Chapter III. Function Spaces, Fibrations and Spectra

In this chapter we shall gather some miscellaneous items. The first and third sections contain some definitions and terminology that will be used later.

1. Function spaces

In this section we work in the category of G-spaces with base point. The group G is arbitrary and need not be finite.

If X and Y are G-spaces we let

$$F(X,Y)$$

denote the space of all (base point preserving) maps from X to Y in the compact-open topology. F(X,Y) is a G-space with the following G-action: If $f\colon X \to Y$ and $g \in G$ we put

$$g(f)(x) = g(f(g^{-1}x)).$$

The set $F(X,Y)^G$ of stationary points of G on F(X,Y) is just the set of equivariant maps from X to Y. Thus we put

(1.1) $$E(X,Y) = F(X,Y)^G.$$

Note that the reduced join

$$X \wedge Y = X \times Y / X \vee Y$$

of G-spaces has a natural G-action induced from the diagonal action on X×Y. Also recall that, for Y locally compact, there is a homeomorphism

(1.2) $$F(X \wedge Y, Z) \xrightarrow{\approx} F(X, F(Y,Z))$$

taking f into $\overline{f}$ defined by $(\overline{f}(x))(y) = f(x \wedge y)$. Note that

$$(g(\overline{f})(x))(y) = (g(\overline{f}(g^{-1}x)))(y)$$
$$= g[(\overline{f}(g^{-1}x))(g^{-1}y)] = g[f(g^{-1}x \wedge g^{-1}y)]$$
$$= g[f(g^{-1}(x \wedge y))] = g(f)(x \wedge y) = (\overline{g(f)}(x))(y),$$

that is, $g(\overline{f}) = \overline{g(f)}$, which means that (1.2) is <u>equivariant</u>.

In particular (1.2) induces a homeomorphism

(1.3) $$E(X \wedge Y,Z) \overset{\approx}{\to} E(X,F(Y,Z)),$$

when Y is locally compact.

If G acts trivially on X, so that $X = X^G$, then clearly $E(X,Y) = F(X,Y^G)$. In particular,

(1.4) $$E(X,F(Y,Z)) \approx F(X,E(Y,Z)) \text{ when } X = X^G.$$

Now the reduced suspension $SX = S \wedge X$ is a G-space, the action on the factor $S = S^1$ being trivial. Similarly, the loop space $\Omega X = F(S,X)$ is a G-space, as above. Thus (1.2) provides the <u>equivariant</u> homeomorphism

(1.5) $$F(SX,Y) \approx F(X,\Omega Y).$$

The comultiplication $SX \to SX \vee SX$ and the loop multiplication on ΩY induce Hopf G-space structures (see Chap. II, §4) on $F(SX,Y)$ and $F(X,\Omega Y)$ and it is well-known, and elementary, that these structures correspond under (1.5). In particular, passing to sets of stationary points, we have the isomorphism

(1.6) $$E(SX,Y) \approx E(X,\Omega Y)$$

of Hopf-spaces.

It is easy to see that (1.6) preserves equivariant homotopies. Thus, denoting equivariant homotopy classes by double square brackets, as before, we have the one-one correspondence

(1.7) $$[[SX;Y]] \leftrightarrow [[X;\Omega Y]]$$

which preserves addition.

$F(SX,\Omega Y)$ possesses two Hopf G-space structures. Let us denote the one induced by comultiplication in SX by $\circ$ and that induced by loop multiplication by $\square$. Then it is well-known, and easily checked, that we have the identity

$$(f\circ g)\square(h\circ k) = (f\square h)\circ(g\square k).$$

This identity is, of course, also satisfied on the fixed point set $E(SX,\Omega Y)$, and also for the induced multiplications on $[[SX;\Omega Y]]$. But the latter set has an identity e for both $\circ$ and $\square$ and we have

$$\alpha\square\beta = (e\circ\alpha)\square(\beta\circ e) = (e\square\beta)\circ(\alpha\square e) = \beta\circ\alpha$$

and $\alpha\square\beta = (\alpha\circ e)\square(e\circ\beta) = (\alpha\square e)\circ(e\square\beta) = \alpha\circ\beta$ so that

(1.8) $$\alpha\circ\beta = \beta\circ\alpha = \alpha\square\beta = \beta\square\alpha$$

on $[[SX;\Omega Y]]$. (The statement on $E(SX,\Omega Y)$ is that the corresponding maps are homotopic.)

It should be noted that when X is <u>locally compact</u>, we can improve these remarks as follows. We have, by (1.3) and (1.4),

$$E(SX,Y) \approx E(S,F(X,Y)) = F(S,E(X,Y)) = \Omega E(X,Y),$$

Also $[[X;Y]] = \pi_0 E(X,Y)$ so that we obtain

(1.9) $$[[X;\Omega Y]] \approx [[SX;Y]] \approx \pi_1(E(X,Y)).$$

Similarly,

(1.10) $$[[X;\Omega^n Y]] \approx [[S^n X;Y]] \approx \pi_n(E(X,Y)).$$

2.* The Puppe sequence

In this section we consider only spaces with base points. Let $f: X \to Y$ be an equivariant map between two G-spaces. Let $C_f = CX \cup_f Y$ be the reduced mapping cone of f with the obvious G-action, and let $j: Y \to C_f$ be the canonical inclusion. It is clear that, for any G-space Z, the sequence

(2.1) $$[[C_f;Z]] \xrightarrow{j^\#} [[Y;Z]] \xrightarrow{f^\#} [[X;Z]]$$

of sets with base points is exact. It can be shown that the mapping cone C_j of j has the same homotopy type as does SX (see Puppe, Math. Zeitschrift, 69 (1958) pp. 299-344). The proof of this is sufficiently canonical to be equivariant and we shall not give the details of this here. Thus C_j has the equivariant homotopy type of SX.

As in [Puppe, loc. cit.] we combine (2.1) with the similar sequence for $Y \xrightarrow{j} C_f \to C_j \sim SX$ and continue this process to finally obtain a long exact sequence

(2.2) $$\ldots \to [[S^n C_f;Z]] \to [[S^n Y;Z]] \to [[S^n X;Z]] \to [[S^{n-1} C_f;Z]] \to \ldots$$

3. G-spectra

In this section we work with the category of spaces (or G-spaces) with base points. By a G-spectrum we mean a collection $\underline{Y} = \{Y_n | n \in Z\}$ of G-spaces, together with equivariant maps

(3.1) $$\varepsilon_n : SY_n \to Y_{n+1}$$

or, by (1.6), of equivariant maps $Y_n \to \Omega Y_{n+1}$. We note that it is sufficient to have Y_n defined for $n \geq n_0$ and let Y_n be a point for $n < n_0$.

If $\underline{\underline{Y}}$ is a G-spectrum and if X is a locally compact G-space, then

$$\underline{\underline{F}}(X,\underline{\underline{Y}})$$

denotes the G-spectrum consisting of the G-spaces $F(X,Y_n)$ and the equivariant maps defined by the composition

$$F(X,Y_n) \to F(X,\Omega Y_{n+1}) \xrightarrow{\approx} F(SX,Y_{n+1}) \xrightarrow{\approx} \Omega F(X,Y_{n+1}).$$

In particular, $\Omega\underline{\underline{Y}}$ is a G-spectrum.

Note that if $\underline{\underline{Y}}$ is a G-spectrum then $\underline{\underline{Y}}^G = \{Y_n^G\}$ is a spectrum. In particular, for X locally compact,

$$\underline{\underline{E}}(X,\underline{\underline{Y}}) = \underline{\underline{F}}(X,\underline{\underline{Y}})^G$$

is a spectrum consisting of the spaces $E(X,Y_n)$.

For a detailed treatment of spectra see G. Whitehead, Generalized homology theories, Trans. A.M.S. 102 (1962), pp. 227-283.

We shall list below some examples of G-spectra:

(1) If Y is a G-space (with base point) and n is an integer, let $Y_n = Y$ and $Y_{n+k} = S^kY$ with the obvious maps $SY_m \to Y_{m+1}$. This forms a G-spectrum $\underline{\underline{S}}(Y,n)$.

(2) If $\rho\colon G \to O(r)$ is a representation of G on $\mathbb{R}^r$ then $\rho \oplus 1$ defines an action (with base point) on S^r and thus defines a G-space S^r_ρ. We denote the G-spectrum $\underline{\underline{S}}(S^r_\rho,r)$ by $\underline{\underline{S}}(\rho)$.

(3) Let $G = Z_2$ and let ρ be the representation defined by the antipodal map in $\mathbb{R}^r$. We denote the G-spectrum $\underline{\underline{S}}(\rho)$ by $\underline{\underline{S}}(r)$. Here the n-th G-space in $\underline{\underline{S}}(r)$, for $n \geq r$, is S^n with a standard involution which leaves S^{n-r} stationary. Thus $\underline{\underline{S}}(r)$ may be called the spectrum of spheres with stationary points of codimension r.

(4) Let $\tilde{\omega} \varepsilon \mathcal{C}_G$ and let Y_n be a G-complex of type $(\tilde{\omega},n)$. Since ΩY_{n+1} has type $(\tilde{\omega},n)$ there is a map $\eta_n\colon Y_n \to \Omega Y_{n+1}$ whose characteristic class

$$\chi^n(\eta_n) \varepsilon H^n_G(Y_n,\tilde{\omega}_n(\Omega Y_{n+1})) \approx \mathrm{Hom}(\tilde{\omega}_n(Y_n),\tilde{\omega}_n(\Omega Y_{n+1}))$$

corresponds to the identity $1\colon \tilde{\omega} \to \tilde{\omega}$ (via given isomorphisms $\omega_n(Y_n) \approx \tilde{\omega}$ and $\tilde{\omega}_n(\Omega Y_{n+1}) \approx \tilde{\omega}$). Thus we obtain a spectrum denoted by $K(\tilde{\omega})$, the Eilenberg-MacLane G-spectrum of $\tilde{\omega}$.

4*. G-fiber spaces

Let $\pi\colon X \to Y$ be an equivariant map between two G-spaces, where G is finite. We shall say that π is a G-fiber map if it has the equivariant homotopy lifting property with respect to G-complexes. That is, if K is a G-complex, $f\colon K \to X$ is equivariant and $F\colon K\times I \to Y$ is equivariant with $F(k,0) = \pi f(k)$, then there exists an equivariant map

$F'\colon K\times I \to X$ with $F = \pi F'$ and $F'(k,0) = f(k)$.

(4.1) <u>Theorem</u>. $\pi: X \to Y$ <u>is a</u> G-<u>fiber map iff</u> $\pi|X^H: X^H \to Y^H$ <u>is a (Serre) fibration for every</u> $H \subset G$.

<u>Proof</u>. If K is any complex then any map $K \to X^H$ has a unique equivariant extension to $f: K\times(G/H) \to X$ (where the action of G on K is trivial). Moreover, an equivariant map $K\times(G/H) \to X$ must take $K\times(H/H)$ into X^H. It follows easily that $X^H \to Y^H$ must be a fibration when $X \to Y$ is a G-fibration.

Suppose that each $X^H \to Y^H$ is a fibration. Let K be a G-complex, $f: K \to X$ equivariant and $F: K\times I \to Y$ equivariant with $F(k,0) = \pi f(k)$ for each $k\epsilon K$. We must construct $F': K\times I \to X$ equivariant with $F = \pi F'$ and $F'(k,0) = f(k)$. This will be done by induction on the skeletons of K. Suppose F' is defined on $K^{n-1}\times I$ and let σ be an n-cell of K. Let $H = G_\sigma$. Now $f: K(\sigma) \to X^H$, $F: K(\sigma)\times I \to Y^H$ and $F': (K^{n-1} \cap K(\sigma))\times I \to X^H$. Since $X^H \to Y^H$ is a fibration we may extend F' to a map $K(\sigma)\times I \to X^H$ covering F. There is then a unique equivariant extension of F' to the cells $g\sigma\times I$ for $g\epsilon G$. If this construction is repeated for each orbit of G on the set of n-cells of K, we obtain the required extension of F' to $K^n\times I \to X$.

As an example, let Y be a G-space such that each Y^H is arcwise connected and let $y_0 \epsilon Y^G$ be a base point. Then the space PY of paths on Y with initial point y_0 is a G-space and the canonical projection $\pi: PY \to Y$ is equivariant. Clearly $(PY)^H = P(Y^H)$ and the restriction $P(Y^H) \to Y^H$ of π is just the path-loop fibration of Y^H. Thus π is a G-fibration.

Suppose now that $\pi: X \to Y$ is a G-fibration. Let $x_0 \epsilon X^G$ be a base point and put $y_0 = \pi(x_0)$. The G-space

$F = \pi^{-1}(y_0)$ is called the fiber of this fibration. As in the non-equivariant theory, we have an exact sequence

(4.2) $$\cdots \to \tilde{\omega}_n(F,x_0) \xrightarrow{i_\#} \tilde{\omega}_n(X,x_0) \xrightarrow{\pi_\#} \tilde{\omega}_n(Y,y_0) \xrightarrow{\partial_\#} \tilde{\omega}_{n-1}(F,x_0) \xrightarrow{i_\#} \cdots$$

In fact, the exactness of this sequence follows from the exactness of the homotopy sequences of the fibrations $X^H \to Y^H$ with fiber F^H. Of course one must show that $i_\#$, $\pi_\#$ and $\partial_\#$, which are defined so that their values on G/H are the corresponding homomorphisms for the fibration $X^H \to Y^H$, are in fact morphisms in $\mathcal{C}_G$. This is left to the reader.