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Covering homology

Morten Brun [a,*], Gunnar Carlsson [b,1], Bjørn Ian Dundas [a,2]

[a] *University of Bergen, Department of Mathematics, Johannes Brunsgate 12, Bergen, Norway*
[b] *Stanford University, Department of Mathematics, Building 380, Sloan Hall, Stanford, CA 94305, United States*

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Abstract

We introduce the notion of *covering homology* of a commutative **S**-algebra with respect to certain families of coverings of topological spaces. The construction of covering homology is extracted from Bökstedt, Hsiang and Madsen's topological cyclic homology. In fact covering homology with respect to the family of orientation preserving isogenies of the circle is equal to topological cyclic homology. Our basic tool for the analysis of covering homology is a cofibration sequence involving homotopy orbits and a restriction map similar to the restriction map used in Bökstedt, Hsiang and Madsen's construction of topological cyclic homology.

Covering homology with respect to families of isogenies of a torus is constructed from iterated topological Hochschild homology. It receives a trace map from iterated algebraic K-theory and there is a hope that the rich structure, and the calculability of covering homology will make it useful in the exploration of J. Rognes' "red shift conjecture".

Keywords: Topological cyclic homology; Smash powers; Equivariant spectra; Chromatic red shift; Burnside–Witt rings

* Corresponding author.
E-mail address: morten.brun@math.uib.no (M. Brun).

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1. Introduction

Topological cyclic homology (*TC*), as defined by Bökstedt, Hsiang and Madsen in [2], is interesting for two reasons: firstly it is a good approximation to algebraic K-theory, secondly it is accessible through methods in stable homotopy theory. Along with motivic homotopy theory, topological cyclic homology is the main source for calculations of algebraic K-theory.

Topological cyclic homology is built from a diagram of categorical fixed point spectra of Bökstedt's topological Hochschild homology. The main reason for the accessibility of *TC* is the so-called "fundamental cofiber sequence" which inductively gives homotopical control of the categorical fixed points. There are many frameworks where people find conceptual reasons for the fundamental cofiber sequence – for instance it can be viewed as a concrete identification of the geometrical fixed points – but regardless of point of view it remains a marvelous fact at a crucial point of the theory.

Just as for other cyclic nerve constructions, if the input is commutative – in our case a connective commutative **S**-algebra A – topological Hochschild homology extends to a functor of spaces taking a space X to a spectrum $\Lambda_X A$ which we call *smash X of A*, or the *Loday functor*. The value at the circle S^1 recovers the usual definition $\Lambda_{S^1} A \simeq THH(A)$. If X is a finite set, $\Lambda_X A$ is just a particular model for the X-fold smash product of A with itself. Our preferred model $\Lambda_X A$ is extracted from Bökstedt's construction of topological Hochschild homology and Street's first construction [19] to enhance the functoriality of homotopy colimits. This functoriality has the side effect that the multiplicative structure of topological Hochschild homology can be realized on our concrete model, and using Bökstedt's construction as our basis, we get that $\Lambda_X A$ is automatically a homotopy functor in both X and A without any cofibrant replacements.

The main reason for our choice of model is that the fundamental cofiber sequence extends in a beautiful manner (see Lemma 5.1.3) giving full homotopy theoretic control over the categorical fixed points. The sequence becomes particularly transparent in the abelian case, which is the interesting part if one is mostly concerned with the case X being a torus (which is the case for iterated topological Hochschild homology): if G is a finite abelian group, X a non-empty free G-space and A a connective commutative **S**-algebra, then (Lemma 5.2.5) there is a cofiber sequence

$$\left[\Lambda_X(A)\right]_{hG} \longrightarrow \left[\Lambda_X(A)\right]^G \longrightarrow \underset{0\neq H\leqslant G}{\underleftarrow{\operatorname{holim}}} \left[\Lambda_{X/H}(A)\right]^{G/H},$$

where $[\Lambda_X(A)]_{hG}$ denotes the homotopy G-orbits, $[\Lambda_X(A)]^G$ the categorical fixed points and the homotopy limit is taken over all nontrivial subgroups of G. In the equivariant world, this could be viewed as an instance of the tom Dieck filtration gotten by taking fixed points of the sequence one gets by smashing $\Lambda_X(A)$ with the cofibration sequence $EG_+ \to S^0 \to \widetilde{EG}$, as discussed in [5], together with identifications, firstly of the geometric H-fixed-points of $\Lambda_X(A)$ and $\Lambda_{X/H}(A)$, and secondly of the categorical fixed points of $\Lambda_X(A)$ and the fixed points obtained by deloopings by representations in a universe. However, we also get that these deloopings are not necessary for developing the theory (with the exception of matters related to transfers, which will be important in a later paper), and we can stay with the concrete functorial model at hand and its associated categorical constructions.

More precisely, by induction on the order of the group, the fundamental cofibration sequence implies that we have full homotopical control over the categorical fixed points $[\Lambda_X A]^G$ if G is a finite group acting freely on X.

As a result of this structure we get that if X is connected, then (Proposition 6.2.4) there is a natural isomorphism

$$\pi_0[\Lambda_X A]^G \cong \mathbb{W}_G(\pi_0 A),$$

where the right hand side is the Burnside–Witt ring of Dress and Siebeneicher [7]; and we recover Hesselholt and Madsen's result $\pi_0[THH(A)]^{C_r} \cong \mathbb{W}_{C_r}(\pi_0 A)$ [10] where C_r is the cyclic group of order r.

Studying systems of coverings, the spectra $[\Lambda_X A]^G$ assemble into a diagram giving rise to the new notion of "covering homology". In the particular case of finite orientation preserving self-coverings of the circle this is Bökstedt, Hsiang and Madsen's topological cyclic homology. If we include reflections we get a definition of topological dihedral homology.

In the special case where X is the n-torus $\mathbb{T}^{\times n}$, the spectrum $\Lambda_{\mathbb{T}^{\times n}} A$ is a model for the n-fold iterated topological Hochschild homology of A. This said, the covering homology is very different from iterated topological cyclic homology, having a vastly richer structure. We give examples at the very end of the paper where we see actions of various Galois groups, units in orders in division algebras (and so Morava stabilizer groups) and in the extreme case, all of $GL_n(\mathbb{Z})$. The study of this structure and concrete calculations will be followed up in a second paper. From this detailed analysis one should hope to glean insight into the chromatic behavior of covering homology.

To give the reader an idea about the structure entering into the construction of covering homology, consider topological Hochschild homology (*THH*) of the spherical group ring $\mathbf{S}[G]$ of an abelian group G. It turns out to be the suspension spectrum $\mathbf{S}[Map(\mathbb{T}, BG)]$ of the free loop space of the classifying space of the group in question, i.e. the space of unbased continuous maps of the circle into BG. The operators used to compute *TC* arise from the evident circle action on the free loop space, as well as from the power maps of various degrees from the circle to itself. This free loop space interpretation of topological Hochschild homology shows that the n-fold iteration of *THH* on $\mathbf{S}[G]$ is equivalent to the suspension spectrum

$$\Lambda_{\mathbb{T}^{\times n}}\big(\mathbf{S}[G]\big) \simeq \mathbf{S}\big[Map\big(\mathbb{T}^{\times n}, B^n G\big)\big]$$

on the unbased mapping space of a higher dimensional torus into the iterated bar construction $B^n G$. This space supports many natural operations beyond the one-variable ones. For example, the group $GL_n(\mathbb{Z})$ acts on the n-torus, and hence on the mapping space from the torus into $B^n G$. In addition, generalizations of the power maps include all possible isogenies of the torus to itself. This will be true in any sufficiently functorial model for *THH*; the important point is that the equivariant structure is "right" – a thing secured by the fundamental cofiber sequence.

This is a brief overview of the paper. In Sections 2–4 we construct the Loday functor $X \mapsto \Lambda_X A$. Section 2 is a guide to the construction with references to related constructions. Section 3 contains combinatorial preliminaries, and in Section 4 we finally define $\Lambda_X A$. In Section 5 we present the fundamental cofibration sequence. In Section 6 this cofibration sequence is used to describe the zeroth homotopy group of fixed points of $\Lambda_X A$ in terms of the Burnside–Witt construction. Finally in the short Section 7 we define covering homology and show how it extends the definition of topological cyclic homology.

1.1. Notation

We let Fin be the category of finite sets and functions. The skeletal subcategory $\mathcal{F}$ of Fin has one object $\underline{n} = \{1, \dots, n\}$ for each non-negative integer n. The category $\mathcal{F}$ is given a strictly associative and unital coproduct by means of concatenation. The subcategory $\mathcal{I}$ of $\mathcal{F}$ consists of all injective functions. We let Γ^{o} be the category of finite sets of the form $n_+ = \{0, 1, \dots, n\}$ and functions fixing the base point 0. The category Δ consists of finite ordered sets of the form $[n] = \{0, 1, \dots, n\}$ and order-preserving functions; Δ^{o} is its opposite. The categories of simplicial sets (called simply *spaces* in most of the paper) and pointed simplicial sets are denoted $\mathcal{S}$ and $\mathcal{S}_*$ respectively, and $\Gamma\mathcal{S}_*$ is the category of Γ-spaces, i.e., of pointed functors $\Gamma^{\mathrm{o}} \to \mathcal{S}_*$. The sphere spectrum is represented by the inclusion $\mathbf{S}$: $\Gamma^{\mathrm{o}} \subseteq \mathcal{S}_*$. If X and Y are pointed spaces, $Map_*(X, Y)$ denotes the space of pointed maps from X to the singular complex of the geometric realization of Y.

2. Guide to the construction

Let X be a simplicial set. In the following sections we give a model, $\Lambda_X A$, for the X-fold smash power of a connective commutative $\mathbf{S}$-algebra A (i.e., a symmetric monoid in $(\Gamma\mathcal{S}_*, \wedge, \mathbf{S})$). We have chosen to work with Γ-spaces, but our constructions work equally well on connective commutative symmetric ring-spectra. We shall call $\Lambda_X A$ the *Loday functor* of A evaluated at X. It is important for the construction that A is *strictly* commutative. The model $\Lambda_X A$ is functorial in both X and A. Therefore, if a group G acts on X, then we can consider the (categorically honest) G-fixed points of $\Lambda_X A$. In the particular situation where G is finite and the action on X is free we have good control on the fixed point spectrum $[\Lambda_X A]^G$. When X is the underlying space of the simplicial circle group $\mathbb{T} = \sin U(1)$, given by the singular complex on the circle group $U(1) = \{x \in \mathbb{C}\colon |x| = 1\}$, the Loday functor evaluated at X is a model for topological Hochschild homology.

2.1. Higher Hochschild homology

As a motivation, consider ordinary Hochschild homology $HH(A) : \Delta^{\mathrm{o}} \to \mathcal{A}b$ of a flat ring A, given in each dimension by

$$HH_q(A) = A^{\otimes q+1}.$$

If A is commutative, then $HH(A)$ is a simplicial commutative ring, and Loday [12] observed that Hochschild homology factors through the category Fin of finite sets:

$$\begin{array}{ccc} \Delta^{\mathrm{o}} & \xrightarrow{HH(A)} & \mathcal{A}b \\ \downarrow{\scriptstyle S^1} & \nearrow & \\ \mathrm{Fin} & & \end{array}$$

(see e.g., [13]). Here $S^1 = \Delta[1]/\partial\Delta[1]$ is the standard simplicial circle. The dotted functor is oftentimes called the *Loday complex*. Let us write simply

$$X \longmapsto \Lambda_X^{\mathbb{Z}} A$$

for this functor. If we extend it to all sets by colimits and to all simplicial sets by applying the functor degreewise, we get a functor $X \mapsto \Lambda_X^{\mathbb{Z}} A$ from the category $\mathcal{S}$ of spaces (simplicial sets) to simplicial abelian groups, with classical Hochschild homology being

$$HH(A) = \Lambda_{S^1}^{\mathbb{Z}} A.$$

Pirashvili [18] uses the notation $H^{[d]}(A, A)$ for the Loday complex for A evaluated on the d-dimensional sphere and calls it "higher Hochschild homology of order d".

There is no obstruction to apply the same construction to symmetric monoids in any symmetric monoidal category, Hochschild homology being the case when one considers $(\mathcal{A}b, \otimes, \mathbb{Z})$.

2.2. Higher topological Hochschild homology

Consider any of the popular symmetric monoidal categories of spectra. Then topological Hochschild homology of a cofibrant $\mathbf{S}$-algebra A is equivalent to the simplicial spectrum gotten by just replacing $\otimes$ with $\wedge$ in the definition of Hochschild homology, and in the commutative case we have a factorization $X \mapsto A \otimes X$ through Fin, where $\otimes$ is the categorical tensor in commutative $\mathbf{S}$-algebras. Extending this functor to the category of spaces just as in the above subsection we obtain for every commutative $\mathbf{S}$-algebra A and every space X the *higher topological Hochschild homology* $A \otimes X$. The above is another way of stating the result of McClure, Schwänzl and Vogt [17]: $THH(A) \simeq A \otimes S^1$. Here THH is Bökstedt's model for topological Hochschild homology. The above equivalence is an equivalence of cyclic spectra. However in the context of $\mathbf{S}$-algebras in the sense of Elmendorf, Kriz, Mandell and May [8] the fixed point spectrum $[A \otimes sd_r S^1]^G$ of $A \otimes sd_r S^1$ with respect to the cyclic group G with $r > 1$ elements does not have the same homotopy type as the G-fixed point spectrum of $sd_r THH(A)$. In the language of McClure et al. it is easy to see that the iterated topological Hochschild homology is:

$$THH\bigl(THH(A)\bigr) \simeq \bigl(A \otimes S^1\bigr) \otimes S^1 \simeq A \otimes \bigl(S^1 \times S^1\bigr).$$

Hence taking the nth iterate of THH is the same as tensoring with the n-torus.

In the situation where $X = \mathbb{T}$ is the circle group and G is a finite subgroup of $\mathbb{T}$ there is a homotopy equivalence between the Loday functor $\Lambda_{\mathbb{T}} A$ of A evaluated at $\mathbb{T}$ and $THH(A)$, and this equivalence is G-equivariant in the sense that it induces an equivalence of H-fixed spectra for every subgroup H of G. In this situation the control on the fixed point space derives from the "fundamental cofiber sequence"

$$THH(A)_{hC_{p^n}} \longrightarrow THH(A)^{C_{p^n}} \longrightarrow THH(A)^{C_{p^{n-1}}}.$$

In Lemma 5.2.5 we generalize the fundamental cofibration sequence to the situation where a finite group G acts freely on a space X. In the toroidal case $X = \mathbb{T}^{\times n}$, the spectrum $\Lambda_X A$ is homotopy equivalent to iterated topological Hochschild homology, but the fact that the complexity of the subgroup lattice of torus increases with dimension makes the fundamental cofibration sequence more involved. This also gives rich and interesting symmetries on the collection of fixed point spectra $\Lambda_{\mathbb{T}^{\times n}}(A)^H$ under varying finite $H \subseteq \mathbb{T}^{\times n}$. This structure gives actions by interesting groups possibly shedding light on the chromatic properties of the spectra, as in Rognes' red shift conjecture.

Although iterated *TC* involves iterations of fixed point spectra of *THH*, we consider the much simpler idea of taking the fixed point under the toroidal action on the iterated *THH*. This gives us in many ways a much cruder invariant, but also a much more computable one – essentially, the difference is that of $[[A \wedge A]^{C_2} \wedge [A \wedge A]^{C_2}]^{C_2}$ (the start of the iterated construction) and $[(A \wedge A) \wedge (A \wedge A)]^{C_2 \times C_2}$ (the start of our construction). Taking $A = \mathbf{S}$ we see that these constructions give different results. Furthermore, the resulting theory displays a vastly richer symmetry, giving rise to a plethora of actions not visible if one only focuses on "diagonal" actions.

The underlying spectra of $X \mapsto \Lambda_X A$ and $A \otimes X$ are equivalent. The problem with the model $A \otimes X$ is that if a finite group G acts on X then we do not fully understand the G-fixed points of this model.

Martin Stolz showed in his thesis that this can be fixed, giving a functor to commutative orthogonal ring spectra with the correct equivariant properties. His approach is similar to the versions of *THH* worked out in the case of associative **S**-algebras by Brun and Lydakis (unpublished) and Kro [11]. However, for this paper we have chosen a more hands-on approach.

3. Encoding coherence with spans

In this section we shall use the category V of spans of finite sets described below, to encode the coherence data for symmetric monoidal categories. More precisely, we shall encode the coherence data by a lax functor from V to the category of categories. Since we are interested in group actions we will also investigate group actions on the morphism sets of V. The reader who is willing to believe in our coherence results may skip this quite technical section. However these coherence results are essential for the construction of the Loday functor Λ in the next section.

3.1. *The category of spans*

The object class of the category V of spans of finite set is the class of finite sets. Given finite sets X and Y the set of morphisms $V(Y, X)$ is the set of equivalence classes $[Y \leftarrow A \rightarrow X]$ of diagrams of finite sets of the form $Y \leftarrow A \rightarrow X$. Here $Y \leftarrow A \rightarrow X$ is equivalent to $Y \leftarrow A' \rightarrow X$ if there exists a bijection $A \rightarrow A'$ making the resulting triangles commute. The composition of two morphisms $[Z \leftarrow B \rightarrow Y]$ and $[Y \leftarrow A \rightarrow X]$ is the morphism $[Z \leftarrow C \rightarrow X]$, where C is the pull-back of the diagram $B \rightarrow Y \leftarrow A$.

Lemma 3.1.1. *Disjoint union of sets is both the product and the coproduct on V.*

Proof. Since V is self-dual, products and coproducts coincide. Thus we need only to prove the statement about coproducts. To be precise, the disjoint union of two maps $[X \leftarrow A \rightarrow Y]$ and $[X' \leftarrow A' \rightarrow B']$ is $[X \sqcup X' \leftarrow A \sqcup A' \rightarrow Y \sqcup Y']$. The required isomorphism $V(X, Y) \times V(X', Y) \cong V(X \sqcup X', Y)$ is given by

$$\big(\big[X \xleftarrow{f} A \xrightarrow{g} Y\big], \big[X' \xleftarrow{f'} A' \xrightarrow{g'} Y\big]\big) \longmapsto \big[X \sqcup X' \xleftarrow{f \sqcup f'} A \sqcup A' \xrightarrow{g+g'} Y\big]$$

with inverse given by the restrictions to the appropriate inverse images

$$\big[X \sqcup X' \xleftarrow{f} A \xrightarrow{g} Y\big] \longmapsto \big(\big[X \longleftarrow f^{-1}(X) \longrightarrow Y\big], \big[X' \longleftarrow f^{-1}\big(X'\big) \longrightarrow Y\big]\big). \qquad \square$$

If G is a group acting from left on a finite set X, then G acts on X considered as an object of V through the functor $g \mapsto g_* = [X \xleftarrow{=} X \xrightarrow{g} X]$. Consequently, G acts on the morphism sets $V(Y, X)$. There is a function

$$\varphi : V(Y, X/G) \longrightarrow V(Y, X)^G$$

defined by the formula

$$\varphi\big([Y \longleftarrow A \longrightarrow X/G]\big) = [Y \longleftarrow A \times_{X/G} X \longrightarrow X].$$

Proposition 3.1.2. *The function φ is bijective.*

Proof. There is a G-bijection $V(Y, X) \xrightarrow{\cong} map(X \times Y, \mathbb{N})$ taking $[Y \xleftarrow{f_1} A \xrightarrow{f_2} X]$ to the function $f : X \times Y \to \mathbb{N}$ with $f(x, y)$ equal to the cardinality of

$$(f_2, f_1)^{-1}(x, y) = \big\{a \in A \mid f_2(a) = x \text{ and } f_1(a) = y\big\}.$$

Under this bijection φ corresponds to the bijection $map(Z/G, \mathbb{N}) \xrightarrow{\cong} map(Z, \mathbb{N})^G$ induced by the projection $Z \to Z/G$ for a G-set Z. □

Corollary 3.1.3. *There is an isomorphism of categories $\psi : V/(X/G) \to (V/X)^G$.*

Proof. The functor ψ takes an object $\alpha \in V(Y, X/G)$ to $\varphi(\alpha) \in V(Y, X)^G$. By the definition of φ it is clear that it is functorial in the sense that the diagram

$$\begin{array}{ccc} V(Z,Y) \times V(Y,X/G) & \longrightarrow & V(Z,X/G) \\ \downarrow{\scriptstyle \mathrm{id}\times\varphi} & & \downarrow{\scriptstyle \varphi} \\ V(Z,Y) \times V(Y,X)^G & \longrightarrow & V(Z,X)^G \end{array}$$

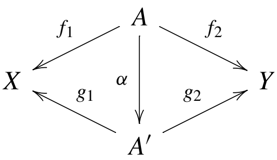
commutes for all finite sets Y and Z. Therefore, on morphisms, we can define the functor ψ to be given by the identity. □

3.2. A lax functor

We work within the framework of Benabou's bicategories [1].

Given a bicategory $\mathcal{B}$ with 0-cells A and B there is a category $\mathcal{B}(A, B)$. The objects of $\mathcal{B}(A, B)$ are the 1-cells from A to B, and the morphisms in $\mathcal{B}(A, B)$ are 2-cells in $\mathcal{B}$.

Let W denote the following bicategory of spans (compare [15, pp. 283–285]). The bicategory W has the class of finite sets as class of 0-cells, and $W(X, Y)$ is the category of spans $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$. A morphism $\alpha : f \to g$ from f to $g = (X \xleftarrow{g_1} A' \xrightarrow{g_2} Y)$ in $W(X, Y)$, that is, a 2-cell α in W, consists of a map $\alpha : A \to A'$ making the diagram



(3.2.0)

commute. The composition functor

$$W(Y,Z)\times W(X,Y)\longrightarrow W(X,Z)$$

takes a pair (g,f) of spans $Y \xleftarrow{g_1} B \xrightarrow{g_2} Z$ and $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$ to the span $g\circ f$ represented by the diagram

$$X \xleftarrow{(g\circ f)_1} A\times_Y B \xrightarrow{(g\circ f)_2} Z$$

where

$$A\times_Y B=\big\{(a,b)\in A\times B\colon\ f_2(a)=g_1(b)\big\}$$

is a functorial choice of fiber product. Here $(g\circ f)_1(a,b)=f_1(a)$ and $(g\circ f)_2(a,b)=g_2(b)$. On morphisms the composition functor is defined by the formula $\alpha\circ\beta=\alpha\times_Y\beta$ for $\alpha:A\to A'$ and $\beta:B\to B'$. The identity functor $\mathbf{1}\to W(X,X)$ is given by the 1-cell $X \xleftarrow{=} X \xrightarrow{=} X$. Using the universal property of pull-backs, we see that composition of 1-cells is associative up to invertible 2-cells. These 2-cells endow W with the structure of a bicategory.

The (strict) bicategory *Cat* has small categories as 0-cells, functors as 1-cells and natural transformations as 2-cells.

A morphism of bicategories, here called a *lax functor*, $F:\mathcal{B}\to\mathcal{C}$ consists of the following data:

- a 0-cell $F(B)$ of $\mathcal{C}$ for every 0-cell B of $\mathcal{B}$,
- a functor $F_{AB}:\mathcal{B}(A,B)\to\mathcal{C}(FA,FB)$ for every pair (A,B) of 0-cells in $\mathcal{B}$. Given a 1-cell f in $\mathcal{B}(A,B)$ we write Ff instead of $F_{AB}(f)$,
- a structure 2-cell $F(g,f):Fg\circ Ff\to F(g\circ f)$ for every pair (g,f) of composable 1-cells in $\mathcal{B}$,
- a structure 2-cell $F_A:\mathrm{id}_{FA}\to F\,\mathrm{id}_A$ for every 0-cell A in $\mathcal{B}$.

This data is subject to the following axioms found in [1]:

(M1) For all composable triples (h, g, f) of 1-cells of B the following diagram commutes:

$$\begin{array}{ccc} Fh \circ (Fg \circ Ff) & \xleftarrow{a_{\mathcal{C}}} & (Fh \circ Fg) \circ Ff \\ \downarrow{\scriptstyle \mathrm{id} \circ F(g,f)} & & \downarrow{\scriptstyle F(h,g) \circ \mathrm{id}} \\ Fh \circ F(g \circ f) & & F(h \circ g) \circ Ff \\ \downarrow{\scriptstyle F(h, g \circ f)} & & \downarrow{\scriptstyle F(h \circ g, f)} \\ F(h \circ (g \circ f)) & \xleftarrow{F(a_{\mathcal{B}})} & F((h \circ g) \circ f). \end{array}$$

Here $a_{\mathcal{B}}$ and $a_{\mathcal{C}}$ are the associativity isomorphisms of $\mathcal{B}$ and $\mathcal{C}$ respectively.

(M2) For every 1-cell $f : A \to B$ of $\mathcal{B}$ the diagrams

$$\begin{array}{ccc} Ff & \xrightarrow{Fr_{\mathcal{B}}} & F(f \circ \mathrm{id}_A) \\ \downarrow{\scriptstyle r_{\mathcal{C}}} & & \uparrow{\scriptstyle F(f, \mathrm{id}_A)} \\ (Ff) \circ \mathrm{id}_{FA} & \xrightarrow{\mathrm{id} \circ F_A} & Ff \circ F(\mathrm{id}_A) \end{array} \qquad \begin{array}{ccc} F(\mathrm{id}_A \circ f) & \xleftarrow{Fl_{\mathcal{B}}} & F(f) \\ \uparrow{\scriptstyle F(\mathrm{id}_A, f)} & & \downarrow{\scriptstyle l_{\mathcal{C}}} \\ F(\mathrm{id}_A) \circ Ff & \xleftarrow{F_A \circ \mathrm{id}} & \mathrm{id}_{FA} \circ Ff \end{array}$$

commute, where r and l are the right and left identities of the bicategories involved.

A *weak functor* $F : \mathcal{B} \to \mathcal{C}$ is a lax functor with the property that the structure 2-cells $F(g, f) : Fg \circ Ff \to F(g \circ f)$ and $F_A : \mathrm{id}_{FA} \to F(\mathrm{id}_A)$ are isomorphisms. Weak functors are called pseudo-functors at many places. A *strict functor* $F : \mathcal{B} \to \mathcal{C}$ is a lax functor with the property that the structure 2-cells $F(g, f) : Fg \circ Ff \to F(g \circ f)$ and $F_A : \mathrm{id}_{FA} \to F(\mathrm{id}_A)$ are identity morphisms.

Definition 3.2.1. The sub-bicategories $iW \subseteq eW \subseteq W$ are defined by allowing only invertible or only epimorphic 2-cells. More precisely, iW and eW have the same 0- and 1-cells as W and the 2-cells of iW and eW from $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$ to $X \xleftarrow{g_1} A' \xrightarrow{g_2} Y$ consist of respectively invertible and epimorphic functions $A \to A'$ making (3.2.0) commute.

We define a lax functor $L : V \to eW$. On objects it is the identity: $L(X) = X$. If $f = [X \xleftarrow{f_1} A \xrightarrow{f_2} Y]$ is a morphism in V we let

$$L(f) = \left(X \xleftarrow{Lf_1} LA \xrightarrow{Lf_2} Y\right),$$

where LA is the image of the map $(f_1, f_2) : A \to X \times Y$, that is,

$$LA = \left\{(x, y) \in X \times Y \colon \exists a \in A \text{ such that } f_1(a) = x, \text{ and } f_2(a) = y\right\}.$$

The maps Lf_1 and Lf_2 are the projection maps to the first and the second factor. We note that this does not depend on the chosen representative for f. The structure map

$$L(g,f): L(g) \circ L(f) \longrightarrow L(g \circ f)$$

is induced by the epimorphism $LA \times_Y LB \to L(A \times_Y B)$ taking $((x,y),(y,z))$ to (x,z).

The functors $\mathrm{Fin}^{\mathrm{o}} \to V$, $(f: X \to Y) \mapsto f^* = [Y \xleftarrow{f} X \xrightarrow{=} X]$ and $\mathrm{Fin} \to V$, $(g: Y \to X) \mapsto g_* = [Y \xleftarrow{=} Y \xrightarrow{g} X]$ allow us to consider both Fin and $\mathrm{Fin}^{\mathrm{o}}$ as subcategories of V. Note that the composite lax functors $f \mapsto L(f^*)$ and $g \mapsto L(g_*)$ are both weak functors.

Definition 3.2.2. Let $\mathcal{B}$ be a category with chosen finite coproducts and let $\mathcal{C} \subseteq \mathcal{B}$ be a subcategory with the property that $\mathcal{C}$ contains all isomorphisms in $\mathcal{B}$ (in particular $\mathcal{C}$ contains all objects of $\mathcal{B}$). We define a weak functor $\mathcal{C}: iW \to \mathit{Cat}$ as follows: Given a finite set X, we let $\mathcal{C}(X) = \mathcal{C}^X$ be the X-fold product of $\mathcal{C}$. Given a 1-cell $f = (X \xleftarrow{f_1} A \xrightarrow{f_2} Y)$, the functor $\mathcal{C}(f): \mathcal{C}(X) \to \mathcal{C}(Y)$ is given by the formula

$$\big(\mathcal{C}(f)(c)\big)(y) = \coprod_{a \in f_2^{-1}(y)} c\big(f_1(a)\big)$$

for both objects and morphisms of $\mathcal{C}(X)$. Given an invertible 2-cell α of the form (3.2.0), we define $\mathcal{C}(\alpha): \mathcal{C}(f) \to \mathcal{C}(f')$ to be the natural transformation with $(\mathcal{C}(\alpha)(c))(y)$ equal to the isomorphism

$$\coprod_{a \in f_2^{-1}(y)} c\big(f_1(a)\big) \cong \coprod_{a' \in (f_2')^{-1}(y)} c\big(f_1'(a')\big),$$

induced by the restriction $f_2^{-1}(y) \xrightarrow{\cong} (f_2')^{-1}(y)$ of α.

A *pointed category* is a category $\mathcal{D}$ together with a chosen null object which is both initial and terminal. A *pointed functor* of pointed categories is a functor $F: \mathcal{D} \to \mathcal{D}'$ preserving null objects. Note that pointed categories are naturally enriched over the category of pointed sets.

If $\mathcal{C}$ and $\mathcal{D}$ are two pointed categories, we get a pointed category $\mathcal{C} \wedge \mathcal{D}$ by smashing the object classes and the individual morphism sets. If $\mathcal{C}$ is a pointed category, we let $\mathcal{C}^e = \mathcal{C}^{\mathrm{o}} \wedge \mathcal{C}$.

We let Cat_* be the 2-category of small pointed categories, pointed functors and natural transformations. The forgetful 2-functor $\mathit{Cat}_* \to \mathit{Cat}$ has a left 2-adjoint

$$\mathit{Cat} \xrightarrow{\mathcal{C} \mapsto \mathcal{C}_+} \mathit{Cat}_*,$$

where $\mathcal{C}_+$ is given by adding an extra object 0 to $\mathcal{C}$ and morphisms $0 \to c$ and $c \to 0$ for every object c in $\mathcal{C}$. Functors and natural transformations are sent to the unique pointed extensions.

Definition 3.2.3. Let $F: iW \to \mathit{Cat}$ be a weak functor of bicategories. We define a weak functor $F_+: eW \to \mathit{Cat}_*$ fitting into the commutative diagram

$$\begin{array}{ccc} iW & \hookrightarrow & eW \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_+} \\ \mathit{Cat} & \xrightarrow{\mathcal{C} \mapsto \mathcal{C}_+} & \mathit{Cat}_*. \end{array}$$

The diagram specifies uniquely the structure 2-cells and the values of F_+ on 0-cells, 1-cells and invertible 2-cells of eW. Thus in order to describe F_+ we need only to specify the values of F_+ on the non-invertible 2-cells of eW. We do this by letting $F_+(\alpha)$ be the null transformation if α is a non-invertible 2-cell of eW.

Note that the axiom (M2) for F_+ is a direct consequence of (M2) for F and that (M1) holds because composing with a non-invertible 2-cell in eW gives a non-invertible 2-cell.

Definition 3.2.4. Let $\mathcal{I}_+ : eW \to Cat_*$ be the lax functor obtained by combining Definition 3.2.2 and Definition 3.2.3 with $\mathcal{C} = \mathcal{I}$, the subcategory of injections in the category $\mathcal{F}$ of finite sets of the form $\{1, \dots, n\}$. Composing with the lax functor $L : V \to eW$ we obtain a lax functor

$$\mathcal{J} = \mathcal{I}_+ L : V \to Cat_*$$

with $\mathcal{J}(X) = (\mathcal{I}^X)_+$. Composing with the inclusion $\mathrm{Fin} \subseteq V$ we obtain a weak functor

$$\mathrm{Fin} \longrightarrow Cat_*, \qquad X \xrightarrow{f} Y \longmapsto \left(\mathcal{I}^X\right)_+ \xrightarrow{\mathcal{J}(f_*)} \left(\mathcal{I}^Y\right)_+,$$
$$i \longmapsto \left\{ y \longmapsto \coprod_{x \in f^{-1}(y)} i(x) \right\},$$

with structure isomorphisms $\mathcal{J}(f_*, g_*) : \mathcal{J}(f_*)\mathcal{J}(g_*) \to \mathcal{J}((fg)_*)$ given by the canonical isomorphism permuting summands. On the other hand, composing with the inclusion $\mathrm{Fin}^{\mathrm{o}} \subseteq V$ we obtain a strict functor

$$\mathrm{Fin}^{\mathrm{o}} \longrightarrow Cat_*, \qquad X \xleftarrow{f} Y \longmapsto \left(\mathcal{I}^X\right)_+ \xrightarrow{\mathcal{J}(f^*)} \left(\mathcal{I}^Y\right)_+,$$
$$i \longmapsto i \circ f.$$

4. The Loday functor

In this section we shall construct the Loday functor $\Lambda_X A$. It is a functor in the unpointed space X and the commutative $\mathbf{S}$-algebra A. When X is the circle group, this is a version of topological Hochschild homology, which we think of as the cyclic nerve of A in the category of $\mathbf{S}$-modules.

The construction of $\Lambda_X A$ is quite technical. It can be summarized in the following steps. The crucial ingredient of Bökstedt's definition of topological Hochschild homology is the stabilizations indexed by the category $\mathcal{I}$ of finite sets of the form $\{1, \dots, n\}$ and injective functions. Firstly, we note that $\mathcal{I}$ is a subcategory of a category $\mathcal{F}$ of finite sets with chosen finite coproducts (given by concatenation of maps between sets of the form $\{1, \dots, n\}$), and we apply Definition 3.2.4 to obtain a lax functor $\mathcal{J} : V \to Cat_*$ from a category of spans $X \leftarrow A \to Y$ of finite sets to the category of pointed categories. Secondly, we construct the left lax transformation G^A associated to a commutative $\mathbf{S}$-algebra A. This is a left lax transformation from the lax functor given by the composition $\mathrm{Fin} \to V \xrightarrow{\mathcal{J}} Cat_*$ to the constant functor from Fin to Cat_* with value $\Gamma\mathcal{S}_*$. Thirdly, using Street's first construction we rectify $\mathcal{J} : V \to Cat_*$ to a strict functor $\widetilde{\mathcal{J}} : V \to Cat_*$. Finally, applying the homotopy colimit construction, we obtain a functor $X \mapsto \Lambda_X A$ from Fin to $\Gamma\mathcal{S}_*$.

4.1. *Left lax transformations*

Bökstedt's construction of topological Hochschild homology involves in simplicial degree k a stabilization by means of a homotopy colimit over a certain category $\mathcal{I}^{k+1}$. When the degree k varies, so does the indexing category $\mathcal{I}^{k+1}$. Since book-keeping becomes rather involved we offer an overview of the functoriality properties of homotopy colimits.

First some notation taken from Street [19].

Given lax functors $F, F' : \mathcal{B} \to \mathcal{C}$ a *left lax transformation* $\sigma : F \to F'$ consists of the data:

- a 1-cell $FA \xrightarrow{\sigma_A} F'A$ for every 0-cell A of $\mathcal{B}$,
- a 2-cell $F'f \circ \sigma_A \xrightarrow{\sigma_f} \sigma_B \circ Ff$ as in the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\sigma_A} & F'(A) \\ {\scriptstyle F(f)}\downarrow & \Downarrow\sigma_f & \downarrow{\scriptstyle F'(f)} \\ F(B) & \xrightarrow{\sigma_B} & F'(B) \end{array}$$

for every 1-cell $f : A \to B$ in $\mathcal{B}$.

The following composition and unit diagrams are required to commute:

$$\begin{array}{ccccc} F'g \circ (F'f \circ \sigma_A) & \xrightarrow{a_{\mathcal{C}}} & (F'g \circ F'f) \circ \sigma_A & \xrightarrow{F'(g,f)\circ\sigma_A} & F'(g \circ f) \circ \sigma_A \\ {\scriptstyle F'g\circ\sigma_f}\downarrow & & & & \downarrow{\scriptstyle \sigma_{g\circ f}} \\ F'g \circ (\sigma_B \circ Ff) & & & & \sigma_C \circ F(gf) \\ {\scriptstyle a_{\mathcal{C}}}\downarrow & & & & \uparrow{\scriptstyle \sigma_C\circ F(g,f)} \\ (F'g \circ \sigma_B) \circ Ff & \xrightarrow{\sigma_g\circ(Ff)} & (\sigma_C \circ Fg) \circ (Ff) & \xrightarrow{a_{\mathcal{C}}} & \sigma_C \circ (Fg \circ Ff) \end{array}$$

and

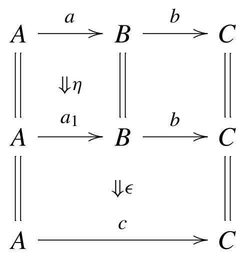
$$\begin{array}{ccc} & & F'(\mathrm{id}_A) \circ \sigma_A \\ & \nearrow{\scriptstyle F'_A\circ\sigma_A} & \downarrow{\scriptstyle \sigma_{\mathrm{id}_A}} \\ \sigma_A & & \\ & \searrow{\scriptstyle \sigma_A\circ F_A} & \\ & & \sigma_A \circ F(\mathrm{id}_A). \end{array}$$

Similarly, a *right lax transformation* $\rho : F \to F'$ consists of the data:

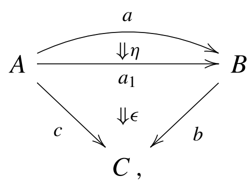
- a 1-cell $FA \xrightarrow{\rho_A} F'A$ for every 0-cell A of $\mathcal{B}$,
- a 2-cell $F'f \circ \rho_A \xleftarrow{\rho_f} \rho_B \circ Ff$ for every 1-cell $f : A \to B$ in $\mathcal{B}$

satisfying the composition and unit conditions analogous to the conditions for left lax transformations.

Remark 4.1.1. Unless otherwise explicitly declared, when we write an unmarked diagram we mean that it commutes (i.e., it is an identity 2-cell). Also, identity cells are occasionally entirely omitted from the notation (being thought geometrically as degenerate cells), so that we may for instance write the combination of "vertical" and "horizontal" compositions of 2-cells

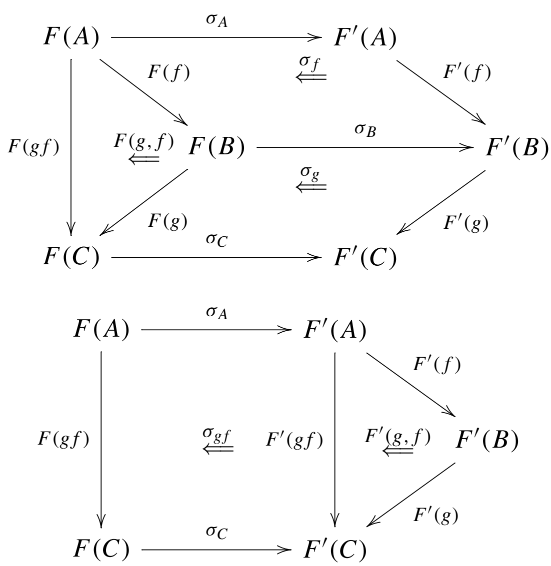


(the upper right hand square is an identity 2-cell) as



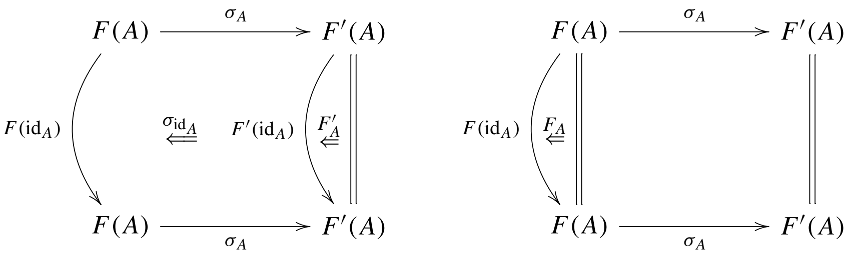
and even $\epsilon \circ \eta$ (Street would write $\epsilon.b\eta$).

Hence, suppressing associativity isomorphisms in $\mathcal{C}$, the diagrams in the definition the left lax transformation $\sigma : F \Rightarrow F'$ amounts to claiming that the two 2-cells controlling how to compose



(4.1.2)

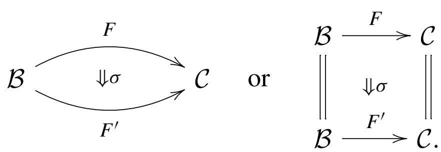
are equal, and that the two 2-cells expressing unitality



(4.1.3)

are equal.

As is customary, we draw a left lax transformation as $\sigma : F \Rightarrow F'$, or as 2-dimensional pictures



4.2. *On the functoriality of homotopy colimits*

Let Cat_* be the category of pointed categories, and let J be a small category. Given functors $E, F : J \to Cat_*$, the concept of a left lax transformation $G : F \to E$ simplifies considerably as compared to the general situation described above, and consists of the following data: for every $i \in J$ we have a functor $G_i : F(i) \to E(i)$ and for every $f : i \to j \in J$ we have a natural transformation $G_f : E(f)G_i \Rightarrow G_j F(f)$ such that if $g : j \to k \in J$ we have that $G_{gf} = G_g G_f$ and $G_{\mathrm{id}_j} = \mathrm{id}_{G_j}$. Given $F : J \to Cat_*$ recall that we write F^e for the functor $j \mapsto F(j)^e = F(j)^{\mathrm{o}} \wedge F(j)$.

The category Bimod^J/E has as objects pairs (F, G) consisting of a functor $F : J \to Cat_*$ taking small pointed categories as values and a left lax transformation $G : F^e \to E$. A morphism $(F, G) \to (F', G')$ in Bimod^J/E consists of a natural transformation $\epsilon : F \Rightarrow F'$ and a modification $\eta : G \Rightarrow G'\epsilon^e$. Recall that η is a modification means that for each $i \in J$ we have a natural transformation $\eta_i : G_i \Rightarrow G'_i \epsilon^e_i$ such that for each $f : i \to j \in J$ we have $\eta_j G_f = G'_f \eta_i : E(f)G_i \Rightarrow G'_j \epsilon^e_j F^e(f) = G'_j (F'(f)\epsilon_i)^e$.

Let K be a pointed category with all small coproducts. In order to define the homotopy colimit we need to choose a coproduct in K. Note that K is automatically tensored over the category of pointed sets: if $k \in K$ and S is a pointed set, then we write $k \wedge S$ for the coproduct of k with itself over the set of non-basepoints of S.

Considering K as a constant functor from J to pointed categories, we use the above construction to give a category Bimod^J/K. The homotopy colimit is the functor

$$\underrightarrow{\mathrm{holim}} : \mathrm{Bimod}^J/K \to \left[J \times \Delta^{\mathrm{o}}, K\right]$$

sending (F, G) to $\underrightarrow{\mathrm{holim}}_F\, G$, which is the functor taking $(j, [q]) \in J \times \Delta^{\mathrm{o}}$ to

$$\bigvee_{x_0,\dots,x_q} G_j(x_0,x_q)\wedge F(j)(x_1,x_0)\wedge\cdots\wedge F(j)(x_q,x_{q-1})$$

where the sum is taken over all $x_0,\dots,x_q\in \mathrm{ob}F(j)$. The simplicial structure is of Hochschild type, where s_k is induced by the identity $S^0\to F(j)(x_k,x_k)$ for $0\leqslant k\leqslant q$ and d_k is induced by the composition $F(j)(x_k,x_{k-1})\wedge F(j)(x_{k+1},x_k)\to F(j)(x_{k+1},x_{k-1})$ for $0<k<q$. The operator d_0 is given by functoriality of G_j in x_0, and d_q is given by functoriality of G_j in x_q.

Actually, this construction is a cyclic bar construction, so it may be misleading to call it a homotopy colimit. However, we have chosen to do so, since in our applications it coincides with a usual homotopy colimit: when $F=F'_+$ factors as $J\xrightarrow{F'}Cat\xrightarrow{\mathcal{C}\mapsto\mathcal{C}_+}Cat_*$ and $G=G'p_2$ factors through the projection p_2 onto the second factor

$$F^e\cong\left(\left(F'\right)^{\mathrm{o}}\times F'\right)_+\xrightarrow{p_2}F'_+\xrightarrow{G'}K,$$

then $(\operatorname{holim}_{\overrightarrow{F}} G)(j)$ is isomorphic to the usual homotopy colimit

$$\underset{\overrightarrow{F'(j)}}{\operatorname{holim}}\, G'=\left\{[q]\longmapsto\bigvee_{F'(j)(x_1,x_0)\times\cdots\times F'(j)(x_q,x_{q-1})} G'_j(x_q)\right\}$$

of the functor $G'_j:F'(j)\to K$.

We work in the pointed setting solely to be able to include trivial modifications η in morphisms $(\epsilon,\eta):(F,G)\to(F',G')$.

If $(\epsilon,\eta):(F,G)\to(F',G')$ is a morphism, we write $\operatorname{holim}_{\overrightarrow{\epsilon}}\eta$ for the corresponding natural transformation $\operatorname{holim}_{\overrightarrow{F}} G\Rightarrow\operatorname{holim}_{\overrightarrow{F'}} G'$ induced by

$$\begin{array}{c} G_j(x_0,x_q)\wedge F(j)(x_1,x_0)\wedge\cdots\wedge F(j)(x_q,x_{q-1}) \\ \Big\downarrow{\scriptstyle \eta_j\wedge\epsilon_j\wedge\cdots\wedge\epsilon_j} \\ G'_j(\epsilon_jx_0,\epsilon_jx_q)\wedge F'(j)(\epsilon_jx_1,\epsilon_jx_0)\wedge\cdots\wedge F'(j)(\epsilon_jx_q,\epsilon_jx_{q-1}). \end{array}$$

4.3. The left lax transformation G^A

Let Σ be the category of finite sets and bijections and choose a functor $S:\Sigma\to\mathcal{S}_*$ with $S(\{1\})=S^1$ which is strong symmetric monoidal with respect to the monoidal structures given by disjoint union of sets and smash products of simplicial sets respectively. The strong symmetric monoidal functor S gives us a left lax transformation

$$\begin{array}{ccc} \mathrm{Fin} & = & \mathrm{Fin} \\ \Big\downarrow{\scriptstyle \Sigma_+} & \overset{S}{\Rightarrow} & \Big\downarrow{\scriptstyle \mathcal{S}_*} \\ Cat_* & = & Cat_* \end{array}$$

with $S(j)=\bigwedge_{x\in X}S^{j(x)}$ for $j\in\Sigma^X$. Here the weak functor $\Sigma:\mathrm{Fin}\to Cat$ is the composition of the inclusion $\mathrm{Fin}\to iW$ and the construction $iW\to Cat$ of Definition 3.2.2 and Σ_+ is constructed as in Definition 3.2.3. The weak functor $\mathcal{S}_*:\mathrm{Fin}\to \mathit{Cat}_*$ is the constant functor with the category $\mathcal{S}_*$ of pointed simplicial sets as value. Likewise, given a commutative **S**-algebra A, we can use the multiplication in A to construct a left lax transformation

$$\begin{array}{ccc} \mathrm{Fin} & = & \mathrm{Fin} \\ \downarrow \Sigma_+ & \overset{A}{\Rightarrow} & \downarrow \mathcal{S}_* \\ \mathit{Cat}_* & = & \mathit{Cat}_* \end{array}$$

with $A(j)=\bigwedge_{x\in X} A(S^{j(x)})$ for $j\in \Sigma^X$. We shall use the left lax transformations S and A to construct a left lax transformation G^A from the weak functor $S\mapsto \mathcal{J}(S)=(\mathcal{I}^S)_+$ from Fin to Cat_* to the constant functor sending everything to $\Gamma\mathcal{S}_*$.

If X and Y are pointed simplicial sets, $map_*(X,Y)$ denotes the Γ-space given by sending $Z\in\Gamma^{\mathrm{o}}$ to $Map_*(X,Z\wedge Y)$.

For a finite set T, let

$$G^A_T:\mathcal{J}(T)\longrightarrow \Gamma\mathcal{S}_*$$

be the pointed functor which to the object $i\in\mathcal{I}^T$ assigns the Γ-space

$$map_*\big(S(i),A(i)\big)=map_*\bigg(\bigwedge_{t\in T} S^{i(t)},\bigwedge_{t\in T} A\big(S^{i(t)}\big)\bigg),$$

and to a morphism $\alpha:i\to j$ assigns the map

$$\begin{array}{c} map_*(\bigwedge_{t\in T} S^{i(t)},\bigwedge_{t\in T} A(S^{i(t)})) \\ \downarrow \\ map_*(\bigwedge_{t\in T} S^{j(t)-\alpha i(t)}\wedge\bigwedge_{t\in T} S^{i(t)},\bigwedge_{t\in T} S^{j(t)-\alpha i(t)}\wedge\bigwedge_{t\in T} A(S^{i(t)})) \\ \downarrow \\ map_*(\bigwedge_{t\in T} S^{j(t)},\bigwedge_{t\in T} A(S^{j(t)})), \end{array}$$

where the first map is induced by the suspension functor $K\mapsto \bigwedge_{t\in T} S^{j(t)-\alpha i(t)}\wedge K$ and the second map is given by the permutation isomorphism

$$\bigwedge_{t\in T} S^{j(t)-\alpha i(t)}\wedge\bigwedge_{t\in T} S^{i(t)}\cong\bigwedge_{t\in T} S^{j(t)}$$

and the map

$$\begin{aligned} \bigwedge_{t\in T} S^{j(t)-\alpha i(t)}\wedge\bigwedge_{t\in T} A\big(S^{i(t)}\big) &\cong \bigwedge_{t\in T}\big(S^{j(t)-\alpha i(t)}\wedge A\big(S^{i(t)}\big)\big) \\ &\longrightarrow \bigwedge_{t\in T} A\big(S^{j(t)-\alpha i(t)}\wedge S^{i(t)}\big)\cong\bigwedge_{t\in T} A\big(S^{j(t)}\big). \end{aligned}$$

Here the two isomorphisms indicated by $\cong$ are permutation isomorphisms and the unmarked arrow is given by the functoriality of A.

If $\phi : S \to T$ is a function of finite sets, there is a natural transformation

$$G^A_\phi : G^A_S \longrightarrow G^A_T \circ \mathcal{J}(\phi)$$

$$\begin{array}{ccc}
\mathcal{J}(S) & \xrightarrow{\quad G^A_S \quad} & \Gamma\mathcal{S}_* \\
{\scriptstyle \mathcal{J}(\phi)}\downarrow & \swarrow\!\!\swarrow {\scriptstyle G^A_\phi} & \nearrow {\scriptstyle G^A_T} \\
\mathcal{J}(T) & &
\end{array}$$

given on $i \in \mathcal{I}^S$ by sending a map $f : S(i) \to A(i)$ to the composite

$$\begin{array}{ccccc}
\bigwedge_{t\in T} S^{\coprod_{s\in\phi^{-1}(t)} i(s)} & & & & \bigwedge_{t\in T} A(S^{\coprod_{s\in\phi^{-1}(t)} i(s)}) \\
\cong\uparrow & & & & \cong\uparrow \\
\bigwedge_{t\in T}\bigwedge_{s\in\phi^{-1}(t)} S^{i(s)} & & \bigwedge_{t\in T}\bigwedge_{s\in\phi^{-1}(t)} A(S^{i(s)}) & \longrightarrow & \bigwedge_{t\in T} A(\bigwedge_{s\in\phi^{-1}(t)} S^{a_{i(s)}}) \\
\cong\uparrow & & \cong\uparrow & & \\
\bigwedge_{s\in S} S^{i(s)} & \xrightarrow{\quad f \quad} & \bigwedge_{s\in S} A(S^{i(s)}), & &
\end{array}$$

where the top vertical maps are structure maps for S, the lower vertical maps are permutation isomorphisms and the unmarked arrow is multiplication in A.

This structure assembles into the fact that G^A is a left lax transformation from the weak functor $S \mapsto \mathcal{J}(S) = (\mathcal{I}^S)_+$ to the constant functor $\mathrm{Fin} \to Cat_*$ sending everything to $\Gamma\mathcal{S}_*$: axiom (4.1.2) follows since for every $\psi : R \to S$ and $\phi : S \to T$ in Fin the difference one gets when one writes out the indexing sets occurring in the definitions of $G^A_\phi G^A_\psi$ and $G^A(\phi\psi)$ is exactly the canonical permutation $\mathcal{J}(\phi_*, \psi_*)$. The unitality condition (4.1.3) degenerates as $\mathcal{J}(\mathrm{id}_R) = \mathrm{id}_{\mathcal{J}(R)}$ and $G^A_{\mathrm{id}_S} = \mathrm{id}_{G^A_S}$.

Note 4.3.1. The reader may wonder why G^A is only defined on Fin (rather than V). The reason is that there exists no extension of G^A to V, for if one did, a factorization $p \circ i$: $\{1\} \subseteq \{1,2\} \twoheadrightarrow \{1\} \in \mathrm{Fin}$ of the identity would then induce a splitting

$$G^{\mathbf{S}}_{i^*}(p^* j) \circ G^{\mathbf{S}}_{p^*}(j) : map_*(S^2, S^2) \longrightarrow map_*(S^4, S^4) \longrightarrow map_*(S^2, S^2)$$

(here $j \in \mathcal{I}^{\{1\}}$ is represented by $\{1,2\} \in \mathcal{I}$), which obviously does not exist since $\mathbb{Z}/2\mathbb{Z} = \pi_1\Omega^4 S^4$ does not contain $\mathbb{Z} = \pi_1\Omega^2 S^2$ as a retract (evaluate the Γ-spaces on S^0).

However, in Section 5, and in subsequent applications, it is crucial for the equivariant structure of the Loday functor that $\mathcal{J}$ is defined on V.

In Section 6.2 we will give a very weak extension of $\pi_0 G^A(S^0)$ to cover also the maps $[T \leftarrow T \times K = T \times K] \in V$ induced by projection onto the first factor.

4.4. Rectifying G^A and the definition of $\Lambda(A)$

We make a brief recollection on the pointed version of Street's first construction [19, p. 226] and make a remark about right cofinality, and then we construct the restriction of the Loday functor of A to finite sets.

Let J be a category and let $F: J \to Cat_*$ be a lax functor. Street's first construction is a functor $\widetilde{F}: J \to Cat_*$ defined as follows: Given an object j of J, the pointed category $\widetilde{F}(j)$ has as set of objects $\bigvee_{j_1 \in \mathrm{ob} J} J(j_1, j)_+ \wedge F(j_1)$. A morphism

$$(\psi, \alpha): (\varphi_1, x_1) \longrightarrow (\varphi_0, x_0)$$

consists of a morphism $\psi: j_0 \to j_1$ with $\varphi_0 = \varphi_1 \psi$ and a morphism $\alpha: x_1 \to F(\psi)(x_0)$ in $F(j_1)$, identifying all $(\psi, 0)$ to the base morphism 0. The composition of two morphisms $(\psi_2, \alpha_2): (\varphi_2, x_2) \to (\varphi_1, x_1)$ and $(\psi_1, \alpha_1): (\varphi_1, x_1) \to (\varphi_0, x_0)$ is the pair $(\psi_2\psi_1, \beta)$ where β is the composition

$$F(\psi_2\psi_1)x_0 \xleftarrow{F(\psi_2,\psi_1)} F(\psi_2)F(\psi_1)x_0 \xleftarrow{F(\psi_2)\alpha_1} F(\psi_2)x_1 \xleftarrow{\alpha_2} x_2.$$

If $\psi: j \to j'$ is a morphism in J, then $\widetilde{F}(\psi): \widetilde{F}(j) \to \widetilde{F}(j')$ is given by the formula $\widetilde{F}(\psi)(\varphi, x) = (\psi\varphi, x)$.

For every object j of J there is a pointed functor $e = e_j: \widetilde{F}(j) \to F(j)$ with $e_j(\varphi_1, x_1) = F(\varphi_1)(x_1)$ and with $e_j(\psi, \alpha)$ equal to the composite

$$F(\varphi_1)x_1 \xrightarrow{F(\varphi_1)\alpha} F(\varphi_1)F(\psi)x_0 \xrightarrow{F(\varphi_1\psi)} F(\varphi_1\psi)x_0 = F(\varphi_0)x_0.$$

These pointed functors assemble to a left lax transformation $e: \widetilde{F} \to F$. Note that for every object $x \in F(j)$ the structure morphism $x \to e_j(\mathrm{id}_j, x) = F(\mathrm{id}_j)(x)$ is an object of the category x/e_j. Given another object $\alpha: x \to e_j(\varphi_0, x_0) = F(\varphi_0)(x_0)$ of the category x/e_j, the morphism (φ_0, α) is the only morphism in x/e_j from $x \to F(\mathrm{id}_j)(x)$ to α. Thus the category x/e_j has an initial element so the pointed functor e_j is right cofinal.

Let $\rho: F \to F'$ be a right lax transformation between lax functors $F, F': J \to Cat_*$. For every object j of J there is a functor $\tilde{\rho}(j): \widetilde{F}(j) \to \widetilde{F'}(j)$ taking a morphism $(\psi, \alpha): (\varphi_1, x_1) \to (\varphi_0, x_0)$ of $\widetilde{F}(j)$ as above to the morphism $\tilde{\rho}(j)(\psi, \alpha) = (\psi, \rho_\psi \circ \rho_{j_1}(\alpha))$ of $\widetilde{F'}(j)$. Here the functor $\rho_{j_1}: F(j_1) \to F'(j_1)$ and the natural transformation $\rho_\psi: \rho_{j_1} F(\psi) \to F'(\psi)\rho_{j_0}$ are part of the right lax transformation ρ. These functors are natural in j, so we have constructed a natural transformation $\tilde{\rho}: \widetilde{F} \to \widetilde{F'}$.

Now we use Street's first construction to build the Loday functor. Let V denote the category of spans of finite sets. In the notation introduced below Definition 3.2.2 we have a lax functor $\mathcal{J}: V \to Cat_*$ such that $\mathcal{J}(f^*) = \mathcal{J}(Y \xleftarrow{f} X \xrightarrow{=} X): (\mathcal{I}^Y)_+ \to (\mathcal{I}^X)_+$ is the functor given by precomposing with $f: X \to Y$ and $\mathcal{J}(f_*) = \mathcal{J}(X \xleftarrow{=} X \xrightarrow{f} Y): (\mathcal{I}^X)_+ \to (\mathcal{I}^Y)_+$ is a functor with

$$\mathcal{J}(f_*)(j)(y) \cong \coprod_{x \in f^{-1}(y)} j(x),$$

for $j \in \mathcal{I}^X$.

Definition 4.4.1. The functor $\widetilde{\mathcal{J}} : V \to Cat_*$ and the left lax transformation $e : \widetilde{\mathcal{J}} \to \mathcal{J}$ are defined by applying the pointed version of Street's first construction with $J = V$ and $F = \mathcal{J}$. We let r be the left lax transformation $r = e^o \wedge e : \widetilde{\mathcal{J}}^e \to \mathcal{J}^e$.

The projection on the second factor

$$\mathcal{J}^e(R) \cong \left(\left(\mathcal{I}^R\right)^{\mathrm{o}} \times \mathcal{I}^R\right)_+ \longrightarrow \left(\mathcal{I}^R\right)_+ = \mathcal{J}(R)$$

defines a strict transformation $p_2 : \mathcal{J}^e \to \mathcal{J}$ of lax functors $V \to Cat_*$. Composing with G^A and the inclusion $\iota : \mathrm{Fin} \to V$, we obtain a left lax transformation

$$\mathcal{G}^A = G^A \circ p_2 \colon (\mathcal{J} \circ \iota)^e = \mathcal{J}^e \circ \iota \quad \Longrightarrow \quad \Gamma\mathcal{S}_*$$

of lax functors $\mathrm{Fin} \to Cat_*$.

Composing with $r = e^{\mathrm{o}} \wedge e$ we get a left lax transformation of strict functors $\mathcal{G}^A \circ r : \widetilde{\mathcal{J}}^e \circ \iota \to \Gamma\mathcal{S}_*$:

$$\begin{array}{ccccc}
\mathrm{Fin} & = & \mathrm{Fin} & = & \mathrm{Fin} \\
\downarrow \iota & & \downarrow \iota & & \\
V & = & V & \overset{\mathcal{G}^A}{\Longrightarrow} & \downarrow \Gamma\mathcal{S}_* \\
\downarrow \widetilde{\mathcal{J}}^e & \overset{r}{\Rightarrow} & \downarrow \mathcal{J}^e & & \\
Cat_* & = & Cat_* & = & Cat_*.
\end{array}$$

Definition 4.4.2. Given a finite set S and a commutative $\mathbf{S}$-algebra A the *Loday functor of A evaluated at S* is the Γ-space $\Lambda_S(A)$ given by the homotopy colimit

$$\underset{\widetilde{\mathcal{J}}(S)}{\underrightarrow{\mathrm{holim}}}\, \mathcal{G}^A_S \circ r_S.$$

By the functoriality of the homotopy colimit explained in Section 4.2, this construction is functorial in $S \in \mathrm{Fin}$, so that we have a functor $\Lambda(A) : \mathrm{Fin} \to \Gamma\mathcal{S}_*$.

Note that we do not use the $\mathbf{S}$-algebra structure on A in the construction of $\Lambda_S(A)$ for a fixed finite set S, and also note that the action of the automorphism group of S on $\Lambda_S(A)$ does not depend on the $\mathbf{S}$-algebra structure.

We have introduced the category V in order to have an isomorphism $\widetilde{\mathcal{J}}(S)^G \cong \widetilde{\mathcal{J}}(S/G)$ whenever G is a finite group acting on a finite set S. This gives us the following crucial lemma.

Lemma 4.4.3. *Let S be a finite set and let G be a group acting on S. For every pointed functor $Z\colon \mathcal{J}(S) = (\mathcal{I}^S)_+ \to \mathcal{S}_*$ the canonical map*

$$\underset{\widetilde{\mathcal{J}}(S)}{\underrightarrow{\mathrm{holim}}}\, Z \circ p_2 \circ r_S \longrightarrow \underset{\mathcal{J}(S)}{\underrightarrow{\mathrm{holim}}}\, Z \circ p_2 \cong \underset{\mathcal{I}^S}{\underrightarrow{\mathrm{holim}}}\, Z$$

induces a weak equivalence on G-fixed points.

Proof. First we treat the case where G is the trivial group. We have seen that the pointed functor $r_S : \widetilde{\mathcal{J}}(S) \to (\mathcal{I}^S)_+$ is right cofinal. Therefore the canonical map

$$\underset{\overrightarrow{\widetilde{\mathcal{J}}(S)}}{\text{holim}}\, Z \circ p_2 \circ r_S \longrightarrow \underset{\overrightarrow{\mathcal{J}(S)}}{\text{holim}}\, Z \circ p_2$$

is an equivalence.

In the case where G is nontrivial we note the identity

$$\Big[\underset{\overrightarrow{\widetilde{\mathcal{J}}(S)}}{\text{holim}}\, Z \circ p_2 \circ r_S\Big]^G = \underset{\overrightarrow{\widetilde{\mathcal{J}}(S)^G}}{\text{holim}} [Z \circ p_2 \circ r_S]^G.$$

Note that $\widetilde{\mathcal{J}}(S)^G$ by construction is a full subcategory of $\widetilde{\mathcal{J}}(S)$, and that Corollary 3.1.3 implies that it is isomorphic to $\widetilde{\mathcal{J}}(S/G)$. We can now reduce to the situation where G is the trivial group via the commutative square

$$\begin{array}{ccc}
[\underset{\overrightarrow{\widetilde{\mathcal{J}}(S)}}{\text{holim}}\, Z \circ p_2 \circ r_S]^G & \longrightarrow & [\underset{\overrightarrow{\mathcal{J}(S)}}{\text{holim}}\, Z \circ p_2]^G \\
\cong \downarrow & & \cong \downarrow \\
\underset{\overrightarrow{\widetilde{\mathcal{J}}(S/G)}}{\text{holim}} [Z \circ p_2 \circ r_S]^G & \longrightarrow & \underset{\overrightarrow{\mathcal{J}(S/G)}}{\text{holim}} [Z \circ p_2]^G.
\end{array} \qquad \square$$

Given an $\mathbf{S}$-module A and a finite set T we let $\bigwedge_T A$ be the $\mathbf{S}$-module $\bigwedge_T A = \bigwedge_{t \in T} A$ (where we use the categorical smash product of $\mathbf{S}$-modules).

Lemma 4.4.4. *For every finite set S the functor $A \mapsto \Lambda_S A$ preserves connectivity of maps of commutative $\mathbf{S}$-algebras, and sends stable equivalences to pointwise equivalences. If A is cofibrant and T is a finite set then there is a chain of stable equivalences between $\Lambda_T A$ and $\bigwedge_T A$.*

Proof. The first statement follows from Bökstedt's lemma (see e.g. [3, Lemma 2.5.1]) since, for given n, there is an $i \in \mathcal{I}^S$ for which

$$G_S^A(K)(i) \longrightarrow \Lambda_S(A)(K)$$

is n-connected. This connectivity follows from the fact that the map $S^1 \wedge A(S^n) \to A(S^{1+n})$ is $2n-1$-connected [14, Proposition 5.21]. As n increases and K varies, the term to the left is just a concrete model for the (derived) S-fold smash of A with itself.

Let $\widetilde{G}_T^A : (\mathcal{I}^T)_+ \to \Gamma\mathcal{S}_*$ be the pointed functor which to the object $i \in \mathcal{I}^T$ assigns the Γ-space

$$K \longmapsto Map_*\bigg(\mathbf{S}(i), \Big(\bigwedge_T A\Big)\big(K \wedge \mathbf{S}(i)\big)\bigg),$$

and with structure maps similar to those of G_T^A. There are natural maps

$$\Lambda_T A(K) \longleftarrow \operatorname*{holim}_{\overrightarrow{i \in \mathcal{I}^T}} G_T^A(i)(K) \longrightarrow \widetilde{\Lambda_T} A(K) := \operatorname*{holim}_{\overrightarrow{i \in \mathcal{I}^T}} \widetilde{G}_T^A(i)(K) \longleftarrow \Big(\bigwedge_T A\Big)(K).$$

The first map is a weak equivalence by Lemma 4.4.3, the third map is always a stable equivalence, and the second map is a stable equivalence if A is cofibrant [14, Proposition 5.22]. □

The smash product of **S**-modules is the coproduct in the category of commutative **S**-algebras, and so the category of commutative **S**-algebras is "tensored" over finite sets through the formula $T \otimes A = \bigwedge_T A$. Using the universal property of the coproduct we see that $S \mapsto S \otimes A$ is a functor from the category of finite sets to the category of commutative **S**-algebras. The following corollary of the proof of Lemma 4.4.4 implies that up to homotopy $\Lambda_S(A)$ is a commutative **S**-algebra.

Corollary 4.4.5. *If A is a commutative* **S***-algebra which is cofibrant as an* **S***-module, then $\Lambda_S(A)$ is stably equivalent as an* **S***-module to the commutative* **S***-algebra $S \otimes A$ via maps that are natural in the finite set S.*

Corollary 4.4.6. *Given an* **S***-module A and finite sets R and S there is an equivalence*

$$\Lambda_{R\times S}(A) \simeq \Lambda_R\big(\Lambda_S(A)\big).$$

4.5. Multiplicative structure

One disadvantage with Bökstedt's formulation of $THH(A)$ for a commutative **S**-algebra A is that the multiplicative structure needs some elaboration. In our model, $\Lambda_X A$ will automatically be an **S**-algebra without any amendments, and it is appropriately functorial in X and A. In particular, our fattening up of Bökstedt's model could be viewed as a way of getting the multiplicative structure in one sweep (though, if one did not care about the equivariant structure to come, one would drop the complicating spans, and work with functors from Fin only). This said, our model is not strictly commutative, so there is still some advantage to the categorical smash product constructions in this regard.

Lemma 4.5.1. *Concatenation in $\mathcal{I}$ gives a left lax transformation with 2-cells consisting of isomorphisms $\sqcup : (\mathcal{J} \wedge \mathcal{J}) \circ \Delta \Rightarrow \mathcal{J}$ of lax functors $V \to Cat_*$, where $\Delta : V \to V \times V$ is the diagonal.*

Proof. We build the 2-cells consisting of isomorphisms from the natural isomorphisms we get in

$$\begin{array}{ccc} \mathcal{I}^X \times \mathcal{I}^X & \longrightarrow & \mathcal{I}^X \\ \downarrow & & \downarrow \\ \mathcal{I}^Y \times \mathcal{I}^Y & \longrightarrow & \mathcal{I}^Y, \end{array}$$

where the vertical maps are induced by $[X \xleftarrow{f} A \xrightarrow{g} Y] \in V$ and the horizontal maps by the concatenation in $\mathcal{I}$, simply (when evaluated on $(i, j) \in \mathcal{I}^X \times \mathcal{I}^X$ and $y \in Y$) by the coherence isomorphism that permutes $\coprod_{a\in(Lg)^{-1}(y)} i(Lf(a)) \sqcup \coprod_{a\in(Lg)^{-1}(y)} j(Lf(a))$ to $\coprod_{a\in(Lg)^{-1}(y)}(i(Lf(a)) \sqcup j(Lf(a)))$. $\square$

Even more simply, we obtain a natural transformation $\widetilde{\sqcup}: (\widetilde{\mathcal{J}} \wedge \widetilde{\mathcal{J}}) \circ \Delta \Rightarrow \widetilde{\mathcal{J}}$: if $X \in \mathrm{ob}V$ then $\widetilde{\sqcup}_X : \widetilde{\mathcal{J}}(X) \wedge \widetilde{\mathcal{J}}(X) \to \widetilde{\mathcal{J}}(X)$ is given by

$$\big((f_1, i_1), (f_2, i_2)\big) \longmapsto \big((f_1 + f_2), (i_1, i_2)\big)$$

(here $f_j : T_j \to X$ are maps in V and $i_j \in (\mathcal{I}^{T_j})_+$, $f_1 + f_2 : T_1 \sqcup T_2 \to X$ is the sum in V, and we have identified $(\mathcal{I}^{T_1\sqcup T_2})_+$ with $(\mathcal{I}^{T_1} \times \mathcal{I}^{T_2})_+$ for the purpose of naming elements). Since concatenation is strictly associative and unital, we see that so is $\widetilde{\sqcup}$.

Note that $X \mapsto \widetilde{\mathcal{J}}(X) \wedge \widetilde{\mathcal{J}}(X)$ is not the rectification of $X \mapsto \mathcal{J}(X) \wedge \mathcal{J}(X)$, so the comparison between these is not the formal one.

However, there is an invertible modification

$$M : e \circ \widetilde{\sqcup} \quad \Rrightarrow \quad \sqcup \circ (e \wedge e) \tag{4.5.1}$$

defined as follows: If $X \in V$ we let $M_X : e_X \circ \widetilde{\sqcup}_X \Rightarrow \sqcup_X \circ (e_X \wedge e_X)$ be the left lax transformation which, when applied to $(([g^1_* f^*_1], i_1), ([g^2_* f^*_2], i_2)) \in \widetilde{\mathcal{J}}(X) \wedge \widetilde{\mathcal{J}}(X)$, is provided by the canonical isomorphism between $L([g^1_* f^*_1] + [g^2_* f^*_2])(i_1, i_2) = \{x \mapsto \coprod_{(s,x)\in L(A_1\sqcup A_2)}(i_1 + i_2)(s)\}$ and $x \mapsto (\coprod_{(t_1,x)\in LA_1} i_1(t_1)) \sqcup (\coprod_{(t_2,x)\in LA_2} i_2(t_2))$.

Let $G^A \wedge G^A$ be the left lax transformation from $(\mathcal{J} \wedge \mathcal{J}) \circ \Delta$ to $\Gamma\mathcal{S}_*$ which evaluated on $(i, j) \in \mathcal{I}^S \times \mathcal{I}^S$ is given by $G^A_S(i) \wedge G^A_S(j)$. As before, $\mathcal{G}^A \wedge \mathcal{G}^A$ is obtained by precomposing with the projection $p_2 : \mathcal{J}^e \to \mathcal{J}$ onto the second factor.

Given objects $i, j \in \mathcal{I}^S$ a natural map $\mu^A_S(i, j) : G^A_S(i) \wedge G^A_S(j) \to G^A_S(i \sqcup j)$ is given by the composite

$$\begin{array}{c} map_*(\bigwedge_{s\in S} S^{i(s)}, \bigwedge_{s\in S} A(S^{i(s)})) \wedge map_*(\bigwedge_{s\in S} S^{j(s)}, \bigwedge_{s\in S} A(S^{j(s)})) \\ \downarrow \\ map_*((\bigwedge_{s\in S} S^{i(s)}) \wedge (\bigwedge_{s\in S} S^{j(s)}), (\bigwedge_{s\in S} A(S^{i(s)})) \wedge (\bigwedge_{s\in S} A(S^{j(s)}))) \\ \downarrow \cong \\ map_*(\bigwedge_{s\in S}(S^{i(s)} \wedge S^{j(s)}), \bigwedge_{s\in S}(A(S^{i(s)}) \wedge A(S^{j(s)}))) \\ \downarrow \\ map_*(\bigwedge_{s\in S}(S^{i(s)\sqcup j(s)}), \bigwedge_{s\in S}(A(S^{i(s)\sqcup j(s)}))) \end{array}$$

where the first map smashes maps together, the second one uses the canonical rearrangements and the third one uses the multiplication in A.

Lemma 4.5.2. *The maps $\mu^A_S(i,j)$ described above assemble to a modification μ^A from $\mathcal{G}^A \wedge \mathcal{G}^A$ to the composite $\mathcal{G}^A \circ \sqcup$*

$$
\begin{array}{ccccc}
\mathrm{Fin} & = & \mathrm{Fin} & = & \mathrm{Fin} \\
\downarrow \iota & & \downarrow \iota & & \\
V & = & V & \overset{\mathcal{G}^A}{\Longrightarrow} & \downarrow \Gamma\mathcal{S}_* \\
\downarrow (\mathcal{J}^e \wedge \mathcal{J}^e)\circ\Delta & \overset{\sqcup}{\Longrightarrow} & \downarrow \mathcal{J}^e & & \\
\mathcal{C}at_* & = & \mathcal{C}at_* & = & \mathcal{C}at_*,
\end{array}
$$

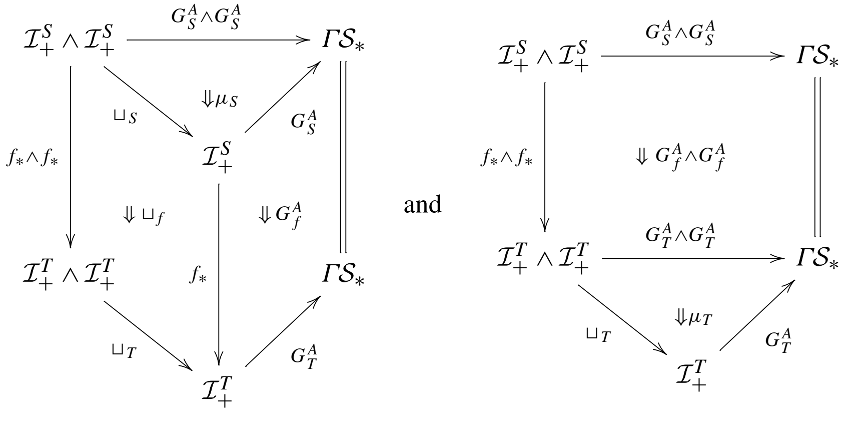
where the left lax transformation $\sqcup$ in the lower left square comes from Lemma 4.5.1.

Proof. As one checks, μ^A_S is a natural transformation, since given inclusions $i \subseteq i'$ and $j \subseteq j'$ in $\mathcal{I}^S$ the diagram

$$
\begin{array}{ccc}
G^A_S(i) \wedge G^A_S(j) & \xrightarrow{\mu^A_S(i,j)} & G^A_S(i \sqcup j) \\
\downarrow & & \downarrow \\
G^A_S(i') \wedge G^A_S(j') & \xrightarrow{\mu^A_S(i',j')} & G^A_S(i' \sqcup j')
\end{array}
$$

commutes.

Checking that this defines a modification amounts to checking that for each $f : S \to T \in \mathrm{Fin}$ the two 2-cells



are equal.

Hence we need to check that for each $i, j \in \mathcal{I}^S$ the diagram

$$\begin{array}{ccccc}
G_S^A(i) \wedge G_S^A(j) & & \xrightarrow{\quad G_f^A(i) \wedge G_f^A(j) \quad} & & G_T^A(f_* i) \wedge G_T^A(f_* j) \\
\Big\downarrow{\scriptstyle \mu_S(i,j)} & & & & \Big\downarrow{\scriptstyle \mu_T(f_* i, f_* j)} \\
G_S^A(i \sqcup j) & \xrightarrow{G_f^A(i \sqcup j)} & G_T^A f_*(i \sqcup j) & \xrightarrow{\sqcup_f(i,j)} & G_T^A(f_* i \sqcup f_* j)
\end{array}$$

commutes, that is that two specific maps from the smash $map_*(\mathbf{S}(i), A(i)) \wedge map_*(\mathbf{S}(j), A(j))$ to $map_*(\mathbf{S}(f_* i \sqcup f_* j), A(f_* i \sqcup f_* j))$ are equal. Writing this out, we are led to consider the commutative diagram

$$\begin{array}{ccccccc}
B(i) \wedge B(j) & \xrightarrow{\cong} & f^{-1}B(i) \wedge f^{-1}B(j) & & \longrightarrow & & B(f_* i) \wedge B(f_* j) \\
\Big\downarrow{\scriptstyle \cong} & & & & & & \Big\downarrow{\scriptstyle \cong} \\
\bigwedge_{s \in S}(B(S^{i(s)}) \wedge B(S^{j(s)})) & & & & & & \bigwedge_{t \in T}(B(S^{f_* i(t)}) \wedge B(S^{f_* j(t)})) \\
\Big\downarrow & & & & & & \Big\downarrow \\
B(i \sqcup j) & \xrightarrow{\cong} & f^{-1}B(i \sqcup j) & \longrightarrow & B(f_*(i \sqcup j)) & \xrightarrow{\cong} & B(f_* i \sqcup f_* j)
\end{array}$$

both for $B = \mathbf{S}$ and for $B = A$. Here, as before, $B(i) = \bigwedge_{s \in S} B(S^{i(s)})$, and we write $f^{-1}B(i)$ as shorthand for $(\bigwedge_{t \in T} \bigwedge_{s \in f^{-1}(t)} B(S^{i(s)}))$, the marked isomorphisms are the rearrangements used in the transformations, and the nonmarked arrows are multiplication. For $B = \mathbf{S}$ all maps involved are isomorphisms. Start with maps $\mathbf{S}(i) \to A(i)$ and $\mathbf{S}(j) \to A(j)$ and move around the diagram (both for $B = \mathbf{S}$ and $B = A$) to give two recipes for the same map $\mathbf{S}(f_* i \sqcup f_* j) \to A(f_* i \sqcup f_* j)$.

Extending along the projection p_2 we get the desired modification. □

Corollary 4.5.3. *Multiplication in A and coherence in $\mathcal{I}$ as in Lemma* 4.5.1 *gives a modification*

$$\begin{array}{ccccc}
(\widetilde{\mathcal{J}}^e \wedge \widetilde{\mathcal{J}}^e) \circ \Delta & \xRightarrow{(r \wedge r)} & (\mathcal{J}^e \wedge \mathcal{J}^e) \circ \Delta & \xRightarrow{\mathcal{G}^A \wedge \mathcal{G}^A} & \Gamma\mathcal{S}_* \\
\Big\Downarrow{\scriptstyle \widetilde{\sqcup}} & \overset{M}{\Leftarrow} & \Big\Downarrow{\scriptstyle \sqcup} & \overset{\mu^A}{\Leftarrow} & \Big\| \\
\widetilde{\mathcal{J}}^e & \xRightarrow{\quad r \quad} & \mathcal{J}^e & \xRightarrow[\mathcal{G}^A]{} & \Gamma\mathcal{S}_*
\end{array}$$

of left lax transformations of lax functors Fin $\to$ *Cat.*

As a consequence of the above corollary we get a natural multiplication map

$$\mu_S^A : \Lambda_S A \wedge \Lambda_S A \longrightarrow \Lambda_S A$$

by using the functoriality of homotopy colimits

$$\left(\tilde{\mu}_S, M_S\mu_S^A\right): \underset{\widetilde{\mathcal{J}}(S)\wedge\widetilde{\mathcal{J}}(S)}{\underrightarrow{\text{holim}}} \mathcal{G}_S^A r_S \wedge \mathcal{G}_S^A r_S \longrightarrow \underset{\widetilde{\mathcal{J}}(S)}{\underrightarrow{\text{holim}}} \mathcal{G}_S^A r_S$$

plus the obvious natural transformation

$$\Big(\underset{\widetilde{\mathcal{J}}(S)}{\underrightarrow{\text{holim}}} \mathcal{G}_S^A r_S\Big) \wedge \Big(\underset{\widetilde{\mathcal{J}}(S)}{\underrightarrow{\text{holim}}} \mathcal{G}_S^A r_S\Big) \longrightarrow \underset{\widetilde{\mathcal{J}}(S)\wedge\widetilde{\mathcal{J}}(S)}{\underrightarrow{\text{holim}}} \mathcal{G}_S^A r_S \wedge \mathcal{G}_S^A r_S.$$

Theorem 4.5.4. *Let A be a connective commutative* $\mathbf{S}$*-algebra. The multiplication*

$$\mu_S^A : \Lambda_S A \wedge \Lambda_S A \longrightarrow \Lambda_S A$$

is associative and unital.

Proof. Proving associativity and unitality is done by direct checking using the explicit formula for μ^A given in the proof of Lemma 4.5.2, and is left as an exercise. □

It may be shown in several ways that $\Lambda_X A$ is an E_∞-spectrum.

4.6. The Loday functor as a functor of unbased spaces

If X is a finite (unbased) space, then we define

$$\Lambda_X(A) = diag^*\big\{[q] \mapsto \Lambda_{X_q}(A)\big\}.$$

If X is any (unbased) space, then we define the *Loday functor* $\Lambda_X(A)$, also called *smash X of A*, to be the filtered colimit

$$\Lambda_X(A) = \varinjlim \Lambda_S A,$$

where S varies over the finite subspaces of X.

More generally, let F be a functor from the category Fin of finite (unbased) sets to the category of $\mathbf{S}$-modules (Γ-spaces). If X is a finite (unbased) space, then we define $F(X) = diag^*\{[q] \mapsto F(X_q)\}$. If X is any (unbased) space, then we define $F(X) = \varinjlim F(S)$ where S varies over the finite subspaces of X. Note that the following lemma in particular applies to the Loday functor. A map of Γ-spaces $X \to Y$ is a pointwise equivalence (resp. pointwise n-connected) if for all finite pointed sets S the map $X(S) \to Y(S)$ is a weak equivalence (resp. n-connected). This is stronger than claiming that the map of associated spectra is a stable equivalence (resp. stably n-connected).

Lemma 4.6.1. *Let $f : X \to Y$ be a map of simplicial sets and let $f_+ : X_+ \to Y_+$ denote the map obtained by adding a disjoint base point.*

1. *If f is a weak equivalence (resp. n-connected), then the induced map*

$$F(f): F(X) \longrightarrow F(Y)$$

is a pointwise equivalence (resp. n-connected).
2. *If f is a weak equivalence (resp. n-connected) after p-completion, then $F(f)^{\wedge}_{p}$ is a pointwise equivalence (resp. n-connected).*
3. *If E is a spectrum and $E \wedge f_+$ is a stable equivalence, then $E \wedge F(f)$ is a stable equivalence.*

Proof. Let LF denote the functor from simplicial sets to pointed simplicial Γ-spaces with

$$(LF)(X)_k = \bigvee_{S_0,\dots,S_k} F(S_k) \wedge \mathrm{Fin}(S_k, S_{k-1})_+ \wedge \cdots \wedge \mathrm{Fin}(S_1, S_0)_+ \wedge \mathcal{S}(S_0, X)_+.$$

The functor $X \mapsto (LF)(X)$ clearly has the stated properties (for spaces B and C we have equivalences $(C_+ \wedge B_+)^{\wedge}_{p} \simeq (C^{\wedge}_{p+} \wedge B^{\wedge}_{p+})^{\wedge}_{p}$ and $\mathcal{S}(C, B)^{\wedge}_{p} \simeq \mathcal{S}(C, B^{\wedge}_{p})$.) The result now is a consequence of the fact that there is a natural pointwise weak equivalence $(LF)(X) \to F(X)$. □

The above proof also gives the following result.

Lemma 4.6.2. *For every space X the map $F(X)^{\wedge}_{p} \to F(X^{\wedge}_{p})^{\wedge}_{p}$ is a weak equivalence.*

4.6.3. Adams operations

Any endomorphism $X \to X$ gives by functoriality rise to an operation $\Lambda_X(A) \to \Lambda_X(A)$.

Definition 4.6.4. The rth Adams operation on $\Lambda_{T^1}(A)$ is the operation induced by the rth power map $z \mapsto z^r$ on the simplicial group $\mathbb{T}^1 = \sin|S^1|$.

Via the weak equivalence $THH(A) \simeq \Lambda_{\mathbb{T}^1}(A)$, the Adams operations on $\Lambda_{\mathbb{T}^1}$ gives an operation on ordinary topological Hochschild homology corresponding to the operation on Hochschild homology and MacLane homology described by McCarthy [16]. There is a difference in description in that he uses the usual simplicial circle S^1, and so has to subdivide to express the operation, whereas in our situation this can be viewed as a reflection of the fact that (the image of the acyclic cofibration) $sd_r S^1 \subseteq \sin|sd_r S^1| \cong \mathbb{T}^1$ is one of the legitimate finite simplicial subsets over which we perform our colimit.

This is general: for any functor F from Fin to, say, spaces, the rth Adams operation on $\pi_q(F(S^1))$ is given through McCarthy's interpretation as the composite

$$\pi_q\big(F\big(S^1\big)\big) \cong \pi_q\big(F\big(sd_r\big(S^1\big)\big)\big) \longrightarrow \pi_q\big(F\big(S^1\big)\big)$$

where the last map is induced by a certain map $sd_r S^1 \to S^1$. This corresponds to our definition, as can be seen from the commutativity of

$$\begin{array}{ccccc}
\sin|sd_r S^1| & \xrightarrow{\cong} & \sin|S^1| & \xrightarrow{z\mapsto z^r} & \sin|S^1| \\
\uparrow{\scriptstyle\simeq} & & & & \uparrow{\scriptstyle\simeq} \\
sd_r S^1 & & \longrightarrow & & S^1.
\end{array}$$

For higher dimensional tori one gets operations for every integral matrix α with nonzero determinant. In this case Lemma 4.6.2 gives $\Lambda_{\mathbb{T}^n}(A)_p \simeq \Lambda_{\mathbb{T}^n_p}(A)_p$ and so we have an action by $GL_n(\mathbb{Z}_p)$ which in the one-dimensional case corresponds to the action by the p-adic units.

5. The fundamental cofibration sequence

In this section we investigate fixed points of the Loday functor $\Lambda_X A$ with respect to group actions on X. This leads to a generalized version of the fundamental cofibration sequence for Bökstedt's topological Hochschild homology. Along the way we describe the norm map from homotopy orbits to homotopy fixed points.

5.1. The norm cofibration sequence

Let G be a finite group. Recall that, if S is a finite set and A is a commutative $\mathbf{S}$-algebra and $j \in \mathcal{I}^S$ we write $A(j)$ for the space $\bigwedge_{s\in S} A(S^{j(s)})$, so that (cf. Section 4.3) $G^A_S(j) = map_*(\mathbf{S}(j), A(j))$.

Definition 5.1.1. A set $\mathcal{F}$ of subgroups of G is a *closed family* if it has the property that if $H \in \mathcal{F}$ and $H \leqslant gKg^{-1}$, for $g \in G$ and a subgroup K of G, then $K \in \mathcal{F}$.

If $\mathcal{F}$ is a closed family and $\mathcal{F}^c$ is the complement of $\mathcal{F}$ in the set of subgroups of G, then $\mathcal{F}^c$ is a family of subgroups in the sense that it is closed under conjugation and passage to subgroups. Thus closed families are complements of families in the usual sense. If G acts on S and $\mathcal{F}$ is a closed family of subgroups in G, we define

$$G^A_S(\mathcal{F})(j) = map_*\left(\bigcup_{H\in\mathcal{F}} \mathbf{S}(j)^H, A(j)\right).$$

This is a left lax transformation from the weak functor $S \mapsto \mathcal{J}(S) = (\mathcal{I}^S)_+$ to the constant functor from finite G-sets to categories sending everything to $\Gamma\mathcal{S}_*$. Just like when we constructed $\Lambda(A)$, we can define a functor $\Lambda^A(\mathcal{F})$ from the category of G-spaces to the category of spectra, with

$$\Lambda^A_S(\mathcal{F}) = \operatorname*{\underrightarrow{holim}}_{\widetilde{\mathcal{J}}(S)} \mathcal{G}^A_S(\mathcal{F}) \circ r_S$$

when S is a finite G-set. Here $\mathcal{G}^A_S(\mathcal{F}) = G^A_S \circ p_2$. In particular, if $\mathcal{F}$ is the empty family of subgroups of G, then $\Lambda^A_S(\mathcal{F}) = *$.

Lemma 5.1.2. *Let N be a normal subgroup in G and let $\mathcal{F}$ be the closed family of subgroups of G consisting of the subgroups containing N. For $j \in \mathcal{I}^S$, we have an equality*

$$G^A_S(\mathcal{F})(j)^N = map_*\big(\mathbf{S}(j)^N, A(j)^N\big)$$

of G/N-spaces. In particular, if $\mathcal{F}$ is the family of all subgroups of G, then $\Lambda^A_S(\mathcal{F}) = \Lambda_S(A)$.

Note that if $\mathcal{G} \subseteq \mathcal{F}$ is an inclusion of closed families of subgroups of G, then the inclusion

$$\bigcup_{H \in \mathcal{G}} \mathbf{S}(j)^H \subseteq \bigcup_{H \in \mathcal{F}} \mathbf{S}(j)^H$$

induces a G-map

$$\mathrm{res}\colon \Lambda_S^A(\mathcal{F}) \longrightarrow \Lambda_S^A(\mathcal{G}).$$

We refer to this map as the map induced by the inclusion of $\mathcal{G}$ in $\mathcal{F}$. If $\mathcal{F}$ is the family of all subgroups in G, then by Lemma 5.1.2 the map induced by the inclusion of $\mathcal{G}$ in $\mathcal{F}$ takes the form

$$\mathrm{res}\colon \Lambda_S(A) \longrightarrow \Lambda_S^A(\mathcal{G}).$$

If the complement of $\mathcal{G}$ in $\mathcal{F}$ is equal to the conjugacy class of a subgroup K in G we shall say that $\mathcal{F}$ and $\mathcal{G}$ are *K-adjacent*. Moreover we let $W_G K = N_G K/K$ denote the *Weyl group* of the subgroup K in G. Here $N_G K$ is the *normalizer* of K in G consisting of those g in G with $gKg^{-1} = K$.

In the next lemma we shall describe the homotopy fiber of the map induced by the inclusion of a K-adjacent family. In the proof we will need a "norm map" $Z_{hG} \to Z^{hG}$ for Z a Γ-space with action of G. In reality the norm is a "weak natural map" in the sense that it consists of a chain of natural maps, where the ones pointing in the "wrong" direction are stable equivalences. Later we shall relate the norm map to a transfer map, and therefore we need to choose a specific representative of it. The discussion below is modeled on Weiss and Williams' paper [20, Section 2]. Given $n \geqslant 0$ we let S^{nG} denote the G-fold smash product of the n-sphere. This is our model for the one-point compactification of the regular representation of G.

The map

$$\alpha\colon G_+ \wedge S^{nG} \longrightarrow Map_*\big(G_+, S^{nG}\big) \tag{5.1.1}$$

with $\alpha(g,x)(g) = x$ and with $\alpha(g,x)(h)$ equal to the base point in S^{nG} if $h \neq g$, and the diagonal inclusion $S^{nG} \to Map_*(G_+, S^{nG})$ induce G-maps

$$\begin{aligned} map_*\big(S^{nG}, Z\big(S^{nG}\big)\big) \;&\cong\; map_*\big(G_+ \wedge S^{nG}, Z\big(S^{nG}\big)\big)^G \\ &\longleftarrow map_*\big(Map_*\big(G_+, S^{nG}\big), Z\big(S^{nG}\big)\big)^G \\ &\longrightarrow map_*\big(S^{nG}, Z\big(S^{nG}\big)\big)^G. \end{aligned}$$

Here G acts by conjugation on the first space, through the left action of G on itself on the second space and through the right action of G on itself on the third space. On the last space G acts trivially. Passing to homotopy colimits we obtain a weak G-map

$$\begin{aligned} \operatorname*{holim}_{\vec{n}} map_*\big(S^{nG}, Z\big(S^{nG}\big)\big) &\xleftarrow{\simeq} \operatorname*{holim}_{\vec{n}} map_*\big(Map_*\big(G_+, S^{nG}\big), Z\big(S^{nG}\big)\big)^G \\ &\longrightarrow \operatorname*{holim}_{\vec{n}} map_*\big(S^{nG}, Z\big(S^{nG}\big)\big)^G \end{aligned}$$

of Γ-spaces. This is an additive transfer associated to G. Observing that there is a chain of stable equivalences between Z and $\operatorname{holim}_{\vec{n}} Map_*(S^{nG}, Z(S^{nG} \wedge -))$ we denote the resulting map in the homotopy category of (naïve) G–Γ-spaces by $V^G : Z \to \operatorname{holim}_{\vec{n}} Map_*(S^{nG}, Z(S^{nG} \wedge -))^G$. The homotopy category we have in mind is the one where we invert the G-maps whose underlying non-equivariant maps are stable equivalences. Since G acts trivially on the target, there is an induced map $\widehat{V}^G_Z : Z_{hG} \to \operatorname{holim}_{\vec{n}} Map_*(S^{nG}, Z(S^{nG} \wedge -))^G$. The *norm map* is the weak map $N : Z_{hG} \to Z^{hG}$ given by the composition

$$\begin{aligned} Z_{hG} \quad &\longrightarrow \quad \big(Map_*(EG_+, Z)\big)_{hG} \\ &\xrightarrow{\widehat{V}^G_{Map_*(EG_+,Z)}} \operatorname*{holim}_{\vec{n}} map_*\big(S^{nG}, Map_*\big(EG_+, Z\big(S^{nG} \wedge -\big)\big)\big)^G \simeq Z^{hG}. \end{aligned}$$

It is an easy matter to check that the composite homomorphism

$$\pi_* Z \longrightarrow \pi_* Z_{hG} \xrightarrow{N} \pi_* Z^{hG} \longrightarrow \pi_* Z$$

is multiplication by $\sum_{g \in G} g \in \pi_0(G_+ \wedge \mathbf{S})$. If Y is a Γ-space with trivial action of G, then there is a commutative diagram of the form

$$\begin{array}{ccccccc} \pi_*(G_+ \wedge Y) & \longrightarrow & \pi_*(G_+ \wedge Y)_{hG} & \xrightarrow{N} & \pi_*(G_+ \wedge Y)^{hG} & \longrightarrow & \pi_*(G_+ \wedge Y) \\ \cong\downarrow & & \cong\uparrow & & \cong\downarrow & & \cong\uparrow \\ \bigoplus_G \pi_* Y & \xrightarrow{\nabla} & \pi_* Y & & \pi_* Y & \xrightarrow{\Delta} & \bigoplus_G \pi_* Y, \end{array}$$

where $\nabla(\{g \mapsto y_g\}) = \sum_g y_g$ and $\Delta(y) = \{g \mapsto y\}$. Since the composition $\bigoplus_G \pi_* Y \to \bigoplus_G \pi_* Y$ from the lower left corner to the lower right corner is the norm map, the composition $\pi_* Y \to \pi_* Y$ must be the identity on $\pi_* Y$. Thus the norm map $N : (G_+ \wedge Y)_{hG} \to (G_+ \wedge Y)^{hG}$ is a stable equivalence. An easy inductive argument on cells shows that the norm map $N : Z_{hG} \to Z^{hG}$ is a weak equivalence when Z is of the form $Z = Map_*(U, X)$ for X a G–Γ-space and U a finite free G-space.

Lemma 5.1.3 *(The norm cofiber sequence). Let G be a finite group, let $\mathcal{G} \subseteq \mathcal{F}$ be K-adjacent families of subgroups of G and let X be a non-empty free G-space. For every commutative* **S**-*algebra A the homotopy fiber of the map*

$$\operatorname{res} : \big[\Lambda^A_X(\mathcal{F})\big]^G \longrightarrow \big[\Lambda^A_X(\mathcal{G})\big]^G$$

induced by the inclusion of $\mathcal{G}$ in $\mathcal{F}$ is equivalent to the homotopy orbit spectrum

$$[\Lambda_{X/K} A]_{hW_G K}.$$

Proof. Since we work stably it suffices to prove the result for every discrete set $X = S$. Moreover, since fixed points, finite homotopy limits and homotopy orbits commute with filtered colimits, it suffices to consider the case where S is a finite set. Let $(\phi, i) \in [\widetilde{\mathcal{J}}(S)]^G$, write $j = r_S(\phi, i) = \mathcal{J}(\phi)i \in \mathcal{J}(S) = (\mathcal{I}^S)_+$ and consider the map

$$map_*\Big(\bigcup_{H\in\mathcal{F}} \big(\mathbf{S}(j)\big)^H, A(j)\Big)^G \longrightarrow map_*\Big(\bigcup_{H\in\mathcal{G}} \big(\mathbf{S}(j)\big)^H, A(j)\Big)^G$$

induced by the inclusion of $\mathcal{G}$ in $\mathcal{F}$. The fiber of this fibration is

$$map_*\big(Z(j), A(j)\big)^G$$

where

$$Z(j) = \Big(\bigcup_{H\in\mathcal{F}} \big(\mathbf{S}(j)\big)^H\Big) \Big/ \Big(\bigcup_{H\in\mathcal{G}} \big(\mathbf{S}(j)\big)^H\Big).$$

Apart from the base point, the G-orbits of $Z(j)$ are all isomorphic to G/K. Therefore we have isomorphisms

$$\begin{aligned} map_*\big(Z(j), A(j)\big)^G &\cong map_*\big(Z(j)^K, A(j)\big)^{N_G K} \\ &\cong map_*\big(Z(j)^K, A(j)^K\big)^{W_G K}. \end{aligned}$$

Since $Z(j)^K$ is a free $W_G K$-space, the map

$$map_*\big(Z(j)^K, A(j)^K\big)^{hW_G K} \longrightarrow map_*\big(Z(j)^K, A(j)^K\big)^{W_G K}$$

is an equivalence. Since $W_G K$ is a finite group and $Z(j)^K$ is a finite free $W_G K$-space, the norm map

$$N : map_*\big(Z(j)^K, A(j)^K\big)_{hW_G K} \longrightarrow map_*\big(Z(j)^K, A(j)^K\big)^{hW_G K}$$

is a stable equivalence, so we can deduce that

$$\underset{[\widetilde{\mathcal{J}}(S)]^G}{\underrightarrow{\operatorname{holim}}}\, map_*(Z \circ r_S, A \circ r_S)^G$$

is equivalent to

$$\underset{[\widetilde{\mathcal{J}}(S)]^G}{\underrightarrow{\operatorname{holim}}}\, map_*\big([Z \circ r_S]^K, [A \circ r_S]^K\big)_{hW_G K}.$$

Note that $H \in \mathcal{G}$ implies that K is a proper subgroup of the subgroup $H \cdot K$ of G generated by H and K. Since K acts freely on S, the space

$$\Big[\bigcup_{H\in\mathcal{G}} \mathbf{S}(j)^H\Big]^K = \bigcup_{H\in\mathcal{G}} \mathbf{S}(j)^{H\cdot K}$$

is at most $\sum_{s\in S} j(s)/2|K|$-dimensional and $S^k \wedge A(j)^K$ is $k-1+\sum_{s\in S} j(s)/|K|$-connected. Using that K-fixed points commute with quotients by sub-K-spaces we can conclude that the map

$$Map_*\big(Z(j)^K, S^k \wedge A(j)^K\big)_{hW_GK} \longrightarrow Map_*\big(\mathbf{S}(j)^K, S^k \wedge A(j)^K\big)_{hW_GK}$$

is $k-1+\sum_{s\in S} j(s)/2|K|$-connected.

These considerations are functorial in (ϕ, i), and since X is non-empty, together with Lemma 4.4.3 and Bökstedt's lemma (see e.g. [3, Lemma 2.5.1]) they give that the homotopy fiber of the map $[Z_S^A(\mathcal{F})]^G \to [Z_S^A(\mathcal{G})]^G$ is equivalent to

$$\underset{[\widetilde{\mathcal{J}}(S)]^G}{\underrightarrow{\operatorname{holim}}}\, map_*(Z\circ r_S, A\circ r_S)^G \simeq \underset{[\widetilde{\mathcal{J}}(S)]^G}{\underrightarrow{\operatorname{holim}}}\, map_*\big([\mathbf{S}\circ r_S]^K, [A\circ r_S]^K\big)_{hW_GK}.$$

The G-maps

$$\begin{aligned}\underset{[\widetilde{\mathcal{J}}(S)]^G}{\underrightarrow{\operatorname{holim}}}\, map_*\big([\mathbf{S}\circ r_S]^K, [A\circ r_S]^K\big) &\longrightarrow \underset{[\widetilde{\mathcal{J}}(S)]^K}{\underrightarrow{\operatorname{holim}}}\, map_*\big([\mathbf{S}\circ r_S]^K, [A\circ r_S]^K\big)\\ &\longrightarrow \underset{\widetilde{\mathcal{J}}(S/K)}{\underrightarrow{\operatorname{holim}}}\, map_*(\mathbf{S}\circ r_S, A\circ r_S)\end{aligned}$$

induced by the inclusion $[\widetilde{\mathcal{J}}(S)]^G \subseteq [\widetilde{\mathcal{J}}(S)]^K$ and the isomorphism $[\widetilde{\mathcal{J}}(S)]^K \cong \widetilde{\mathcal{J}}(S/K)$ are equivalences by Lemma 4.4.3, and hence we get that the homotopy fiber is equivalent to

$$[\Lambda_{S/K}A]_{hW_GK}. \qquad \square$$

Corollary 5.1.4. *Let G be a finite group and let X be a non-empty free G-space. For every closed family $\mathcal{F}$ of subgroups of G the functor $A \mapsto [\Lambda_X^A(\mathcal{F})]^G$ preserves connectivity of maps of commutative $\mathbf{S}$-algebras and has values in very special Γ-spaces. Furthermore we have a natural equivalence $([\Lambda_X^A(\mathcal{F})]^G)\hat{}_p \simeq [\Lambda_X^A(\mathcal{F})\hat{}_p]^G$.*

Proof. We make induction on the partially ordered set of closed families ordered by inclusion. If $\mathcal{F}$ is empty, then $[\Lambda_X(\mathcal{F})]^G$ is contractible, and thus the result holds. Otherwise we may choose a minimal subgroup K in $\mathcal{F}$ and a closed family $\mathcal{G} \subseteq \mathcal{F}$ of subgroups of G so that $\mathcal{G}$ is K-adjacent to $\mathcal{F}$. By Lemma 5.1.3 there is a cofibration sequence

$$\big[\Lambda_{X/K}(A)\big]_{hW_GK} \longrightarrow \big[\Lambda_X^A(\mathcal{F})\big]^G \longrightarrow \big[\Lambda_X^A(\mathcal{G})\big]^G.$$

By Lemma 4.4.4 the functor $A \mapsto [\Lambda_{X/K}(A)]$ preserves connectivity, and thus also the functor $A \mapsto [\Lambda_{X/K}(A)]_{hW_GK}$ preserves connectivity. Together with the inductive assumption and the five lemma, this implies that the functor $A \mapsto [\Lambda_X^A(\mathcal{F})]^G$ preserves connectivity. Since completion commutes with homotopy orbits the second statement is proved in the same way. $\square$

Corollary 5.1.5. *Let $\mathcal{G} \subseteq \mathcal{F}$ be closed families of subgroups of a finite group G. For every commutative $\mathbf{S}$-algebra A and non-empty free G-space X the map $[\Lambda_X^A(\mathcal{F})]^G \to [\Lambda_X^A(\mathcal{G})]^G$ induced by the inclusion of $\mathcal{G}$ in $\mathcal{F}$ is 0-connected.*

Proof. Choose adjacent closed families $\mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_n$ of subgroups of G with $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_n = \mathcal{F}$ and apply Lemma 5.1.3 □

5.2. The restriction map

Given a normal subgroup N of G we let $p_S^N : S \to S/N$ be the projection and we let $\mathcal{F}$ denote the closed family consisting of all subgroups of G containing N. Notice that Corollary 3.1.3 implies that the induced functor $(p_S^N)^* : \widetilde{\mathcal{I}}(S/N) \to \widetilde{\mathcal{I}}(S)$ induces an isomorphism $(p_S^N)^* : \widetilde{\mathcal{I}}(S/N) \xrightarrow{\cong} \widetilde{\mathcal{I}}(S)^N$.

Lemma 5.2.1. *Let N be a normal subgroup of a finite group G and let S be a finite G-set. For every commutative* $\mathbf{S}$*-algebra A the G/N-spaces $[\Lambda_S^A(\mathcal{F})]^N$ and*

$$\underset{\widetilde{\mathcal{J}}(S/N)}{\underrightarrow{\operatorname{holim}}} \big[\mathcal{G}_S^A(\mathcal{F}) \circ r_S \circ \big(p_S^N\big)^*\big]^N$$

are naturally isomorphic.

Proof. The G/N-space $[\Lambda_S^A(\mathcal{F})]^N$ is defined to be the N-fix points of the homotopy colimit $\operatorname{holim}_{\overrightarrow{\widetilde{\mathcal{J}}(S)}} \mathcal{G}_S^A(\mathcal{F}) \circ r_S$. Homotopy colimits and fixed points commute in the sense that this N-fix point space is isomorphic to the homotopy colimit $\operatorname{holim}_{\overrightarrow{\widetilde{\mathcal{J}}(S)^N}} [\mathcal{G}_S^A(\mathcal{F}) \circ r_S]^N$. The result now follows from the fact that $(p_S^N)^* : \widetilde{\mathcal{J}}(S/N) \xrightarrow{\cong} \widetilde{\mathcal{J}}(S)^N$ is an isomorphism. □

Let $J \in \widetilde{\mathcal{J}}(S/N)^e$ and let $j = (p_2 \circ r_S \circ (p_S^N)^*)J$. The isomorphisms $\mathbf{S}(j)^N \cong (\mathbf{S} \circ p_2 \circ r_{S/N})J$ and $A(j)^N \cong (A \circ p_2 \circ r_{S/N})J$ induce a natural isomorphism of the form

$$\begin{aligned}
\big[\big(\mathcal{G}_S^A(\mathcal{F}) \circ r_S \circ \big(p_S^N\big)^*\big)(J)\big]^G &= map_*\Big(\bigcup_{N \leqslant H} \mathbf{S}(j)^H, A(j)\Big)^G \\
&= map_*\Big(\bigcup_{N \leqslant H} \mathbf{S}(j)^H, \bigcup_{N \leqslant H} A(j)^H\Big)^G \\
&= map_*\big(\mathbf{S}(j)^N, A(j)^N\big)^{G/N} \\
&\cong \big[\big(\mathcal{G}_{S/N}^A \circ r_{S/N}\big)(J)\big]^{G/N}.
\end{aligned}$$

Thus the inclusion of $\mathcal{F}$ in the family of all subgroups of G induces a natural transformation of the form

$$\big[\mathcal{G}_S^A \circ r_S \circ \big(p_S^N\big)^*\big]^G \implies \big[\mathcal{G}_S^A(\mathcal{F}) \circ r_S \circ \big(p_S^N\big)^*\big]^G \cong \big[\mathcal{G}_{S/N}^A \circ r_{S/N}\big]^{G/N} \tag{5.2.1}$$

of functors $\widetilde{\mathcal{J}}(S/N) \to \Gamma\mathcal{S}_*$ given on J by restricting to N-fixed points. This induces a modification, and hence a natural map

$$R_N^G : \Big[\underset{\widetilde{\mathcal{J}}(S)}{\underrightarrow{\operatorname{holim}}} \big[\mathcal{G}_S^A \circ r_S\big]\Big]^G \longrightarrow \Big[\underset{\widetilde{\mathcal{J}}(S/N)}{\underrightarrow{\operatorname{holim}}} \mathcal{G}_{S/N}^A \circ r_{S/N}\Big]^{G/N},$$

that is, a map

$$R_N^G : \left[\Lambda_S(A)\right]^G \longrightarrow \left[\Lambda_{S/N}(A)\right]^{G/N}.$$

If G is a finite group and X is a simplicial G-set, let $\mathcal{F}_G^X$ be the filtered category of finite G-subspaces of X and inclusions. Notice that $\mathcal{F}_G^X \subseteq F_{\{1\}}^X$ is right cofinal, and so colimits over F_G^X and $F_{\{1\}}^X$ are isomorphic, and the isomorphism is G-equivariant.

Definition 5.2.2. Let A be a commutative **S**-algebra, G a finite group acting on a simplicial set X and N a normal subgroup of G. The *restriction map*

$$R_N^G : \left[\Lambda_X(A)\right]^G \longrightarrow \left[\Lambda_{X/N}(A)\right]^{G/N}$$

is the composite

$$\begin{aligned} \left[\Lambda_X(A)\right]^G &\cong \varinjlim_{S\in\mathcal{F}_G^X} \left[\Lambda_S(A)\right]^G \\ &\longrightarrow \varinjlim_{S\in\mathcal{F}_G^X} \left[\Lambda_{S/N}(A)\right]^{G/N} \cong \varinjlim_{U\in\mathcal{F}_{G/N}^{X/N}} \left[\Lambda_U(A)\right]^{G/N} \cong \left[\Lambda_{X/N}(A)\right]^{G/N}. \end{aligned}$$

Here the first isomorphism is given by right cofinality of $\mathcal{F}_G^X$ in $\mathcal{F}_{\{1\}}^X$. The arrow is induced by the maps R_N^G discussed above. The next isomorphism is due to the right cofinality of the functor $S \mapsto S/N$, and the last isomorphism is due to the fact that G/N-fixed points commute with filtered colimits.

The restriction map R_N^G is natural in the commutative **S**-algebra A and in the G-space X. Moreover, Corollary 5.1.5 implies that R_N^G is 0-connected.

Definition 5.2.3. Let A be a commutative **S**-algebra, G a finite group acting on a simplicial set X and N a subgroup of G. The *Frobenius map*

$$F_N^G : \left[\Lambda_X(A)\right]^G \subseteq \left[\Lambda_X(A)\right]^N$$

is the inclusion of fixed points.

The following lemma is direct from the definition, but is important for future reference:

Lemma 5.2.4. *Let A be a commutative* **S***-algebra, X a G-space with G a finite group, and let $H \subseteq N \subseteq G$ be normal subgroups. Then*

$$R_N^G = R_{N/H}^{G/H} R_H^G,$$

and the diagram

$$\begin{array}{ccc}
[\Lambda_X(A)]^G & \xrightarrow{R^G_H} & [\Lambda_{X/H}(A)]^{G/H} \\
{\scriptstyle F^G_N}\downarrow & & \downarrow{\scriptstyle F^{G/H}_{N/H}} \\
[\Lambda_X(A)]^N & \xrightarrow{R^N_H} & [\Lambda_{X/H}(A)]^{N/H}
\end{array}$$

commutes.

Most importantly for our applications, adapting techniques from tom Dieck [6], we get the *fundamental cofiber sequence*:

Lemma 5.2.5. *Let G be a finite abelian group, X a non-empty free G-space and A a commutative* $\mathbf{S}$*-algebra. The homotopy fiber of the map*

$$\big[\Lambda_X(A)\big]^G \longrightarrow \operatorname*{holim}_{\overleftarrow{0\neq H\leqslant G}} \big[\Lambda_{X/H}(A)\big]^{G/H}$$

induced by the restriction maps is connected by a chain of natural maps that are stable equivalences to the homotopy orbit spectrum $[\Lambda_S(A)]_{hG}$.

Proof. Let $\mathcal{F} = \{H\colon\ 0 \neq H \leqslant G\}$. In order to apply Lemma 5.1.3 it suffices to show that

$$\big[\Lambda_X(\mathcal{F})\big]^G \cong \operatorname*{holim}_{\overrightarrow{[\widetilde{\mathcal{J}}(S)]^G}} map_*\bigg(\bigcup_{0\neq H\leqslant G} (\mathbf{S}\circ r_S)^H, A\circ r_S\bigg)^G$$

is equivalent to

$$\Big[\operatorname*{holim}_{\overleftarrow{0\neq H\leqslant G}} \Lambda_{S/H}(A)\Big]^G = \operatorname*{holim}_{\overleftarrow{0\neq H\leqslant G}} \big[\Lambda_{X/H}(A)\big]^{G/H}.$$

First we notice that the natural map

$$\begin{aligned}
&\operatorname*{holim}_{\overrightarrow{[\widetilde{\mathcal{J}}(S)]^G}} map_*\bigg(\bigcup_{0\neq H\leqslant G} (\mathbf{S}\circ r_S)^H, A\circ r_S\bigg)^G \\
&\quad\longrightarrow \Big[\operatorname*{holim}_{\overleftarrow{0\neq H\leqslant G}} \operatorname*{holim}_{\overrightarrow{[\widetilde{\mathcal{J}}(S)]^G}} map_*\big((\mathbf{S}\circ r_S)^H, A\circ r_S\big)\Big]^G \\
&\quad\cong \Big[\operatorname*{holim}_{\overleftarrow{0\neq H\leqslant G}} \operatorname*{holim}_{\overrightarrow{[\widetilde{\mathcal{J}}(S)]^G}} map_*\big((\mathbf{S}\circ r_S)^H, (A\circ r_S)^H\big)\Big]^G
\end{aligned}$$

is an equivalence (again by Lemma 4.4.3, and since G is finite). The last term is isomorphic to

$$\Big[\operatorname*{holim}_{\overleftarrow{0\neq H\leqslant G}} \operatorname*{holim}_{\overrightarrow{[\widetilde{\mathcal{J}}(S/G)]}} \mathcal{G}^A_{S/H}\circ r_{S/H}\circ \big(p^G_H\big)^*\Big]^G$$

which again is isomorphic to

$$\Big[\underset{0\neq H\leqslant G}{\operatorname{holim}} \Lambda_{S/H}(A)\Big]^G.$$

It is now clear that the induced maps are the restriction maps. □

6. The Burnside–Witt construction

Hesselholt and Madsen prove in [10] that if A is a discrete commutative ring and $TR(A)$ is the homotopy inverse limit over the restriction maps between the fixed points of the (one-dimensional) topological Hochschild homology under finite subgroups of the circle, then $\pi_0 TR(A)$ is isomorphic to the ring of Witt vectors over A.

In this section we prove an analogous result for arbitrary finite groups. Let G be a finite group, let X be a connected free G-space and let A be a commutative $\mathbf{S}$-algebra A. There is a canonical isomorphism of the form

$$\pi_0 \Lambda_X(A)^G \cong \mathbb{W}_G(\pi_0 A),$$

where the latter ring is Dress and Siebeneicher's "Burnside–Witt"-ring [7] of the commutative ring $\pi_0 A$ over the group G.

6.1. The Burnside–Witt ring

We shall review some elementary facts about the G-typical Burnside–Witt ring $\mathbb{W}_G(B)$ of a commutative ring B. For more details on the Burnside–Witt construction, the reader may consult for instance [9] and [4] in addition to [7]. The underlying set of $\mathbb{W}_G(B)$ is the set

$$\Big[\prod_{H\leqslant G} B\Big]^G$$

where G acts on the product by taking the factor of B corresponding to a subgroup H of G identically to the factor corresponding to gHg^{-1}. For every subgroup $K\leqslant G$ there is a ring homomorphism $\phi_K : \mathbb{W}_G(B)\to B$ taking $x=(x_H)_{H\leqslant G}$ to

$$\phi_K(x)=\sum_{[H]}\big|(G/H)^K\big|x_H^{|H|/|K|},$$

where the sum runs over the conjugacy classes of subgroups of G. Note that the factor $|(G/H)^K|$ is zero unless K is subconjugate to H (i.e., K is conjugate to a subgroup of H). The endofunctor $B\mapsto \mathbb{W}_G(B)$ on the category of commutative rings is uniquely determined by the following facts: Firstly, the underlying set of $\mathbb{W}_G(B)=[\prod B]^G$ is functorial in B and secondly, ϕ_K is a natural ring homomorphism for every subgroup K of G. Dress and Siebeneicher established the existence of the ring structure on $\mathbb{W}_G(B)$ by making a detailed study of the combinatorics of finite G-sets. Below we use the functor Λ to give a different proof of the existence of the ring structure on $\mathbb{W}_G(B)$.

Note that when the underlying additive group of B is torsion free the map

$$\phi : \mathbb{W}_G(B) \longrightarrow \Big[\prod_{H \leqslant G} B\Big]^G$$

with $\phi(x)_K = \phi_K(x)$ is injective. For every closed family $\mathcal{F}$ of subgroups of G we define $\mathbb{W}_{\mathcal{F}}(B)$ to be the set

$$\mathbb{W}_{\mathcal{F}}(B) = \Big[\prod_{H \in \mathcal{F}} B\Big]^G.$$

6.2. The Teichmüller map

We shall show by induction on the size of $\mathcal{F}$, that for every connected free G-space X there is a bijection $\mathbb{W}_{\mathcal{F}}(\pi_0 A) \to \pi_0 \Lambda_X^A(\mathcal{F})^G$. In order to do this we need to recollect some structure on $\Lambda_X^A(\mathcal{F})^G$.

Suppose that S is a finite free G-set, let K be a subgroup of G, let $j \in [\mathcal{I}^S]^G$ and let L be a pointed space. Consider the maps

$$\begin{aligned} Map_*\Big(\bigcup_{H\in\mathcal{F}} \mathbf{S}(j)^H, A(j)\wedge L\Big)^K &\cong Map_*\Big(G_+ \wedge_K \bigcup_{H\in\mathcal{F}} \mathbf{S}(j)^H, A(j)\wedge L\Big)^G \\ &\longleftarrow Map_*\Big(Map_*\Big(G_+, \bigcup_{H\in\mathcal{F}} \mathbf{S}(j)^H\Big)^K, A(j)\wedge L\Big)^G \\ &\longrightarrow Map_*\Big(\bigcup_{H\in\mathcal{F}} \mathbf{S}(j)^H, A(j)\wedge L\Big)^G, \end{aligned}$$

where the isomorphism is given by the adjunction between restriction and induction, the map pointing left is induced by the G-map $\alpha : G_+ \wedge Z \to Map_*(G_+, Z)$ introduced in (5.1.1) with $Z = \bigcup_{H\in\mathcal{F}} \mathbf{S}(j)^H$ and the map pointing right is induced by the diagonal inclusion $Z \to Map_*(G_+, Z)$. Taking homotopy colimits over the above maps we obtain a weak map V_K^G of the form $\Lambda_S^A(\mathcal{F})^H \xleftarrow{\simeq} \widetilde{\Lambda_S^A}(\mathcal{F})^G \to \Lambda_S(A)^G$, where $\widetilde{\Lambda_S^A}(\mathcal{F})^G(L)$ is the homotopy colimit of the middle term in the diagram displayed above. Extending from finite sets S to G-spaces X we denote the induced homomorphism on π_0 by

$$V_K^G : \pi_0 \Lambda_X^A(\mathcal{F})^K \longrightarrow \pi_0 \Lambda_X^A(\mathcal{F})^G,$$

and call it the *additive transfer*. Clearly, if $\mathcal{G} \subseteq \mathcal{F}$ is an inclusion of closed families of subgroups of G, then the induced maps defined in Section 5.1 commute with additive transfers:

$$\begin{array}{ccc} \pi_0 \Lambda_X^A(\mathcal{F})^K & \xrightarrow{V_K^G} & \pi_0 \Lambda_X^A(\mathcal{F})^G \\ {\scriptstyle \text{res}}\downarrow & & \downarrow{\scriptstyle \text{res}} \\ \pi_0 \Lambda_X^A(\mathcal{G})^K & \xrightarrow{V_K^G} & \pi_0 \Lambda_X^A(\mathcal{G})^G. \end{array}$$

The homomorphism V_K^G is the additive transfer associated to the inclusion $K \leqslant G$.

We also need a kind of multiplicative transfer. Given a pointed space L, we can consider the H-fold smash power $L^{\wedge H}$, where H acts by permuting the smash factors. Let S be a finite set, let $p_H : H \times S \to S$ be the projection onto the second factor, and let $P_S = \mathcal{J}(p_H^*) : \mathcal{J}(S) \to \mathcal{J}_H(S) = \mathcal{J}(H \times S)$ be the functor sending $i \in \mathcal{J}(S)$ to $\mathcal{J}(p_H^*)i = ip_H$. Varying S, we get a right lax transformation $P : \mathcal{J} \to \mathcal{J}_H = \mathcal{J} \circ (H \times -)$. Strictifying, as described in Section 4.4, we get a natural transformation $\widetilde{\mathcal{J}} \to \widetilde{\mathcal{J}_H}$ sending (ϕ, i) to (ϕ, ip_H). If we compose with the natural transformation $\widetilde{\mathcal{J}_H} \to \widetilde{\mathcal{J}} \circ (H \times -)$ sending (ϕ, j) to $(\mathrm{id}_H \times \phi, j)$ we get a natural transformation $\widetilde{\mathcal{J}} \to \widetilde{\mathcal{J}} \circ (H \times -)$ which, by abuse of notation, we call $\widetilde{P}$ with $\widetilde{P}_S(\phi, i) = (\mathrm{id}_H \times \phi, ip_H)$. The structure on $\mathcal{J}$ gives an invertible modification $Pe \cong e\widetilde{P}$ (both compare with the left lax transformation sending (ϕ, i) to $\mathcal{J}(p_H^* \phi)i$).

The map $G_S^A(L) \Rightarrow G_{H\times S}^A(L^{\wedge H})P_S$ taking the H-fold smash power of a map gives a modification as S varies, and so

$$G_S^A(L)p_2r_S \quad \Longrightarrow \quad G_{H\times S}^A\big(L^{\wedge H}\big)P_S p_2 r_S = G_{H\times S}^A\big(L^{\wedge H}\big)p_2 P_S^e r_S \cong G_{H\times S}^A\big(L^{\wedge H}\big)p_2 r_S \tilde{P}_S^e$$

defines a modification as S varies. Taking homotopy colimits, we get the first map in

$$\operatorname*{\underrightarrow{holim}}_{\widetilde{\mathcal{J}}(S)} G_S^A(L)p_S r_S \longrightarrow \operatorname*{\underrightarrow{holim}}_{\widetilde{\mathcal{J}}(S)} G_{H\times S}^A\big(L^{\wedge H}\big)p_2 r_S \tilde{P}_S^e \longrightarrow \operatorname*{\underrightarrow{holim}}_{\widetilde{\mathcal{J}}(H\times S)} G_{H\times S}^A\big(L^{\wedge H}\big)p_2 r_S;$$

the second map comes from the functoriality of the homotopy colimit. We note that this composite lands in the H-fixed points, and so defines a natural map

$$\Lambda_S(A)(L) \longrightarrow \big[\Lambda_{H\times S} A\big(L^{\wedge H}\big)\big]^H.$$

Thus, if S comes with an H-action $\mu_H : H \times S \to S$ there is a map

$$\Lambda_S(A)(L) \longrightarrow \big[\Lambda_{H\times S}(A)\big(L^{\wedge H}\big)\big]^H \longrightarrow \big[\Lambda_S(A)\big(L^{\wedge H}\big)\big]^H,$$

where the last map is induced by μ^H, which is an H-map since the action of H on $H \times S$ is on the first factor only. When $L = S^0$ we denote the induced map on π_0 by

$$\Delta_H : \pi_0 \Lambda_S(A) \longrightarrow \pi_0 \Lambda_S(A)^H,$$

and call it the *multiplicative transfer*. The following lemma summarizes the properties of the additive and multiplicative transfers that we shall need. Recall that $F_H^G : \Lambda_S(A)^G \to \Lambda_S(A)^H$ is the inclusion of fixed points.

Lemma 6.2.1. *Let H and K be subgroups of G and let X be a free G-space. Let $\phi_{K,H}: \pi_0\Lambda_X(A) \to \pi_0\Lambda_X(A)$ be the map defined by $\phi_{K,H}(x) = |(G/H)^K| x^{|H|/|K|}$ and let q be the quotient map $q: X \to X/K$. The following diagram commutes:*

$$\begin{array}{ccccc} \pi_0\Lambda_X(A) & \xrightarrow{\phi_{K,H}} & \pi_0\Lambda_X(A) & \xrightarrow{\pi_0\Lambda_q(A)} & \pi_0\Lambda_{X/K}(A) \\ \downarrow{\scriptstyle \Delta_H} & & & & \uparrow{\scriptstyle R^G_K} \\ \pi_0\Lambda_X(A)^H & \xrightarrow{V^G_H} & \pi_0\Lambda_X(A)^G & \xrightarrow{F^G_K} & \pi_0\Lambda_X(A)^K. \end{array}$$

Proof. First, note that $X \mapsto \pi_0\Lambda_X(A)^G$ can be "calculated degreewise" to see that it suffices to consider the case where X is a finite free G-set.

Let $j = (\phi, i) \in \widetilde{\mathcal{J}}(X)$ and let α in $Map_*(\mathbf{S}(e(j)), A(e(j)))$ represent an element of $\pi_0\Lambda_X(A)$.

Let $\mu^H : H \times X \to X$ be the action map and let $p_H : H \times X \to X$ be the projection. We consider $H \times X$ as an H-set with H acting only on the first factor, so that μ^H is an H-map, but p_H is not. Working out the definition of Δ_H, we get that the image of α lies in the $j' = (\mu^H_*(\mathrm{id}_H \times \phi), p^*_H i)$ summand, and (under natural isomorphism $\mathcal{J}(\mu^H_*)\mathcal{J}(p^*_H)\mathcal{J}(\phi)i \cong ej'$) is given by the composite

$$\mathbf{S}\big(e\mu^H_* p^*_H j\big) \xleftarrow{\cong} \mathbf{S}\big(ep^*_H j\big) \xrightarrow{\alpha^{\wedge H}} A\big(ep^*_H j\big) \longrightarrow A\big(e\mu^H_* p^*_H j\big),$$

where outer maps are induced by the multiplication in the $\mathbf{S}$-algebras $\mathbf{S}$ and A respectively.

The map $\Delta_H(\alpha)$ is an H-map. In order to apply V^G_H to this element of $\Lambda_X(A)^H$ we need to represent it by a stabilization of the form $\mathbf{S}(i) \to A(i)$, where i is an element of $[I^X]^G$. We let $\mu : G \times X \to X$ be the action map and we let $p : G \times X \to X$ be the projection. We consider $G \times X$ as a G-set with G acting only on the first factor.

The inclusion $H \subseteq G$ induces a map $f : (\mu^H_*(\mathrm{id}_H \times \phi), p^*_H i) \to (\mu_*(\mathrm{id}_G \times \phi), p^* i)$. Here we use that if $\psi : H \times X_1 \to G \times X_1$ is the inclusion, then there is a preferred map $\psi_* p^*_H i = \psi_*\psi^* p^* i \to p^* i$. The maps $\Delta(\alpha)$ and $\mathcal{G}^A_f \Delta(\alpha)$ represent the same element in $\pi_0\Lambda_X(A)^H$.

There is a shear isomorphism $[G_+ \wedge_H \mathbf{S}(ep^* j)]^K \xrightarrow{\cong} [G/H_+]^K \wedge \mathbf{S}(ep^* j)^K$ induced by the map $(g, z) \mapsto (g, gz)$. For j sufficiently large, the composite

$$w : [G/H_+]^K \wedge \mathbf{S}\big(ep^* j\big)^K \longrightarrow \big[G_+ \wedge_H \mathbf{S}\big(ep^* j\big)\big]^K \longrightarrow \big[Map_*\big(G_+, \mathbf{S}\big(ep^* j\big)\big)^H\big]^K$$

of the inverse of the shear map and the map induced by (5.1.1) has connectivity higher than the dimension of $\mathbf{S}(ep^* j)^K$.

Via the isomorphism $q^* : \widetilde{\mathcal{J}}(X/K) \to [\widetilde{\mathcal{J}}(X)]^K$ the element $(R^K_G \circ F^G_K \circ V^G_H \circ \Delta_K)([\alpha])$ is represented in the $(q^*)^{-1}(\mu_*(\mathrm{id}_G \times \phi), p^* i)$th summand by a stable splitting of w in the diagram

$$\begin{array}{ccccc} & & [G/H_+]^K \wedge \mathbf{S}(\mu_* p^*\phi(i))^K & \xrightarrow{\mathrm{id}\wedge \mathcal{G}^A_f \Delta(\alpha)} & [G/H_+]^K \wedge [A(\mu_* p^*\phi(i))]^K \\ & & \downarrow{\scriptstyle w} & & \downarrow \\ \mathbf{S}(\mu_* p^*\phi(i)) & \longrightarrow & [Map_*(G_+, \mathbf{S}(\mu_* p^*\phi(i)))^H]^K & & A(\mu_* p^*\phi(i))^K. \end{array}$$

Here the left horizontal map is the inclusion of constant maps on G and the right vertical map is the shear isomorphism composed by the map induced by acting by G.

We finish the proof by noting that via the isomorphism $q^*: \widetilde{\mathcal{J}}(X/K) \to [\widetilde{\mathcal{J}}(X)]^K$ the element $\Lambda_q(\alpha^{|H|/|K|})$ is represented by $[\mathcal{G}_f^A \Delta(\alpha)]^K : \mathbf{S}(\mu_* p^* \phi(i))^K \to A(\mu_* p^* \phi(i))^K$. □

Lemma 6.2.2. *Let $\mathcal{F}$ be a closed family of subgroups of G and let K be a minimal element of $\mathcal{F}$. For every connected free G-space X the following diagram commutes:*

$$\begin{array}{ccccc}
\pi_0\Lambda_X(A) & \xrightarrow{\Delta_K} & \pi_0\Lambda_X(A)^K & \xrightarrow{V_K^G} & \pi_0\Lambda_X(A)^G \\
{\scriptstyle \pi_0\Lambda_q(A)}\downarrow\cong & \swarrow{\scriptstyle R_K^K} & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} \\
\pi_0\Lambda_{X/K}(A) & \xleftarrow[\cong]{\rho} & \pi_0\Lambda_X^A(\mathcal{F})^K & \xrightarrow{V_K^G} & \pi_0\Lambda_X^A(\mathcal{F})^G \\
 & {\scriptstyle \mathrm{can}}\searrow\cong & & \nearrow & \\
 & & \pi_0\Lambda_{X/K}(A)_{hW_GK}. & &
\end{array}$$

Here $q: X \to X/K$ is the quotient map, the map ρ is induced by the natural isomorphism in (5.2.1) *and the unlabeled map is given by the fundamental cofibration sequence in Lemma* 5.2.5.

Proof. Apart from the lower part of the diagram everything follows directly from the definitions. We need to jump back to the proof of Lemma 5.1.3, where we in particular considered the spaces $Z(j)$. The commutativity of the lower part of the diagram follows from the commutativity of the diagram

$$\begin{array}{ccc}
map_*(Z(j), A(j))^K & \xrightarrow{V_K^G} & map_*(Z(j), A(j))^G \\
\cong\downarrow & & \cong\downarrow \\
map_*(Z(j)^K, A(j)^K) & \xrightarrow{V_0^{W_GK}} & map_*(Z(j)^K, A(j)^K)^{W_GK} \\
{\scriptstyle \mathrm{can}}\downarrow & & \uparrow{\scriptstyle \mathrm{can}} \\
map_*(Z(j)^K, A(j)^K)_{hW_GK} & \xrightarrow{N} & map_*(Z(j)^K, A(j)^K)^{hW_GK}
\end{array}$$

in the homotopy category of the category of Γ-spaces. Here N is the norm map from Section 5.1, and the commutativity of this diagram is a direct consequence of the definitions. □

We define the *extended Teichmüller map* $\tau_A^{\mathcal{F}}: \mathbb{W}_{\mathcal{F}}(\pi_0\Lambda_X(A)) \to \pi_0\Lambda_X^A(\mathcal{F})^G$ to be the composition

$$\mathbb{W}_{\mathcal{F}}\big(\pi_0\Lambda_X(A)\big) \longrightarrow \pi_0\Lambda_X(A)^G \xrightarrow{\mathrm{res}} \pi_0\Lambda_X^A(\mathcal{F})^G,$$

where the first map takes $x = (x_H)_{H \leqslant G}$ to $\sum_{[K]} V_K^G \Delta_K(x_K)$, the sum runs over the conjugacy classes of subgroups in $\mathcal{F}$ and the second map is the restriction map from Section 5.1.

Taking $\mathcal{F}$ to be the family of all subgroups of G we obtain the extended Teichmüller map $\tau_A^G : \mathbb{W}_G(\pi_0 \Lambda_X(A)) \to \pi_0 \Lambda_X(A)^G$.

Lemma 6.2.3. *Let $\mathcal{F}$ be a closed family of subgroups of G and let K be a minimal element of $\mathcal{F}$. If the underlying abelian group of $\pi_0 A$ is torsion free, then the composite*

$$\pi_0 \Lambda_X(A) \longrightarrow \mathbb{W}_{\mathcal{F}}\big(\pi_0 \Lambda_X(A)\big) \xrightarrow{\tau_A^{\mathcal{F}}} \pi_0 \Lambda_X^A(\mathcal{F})^G$$

is injective. Here the left map is the diagonal inclusion in the factor corresponding to K.

Proof. By Lemma 6.2.1 the composite

$$\begin{aligned}\pi_0 \Lambda_X(A) \xrightarrow{\Delta_K} \pi_0 \Lambda_X(A)^K \xrightarrow{V_K^G}\ & \pi_0 \Lambda_X(A)^G \\ \xrightarrow{\text{res}}\ & \pi_0 \Lambda_X^A(\mathcal{F})^G \\ \xrightarrow{F_K^G}\ & \pi_0 \Lambda_X^A(\mathcal{F})^K = \pi_0 \Lambda_X(A)^K \\ \xrightarrow{R_0^K}\ & \pi_0 \Lambda_{X/K}(A) \cong \pi_0 \Lambda_X(A)\end{aligned}$$

is equal to $\phi_{K,K}$, that is, it is multiplication by the cardinality of $W_G K$. □

The existence of the endofunctor $B \mapsto \mathbb{W}_G(B)$ is a corollary of the proof of the following proposition.

Proposition 6.2.4. *Let G be a finite group, X a connected free G-space and A a commutative* **S***-algebra. The extended Teichmüller map $\tau_A^G : \mathbb{W}_G(\pi_0 A) \to \pi_0[\Lambda_X(A)]^G$ is an isomorphism of rings.*

Proof. We first consider the case where $\pi_0 A$ is torsion free. If $\mathcal{F} = \{G\}$, then it is a consequence of Lemma 6.2.2 and Lemma 5.1.3 that τ_A^G is bijective since $\Lambda_X^A(\emptyset) = *$. If $\mathcal{G} \subseteq \mathcal{F}$ are K-adjacent closed families of subgroups of G, then by Lemma 6.2.2 there is a commutative diagram of the form

$$\begin{array}{ccccc} \pi_0 \Lambda_X(A) & \longrightarrow & \mathbb{W}_{\mathcal{F}}(A) & \longrightarrow & \mathbb{W}_{\mathcal{G}}(A) \\ \cong \downarrow & & \tau_A^{\mathcal{F}} \downarrow & & \tau_A^{\mathcal{G}} \downarrow \\ \pi_0 \Lambda_{X/K}(A)_{hW_G K} & \longrightarrow & \pi_0 \Lambda_X^A(\mathcal{F})^G & \xrightarrow{\text{res}} & \Lambda_X(\mathcal{G})^G. \end{array}$$

Here the upper left map is the diagonal inclusion in the factor of the product corresponding to the conjugacy class of K and the upper right map is the projection away from that factor. If $\tau_A^{\mathcal{G}}$ is a bijection, then using Lemma 6.2.3 and the five lemma it follows readily that also $\tau_A^{\mathcal{F}}$ is a bijection. Thus, by induction $\tau_A^{\mathcal{F}}$ is a bijection for every $\mathcal{F}$, and in particular τ_A^G is a bijection.

When $\pi_0 A$ has torsion one proceeds as follows.

First notice that τ_A^G is natural in A. Let $P_. \xrightarrow{\sim} \pi_0 A$ be a simplicial resolution of the discrete ring $\pi_0 A$ such that P_q is torsion free for all q. Then $HP_. \to H\pi_0 A$ is a fibration, and therefore

$Q_\cdot = HP_\cdot \times_{H\pi_0 A} A$ is a simplicial resolution of A. Note that $\pi_0(Q_q) \cong P_q$ for every q. It is easy to see that $\pi_0\{[q] \mapsto \mathbb{W}_G(\pi_0(Q_q))\} \cong \mathbb{W}_G(\pi_0 A)$, and likewise $\pi_0\{[q] \mapsto \pi_0[\Lambda_X(Q_q)]^G\} \cong \pi_0[\Lambda_X(A)]^G$ (for the last piece one must show that $A \mapsto [\Lambda_X(A)]^G$ may be "calculated degree-wise", a fact which is readily proved by induction using the fundamental cofibration sequence). Thus, for each q, $\tau^G_{Q_q}$ is an isomorphism, and so $\tau^G_A = \pi_0 \tau^G_{Q_\cdot}$ is an isomorphism.

Now it follows from Lemma 6.2.1 that the composition

$$\mathbb{W}_G(\pi_0 A) \cong \mathbb{W}_G\big(\pi_0 \Lambda_X(A)\big) \xrightarrow{\tau^G_A} \pi_0 \Lambda_X(A)^G \xrightarrow{F^G_H} \pi_0 \Lambda_X(A)^H \xrightarrow{R^H_0} \pi_0 \Lambda_{X/H}(A) \cong \pi_0(A)$$

is equal to ϕ_H. Taking A to be the Eilenberg–MacLane spectrum on a commutative ring B we see that the ring structure on $\mathbb{W}_G(\pi_0 A)$ obtained from the bijection to $\pi_0 \Lambda_X(A)^G$ satisfies the two criteria mentioned in Section 6.1 determining the ring structure on the ring $\mathbb{W}_G(B)$ of Witt vectors on B uniquely. Thus we may conclude the existence of the ring $\mathbb{W}_G(B)$ of Witt vectors and moreover that $\pi_0 \Lambda_X(A)^G$ is isomorphic to $\mathbb{W}_G(\pi_0 A)$. □

Dress and Siebeneicher also defined Burnside–Witt rings over profinite groups. In order to recall their construction we note that, if N is a normal subgroup of a finite group G and if $f : G \to G' = G/N$ is the quotient homomorphism, then there is a natural (surjective) ring homomorphism $R^G_N : \mathbb{W}_G(B) \to \mathbb{W}_{G'}(B)$ taking $x = (x_H)_{H \leqslant G}$ to $y = (y_{H'})_{H' \leqslant G'}$ where $y_{H'} = x_{f^{-1}H}$.

Lemma 6.2.5. *Let A be a commutative $\mathbf{S}$-algebra, let N be a normal subgroup of a finite group G, let X be a G-space and let $q : X \to X/N$ be the quotient map. For every subgroup K of G containing N the diagram*

$$\begin{array}{ccc}
\pi_0 \Lambda_X A & \xrightarrow{\pi_0 \Lambda_q A} & \pi_0 \Lambda_{X/N} A \\
\downarrow \Delta_K & & \downarrow \Delta_{K/N} \\
\pi_0 \Lambda_X A^K & \xrightarrow{R^K_N} & \pi_0 \Lambda_{X/N} A^{K/N} \\
\downarrow V^G_K & & \downarrow V^{G/N}_{K/N} \\
\pi_0 \Lambda_X A^G & \xrightarrow{R^G_N} & \pi_0 \Lambda_{X/N} A^{G/N}
\end{array}$$

commutes.

Given a profinite group G, we let $\mathbb{W}_G(B)$ be the projective limit of the rings $\mathbb{W}_{G/U}(B)$ with respect to the homomorphisms $R^{G/U}_{V/U}$ for $U \leqslant V$ open (i.e. finite index) normal subgroups of G. Let X be a G-space with the property that X/U is a free connected G/U-space for every open normal subgroup U in G. (The guiding example we have in mind is when $G = (\mathbb{Z}^\wedge)^{\times n}$ is the profinite completion of $\mathbb{Z}^n$ or $G = (\mathbb{Z}_p)^{\times n}$ is the p-completion of $\mathbb{Z}^n$ and $X = G \times_{\mathbb{Z}^n} \mathbb{R}^n$.) Then it follows from Lemma 6.2.5 that the diagram

$$\begin{array}{ccc} \mathbb{W}_{G/U}(\pi_0\Lambda_{X/U}A) & \xrightarrow{\tau_A^{G/U}} & \pi_0\Lambda_{X/U}A^{G/U} \\ \Big\downarrow{\scriptstyle R^{G/U}_{V/U}\circ\mathbb{W}_{G/U}(\pi_0\Lambda_q A)} & & \Big\downarrow{\scriptstyle R^{G/U}_{V/U}} \\ \mathbb{W}_{G/V}(\pi_0\Lambda_{X/V}A) & \xrightarrow{\tau_A^{G/V}} & \pi_0\Lambda_{X/V}A^{G/V} \end{array}$$

commutes for $U \leqslant V$ open normal subgroups of G. Since the Mittag-Leffler condition is satisfied we conclude that there are isomorphisms

$$\mathbb{W}_G(\pi_0 A) \cong \lim_U \mathbb{W}_{G/U}(\pi_0 A) \cong \lim_U \pi_0\Lambda_{X/U}A^{G/U} \cong \pi_0 \operatorname{holim}_{\overleftarrow{U}} \Lambda_{X/U}A^{G/U}$$

where the limits are taken over restriction maps. We summarize the above discussion with a very special case.

Given a prime p, consider the set $\mathcal{O}_p$ consisting of subgroups $U \subseteq \mathbb{Z}^n$ with index a power of p. Choose a free contractible $\mathbb{Z}^n$-space E (e.g., $\sin\mathbb{R}^n$), and consider the diagram of spaces E/U where U varies over $\mathcal{O}_p$. Notice that this diagram is equivalent to a diagram of isogenies of the n-torus.

Corollary 6.2.6. *Let A be a connective commutative $\mathbf{S}$-algebra. Then there is a natural ring isomorphism between Dress and Siebeneicher's Burnside–Witt ring $\mathbb{W}_{\mathbb{Z}_p^{\times n}}(\pi_0 A)$ and the commutative ring $\pi_0 \operatorname{holim}_{\overleftarrow{U\in\mathcal{O}_p}}(\Lambda_{E/U}A)^{\mathbb{Z}^n/U}$. Under this isomorphism, Dress and Siebeneicher's Frobenius and Verschiebung maps f_U and v_U* [7, (1.21), (1.22)] *correspond to F_U^G and V_U^G.*

Note that we have a cofinal subsystem given by the powers of p, so that both groups in the corollary are isomorphic to $\pi_0 \operatorname{holim}_{\overleftarrow{k}}(\Lambda_{E/p^k\mathbb{Z}^n}A)^{\mathbb{Z}^n/p^k\mathbb{Z}^n}$.

7. Covering homology

In this section we define covering homology through appropriate homotopy limits of the restriction and Frobenius maps. The name derives from the fact that in our main examples the limit is indexed by systems of self-coverings.

7.1. Spaces with finite actions

We shall use a category $\mathcal{E}$ to index the homotopy limit defining covering homology. The objects of $\mathcal{E}$ consist of triples (G, H, X) where G is a simplicial group, H is a discrete subgroup of G and X is a non-empty G-space with the property that the image of H in $\operatorname{Aut}(X)$ is finite. A morphism in $\mathcal{E}$ from (G, H, X) to (G', H', X') consists of a pair (φ, f) where $\varphi : G \to G'$ is a group homomorphism such that $H' \subseteq \varphi(H)$ and $f : X \to \varphi^* X'$ is a G-map. The composition in $\mathcal{E}$ is given by composing in each of the two entries.

Covering homology is built from functors into $\mathcal{E}$. Before we give examples of functors into $\mathcal{E}$ we recall the definition of the twisted arrow category.

Definition 7.1.1. If $\mathcal{C}$ is a category, then the *twisted arrow category* $\mathcal{A}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are arrows in $\mathcal{C}$. A morphism from $d : x \to y$ to $b : z \to w$ in $\mathcal{A}(\mathcal{C})$ is a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{c} & z \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle b} \\ y & \xleftarrow{a} & w \end{array}$$

in $\mathcal{C}$, i.e., every equation $abc = d$ represents an arrow (a, c) from d to b, and composition is horizontal composition of squares: $(a_1, c_1)(a_0, c_0) = (a_0 a_1, c_1 c_0)$.

The shorthand notation $a^* = (a, \mathrm{id})$ and $c_* = (\mathrm{id}, c)$ is usual, giving formulae like $a^* c_* = c_* a^*$, $(ab)^* = b^* a^*$ and $b_* c_* = (bc)_*$.

Given a simplicial group G we let $I(G)$ be the monoid of isogenies of G, that is, group-endomorphisms of G with finite discrete kernel and cokernel.

Definition 7.1.2. Let G be a simplicial group and consider $I(G)$ as a category with one object. Let $\mathcal{A}_G$ be the subcategory of $\mathcal{A}(I(G))$ containing all objects and with morphism set $\mathcal{A}_G(\delta, \beta)$ equal to the set of pairs (γ, α) of isogenies of G with the property that $\operatorname{Ker}\beta \subseteq \gamma(\operatorname{Ker}\delta)$. The functor $S_G : \mathcal{A}_G \to \mathcal{E}$ takes an object $\alpha : G \to G$ of $\mathcal{A}_G$ to the triple $S_G(\alpha) = (G, \operatorname{Ker}(\alpha), X)$, where $X = G$ is the G-space where G acts on itself by translation. A morphism

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & G \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \beta} \\ G & \xleftarrow{\alpha} & G \end{array}$$

in $\mathcal{A}_G$ is taken to the morphism

$$S_G(\alpha, \gamma) = (\gamma, f) : \big(G, \operatorname{Ker}(\delta), X\big) \longrightarrow \big(G, \operatorname{Ker}(\beta), X\big)$$

in $\mathcal{E}$, where $f : X \to \gamma^* X$ is equal to γ considered as a map of G-spaces.

Given a set X we let EX denote the contractible simplicial set defined by the formula $EX_k = Map([k], X)$, where $[k] \in \Delta$ is considered as a set with $k + 1$ elements. Note that there is an inclusion $X \cong EX_0 \subseteq EX$. If $X = K$ is a discrete group, then $EK = EX$ is a simplicial group containing K. In particular, K acts freely on EK. Note that if K is abelian, then EK is abelian.

Given an inclusion $H \subseteq K$ of discrete abelian groups, let $\mathcal{C}(H, K)$ be the monoid of group automorphisms $a : K \to K$ with the property that $a(H)$ is a subgroup of finite index in H. There is a monoid homomorphism $\varphi : \mathcal{C}(H, K) \to I(EK/H)$, taking $a \in \mathcal{C}(H, K)$ to the surjection

$$\varphi(a) : EK/H \longrightarrow EK/H$$

induced by a.

In the situation of Definition 7.1.2, let $G = EK/H$. Since the functor $\mathcal{A}(\varphi) : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(I(G))$ factors through the subcategory $\mathcal{A}_G$ we can make the following definition.

Definition 7.1.3. Let $H \subseteq K$ be an inclusion of discrete abelian groups. Given a submonoid $\mathcal{C}$ of $\mathcal{C}(H, K)$ we let $S(\mathcal{C}) : \mathcal{A}(\mathcal{C}) \to \mathcal{E}$ be the functor taking $c \in \mathcal{C}$ to the triple $S(\mathcal{C})(c) = (EK/H, \operatorname{Ker}(\varphi(c)), EK/H)$.

7.2. Covering homology

To every commutative **S**-algebra A there is an associated functor ΛA from the category $\mathcal{E}$ of Section 7.1 to the category of spectra, taking an object (G, H, X) to $[\Lambda_X A]^H$. For a morphism $(\varphi, f): (G, H, X) \to (G', H', X')$ in $\mathcal{E}$ we write $K = \operatorname{Ker}(\varphi) \cap H$, and by abuse of notation we write Λ_f for the map

$$[\Lambda_{X/K} A]^{H/K} \longrightarrow \big[\Lambda\big[\varphi^* X'\big]A\big]^{H/K} = \big[\Lambda\big[X'\big]A\big]^{\varphi(H)}$$

induced by f. We define the map $\Lambda A(f, \varphi)$ to be the composition

$$[\Lambda_X A]^H \xrightarrow{R_K^H} [\Lambda_{X/K} A]^{H/K} \xrightarrow{\Lambda_f} \big[\Lambda\big[X'\big]A\big]^{\varphi(H)} \xrightarrow{F_{H'}^{\varphi(H)}} \big[\Lambda\big[X'\big]A\big]^{H'}.$$

In order to check that ΛA is a functor, let

$$(\varphi, f): (G, H, X) \longrightarrow \big(G', H', X'\big)$$

and

$$(\psi, g): \big(G', H', X'\big) \longrightarrow \big(G'', H'', X''\big)$$

be morphisms in $\mathcal{E}$. Let $K = \operatorname{Ker}(\varphi) \cap H$, $K' = \operatorname{Ker}(\psi) \cap H'$ and $K'' = \operatorname{Ker}(\psi\varphi) \cap H$. By the naturality and transitivity properties of the maps R and F stated in Lemma 5.2.4 the diagram

$$\begin{array}{ccccccc}
[\Lambda_X A]^H & & & & & & \\
\downarrow R & \searrow^{R} & & & & & \\
[\Lambda_{X/K} A]^{H/K} & \xrightarrow{R} & [\Lambda_{X/K''} A]^{H/K''} & & & & \\
\downarrow \Lambda_f & & \downarrow \Lambda_f & \searrow^{\Lambda_{gf}} & & & \\
[\Lambda_{X'} A]^{H/K} & \xrightarrow{R} & [\Lambda_{X'/K'} A]^{H/K''} & \xrightarrow{\Lambda_{g/K'}} & [\Lambda_{X''} A]^{H/K''} & & \\
\downarrow F & & \downarrow F & & \downarrow F & \searrow^{F} & \\
[\Lambda_{X'} A]^{H'} & \xrightarrow{R} & [\Lambda_{X'/K'} A]^{H'/K'} & \xrightarrow{\Lambda_{g/K'}} & [\Lambda_{X''} A]^{H'/K'} & \xrightarrow{F} & [\Lambda_{X''} A]^{H''}
\end{array}$$

commutes.

Definition 7.2.1. Let A be a commutative **S**-algebra and let $S: \mathcal{A} \to \mathcal{E}$ be a functor from an arbitrary category $\mathcal{A}$. The *covering homology* of A with respect to the functor S is

$$TC_S(A) := \operatorname*{holim}_{\overleftarrow{\mathcal{A}}} \Lambda A \circ S.$$

Let G be a simplicial group and let $I(G)$ be the monoid of isogenies of G, that is, group-endomorphisms α of G with finite and discrete kernel and cokernel. Using the functor $S_G : \mathcal{A}_G \to \mathcal{E}$ given in Definition 7.1.2 we obtain a covering homology $A \mapsto TC_{S_G}(A)$ associated to every simplicial group. The fact that surjective isogenies are simplicial versions of finite covering maps is the reason for our choice of the name "covering homology". Note that given α and β in $I(G)$, the map Λ_α is the composite

$$[\Lambda_{G/\operatorname{Ker}\alpha} A]^{\operatorname{Ker}(\beta\alpha)/\operatorname{Ker}\alpha} \longrightarrow \big[\Lambda\big[\alpha^* G\big]A\big]^{\operatorname{Ker}(\beta\alpha)/\operatorname{Ker}\alpha} = [\Lambda_G A]^{\operatorname{Ker}(\beta)}.$$

If α is surjective, the map $G/\operatorname{Ker}(\alpha) \to G$ induced by α is an isomorphism, and thus Λ_α is an isomorphism in this case.

We shall occasionally use the notations $R_\gamma = \Lambda A(\gamma_*)$ and $F^\alpha = \Lambda(\alpha^*)$ for ΛA applied to morphisms of the form

$$\gamma_* = \begin{array}{ccc} G & \xrightarrow{\gamma} & G \\ \big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle\beta} \\ G & \xleftarrow{=} & G \end{array} \quad \text{and} \quad \alpha^* = \begin{array}{ccc} G & \xrightarrow{=} & G \\ \big\downarrow{\scriptstyle\delta} & & \big\downarrow{\scriptstyle\beta} \\ G & \xleftarrow{\alpha} & G \end{array}$$

in $\mathcal{A}_G$.

Example 7.2.2. The situation where G is the simplicial n-torus $\mathbb{T}^{\times n} = \sin(\mathbb{R}^n/\mathbb{Z}^n)$ is particularly interesting. There is an isomorphism

$$Hom\big(\mathbb{Z}^n, \mathbb{Z}^n\big) \longrightarrow Hom\big(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^n/\mathbb{Z}^n\big)$$

taking a to the map induced by $\mathbb{R} \otimes_{\mathbb{Z}} a : \mathbb{R}^n \to \mathbb{R}^n$, and the induced homomorphism

$$Hom\big(\mathbb{Z}^n, \mathbb{Z}^n\big) \subseteq Hom\big(\mathbb{T}^{\times n}, \mathbb{T}^{\times n}\big), \quad a \longmapsto \sin(\mathbb{R} \otimes_{\mathbb{Z}} a),$$

is injective. This allows us to consider the monoid $\mathcal{M}_n$ of injective linear endomorphisms of $\mathbb{Z}^n$ as a submonoid of $I(G)$. Thus, given a submonoid $\mathcal{C}$ of $\mathcal{M}_n$, we can consider the covering homology $A \mapsto TC_{S(\mathcal{C})}(A)$.

In the special case where $n = 1$ and where $\mathcal{C} = \mathcal{M}_1 = (\mathbb{Z} \setminus \{0\}, \cdot)$, the covering homology $TC_{S(\mathcal{M}_1)}(A)$ gives us what might be called *topological dihedral homology*. If $\mathcal{C}$ is the submonoid $(\mathbb{N}_{>0}, \cdot)$, giving just orientation-preserving coverings of the circle, the covering homology $TC_{S(\mathcal{C})}(A)$ is weakly equivalent to Bökstedt, Hsiang and Madsen's topological cyclic homology.

Let G be a simplicial group, let $\mathcal{C}$ be a submonoid of $I(G)$ and let $\mathcal{A}_\mathcal{C} := \mathcal{A}(\mathcal{C}) \cap \mathcal{A}_G$. If $\varphi : G \to G$ is a group automorphism with the property that $\varphi x \varphi^{-1} \in \mathcal{C}$ for every $x \in \mathcal{C}$, then we define the functor

$$c_\varphi : \mathcal{A}_\mathcal{C} \longrightarrow \mathcal{A}_\mathcal{C}, \quad x \longmapsto \varphi x \varphi^{-1},$$

and the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \varphi x \varphi^{-1}} \\ G & \xleftarrow{\varphi^{-1}} & G, \end{array}$$

specifies a natural transformation η_φ from the inclusion $j\colon \mathcal{A}_\mathcal{C} \subseteq \mathcal{A}_G$ to the functor $j \circ c_\varphi$. Since $(\varphi, \varphi^{-1}) = \varphi^* \circ \varphi_*^{-1}$ we have

$$\Lambda A(\varphi, \varphi^{-1}) = \Lambda A(\varphi^*) \circ \Lambda A(\varphi_*^{-1}) = R_\varphi \circ F_{\varphi^{-1}} : \Lambda_G A \longrightarrow \Lambda_G A.$$

Let $S_\mathcal{C}$ be the composite

$$\mathcal{A}_\mathcal{C} \xrightarrow{j} \mathcal{A}_G \xrightarrow{S_G} \mathcal{E}.$$

Definition 7.2.3. Let G be a simplicial group, let $\mathcal{C}$ be a submonoid of $I(G)$ and let I be a group contained in $I(G)$ with the property that $\varphi x \varphi^{-1} \in \mathcal{C}$ for every $x \in \mathcal{C}$ and every $\varphi \in I$. We define an action of I on $TC_{S_\mathcal{C}}(A)$ by letting the action of $\varphi \in I$ be given by the endomorphism

$$\underleftarrow{\operatorname{holim}}\, \Lambda A \circ S_\mathcal{C} \xrightarrow{\operatorname{holim}_{\leftarrow} \Lambda A \circ S_G \circ \eta_\varphi} \underleftarrow{\operatorname{holim}}\, \Lambda A \circ S_\mathcal{C} \circ c_\varphi \longrightarrow \underleftarrow{\operatorname{holim}}\, \Lambda A \circ S_\mathcal{C}.$$

Example 7.2.4. Let $G = \mathbb{T}^{\times n}$, let p be a prime and let $\mathcal{C}$ denote the submonoid of $I(G)$ consisting of isogenies of the form

$$p^r : \mathbb{T}^{\times n} \longrightarrow \mathbb{T}^{\times n}, \qquad (x_1, \ldots, x_n) \longmapsto p^r(x_1, \ldots, x_n) = (p^r x_1, \ldots, p^r x_n)$$

for $r \geqslant 0$ corresponding to the submonoid of $\mathcal{M}_n$ consisting of endomorphisms of $\mathbb{Z}^n$ of the form $p^r : \mathbb{Z}^n \to \mathbb{Z}^n$. The group of automorphisms of $\mathbb{Z}^n$ is contained in $\mathcal{M}_n$. Since the endomorphisms in $\mathcal{C}$ correspond to multiplication by a number, they are fixed under conjugation by elements in the group $GL_n\mathbb{Z}$ of group automorphisms of $\mathbb{Z}^n$. Thus Definition 7.2.3 specifies an action of I on $TC_{S_\mathcal{C}} A$. In this particular situation the category

$$\cdots \underset{p_*}{\overset{p^*}{\rightrightarrows}} (p^2) \underset{p_*}{\overset{p^*}{\rightrightarrows}} (p) \underset{p_*}{\overset{p^*}{\rightrightarrows}} (1),$$

with (p^n) equal to the endomorphism of G given by multiplication by p^n, is cofinal in the category $\mathcal{A}_\mathcal{C}$. Thus $TC_{S_\mathcal{C}}$ is homotopy equivalent to the homotopy limit of the diagram

$$\cdots \underset{R_p}{\overset{F_p}{\rightrightarrows}} (\Lambda_{\mathbb{T}^{\times n}} A)^{C_{p^2}^{\times n}} \underset{R_p}{\overset{F_p}{\rightrightarrows}} (\Lambda_{\mathbb{T}^{\times n}} A)^{C_p^{\times n}} \underset{R_p}{\overset{F_p}{\rightrightarrows}} \Lambda_{\mathbb{T}^{\times n}} A.$$

If $n = 1$ we may consider the action of $\{\pm 1\} = GL_1(\mathbb{Z})$, and the homotopy fixed point spectrum picks up the "part relevant to p" of the topological dihedral homology in Example 7.2.2.

Working with the p-complete torus instead, we get operations by all of $GL_n(\mathbb{Z}_p)$. Note that if $n = 1$ the map from $TC(A)\hat{_p} \simeq TC_{S_\mathcal{C}}(A)\hat{_p}$ to $THH(A)\hat{_p} \simeq \Lambda_{(\mathbb{T}^1)\hat{_p}}(A)\hat{_p}$ then sends the operation of a p-adic unit on $TC(A)\hat{_p}$ to the corresponding Adams operation on $THH(A)\hat{_p}$, as discussed in Section 4.6.3.

Example 7.2.5. Let $R \subseteq B$ be an inclusion of (discrete) commutative rings and let M be a flat R-module. In the context of Definition 7.1.3, let $H = M = R \otimes_R M$, let $K = B \otimes_R M$ and let $H \subseteq K$ be the inclusion of discrete abelian groups induced by the R-module homomorphism $R \to B$. Applying Definition 7.1.3 to a submonoid $\mathcal{C}$ of $\mathcal{C}(M, B \otimes_R M)$ we obtain a functor $S(\mathcal{C}) : \mathcal{A}(\mathcal{C}) \to \mathcal{E}$, and we may form the covering homology $A \mapsto TC_{S(\mathcal{C})}(A)$. If I is a group contained in $\mathcal{C}(M, B \otimes_R M)$ with the property, that $\varphi x \varphi^{-1} \in \mathcal{C}$ for every $x \in \mathcal{C}$ and every $\varphi \in I$, then Definition 7.2.3 specifies an action of I on $TC_{S(\mathcal{C})}(A)$.

Let us emphasize that if $R \subseteq B$ is the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$ and if $M = \mathbb{Z}^n$, then $G = E(\mathbb{Q} \otimes_{\mathbb{Z}} M)/(\mathbb{Z} \otimes_{\mathbb{Z}} M)$ is a model for the classifying space $B\mathbb{Z}^n$. In fact, the homomorphisms

$$\mathbb{R}^n/\mathbb{Z}^n \longleftarrow \left(\mathbb{R}^n \times \left|E\left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n\right)\right|\right)/\mathbb{Z}^n \longrightarrow \left|E\left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n\right)\right|/\mathbb{Z}^n \cong |G|$$

of topological abelian groups are homotopy equivalences. In Example 7.2.2 we have seen that under these equivalences $\mathcal{M}_n = \mathcal{C}(\mathbb{Z}^n, \mathbb{Q}^n)$ corresponds to the monoid of isogenies of the n-torus $\mathbb{R}^n/\mathbb{Z}^n$. The spectrum $TC_{S(\mathcal{C}(\mathbb{Z}^n, \mathbb{Q}^n))}(A)$ is related to iterated topological cyclic homology. In fact, in Example 7.2.2 we have seen that when $n = 1$ and $\mathcal{C}$ is the submonoid $(\mathbb{N}_{>0}, \cdot)$ of $\mathcal{C}(\mathbb{Z}, \mathbb{Q}) = (\mathbb{Z} \setminus \{0\}, \cdot)$, then $TC_{S(\mathcal{C})}(A)$ is weakly equivalent to Bökstedt, Hsiang and Madsen's topological cyclic homology. Note that Definition 7.2.3 gives an action of $I = \{-1, +1\}$ on $TC_{S(\mathcal{C})}(A)$ whose homotopy fixed point spectrum is the topological dihedral homology of Example 7.2.2.

Consider the situation where B is the quotient field of an integral domain R and $M = \mathcal{O}$ is a possibly non-commutative R-algebra. In this situation we can choose the monoid $\mathcal{C}$ to be the intersection of $\mathcal{C}(\mathcal{O}, B \otimes_R \mathcal{O})$ and image of the homomorphism $\psi : \mathcal{O} \to End_{\mathbb{Z}}(B \otimes_R \mathcal{O})$ from $\mathcal{O}$ to the monoid $End_{\mathbb{Z}}(B \otimes_R \mathcal{O})$ of group-endomorphisms of $B \otimes_R \mathcal{O}$ with $\psi(x)(b \otimes y) = b \otimes yx$. If $f : \mathcal{O} \to \mathcal{O}$ is an R-algebra automorphism, then the diagram

$$\begin{array}{ccc}
B \otimes_R \mathcal{O} & \xrightarrow[\cong]{B \otimes_R f} & B \otimes_R \mathcal{O} \\
{\scriptstyle \psi(x)} \downarrow & & \downarrow {\scriptstyle \psi(f(x))} \\
B \otimes_R \mathcal{O} & \xrightarrow[\cong]{B \otimes_R f} & B \otimes_R \mathcal{O}
\end{array}$$

commutes. This implies that if we let I be the group of R-algebra automorphisms of $B \otimes_R \mathcal{O}$, of the form $\varphi = B \otimes_R f$, then $\varphi \psi(x) \varphi^{-1} \in \mathcal{O}$ for every $\varphi \in I$ and every $\psi(x) \in \mathcal{C}$. Thus the group of R-algebra automorphisms of $\mathcal{O}$ acts on $TC_{S(\mathcal{C})}(A)$.

Explicit examples are listed in the table below, where G is a finite group, $K \subseteq L$ is a finite Galois extension of (local) number fields and $\mathcal{O}(K) \subseteq \mathcal{O}(L)$ is the induced inclusion of rings of integers.

R	B	$\mathcal{O}$	$\mathcal{C}$	$\mathrm{Aut}_R(\mathcal{O})$
$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}^n$	$(\mathbb{Z} \setminus \{0\})^n$	Σ_n
$\mathbb{Z}\hat{_p}$	$\mathbb{Q}_p$	$(\mathbb{Z}\hat{_p})^n$	$(\mathbb{Z}\hat{_p} \setminus \{0\})^n$	Σ_n
$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z}[G]$	$\mathbb{Z}[G] \cap \mathbb{Q}[G]^*$	$\mathrm{Aut}(G)$
$\mathcal{O}(K)$	K	$\mathcal{O}(L)$	$\mathcal{O}(L) \cap L^*$	$Gal(L/K)$

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