HISTORY OF THE KERVAIRE INVARIANT PROBLEM

WILLIAM BROWDER

The history of this invariant could very well be considered to start with the paper of Pontryagin in 1938, where he introduced Framed Bordism (as it is now known) as a tool to calculate homotopy groups of spheres, using smooth manifolds. He proved that the second stable homotopy group of the n-sphere was zero, but this was soon shown to be incorrect by algebraic methods. The problem was the absence of the Kervaire invariant.

For oriented closed manifolds of dimension $4k$ the middle dimensional intersection pairing defines a nonsingular symmetric bilinear form over the integers, whose signature gives a famous algebraic invariant often called the index of the manifold. For dimensions $4k+2$ the intersection bilinear form is skew symmetric, and thus can be put in canonical form, so no apparent such invariants exist. But if, in the mod 2 version, we can enrich the intersection form to be associated to a quadratic form, that quadratic form has an invariant, called the Arf invariant after its discoverer.

I like to call this invariant the democratic invariant as it can be defined as follows:

Let V be a vector space of finite dimension over the integers mod 2, and let $q : V \longrightarrow Z/2$ be a quadratic form with associated bilinear form f (i.e., $q(a+b) = q(a) + q(b) + f(a,b)$). Consider q to be a vote between 0 and 1 (the candidates) among the elements of V (voters) and the Arf invariant of q is defined to be the winner of the election. If the bilinear form f is nonsingular the election is decisive. However, in the general case the election is a tie if and only if there is some element r of V such that $f(r,x) = 0$ for all x in V, but $q(r) = 1$. (The election reaches a clear result unless some radical element votes positively).

Pontryagin had failed to note that an underlying obstruction to the process he was carrying out in dimension 2 was quadratic rather than linear, so that its Arf invariant was an obstruction for his argument, but he corrected this mistake in a later paper in 1955.

This might be considered the prehistory of the topological invariant, and in my view the history properly begins with the paper of Kervaire in 1960 where he constructed a PL 10-manifold which was not of the homotopy type of a smooth manifold. In it he constructed a cohomology operation from dimension 5 to 10, for a 4-connected closed 10-manifold which could be framed (stable tangent bundle trivial) on the complement of a point, and this operation was quadratic. In his example the Arf invariant was non trivial, while for any smooth manifold of that type, he proved it would be trivial because of the vanishing of some homotopy.

Kervaire's operation is defined in analogous circumstances for all dimensions and in the famous 1962 paper of Kervaire and Milnor defined the middle dimensional surgery obstruction in dimensions of the form $4k+2$. It followed from the surgery theory developed there that this defined a Framed Bordism invariant. You could do surgery on the framed manifold to make it $2k$-connected to define the invariant and do surgery on a framed bordism to make the bordism similarly connected to prove it well defined.

If a PL manifold M has a trivial tangent bundle (or more properly microbundle) over the complement of a point it is in fact smoothable away from that point, from the theory of smoothing of PL manifolds of Mazur and Hirsch-Mazur. If M is of dimension $2n$ and $(n-1)$-connected, then its n^{th} homology has a basis of embedded spheres, and the normal bundle of each of these spheres is stably trivial. When $n = 1$, 3 or 7 it is trivial, but in other dimensions there are nontrivial possibilities, and if n is odd there is a single nontrivial possibility, namely the tangent bundle to the n-sphere. The quadratic form in these cases is simply given by whether this normal bundle is trivial or not. (For $n = 1$, 3 or 7, the definition of a quadratic form is related to the framing and is not homotopy invariant). The cohomology operation of Kervaire detects this non-triviality, and its model is actually the Thom complex of this tangent bundle.

If the Arf invariant of this form is zero, one can find enough embedded products $S^n \times R^n$ representing a basis of the middle cohomology to carry out surgery to make M into a homotopy sphere, and otherwise you cannot.

Thus the question of whether or not a framed manifold could have a nonzero Kervaire invariant then became a central question for differential topology, equivalent to the calculation of the subgroup of homotopy spheres which bounded framed manifolds in dimension $4k + 1$. The answer was yes for dimensions 2, 6 and 14 because of the parallelizability of the spheres of dimensions 1, 3 and 7, but remained open for other dimensions of the form $4k + 2$.

E. H. Brown in 1965 showed that for dimension $8k + 2$ Spin manifolds, the cohomology operation could be defined on the middle dimension without assuming $4k$-connectivity, so that the Kervaire Invariant could be made a Spin bordism invariant, using surgery only to make the manifold simply connected. Subsequently, he and F. P. Peterson in 1966 used this to show the Kervaire Invariant vanished on Framed Bordism in dimension $8k + 2$.

Then in 1968, I proved that the Kervaire Invariant vanished on Framed Bordism in dimensions different from $2^n - 2$, and related possible nonvanishing in those dimensions to the existence of elements in the homotopy of spheres related to certain elements in the Adams spectral sequence. It turned out that this element had already been constructed by Mahowald and Tangora in dimension 30 and such an element was later constructed in dimension 62 by Barratt, Jones and Mahowald.

My method was to define the quadratic form by means of a functional Steenrod operation on a subgroup of the middle cohomology mod 2 which allowed me to define the form on a manifold M which had been "oriented" in a theory in which the $2k+2$ Wu class was zero, a condition satisfied by any $4k+2$ manifold. (The Wu classes are defined using the Steenrod operations in M and are directly related to the Stiefel-Whitney classes). A subtlety was that everything depended on how you made this Wu class vanish, how you chose the "orientation". This definition allowed one to define the Kervaire invariant immediately on the framed (or otherwise "oriented") manifold without doing any surgery or other geometrical operation, and so gave a definition in a purely homotopy theoretical context of spaces satisfying Poincaré duality.

For a smooth manifold M, my definition can be translated into a condition on extending vector fields on submanifolds of M representing the middle dimensional mod 2 cohomology.

A simpler proof of my theorem on Framed Bordism was given by Jones and Rees, and Jones gave a beautiful construction of the 30 dimensional manifold representing the Mahowald-Tangora homotopy element.

After the results in dimensions 30 and 62, attention turned to dimension 126, the first open case, but this has resisted concerted attempts by many strong homotopy theorists and still is unknown. Many had tried to prove that all of these possible elements (or manifolds) existed but, conscious of the Hopf invariant 1 results (only three possible dimensions 1, 3 and 7), some began to try to prove that they did not exist beyond some dimension. Now this has been carried out by Hill, Hopkins and Ravenel for dimensions greater than 126.

Much other work has been done on the Kervaire Invariant because of its importance in surgery theory, e.g., Sullivan (product formula), Ranicki (algebraic surgery), and others.

References

[1] Pontryagin, L. S., *Gladkie mnogoobraziya i ih primeneniya v teorii gomotopiä.* (Russian) [Smooth manifolds and their applications in homotopy theory.], Trudy Mat. Inst. im. Steklov. no. 45., Izdat. Akad. Nauk SSSR, Moscow, 1955. 139 pp.

[2] Kervaire, Michel A., *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. 34 1960, 257–270.

[3] Kervaire, Michel A.; Milnor, John W., Groups of homotopy spheres. I, Ann. of Math. (2) 77 1963, 504–537.

[4] Brown, Edgar H., Jr., *Note on an invariant of Kervaire*, Michigan Math. J. 12 1965, 23–24.

[5] Brown, Edgar H., Jr.; Peterson, Franklin P., *The Kervaire invariant of* $(8k+2)$*-manifolds*, Bull. Amer. Math. Soc. 71 1965, 190–193.

[6] Browder, William, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math. (2) 90 1969, 157–186.

[7] Browder, William, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65. Springer-Verlag, New York-Heidelberg, 1972. ix+132 pp.

[8] Mahowald, Mark; Tangora, Martin, *Some differentials in the Adams spectral sequence*, Topology 6 1967, 349–369.

[9] Barratt, M. G.(1-NW); Jones, J. D. S.(4-WARW); Mahowald, M. E.(1-NW), *Relations amongst Toda brackets and the Kervaire invariant in dimension* 62, J. London Math. Soc. (2) 30 (1984), no. 3, 533–550.

[10] Jones, John; Rees, Elmer, *Kervaire's invariant for framed manifolds. Algebraic and geometric topology* (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, pp. 141–147, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

[11] Jones, John D. S., *The Kervaire invariant of extended power manifolds*, Topology 17 (1978), no. 3, 249–266.

[12] Jones, John; Rees, Elmer, *Kervaire's invariant for framed manifolds. Algebraic and geometric topology* (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1, pp. 141–147, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

[13] Brown, Edgar H., Jr., *The Kervaire invariant and surgery theory. (English summary) Surveys on surgery theory,* Vol. 1, 105–120, Ann. of Math. Stud., 145, Princeton Univ. Press, Princeton, NJ, 2000.

[14] Barratt, M. G.; Jones, J. D. S.; Mahowald, M. E., *The Kervaire invariant and the Hopf invariant*, Algebraic topology (Seattle, Wash., 1985), 135–173, Lecture Notes in Math., 1286, Springer, Berlin, 1987.

[15] Barratt, M. G.; Jones, J. D. S.; Mahowald, M. E., *The Kervaire invariant problem. Proceedings of the Northwestern Homotopy Theory Conference*, (Evanston, Ill., 1982), 9–22, Contemp. Math., 19, Amer. Math. Soc., Providence, RI, 1983.

[16] Brown, Edgar H., Jr., *The Arf invariant of a manifold. 1969 Conf. on Algebraic Topology*, (Univ. of Illinois at Chicago Circle, Chicago, Ill., 1968) pp. 9–18 Univ. of Illinois at Chicago Circle, Chicago, Ill.