Chapter IV. Generalized Equivariant Cohomology

In this chapter we show how to construct generalized equivariant cohomology theories, using G-spectra. We then show how any generalized theory is connected by a spectral sequence to the "classical" theory of Chapter I.

1. Equivariant cohomology via G-spectra

We work with the category of spaces with base points in this section. Let Y be a G-spectrum. Then for any G-space X we have homomorphisms

$$\eta_k\colon [[S^{k-n}X;Y_k]] \overset{S}{\to} [[S^{k-n+1}X;SY_k]] \xrightarrow{\varepsilon_{k\#}} [[S^{k-n+1}X;Y_{k+1}]].$$

Thus, with these maps, the groups $[[S^{k-n}X;Y_k]]$ form a direct system and we define

$$(1.1) \qquad \tilde{H}^n_G(X;\underline{Y}) = \lim_k [[S^{k-n}X;Y_k]] = \lim_k [S^kX;Y_{m+k}]].$$

Note that if X is locally compact then this is the same as

$$(1.2) \qquad \pi_{-n}(\underline{E}(X,\underline{Y})) = \lim_k \pi_{k-n}(E(X,Y_k)).$$

Note that $[[S^kX;Y_{n+k}]] \approx [[X;\Omega^kY_{n+k}]]$. If $A \subset X$ is invariant under G, then for any G-space W there is the exact sequence

$$[[X \cup C_A;W]] \to [[X;W]] \to [[A;W]]$$

of (2.1) in Chapter III. If (X,A) is a pair of G-complexes, then $X \cup C_A$ has the same equivariant homotopy type as does X/A. Thus, taking $W = \Omega^kY_{n+k}$, and passing to the limit over k, we obtain the exact sequence

$$(1.3) \qquad \tilde{H}^n_G(X/A;\underline{Y}) \to \tilde{H}^n_G(X;\underline{Y}) \to \tilde{H}^n_G(A;\underline{Y})$$

on the category $\mathscr{L}_0$ of G-complexes with base point.

Using the natural homeomorphism $S^{k-n}X \approx S^{k-(n+1)}SX$ we obtain a natural isomorphism $s_k\colon [[S^{k-n}X;Y_k]] \xrightarrow{\approx} [[S^{k-(n+1)}SX;Y_k]]$. These commute with the η_k and hence define a natural isomorphism

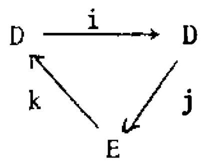
$$S^*\colon \tilde{H}^n_G(X;\underline{\underline{Y}}) \to \tilde{H}^{n+1}_G(SX,\underline{\underline{Y}}).$$

We have shown that $\tilde{H}^*_G(X;\underline{\underline{Y}})$ defines an equivariant cohomology theory on $\mathcal{B}_0$.

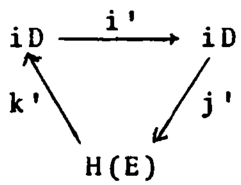
2. Exact couples

In this section we provide some background from the theory of exact couples. Let

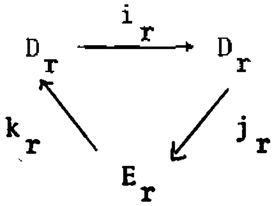
(2.1)



be an exact couple where E and D are bigraded, k has total degree 1 and i and j have total degree 0. Note that $(jk)^2 = 0$ and let H(E) be the homology of E with respect to the differential jk. The derived couple of (2.1) is



where $i' = i|iD$, j' is induced by ji^{-1} and k' is induced by k. Let $D_1 = D$ and $E_1 = E$. Iterating the above procedure we obtain the (r-1)st derived couple



where $E_r = H(E_{r-1})$ and $D_r = iD_{r-1} = i^{r-1}D$.

We shall now assume that

$$(2.2) \qquad \begin{cases} \deg i = (-1,1) \\ \deg j = (0,0) \\ \deg k = (1,0) \end{cases}$$

and it is then easy to check that

$$(2.3) \qquad \begin{cases} \deg i_r = (-1,1) \\ \deg j_r = (r-1,1-r) \\ \deg k_r = (1,0) \, . \end{cases}$$

We let $d_r = j_r k_r$ which has degree $(r,1-r)$. The system $\{E_r^{p,q}\}$ together with the differentials d_r then form a spectral sequence.

We shall now assume that, for some integer N,

$$(2.4) \qquad \begin{cases} E^{p,q} = 0 & \text{for } p < 0 \text{ and for } p > N \\ D^{p,q} = 0 & \text{for } p < 0. \end{cases}$$

From the exact sequence

$$\cdots \to D^{p,q} \xrightarrow{j} E^{p,q} \xrightarrow{k} D^{p+1,q} \xrightarrow{i} D^{p,q+1} \xrightarrow{j} E^{p,q+1} \to \cdots$$

we see that

$$i\colon D^{p+1,q} \xrightarrow{\approx} D^{p,q+1} \qquad \text{for } p > N.$$

For $n = p+q$ we let J^n be a group which is isomorphic to $D^{p,q+1}$ for $p > N$ and let $\theta^{p,q+1}\colon J^n \to D^{p,q+1}$ be some isomorphism chosen so that

$$(2.5) \qquad \begin{array}{ccc} & \xrightarrow{\theta^{p+1,q}} & D^{p+1,q} \\ J^n & & \Big\downarrow i \\ & \xrightarrow[\theta^{p,q+1}]{} & D^{p,q+1} \end{array}$$

commutes. Following θ by iterates of i we have homomorphisms $\theta^{p,q+1}: J^n \to D^{p,q+1}$ defined for all p (with $n = p+q$) such that (2.5) commutes.

If $r > N$ we see that $d_r = 0$, since $E_r^{p,q} = 0$ for $p < 0$ and for $p > N$. Thus

$$E_r^{p,q} \approx E_{r+1}^{p,q} \approx \ldots$$

for $r > N$ and we let $E_\infty^{p,q}$ denote the common value. The (r-1)st derived couple has the form

$$\ldots i^{r-1}D^{p,q} \xrightarrow{j_r} E_r^{p,q} \xrightarrow{k_r} i^{r-1}D^{p+r,q-r+1} \xrightarrow{i} i^{r-1}D^{p+r+1,q-r} \to \ldots$$

Now $i^{r-1}D^{p,q} \subset D^{p-r+1,q+r-1} = 0$ for r sufficiently large and $i^{r-1}D^{p+r,q-r+1} = \operatorname{Im} \theta^{p+1,q} \subset D^{p+1,q}$ for r sufficiently large. Thus, for r large, this exact sequence has the form

$$(2.6) \qquad 0 \to E_\infty^{p,q} \to \operatorname{Im} \theta^{p+1,q} \xrightarrow{i} \operatorname{Im} \theta^{p,q+1} \to 0.$$

That is, we have an exact sequence

$$(2.7) \qquad 0 \to E_\infty^{p,q} \to \frac{J^{p+q}}{\ker\theta^{p+1,q}} \xrightarrow{i} \frac{J^{p+q}}{\ker\theta^{p,q+1}} \to 0.$$

Put

$$(2.8) \qquad J^{p,q} = \ker\{\theta^{p,q+1}: J^{p+q} \to D^{p,q+1}\}$$

so that (2.7) provides the isomorphism

$$(2.9) \qquad E_\infty^{p,q} \approx J^{p,q}/J^{p+1,q-1} .$$

Thus we have that the spectral sequence $E_r^{p,q}$ converges to the graded group associated with the (finite) filtration

$$\dots \supset J^{p,q} \supset J^{p+1,q-1} \supset \dots$$

of $J^{p+q} = D^{M+1,p+q-M}$ for $M \geq N$.

3. The spectral sequence of a filtered G-complex

Let K be a G-complex and let $\{K_r\}$ be a sequence of G-subcomplexes such that

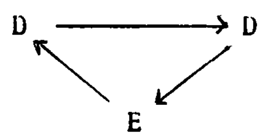
$$(3.1) \qquad \begin{cases} K_r \subset K_{r+1} \\ K_{-1} = \emptyset \\ K_N = K \end{cases}$$

where N is some given integer.

Let $\{\mathcal{H}^*, \delta^*\}$ be <u>any</u> equivariant cohomology theory and put

$$(3.2) \qquad \begin{cases} E^{p,q} = \mathcal{H}^{p+q}(K_p, K_{p-1}) \\ D^{p,q} = \mathcal{H}^{p+q-1}(K_{p-1}). \end{cases}$$

Then the exact cohomology sequence of the pair (K_p, K_{p-1}) provides an exact couple



as in section 2.

The differential d_1 is the composition

$$(3.3) \qquad E_1^{p,q} = \mathcal{H}^{p+q}(K_p, K_{p-1}) \to \mathcal{H}^{p+q}(K_p) \xrightarrow{\delta} \mathcal{H}^{p+q+1}(K_{p+1}, K_p) = E_1^{p+1,q}.$$

And the spectral sequence converges to the graded group associated with the filtration

$$J^{p,q} = \ker\{\mathcal{H}^{p+q}(K) \to \mathcal{H}^{p+q}(K_{p-1})\}$$

of $J^{p+q} = \mathcal{H}^{p+q}(K)$.

4. The main spectral sequence

Let $\{\mathcal{H}^*, \delta^*\}$ be any equivariant cohomology theory and let K be a G-complex of dimension $N < \infty$. If K is not finite then we shall assume that $\mathcal{H}^*$ also satisfies the axiom:

(A) If S is a discrete G-set with orbits S_α then $\prod i_\alpha^*: \mathcal{H}^n(S) \to \prod \mathcal{H}^n(S_\alpha)$ is an isomorphism, where $i_\alpha: S_\alpha \to S$ is the inclusion.

Letting $K_p = K^p$, the p-skeleton of K, the preceding section provides a spectral sequence with

$$E_1^{p,q} = \mathcal{H}^{p+q}(K^p, K^{p-1}) \approx \mathcal{H}^{p+q}(K^p/K^{p-1}).$$

Now

$$K^p/K^{p-1} \approx S^p C_p^+$$

the p-th reduced suspension of the discrete G-set C_p^+ where C_p stands for the set of all p-cells of K. Thus

$$E_1^{p,q} \approx \tilde{\mathcal{H}}^{p+q}(S^p C_p^+) \approx \tilde{\mathcal{H}}^q(C_p^+) \approx \mathcal{H}^q(C_p).$$

Now let $h^q \in \mathcal{C}_G$ denote the coefficient system of Chapter I, section 4, example (1). That is

$$h^q(G/H) = \mathcal{H}^q(G/H) = \tilde{\mathcal{H}}^q((G/H)^+).$$

We shall define an isomorphism

$$(4.1) \qquad \alpha: \tilde{\mathcal{H}}^q(C_q^+) \xrightarrow{\approx} C_G^p(K; h^q)$$

as follows:

For $\sigma \in C_p$ let

$$i_\sigma: (G/G_\sigma)^+ \to C_p^+$$

be the equivariant map defined by $i_\sigma(gG_\sigma) = g\sigma \in C_p$. Also let

$$j_\sigma: C_p^+ \to (G/G_\sigma)^+$$

be defined by $j_\sigma(g\sigma) = gG_\sigma$ and $j_\sigma(\tau)$ = base point if τ is not in the orbit of σ. Note that

$$(4.2)\qquad \begin{cases} i_{g\sigma} = i_\sigma \hat{g} \\ j_{g\sigma} = \hat{g}^{-1} j_\sigma \\ j_\sigma i_\sigma = 1 \\ j_\tau i_\sigma = 0 \text{ (the base point) if } \tau \notin G(\sigma), \end{cases}$$

where $\hat{g} = R_g \colon G/G_{g\sigma} = G/gG_\sigma g^{-1} \to G/G_\sigma$. Also note that $i_\sigma j_\sigma$ is the identity on $G(\sigma)$ and collapses everything else to the base point.

We have the induced maps

$$\begin{cases} i_\sigma^* \colon \tilde{\mathcal{H}}^q(C_p^+) \to \tilde{\mathcal{H}}^q((G/G_\sigma)^+) = h^q(G/G_\sigma) \\ j_\sigma^* \colon \tilde{\mathcal{H}}^q((G/G_\sigma)^+) \to \tilde{\mathcal{H}}^q(C_p^+). \end{cases}$$

Define, for $\lambda \varepsilon \tilde{\mathcal{H}}^q(C_p^+)$ and $\sigma \varepsilon C_p$,

$$(4.3)\qquad \alpha(\lambda)(\sigma) = i_\sigma^*(\lambda).$$

To check that $\alpha(\lambda)$ is equivariant we compute

$$\begin{aligned} \alpha(\lambda)(g\sigma) = i_{g\sigma}^*(\lambda) &= (i_\sigma \hat{g})^*(\lambda) \\ &= \hat{g}^* i_\sigma^*(\lambda) = \hat{g}^*(\alpha(\lambda)(\sigma)) \end{aligned}$$

as was to be shown. (See Chapter I, sections 5 and 6.)

We must check that α is an isomorphism. We shall show that its inverse is given by the map

$$\beta \colon C_G^p(K, h^q) \to \tilde{\mathcal{H}}^q(C_p^+)$$

defined as follows: Let $f \varepsilon C_G^p(K, h^q)$. Note that

$$j_{g\sigma}^*(f(g\sigma)) = (\hat{g}^{-1} j_\sigma)^*(\hat{g}^*(f(\sigma))) = j_\sigma^*(f(\sigma)).$$

Let $T \subset C_p$ be a system of representatives of the orbits of G on the set C_p and define

(4.4) $$\beta(f) = \prod_{\sigma\in T} j_\sigma^*(f(\sigma)).$$

Now we compute

$$\alpha(\beta(f))(\sigma) = i_\sigma^*(\beta(f)) = i_\sigma^*(\prod_{\tau\in T} j_\tau^*(f(\tau)))$$
$$= i_\sigma^* j_\sigma^*(f(\sigma)) = (j_\sigma i_\sigma)^*(f(\sigma)) = f(\sigma)$$

so that $\alpha\beta = 1$. Also

$$\beta(\alpha(\lambda)) = \prod_{\sigma\in T} j_\sigma^*(\alpha(\lambda)(\sigma))$$
$$= \prod_{\sigma\in T} j_\sigma^*(i_\sigma^*(\lambda)) = \prod_{\sigma\in T}(i_\sigma j_\sigma)^*(\lambda) = \lambda$$

so that $\beta\alpha = 1$. Thus α is an isomorphism as was to be shown.

Now we claim that under the isomorphism

$$E_1^{p,q} \approx \tilde{\mathcal{H}}^q(C_p^+) \xrightarrow{\alpha} C_G^p(K;h^q)$$

the differential d_1 becomes, up to sign, the coboundary.

We first remark that, up to sign, $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$ may be identified with the homomorphism

$$\tilde{\mathcal{H}}^{p+q}(K^p/K^{p-1}) \xrightarrow{\approx} \tilde{\mathcal{H}}^{p+q+1}(S(K^p/K^{p-1})) \xrightarrow{\psi_p^*} \tilde{\mathcal{H}}^{p+q+1}(K^{p+1}/K^p)$$

where $\psi_p\colon K^{p+1}/K^p \to S(K^p/K^{p-1})$ is an equivariant map defined as follows: If σ is a (p+1)-cell and $f_\sigma\colon S^p \to K^p$ is a characteristic map (chosen equivariantly) we follow f_σ by collapsing $K^p \to K^p/K^{p-1}$ and suspending $S^{p+1} \to S(K^p/K^{p-1})$ (unreduced on the left, reduced on the right). Then the cell $\sigma/\dot{\sigma} \subset K^{p+1}/K^p$ is identified with S^{p+1} in a canonical way (taking the base point into the north pole of S^{p+1}). The resulting maps $\sigma/\dot{\sigma} \to S(K^p/K^{p-1})$ are put together to form the map $\psi_p\colon K^{p+1}/K^p \to S(K^p/K^{p-1})$. The verification of this relies on the fact that in the Puppe sequence for the inclusion $i\colon K^p/K^{p-1} \to K^{p+1}/K^{p-1}$ the map $C_i \to S(K^p/K^{p-1})$ may be identified with ψ_{p+1}. The details will be left to the reader.

Now $K^{p+1}/K^p \approx S^{p+1}C^+_{p+1}$ and $S(K^p/K^{p-1}) \approx S^{p+1}C^+_p$ so that the map ψ_p is described by the induced maps $\sigma/\dot{\sigma} \subset S^{p+1}C^+_p \to S^{p+1}C^+_p = \bigvee_\tau S(\tau/\dot{\tau}) \to S(\tau/\dot{\tau})$ (where $\sigma\varepsilon C_{p+1}, \tau\varepsilon C_p$). It is easy to see that, in fact, this map has degree $[\tau\colon \sigma]$ (see Chapter I, section 1).

Thus d_1 is induced, up to sign, by

$$\eta_p^*\colon \tilde{\mathcal{H}}^{q+1}(SC^+_p) \to \tilde{\mathcal{H}}^{q+1}(SC^+_{p+1})$$

where $\eta_p\colon SC^+_{p+1} = \bigvee_\sigma S_\sigma \to \bigvee_\tau S_\tau = SC^+_p$ is an equivariant map such that the induced map $S_\sigma \to S_\tau$ has degree $[\tau\colon \sigma]$. (Here we use S_σ to stand for a copy of the circle indexed by the cell σ.)

We claim that the following diagram commutes

$$\begin{array}{ccc} \tilde{\mathcal{H}}^{q+1}(SC^+_p) & \xrightarrow{\eta_p^*} & \tilde{\mathcal{H}}^{q+1}(SC^+_{p+1}) \\ \downarrow{\scriptstyle \alpha S^{-1}} & & \downarrow{\scriptstyle \alpha S^{-1}} \\ C^p_G(K;h^q) & \xrightarrow{\delta^p} & C^{p+1}_G(K;h^q) \end{array} \tag{4.5}$$

where we use S to denote the suspension isomorphism. The proof is straightforward but will involve some cumbersome details. First, suppose σ is a $(p+1)$-cell and τ is a p-cell of K with $K(\tau) \subset K(\sigma)$. Then let θ^τ_σ denote the equivariant map $G/G_\sigma \to G/G_\tau$ induced by inclusion $G_\sigma \subset G_\tau$. Using $[\tau\colon \sigma]$ to denote maps of <u>degree</u> $[\tau\colon \sigma]$ we note that the diagram

$$\begin{array}{ccccc} S(G/G_\sigma)^+ & \xrightarrow{Si_\sigma} & SC^+_{p+1} & \xrightarrow{\eta_p} & SC^+_p \\ \downarrow{\scriptstyle \bigvee[\tau:\,\sigma]} & & & & \uparrow{\scriptstyle \bigvee Si_\tau} \\ \bigvee_{\tau\varepsilon T} S(G/G_\sigma)^+ & & \xrightarrow{\bigvee S\theta^\tau_\sigma} & & \bigvee_{\tau\varepsilon T} S(G/G_\tau)^+ \end{array}$$

of equivariant maps commutes, where T is the set of all p-cells τ with $K(\tau) \subset K(\sigma)$.

The induced diagram in cohomology is

$$\begin{array}{ccccc}
\tilde{\mathcal{H}}(S(G/G_\sigma)^+) & \xleftarrow{(Si_\sigma)^*} & \tilde{\mathcal{H}}(SC_{p+1}^+) & \xleftarrow{\eta_p^*} & \tilde{\mathcal{H}}(SC_p^+) \\
\Big\uparrow \sum_\tau [\tau:\ \sigma] & & & & \Big\downarrow \sum_\tau (Si_\tau)^* \\
\sum_{\tau \in T} \tilde{\mathcal{H}}(S(G/G_\sigma)^+) & & \xleftarrow{\quad \Sigma(S\theta_\sigma^\tau)^* \quad} & & \sum_{\tau \in T} \tilde{\mathcal{H}}(S(G/G_\tau)^+)
\end{array}$$

Since $(S\varphi)^* = S \circ \varphi^* \circ S^{-1}$ we obtain from this diagram that

$$(4.6) \qquad (Si_\sigma)^* \eta_p^* = S[\sum_\tau [\tau:\ \sigma](i_\tau \theta_\sigma^\tau)^*]S^{-1}.$$

Now let us verify that (4.5) commutes. Let $\lambda \varepsilon \tilde{\mathcal{H}}^{q+1}(SC_p^+)$ and, as usual, let σ be a (p+1)-cell of K. Then

$$(4.7) \qquad \begin{aligned} \alpha(S^{-1}(\eta_p^*(\lambda)))(\sigma) &= i_\sigma^*(S^{-1}(\eta_p^*(\lambda))) \\ &= S^{-1}(Si_\sigma)^*\eta_p^*(\lambda) \\ &= \sum_\tau [\tau:\ \sigma](i_\tau \theta_\sigma^\tau)^*(S^{-1}\lambda). \end{aligned}$$

(The last equality comes from (4.6).) On the other hand

$$\delta^p(\alpha S^{-1}(\lambda))(\sigma) = \sum_\tau [\tau:\ \sigma](\theta_\sigma^\tau)^*(\alpha(S^{-1}(\lambda))(\tau))$$

directly from the definition of δ^p. This may be further simplified to

$$\sum_\tau [\tau:\ \sigma](\theta_\sigma^\tau)^* i_\tau^*(S^{-1}\lambda),$$

the same as in (4.7). This shows that (4.5) commutes and hence, finally, that $d_1\colon E_1^{p,q} \to E_1^{p+1,q}$ becomes the coboundary under our isomorphism with $C_G^*(K;h^q)$. Thus we have

$$(4.8) \qquad E_2^{p,q} \approx H_G^p(K;h^q).$$

As noted before, the spectral sequence converges (when $\dim K < \infty$) to the graded group associated with some filtration of $\mathcal{H}^{p+q}(K)$.

5. The "classical" uniqueness theorem

Suppose that $\mathcal{H}^*$ is an equivariant cohomology theory satisfying the dimension axiom (4) of section 2, Chapter I. Let $h \in \mathcal{C}_G$ denote the "coefficients" of this theory. That is $h(G/H) = \mathcal{H}^0(G/H)$, and so on. Let K be a finite dimensional G-complex. If K is infinite we assume that (A) of the last section is satisfied.

In this case the spectral sequence of the last section degenerates for $r \geq 2$. In fact

$$E_2^{p,q} = \begin{cases} H_G^p(K;h); & q = 0 \\ 0 & ; q \neq 0 \end{cases}$$

It follows that, in fact,

$$(5.1) \qquad \mathcal{H}^p(K) \approx H_G^p(K;h)$$

and naturality is not hard to verify. Thus this is the <u>only</u> equivariant classical cohomology theory having coefficients h. The reader should note that, for general $h \in \mathcal{C}_G$, h is indeed the coefficient system of the cohomology theory $H_G^*(K;h)$. That is, there is a natural isomorphism

$$h(G/H) \approx H_G^0(G/H;h).$$