



Derived Completions in Stable Homotopy theory

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Abstract

We introduce a notion of derived completion applicable to arbitrary homomorphisms of commutative S -algebras, and work out some of its properties, including invariance results, a spectral sequence proceeding from purely algebraic information to the geometric results, and analysis of relationships with earlier constructions. We also provide some examples. The construction has applications in algebraic K -theory.

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1. Introduction

It has long been recognized that the development of a theory of ring and module spectra, which bears the same relationship to the category of spectra as the ordinary theory of rings and modules has to the category of abelian groups, is a very desirable thing. A number of such theories exist. The different approaches [10,12] solve the problem in a satisfactory way, and more recently (see [14]) versions using orthogonal spectra and Γ -spaces are also available. The goal of transporting constructions which are available for ordinary rings and modules to this new category of ring spectra is also a very worthwhile one. Some of these constructions have already been made by the authors of [10, 12]. In this paper, we will use the notion of an S -algebra (as in [10]) as the spectrum version of a ring. Given an S -algebra A and module spectra M and N , one can construct spectra $M \wedge_A N$ and $\mathrm{Hom}_A(M, N)$, analogous to the constructions $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$ for rings A and modules M and N . From the point of view of topologists, the most important constructions to transport to the category of spectra are those which are *homotopy invariant*, i.e. those for which module or ring homomorphisms which induce weak equivalences on underlying spectra induce equivalences on the constructions. For this reason, one considers only *derived constructions*, i.e. constructions which on the algebraic side would always replace a module by a projective resolution for it, and would replace a ring by a levelwise free simplicial ring. In the context of these categories of modules, this means that one replaces rings and modules by *cofibrant* and/or *fibrant* objects in the categories of S -algebras and module spectra. Our goal in this paper is to introduce and study the derived version of the completion construction for modules over a commutative ring.

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We describe the main results. Let $f : A \rightarrow B$ be a homomorphism of commutative S -algebras, and let M be an A -module spectrum. Then we define a cosimplicial A -module spectrum $\mathcal{T}_A(M; B)$, and the *derived completion of M along the homomorphism f* , denoted by M_B^\wedge , as the total spectrum of $\mathcal{T}_A(M; B)$. Here are the properties of this construction that we will prove in this paper.

- The construction $M \rightarrow M_B^\wedge$ is functorial for homomorphism of module spectra over A .
- The construction is functorial in B in the sense that if $B \rightarrow C$ is a homomorphism of commutative A -algebras, we obtain an induced homomorphism $M_B^\wedge \rightarrow M_C^\wedge$.
- The construction is functorial in A in the following sense. Let $A \rightarrow B \rightarrow C$ be a diagram of commutative S -algebras, and let M be a B -module. Let $\rho_A(M)$ denote the spectrum regarded as an A -module spectrum by restriction of scalars along the ring homomorphism $A \rightarrow B$. Then there is an induced map $\rho_A(M)_C^\wedge \rightarrow M_C^\wedge$.
- For any commutative ring A , A may be regarded as a commutative S -algebra via the Eilenberg–MacLane spectrum $\mathbb{H}(A)$, and an A -module M can be regarded as a module spectrum $\mathbb{H}(M)$. Thus, for a homomorphism of commutative rings $f : A \rightarrow B$, it is possible to construct the derived completion $\mathbb{H}(M)_{\mathbb{H}(B)}^\wedge$. For general A , B and M , it is possible that although $\mathbb{H}(A)$, $\mathbb{H}(B)$, and $\mathbb{H}(M)$ have no higher homotopy, the derived completion will. However, if A is Noetherian, M is finitely generated, and f is surjective, the derived completion of $\mathbb{H}(M)$ coincides with the Eilenberg–MacLane construction $\mathbb{H}(M_I^\wedge)$, where M_I^\wedge is the usual algebraic completion construction at the ideal $I = \text{Ker}(f)$.
- One can show that in a sense, the completion “depends only on $\pi_0(B)$ for (-1) -connected S -algebras”. The precise statement, which is our [Theorem 6.1](#), is that if we are given a diagram $f : A \rightarrow B \rightarrow C$ of (-1) -connected commutative S -algebras with $\pi_0 B \rightarrow \pi_0 C$ an isomorphism, and an A -module spectrum M , then the map $M_B^\wedge \rightarrow M_C^\wedge$ is a weak equivalence of spectra.
- For any homomorphism $f : A \rightarrow B$ of (-1) -connected commutative S -algebras, and any connective A -module spectrum M (i.e. $\pi_s M = 0$ for s sufficiently small), there is a spectral sequence whose E_2 -term depends only on the structure of $\pi_* M$ as a module over $\pi_0 A$ and of $\pi_0 B$ as an algebra over $\pi_0 A$, and which converges to $\pi_* M_B^\wedge$. This is our [Theorem 7.1](#).

By far the most important use of the notion of completion in homotopy has been in p -adically or profinitely completing spaces. These constructions are in a sense “tame”, in that for spectra with finitely generated homotopy groups, the homotopy groups of the completion can be obtained by algebraically completing the homotopy groups at primes or profinitely, and in general are viewed as a simplification of the homotopy type. When the homotopy groups are not finitely generated, one has situations where there is a single derived functor of completion which contributes to the homotopy groups of the completion. However, when applied to more complicated rings, our construction can construct interesting homotopy types from discrete rings, even for finitely generated modules over the ring. For example, let A be the group ring of the discrete group $\mathbb{Z}/p^\infty\mathbb{Z} = \bigcup_n \mathbb{Z}/p^n\mathbb{Z}$, and \mathbb{F}_p as an A -algebra via augmentation followed by reduction mod p . If we form the derived completion of A itself as an A -module (using the Eilenberg–MacLane construction as above), we obtain the p -adically completed group ring on the singular chains on the circle group, regarded as a simplicial ring. What has in effect happened is that the derived completion construction on a discrete ring has created an S -algebra (which is much like a topological ring), which coincides with our geometric notion of “filling in the gaps” in the group $\mathbb{Z}/p^\infty\mathbb{Z}$, viewed as embedded in the circle group. So in this case, the completion produces interesting homotopy types related to embeddings of the ring within topological rings. This phenomenon has some similarity with the behavior of Quillen’s plus construction, which replaces the classifying space of a discrete group with a homologically equivalent space. In [13], Lawson has studied the completion process in the context of the pro- p completion of a finitely generated nilpotent group Γ , and has shown that the homotopy groups of the completion are strongly related with the homology groups of stable (with respect to dimension) representation varieties for Γ .

We are developing this material for use in applications in algebraic K -theory, specifically in order to understand the descent problem for the K -theory of fields (see [6]). The goal is to obtain a homotopy theoretic model for the K -theory spectrum of a field F which depends only on the absolute Galois group G_F of the F . It turns out that a model which is often correct can be constructed out of the S -algebra associated with the symmetric monoidal category of finite dimensional complex representations of G_F precisely by performing the derived completion described in this paper. Although this is our main application, we hope and expect that our constructions will find use in other contexts.

At least three other notions of derived completion have appeared in the literature, in [11,8,3]. The goal of this paper is to record a precise version of the construction which we can use in future work, but we do make some comments on the relationship of the present construction with those in [11,8].

The outline of the paper is as follows. Section 2 develops the preliminary technical material we will require, including material on the theory of S -algebras and module spectra, as well as the theory of Barr and Beck on simplicial and cosimplicial approximations of spectra using “triple” or “monads”. Section 3 then defines our derived completion, and develops its elementary properties. In Section 4, we study the behavior of this construction on discrete rings, and show that it coincides with ordinary completion for finitely generated modules over a Noetherian ring. In Section 5, we compare our construction with two other completions, those constructed in [11,8]. Section 6 proves our invariance result for S -algebra homomorphisms inducing an isomorphism on π_0 , and Section 7 constructs the spectral sequence discussed above. Finally, in Section 8, we develop some interesting examples.

2. Preliminaries

2.1. S -algebras and module spectra

As mentioned in the introduction, there are a number of constructions of categories of spectra which admit a coherently associative and commutative smash product. One consequence of these constructions is that one can develop a theory of “ring spectra” as the category of monoid objects in a category of spectra relative to the smash product. As mentioned in the introduction, we elect to work with S -algebras as our notion of ring spectrum. In a similar way, one can define module spectra M over an S -algebra R as spectra equipped with a map $R \wedge M \rightarrow M$ so that the standard algebraic diagrams commute. One can also define the notion of a commutative S -algebra, in terms of the commutativity of an obvious diagram. The commutative S -algebras form a category in their own right, we will denote it by Alg_S . More generally, if A is a commutative S -algebra, we can also construct the category Alg_A of commutative A -algebra spectra. Given any commutative S -algebra A , one can also define a category Mod_A of module spectra over A . Note that because A is commutative, we do not have to specify whether the module is a right or left module. For any commutative S -algebra there exist relative notions of smash products (analogous to tensor products over a ring) and Hom-spectra (analogous to Hom-modules in algebra). See [10] or [12] for the particulars of these theories. We will henceforth work with the S -algebra version of this theory as constructed in [10]. The following proposition summarizes the properties of the categories Alg_A and Mod_A which we will need. The results can all be found in [10], pp. 140–148.

Proposition 2.1. *For any commutative S -algebra A , the categories Alg_A and Mod_A can both be equipped with the structure of a Quillen model category (see [7]) with the following properties.*

1. *A morphism in Alg_A or Mod_A is a weak equivalence if and only if its underlying map of spectra is one.*
2. *All objects in Alg_A or Mod_A are fibrant.*
3. *In each category, there is a functorial way to replace each morphism f with a decomposition $f = p \circ i$ with p being a fibration and i , a cofibration which is also a weak equivalence. Similarly for p , a fibration and a weak equivalence and i , a cofibration. In particular, there exist functorial cofibrant and fibrant replacements in Mod_A and Alg_A .*
4. *For any homomorphism $A \rightarrow B$ of commutative S -algebras, the functor $M \rightarrow B \wedge_A M$ is a triple in the sense of [2]. See Section 2.3 below.*
5. *There exists a triple Sym_A on the category Mod_A so that a commutative A -algebra spectrum is precisely the same thing as an algebra over the triple Sym_A , in the sense of [2].*

In particular, the category of spectra is equivalent to the category Mod_S , where S is the sphere spectrum, and so the results apply to the category of spectra.

The smash products and Hom constructions are homotopy invariant when the argument modules are cofibrant and/or fibrant, in the sense made precise in the following proposition.

Proposition 2.2. *Let A be a commutative S -algebra. Let $f : M \rightarrow M'$ be a weak equivalence of cofibrant A -module spectra, and let N be any fibrant A -module spectrum. Then the induced maps*

$$f \wedge_A id_N : M \wedge_A N \rightarrow M' \wedge_A N$$

and

$$\text{Hom}_A(f, N) : \text{Hom}_A(M', N) \rightarrow \text{Hom}_A(M, N)$$

are both equivalences. Also, if M is cofibrant, and $g : N \rightarrow N'$ is a weak equivalence, then the natural map

$$\text{Hom}_A(M, g) : \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, N')$$

is a weak equivalence.

The smash product and Hom constructions also behave well on cofibrations of module spectra, in the following sense.

Proposition 2.3. *Let A be a commutative S -algebra, and let $f : M \rightarrow M'$ be a cofibration of A -module spectra, and let N be an A -module spectrum. The induced maps $f \wedge_A Id_N$ and $\text{Hom}_A(f, N)$ are cofibrations and fibrations, respectively.*

These results allow one to develop spectral sequences as computational tools.

Proposition 2.4. *Suppose A is a commutative S -algebra, M is a cofibrant A -module spectrum, and N is an A -module spectrum. Then there is a spectral sequence with $E_2^{pq} = \text{Tor}_{pq}^{\pi_*(A)}(\pi_*(M), \pi_*(N))$, converging to $\pi_{p+q}(M \wedge_A N)$. The superscript p refers to homological degree, and q refers to internal degree.*

Corollary 2.5. *Suppose A is a (-1) -connected commutative S -algebra, M is an s -connected cofibrant A -module spectrum, and N is a t -connected A -module spectrum. Then $M \wedge_A N$ is $(s + t + 1)$ -connected.*

2.2. Cosimplicial S -algebras and module spectra

Let A be a commutative S -algebra, and let Mod_A denote the category of module spectra over A . Since Mod_A is a Quillen model category the usual notions of homotopy colimits, homotopy limits, the total A -module spectrum of a cosimplicial object in Mod_A as well as its finite stage approximations Tot , and fibrancy of a cosimplicial A -module spectrum all make sense, and they share the properties of the corresponding notions in the category of simplicial sets and the category of spectra. Every cosimplicial object in Mod_A is functorially equivalent to a fibrant one, and we write $(-)\text{fib}$ for this functor. We recall some of the important properties.

Proposition 2.6. *Let $F : \Delta \rightarrow \text{Mod}_A$ denote a fibrant cosimplicial object in Mod_A . Then there is a canonical natural equivalence*

$$\text{Tot}(F) \rightarrow \underset{\Delta}{\text{holim}} F.$$

Moreover, if we let $\Delta^{(n)}$ denote the full subcategory on the objects of cardinality less than or equal to $n + 1$, then there is a natural equivalence

$$\text{Tot}^n(F) \rightarrow \underset{\Delta^{(n)}}{\text{holim}} F.$$

Proposition 2.7. *Let $F : \underline{C} \times \underline{D} \rightarrow \text{Mod}_A$ be a functor, where \underline{C} and \underline{D} are small categories. Then for any object $c \in \underline{C}$, we may construct the homotopy inverse limit object $F_c = \underset{c \times \underline{D}}{\text{holim}} F \mid c \times \underline{D}$. The construction $c \rightarrow F_c$ is*

functorial in c , and we obtain a natural equivalence

$$\underset{\underline{C} \times \underline{D}}{\text{holim}} F \cong \underset{\underline{C}}{\text{holim}} F_c.$$

Proposition 2.8. *Suppose we have a sequence $X \rightarrow Y \rightarrow Z$ of fibrant cosimplicial objects in Mod_A , which is levelwise a cofibration sequence. Then the sequences*

$$\text{Tot}^n X^\cdot \rightarrow \text{Tot}^n Y^\cdot \rightarrow \text{Tot}^n Z^\cdot$$

and

$$\text{Tot} X^\cdot \rightarrow \text{Tot} Y^\cdot \rightarrow \text{Tot} Z^\cdot$$

are cofibration sequences up to homotopy in Mod_A .

Proposition 2.9. *Let X^\cdot be a fibrant cosimplicial object in Mod_A , and let M denote an object in Mod_A . Then we may form the new cosimplicial object $M \wedge_A X^\cdot$, and there is a natural equivalence in Mod_A*

$$M \wedge_A \text{Tot}^n X^\cdot \cong \text{Tot}^n((M \wedge_A X^\cdot)_{\text{fib}}).$$

Proof. Follows easily from (see [10] or [12]) the fact that forming smash products over A with a fixed module commutes with homotopy pullbacks of diagrams of the form $X \rightarrow Y \leftarrow Z$. We omit the details. \square

It will also be useful to use the idea of an *augmented cosimplicial object* in a category. Let Δ_{aug} denote the category obtained by adjoining to Δ the single object -1 , which represents the empty set, regarded as a totally ordered set. This description also makes it clear how to define the morphisms in Δ_{aug} . By an augmented cosimplicial object in a category \underline{C} , we mean a covariant functor from Δ_{aug} with values in \underline{C} . If we have any abelian group-valued functor A on \underline{C} , we obtain from any cosimplicial object in \underline{C} an augmented cosimplicial abelian group. As with cosimplicial abelian groups, we may then associate a cochain complex (starting in codimension -1) to this cosimplicial abelian group. We summarize the properties of augmented cosimplicial spectra.

Proposition 2.10. *Suppose we are given an augmented cosimplicial object X^\cdot in a category \underline{C} , and let ρX^\cdot denote the cosimplicial object obtained by restriction to $\Delta \subseteq \Delta_{\text{aug}}$. Suppose also that \underline{C} is complete, hence admits a notion of “total object” $\text{Tot} \rho X^\cdot$. Then the coface map $\delta^0 : X^{-1} \rightarrow X^0$ induces a morphism $\theta : X^{-1} \rightarrow \text{Tot}(\rho X^\cdot)$.*

The following result gives a criterion which guarantees that the map η is a weak equivalence, when the underlying category \underline{C} is the category Mod_A .

Proposition 2.11. *Suppose X^\cdot is an augmented cosimplicial object in the model category Mod_A and that the cosimplicial object ρX^\cdot is a fibrant cosimplicial object in Mod_A . For each t , we obtain an augmented cosimplicial abelian group $\pi_t(X^\cdot)$, to which we associate a cochain complex $C^*(t)$ as above. If each of the cochain complexes $C^*(t)$ has trivial cohomology, then the map $\theta : X^{-1} \rightarrow \text{Tot}(\rho X^\cdot)$ is a weak equivalence in Mod_A .*

Proof. Straightforward verification using the homotopy spectral sequence of a cosimplicial space, see [5], Chapter X. \square

Finally, we will require a comparison theorem for bicosimplicial A -modules.

Proposition 2.12. *Suppose that we have a map of bicosimplicial A -modules $f^\cdot : X^\cdot \rightarrow Y^\cdot$. Suppose that for each $p \geq 0$, the map $\text{Tot}(X_{\text{fib}}^{p,\cdot}) \rightarrow \text{Tot}(Y_{\text{fib}}^{p,\cdot})$ is a weak equivalence. Then the natural map $\text{Tot}(\Delta(X^\cdot)_{\text{fib}}) \rightarrow \text{Tot}(\Delta(Y^\cdot)_{\text{fib}})$ is a weak equivalence, where Δ denotes the diagonal cosimplicial space. Similarly if we study the levelwise simplicial objects obtained by holding q fixed.*

Proof. Straightforward, and we omit it. \square

2.3. The theory of Barr and Beck

Our theory of completion will make use of the theory of the *cosimplicial object of a triple*, a notion discussed by Barr and Beck [2]. We will need to use some of the comparison theorems from that paper, so we review that theory here. For the definition of a triple and algebras over a triple, see [19].

Suppose a category \underline{C} is equipped with a triple T , and $c \in \underline{C}$. Then we may define the cosimplicial resolution of c relative to T to be the cosimplicial object $T^\cdot(c)$ defined by $T^k(c) = T^{k+1}(c)$, and where the cofaces and codegeneracies are defined by

$$\delta^s : T^{k+1}(c) \rightarrow T^{k+2}(c) = T^s(\eta(T^{k+1-s}(c)))$$

and

$$\sigma^t : T^{k+1}(c) \rightarrow T^k(c) = T^t(\mu(T^{k-t-1}(c))).$$

Note that $T^\cdot(c)$ extends canonically to an augmented cosimplicial object $T_{\text{aug}}^\cdot(c)$ by setting $T_{\text{aug}}^{-1} = c$, and letting $\delta^0 : T_{\text{aug}}^{-1} \rightarrow T_{\text{aug}}^0$ be the canonical inclusion $\eta : c \rightarrow T(c)$. This means that we have a natural map $\theta : c \rightarrow \text{Tot}(T^\cdot(c))$.

Suppose now that the category \underline{C} is Mod_A . For any A -module spectrum M , will denote by $T_{\text{fib}}^\cdot(M)$ the functorial fibrant replacement for $T^\cdot(M)$. We will need criteria which guarantee that the map $\theta_{\text{fib}}(M) : M \rightarrow \text{Tot}(T_{\text{fib}}^\cdot(M))$ is a weak equivalence in Mod_A , where $\theta_{\text{fib}}(M)$ is the composite $M \xrightarrow{\theta} \text{Tot}(T^\cdot(M)) \rightarrow \text{Tot}(T_{\text{fib}}^\cdot(M))$. Barr and Beck now prove the following result.

Theorem 2.13. *Suppose that M is equipped with a T -algebra structure. Then $\theta_{\text{fib}}(M)$ is a weak equivalence.*

We will derive a useful corollary of this result, using the following lemma.

Lemma 2.14. *Let S and T be two triples on the model category Mod_A , and suppose we are given a natural transformation $\nu : S \rightarrow T$ of triples. We may construct the bicosimplicial A -module C^\cdot given by $C^{pq} = T_{\text{fib}}^p(S_{\text{fib}}^q(M))$. We may also regard the cosimplicial spectrum T_{fib}^\cdot as a bicosimplicial spectrum, constant in the q -direction, and denote it by T_0^\cdot . Then the evident bisimplicial map $\theta^T : T_0^\cdot \rightarrow C^\cdot$ induces an equivalence on total A -modules.*

Proof. By applying Proposition 2.12, it will suffice to prove that each of the natural maps

$$T^p(M) \rightarrow \text{Tot}(T^p S^q(M)_{\text{fib}})$$

is a weak equivalence of A -module spectra. By 2.11, we need only verify that the cohomology of the cochain complex A^* attached to the augmented cosimplicial group $\pi_i(T^p((S_{\text{fib}}^\cdot(M)_{\text{aug}})))$ is trivial. However, for each $q \geq 0$, we have the map $h^q : T^p(S^q(M)) \rightarrow T^p(S^{q-1}(M))$ given by the composite

$$T^p(S^q(M)) \xrightarrow{T^p(\nu(S^{q-1}(M)))} T^p(TS^{q-1}(M)) \xrightarrow{T^{p-1}(\mu^T(S^{q-1}(M)))} T^p(S^{q-1}(M)).$$

As in the proof in [2] of Theorem 2.13, the operators $\pi_1(h^q)$ yield a contracting homotopy for the cochain complex A^* , which gives the result.

We also obtain the following.

Proposition 2.15. *Let S and T be two triples on the model category Mod_A , and suppose we are given a natural transformations $\nu : S \rightarrow T$ and $\mu : T \rightarrow S$ of triples. Then the natural map*

$$\text{Tot } S_{\text{fib}}^\cdot(M) \rightarrow \text{Tot } T_{\text{fib}}^\cdot(M)$$

is an equivalence, where S^\cdot and T^\cdot denote the cosimplicial resolutions of M associated with S and T , respectively.

Proof. This follows directly from Theorem 5.3 in [16]. It is also a straightforward consequence of the work in [2] or [4]. \square

We will also find it useful to discuss the simplicial resolution of a T -algebra X , for any triple T .

Definition 2.16. Let T be a triple on a category \underline{C} , and let X be any T -algebra, with structure morphism $\alpha : TX \rightarrow X$. The simplicial resolution of X relative to T is the simplicial object $T_\cdot(X)$, defined on objects by

$$T_k(X) = \underbrace{T \circ T \circ \dots \circ T}_{k+1 \text{ factors}}(X) \quad \text{for } k \geq 0$$

and on morphisms by

$$\begin{cases} d_i = T^i \mu(T^{k-i-1}(X)) & \text{for } 0 \leq i \leq k-1 \\ d_k = T^k(\alpha) \\ s_i = T^{i+1} \eta(T^{k-i}(X)) & \text{for } 0 \leq i \leq k. \end{cases}$$

The structure map α gives a map $\varepsilon(X) : |T_\cdot(X)| \rightarrow X$, where X denotes the constant simplicial object with value X .

Proposition 2.17. *Let T denote a triple on the category of spectra. Then for any T -algebra spectrum X , the map $\varepsilon(X)$ induces an isomorphism on homotopy groups.*

Proof. The structure map $\alpha : TX \rightarrow X$ provides an extension of the functor T to the larger category $\Delta_{\text{aug}}^{\text{op}}$, which we will denote by T^{aug} . We will refer to a functor from $\Delta_{\text{aug}}^{\text{op}}$ to a category \underline{C} as an *augmented simplicial object* in \underline{C} . For any augmented simplicial abelian group A ., we may construct the associated chain complex $C_*(A)$., with the alternating sum of face maps as boundary operator. For any augmented spectrum X ., we denote by $\rho(X)$ its restriction to the subcategory $\Delta^{\text{op}} \subseteq \Delta_{\text{aug}}^{\text{op}}$, so $\rho(X)$ is a simplicial spectrum. The augmented spectrum gives rise to a map $\nu(X) : \rho(X) \rightarrow X_{-1}$., by analogy with the cosimplicial case, it is now easy to verify that given any augmented simplicial spectrum X ., if it is the case that for each i ., the chain complex $C_*\pi_s(X)$ has trivial homology, then the map $\nu(X)$ is a weak equivalence. For any triple T on the category of spectra, and every integer t ., the complex $C_*(\pi_t T^{\text{aug}}(X))$ has trivial homology, since the operators

$$\pi_t \eta(T^{k+1}(X)) : \pi_t T_k^{\text{aug}}(X) \rightarrow \pi_t T_{k+1}^{\text{aug}}(X)$$

provide a contracting homotopy for it. Note that the operator even makes sense for $k = -1$., with $T^0(X) = X$. This gives the result. \square

3. Definitions

Let A denote a commutative S -algebra and B a commutative A -algebra spectrum. The construction in [10,12] of smash products over A make the functor $T_A(-; B) = B \wedge_A -$ into a triple on the category Mod_A . Consequently, we may construct the cosimplicial resolution of $T_A(-; B)$ as a functor on the category of A -module spectra to the category of cosimplicial A -module spectra. We will write $T'_A(-; B)$ for the functorial fibrant replacement of this cosimplicial object in Mod_A .

We recall from [18] that for any commutative S -algebra A , there is a closed model structure on the category of A -algebras. Moreover, one may functorially replace any A -algebra by a weakly equivalent A -algebra which is cofibrant.

Definition 3.1. Let A denote a commutative S -algebra, B a commutative A -algebra, and M an A -module. We define the *derived completion* of M at the A -algebra B to be the total spectrum of the cosimplicial spectrum $T'_A(M; \tilde{B})$., where \tilde{B} is the cofibrant replacement for B in the category of commutative A -algebras. We write M_B^\wedge for $\text{Tot}(T'_A(M; B))$.

The following propositions summarize the most important properties of this construction.

Proposition 3.2. *Let $A \rightarrow B \rightarrow C$ be a diagram of commutative S -algebras. Then the following statements all hold.*

1. *The construction $M \rightarrow M_B^\wedge$ is functorial for homomorphisms of A -modules. The map on B -completions induced by a homomorphism $f : M \rightarrow N$ of A -modules will be denoted by f_B^\wedge .*
2. *Let $f : M \rightarrow N$ be a homomorphism of A -modules, where A is a commutative S -algebra, and let B denote a commutative A -algebra. Suppose that $\text{id}_B \wedge_A f$ is a weak equivalence of B -module spectra. Then the map on completions f_B^\wedge is also a weak equivalence of spectra.*
3. *Let $M \rightarrow N \rightarrow P$ be a cofibration sequence of A -module spectra, where A is a commutative S -algebra. Let B be a commutative A -algebra spectrum. Then the sequence $M_B^\wedge \rightarrow N_B^\wedge \rightarrow P_B^\wedge$ is a cofibration sequence up to homotopy.*
4. *Let $A \rightarrow B \rightarrow C$ be a diagram of commutative S -algebras, and let M denote a B -module spectrum. M may be regarded as an A -module spectrum M^A . Suppose that the natural map $C \wedge_A M^A \rightarrow C \wedge_B M$ is a weak equivalence of spectra. Then the natural map $(M^A)_C^\wedge \rightarrow M_C^\wedge$ is also an equivalence of spectra.*
5. *There is a natural transformation $\eta : M \rightarrow M_B^\wedge$.*
6. *Suppose $A \rightarrow B$ is a homomorphism of commutative S -algebras. Suppose M is an A -module spectrum, for which the A -module structure admits a B -module structure extending the given A -module structure. Then the natural map $\eta : M \rightarrow M_B^\wedge$ is an equivalence of spectra.*

Proof. (1) is immediate from the constructions. (2) is an immediate consequence of the standard fact that a levelwise equivalence of fibrant spectra induces a weak equivalence of Total spectra. (3) and (4) follow from the fact that smash products with a fixed module preserve homotopy cofibration sequences, and that the total spectrum carries levelwise homotopy cofibration sequences to homotopy cofibration sequences. (5) follows from 2.13. (6) is a general property of the cosimplicial resolution of a triple. \square

Remark 3.1. We note that although we define the derived completion of an A -module spectrum M at a homomorphism $f : A \rightarrow B$ of commutative S -algebras to be the total spectrum of $\mathcal{T}_A(M; B)$, it is useful to recall that it is therefore the total space of the tower of fibrations

$$\cdots \rightarrow \text{Tot}^n(\mathcal{T}_A(M; B)) \rightarrow \text{Tot}^{n-1}(\mathcal{T}_A(M; B)) \rightarrow \cdots \rightarrow \text{Tot}^1(\mathcal{T}_A(M; B)) \rightarrow \text{Tot}^0(\mathcal{T}_A(M; B))$$

and that the tower of fibrations is actually functorial as well. This tower of fibrations gives filtrations on the homotopy groups and other invariants, and will likely be interesting in future K -theoretic applications of these ideas.

4. The case of rings

For any ring A , we may construct the Eilenberg–MacLane spectrum $\mathbb{H}(A)$. It is direct from the constructions of [10,12] that $\mathbb{H}(A)$ is in a canonical way a S -algebra, and that if the ring is commutative, $\mathbb{H}(A)$ is a commutative S -algebra. Left and right modules over the ring A yield, via the \mathbb{H} construction, left and right module spectra over the S -algebra A . The goal of this section is to analyze how the derived completion construction given in the preceding section applies to S -algebras obtained via this construction. Specifically, we will show that it coincides with ordinary completion for finitely generated modules over commutative Noetherian rings, in the sense, $\mathbb{H}(M)_{\mathbb{H}(A/I)}^{\wedge} \cong \mathbb{H}(M_I^{\wedge})$ for any finitely generated A -module M and any ideal $I \subseteq A$, and where M_I^{\wedge} denotes the usual algebraic completion operation at the ideal I . Note that for a commutative ring A , any left A -module can be canonically regarded as a right A -module as well, so we will simply refer to A -modules without specifying right or left modules.

Proposition 4.1. *Let A be a ring, M and N be A -modules. Let $\overline{\mathbb{H}(M)}$ denote a cofibrant replacement of the $\mathbb{H}(A)$ -module spectrum $\mathbb{H}(M)$. Then*

$$\pi_*(\overline{\mathbb{H}(M)} \wedge_{\mathbb{H}(A)} \mathbb{H}(N)) \cong \text{Tor}_*^A(M, N)$$

as graded groups. $M \wedge_A N$ denotes the spectrum level construction of “smash product over A ”, and Tor denotes the usual algebraic derived functors.

Proof. This results follows from the Künneth spectral sequence whose E_2 -term is $\text{Tor}_*^A(M, N)$, and which collapses for dimensional reasons. \square

This means that for any family of k left A -modules M_1, M_2, \dots, M_k , we may define groups

$$\text{MultiTor}_i^A(M_1, M_2, \dots, M_k)$$

as the i th homology of the complex $R(M_1) \otimes_A R(M_2) \otimes_A \cdots \otimes_A R(M_k)$, where $R(M_j)$ denotes an A -projective resolution of M_j .

Proposition 4.2. *Let A be a commutative S -algebra, and let M_1, M_2, \dots, M_k denote a family of left A -modules. The Künneth spectral sequence generalizes to a spectral sequence with E_2 -term*

$$\text{MultiTor}_*^A(M_1, M_2, \dots, M_k)$$

converging to $\pi_*(M_1 \wedge_A M_2 \wedge_A \cdots \wedge_A M_k)$.

Proposition 4.3. *Let A be a commutative ring, and let $B = A/I$, where I is an ideal in A . Suppose further that M is a B -module, which we regard as an A -module by restriction of scalars. Then the natural map $M \rightarrow M_B^{\wedge}$ is an equivalence.*

Proof. This is immediate from Proposition 3.2, (6). \square

What we have now shown is that for any B -module M , the derived completion M_B^\wedge is equivalent to M itself, regarded as a module spectrum over A . We wish to extend this to a result valid for all finitely generated A -modules over a Noetherian ring. We recall first that the definition of the I -adic completion of an A -module M is

$$\lim_{\leftarrow k} M/I^k M.$$

We may also consider the homotopy inverse limit of the pro- A -module $\{M/I^k M\}_{k \geq 0}$. Recall from [5] that for any inverse system of spectra $\{X_k\}_{k \geq 0}$, there is a short exact sequence

$$0 \rightarrow \lim^1_{\leftarrow k} \{\pi_{i-1} X_k\}_{k \geq 0} \rightarrow \pi_i \operatorname{holim}_{\leftarrow k} X_k \rightarrow \lim_{\leftarrow k} \pi_i X_k \rightarrow 0.$$

In our case, the inverse system $\{\pi_{i-1} X_k\}_{k \geq 0}$ consists entirely of surjective maps, it is a standard result that its \lim^1 -term vanishes, leaving us with the isomorphism

$$\pi_i(\operatorname{holim}_{\leftarrow k} M/I^k M) \cong \lim_{\leftarrow k} M/I^k M \quad \text{for } i = 0$$

and $\pi_i(\operatorname{holim}_{\leftarrow k} M/I^k M) \cong 0$ for $i \neq 0$. In other words

$$\operatorname{holim}_{\leftarrow k} M/I^k M \cong \mathbb{H}(M_I^\wedge).$$

We now wish to show that $M_B^\wedge \cong \operatorname{holim}_{\leftarrow k} M/I^k M$. Consider the inverse system of cosimplicial A -module spectra $\{\mathcal{T}_A(M/I^k M; B)\}_{k \geq 0}$. We have the natural map

$$\theta : \mathcal{T}_A(M; B) \rightarrow \{\mathcal{T}_A(M/I^k M; B)\}_{k \geq 0}$$

where $\mathcal{T}_A(M; B)$ is considered as a constant pro-cosimplicial A -module spectrum. By taking total spectra and homotopy inverse limits (in the k -direction), we obtain a map of an A -module spectra $M_B^\wedge \rightarrow \operatorname{holim}_{\leftarrow k} \operatorname{Tot} \mathcal{T}_A(M/I^k M; B)$. By 4.3, we have that $\operatorname{Tot} \mathcal{T}_A(M/I^k M; B) \cong M/I^k M$, so we obtain a natural map

$$\lambda : M_B^\wedge \rightarrow \operatorname{holim}_{\leftarrow k} M/I^k M \cong \mathbb{H}(M_I^\wedge).$$

Our result is now

Theorem 4.4. *For a Noetherian commutative ring A , and a finitely generated A -module M , the map $\lambda : M_B^\wedge \rightarrow \mathbb{H}(M_I^\wedge)$ described above is an equivalence.*

Proof. It is standard that homotopy inverse limits of cosimplicial spectra commute with taking total spectra, so it is enough to verify that the maps

$$\mathcal{T}_A^i(M; B) \rightarrow \operatorname{holim}_{\leftarrow k} \mathcal{T}_A^i(M/I^k M; B)$$

are equivalences for each i . In order to prove this, it will suffice to show that the pro- A -module $\{\mathcal{T}_A^i(M/I^k M; B)\}_{k \geq 0}$ is isomorphic, as pro-abelian groups, to the constant pro-abelian group with value $\mathcal{T}_A^i(M; B)$. By 4.2, this means that it will suffice to show that the pro-abelian group

$$\{\operatorname{MultiTor}_*^A(\underbrace{B, B, \dots, B}_s \text{ factors}, M/I^k M)\}_{k \geq 0}$$

is isomorphic to the constant pro-abelian group with value $\operatorname{MultiTor}_*^A(\underbrace{B, B, \dots, B}_s \text{ factors}, M)$. For any A -module M , we

let $\underline{I}M$ denote the pro- A -module $\{I^k M\}_{k \geq 0}$. Recall (see [1]) that the category of pro- A -modules is itself an abelian

category, and so the notion of exact sequence has meaning. The construction $M \rightarrow \underline{I}M$ gives a functor from the category of finitely generated A -modules to the category of pro- A -modules, which is exact in the sense that it carries exact sequences to exact sequences. The exactness follows from the usual proof of exactness of completion for finitely generated modules over a Noetherian ring (see [9]). This same exactness also shows that for complexes of finitely generated A -modules, homology commutes with the functor \underline{I} , so that we have

$$\text{MultiTor}_*^A(\underbrace{B, B, \dots, B}_s \text{ factors}, \underline{I}M) \cong \underline{I}\text{MultiTor}_*^A(\underbrace{B, B, \dots, B}_s \text{ factors}, M).$$

It is now clear that $\text{MultiTor}_*^A(\underbrace{B, B, \dots, B}_s \text{ factors}, M)$ is a B -module, and therefore I acts trivially on it, which means that the pro-abelian group $\underline{I}\text{MultiTor}_*^A(\underbrace{B, B, \dots, B}_s \text{ factors}, M)$ is pro-isomorphic to the zero group. We have an exact sequence

of pro-abelian groups

$$\underline{I}M \rightarrow M \rightarrow M/\underline{I}M$$

so we may conclude that the map $M \rightarrow M/\underline{I}M$ induces an isomorphism on homology as pro-groups. This is the required result. \square

5. Comparison with other constructions

In this section we will prove some results comparing the present construction with those of Greenlees and May [11] and Dwyer, Greenlees and Iyengar [8].

5.1. The Greenlees–May completion

This completion construction considers the situation of a commutative ring A and a finitely generated ideal I . It begins with a purely algebraic construction of derived functors of completion, and then produces a spectrum level construction. We will compare the derived functors defined in [11] with the homotopy groups of our derived completion, and proved that they agree. It appears likely that the methods of Section 7 would allow a comparison for the spectrum level construction, but we do not carry out that here. We will write $B = A/I$, and let S denote any finite generating set for I . In [11], the authors construct a chain complex C_*^S , essentially a colimit of Koszul complexes based on S , and define their derived completion of an A -module M , which we will denote by M_S^{gm} , by

$$M_S^{gm} = \text{Hom}_A(C_*^S, M).$$

The construction depends on S , but there are canonical isomorphisms between the homology groups constructed using different finite generating sets S . These are the derived functors of completion at I applied to the module M . There is a natural chain map $C_*^S \xrightarrow{\gamma} A_*$, where A_* denotes the chain complex, concentrated in degree 0, with zeroth module equal to A . The map γ now induces a natural map

$$\xi : M \rightarrow M_S^{gm}$$

where M denotes the module M regarded as a chain complex concentrated in degree zero. We now have the following.

Lemma 5.1. *Let M be any B -module, regarded as an A -module by restriction of scalars. Then the natural map $\xi : M \rightarrow M_S^{gm}$ is a quasi-isomorphism of chain complexes*

Proof. Immediate consequence of Lemma 1.3 of [11]. \square

It is also immediate from the definitions that the construction $M \rightarrow M_S^{gm}$ extends to chain complexes of A -modules. The following is now easy to prove.

Lemma 5.2. *The construction $C_* \rightarrow (C_*)_S^{gm}$ preserves quasi-isomorphisms. Given a chain complex C_* which is bounded below, and so that the A -module structure on $H_i(C_*)$ extends to a B -module structure for all i , then the natural map $\xi : C_* \rightarrow (C_*)_S^{gm}$ is a quasi-isomorphism.*

Proof. The first statement is straightforward from the definitions. The second one follows from the existence of a “Postnikov tower” for any bounded below chain complex, where the relative terms are resolutions of the A -modules $H_i(C_*)$. The first statement implies that if $R_*(M)$ is a resolution of an A -module M , then the map $R_*(M) \rightarrow M$ induces an equivalence $(R_*(M))_S^{gm} \rightarrow M_S^{gm}$. The result now follows from Lemma 5.1 above, together with the evident fact that the construction carries exact sequences of chain complexes to exact sequences.

In order to compare M_S^{gm} with M_B^\wedge , we will need to reinterpret M_B^\wedge as a chain complex. We first begin by constructing a cofibrant version \mathcal{B}_* of B . This can be done, for example, by using the free commutative A -algebra functor on sets. This is a triple on the category of sets, and for any A -algebra yields a simplicial resolution of B , which is levelwise free as an A -module. It is also a simplicial commutative ring. The cosimplicial spectrum $\mathcal{T}_A(M; \mathcal{B}_*)$ which constructs M_B^\wedge is now a cosimplicial simplicial abelian group, and its total space is a simplicial abelian group. As such, it corresponds to a chain complex. On the other hand, one can define the notion of the chain complex associated with a spectrum. One begins with the associated chain complex functor C_* for the notion of spaces one is dealing with, the singular complex in the case of topological spaces, and the simplicial chains for simplicial sets. For any spectrum $\mathcal{X} = \{X_i\}_{i \geq 0}$, with structure maps $\sigma_i : \Sigma X_i \rightarrow X_{i+1}$, one defines $ch(\mathcal{X})$ to be the colimit of the system

$$\cdots \rightarrow \Sigma^{-i} C_*(X_i) \rightarrow \Sigma^{-(i+1)} C_*(X_{i+1}) \rightarrow \Sigma^{-(i+2)} C_*(X_{i+2}) \rightarrow \cdots .$$

Now, the cosimplicial spectrum corresponds to a cosimplicial chain complex, by applying ch levelwise. We write $E(k, M)_*$ for the chain complex in codimension k . It is defined by

$$E(k, M)_* = ch \left(\bigwedge_k \mathcal{B} \wedge M \right).$$

For any cosimplicial chain complex $C(-)_*$, we define a chain complex $\mathcal{DC}(-)_*$ by

$$\mathcal{DC}(-)_* = \prod_k \Sigma^{-k} C(k)_*$$

on the level of graded groups (Σ^t just denotes a shift of degrees by t), and the boundary map comes from the bicomplex structure on this graded group, with one boundary being the one existing on each of the complexes $\Sigma^{-k} C(k)_*$, and the other being the alternating sum of the coface maps.

Proposition 5.3. *Let $k \rightarrow A(k)$. denote any cosimplicial simplicial abelian group. Then there are canonical isomorphisms*

$$\pi_s(\text{Tot}_k A(k)) \cong H_s(\mathcal{Dch}(A(-)))$$

for all s .

Proof. This is standard. See for example [5]. \square

We now write $\mathcal{E}_*(M)$ for the chain complex $\mathcal{DE}(-, M)_*$. The construction \mathcal{E}_* is clearly functorial in M , and $H_*(\mathcal{E}_*(M))$ is canonically isomorphic to $\pi_*(M_B^\wedge)$. We note that \mathcal{E}_* is a chain complex of A -modules, and that the standard map from the constant cosimplicial A -module with value M to $\mathcal{T}_A(M; \mathcal{B}_*)$ induces a natural transformation $\theta : M_* \rightarrow \mathcal{E}_*(M)$, where M_* denotes M regarded as a chain complex concentrated in degree zero.

We wish to compare the groups $H_*(M_S^{gm})$ and $H_*(\mathcal{E}_*(M))$. The idea will be to compare each of the complexes in question with the complex $\mathcal{E}_*(M)_S^{gm}$. The natural transformation θ produces a natural map $\Theta : M_S^{gm} \rightarrow \mathcal{E}_*(M)_S^{gm}$, and the transformation ξ above yields a natural map $\Xi : \mathcal{E}_*(M) \rightarrow \mathcal{E}_*(M)_S^{gm}$. We now have the following result.

Proposition 5.4. *The natural maps Θ and Ξ are quasi-isomorphisms. Therefore the groups $H_*(M_S^{gm})$ and $H_*(\mathcal{E}_*(M))$ are canonically isomorphic.*

Proof. We begin with Ξ . We note first that $\mathcal{E}_*(M)$ is itself obtained as the total complex of a bicomplex, and can be filtered by the codimension k . Note that this is a decreasing filtration. The subquotients are the complexes $ch(\bigwedge_k \mathcal{B} \wedge M)$. We obtain a filtration on $\mathcal{E}_*(M)_S^{gm}$ which is compatible with the filtration on $\mathcal{E}_*(M)$ under the natural

map Ξ . Both complexes are obtained as inverse limits of the quotients by the terms in the filtrations, so in order to prove the result it will suffice to show that the natural map

$$ch \left(\bigwedge_k \mathcal{B} \wedge M \right) \rightarrow \left(ch \left(\bigwedge_k \mathcal{B} \wedge M \right) \right)_S^{gm}$$

is a quasi-isomorphism for all k . The reduction uses the \lim^1 sequences for the homology of inverse limits of chain complexes. But now, in view of Lemma 5.2 above, it will suffice to show that the A -module structure on $H_i(ch(\bigwedge_k \mathcal{B} \wedge M))$ extends to a B -module structure. But this is clear, since the homology of these complexes are the Multitor groups of M with coefficients in B , and the elements of B act on B by multiplication. This shows that Ξ is a quasi-isomorphism. For Θ , we note that the Greenlees–May complex C_*^S is a colimit of subcomplexes $C_*^S(k)$, which are quasi-isomorphic to the Koszul complexes $\mathcal{K}(k)$ based on the set $\{s^k\}_{s \in S}$. It follows that M_S^{gm} is expressed as the inverse limit of complexes $M_S^{gm}(k)$, which are quasi-isomorphic to $\text{Hom}_A(\mathcal{K}(k), M)$, and similarly for chain complexes. Therefore, there is also a description of $\mathcal{E}_*(M)_S^{gm}$ as an inverse limit of quotient complexes $\mathcal{E}_*(M)_S^{gm}(k)$, which is compatible (under the map Θ) with the inverse limit description of M_S^{gm} . It therefore suffices to show that the natural map

$$\text{Hom}_A(\mathcal{K}(k), M) \rightarrow \text{Hom}_A(\mathcal{K}(k), \mathcal{E}_*(M))$$

is a quasi-isomorphism. To see this, we note that there is a canonical isomorphism of complexes $\text{Hom}_A(\mathcal{K}(k), \mathcal{E}_*(M)) \rightarrow \mathcal{E}_*(\text{Hom}_A(\mathcal{K}(k), M))$. We also note that the complex $\text{Hom}_A(\mathcal{K}(k), M)$ admits a finite Postnikov tower, with the relative quotients being quasi-isomorphic to resolutions of the various A -modules $H_i(\text{Hom}_A(\mathcal{K}(k), M))$. This therefore induces a similar tower on $\mathcal{E}_*(\text{Hom}_A(\mathcal{K}(k), M))$, and it therefore suffices to show that the natural map $R_*(H_i) \rightarrow \mathcal{E}_*(R_*(H_i))$ is a quasi-isomorphism for all i , where

$$H_i = H_i(\text{Hom}_A(\mathcal{K}(k), M)).$$

Since a resolution of an A -module is quasi-isomorphic to that module regarded as a constant chain complex, we have reduced the problem to verifying that $H_i \rightarrow \mathcal{E}_*(H_i)$ is a quasi-isomorphism for all i , or equivalently that $H_i \rightarrow (H_i)_B^\wedge$ is an equivalence. The action of any element s^k on $\mathcal{K}(k)$ is easily seen to be chain null homotopic, and the similar result follows for $P = \text{Hom}_A(\mathcal{K}(k), M)$. Each H_i therefore has the property that every element of the form s^k , for $s \in S$, acts by zero on H_i . It follows that there exists an integer l so that the ideal I^l acts trivially on H_i . Now consider the filtration $\{I^s H_i\}$ on H_i . It is a finite filtration, and the subquotients $I^s H_i / I^{s+1} H_i$ are A -modules so that I acts trivially. Using the exactness property of our derived completion construction, it suffices to prove that the natural maps

$$I^s H_i / I^{s+1} H_i \rightarrow (I^s H_i / I^{s+1} H_i)_B^\wedge$$

are equivalences. But this is Proposition 4.3. \square

The following corollary is now immediate.

Corollary 5.5. *Let A be any commutative ring, I any finitely generated ideal, with finite generating set S . Let M be any A -module. Then the Greenlees–May completion M_S^{gm} and the derived completion $M_{A/I}^\wedge$ are canonically equivalent in the homotopy category of A -module spectra, where A is regarded as an S -algebra via the Eilenberg–MacLane construction.*

5.2. The Dwyer–Greenlees–Iyengar completion

In [8], the authors construct a version of completion which is inspired by Morita theory. The idea is as follows. Let A and B be commutative S -algebras, and let $f : A \rightarrow B$ be a homomorphism of commutative S -algebras. One can then construct an S -algebra $\mathcal{E} = \mathcal{E}_A(B) = \text{Hom}_A(B, B) = \text{End}_A(B)$, where the Hom constructions are in the category of A -modules. B now becomes a left \mathcal{E} -module. Further, for any A -module M , we may form $B \underset{A}{\wedge} M$, and it becomes a left \mathcal{E} -module as well. One now constructs $\text{Hom}_{\mathcal{E}}(B, B \underset{A}{\wedge} M)$, and note that there is a natural homomorphism $M \rightarrow \text{Hom}_{\mathcal{E}}(B, B \underset{A}{\wedge} M)$, whose adjoint is the map

$$B \wedge M \rightarrow B \underset{A}{\wedge} M$$

of left \mathcal{E} -modules, where the action on $B \wedge M$ is on the first factor. We will call the spectrum $\text{Hom}_{\mathcal{E}}(B, B \wedge_A M)$, the *DGI completion* of A at the homomorphism f . The main subject of [8] is duality theory in module spectra, and this notion of completion is best understood in the context where B satisfies certain finiteness conditions, which is where the duality theory of [8] takes place. We will see that in our situation, there is a related (more restrictive) finiteness condition, under which we will prove that the DGI completion agrees with our derived completion.

Given A, B, M, f , and \mathcal{E} as above, we construct the cosimplicial A -module spectrum $\mathcal{T}_A^{\cdot}(B; M)$ as in Section 3. We can now construct the cosimplicial B -module spectrum $B \wedge_A \mathcal{T}_A^{\cdot}(B; M)$, to obtain a cosimplicial \mathcal{E} -module \mathcal{M} , and construct the cosimplicial \mathcal{E} -module spectrum

$$\text{Hom}_{\mathcal{E}}(B, \mathcal{M}).$$

Of course, there is the canonical map $M \rightarrow \mathcal{T}_A^{\cdot}(B; M)$, consequently the induced map $B \wedge_A M \rightarrow \mathcal{M}$, and therefore a natural map

$$\alpha : \text{Hom}_{\mathcal{E}}(B, B \wedge_A M) \rightarrow \text{Hom}_{\mathcal{E}}(B, \mathcal{M})$$

where the left hand side is the DGI completion of M .

Proposition 5.6. *The map $\text{Tot}(\alpha)$ of total spectra induced by α is an equivalence of spectra.*

Proof. One readily checks that there is an equivalence

$$\text{Hom}_{\mathcal{E}}(B, \text{Tot } \mathcal{M}) \rightarrow \text{Tot}(\text{Hom}_{\mathcal{E}}(B, \mathcal{M}))$$

and reduces to proving that the natural map $B \wedge_A M \rightarrow \text{Tot}(\mathcal{M})$ is an equivalence. This follows readily from [Theorem 2.13](#). \square

Since we have a natural map from an A -module M to its DGI completion, we obtain a natural map

$$\rho : \mathcal{T}_A^{\cdot}(M; B) \rightarrow \text{Hom}_{\mathcal{E}}(B, \mathcal{M})$$

and the corresponding map of total spectra from M_B^{\wedge} to $\text{Tot}(\text{Hom}_{\mathcal{E}}(B, \mathcal{M}))$, which in the homotopy category can be interpreted as a map from M_B^{\wedge} to the DGI completion of M , according to [Proposition 5.6](#).

Lemma 5.7. *Suppose that B is finitely built from A in the sense of [8], and that the A -module M is also finitely built from A . Then the map ρ defined above is an equivalence. In other words, when B is finitely built from A , our derived completion agrees with the DGI completion.*

Proof. We sketch a proof. One can write $\text{Hom}_{\mathcal{E}}(B, \text{Tot } \mathcal{M})$ as the total spectrum of the cosimplicial spectrum

$$k \mapsto \text{Hom}_{\mathcal{E}}(B, B \wedge_A \mathcal{T}_A^k(M; B))$$

and in order to prove the result it will clearly suffice to prove that the natural maps

$$\mathcal{T}_A^k(M; B) \rightarrow \text{Hom}_{\mathcal{E}}(B, B \wedge_A \mathcal{T}_A^k(M; B))$$

are equivalences. From the definition of \mathcal{T}_A^{\cdot} , it will now suffice to show that for any module N which is finitely built from A , the natural map

$$B \wedge_A N \rightarrow \text{Hom}_{\mathcal{E}}(B, B \wedge_A B \wedge_A N)$$

is an equivalence. This follows from the three facts given below, which are easily obtained using the results of [8].

- The correspondence D defined by $D(N) = \text{Hom}_A(N, A)$ is a contravariant equivalence of categories from the category of modules which are finitely built over A to itself. There is a canonical equivalence $N \cong D^2(N)$.
- Given any A -module N which is finitely built from A , there is a canonical equivalence

$$\text{Hom}_{\mathcal{E}}(B, B \wedge_A D(N)) \rightarrow \text{Hom}_{\mathcal{E}}(B \wedge_A N, B).$$

- For any A -module N which is finitely built from A , there is an equivalence of left \mathcal{E} -modules

$$B \wedge_A D(B \wedge_A N) \cong \mathcal{E} \wedge_A D(N)$$

where the \mathcal{E} -action on the left hand factor is on the left hand factor B , and on the right hand side by left multiplication on the \mathcal{E} -factor. \square

We now extend this lemma into the main result of this section. We recall the notion of *proxy finiteness* from [8]. Let A be a commutative S -algebra. The A -module B is said to be proxy finite if there is an A -module K so that K is finitely built from A , and so that K is built from B . The DGI completion uses arbitrary A -modules as input, while our construction requires a commutative A -algebra. Consequently, this notion as it stands is not useful for us. We define the following replacement notion which is needed in our context.

Definition 5.8. Let A be a commutative S -algebra, and let B be a commutative A -algebra, with structure homomorphism $f : A \rightarrow B$. We say B is *algebra proxy finite* over A if there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & \overline{B} \\ & \searrow f & \downarrow \\ & & B \end{array}$$

of commutative S -algebras, where the A -module \overline{B} is finitely built from A , and built from B .

The following lemma makes this notion useful.

Lemma 5.9. *Let*

$$\begin{array}{ccc} A & \longrightarrow & \overline{B} \\ & \searrow & \downarrow \\ & & B \end{array}$$

be a commutative diagram of commutative S -algebras as above, and suppose \overline{B} is built from B . Then for any A -module M , the natural map

$$M_B^\wedge \rightarrow M_B^\wedge$$

is an equivalence of A -module spectra.

Proof. We consider the bicosimplicial spectrum \mathcal{B}^\cdot defined by

$$(k, l) \mapsto \mathcal{T}_A^k(\mathcal{T}_A^l(M; \overline{B}); B).$$

It is equipped with a natural map

$$\eta : \mathcal{T}_A^\cdot(M; \overline{B}) \rightarrow \mathcal{B}^\cdot$$

when $\mathcal{T}_A^\cdot(M; \overline{B})$ is regarded as a cosimplicial spectrum constant in the k -direction. We first observe that η induces an equivalence on total spectra. To verify this only required to show that for any \overline{B} -module N , the natural map $N \rightarrow N_B^\wedge$ is an equivalence. This follows from the fact that the total spectrum construction respects cofibration sequences, together with Proposition 3.2, part 5. There is also a natural map $\nu : \mathcal{T}_A^\cdot(M; B) \rightarrow \mathcal{B}^\cdot$, where \mathcal{T}_A^\cdot is regarded as a bicosimplicial spectrum constant in the l -direction. It too induces an equivalence on total spectra. It is easy to see that to verify this only required to show that for any A -module N , the natural map

$$B \wedge_A N \rightarrow \text{Tot}(B \wedge_A \mathcal{T}_A^\cdot(N; \overline{B}))$$

is an equivalence of spectra. But, there is an evident equivalence of cosimplicial spectra

$$B \wedge_A \mathcal{T}_A^\cdot(N; \overline{B}) \simeq \mathcal{T}_A^\cdot(B \wedge_A N; \overline{B})$$

and we are reduced to showing that the map

$$\eta : B \wedge_A N \rightarrow \text{Tot } \mathcal{T}_A(B \wedge_A N; \overline{B}) \cong (B \wedge_A N) \wedge_B$$

is an equivalence. But, since B is an \overline{B} -algebra spectrum, it follows directly that $B \wedge_A N$ admits a \overline{B} -module structure extending the A -module structure, and the result now follows directly from Proposition 3.2, part 5. It is finally not difficult to check that we have a commutative diagram in the homotopy category

$$\begin{array}{ccc} M \wedge_B & \longrightarrow & M \wedge_B \\ & \searrow \eta & \downarrow v \\ & & \text{Tot } \mathcal{B} \end{array}$$

where the horizontal map is the usual map arising from the functoriality of the completion construction. This gives the result. \square

We now draw our main conclusion.

Theorem 5.10. *Let $f : A \rightarrow B$ be a homomorphism of commutative S -algebras, and suppose that B is algebra proxy finite. Then for any A -module M which is finitely built from A , the map ρ defined above from $M \wedge_B$ to the DGI completion of M is an equivalence.*

Proof. Follows directly from Lemma 5.9 above, together with Theorem 4.10 of [8], from which it follows that the DGI completion has an invariance property similar to that proved in Lemma 5.9 above. \square

6. Main isomorphism theorem

In this section we will prove the following theorem about derived completions.

Theorem 6.1. *Suppose that we have a diagram $A \rightarrow B \xrightarrow{f} C$ of commutative S -algebras. Suppose further that $A, B,$ and C are all (-1) -connected, and that the homomorphism $\pi_0(f)$ is an isomorphism. Suppose further that the natural homomorphisms $\pi_0(A) \rightarrow \pi_0(B)$ and $\pi_0(A) \rightarrow \pi_0(C)$ are surjections. Then for any connective left A -module spectrum M , the natural homomorphism $M_B \rightarrow M_C$ is an equivalence of A -module spectra.*

The proof of this theorem requires some preliminary technical work on cosimplicial spaces and spectra. Let X' denote any fibrant cosimplicial space (or, more generally, a fibrant cosimplicial spectrum or A -module, where A is a commutative S -algebra), i.e. a space-valued functor from the category Δ whose objects are the totally ordered sets $\underline{k} = \{0, 1, \dots, k\}$ and whose morphisms are the order-preserving maps of sets. From Proposition 2.6, we have that $\text{Tot}(X')$ is weakly equivalent to $\text{holim}_{\Delta} X'$, and similarly that $\text{Tot}^n(X') \cong \text{holim}_{\Delta^{(n)}} X'$, where for any non-negative integer,

$\Delta^{(n)}$ denotes the full subcategory on the subsets of cardinality $\leq n$. Also, let \mathcal{D}_n denote the partially ordered set of non-empty subsets of the set \underline{n} , regarded as a category with a unique morphism from S to T whenever $S \subseteq T$, and so that $\text{Hom}_{\mathcal{D}_n}(S, T) = \emptyset$ whenever $S \not\subseteq T$. For any subset $S \subseteq \underline{n}$ of cardinality $s + 1$, let $\pi_n(S) = \underline{s}$. Since S inherits a total ordering from that of \underline{n} , there is a unique order-preserving bijective map $\xi_S : S \rightarrow \pi_n(S)$. For any inclusion $S \subseteq T$ in \mathcal{D}_n , we let $\pi_n(S \subseteq T)$ denote the unique morphism in $\Delta^{(n)}$ which makes the diagram

$$\begin{array}{ccc} S & \xrightarrow{\xi_S} & \pi_n(S) \\ \downarrow S \subseteq T & & \downarrow \pi_n(S \subseteq T) \\ T & \xrightarrow{\xi_T} & \pi_n(T) \end{array}$$

commute. The two definitions make π_n into a functor from \mathcal{D}_n to $\Delta^{(n)}$. Our next goal is now to prove that the natural pullback map

$$\pi_n^* : \text{holim}_{\Delta^{(n)}} X' \rightarrow \text{holim}_{\mathcal{D}_n} X' \circ \pi_n$$

is an equivalence. In order to do this, we recall that by [5], Theorem 9.2, it will suffice to show that for any object $\underline{k} \in \Delta^{(n)}$, the category $\pi_n \downarrow \underline{k}$ has a contractible nerve. Recall that $\pi_n \downarrow \underline{k}$ denotes the category whose objects are pairs (S, θ) , where $S \in \mathcal{D}_n$ and $\theta : \pi_n(S) \rightarrow \underline{k}$ is a morphism in $\Delta^{(n)}$, and where a morphism from (S, θ) to (S', θ') in $\pi_n \downarrow \underline{k}$ is a morphism φ from S to S' in \mathcal{D}_n making the diagram

$$\begin{array}{ccc} \pi_n(S) & \xrightarrow{\pi_n(\varphi)} & \pi_n(S') \\ & \searrow \theta & \downarrow \theta' \\ & & \underline{k} \end{array}$$

commute. We will construct a category equivalent to $\pi_n \downarrow \underline{k}$ which is readily analyzed. For any finite totally ordered set S , and any positive integer j , a j -fold interval decomposition of S is a partition

$$S = S_0 \coprod S_1 \coprod \cdots \coprod S_j$$

of S , so that whenever $0 \leq i < i' \leq j$, then for any $x \in S_i$ and $y \in S_{i'}$, $x < y$. Note that some of the sets S_i may be empty. We let P_n^k be the partially ordered set whose objects are pairs $(S, \{S_j\}_{0 \leq j \leq k})$, where S is a non-empty subset of \underline{n} , $\{S_j\}_{0 \leq j \leq k}$ is a k -fold interval decomposition of S , and where $(S, \{S_j\}_{0 \leq j \leq k}) \leq (T, \{T_j\}_{0 \leq j \leq k})$ if and only if $S_j \subseteq T_j$ for all $0 \leq j \leq k$.

Proposition 6.2. *There is a natural equivalence of categories from the category attached to the poset P_n^k to the category $\pi_n \downarrow \underline{k}$. Consequently, in order to verify that $\pi_n \downarrow \underline{k}$ has contractible nerve, it suffices to show that P_n^k does.*

Proof. Define $\alpha : P_n^k \rightarrow \pi_n \downarrow \underline{k}$ by setting $\alpha(S, \{S_j\}_{0 \leq j \leq k})$ equal to the pair (S, θ) , where $\theta : S \rightarrow \underline{k}$ is the unique order-preserving map described by $\theta(x) = j$ if and only if $x \in S_j$. It is easy to check that $\pi_n \downarrow \underline{k}$ is a partially ordered set in the sense that for any pair of objects x and y , $\text{Hom}_{\pi_n \downarrow \underline{k}}(x, y)$ is either empty or consists of a single element. One easily sees that the construction α respects the partial orderings, and hence creates a functor. This process is clearly completely reversible (in fact, it is an isomorphism of categories), and the result follows. \square

Proposition 6.3. *$N.P_n^k$ is contractible whenever $k \leq n$.*

Proof. We proceed by induction on k . Of course, if $k = 0$, we have that $P_n^0 \cong \mathcal{D}_n$, and the latter poset has a maximal element, whence its nerve is contractible. Now, suppose we know the result to be true for all $k \leq K$, and we wish to prove it for K . Let $n \geq K$. The poset P_n^K contains $K + 1$ minimal elements $\{x_0, x_1, \dots, x_K\}$, where x_j denotes the element whose underlying set is $\{n\}$, and whose partition places n in the j th interval. Let $\tilde{P}_n^K \subseteq P_n^K$ denote the partially ordered subset $P_n^K - \{x_0, x_1, \dots, x_K\}$. We also have the obvious embedding $i : P_{n-1}^K \hookrightarrow \tilde{P}_n^K$. We claim that this inclusion induces a homotopy equivalence on nerves. To see this, it suffices to construct an order-preserving map $f : \tilde{P}_n^K \rightarrow P_{n-1}^K$ so that $f \circ i = id$, and so that $i \circ f(x) \leq x$ for all $x \in \tilde{P}_n^K$. But now we note that the map given by $(S, \{S_j\}) \rightarrow (S - \{n\}, \{S_j - \{n\}\})$ provides the required map of partially ordered sets. For each $0 \leq j \leq K$, we let $Q_j \subseteq \tilde{P}_n^K$ denote the subset $\{y \in \tilde{P}_n^K \mid y \geq x_j\}$. As in [15], one now verifies that we have a decomposition

$$N.P_n^K \cong N.\tilde{P}_n^K \cup CN.Q_0 \cup CN.Q_1 \cup \cdots \cup CN.Q_K.$$

We next observe that the inclusion $Q_0 \hookrightarrow \tilde{P}_n^K$ induces an equivalence on nerves. To see this, we observe that the restriction of the poset map f constructed above to Q_0 is an isomorphism of partially ordered sets. It follows that the simplicial set $N.\tilde{P}_n^K \cup CN.Q_0$ is contractible. Now, $N.P_n^K$ is obtained from $N.\tilde{P}_n^K \cup CN.Q_0$ by adjoining cones to the subspaces $N.Q_1, N.Q_2, \dots, N.Q_n$. This means that in order to prove the contractibility of $N.P_n^K$, it will suffice to prove the contractibility of the sets $N.Q_j$, for $j = 1, 2, \dots, n$. We claim that the partially ordered set Q_j is isomorphic to P_{n-1}^{K-j} . Since we have $K - j \leq n - 1$ whenever $j \geq 1$, we have that $N.P_{n-1}^{K-j}$ is contractible by the inductive hypothesis, and the result would follow. To establish the isomorphism $Q_j \cong P_{n-1}^{K-j}$, we first note that $(S, \{S_i\}_{0 \leq i \leq K}) \in Q_j$ if and only if $n \in S_{K-j}$. Therefore, for $(S, \{S_i\}_{0 \leq i \leq K}) \in Q_j$ we have that $S_l = \emptyset$ for $l > K - j$. The isomorphism of partially ordered sets is now given by $(S, \{S_i\}_{0 \leq i \leq K}) \rightarrow (S - \{n\}, \{S_i - \{n\}\}_{0 \leq i \leq K-j})$. \square

Consider any spectrum-valued functor F on \mathcal{D}_n , and for any spectrum X , we let F_X denote the constant functor on \mathcal{D}_n with value X . Suppose that we are given a natural transformation $F_X \rightarrow F$. We are interested in constructing a useful model for the homotopy fiber of the natural map

$$\operatorname{holim}_{\mathcal{D}_n} F_X \rightarrow \operatorname{holim}_{\mathcal{D}_n} F.$$

We let I denote the partially ordered set of non-empty subsets of $\underline{1}$, i.e. the category pictured by the diagram

$$\{0\} \longrightarrow \{0, 1\} \longleftarrow \{1\}$$

and let J denote the full subcategory on the two objects $\{0\}$ and $\{0, 1\}$. Consider the category I^{n+1} , and its full subcategory J^{n+1} . For any object $\sigma = (S_0, S_1, \dots, S_n)$ of J^{n+1} , we let $\psi(\sigma) \subseteq \underline{n}$ denote the subset $\{j \in \underline{n} \mid S_j = \{0, 1\}\}$. We now define a spectrum-valued functor \bar{F} on I^{n+1} by the formulas

$$\begin{cases} \bar{F}(0, 0, \dots, 0) = X \\ \bar{F}(\sigma) = F(\psi(\sigma)) \quad \text{for any } \sigma \in J^{n+1} - (0, 0, \dots, 0) \\ \bar{F}(\xi) = * \quad \text{for any } \xi \in I^{n+1} - J^{n+1}. \end{cases}$$

The behavior on morphisms is evident, when we recall that we are given a natural transformation from the constant functor F_X to F .

Proposition 6.4. *There is a natural equivalence (in the homotopy category) ε_n from the homotopy fiber Φ of the natural map $\operatorname{holim}_{\mathcal{D}_n} F_X \rightarrow \operatorname{holim}_{\mathcal{D}_n} F$ to $\operatorname{holim}_{J^{n+1}} \bar{F}$.*

Proof. Φ can clearly be interpreted as the homotopy inverse limit over the category $\mathcal{D}_n \times I$ of the functor G defined by

$$\begin{cases} G(S, \{0\}) = X \\ G(S, \{0, 1\}) = F(S) \\ G(S, \{1\}) = *. \end{cases}$$

Define \mathcal{E}_n to be the partially ordered set obtained from the product poset $\mathcal{D}_n \times I$ by identifying the subset $\{0\} \times \mathcal{D}_n$ to a single point ϵ . This means that the object set of \mathcal{E}_n is $\{\epsilon\} \cup \mathcal{D}_n \times \{0, 1\} \cup \mathcal{D}_n \times \{1\}$, that $\epsilon \leq (S, \{0, 1\})$ for all $S \in \mathcal{D}_n$, that ϵ is incomparable with any element of the form $(S, \{1\})$, and that the ordering relationships between any elements of $\mathcal{D}_n \times \{\{0, 1\}, \{1\}\}$ are identical to those in the original set $\mathcal{D}_n \times I$. There is a natural projection $\pi : \mathcal{D}_n \times I \rightarrow \mathcal{E}_n$. It is clear from the definition that the functor G factors over π , i.e. that there is a spectrum-valued functor \bar{G} on \mathcal{E}_n so that $G = \bar{G} \circ \pi$. On the other hand, we may also define a functor $\rho : I^{n+1} \rightarrow \mathcal{E}_n$ as follows. For any element $v = \{S_0, S_1, \dots, S_n\}$ of $I^{n+1} - J^{n+1}$, we define $\theta(v) \in \mathcal{D}_n$ to be the subset $\{j \in \underline{n} \mid S_j = \{1\} \text{ or } \{0, 1\}\}$. The functor ρ is now defined by

$$\begin{cases} \rho(\{0\}, \{0\}, \dots, \{0\}) = \epsilon \\ \rho(S_0, S_1, \dots, S_n) = \psi(S_0, S_1, \dots, S_n) \times \{0, 1\} \quad \text{for any } (S_0, S_1, \dots, S_n) \in J^{n+1} - (\emptyset, \emptyset, \dots, \emptyset) \\ \rho(S_0, S_1, \dots, S_n) = \theta(S_0, S_1, \dots, S_n) \times \{1\} \quad \text{for any } (S_0, S_1, \dots, S_n) \in I^{n+1} - J^{n+1}. \end{cases}$$

Behavior on morphisms is determined since all categories in question are partially ordered sets, and one readily checks that the ordering is respected. One now checks directly that $G \circ \rho$ is identical to the functor \bar{F} defined above. It is also direct to check that for any object $x \in \mathcal{E}_n$, the categories $\pi \downarrow x$ and $\rho \downarrow x$ have contractible nerves. The result now follows from [5], Theorem 9.2. \square

Now let A be a commutative S -algebra, B a commutative A -algebra spectrum, and let Ψ denote the I -diagram

$$A \longrightarrow B \longleftarrow *$$

of A -module spectra, i.e $\{0\} \rightarrow A$, $\{0, 1\} \rightarrow B$, and $\{1\} \rightarrow *$. Then we can define an I^{n+1} -diagram Ψ^n by

$$\Psi^n(S_0, S_1, \dots, S_n) = \Psi(S_0) \wedge_A \Psi(S_1) \wedge_A \dots \wedge_A \Psi(S_n).$$

This is a diagram of A -bimodule spectra. We record the following lemma concerning homotopy inverse limits of these I -diagrams.

Lemma 6.5. *Let M denote any A -module. Then*

$$\operatorname{holim}_{\leftarrow I} (\Psi(-) \wedge_A M) \cong (\operatorname{holim}_{\leftarrow I} \Psi) \wedge_A M.$$

Proof. We note that by the definitions of homotopy inverse limits, we find that $\operatorname{holim}_{\leftarrow I} \Psi$ is the homotopy fiber of the spectrum map $A \rightarrow B$. Consequently, the result reduces to the fact that given any map $f : P \rightarrow Q$ of A -module spectra, and any A -module spectrum M , we have an equivalence $\operatorname{Fib}(f) \wedge_A M \cong \operatorname{Fib}(f \wedge_A id_M)$. But this is clear from the results of [10,12]. \square

The following result will be the key in proving our main theorem.

Proposition 6.6. *Let A and Ψ be as above, and let M be an A -module spectrum.. Then we have a natural equivalence*

$$\operatorname{holim}_{\leftarrow I^{n+1}} M \wedge_A \Psi^n \cong M \wedge_A \underbrace{\operatorname{holim}_{\leftarrow I} \Psi \wedge_A \cdots \wedge_A \operatorname{holim}_{\leftarrow I} \Psi}_{n+1 \text{ factors}}.$$

Proof. By induction on n . The result is trivially true for $n = 0$. Suppose that it holds for $n = N$, and we wish to prove it for $n = N + 1$. We have the string of equivalences

$$\begin{aligned} & \operatorname{holim}_{(S_0, \dots, S_{N+1}) \in I^{N+2}} (M \wedge_A \Psi^{N+2}(S_0, \dots, S_{N+1})) \\ & \cong \operatorname{holim}_{S_0 \in I} \left[\operatorname{holim}_{(S_1, \dots, S_{N+1}) \in I^{N+1}} M \wedge_A \Psi(S_0) \wedge_A \Psi^{N+1}(S_1, \dots, S_{N+2}) \right] \\ & \cong \operatorname{holim}_{S_0 \in I} M \wedge_A \Psi(S_0) \wedge_A \underbrace{\left[\operatorname{holim}_{\leftarrow I} \Psi \wedge_A \cdots \wedge_A \operatorname{holim}_{\leftarrow I} \Psi \right]}_{N+1 \text{ factors}} \\ & \cong M \wedge_A \left[\operatorname{holim}_{\leftarrow I} \Psi \right] \wedge_A \underbrace{\left[\operatorname{holim}_{\leftarrow I} \Psi \wedge_A \cdots \wedge_A \operatorname{holim}_{\leftarrow I} \Psi \right]}_{N+1 \text{ factors}} \\ & \cong M \wedge_A \underbrace{\left[\operatorname{holim}_{\leftarrow I} \Psi \wedge_A \cdots \wedge_A \operatorname{holim}_{\leftarrow I} \Psi \right]}_{N+2 \text{ factors}}. \end{aligned}$$

The first equivalence is a “Fubini” type theorem for homotopy inverse limits, the second is an application of the inductive hypothesis, and the third is an application of 6.5. \square

Corollary 6.7. *Let $f : A \rightarrow B$ be a homomorphism of commutative S -algebras, and let M be a A -module spectrum. Let \mathcal{I} denote the homotopy fiber of the homomorphism f . \mathcal{I} is an A -module spectrum. Then there is a homotopy*

fibration sequence of A -module spectra

$$\mathcal{I}_A^{\wedge(k+1)} \wedge_A M \rightarrow M \rightarrow \text{Tot}^k T_A(M; B).$$

Moreover, these sequences fit together in homotopy commutative diagrams

$$\begin{array}{ccccc} \mathcal{I}_A^{\wedge(k+2)} \wedge_A M & \longrightarrow & M & \longrightarrow & \text{Tot}^{k+1}(T_A(M; B)) \\ \downarrow & & \downarrow \text{id}_M & & \downarrow \\ \mathcal{I}_A^{\wedge(k+1)} \wedge_A M & \longrightarrow & M & \longrightarrow & \text{Tot}^k(T_A(M; B)) \end{array} \tag{6.1}$$

where the right hand vertical map is the usual projection, and the left hand vertical map is the composite

$$\mathcal{I}_A^{\wedge(k+2)} \cong \mathcal{I}_A \wedge_A \mathcal{I}_A^{\wedge(k+1)} \xrightarrow{j \wedge_A \text{id}} A \wedge_A \mathcal{I}_A^{\wedge(k+1)} \cong \mathcal{I}_A^{\wedge(k+1)} \tag{6.2}$$

$j : \mathcal{I} \rightarrow A$ being the evident inclusion.

This result implies the following technical corollary.

Corollary 6.8. *Suppose that we have a homomorphism $f : A \rightarrow B$ of (-1) -connected commutative S -algebras, and that $\pi_0(f)$ is surjective. Suppose that M is a k -connected A -module spectrum. Then each of the spectra $\text{Tot}^n(\mathcal{T}_A(M; B))$ is k -connected, and $M_B = \text{Tot}(\mathcal{T}_A(M; B))$ is at least $k - 1$ -connected.*

Proof. We prove the first part by induction on n . The result holds for $n = 0$ since $\text{Tot}^0(\mathcal{T}_A(M; B)) \cong B \wedge_A M$. The Künneth spectral sequences for smash products show that the connectivity $B \wedge_A M$ is at least k . For higher values of n , we suppose the result known for the value $n - 1$. By the commutative diagram (6.1) in Corollary 6.7 above, we have an identification of the homotopy fiber of the map $\text{Tot}^n(\mathcal{T}_A(M; B)) \rightarrow \text{Tot}^{n-1}(\mathcal{T}_A(M; B))$ with the homotopy cofiber of the map $I_A^{\wedge(n+1)} \rightarrow I_A^{\wedge n}$ in diagram (6.2) above. Since both $I_A^{\wedge(n+1)}$ and $I_A^{\wedge n}$ are (-1) -connected, so is the homotopy fiber of the map $\text{Tot}^n(\mathcal{T}_A(M; B)) \rightarrow \text{Tot}^{n-1}(\mathcal{T}_A(M; B))$. It now readily follows that $\text{Tot}^n(\mathcal{T}_A(M; B))$ is (-1) -connected. The result for $\text{Tot}(\mathcal{T}_A(M; B))$ now follows from the \lim^1 -exact sequence for homotopy fibrations (see [5], Ch X, Section 3). \square

We will also need the following technical lemma.

Lemma 6.9. *Let $A \rightarrow B \xrightarrow{f} C$ be a diagram of commutative S -algebras, and let M denote an A -module spectrum. Suppose further that there is a homomorphism of commutative A -algebra spectra $s : C \rightarrow B$ such that $s \circ f = \text{id}_B$. Then the natural map $M_B^\wedge \rightarrow M_C^\wedge$ is an equivalence of spectra.*

Proof. Follows directly from Proposition 2.15, applied to the triples $S = B \wedge_A -$ and $C \wedge_A -$. \square

Proof of Theorem 6.1. We consider first the case when the diagram of S -algebra homomorphisms is $A \rightarrow A \rightarrow C$, i.e. $A = B$, and where the homotopy fiber \mathcal{I} of the map $f : A = B \rightarrow C$ is 0-connected, i.e. when $\pi_0(f)$ is an isomorphism and $\pi_1(f)$ is surjective. It follows from Propositions 6.4 and 6.6 that the homotopy fiber \mathcal{F} of the map $M \rightarrow \text{holim}_{\Delta^n} \mathcal{T}_A(M; B) \cong \text{Tot}^n \mathcal{T}_A(M; B)$ is equivalent to $\underbrace{\mathcal{I} \wedge_A \cdots \wedge_A \mathcal{I}}_{n+1 \text{ factors}}$. Since \mathcal{I} is 0-connected, it follows by an easy

application of Corollary 2.5 that \mathcal{F} is n -connected. This gives the result in this case, since when one passes to the homotopy limit over n the connectivity of the map goes to infinity, and since $M_A^\wedge \cong M$.

We next consider the case where A is not necessarily equal to B , but where we have that the homotopy fiber \mathcal{I} of the map $f : B \rightarrow C$ is 0-connected. We construct the bicosimplicial spectrum \mathcal{C}^\cdot given by $\mathcal{C}^{pq} = T_A^p(T_A^q(M; B); C)$,

with the obvious coface and codegeneracy maps. Lemma 2.14 asserts that the natural map

$$T_A(M; C) \rightarrow C^{\cdot}$$

(induced by $\eta : M \rightarrow T_A(M; B)$) induces an equivalence on total spectra, where $T_A(M; C)$ is regarded as a bicosimplicial spectrum constant in the q -direction. We also have the natural map $\eta : T_A(M; B) \rightarrow C^{\cdot}$ obtained by regarding $T_A(M; B)$ as a bicosimplicial spectrum constant in the p -direction. Since $\text{Tot}(C^{\cdot}) \cong M_C^{\wedge}$, and this equivalence is compatible with the standard map $M_B^{\wedge} \rightarrow M_C^{\wedge}$, it suffices to show that η induces a weak equivalence on total spectra. For this it suffices to show that each of the maps

$$T_A^q(M; B) \rightarrow \text{Tot}(T_A(T_A^q(M; B)); C)$$

is a weak equivalence of spectra. Since each of the A -module spectra

$$T_A^q(M; B) = \underbrace{B \wedge_A \cdots \wedge_A B \wedge_A M}_{q+1 \text{ factors}}$$

is equipped with a B -module structure restricting to its given A -module structure, it will now suffice to prove that the natural map $M_B^{\wedge} \rightarrow M_C^{\wedge}$ is an equivalence for any B -module spectrum M . For a B -module spectrum M , we may construct the cosimplicial spectrum $T_B(M; C)$. From the hypothesis on f , and by the previous case, we find that the connectivity of the natural maps $M \rightarrow \text{Tot}^n(T_B(M; C))$ goes to infinity with n . We will now apply the functors $T_A(-; B)$ and $T_A(-; C)$ levelwise to the cosimplicial spectrum $T_B(M; C)$ to obtain bicosimplicial spectra, with a natural bicosimplicial map from the first to the second induced by the A -algebra spectrum map $B \rightarrow C$. We obtain a commutative diagram

$$\begin{array}{ccc} M_B^{\wedge} = \text{Tot}_{p \in \Delta} T_A^p(M; B) & \longrightarrow & M_C^{\wedge} = \text{Tot}_{p \in \Delta} T_A^p(M; C) \\ \downarrow & & \downarrow \\ \text{Tot}_{p \in \Delta} \text{Tot}_{q \in \Delta} T_A^p(T_B^q(M; C); B) & \longrightarrow & \text{Tot}_{p \in \Delta} \text{Tot}_{q \in \Delta} T_A^p(T_B^q(M; C); C). \end{array} \tag{6.3}$$

We wish to prove that the upper horizontal arrow is an equivalence of spectra. We will do this by showing that the two vertical arrows and the lower horizontal arrow are equivalences of spectra. We first note that by the increasing connectivity (with n) of the maps $M \rightarrow \text{Tot}^n T_B^q(M; C)$ and Corollary 6.8, we find that the natural maps

$$\text{Tot}_{p \in \Delta} T_A^p(M; B) \rightarrow \text{holim}_{\leftarrow_n} \text{Tot}_{p \in \Delta} T_A^p(\text{Tot}^n T_B^q(M; C); B)$$

and

$$\text{Tot}_{p \in \Delta} T_A^p(M; C) \rightarrow \text{holim}_{\leftarrow_n} \text{Tot}_{p \in \Delta} T_A^p(\text{Tot}^n T_B^q(M; C); C).$$

are equivalences of spectra. By Proposition 2.9, we have that

$$T_A^p(\text{Tot}^n T_B^q(M; C); B) \cong \text{Tot}_{q \in \Delta}^n T_A^p(T_B^q(M; C); B)$$

and

$$T_A^p(\text{Tot}^n T_B^q(M; C); C) \cong \text{Tot}_{q \in \Delta}^n T_A^p(T_B^q(M; C); C).$$

Therefore, the maps

$$\text{Tot}_{p \in \Delta} T_A^p(M; B) \rightarrow \text{holim}_{\leftarrow_n} \text{Tot}_{p \in \Delta} \text{Tot}_{q \in \Delta}^n T_A^p(T_B^q(M; C); B) \cong \text{Tot}_{p \in \Delta} \text{Tot}_{q \in \Delta} T_A^p(T_B^q(M; C); B)$$

and

$$\text{Tot}_{p \in \Delta} T_A^p(M; C) \rightarrow \text{holim}_{\leftarrow n} \text{Tot}_{p \in \Delta} \text{Tot}_{q \in \Delta}^n T_A^p(T_B^q(M; C); C) \cong \text{Tot}_{p \in \Delta} \text{Tot}_{q \in \Delta} T_A^p(T_B^q(M; C); C)$$

are also weak equivalences of spectra. These are the vertical arrows in Diagram (6.3). We must also check that the lower horizontal arrow in Diagram (6.3) is an equivalence of spectra. This arrow is of course induced by a map of bicosimplicial spaces, so we may verify it by proving that it is a levelwise equivalence. Specifically, if we can show that for each q , the map

$$\text{Tot}_{p \in \Delta} T_A^p(T_B^q(M; C); B) \rightarrow \text{Tot}_{p \in \Delta} T_A^p(T_B^q(M; C); C)$$

is an equivalence of spectra, then the lower horizontal map in Diagram (6.3) will be an equivalence. But, $T_B^q(M; C)$ admits a C -module structure (and therefore also a B -module structure), so by Proposition 3.2,(6), we have

$$\text{Tot}_{p \in \Delta} T_A^p(T_B^q(M; C); B) \cong \text{Tot}_{p \in \Delta} T_A^p(T_B^q(M; C); C)$$

which gives the result, i.e. that the statement of the theorem holds in the case when \mathcal{I} is 0-connected.

The general case, i.e. without the assumption that $\pi_1 B \rightarrow \pi_1 C$ is surjective, can now be handled as follows. Let $\{f_\alpha\}_{\alpha \in a}$ be a generating set for $\pi_1 C$. Let $Z = \bigvee_{\alpha \in A} S_\alpha^1$ denote the suspension spectrum of a bouquet of circles parametrized by A , and let $\theta : Z \rightarrow C$ denote the map whose restriction to S_α^1 is the map f_α . Let $B(Z)$ denote the free B -module on the spectrum Z . Then we obtain a map of commutative A -algebra spectra $g : \text{Sym}_B(B(Z)) \rightarrow C$. One readily checks that $\pi_0(g)$ is an isomorphism and that $\pi_1(g)$ is surjective, so we may conclude that $M_{\text{Sym}_B(B(Z))} \rightarrow M_C$ is an equivalence of spectra. On the other hand, the inclusion $B \rightarrow \text{Sym}_B(B(Z))$ admits a section, and so the natural map $M_B \rightarrow M_{\text{Sym}_B(B(Z))}$ is also an equivalence by 6.9. This concludes the proof of Theorem 6.1. \square

We have been discussing the invariance of completion along an S -algebra homomorphism $f : A \rightarrow B$ under changes in the target S -algebra B . We will also need a similar result which will demonstrate invariance under changes in A under suitable circumstances. For a homomorphism of commutative S -algebras $f : A \rightarrow B$, and a B -module spectrum M , we will denote by $\rho^A M$ the result of regarding M as an A -module spectrum via the S -algebra homomorphism f .

Theorem 6.10. *Let $A \xrightarrow{f} B \rightarrow C$ be a diagram of (-1) -connected commutative S -algebras, and let M be a B -module spectrum. Suppose that the homomorphism $\pi_0 f : \pi_0 A \rightarrow \pi_0 B$ is an isomorphism. Then the natural homomorphism $(\rho^A M)_C^\wedge \rightarrow M_C^\wedge$ is a weak equivalence of spectra.*

Proof. Let T_A and T_B denote the triples $M \mapsto C \wedge_A M$ and $M \mapsto C \wedge_B M$, respectively. We consider the bicosimplicial A -module spectrum

$$(p, q) \mapsto T_B^p T_A^q(M)$$

as usual, and we have by Lemma 2.14 that its total space is equivalent to M_C^\wedge . In order to check the equivalence we are interested in, it will be sufficient to prove that for each q , the total spectrum of the cosimplicial spectrum

$$p \mapsto T_B^p T_A^q(M)$$

is weakly equivalent to $T_A^q(M)$ under the natural map $\eta_B(T_A^q(M))$. As in the proof of Theorem 6.1, we let \mathcal{I} denote the homotopy fiber of the homomorphism $B \rightarrow C$. \mathcal{I} is a B -module. Then Corollary 6.7 shows that it will suffice to prove that the inverse system of spectra

$$\dots \rightarrow \mathcal{I}^{\wedge^{k+1}} \wedge_B T_A^p(M) \rightarrow \mathcal{I}^{\wedge^k} \wedge_B T_A^p(M) \rightarrow \mathcal{I}^{\wedge^{k-1}} \wedge_B T_A^p(M) \rightarrow \dots$$

has contractible homotopy inverse limit. It is clear that to verify this, it will suffice to show that for any B -module spectrum N , the inverse system of spectra

$$\dots \rightarrow \mathcal{I}^{\wedge^{k+1}} \wedge_B C \wedge_A N \rightarrow \mathcal{I}^{\wedge^k} \wedge_B C \wedge_A N \rightarrow \mathcal{I}^{\wedge^{k-1}} \wedge_B C \wedge_A N \rightarrow \dots$$

has contractible homotopy inverse limit. In order to prove this, we observe that there is a standard equivalence

$$C \wedge_A N \cong C \wedge_B B \wedge_A B \wedge_B M$$

of B -module spectra. The idea will be to replace $B \wedge_A B$ by its cosimplicial approximation $\mathcal{T}_{B \wedge_A B}^{\cdot}(B \wedge_A B; B)$, where B is a commutative $B \wedge B$ -algebra via the multiplication map $B \wedge B \rightarrow B$. Since we are assuming that $\pi_0 f$ is an isomorphism, it follows that the map $\pi_0 B \wedge_A B \rightarrow \pi_0 B$ is an isomorphism, and therefore by Theorem 6.1 that the natural map η gives an equivalence $B \wedge_A B \rightarrow (B \wedge_A B)_{\hat{B}}$. Also, $\pi_1 B \wedge_A B \rightarrow \pi_1 B$ is surjective since the multiplication map has a section. It follows that the natural map $B \wedge_A B \rightarrow \text{Tot}^i(\mathcal{T}_{B \wedge_A B}^{\cdot}(B \wedge_A B; B))$ is at least $(i - 1)$ -connected. Letting $T^i = \text{Tot}^i(\mathcal{T}_{B \wedge_A B}^{\cdot}(B \wedge_A B; B))$, it now follows from the Künneth spectral sequence that the natural map $C \wedge_A N \rightarrow C \wedge_B T^i \wedge_B N$ is $(i - 1)$ -connected. Consequently, we have an equivalence

$$C \wedge_A N \cong \text{Tot}(k \mapsto C \wedge_B \mathcal{T}_{B \wedge_A B}^k(B \wedge_A B; B) \wedge_B N).$$

Therefore, it will now suffice to prove that the homotopy inverse limit of the system

$$\begin{aligned} \dots &\rightarrow \mathcal{I}_B^{\wedge^{k+1}} \wedge_B C \wedge_B \mathcal{T}_{B \wedge_A B}^k(B \wedge_A B; B) \wedge_B N \rightarrow \mathcal{I}_B^{\wedge^k} \wedge_B C \wedge_B \mathcal{T}_{B \wedge_A B}^k(B \wedge_A B; B) \wedge_B N \rightarrow \\ &\rightarrow \mathcal{I}_B^{\wedge^{k-1}} \wedge_B C \wedge_B \mathcal{T}_{B \wedge_A B}^k(B \wedge_A B; B) \wedge_B N \rightarrow \dots \end{aligned}$$

is contractible. Now, $\mathcal{T}_{B \wedge_A B}^k(B \wedge_A B; B)$ is a $(k + 1)$ -fold smash product over $B \wedge_A B$ of copies of B . By including B into the first tensor factor, we find that it can be viewed as a B -algebra, for which we write Λ . Λ can be viewed as a B - B -bimodule, with both right and left actions coming from the (left) action of B on Λ . We can now resolve Λ by free B -modules (each of which is regarded as a bimodule via the left action, and which can be taken to be free on a wedge of spheres of the same dimension), and it is now easy to see that it will suffice to show that the inverse system

$$\dots \rightarrow \mathcal{I}_B^{\wedge^{k+1}} \wedge_B C \wedge_B F \wedge_B N \rightarrow \mathcal{I}_B^{\wedge^k} \wedge_B C \wedge_B F \wedge_B N \rightarrow \mathcal{I}_B^{\wedge^{k-1}} \wedge_B C \wedge_B F \wedge_B N \rightarrow \dots$$

has contractible homotopy inverse limit, where F is any free B -module on a bouquet of spheres of a fixed dimension k , regarded as a B - B -bimodule, with both actions agreeing with the left action on F . But to check this, it will suffice to show that the map

$$\mathcal{I}_B^{\wedge^k} \wedge_B C \wedge_B F \wedge_B N \rightarrow \mathcal{I}_B^{\wedge^{k-1}} \wedge_B C \wedge_B F \wedge_B N$$

is null homotopic as a map of spectra. But this question clearly can be reduced to the case where $F = B$, so we will have to show that the map

$$\mathcal{I}_B^{\wedge^k} \wedge_B C \wedge_B N \rightarrow \mathcal{I}_B^{\wedge^{k-1}} \wedge_B C \wedge_B N$$

is null homotopic. This can clearly be reduced to the case where $k = 1$, so we need to check that the map

$$\mathcal{I}_B \wedge_B C \wedge_B N \rightarrow B \wedge_B C \wedge_B N \cong C \wedge_B N$$

is null. But we have the cofiber sequence

$$\mathcal{I}_B \wedge_B C \wedge_B N \rightarrow C \wedge_B N \rightarrow C \wedge_B C \wedge_B N$$

and the second map is the inclusion of a wedge summand, since we have the retraction

$$\mu \wedge id_N : C \wedge_B C \wedge_B N \rightarrow C \wedge_B N$$

where $\mu : C \wedge_B C \rightarrow C$ is the multiplication map. This gives the result. \square

7. Algebraic to geometric spectral sequence

In this section, we will provide a computational device which will permit the computation of homotopy groups of derived completions of module spectra over commutative S -algebras in terms of derived completions over actual rings. Here is the statement of the main theorem of this section.

Theorem 7.1. *Let $A \rightarrow B$ be a map of commutative (-1) -connected S -algebras, with $\pi_0 A \rightarrow \pi_0 B$ surjective, and let M be a connective A -module spectrum. $\pi_0 A$ is a commutative ring, $\pi_0 B$ is a commutative $\pi_0 A$ -algebra, and for each i , $\pi_i M$ is a $\pi_0 A$ -module. We may therefore construct the derived completion spectrum $(\pi_i M)_{\pi_0 B}$ for each i . There is a second quadrant spectral sequence with $E_1^{pq} = \pi_{q+2p}((\pi_{-p} M)_{\pi_0 B}^\wedge)$, converging to $\pi_{p+q}(M_B)$.*

We will require some preliminary technical work on Postnikov decompositions of S -algebras and module spectra before we can present the proof of this result. We recall from [17] and [18] that the category Alg_S admits the structure of a Quillen model category, in which the weak equivalences are the homomorphisms of S -algebras inducing isomorphisms on homotopy groups. There is also a “free commutative S -algebra” on a spectrum X , which we denote by $\text{Sym}(X)$. This construction satisfies the adjointness relationship

$$\text{Hom}_{\text{Alg}_S}(\text{Sym}(X), A) \cong \text{Hom}_{\text{Mod}_S}(X, A).$$

Sym is a triple on the category of spectra, and commutative S -algebras are exactly the algebras over the triple Sym . For any commutative S -algebra A , we may as in Definition 2.16 construct its simplicial resolution $\text{Sym}(A)$, relative to the triple Sym . Moreover, Proposition 2.17 shows that the natural map $\text{Sym}(A) \rightarrow A$ is a weak equivalence. In [18], it is shown that $\text{Sym}(A)$ is a cofibrant object in the model structure on the category Alg_S .

Given two commutative S -algebras A and B , the set $\text{Hom}_{\text{Alg}_S}(A, B)$ is equipped with the structure of a space, as a subspace of the zeroth space of the spectrum $\text{Hom}_{\text{Mod}_S}(A, B)$. This construction is not homotopy invariant, but the result of replacing A by any weakly equivalent cofibrant object does yield a homotopy invariant notion. Of course, we may take the cofibrant replacement to be $\text{Sym}(A)$. We now need a result about the space of S -algebra maps to Eilenberg–MacLane spectra.

Proposition 7.2. *Let A be a (-1) -connected commutative S -algebra. Let B denote any commutative ring, and $\mathbb{H}(B)$ the corresponding Eilenberg–MacLane spectrum. The canonical map from the space $\text{Hom}_{\text{Alg}_S}(\text{Sym}(A), \mathbb{H}(B)) \rightarrow \text{Hom}(\pi_0(A), B)$ is an equivalence, i.e. $\text{Hom}_{\text{Alg}_S}(\text{Sym}(A), \mathbb{H}(B))$ is a space whose components are in bijective correspondence with the ring homomorphisms from $\pi_0(A)$ to B , and so that each component is contractible.*

Proof. The adjunction $\text{Hom}_{\text{Alg}_S}(\text{Sym}(X), A) \cong \text{Hom}_{\text{Mod}_S}(X, A)$ shows that the space

$$\text{Hom}_{\text{Alg}_S}(\text{Sym}(A), \mathbb{H}(B))$$

is equivalent to the total space of the cosimplicial space which in level k is the space of spectrum maps from $\text{Sym}^k(A)$ to $\mathbb{H}(B)$. Since the $k + 1$ -skeleton $Sk^{(k+1)}\text{Sym}(A)$ is obtained from the k -skeleton by attaching cells in dimensions $k + 1$ and higher, it follows that the restriction map

$$\text{Hom}_{\text{Alg}_S}(\text{Sym}(A), \mathbb{H}(B)) \rightarrow \text{Hom}_{\text{Alg}_S}(Sk^1\text{Sym}(A), \mathbb{H}(B)) \stackrel{\text{def } n}{=} T^1$$

is an equivalence. Now, T^1 is clearly the homotopy equalizer of the two maps

$$\varphi, \psi : \text{Hom}_{\text{Mod}_S}(A, \mathbb{H}(B)) \rightarrow \text{Hom}_{\text{Mod}_S}(\text{Sym}(A), \mathbb{H}(B))$$

defined by

$$\begin{cases} \varphi(f) = f \circ \alpha_A \text{ where } \alpha_A : \text{Sym}(A) \rightarrow A \text{ is the structure map for } A \\ \psi(f) = \alpha_B \circ \text{Sym}(f) \text{ where } \alpha_{\mathbb{H}(B)} : \text{Sym}(\mathbb{H}(B)) \rightarrow \mathbb{H}(B) \text{ is the structure map for } \mathbb{H}(B). \end{cases}$$

The space $\text{Hom}_{\text{Mod}_S}(A, \mathbb{H}(B))$ is clearly equivalent to the discrete space $\text{Hom}_{Ab}(\pi_0(A), B)$. Similarly, $\text{Hom}_{\text{Mod}_S}(\text{Sym}(A), \mathbb{H}(B))$ is equivalent to the discrete space

$$\text{Hom}_{Ab}(\pi_0(\text{Sym}(A)), B) = \text{Hom}_{Ab}(\text{Sym}(\pi_0(A)), B).$$

The homotopy equalizer is now equivalent to the equalizer of the pair of set maps

$$\Phi, \Psi : \text{Hom}_{Ab}(\pi_0(A), B) \rightarrow \text{Hom}_{Ab}(\text{Sym}(\pi_0(A)), B)$$

defined by

$$\begin{cases} \Phi(f) = f \circ \alpha_{\pi_0 A} \text{ where } \alpha_{\pi_0 A} : \text{Sym}(\pi_0 A) \rightarrow \pi_0 A \text{ is the structure map for the ring } \pi_0 A. \\ \Psi(f) = \alpha_B \circ \text{Sym}(f) \text{ where } \alpha_B : \text{Sym}(B) \rightarrow B \text{ is the structure map for the ring } B. \end{cases}$$

This set is the set of all abelian group homomorphisms $f : \pi_0 A \rightarrow B$ making the diagram

$$\begin{array}{ccc} \text{Sym}(\pi_0 A) & \xrightarrow{\text{Sym}(f)} & \text{Sym}(B) \\ \alpha_{\pi_0 A} \downarrow & & \downarrow \alpha_B \\ \pi_0 A & \xrightarrow{f} & B \end{array}$$

commute. This is clearly the set of ring homomorphisms from $\pi_0 A$ to B . \square

Corollary 7.3. *There is a canonical homotopy equivalence class of homomorphisms of commutative S -algebras $\pi_A : A \rightarrow \mathbb{H}(\pi_0 A)$ which induces the identity on π_0 .*

We also have the following results on A -module spectra. They follow immediately from Proposition 3.9 of [8].

Proposition 7.4. *Let A be a (-1) -connected commutative S -algebra, and let M be a cofibrant $(k - 1)$ -connected A -module spectrum. Let N be any module over the ring $\pi_0 A$, and $\mathbb{H}(N, k)$ the k -dimensional Eilenberg–MacLane spectrum for N , regarded as an A -module spectrum via the homomorphism $A \rightarrow \pi_0 A$. The zeroth space of the function spectrum $\text{Hom}_A(M, \mathbb{H}(N, k))$ is homotopy equivalent to the set $\text{Hom}_{\pi_0 A}(\pi_k M, N)$ via the natural map which assigns to any map its induced map on π_k .*

Corollary 7.5. *Let A be a (-1) -connected commutative S -algebra. Let M be a module spectrum over A , and suppose that the underlying spectrum of M is an Eilenberg–MacLane spectrum, say of dimension n . $\pi_n M$ is a module over the ring $\pi_0 A$. We let Π_M denote the A -module spectrum obtained from $\pi_n M$ by pullback along the ring homomorphism π_A . Then, M is equivalent as an A -module spectrum to Π_M .*

These results allow us to construct a Postnikov tower in the category of connective modules over A .

Corollary 7.6. *Let A be a (-1) -connected commutative S -algebra, and let M be a k -connected A -module spectrum, for some integer k . Then there is a tower of fibrations of A -module spectra*

$$\dots M[s] \rightarrow M[s] \rightarrow \dots \rightarrow M[k + 1] \rightarrow M[k]$$

together with A -module spectrum maps $M \xrightarrow{f_s} M[s]$ making all the triangles

$$\begin{array}{ccc} M & \xrightarrow{f_s} & M[s] \\ & \searrow f_{s-1} & \downarrow \\ & & M[s - 1] \end{array}$$

commute, with the following properties.

- f_s induces an isomorphism on π_i for $i \leq s$.
- $\pi_i M[s] = 0$ for $i > s$.
- The homotopy fiber of the map $M[s] \rightarrow M[s - 1]$ is an Eilenberg–MacLane spectrum in dimension s .
- Using the maps $\{f_s\}_s$, $\text{holim}_s M[s]$ is naturally equivalent to M

Moreover, the tower is natural in the homotopy category.

Proof. Immediate from Corollary 7.5 above. \square

We will use the Postnikov tower to construct the spectral sequence. A preliminary observation is the following.

Proposition 7.7. *Let*

$$\cdots \rightarrow M_s \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$$

be any diagram of module spectra over a commutative S -algebra A , and let P denote any cofibrant A -module spectrum. Suppose further that the connectivity of the M_s goes to infinity with s , and that P is connective, i.e. is k -connected for some k . Then the natural map

$$P \wedge_A (\text{holim}_s M_s) \longrightarrow \text{holim}_s (P \wedge_A M_s)$$

is an equivalence of spectra.

Proof. Follows directly from the fact that $P \wedge_A$ preserves connectivity when P is cofibrant. \square

Corollary 7.8. *Suppose that A is a (-1) -connected commutative S -algebra, and that we have a homomorphism $f : A \rightarrow B$ of commutative S -algebras. Suppose further that B is (-1) -connected and cofibrant. For any diagram*

$$\cdots \rightarrow M_s \rightarrow \cdots \rightarrow M_1 \rightarrow M_0$$

of connective A -module spectra, for which the connectivity of M_s goes to infinity with s , the natural map

$$(\text{holim}_s M_s)_B^\wedge \rightarrow \text{holim}_s (M_s)_B^\wedge$$

is a weak equivalence of spectra. In particular, if $\{M[s]\}_s$ is the Postnikov tower for an A -module spectrum M , then $M_B^\wedge \cong \text{holim}_s M[s]_B^\wedge$. Further, the homotopy fiber of the map $M[s]_B^\wedge \rightarrow M[s-1]_B^\wedge$ is equivalent to $(F_s)_B^\wedge$, where F_s is the homotopy fiber of the map $M[s] \rightarrow M[s-1]$.

The spectral sequence in question is simply the spectral sequence on homotopy groups attached to an inverse system of spectra. This is a standard result which is discussed in [5], Chapter IX.

Proposition 7.9. *Let $\cdots X_s \rightarrow X_{s-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0$ be an inverse system of connective spectra, and let F_s denote the homotopy fiber of the map $X_s \rightarrow X_{s-1}$. Suppose further that the connectivity of the spaces goes to infinity with s . Then there exists a left half plane spectral sequence with $E_1^{p,q}$ -term $\pi_{p+q} F_{-p}$, converging to $\pi_{p+q} (\text{holim}_s X_s)$.*

Proof of Theorem 7.1. We apply Proposition 7.9 and Corollary 7.8 to the inverse system

$$\cdots \rightarrow M[s]_B^\wedge \rightarrow M[s-1]_B^\wedge \rightarrow \cdots \rightarrow M[k+1]_B^\wedge \rightarrow M[k]_B^\wedge$$

where k is the connectivity of M . The homotopy fiber of the map $M[s]_B^\wedge \rightarrow M[s-1]_B^\wedge$ is $(F_s)_B^\wedge$, where as above F_s denotes the homotopy fiber of the map $M[s] \rightarrow M[s-1]$. Since $\{M[s]\}_s$ is the Postnikov tower for M , F_s is an s -dimensional Eilenberg–MacLane spectrum with $\pi_s(F_s) \cong \pi_s(M)$. Consequently, by Corollary 7.5, it is equivalent to a module over the ring $\pi_0(A)$, and by Theorem 6.10, it follows that the derived completion of F_s at the S -algebra homomorphism $A \rightarrow B$ is equivalent to its derived completion at the ring homomorphism $\pi_0 A \rightarrow \pi_0 B$. The theorem now follows. \square

We can now use this spectral sequence to obtain two useful corollaries. We first have an extension of Proposition 3.2, item 6.

Corollary 7.10. *Let $B \rightarrow C$ be a homomorphism of (-1) -connected commutative S -algebras, and suppose $\pi_0 B \rightarrow \pi_0 C$ is surjective. Let M be a B -module spectrum, and suppose that for each i , the $\pi_0 B$ -module structure on $\pi_1 M$ extends to a $\pi_0 C$ -action. Then the natural map $\eta : M \rightarrow M_C^\wedge$ is an equivalence.*

Proof. Follows directly from Proposition 3.2, item 6, and Theorem 7.1. \square

We also have the following corollary, which will be useful in the K -theoretic applications.

Corollary 7.11. *Let*

$$A \longrightarrow B \longrightarrow C$$

be a diagram of (-1) -connected commutative S -algebras, and suppose that $\pi_0 A \rightarrow \pi_0 B$ and $\pi_0 B \rightarrow \pi_0 C$ are surjective. For any B -module M , let $\rho_A(M)$ denote M regarded as an A -module by restriction of scalars. Then the natural map $(\rho_A(M))_C^\wedge \rightarrow M_C^\wedge$ is a weak equivalence.

Proof. It follows from [Theorem 7.1](#) that it suffices to consider the case where A , B , and C are all obtained by applying the Eilenberg–MacLane construction to diagram of actual commutative rings, and that M is obtained as an actual module over the ring B . Exactly as in the proof of [Theorem 6.10](#), it suffices to prove that the natural map

$$C \underset{A}{\wedge} M \longrightarrow (C \underset{A}{\wedge} M)_C^\wedge$$

is an equivalence. Here the completion is taking place along the homomorphism $B \rightarrow C$, and the B -module structure on $C \underset{A}{\wedge} M$ is obtained from the original B -action on M , not via restriction of scalars from the extended left C -action.

That is, we have $b \cdot (c \wedge m) = c \wedge b \cdot m$. Now, the homotopy groups of $C \underset{A}{\wedge} M$ are the groups $\text{Tor}_*^A(M, C)$, via the (collapsing) Künneth spectral sequence. We must only show that the B -action on these Tor-groups obtained from the given B -action on M extends over C . But from the surjectivity of the homomorphism $A \rightarrow B$, the B -action can be computed as the action of A/J , where J is the kernel of the ring homomorphisms $A \rightarrow B$, and hence is determined by the A -action. It is clear from the definition that the A -action extends over C , and hence that it vanishes on the I , the kernel of the ring homomorphism $A \rightarrow C$. But, $A/I \cong B/I'$, where I' is the kernel of $B \rightarrow C$, which gives the result. \square

8. Examples

In this section, we will discuss a number of examples of this construction.

Example 8.1. We consider the case where the S -algebras A and B are the Eilenberg–MacLane spectra for the rings \mathbb{Z} and \mathbb{F}_p , and where the homomorphism $f : A \rightarrow B$ is induced by reduction mod p . For a finitely generated A -module M , [Theorem 4.4](#) tells us that the derived completion of M along f is simply the Eilenberg–MacLane spectrum for the completed module M_p^\wedge .

Example 8.2. Let $A = S^0$, the sphere spectrum, and as in the previous example let B be the Eilenberg–MacLane spectrum for the ring \mathbb{F}_p . Let f be the composition of the natural map the the Eilenberg–MacLane spectrum for \mathbb{Z} with mod- p reduction. Any spectrum is in a natural way a module over the commutative S -algebra S^0 , and its completion along this map is the usual p -adic completion. The tower of fibrations mentioned in [Remark 3.1](#) is in this case identical to the tower in the Adams spectral sequence.

Example 8.3. Let A be the S -algebra ku , i.e. connective complex K -theory, and let B be the Eilenberg–MacLane spectrum for the ring \mathbb{Z} . We view ku as the spectrum associated with the topological symmetric monoidal category of finite dimensional complex vector spaces, and we define f to be the homomorphism of S -algebras induced by the functor which sends every complex vector space to its dimension. Since $\pi_0 f$ is an isomorphism, [Theorem 7.1](#) shows that in this case, ku_B^\wedge is equivalent to ku itself. The tower of fibrations mentioned in [Remark 3.1](#) is in this case the *accelerated Postnikov tower* for ku , with the i th spectrum in the tower of fibrations equivalent to the Postnikov cover $ku[2i, \dots, +\infty)$

Example 8.4. Let A denote the Eilenberg–MacLane spectrum for the representation ring $R[\mathbb{Z}_p]$, where \mathbb{Z}_p denotes the group of p -adic integers. For a profinite group G , the representation ring $R[G]$ is defined to be the direct limit of $R[G/N]$, where N ranges over all the finite index normal subgroups of the group G . As a ring, one readily sees that $R[\mathbb{Z}_p] \cong \mathbb{Z}[\mathbb{Z}/p^\infty\mathbb{Z}]$, where $\mathbb{Z}[-]$ denotes integral group ring, and where $\mathbb{Z}/p^\infty\mathbb{Z}$ denotes the union $\bigcup_n \mathbb{Z}/p^n\mathbb{Z}$. We let $C = \mathbb{F}_p$, and define $f : A \rightarrow C$ to be the composite $R[\mathbb{Z}_p] \rightarrow R[\{e\}] = \mathbb{Z} \rightarrow \mathbb{F}_p$. We will let the module M be the ring A itself. We let S^1 denote the simplicial group of singular chains on the circle group, with multiplication and inverse induced by those on the topological group. There is of course a natural homomorphism of simplicial

groups $\mathbb{Z}/p^\infty\mathbb{Z} \rightarrow S^1$, induced by the identification of $\mathbb{Z}/p^\infty\mathbb{Z}$ with the group of p -power torsion points on S^1 , and where $\mathbb{Z}/p^\infty\mathbb{Z}$ is regarded as a discrete simplicial group. We define B to be the simplicial group ring $\mathbb{Z}[S^1]$, which is a simplicial algebra $\mathbb{Z}[\mathbb{Z}/p^\infty\mathbb{Z}]$. Such a simplicial algebra may be regarded as a commutative S -algebra, in fact a commutative algebra spectrum over the S -algebra $\mathbb{H}(A)$. We let $\rho(B)$ denote B regarded as an A -module spectrum via restriction of scalars along the inclusion $\mathbb{H}(A) \rightarrow B$. We then have a composite

$$\mathbb{H}(A)^\wedge_C \rightarrow \rho(B)^\wedge_C \rightarrow B^\wedge_C. \tag{8.4}$$

Note that the right hand spectrum is obtained by completion over the ground S -algebra B , and the two leftmost spectra are completions over the ground S -algebra $\mathbb{H}(A)$. The right hand map is induced by the natural morphism of cosimplicial spectra $\mathcal{T}_{\mathbb{H}(A)}(\rho(B); C) \rightarrow \mathcal{T}_B(B; C)$, so the composite is induced by the natural morphism $\mathcal{T}_{\mathbb{H}(A)}(A; C) \rightarrow \mathcal{T}_B(B; C)$. One can now check from the definitions that

$$\pi_s(\mathcal{T}_A^k(A; C)) \cong \text{MultiTor}_s^A(\underbrace{\mathbb{F}_p, \mathbb{F}_p, \dots, \mathbb{F}_p}_{k+1 \text{ factors}}) \cong H_s(\underbrace{B\mathbb{Z}/p^\infty\mathbb{Z} \times \dots \times B\mathbb{Z}/p^\infty\mathbb{Z}}_{k \text{ factors}}, \mathbb{F}_p)$$

and

$$\pi_s(\mathcal{T}_B^k(B; C)) \cong H_s(\underbrace{BS^1 \times \dots \times BS^1}_{k \text{ factors}}, \mathbb{F}_p).$$

It is easy to check that the induced homomorphism $\pi_s(\mathcal{T}_A^k(A; C)) \rightarrow \pi_s(\mathcal{T}_B^k(B; C))$ is (under the isomorphisms above) identified with $H_s(\underbrace{Bi \times \dots \times Bi}_{k \text{ factors}}, \mathbb{F}_p)$, where $i: \mathbb{Z}/p^\infty\mathbb{Z} \rightarrow S^1$ is the inclusion. It is a well known fact that

this map is an isomorphism, which shows that the map in (8.4) above is an equivalence of spectra. In order to identify $\mathbb{H}(A)^\wedge_C$, then, it will suffice to identify B^\wedge_C . To do this, we consider the homomorphism of commutative S -algebras $g: S^0 \rightarrow \mathbb{H}(Z) \rightarrow \mathbb{Z}[S^1]$. As before, let $\rho(\mathbb{Z}[S^1])$ denote $\mathbb{Z}[S^1]$ regarded as an S^0 -module via restriction of scalars along g . Then we obtain a natural map $\rho((\mathbb{Z}[S^1])^\wedge_{\mathbb{H}(\mathbb{F}_p)}) \rightarrow \mathbb{Z}[S^1]^\wedge_{\mathbb{H}(\mathbb{F}_p)}$. Again, the left hand completion is over the commutative S -algebra S^0 , and the right one is over the S -algebra $\mathbb{Z}[S^1]$. There is a homomorphism of algebraic to geometric spectral sequences (7.1), which is an isomorphism since $\pi_0 S^0 \rightarrow \pi_0 \mathbb{Z}[S^1]$. Consequently, the desired completion is the p -adic completion of the spectrum $\mathbb{Z}[S^1]$. The resulting homotopy groups are the groups $H_i(S^1, \mathbb{Z}_p)$, so we finally obtain the formula

$$\begin{aligned} l\pi_i(A^\wedge_{\mathbb{H}(\mathbb{F}_p)}) &\cong \mathbb{Z}_p \quad \text{for } i = 0, 1 \\ \pi_i(A^\wedge_{\mathbb{H}(\mathbb{F}_p)}) &= 0 \quad \text{otherwise.} \end{aligned}$$

This result has been extended to a statement about finitely generated nilpotent groups by T. Lawson in [13]. A construction referred to as *deformation K-theory* can be made which incorporates the topology on spaces of representations, and which applies to any discrete group. In the case of \mathbb{Z} , it produces a spectrum with homotopy groups given by $\mathbb{Z}[S^1] \otimes \pi_*ku$. What Lawson proves is that for a finitely generated nilpotent group Γ , the derived completion of the representation ring of the pro- p completion of Γ at the homomorphism given by mod p augmentation is equivalent to the p -adic completion of the deformation k -theory of Γ . This gives a link between this completion process on representation rings of pro- p quotients of Γ with the geometry of representation varieties of Γ .

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References

[1] J.F. Adams, J.-P. Haeberly, S. Jackowski, J.P. May, A generalization of the Atiyah–Segal completion theorem, *Topology* 27 (1) (1988) 1–6.

- [2] M. Barr, J. Beck, Acyclic models and triples, in: 1966 Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, pp. 336–343.
- [3] A.K. Bousfield, Homotopical localizations of spaces, *Amer. J. Math.* 119 (6) (1997) 1321–1354.
- [4] A.K. Bousfield, Cosimplicial Resolutions and Homotopy Spectral Sequences in model categories, in: *Geometry and Topology*, vol. 7, 2003, pp. 1001–1053.
- [5] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions and Localizations, in: *Lecture Notes in Mathematics*, vol. 304, Springer-Verlag, Berlin, New York, 1972, v+348 pp.
- [6] G. Carlsson, Recent Developments in Algebraic K -Theory, in: *Current Developments in Mathematics*, vol. 2001, International Press, 2002, pp. 1–40.
- [7] W.G. Dwyer, J. Spaliński, Homotopy theories and model categories, in: *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995, pp. 73–126.
- [8] W.G. Dwyer, J.P.C. Greenlees, S. Iyengar, Duality in algebra and topology, *Adv. Math.* 200 (2) (2006) 357–402.
- [9] D. Eisenbud, Commutative Algebra. With a View Toward Algebraic Geometry, in: *Graduate Texts in Mathematics*, vol. 150, Springer-Verlag, New York, 1995.
- [10] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, Rings, Modules, and Algebras in Stable Homotopy Theory. With an Appendix by M. Cole, in: *Mathematical Surveys and Monographs*, vol. 47, American Mathematical Society, Providence, RI, 1997.
- [11] J.P.C. Greenlees, J.P. May, Completions in algebra and topology, in: *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995, pp. 255–276.
- [12] M. Hovey, B. Shipley, J. Smith, Symmetric spectra, *J. Amer. Math. Soc.* 13 (1) (2000) 149–208.
- [13] T. Lawson, Completed representation ring spectra of nilpotent groups, *Algebr. Geom. Topol.* 6 (2006) 253–286.
- [14] M.A. Mandell, J.P. May, S. Schwede, B. Shipley, Model categories of diagram spectra, *Proc. London Math. Soc.* (3) 82 (2) (2001) 441–512.
- [15] D.G. Quillen, Finite generation of the groups K_j of rings of algebraic integers, in: *Algebraic K -Theory, I: Higher K -Theories* (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), in: *Lecture Notes in Math.*, vol. 341, Springer, Berlin, 1973, pp. 179–198.
- [16] A. Radulescu-Banu, Cofibrance and completion, Thesis, M.I.T. [arXiv.org:math.AT/0612203](https://arxiv.org/abs/math/0612203), 1999.
- [17] S. Schwede, B.E. Shipley, Algebras and modules in monoidal model categories, *Proc. London Math. Soc.* (3) 80 (2000) 491–511.
- [18] B.E. Shipley, A convenient model category for commutative ring spectra, in: *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K -Theory*, in: *Contemp. Math.*, vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 473–483.
- [19] Weibel, A. Charles, An Introduction to Homological Algebra, in: *Cambridge Studies in Advanced Mathematics*, vol. 38, Cambridge University Press, Cambridge, ISBN: 0-521-43500-5, 1994, xiv+450 pp. ISBN: 0-521-55987-1.