Generalizations of the Kervaire invariant

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1. Introduction

M. Kervaire in [4], F. Peterson and myself in [1], and W. Browder in [3] defined numerical invariants for various classes of $2n$-manifolds roughly as follows: Suppose M is a closed compact $2n$-manifold with some additional structure T. In [4] T is a framing of the normal bundle of M and n is odd, in [1] it is a spin structure and $n \equiv 1 \bmod 4$ and in [3] it is a v_{n+1} structure. Using T one constructs a function

$$\varphi\colon H^n(M; \mathbf{Z}_2) \longrightarrow \mathbf{Z}_2$$

satisfying

$$\varphi(u + v) = \varphi(u) + \varphi(v) + (u \cup v)\,(M)\,. \tag{1.1}$$

The Kervaire invariant of (M, T) is defined to be the Arf invariant of φ. $\operatorname{Arf}\varphi = 0$ or 1 and is 0 if and only if φ is zero on more than half the elements of $H^n(M; \mathbf{Z}_2)$.

In this paper we describe a general technique for constructing functions satisfying (1.1) and hence of obtaining generalizations of the Kervaire invariant. This technique gives, as special cases, the functions defined in [1], [3] and [4]. In the remainder of this section we outline in detail our techniques and state our results. Some of these results appear in [2]. Our algebraic results concerning the Arf invariant are stated in Theorem 1.20 and proved in § 3. The proofs of all other lemmas and theorems in this section are either given immediately or in § 2.

All homology and cohomology groups will be with $\mathbf{Z}_2$ coefficients unless otherwise stated. Usually spaces will have base points. $[X, Y]$ denotes the set of homotopy classes of maps from X to Y. S denotes suspension and

$$\{X, Y\} = \lim\,[S^k X, S^k Y]\,.$$

K_n will denote $K(\mathbf{Z}_2, n)$.

We first describe how $\{\ \ \}$ gives rise to quadratic functions. Let X be a CW complex with base point of dimension $2n$. We define a function

$$F\colon H^n(X) \times H^{2n}(X) \longrightarrow \{X, K_n\}$$

as follows:

[1] This work was carried out while the author was supported by NSF grant GP 21510.

LEMMA (1.2). $\{S^{2n}, K_n\} \approx \mathbf{Z}_2$.

Let $\mu \in \{S^{2n}, K_n\}$ be the generator. We view $u \in H^n(X)$ as a map $u: X \to K_n$. If $v \in H^{2n}(X)$, there is a map $g_v: X \to S^{2n}$ such that $g_v^*(s_{2n}) = v$, where $s_{2n} \in H^{2n}(S^{2n})$ is the generator. Define $F(u, v)$ by

$$F(u, v) = \{u\} + \{\mu g_v\} .$$

PROPOSITION (1.3). *If X is a $2n$-dimensional* CW *complex, then*

$$F: H^n(X) \times H^{2n}(X) \longrightarrow \{X, K_n\}$$

is bijective and

$$F(u, v) + F(u', v') = F(u + u', u \cup u' + v + v') .$$

Remark. If $u^2 \neq 0$, $F(u, 0)$ has order four.

Let $j: \mathbf{Z}_2 \to \mathbf{Z}_4$ be the homomorphism sending 1 to 2. Suppose $h: \{X, K_n\} \to \mathbf{Z}_4$ is a homomorphism. Since $hF(0, v)$ is linear in v, $hF(0, v) = jv(x)$ for some $x \in H_{2n}(X)$. Let $\varphi_h: H^n(X) \to \mathbf{Z}_4$ be given by

$$\varphi_h(u) = hF(u, 0) .$$

Then (1.3) yields:

LEMMA (1.4). $\varphi_h(u + v) = \varphi_h(u) + \varphi_h(v) + j(u \cup v)(x)$. *Furthermore, if φ and x satisfy the above formula, $\varphi = \varphi_h$ for some h.*

Definition (1.5). An m-Poincaré triple (X, ζ, α) is

(i) A CW complex X with finitely generated homology (X is without base point).

(ii) A fibration ζ over X with fibres homotopy equivalent to S^{k-1}, k large, (e.g. $k > m + 1$).

(iii) $\alpha \in \pi_{m+k}(T(\zeta))$ ($T(\zeta)$ = Thom space) such that an $m + k$ Spanier-Whitehead S-duality is given by

$$S^{m+k} \xrightarrow{\alpha} T(\zeta) \xrightarrow{\Delta} T(\zeta) \wedge X^+$$

where Δ is the diagonal map.

Recall $\Delta\alpha$ being an S-duality means that if $\bar{s}_{m+k} \in H_{m+k}(S^{m+k}; \mathbf{Z})$ is a generator and $y = (\Delta\alpha)_* \bar{s}_{m+k}$,

$$/y: H^{q+k}(T(\zeta); \mathbf{Z}) \approx H_{m-q}(X; \mathbf{Z})$$

for all q. Let $U_k \in H^k(T(\zeta))$ be the Thom class and let $x = U_k/y \in H_m(X)$. It follows from the above isomorphism and the Thom isomorphism that

$$x \cap : H^q(X) \approx H_{m-q}(X) .$$

Furthermore, for any CW complex Y,

$$\text{(1.6)} \qquad A_\alpha: \{X^+, Y\} \approx \{S^{m+k}, T(\zeta) \wedge Y\} ,$$

where $A_\alpha\{f\} = \{(\mathrm{id} \wedge f)\Delta\alpha\}$.

Suppose (X, ζ, α) is a $2n$-Poincaré triple. Let X_i be the connected components of X and let a_i: $S^k \to T(\zeta \mid X_i) \subset T(\zeta)$ be the inclusion of a fibre. Let $\lambda_i \in \{S^{2n+k}, T(\zeta) \wedge K_n\}$ be the image of the generator μ under

$$\{S^{2n}, K_n\} = \{S^{2n+k}, S^k \wedge K_n\} \xrightarrow{a_{i*}} \{S^{2n+k}, T(\zeta) \wedge K_n\} .$$

THEOREM (1.7). *If (X, ζ, α) is a $2n$-Poincaré triple with fundamental class $x \in H_{2n}(X)$, the functions φ: $H^n(X) \to \mathbf{Z}_4$ satisfying*

$$\varphi(u + v) = \varphi(u) + \varphi(v) + j(u \cup v)(x) \tag{1.8}$$

are in one-to-one correspondence with homomorphisms

$$h: \{S^{2n+k}, T(\zeta) \wedge K_n\} \longrightarrow \mathbf{Z}_4$$

such that $h(\lambda_i) = 2$ under the correspondence $\varphi(u) = hA_\alpha F(u, 0)$ where F and A_α are given in (1.3) *and* (1.6), *respectively.*

Proof. Note A_α in (1.6) is natural in Y and it can be applied to the commutative diagram,

$$\begin{array}{ccc} \{X_i, S^{2n}\} & \overset{A_\alpha}{\approx} & \{S^{2n+k}, T(\zeta \mid X_i) \wedge S^{2n}\} \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} \\ \{X, K_i\} & \overset{A_\alpha}{\approx} & \{S^{2n+k}, T(\zeta \mid X_i) \wedge K_n\} . \end{array}$$

If g: $X_i \to S^{2n}$, $\mu_* A_\alpha\{g\} = (g^* s_{2n})(x)\lambda_i$. Hence $A_\alpha F(0, v) = v(x)\lambda_i$ if $v \in H^n(X_i)$. (1.7) now follows from (1.3) and (1.4).

Theorem (1.7) suggests a method of constructing quadratic functions for manifolds in a cobordism theory as follows: Suppose $MG = \{MG_k\}$ is a Thom spectrum. Let $\lambda \in \{S^{2n+k}, MG_k \wedge K_n\}$ be the image of $\mu \in \{S^{2n+k}, S^k \wedge K_n\}$ under the inclusion of a fibre. Choose a homomorphism

$$h: \{S^{2n+k}, MG_k \wedge K_n\} \longrightarrow \mathbf{Z}_4$$

such that $h(\lambda) = 2$. If X is a $2n$-manifold, ζ is its normal bundle, α is obtained from the Thom construction and W: $T(\zeta) \to MG_k$ comes from a G structure on ζ, then

$$\varphi(u) = hW_* A_\alpha(F(u, 0))$$

gives a function on $H^n(X)$ satisfying the formula in (1.3). This method works so long as $\lambda \neq 0$. We would of course choose h above for each k in a consistent way. This amounts to choosing

$$h: H_{2n}(K_n: MG) \to \mathbf{Z}_4 .$$

Our next result describes when $\lambda \neq 0$ and h exists.

Suppose $Y = \{Y_k, \mu_k\}$ is a spectrum such that Y_k is $k-1$ connected and $H^0(Y) \approx \mathbf{Z}_2$. Let $U \in H^0(Y)$ be a generator and let $p: S \to Y$ be a map of the sphere spectrum into Y such that $p^*U \in H^0(S)$ is the generator. Let $\lambda \in H_{2n}(K_n, Y)$ be the element represented by

$$S^{2n+k} \xrightarrow{\mu} S^k \wedge K_n \xrightarrow{p_k \wedge \mathrm{id}} Y_k \wedge K_n .$$

PROPOSITION (1.9). *$\lambda \neq 0$ if and only if $\chi(Sq^{n+1})U = 0$, where χ is the canonical antiautomorphism of the Steenrod algebra. Furthermore, if $\lambda \neq 0$, it is at most divisible by 2.*

Definition (1.10). A Wu n-spectrum is a pair (Y, h) where Y is as above, $\chi(Sq^{n+1})U = 0$ and $h: H_{2n}(K_n, Y) \to \mathbf{Z}_4$ is a homomorphism such that $h(\lambda) = 2$. A (Y, h) orientation of a fibration ζ is a map $W: T(\zeta) \to Y_k$ such that W^*U_k is the Thom class of $T(\zeta)$.

COROLLARY (1.11). *If (X, ζ, α) is a $2n$-Poincaré triple, W is (Y, h) orientation of ζ where (Y, h) is a Wu n-spectrum and $\varphi: H^n(X) \to \mathbf{Z}_4$ is given by*

$$\varphi(u) = h\ W_* A_\alpha F(u, 0) ,$$

then

$$\varphi(u + v) = \varphi(u) + \varphi(v) + j(u \cup v)(x)$$

where $x \in H_{2n}(X)$ is the fundamental class.

We next describe the geometrical properties of φ. Suppose $f: S^n \to X$ and $f^*\zeta$ is trivial. Up to homotopy type $T(f^*\zeta) = S^k \vee S^{n+k}$. Let $\hat{f}: S^{n+k} \to T(\zeta)$ be the inclusion of $T(f^*\zeta)$ in $T(\zeta)$ restricted to S^{n+k}. Consider the commutative diagram:

$$\begin{array}{ccccc}
\{X^+, S^n\} & \overset{A_\alpha}{\approx} & \{S^{2n+k}, T(\zeta) \wedge S^n\} & \xrightarrow{W_*} & \{S^{2n+k}, Y_k \wedge S^n\} \\
\Big\downarrow (s_n)_* & & & & \Big\downarrow \\
\{X^+, K_n\} & \overset{A_\alpha}{\approx} & \{S^{2n+k}, T(\zeta) \wedge K_n\} & \xrightarrow{W_*} & \{S^{2n+k}, Y_k \wedge K_n\} .
\end{array}$$

Let $\tilde{f} \in \{X^+, S^n\}$ be such that $A_\alpha \tilde{f} = \hat{f}$, that is, $\tilde{f}$ is the S-dual of f. Let $\bar{f} = s_{n*}\tilde{f}$ and let $\beta_f = W_* A_\alpha(\bar{f})$.

THEOREM (1.12). *$\bar{f} = F(u, v)$ where u is the Poincaré dual of the element in $H_n(X)$ represented by $f: S^n \to X$ and*

$$\varphi(u) = jv(x) + h(\beta_f) .$$

Hence, if $h(\beta_f) = 0$, $\varphi(u)$ is the obstruction to desuspending $\bar{f}$ to a map of X^+ to K_n.

Proof. Let $B: \{X^+, K_n\} \to H^n(X)$ be given by $S^l B(g) = g^* S^l \iota_n$ where

$g\colon S^l X^+ \to S^l K_n$. $BF(u, v) = u$ is the S-dual of the element in $H_{n+k}(T(\zeta))$ represented by $\hat{f}$. Hence u is the Poincaré dual of $f(S^n) \in H_n(X)$. Since $\bar{f} = F(u, v)$,

$$\begin{aligned} h(\beta_f) &= hW_* A_\alpha F(u, v) \\ &= hW_* A_\alpha\big(F(u, 0) + F(0, v)\big) \\ &= \varphi(u) + jv(x) . \end{aligned}$$

COROLLARY (1.13). *If (X, ζ, α) is a smooth (or PL) manifold, its normal bundle and its Thom map, $f\colon S^n \to X$ is a smooth embedding, ν is the normal bundle of $f(S^n)$ in X, ν is stably trivial, and $u \in H^n(X)$ is the Poincaré dual of $f(S^n)$, then $\varphi(u) = \varepsilon + h(\beta_f)$ where*

$$\begin{aligned} \varepsilon &= 0 \ \text{ if } \nu \text{ is trivial} \\ &= 1 \ \text{ if } \nu \text{ is not trivial and } n \text{ is odd} \\ &= \text{Euler number of } \nu \bmod 4 \text{ if } n \text{ is even} . \end{aligned}$$

Remark. Suppose $Y = MG$ for some Thom spectrum and W comes from a map $(g', g)\colon (\zeta, X) \to (\xi_K, BG)$ where ξ_k is the canonical bundle over BG. If $gf = 0$, $W\hat{f} = Vi \in \{S^{n+k}, MG_k\}$ where $V\colon S^{n+k} \to S^k$ and $i\colon S^k \to MG_k$. From the proof of (1.9) one sees that $\beta_f = m\lambda$ where $m =$ Hopf invariant of V. Hence if $n \neq 1, 3, 7$, $h(\beta_f) = 0$.

We next consider the situation in which X is a Poincaré space boundary. Suppose $X \subset Y$, η is a spherical fibration over Y, $\eta \mid X = \zeta$, $\alpha' \in \pi_{2n+k+1}(T(\eta)/T(\zeta))$ maps into $S\alpha$ under the map $T(\eta)/T(\zeta) \to ST(\zeta)$ and if $y \in H_{2n+1}(Y, X)$ is the element corresponding to α',

$$y \cap : H^n(Y) \approx H_{n+1}(Y, X) .$$

LEMMA (1.14). *If $i\colon X \to Y$ is the inclusion, $\varphi i^* = 0$.*

Suppose (X_i, ζ_i, α_i), $i = 1, 2$, are $2n$-Poincaré triples, W_i are (Y, h) orientations of ζ_i, k_i is the fibre dimension of ζ_i,

$$g\colon \zeta_1 + 0^{l_1} \longrightarrow \zeta_2 + 0^{l_2}$$

where $k_1 + l_1 = k_2 + l_2$ and 0^{l_i} is the trivial $S^{l_i - 1}$ bundle over X_i and g covers $f\colon X_1 \to X_2$. W_i define (Y, h) orientations $\bar{W}_i$ of $\zeta_i + 0^{l_i}$. Let φ_i be the quadratic functions on $H^n(X_i)$. The following is immediate.

PROPOSITION (1.15). *If $\bar{W}_1 = T(g)^* \bar{W}_2$ and $T(g)_* S^{l_1}\alpha_1 = S^{l_2}\alpha_2$, then $\varphi_1(f^* u) = \varphi_2(u)$, $u \in H^n(X_2)$.*

We next examine how φ depends on the choice of W and α. Let $W(n) = \{W_k(n)\}$ be the Ω-spectrum where $W_k(n)$ is the fibration over K_k with fibre K_{k+n} and k-invariant $\chi(Sq^{n+1})\, l_k$. Note by (1.9) there is an h making $(W(n), h)$

a Wu n-spectrum. If Y is a spectrum as in (1.10) there is a map $\tau: Y \to W(n)$ such that $(Y, h\tau_*)$ is a Wu n-spectrum. Also if (X, ζ, α) is a $2n$-Poincaré triple, ζ is $W(n)$ orientable since $\chi(Sq^{n+1})$ of the Thom class of ζ is the Wu class v_{n+1}.

If $W_1, W_2: T(\zeta) \to W_k(n)$ are $(W(n), h)$ orientations, they differ by an element

$$d(W_1, W_2) \in H^n(X) \approx H^{n+k}(T(\zeta)) .$$

PROPOSITION (1.16). *If φ_1 and φ_2 are the quadratic functions from (X, ζ, α) and W_1 and W_2, respectively, then*

$$\varphi_1(u) = \varphi_2(u) + j(u \cup d(W_1, W_2))(x) .$$

Note if φ_1 and φ_2 are two quadratic functions for (X, ζ, α), $\varphi_1 - \varphi_2$ is linear and hence $\varphi_1 - \varphi_2 = j(y \cup \ )$ for some $y \in H^n(X)$. This yields:

COROLLARY (1.17). *The $(W(n), h)$ orientations of (X, ζ, α) are in one-to-one correspondence with functions on $H^n(X)$ satisfying* (1.8).

Let G_k be the H space of unbased maps of S^{k-1} to itself of degree one. Recall if (X, ζ, α_i), $i = 1, 2$, are Poincaré triples, there is a map $g: X \to G_k$ which defines an automorphism $\bar{g}: \zeta \to \zeta$ such that $T(\bar{g})\alpha_2 = \alpha_1$ (see proof of (1.17)). Let $u_i \in H^i(G_k)$ be the classes which transgress to $w_{i+1} \in H^{i+1}(BG_k)$, the $i+1$ Stiefel-Whitney class. If φ_1, φ_2, and φ_3 are the quadratic functions associated to (X, ζ, α_1, W), (X, ζ, α_2, W), and $(X, \zeta, \alpha_1, T(g)^* W)$, respectively, $\varphi_2 = \varphi_3$ by (1.15) and hence

$$\varphi_1 = \varphi_2 + j(x \cup \ )$$

where $x = d(T(g^*) W, W)$.

THEOREM (1.18). *If (X, ζ, α_i), $i = 1, 2$ are $2n$-Poincaré triples, W is a $(W(n), h)$ orientation of ζ, φ_i are the associated quadratic functions and $g: X \to G_k$ is a map such that $T(\bar{g})\alpha_1 = \alpha_2$, then*

$$\varphi_1(u) = \varphi_2(u) + j(x \cup u)$$

where

$$x = d(W, T(g)^* W) = \sum v_{n+1-2^i} \cup g^* u_{2^i-1}$$

where $v_i = v_i(\zeta)$ are the Wu classes.

To obtain numerical invariants for (X, ζ, α, W) we construct an algebraic invariant as follows:

Definition (1.19). Suppose V is a finite dimensional vector space over $\mathbf{Z}_2$. A function $\varphi: V \to \mathbf{Z}_4$ is quadratic if

$$\varphi(u + v) = \varphi(u) + \varphi(v) + jt(u, v)$$

where $t: V \otimes V \to \mathbf{Z}_2$ is a bilinear pairing. φ is *nonsingular* if t is. If $\varphi_i: V_i \to \mathbf{Z}_4$ are two such functions, φ_1 is *isomorphic* to φ_2 if there is a linear isomorphism $T: V_1 \to V_2$ such that $\varphi_1 = \varphi_2 T$. $(\varphi_1 + \varphi_2): V_1 \oplus V_2 \to \mathbf{Z}_4$ is defined by $(\varphi_1 + \varphi_2)(u, v) = \varphi_1(u) + \varphi_2(v)$. $(-\varphi_1)(u) = -\varphi_1(u)$. $\varphi_1\varphi_2: V_1 \otimes V_2 \to \mathbf{Z}_4$ is the unique quadratic function such that $\varphi_1\varphi_2(u \otimes u) = \varphi_1(u)\varphi_2(v)$.

THEOREM (1.20). *There is a unique function σ from non-singular quadratic functions as in* (1.9) *to $\mathbf{Z}_8$ satisfying*

(i) *If* $\varphi_1 \approx \varphi_2$, $\sigma(\varphi_1) = \sigma(\varphi_2)$.

(ii) $\sigma(\varphi_1 + \varphi_2) = \sigma(\varphi_1) + \sigma(\varphi_2)$.

(iii) $\sigma(-\varphi_1) = -\sigma(\varphi_1)$.

(iv) $\sigma(\gamma) = 1$ *where* $\gamma: \mathbf{Z}_2 \to \mathbf{Z}_4$ *by* $\gamma(0) = 0$, $\gamma(1) = 1$.

Furthermore σ satisfies:

(v) $\sigma(\varphi_1\varphi_2) = \sigma(\varphi_1)\sigma(\varphi_2)$.

(vi) *If* $\varphi: V \to Z_4$, $\sigma(\varphi) = (\dim V) \bmod 2$.

(vii) *If* $\varphi = j\varphi'$,

$$\sigma(\varphi) = l(\mathrm{Arf}\,\varphi')$$

where $l: \mathbf{Z}_2 \to \mathbf{Z}_8$ *is the homomorphism sending* 1 *to* 4.

(viii) *If U is a finitely generated free abelian group,* $\theta: U \otimes U \to \mathbf{Z}$ *is a symmetric, unimodular bilinear form,* $\psi(u) = \theta(u, u)$ *and* $\varphi: U/2U \to \mathbf{Z}_4$ *is defined by* $\varphi(u) = \psi(u) \bmod 4$, *then φ is quadratic and*

$$\sigma(\varphi) = (\textit{signature}\ \psi) \bmod 8.$$

(ix) *Suppose* $t: V \otimes V \to \mathbf{Z}_2$ *is the bilinear form associated to* $\varphi: V \to \mathbf{Z}_4$,

$$V_1 \xrightarrow{\ \nu\ } V \xrightarrow{\ \delta\ } V_2$$

is an exact sequence of $\mathbf{Z}_2$ vector spaces, and $t': V_1 \otimes V_2 \to \mathbf{Z}_2$ *is a nonsingular bilinear form such that* $t'(u, \delta v) = t(\nu u, v)$. *If* $\varphi\nu = 0$, $\sigma(\varphi) = 0$.

(x) *If* $\varphi_1, \varphi_2: V \to \mathbf{Z}_4$ *have the same bilinear form t, then* $\varphi_2(u) = \varphi_1(u) + jt(u, x)$ *for some x and*

$$\sigma(\varphi_1) - \sigma(\varphi_2) = m(\varphi_1(x))$$

where $m: \mathbf{Z}_4 \to \mathbf{Z}_8$ *sends* 1 *to* 2.

(xi) $\sigma(\varphi)$ *is related to φ by the formula*

$$\sum\nolimits_{u \in V} i^{\varphi(u)} = \sqrt{2}^{\dim V} e^{\frac{\pi i \sigma(\varphi)}{4}}$$

where $i = \sqrt{-1}$.

Definition (1.21). If (X, ζ, α) is a $2n$-Poincaré triple, (Y, h) is a Wu

n-spectrum and W is a Y oriention of ζ, we define the Kervaire invariant of (X, ζ, α, W) to be

$$K(X, \zeta, \alpha, W) = \sigma(\varphi) \in \mathbf{Z}_8$$

where $\varphi: H^n(X) \longrightarrow \mathbf{Z}_4$ is given by

$$\varphi(u) = hW_* A_\alpha F(u, 0) .$$

(1.20) and (1.14) yield:

COROLLARY (1.22). *If MG is a Thom spectrum and (MG, h) is a Wu n-spectrum, K defines a homorphism*

$$K: \Omega_{2n}(G) \longrightarrow \mathbf{Z}_8$$

where $\Omega_{2n}(G)$ denotes the cobordism group based on G orientable manifolds.

Suppose $(X_i, \zeta_i, \alpha_i, W_i)$ are $2n$-Poincaré triples with $(W(n), h)$ orientations and $(g, f): (\zeta_1, X_1) \longrightarrow (\zeta_2, X_2)$ is a fibre map, ζ_1 and ζ_2 have some fibre dimension and g is homotopy equivalence on fibres. Then $H^n(X_1)$ splits over the cup product pairing into $f^* H^n(X_2) \oplus V$. Furthermore, f^* is a monomorphism (see [3]). (1.15) gives:

COROLLARY (1.23). *If* $T(g)^* W_2 = W_1$ *and* $T(g)_* \alpha_1 = \alpha_2$,

$$\sigma(\varphi_1 | V) = K(X_1, \zeta_1, \alpha_1, W_1) - K(X_2, \zeta_2, \alpha_2, W_2) .$$

If the Wu classes $v_j(\zeta_2) = 0$ for $j = n - (2^j - 1)$, then the above is independent of the choice of α_1 and α_2. If, in the above, one replaces W_1 by $T(g)^* W_2$, then $\sigma(\varphi | V)$ is independent of the choice of W_2.

Remark. If X_2 is 1-connected, n is odd, and (X_1, ζ_1, α_1) is a smooth or PL manifold, its normal bundle, and its Thom construction, then $\sigma(\varphi | V)$ is the surgery obstruction to making f a homotopy equivalence [3].

We conclude this section with some examples of cobordism theories in which K is defined.

Example (1.24). If $G_k = \{e\}$, $MG_k = S^k$, and (S, h) is a $W(n)$ spectrum for all n with h the unique map taking $H_{2n}(K_n, S) \approx \mathbf{Z}_2 \longrightarrow \mathbf{Z}_4$ such that $\lambda \longrightarrow 2$. $\Omega_{2n}(e)$ is framed cobordism and

$$K: \Omega_{2n}(e) \longrightarrow \mathbf{Z}_8$$

is 0 or 4 according as the Kervaire invariant [4] is 0 or 1. φ in this case has a somewhat simpler form, namely if $u \in H^n(X)$, $\varphi(u) \in \{S^{2n}, K_n\} \approx \mathbf{Z}_2 \subset \mathbf{Z}_4$ is the composition

$$S^{2n+k} \xrightarrow{\alpha} T(\zeta) \xrightarrow{\Delta} T(\zeta) \wedge X^+ \xrightarrow{T \wedge u} S^k \wedge K_n$$

where T comes from a framing of ζ.

Example (1.25). If $G_k = \text{Spin}_k$, there is a Wu n-spectrum $(M\text{Spin}, h)$ for $n \equiv 1 \bmod 4$, since

$$\chi(Sq^{n+1}) = \chi(Sq^{n-1})Sq^2 + \chi(Sq^n)Sq^1$$

and $Sq^i U_k = W_i U_k = 0$, $i = 1, 2$. For certain choices of h, the K obtained is equivalent to the Kervaire invariant defined in [1].

Example (1.26). If $G_k = SU_k$, the situation is as in (1.25) except h is unique because $H_n(BSU) = 0$.

Example (1.27). Suppose $\{MG_k\}$ is a sequence of classifying spaces. Let $BG_k(v_{n+1})$ be the fibration over BG_k with fibre K_n and k-invariant v_{n+1}. Let $MG_k(v_{n+1})$ be the associated spectrum. Clearly there is an h in this case. These are the cobordism theories studied in [2], particularly $BO_k(v_{n+1})$. Suppose M is a $2n$-manifold with a v_{n+1} orientation. Each choice of h for $MG(v_{n+1})$ gives a function

$$\varphi_h\colon H^n(M) \longrightarrow \mathbf{Z}_4 .$$

Let $L = \{u \mid \varphi_n(u)$ is independent of $h\}$. Then $\varphi \mid L$ is the quadratic function studied in [2]. In [2] the Kervaire invariant of M is defined if φ is zero on the radical of L and is $\sigma(\varphi \mid L/R)$, where R is the radical. In general this will be different from the invariant we have defined.

Example (1.28). If $G_k = \mathbf{Z}_2$ and $n = 1$, $\Omega_2(\mathbf{Z}_2)$ is the cobordism group of surfaces immersed in $\mathbf{R}^3$. (A cobordism is a 3-manifold immersed in $\mathbf{R}^3 \times I$.) $H_2(K_1, M(\mathbf{Z}_2)) \approx \mathbf{Z}_4$, so that h is unique up to a sign. One may prove:

$$K\colon \Omega_2(\mathbf{Z}_2) \approx \mathbf{Z}_8 .$$

φ has the following geometric interpretation suggested to me by Dennis Sullivan. Let $i\colon S \longrightarrow \mathbf{R}^3$ be an immersion of a compact, closed surface S. If $u \in H^1(S)$, choose an embedded circle $S^1 \subset S$ representing the dual of u. Let T be a tubular neighborhood of S^1 in S. $i(T)$ is a twisted strip in $\mathbf{R}^3$. Then $\varphi(u)$ is the number of half twists of $i(T)$. (Möbius band has ± 1 half twists depending on whether its twist is right- or left-handed.) The number of half twists of $i(T)$ only makes sense modulo four because one must frame the normal bundle of $i(S^1)$ in $\mathbf{R}^3$ in order to count twists. This framing is determined up to 720° since $\pi_1(SO_3) = \mathbf{Z}_2$. In this situation one easily sees that $K((S, i)) = \sigma(\varphi)$ is the surgery obstruction to making S immersion cobordant to S^2.

Example (1.29). If $G_k = SO_k$ and n is even, there is an h making (MSO_k, h) a Wu n-spectrum for all even n which may be chosen as follows: Let $\bar{U}_k\colon MSO_k \longrightarrow K(\mathbf{Z}, k)$ be the Thom class and let $p_2\colon K(\mathbf{Z}_2, n) \longrightarrow K(\mathbf{Z}_4, 2n)$ represent the Pontrjagin (cohomology) square. Let h be

$$(p_{2*}\bar{U}_*)\colon H_{2n}(K_n, MSO) \longrightarrow H_{2n}(K(\mathbf{Z}_4, 2n); \mathbf{Z}) = \mathbf{Z}_4 .$$

With this choice of h, $\varphi(u)$ is $p_2(u)(\bar{x})$ where $\bar{x} \in H_{2n}(X; \mathbf{Z})$ is the fundamental class. S. Morita has shown in [5] that $\sigma(\varphi)$, in this case, is the index of X modulo 8. Hence

$$K\colon \Omega_{4l}(SO) \to \mathbf{Z}_8$$

is the index modulo 8.

2. Proofs

Let N be a large integer and $p\colon E_N \to K_N$ be the fibration with fibre K_{N+n} and k-invariant $Sq^{n+1}\iota_N$. Let $R\colon S^N K_n \to E_{N+n}$ be a map such that $pR = S^N\iota_N$. Let $E_k = \Omega^{N-k}E_N$ and let $R_k\colon S^k K_n \to E_{k+n}$ be the adjoint of R. Since $Sq^{n+1}\iota_n = 0$, there is a homotopy equivalence $\iota_n \times \iota_{2n}\colon E_n \to K_n \times K_{2n}$.

The following is well known:

LEMMA (2.1). *If $\mu\colon E_n \times E_n \to E_n$ is the loop multiplication map,*

$$\mu^*(\iota_{2n}) = \iota_{2n} \otimes 1 + 1 \otimes \iota_{2n} + \iota_n \otimes \iota_n .$$

The following is easily checked:

LEMMA (2.2). *If $m > 1$ and Y is a* CW *complex such that* $\dim Y \leqq 2n + m$,

$$\{Y, S^m \wedge K_n\} = [Y, S^m \wedge K_n]$$

and

$$[Y, S^m \wedge K_n] \overset{R_{m*}}{\approx} [Y, E_{n+m}] .$$

Proof of (1.2). By (2.1) and (2.2)

$$\{S^{2n}, K_n\} \approx [S^{2n+2}, E_{n+2}] \approx \mathbf{Z}_2 .$$

Proof of (1.3). Let T be the isomorphism

$$\{X, K_n\} = [S^2X, S^2K_n] \approx [S^2X, E_{n+2}] \approx [X, E_n] \approx [X, K_n \times K_{2n}] .$$

If $u \in H^n(X)$, $v \in H^{2n}(X)$, and $g\colon X \to S^{2n}$ is a map such that $g^*s_{2n} = v$,

$$TF(u, v) = T(\{u\} + \{\mu g\}) = u \times v$$

(where $\mu \in \{S^{2n}, K_n\}$ is the generator). With respect to the H space structure on $K_n \times K_{2n}$ coming from E_n, (2.1) yields

$$u \times v + u' \times v' = (u + u') \times (u \cup u' + v + v') .$$

Proof of (1.9). We wish to show

$$\{S^{2n}, K_n\} \to \{S^{2n+k}, S^k \wedge K_n\} \to \{S^{2n+k}, Y_k \wedge K_n\}$$

is nonzero if and only if $\chi(Sq^{n+1})U_k \neq 0$. Since

$$\lim \{S^{2n+k}, Y^a \wedge K_n\} \approx \{S^{2n+k}, Y_k \wedge K_n\}$$

where Y^a are the finite subcomplexes of Y_k, we may assume Y_k is a finite complex. Let Y^* be an m S-dual of Y_k. Let $g\colon Y^* \to S^{m-k}$ be the S-dual of the generator $v\colon S^k \to Y_k$. S-duality gives a commutative diagram

$$\begin{array}{ccc} \{S^{2n+k}, S^k \wedge K_n\} & \xrightarrow{p_*} & \{S^{2n+k}, Y_k \wedge K_n\} \\ \wr & & \wr \\ \{S^{2n+m}, S^m \wedge K_n\} & \xrightarrow{g^*} & \{S^{2n+k} \wedge Y^*, S^m \wedge K_n\} \\ \wr & & \wr \\ [S^{2n+m}, E_{n+m}] & \xrightarrow{g^*} & [S^{2n+k} Y^*, E_{n+m}] . \end{array}$$

The fibration $K_{m+2n} \to E_{n+m} \to K_m$ gives exact sequences:

$$\begin{array}{ccccccc} & \longrightarrow & H^{2n+m}(S^{2n+m}) \approx [S^{2n+m}, E_{n+m}] & \longrightarrow & & & H^m(S^{2n+m}) \\ & & \wr\wr & & \Big\downarrow g^* & & \Big\downarrow \\ H^{m+n-1}(Y^*) & \xrightarrow{Sq^{n+1}} & H^{2n+m}(Y^*) & \longrightarrow & [S^{2n+k} Y^*, E_{n+m}] & \longrightarrow & H^m(Y^*) . \end{array}$$

Hence p_* above is nonzero if and only if $Sq^{n+1} H^{m+n-1}(Y^*) = 0$, if and only if $\chi(Sq^{n+1})H^k(Y_k) = 0$. ($\chi(Sq^{n+1})$ corresponds to Sq^{n+1} under S-duality.)

Proof of (1.13). In this situation $T(f^*\zeta)$ is the S-dual of $T(\nu)$. S-duality gives a commutative diagram:

$$\begin{array}{ccc} \{T(\nu), K_n\} & \overset{d}{\approx} & \{S^{2n+k}, T(f^*\zeta) \wedge K_n\} \\ \Big\downarrow T^* & & \Big\downarrow i_* \\ \{X^+, K_n\} & \overset{A_\alpha}{\approx} & \{S^{2n+k}, T(\zeta) \wedge K_n\} \end{array}$$

where $T\colon X^+ \to T(\nu)$ is the Thom construction. If V is the Thom class of ν, $T^*V = F(u, 0)$ where u is the Poincaré dual of $f(S^n) \in H_n(X)$. $T(f^*\zeta) = S^k \vee S^{n+k}$. Hence $dV = \alpha_1 + \alpha_2$ where $\alpha_1 \in \{S^{2n+k}, S^k \wedge K_n\}$ and $\alpha_2 \in \{S^{2n+k}, S^{n+k} \wedge K_n\}$. Note α_2 is the generator because V is the Thom class.

$$\begin{aligned} \varphi(u) &= hW_*A_\alpha F(u, 0) \\ &= hW_*i_*(\alpha_1 + \alpha_2) \\ &= hW_*i_*\alpha_1 + h(\beta_f) . \end{aligned}$$

Thus $\varphi(u) - h(\beta_f) = 0$ or 2 according as $\alpha_1 = 0$ or the generator of $\{S^{2n+k}, S^k \wedge K_n\}$. Note α_1 depends only on ν; hence to compute $\varphi(u)$ we can choose any X^+ containing S^n with normal bundle ν.

Suppose ν is trivial. Referring to (1.12), $\tilde{f} \in \{X^+, S^n\}$ desuspends to $X^+ \to T(\nu) \to S^n$ and hence $\tilde{f}$ desuspends to u. Hence by (1.12), $\varphi(u) - h(\beta_f) = 0$.

Suppose ν is not trivial and n is odd. Let $f\colon S^n \to S^n \times S^n$ be the diagonal. Then ν is the normal bundle of $f(S^n)$. Take $Y_k = S^k$. Then $\beta_f = 0$, $u = s_n \otimes 1 + 1 \otimes s_n$ and

$$\varphi(u) = \varphi(s_n \otimes 1) + \varphi(1 \otimes s_n) + j(s_n \otimes s_n)(S^n \times S^n) = 2 .$$

$\varphi(s_n \otimes 1) = 0$ because $(S^n, *) \subset S^n \times S^n$ has trivial normal bundle.

The same argument as above works for n even if one takes f to be a multiple of the diagonal.

Proof of (1.14). We use $T(\eta) \cup \widehat{T(\zeta)}$, the mapping cone of $T(\zeta) \subset T(\eta)$, instead of $T(\eta)/T(\zeta)$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & T(\eta) \cup \widehat{T(\zeta)} & \longrightarrow & T(\eta) \wedge (Y \cup \hat{X}) & & \\
 & {\scriptstyle\alpha'}\nearrow & \Big| & & \Big\downarrow & & \\
S^{2n+k+1} & & \Big| & & T(\eta) \wedge SX^+ & \longrightarrow & T(\eta) \wedge SY^+ \xrightarrow{\ c\ } Y^k \wedge SK_n \\
 & {\scriptstyle\alpha}\searrow & \Big\downarrow & & \Big\uparrow & & \\
 & & ST(\zeta) & \longrightarrow & T(\zeta) \wedge SX^+ & &
\end{array}$$

where $c = W \wedge Sv$, $v\colon Y^+ \to K_n$ and the unlabeled maps are the obvious maps. If $\beta\colon S^{2n+k+1} \to Y_k \wedge SK_n$ denotes the composition of the bottom line, $\varphi(i^*v) = h(\beta)$. The top line is zero because $Y \cup \hat{X} \to SX^+ \to SY^+$ is zero. Hence $\varphi(i^*v) = 0$.

Proof of (1.16). If W_1 and W_2 are $(W(n), h)$ orientations of ζ and $v = d(W_1, W_2)$, W_2 is the composition

$$T(\zeta) \xrightarrow{\ \Delta\ } T(\zeta) \times T(\zeta) \xrightarrow{W_1 \times (vU_k)} W_k(n) \times K_{n+k} \xrightarrow{\ \mu\ } W_k(n)$$

where μ is the action of K_{n+k} on $W_k(n)$. Consider:

$$\begin{array}{ccccc}
 & & (T(\zeta) \wedge X^+) \vee (T(\zeta) \wedge X^+) & & \\
 & {\scriptstyle\alpha'}\nearrow & \| & \nwarrow{\scriptstyle a} & \\
 & & (T(\zeta) \vee T(\zeta)) \wedge X^+ & & \\
 & & \cap & & \\
S^{2n+k} & \xrightarrow{\ \Delta\alpha\ } & (T(\zeta) \times T(\zeta)) \wedge X^+ & \xrightarrow{\ b\ } & W_k(n) \wedge K_n
\end{array}$$

where α' is a lifting of $\Delta\alpha$, $b = \mu(W_1 \times vU_k) \wedge u$, $a = (W_1 \wedge u) \vee c$, and $c = i(vU_k) \wedge u$, $i\colon K_{n+k} \to W_k(n)$ the inclusion. $\alpha' = \alpha_1 + \alpha_2$ where α_1 and α_2 are on the two factors of the wedge.

$$\begin{aligned}
\varphi_2(u) &= h(b\Delta\alpha) \\
&= h(a\alpha_1) + h(a\alpha_2) \\
&= \varphi_1(u) + h(a\alpha_2) .
\end{aligned}$$

The generator of $\pi_{2n+k}(K_{n+k} \wedge K_n)$ goes into the generator of $\{S^{2n+k}, W_k(n) \wedge K_n\}$ (see proof of (1.3)). Hence

$$\begin{aligned}
h(a\alpha_2) &= j(vU_k \otimes u)(S^{2n+k}) \\
&= j(v \cup u)(x) .
\end{aligned}$$

Proof of (1.18). Let l be a large integer, F_l the H-space of degree one, based maps of S^l to itself and G_l the unbased, degree one maps of S^{l-1} to itself. The unreduced suspension gives a map $\rho\colon G_l \to F_l$. Consider

$$[X, G_l] \approx [X, F_l] \subset [S^l X^+, S^l] \approx \{X^+, S^0\} \overset{A_\alpha}{\approx} \{S^{2n+k}, T(\zeta)\}$$

where A_α is S-duality with respect to α. Under this map $[X, G_l]$ maps to those $\bar{\alpha}$'s such that $(X, \zeta, \bar{\alpha})$ is a Poincaré triple. Hence if (X, ζ, α_i), $i = 1, 2$ are Poincaré triples, there is a $g\colon X \to G_l$ which goes to α_2 under A_{α_1}. g gives a stable automorphism of ζ by

$$\mathrm{id} \times \bar{g}\colon \zeta + 0^l \to \zeta + 0^l$$

where 0^l is the trivial bundle $X \times S^{l-1}$ and $\bar{g}(x, s) = \big(x, g(x)(s)\big)$. $T(\mathrm{id} \times \bar{g})$ is the composition

$$\hat{g}\colon S^l T(\zeta) \xrightarrow{S^l\Delta} S^l\big(T(\zeta) \wedge X^+\big) = T(\zeta) \wedge S^l X^+ \xrightarrow{\mathrm{id} \wedge \tilde{g}} T(\zeta) \wedge S^l = S^l T(\zeta)$$

where $\tilde{g} = \rho g$. By the definition of g, $\hat{g}_* \alpha_1 = \alpha_2$. If W is $W(n)$ orientation for $T(\zeta)$,

$$W'\colon S^l T(\zeta) \xrightarrow{\hat{g}} S^l T(\zeta) \xrightarrow{S^l W} S^l W_k(n) \longrightarrow W_{k+l}(n)$$

is the new orientation produced by g. We wish to determine $d(W, W')$.

Choose a base point for X. Identifying $S^l X^+$ with $S^l \vee S^l X$, the adjoint of $\rho g\colon X^+ \to F_l$ gives a map $g'\colon S^l X \to S^l$. Identify $T(\zeta) \wedge S^l X^+$ with

$$T(\zeta) \wedge (S^l \vee S^l X) = \big(T(\zeta) \wedge S^l\big) \vee \big(T(\zeta) \wedge S^l X\big) .$$

Under this identification, $\mathrm{id} \wedge \tilde{g}$ becomes $(\mathrm{id} \wedge \mathrm{id}) \vee (\mathrm{id} \wedge g')$.

$$T(\zeta) \wedge S^l X \xrightarrow{\mathrm{id} \wedge g'} T(\zeta) \wedge S^l = S^l T(\zeta) \xrightarrow{W} W_{k+l}(n)$$

factors through the inclusion of the fibre $i\colon K_{k+l+n} \to W_{k+l}(n)$. Let $W(\mathrm{id} \wedge g') = iv$, $v\colon T(\zeta) \wedge S^l X \to K_{k+l+n}$. v will have the form

$$v = \sum x_i U_k \otimes S^l y_i$$

and $d(W, W') = \sum x_i y_i$. By the fibre space definition of functional operations

$$v \in \chi(Sq^{n+1})_{\mathrm{id} \wedge g'}(U_k \otimes s_l) .$$

One easily checks that this operation and $\chi(Sq^i)_{g'}(s_l)$, $i > 0$, has zero indeterminacy. Using the Cartan formula, $\chi(Sq^i)U_k = v_i U_k$ and the exact sequence definition of functional operations one easily checks that

$$v = \sum v_{n+1-i}\, U_k \otimes \chi(Sq^i)_{g'}(s_l) .$$

We complete the proof of (1.18) by proving

LEMMA (2.3).

$$\begin{aligned}\chi(Sq^i)_{g'}(s_l) &= S^l(u_{2^j-1}) && \text{if } i = 2^j \\ &= 0 && \text{if } i \neq 2^j .\end{aligned}$$

Proof. Let γ be the S^{l-1} fibration over SX defined by $g\colon X \to G_l$. It is well known that

$$T(\gamma) = S^l \cup g' S^l X$$

where S^l corresponds to the "fibre" of $T(\gamma)$. Hence

$$S(\chi(Sq^i)_{g'}(s_l))\, U_l = \chi(Sq^i)\, U_l$$

where U_l is the Thom class of $T(\gamma)$. $\chi(Sq^i)\, U_l = v_i(\gamma)\, U_l$. Let $\hat{g}\colon SX \to BG_l$ be the classifying map of γ and $\bar{v}_i \in H^i(BG_l)$ the Wu classes. Then

$$S(\chi(Sq^i) g(s_l)) = \hat{g}^* \bar{v}_i .$$

$\chi(Sq^{2^i}) = Sq^{2^i}$ + decomposables and $\chi(Sq^j)$ is decomposable for $j \neq 2^i$. Hence $\bar{v}_j$ is decomposable for $j \neq 2^i$ and is W_{2^i} + decomposables for $j = 2^i$. $H^*(SX)$ has zero cup products. Hence $\hat{g}^*(v_j) = 0, j \neq 2^i$ and w_{2^i} for $j = 2^i$. w_j suspends to $u_{j-1} \in H^*(G_l)$ and the lemma is proved.

3. Proof of Theorem (1.20)

Suppose V is a finite dimensional vector space over $\mathbf{Z}_2$ and $\varphi\colon V \to \mathbf{Z}_4$ is any function. Let $\lambda\varphi$ be the complex number defined by

$$\lambda(\varphi) = \sum\nolimits_{u \in V} i^{\varphi(u)} .$$

$\lambda(\varphi)$ is well-defined since $i^4 = 1$.[2]

LEMMA (3.1). *If φ is linear,*

$$\begin{aligned}\lambda(\varphi) &= 2^{\dim V} && \text{if } \varphi = 0 \\ &= 0 && \text{if } \varphi \neq 0 .\end{aligned}$$

Proof. For any $v \in V$

$$\sum\nolimits_u i^{\varphi(u)} = \sum\nolimits_u i^{\varphi(u+v)} = i^{\varphi(v)} \sum i^{\varphi(u)} .$$

Hence $\lambda(\varphi) = i^{\varphi(v)} \lambda(\varphi)$. Hence $\lambda(\varphi) = 0$ or $\varphi(v) = 0$ for all v.

The following is immediate

LEMMA (3.2).

$$\begin{aligned}\lambda(\varphi_1 + \varphi_2) &= \lambda(\varphi_1)\lambda(\varphi_2) \\ \lambda(-\varphi) &= \overline{\lambda(\varphi)} .\end{aligned}$$

LEMMA (3.3). *If φ is quadratic and non-singular, $\lambda(\varphi)^8$ is a positive real number.*

Proof. Let $t\colon V \otimes V \to \mathbf{Z}_2$ be the bilinear form of φ. For some $x \in V$,

[2] This method of constructing an algebraic invariant was suggested to me by Paul Monsky.

$$t(u, u) = t(u, x)$$

for all u.

$$0 = \varphi(2u) = 2\varphi(u) + jt(u, u) .$$

Hence $2\varphi(u) = jt(u, x)$. Therefore

$$\begin{aligned} \varphi(u + v) + \varphi(v) &= \varphi(u) + 2\varphi(v) + jt(u, v) \\ &= \varphi(u) + jt(u + x, v); \end{aligned}$$

$$\begin{aligned} \lambda(2\varphi) &= \textstyle\sum_{u,v} i^{\varphi(u)+\varphi(v)} \\ &= \textstyle\sum_{u,v} i^{\varphi(u+v)+\varphi(v)} \\ &= \textstyle\sum_u i^{\varphi(u)} \sum_v i^{jt(u+x,v)} . \end{aligned}$$

$t(u + x, v)$ is linear in v and zero for all v if and only if $u = x$. Therefore

$$\lambda(2\varphi) = 2^{\dim V} i^{\varphi(x)} .$$

Hence

$$\lambda(\varphi)^8 = \lambda(2\varphi)^4 = 2^{4 \dim V} .$$

Therefore

$$\lambda(\varphi) = \sqrt{2}^{\dim V} e^{\frac{\pi i}{4} n}$$

for some $n \in \mathbf{Z}_8$. Define $\sigma(\varphi) = n$. By the definition of σ and (3.2), it satisfies (i), (ii), (iii), and (xi) (1.20). If $\gamma\colon \mathbf{Z}_2 \to \mathbf{Z}_4$ by $\gamma(0) = 0$, $\sigma(\gamma) = 1$, and hence (iv) if satisfied. We next show that σ is unique. This follows by induction on dim V and the following lemma.

LEMMA (3.4). *If φ is non-singular quadratic,*

$$\gamma + \varphi \approx \varepsilon_1\gamma + \varepsilon_2\gamma + \varphi'$$

where $\varepsilon_i = \pm 1$.

Proof. Choose $u, v \in V$ such that $t(u, v) = 1$ and $u = v$ if possible. Let $V_1 = \{u, v\}$, $V_2 = \{w \in V \mid t(u, w) = t(v, w) = 0\}$. V is a direct sum of V_1 and V_2 and $\varphi \mid V \approx \varphi \mid V_1 + \varphi \mid V_2$. If $u = v$, $\varphi \mid V_1 \approx \pm \gamma$. If $u \neq v$, let U_1, U_2 and U_3 be the subspaces of $\mathbf{Z}_2 + V_1$ spanned by $(1, v)$, $(1, u)$, and $(1, u + v)$ respectively. One easily checks that

$$\gamma + \varphi \mid V_1 = \sum (\gamma + \varphi) \mid U_i$$

and $(\gamma + \varphi) \mid U_i = \pm \gamma$.

The multiplicativity of σ easily follows from (3.4) and the fact that $\gamma\varphi = \varphi$ for any φ.

If $\varphi = j\varphi'$, $\lambda(\varphi) = (n - m)$ where n and m are the number of elements in V at which φ' is 0 and 1, respectively. Hence $\sigma(\varphi)$ is 0 or 4 according as Arf φ' is 0 or 1.

To prove (1.20) (ix), there is no loss in generality if we assume

$$0 \longrightarrow V_1 \xrightarrow{\nu} V_1 \xrightarrow{\delta} V_2 \longrightarrow 0$$

is exact. Choose a map β: $V_2 \to V$ such that $\delta\beta = \text{id}$. If $u \in V_1$, and $\varphi\nu = 0$

$$\varphi\big((\nu) + \beta(v)\big) = \varphi\big(\beta(u)\big) + jt\big(\nu(u), \beta(v)\big)$$
$$= \varphi\big(\beta(u)\big) + jt'(u, v) .$$

Hence

$$\lambda(\varphi) = \sum i^{\varphi(\nu(u)+\beta(v))}$$
$$= \sum_u i^{\varphi(\beta(u))} \sum_v i^{jt'(u,v)}$$
$$= \text{positive real number}$$

since $t'(u, v)$ is linear in v and zero for all v if and only if $u = 0$. Hence if $\varphi\nu = 0$, $\sigma(\varphi) = 0$.

If φ_1, φ_2: $V \to \mathbf{Z}_4$, and $\varphi_2(u) = \varphi_1(u) + jt(u, x)$ for some x as in (x),

$$\lambda(\varphi_2) = \sum i^{\varphi_1(u)+jt(u,x)}$$
$$= \sum i^{\varphi_1(u+x)-\varphi_1(x)}$$
$$= i^{-\varphi_1(x)} \lambda(\varphi_1) .$$

Hence $\sigma(\varphi_2) = \sigma(\varphi_1) - l\varphi_1(x)$ where l: $\mathbf{Z}_4 \to \mathbf{Z}_8$ takes 1 to 2.

Finally we prove (1.20) (vii). (viii) is obviously true if θ is a diagonal form. Recall the Grothendieck group of symmetric unimodular bilinear forms over the integers is isomorphic to $\mathbf{Z} + \mathbf{Z}$ and the isomorphism is given by the rank + the signature [6]. This means that if θ is such a form, there are forms D, θ_1, and θ_2, where D is diagonal, such that $\theta + \big(\theta_1 + (-\theta_1)\big) \approx D + \big(\theta_2 + (-\theta_2)\big)$. The desired result now follows.

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(Received September 14, 1971)