorphisms. The remaining claims involve only products, equalisers and coequalisers, all of which are created in $\mathcal{U}$. Moreover, the proofs given for Theorem 3.1 also use only these constructs. Thus, everything goes through as before.

We now return to (d). Consider injective maps $i : A \to B$ and $j : C \to D$. By inspection of the construction we see that $i \vee j : A \vee C \to B \vee D$ is injective. Part (a) tells us that monomorphisms are precisely the injective maps, so coproducts preserve monomorphisms. Now suppose that i and j are regular monomorphisms, or in other words closed inclusions. We have a commutative diagram as follows, in $i \amalg j$ is a closed inclusion, and q is also a closed map by Remark 2.39.

$$\begin{array}{ccc} A \amalg C & \xrightarrow{i \amalg j} & B \amalg D \\ {\scriptstyle p} \downarrow & & \downarrow {\scriptstyle q} \\ A \vee C & \xrightarrow[i \vee j]{} & B \vee D \end{array}$$

Given a closed subset $F \subseteq A \vee C$, one checks directly (separating the cases $0 \in F$ and $0 \notin F$) that $(i \vee j)(F) = q((i \amalg j)(p^{-1}(F)))$, which is again closed. If follows that $i \vee j$ is a closed inclusion, as claimed. □

Proposition 4.9. *The maps $X \xrightarrow{i} X \vee Y \xleftarrow{j} Y$ and $X \vee Y \xrightarrow{k} X \times Y$ are closed inclusions.*

Proof. Let $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ be the projections. Note that $pki = 1_X$ and $qkj = 1_Y$. This means that i, j, ki and kj are all split monomorphisms, hence regular monomorphisms, hence closed inclusions. If $A \subseteq X \vee Y$ is closed then $B = i^{-1}(A)$ is closed in X and $C = j^{-1}(A)$ is closed in Y, so $k(A) = ki(B) \cup kj(C)$ is closed in $X \times Y$. Thus k is closed as claimed. □

5. The smash product

As usual, we define $X \wedge Y = (X \times Y)/(X \vee Y)$ and $S^0 = \{0, 1\}$ (with 0 as the basepoint). The underlying sets satisfy

$$(X \wedge Y) \setminus \{0\} = (X \setminus \{0\}) \times (Y \setminus \{0\}).$$

Proposition 5.1. *There are natural homeomorphisms* $S^0 \wedge X = X = X \wedge S^0$ *and* $X \wedge Y = Y \wedge X$ *and* $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$.

Proof. The only point requiring a little attention is the associativity isomorphism. Let $q_{X,Y} : X \times Y \to X \wedge Y$ be the quotient map. Using Proposition 2.20 we see that the composite

$$X \times Y \times Z \xrightarrow{q_{X,Y} \times 1} (X \wedge Y) \times Z \xrightarrow{q_{X \wedge Y, Z}} (X \wedge Y) \wedge Z$$

is a quotient map. This identifies $(X \wedge Y) \wedge Z$ with $(X \times Y \times Z)/A$, where

$$A = \{(x, y, z) \in X \times Y \times Z \mid x = 0_X \text{ or } y = 0_Y \text{ or } z = 0_Z\}.$$

We identify $X \wedge (Y \wedge Z)$ with the same space, by a symmetrical argument. □

Proposition 5.2. *The square*

$$\begin{array}{ccc} X \vee Y & \xrightarrow{k} & X \times Y \\ \downarrow & & \downarrow q \\ 0 & \longrightarrow & X \wedge Y \end{array}$$

is both a pushout and a pullback. Moreover k *is a closed inclusion, and* q *is a closed quotient map.*

Proof. The map k is a closed inclusion by Proposition 4.9. The map q is a quotient map by definition, and is closed by Remark 2.39. The square is a pushout by definition, and a pullback by inspection or by Proposition 2.35. □

Proposition 5.3. *Let* X *be a based space. Then the functor* $Y \mapsto X \wedge Y$ *preserves equalisers (but not pullbacks or products).*

Proof. Consider a pair of arrows $f, g : V \to W$ with equaliser $j : U \to V$. Consider a point $b \in X \wedge V$ with $(1 \wedge f)(b) = (1 \wedge g)(b)$. If $b = 0$ then $b = (1 \wedge j)(0)$. Otherwise $b = x \wedge v$ for a unique pair $(x, v) \in (X \setminus \{0\}) \times (V \setminus \{0\})$, in which case the equation $(1 \wedge f)(b) = (1 \wedge g)(b)$ gives $x \wedge f(v) = x \wedge g(v)$. As $x \neq 0$ this gives $f(v) = g(v)$ (even if $f(v) = 0$ or $g(v) = 0$). This means that $v = j(u)$ for a unique $u \in U$, so $b = (j \wedge 1)(x \wedge u)$. This shows that $1 \wedge j$ is the equaliser of $1 \wedge f$ and $1 \wedge g$ in the category of sets. Now consider the diagram

$$\begin{array}{ccc} X \times U & \xrightarrow{1 \times j} & X \times V \\ p \downarrow & & \downarrow q \\ X \wedge U & \xrightarrow[1 \wedge j]{} & X \wedge V \end{array}$$

It is formal that products preserve regular monomorphisms, so $1 \times j$ is a closed inclusion. The map q is closed by Remark 2.39. Given a closed subset $F \subseteq X \wedge U$ we have $F = pp^{-1}(F)$ (because p is surjective) so $(1 \wedge j)(F) = (1 \wedge j)pp^{-1}(F) = q((1 \times j)(p^{-1}(F)))$, which is closed. Thus $1 \times j$ is a closed inclusion, so it is the equaliser in $\mathcal{T}$.

On the other hand, if we let n denote a set with n points then smashing with 2_+ does not preserve the pullback of the maps $1_+ \to 0 \leftarrow 1_+$. □

Corollary 5.4. *The functor* $Y \mapsto X \wedge Y$ *preserves closed inclusions.*

Proof. Closed inclusions are the same as regular monomorphisms, or in other words maps that can be written as an equaliser of some fork. □

Definition 5.5. We put $F(X, Y) = \{f \in C(X, Y) \mid f(0_X) = 0_Y\}$, topologised as a closed subspace of $C(X, Y)$. We take the constant map with value 0_Y as a basepoint in $F(X, Y)$.

Remark 5.6. To see that $F(X, Y)$ is closed one can go back to Definition 2.8 and note that $F(X, Y) = W(\{0_X\}, \{0_Y\}^c)^c$. More abstract arguments are also possible.

Proposition 5.7. *There are natural homeomorphisms* $F(X, F(Y, Z)) = F(X \wedge Y, Z)$.

Proof. As $F(Y,Z)$ is a subspace of $C(Y,Z)$, we see that $F(X,F(Y,Z))$ is a subset of $F(X,C(Y,Z))$, which is a subset of $C(X,C(Y,Z))$, which bijects naturally with $C(X\times Y,Z)$. A function $f\colon X\times Y\to Z$ corresponds to an element of $F(X,F(Y,Z))$ iff (a) each of the functions $f(x,-)\colon Y\to Z$ preserves basepoints, and (b) the map $f(0_X,-)\colon Y\to Z$ is the zero map. These mean that $f(x,y)=0_Z$ if $x=0_X$ or $y=0_Y$, or in other words that $f(X\vee Y)=0_Z$, so f factors through $(X\times Y)/(X\vee Y)=X\wedge Y$. We thus arrive at a bijection $F(X,F(Y,Z))=F(X\wedge Y,Z)$. One can show using the Yoneda Lemma that this is in fact a homeomorphism, as in the proof of Proposition 2.12. ☐

Corollary 5.8. *The functor $(-)\wedge Y$ is left adjoint to $F(Y,-)$. Thus*

(a) *$(-)\wedge Y$ preserves all colimits in $\mathcal{T}$, and thus preserves regular epimorphisms.*
(b) *$F(Y,-)$ preserves all limits in $\mathcal{T}$, and thus preserves regular monomorphisms.* ☐

5.1. **Mapping spaces and filtered colimits.**

Lemma 5.9. *If X is compact, then the functor $F(X,-)$ preserves strongly filtered colimits of closed inclusions.*

Proof. This can be deduced from Lemma 3.8, or proved by the same line of argument. ☐

Lemma 5.10. *The unit map $\eta\colon A\to F(B,B\wedge A)$ is a closed embedding unless $B=0$.*

Proof. Suppose that $B\neq 0$, and choose $b\in B$ with $b\neq 0$. Define $i\colon S^0\to B$ by $i(0)=0$ and $i(1)=b$; this is a closed inclusion. One checks that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta} & F(B,B\wedge A) \\ & \searrow^{i\wedge 1} & \downarrow{\scriptstyle i^*} \\ & & B\wedge A. \end{array}$$

The map $i\wedge 1$ is a closed inclusion by Corollary 5.4, so η is a closed inclusion by Proposition 2.31(d). ☐

Corollary 5.11. *If $f\colon A\wedge B\to C$ is a closed inclusion and $B\neq 0$ then the adjoint map $f^{\#}\colon A\to F(B,C)$ is a closed inclusion.*

Proof. The map $f^{\#}$ is the composite

$$A\xrightarrow{\eta} F(B,A\wedge B)\xrightarrow{f_*} F(B,C).$$

We have just shown that η is a closed inclusion and $F(B,-)$ preserves closed inclusions so f_* is a closed inclusion; the claim follows. ☐

Corollary 5.12. *For any A, the maps $\Omega^n\Sigma^n A\to\Omega^{n+1}\Sigma^{n+1}A\to QA$ are closed inclusions.*

Proof. After replacing A by $\Sigma^n A$ and taking $B=S^1$, the lemma tells us that $\Sigma^n A\to\Omega\Sigma^{n+1}A$ is a closed inclusion. The functor Ω^n preserves regular monos, so $\Omega^n\Sigma^n A\to\Omega^{n+1}\Sigma^{n+1}A$ is a closed inclusion. As $QA=\varinjlim_n\Omega^n\Sigma^n A$, the rest follows from Lemma 3.3. ☐

Corollary 5.13. *If $A\to B$ is a closed embedding, then the diagram*

$$\begin{array}{ccccc} \Omega^n\Sigma^n A & \rightarrowtail & \Omega^{n+1}\Sigma^{n+1}A & \rightarrowtail & QA \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^n\Sigma^n B & \rightarrowtail & \Omega^{n+1}\Sigma^{n+1}B & \rightarrowtail & QB \end{array}$$

consists of pullback squares of closed inclusions.

Proof. We know by Corollary 5.12 that the horizontal maps are closed inclusions. Corollary 5.4 tells us that $\Sigma^n A \to \Sigma^n B$ is a closed inclusion, and it follows that $\Omega^n\Sigma^n A \to \Omega^n\Sigma^n B$ is a closed inclusion. The left hand square is a pullback of sets, by inspection. The claim now follows from Lemma 3.9. ☐

6. Examples

6.1. $\mathbb{R}^\infty$**.** We regard $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ in the obvious way, and put $\mathbb{R}^\infty = \varinjlim_n \mathbb{R}^n$. Each $\mathbb{R}^n$ is a metric space and thus CGWH, so $\mathbb{R}^\infty$ is CGWH. If we let K_n be the closed ball of radius n in $\mathbb{R}^n$ then K_n is compact and $\mathbb{R}^\infty = \bigcup_n K_n$. Thus $\mathbb{R}^\infty$ is σ-compact (ie a countable union of compact sets). It follows that every open cover has a countable subcover. Let U be an open set with compact closure. Then $U \subseteq \mathbb{R}^n$ for some n and $U = U \cap \mathbb{R}^{n+1}$ is open in $\mathbb{R}^{n+1}$; it follows that $U = \emptyset$. This shows that $\mathbb{R}^\infty$ is not locally compact.

6.2. $[0, \omega_1)$**.** Let ω_1 be the first uncountable ordinal, and give the set $[0, \omega_1)$ the order topology. This is an open subspace of the compact Hausdorff space $[0, \omega_1]$, so it is locally compact Hausdorff and thus CGWH. It is easy to see that a subset is countable iff bounded iff precompact. It follows easily that $[0, \omega_1)$ is neither separable nor σ-compact. The open sets $[0, \alpha)$ (with α countable) form an open cover with no countable subcover, so $[0, \omega_1)$ is not Lindelöf. I think that every countable open cover has a finite subcover, ie $[0, \omega_1)$ is countably compact.

6.3. $C(I, I)$**.** The space $C(I, I)$ is separable, because the set of piecewise linear functions with rational breakpoints and slopes is a countable dense subset. Put

$$F_n = \{f : I \to I \mid f(0) = 0 \text{ and } f(t) = 1 \text{ for } t \geq 1/n\}.$$

This is a decreasing sequence of closed sets with empty intersection. It follows that the sets $U_n = F_n^c$ form a countable open cover with no finite subcover, showing that $C(I, I)$ is not countably compact and thus not σ-compact.

6.4. $(\beta\mathbb{N}) \setminus \{\omega\}$**.** Let $\beta\mathbb{N}$ be the Stone-Cech compactification of $\mathbb{N}$, and choose a point $\omega \in (\beta\mathbb{N}) \setminus \mathbb{N}$ (so ω is a free ultrafilter on $\mathbb{N}$). Put $X = (\beta\mathbb{N}) \setminus \{\omega\}$; this is clearly a locally compact Hausdorff space. For any $T \subseteq \mathbb{N}$ we put $V(T) = \{\alpha \in \beta\mathbb{N} \mid T \in \alpha\}$. These sets satisfy $V(S \cup T) = V(S) \cup V(T)$ and $V(S \cap T) = V(S) \cap V(T)$ and $V(T) \cap \mathbb{N} = T$. They form a basis for the topology on $\beta\mathbb{N}$, consisting of compact clopen sets. It follows that $\{V(T) \mid T \notin \omega\}$ is a basis of open sets in X, which is closed under finite unions and intersections. It follows in turn that every compact subspace of X is contained in some $V(T)$ with $T \notin \omega$.

Lemma 6.1. *Consider a descending chain $T_0 \supseteq T_1 \supseteq \dots$ of subsets of $\mathbb{N}$. Then either $T_n = \emptyset$ for some n, or $X \cap \bigcap_n V(T_n) \neq \emptyset$.*

Proof. First suppose that $\bigcap_n T_n$ is nonempty, containing a number a say. The image of a under the embedding $\mathbb{N} \to \beta\mathbb{N}$ is the ultrafilter $\alpha = \{T \subseteq \mathbb{N} \mid a \in T\}$, which certainly lies in $\bigcap_n V(T_n)$. Moreover, as $\omega \in (\beta\mathbb{N}) \setminus \mathbb{N}$ we have $\alpha \neq \omega$ and so $\alpha \in X \cap \bigcap_n V(T_n)$, as required.

Now suppose that $\bigcap_n T_n = \emptyset$. If T_n is finite for some n, it is easy to see that $T_n = \emptyset$ for some larger n. Thus, we may assume that T_n is always infinite. It follows that we can choose numbers $a_1, a_2, \dots, b_1, b_2, \dots$, all of them distinct, such that $a_n, b_n \in T_n$. Put $A = \{a_n \mid n > 0\}$ and $B = \{b_n \mid n > 0\}$ so the sets $A \cap T_n$ and $B \cap T_n$ are nonempty for all n. As $A \cap B = \emptyset$ and ω is an ultrafilter we must have $A \notin \omega$ or $B \notin \omega$; we assume wlog that $A \notin \omega$. Now put

$$\phi = \{S \subseteq \mathbb{N} \mid S \supseteq A \cap T_n \text{ for some } n\}.$$

This is a proper filter, so we can choose an ultrafilter α with $\phi \subseteq \alpha$. Clearly $A \in \phi$ and $A \notin \omega$ so $\alpha \neq \omega$ so $\alpha \in X$. For each n we have $T_n \in \phi$ so $T_n \in \alpha$ so $\alpha \in V(T_n)$. This shows that $\alpha \in X \cap \bigcap_n V(T_n)$ as claimed. ☐

Claim 6.2. *X is not σ-compact.*

Proof. Any compact subset of X is contained in $V(T)$ for some T with $T \notin \omega$. It will thus suffice to show that for any chain $T_1 \subseteq T_2 \subseteq \dots$ with $T_n \notin \omega$ for all n, we have $\bigcup_n V(T_n) \neq X$. As $V(T)^c = V(T^c)$, it is equivalent to show that $X \cap \bigcap_n V(T_n^c) \neq \emptyset$. This will follow from Lemma 6.1, if we can show that $T_n^c \neq \emptyset$, or equivalently $T_n \neq \mathbb{N}$. This holds because $T_n \notin \omega$, whereas $\mathbb{N}$ lies in every ultrafilter. ☐

Corollary 6.3. *X is not Lindelöf.*

Proof. The sets $V(T)$ (with $T \notin \omega$) form an open cover of X. If X were Lindelöf, then there would be a countable subcover, and as the sets $V(T)$ are compact, this would mean that X was σ-compact, giving a contradiction. $\square$

Claim 6.4. *Let U be an open subset of $I \times X$ containing $0 \times X$. Then there exists $m \in \mathbb{N}$ such that $[0, 2^{-m}) \times \mathbb{N} \subseteq U$.*

Proof. Put

$$T_m := \{n \in \mathbb{N} \mid [0, 2^{-m}) \times \{n\} \subseteq U\}.$$

We clearly have $T_m \subseteq T_{m+1}$. For any $n \in \mathbb{N}$ we know that U is open and contains $(0, n)$, so we have $[0, 2^{-m}) \times \{n\} \subseteq U$ for $m \gg 0$; this shows that $\bigcup_m T_m = \mathbb{N}$, so $\bigcap_m T_m^c = \emptyset$. More generally, consider $\alpha \in X$. We again know that U is a neighbourhood of $(0, \alpha)$, so there exists $S \in \alpha$ and $m \in \mathbb{N}$ with $[0, 2^{-m}) \times V(S) \subseteq U$. As $V(S) \cap \mathbb{N} = S$ we deduce that $[0, 2^{-m}) \times S \subseteq U$ and so $S \subseteq T_m$, so $V(S) \subseteq V(T_m)$, so $\alpha \in V(T_m)$. This shows that $X \subseteq \bigcup_m V(T_m)$, so $X \cap \bigcap_m V(T_m^c) = \emptyset$. Lemma 6.1 thus tells us that for $m \gg 0$ we have $T_m^c = \emptyset$ and so $T_m = \mathbb{N}$. This means that $[0, 2^{-m}) \times \mathbb{N} \subseteq U$, as required. $\square$

Corollary 6.5. *The space $Y := (I \times X)/(0 \times X) = I \wedge X_+$ is not locally compact.*

Proof. Let N be a neighbourhood of the cone point in Y, and let $\widetilde{N}$ be its preimage in $I \times X$. Then claim 6.4 tells us that $[0, 2^{-m}) \times X \subseteq \widetilde{N}$ for some m. It follows that the set $L = \{2^{-m-1} \wedge x \mid x \in X\}$ is contained in N. Moreover, L is closed in Y and is not compact, so N cannot be compact. $\square$

Corollary 6.6. *Put*

$$\widetilde{K} = \{(t, n) \in I \times \mathbb{N} \mid t \leq 2^{-n}\} \subset I \times X,$$

and let K be the image of $\widetilde{K}$ in Y. Then K is compact, but there is no compact set $L \subseteq X$ with $K \subseteq I \wedge L_+$.

Proof. Let $\{U_i\}_{i \in I}$ be a family of open sets in Y that cover K. Then some set U_{i_0} contains the basepoint, and thus (by Claim 6.4) contains $[0, 2^{-m}) \wedge X_+$ for some m. It follows easily that $K \setminus U_{i_0}$ is compact, and thus is covered by some finite list $U_{i_1}, \ldots, U_{i_r}$, so K is covered by $U_{i_0}, \ldots, U_{i_r}$. Thus, K is compact as claimed.

If $K \subseteq I \wedge L_+$ with L closed we see that $\mathbb{N} \subseteq L$, but $\mathbb{N}$ is dense so we must have $L = X$, so in particular L is not compact. $\square$

6.5. **A non-regular space.** Let ζ be the lattice of closed sets for the usual topology on $\mathbb{R}$. For $k \geq 0$, write $S_k = \{1/n \mid n > k\}$, and write $S = S_0$. Define

$$\zeta' = \{F \cup T \mid F \in \zeta \text{ and } T \subseteq S\} \tag{1}$$

$$= \{F \cup T \mid F \in \zeta \text{ and } T \subseteq S_k\} \qquad \text{(for any } k \geq 0) \tag{2}$$

$$= \{G \subseteq \mathbb{R} \mid G \cap S^c \in \zeta|_{S^c}\} \tag{3}$$

(where $\zeta|_{S^c}$ means the ordinary subspace topology on S^c).

Proposition 6.7. *ζ' is a compactly generated weak Hausdorff topology on $\mathbb{R}$. Moreover,*

$$c(\zeta') = \{K \in c(\zeta) \mid K \cap S \text{ is finite }\} = \{K \in c(\zeta) \mid K \cap S_k = \emptyset \text{ for } k \gg 0\},$$

and for any $K \in c(\zeta')$ we have $\zeta'|_K = \zeta|_K$. However, ζ' is not regular (so there is a closed set and a point which cannot be separated by disjoint neighbourhoods).

Proof. The last description of ζ' makes it clear that it is a topology. As $\zeta \subseteq \zeta'$, we see that ζ' is Hausdorff, and that $c(\zeta') \subseteq c(\zeta)$. Suppose that $K \in c(\zeta')$. For any $T \subseteq S$ we have $T \in \zeta'$, so $T \cap K$ is closed in K. Thus $S \cap K$ is compact and discrete, hence finite as claimed; thus $S_k \cap K = \emptyset$ for $k \gg 0$. Using this and the second description of ζ', we see that $\zeta'|_K = \zeta|_K$.

Conversely, suppose that $K \in c(\zeta)$ and that $K \cap S_k = \emptyset$ for some k. Then the second description of ζ shows that $\zeta|_K = \zeta'|_K$, so that K is also compact under ζ'. This verifies all the claims about $c(\zeta')$.

We next prove that ζ' is compactly generated. Let F be compactly closed for ζ'; we must show that $F \in \zeta'$. Let G be the closure of F in the ordinary topology ζ (so $F \subseteq G$). If $G = F$, then $F \in \zeta \subset \zeta'$, so we are done; so suppose that $G \neq F$. We claim that $G = F \cup \{0\}$ and $0 \notin F$. Indeed, suppose that $0 \neq x \in G$. If x has the form $1/n$ then set $k = n$, otherwise set $k = 0$. We can then choose a sequence x_i in $F \setminus S_k$ converging to x in the usual metric. Moreover, $K = \{x_i \mid i \geq 0\} \cup \{x\}$ is compact under ζ', so by assumption $F \cap K$ is closed in $\zeta'|_K = \zeta|_K$. It follows that $x \in F$. This (with $G \neq F \subset G$) implies immediately that $G = F \cup \{0\}$ and $0 \notin F$. If $(-1/n, 1/n) \cap F$ is not contained in S for any n, then we may choose $x_n \in (-1/n, 1/n) \cap F$ (so that $x_n \to 0$) and proceed as as above to deduce that $0 \in F$, contrary to assumption; thus $T = (-1/n, 1/n) \cap F \subseteq S$ for some n. Write $F' = F \setminus (-1/n, 1/n) = G \setminus (-1/n, 1/n)$, so that $F' \in \zeta$. Thus $F = F' \cup T \in \zeta'$ as required.

Finally, it is easy to see that the closed sets $\{0\}$ and S cannot be separated by disjoint open neighbourhoods in ζ'. Thus, ζ' is not regular. □

6.6. **Bad sequential colimits.**

Proposition 6.8. *There is a diagram*

$$\begin{array}{ccccccc}
W_0 & \twoheadrightarrow & W_1 & \twoheadrightarrow & W_2 & \twoheadrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X_0 & \twoheadrightarrow & X_1 & \twoheadrightarrow & X_2 & \twoheadrightarrow & \cdots
\end{array}$$

in which the vertical maps are closed inclusions, but the induced map $\varinjlim_n W_n \to \varinjlim_n X_n$ *is not injective. (It follows that finite limits do not commute with sequential colimits in general.)*

Proof. Put $X' = [0,1] \times \{0,1\}$ and $W' = \{0\} \times \{0,1\}$. Define an equivalence relation R_n on X' by $(s,a) R_n (t,b)$ if $s = t$ and ($a = b$ or $s \geq 2^{-n}$). Put $X_n = X'/R_n$ and $W_n = W'$. One finds that the induced map $W_n \to X_n$ is a closed inclusion, but $\varinjlim_n W_n = W'$ and $\varinjlim_n X_n = I$ but the induced map $W' \to I$ sends both points of W' to 0. □

6.7. **Irregular evaluation.**

Proposition 6.9. *There is a path-connected based space X for which the evaluation map $\epsilon\colon PX \to X$ is not a quotient map.*

Proof. Put

$$\begin{aligned}
a &= (0,0,0) \\
b_n &= (0,n,1) \text{ for } n < \infty \\
b_\infty &= (0,-1,1) \\
c_n &= (2^{-n},0,2) \text{ for } n < \infty \\
c_\infty &= (0,0,2).
\end{aligned}$$

Note that

(a) $d(b_i, b_j) > 1$ whenever $i \neq j$
(b) $c_n \to c_\infty$ as $n \to \infty$
(c) The line segments $(a, b_i]$ and $(b_i, c_i]$ are all disjoint.

Let X be the union of all the line segments $[a, b_i]$ and $[b_i, c_i]$, and take a as the basepoint. Put $U = (b_\infty, c_\infty]$, which is not open in X because $c_n \to c_\infty$. Put $V = \epsilon^{-1} U \subseteq PX$. I'm fairly sure that this is open in PX, proving the claim. □

6.8. **Discontinuity of the Pontrjagin-Thom construction.** Given a locally compact Hausdorff space U, we write U_∞ for the one-point compactification. Given an open embedding $i\colon U \to V$ of such spaces, we define $i^\bullet\colon V_\infty \to U_\infty$ by $i^\bullet(i(u)) = u$ and $i^\bullet(v) = \infty$ for v not in the image of i. We write $\mathrm{Emb}(U,V)$ for the space of open embeddings from U to V, with the Kellification of the subspace topology inherited from $C(U,V)$. We then define a function $\phi\colon \mathrm{Emb}(U,V) \to F(V_\infty, U_\infty)$ by $\phi(i) = i^\bullet$. I think that this is often continuous, for example when U and V are manifolds. Here, however, we will give an example where ϕ is *not* continuous.

First, we take $U = V = \prod_{k=0}^{\infty}\{0,1\}$. We define $f_k \colon U \to U$ by

$$f_k(x)_i = \begin{cases} x_i & \text{if } i < k \\ 0 & \text{if } i = k \\ x_{i-1} & \text{if } i > k. \end{cases}$$

This is isomorphic to the product of 1_U with the inclusion $\{0\} \to \{0,1\}$, so it is an open embedding. I claim that $f_k \to 1$ in $C(U,U)$. To see this, note that $C(U,U) \simeq \prod_i C(U,\{0,1\})$, and convergence in product spaces is detected termwise, so it suffices to show that $\pi_i \circ f_k \to \pi_i$ as $k \to \infty$, for all $i \geq 0$. This is clear because $\pi_i \circ f_k = \pi_i$ when $k > i$.

Now put $e_k = 1$ for all k, giving an element $e \in U$. For any $x \in U$ we have $f_k(x)_k = 0 \neq e_k$, so e is not in the image of f_k, so $f_k^\bullet(e) = \infty$. Thus $f_k^\bullet(e) \not\to e = 1^\bullet(e)$, so $f_k^\bullet \not\to 1^\bullet$.

6.9. **Bad pullbacks.** We remarked earlier that although $\mathcal{U}$ is regular (so regular epis are preserved by pullback), coequaliser diagrams need not be preserved by pullback. We now give an example of this behaviour. Let X and Y be CGWH spaces, and let U be a dense open subspace of Y with $Z = Y \setminus U$. Consider the following diagram, in which all the maps are the evident inclusions and projections:

$$\begin{array}{ccccc} \emptyset & \rightrightarrows & X \times Z & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X^2 \times U & \rightrightarrows & X \times Y & \longrightarrow & Y. \end{array}$$

We claim that the bottom line is a coequaliser. To see this, let R be the smallest equivalence relation on $X \times Y$ such that $(x,u)R(x',u)$ whenever $x, x' \in X$ and $u \in U$. As subsets of $X^2 \times Y^2 = (X \times Y)^2$, one can check that

$$R = (\Delta_X \times \Delta_Y) \cup (X^2 \times \Delta_U).$$

The set $\overline{R} = X^2 \times \Delta_Y$ is a closed equivalence relation and R is dense in $\overline{R}$ so $\overline{R}$ is the smallest closed equivalence relation containing R. Clearly also $(X \times Y)/\overline{R} = Y$, and it follows that the bottom line is a coequaliser as claimed. The top row is obtained by pulling back the bottom row along the map $Z \to Y$. Although the right hand map on the top row is a regular epimorphism (as required for regularity) the row itself is not a coequaliser diagram.

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