Chapter II. Equivariant Obstruction Theory

In this chapter we shall assume that the reader is reasonably familiar with obstruction theory on CW-complexes. We shall attempt to strike a reasonable balance between giving no details on the one hand and developing the theory from scratch on the other by making use of the results, without proof, of the classical theory.

1. The obstruction cocycle

In this section $n \geq 1$ will be an integer, fixed throughout the discussion. Let K be a G-complex and L a G-subcomplex. Let Y be a G-space. We shall assume, for simplicity, that the set Y^H of stationary points of H on Y is non-empty, arcwise connected and n-simple for each subgroup $H \subset G$, (We note here that the theory could be generalized to relative CW-complexes (K,L) with no trouble.)

Assume that we are given an equivariant map $\varphi: K^n \cup L \to Y$. Let σ be an (n+1)-cell of K and let $f_\sigma: S^n \to K^n$ be a characteristic map for σ (note that the characteristic maps may be chosen equivariantly).

The subgroup G_σ leaves $K(\sigma)$, and hence Im f_σ, stationary. It follows that G_σ leaves $\mathrm{Im}(\varphi \circ f_\sigma)$ stationary. That is,

$$(\varphi \circ f_\sigma)(S^n) \subset Y^{G_\sigma}.$$

Thus $\varphi \circ f_\sigma$ defines an element $c_\varphi(\sigma) \in \pi_n(Y^{G_\sigma})$, and clearly $c_\varphi(\sigma) = 0$ if σ is in L. But, with $\tilde{\omega}_n(Y)$ defined as in example (3) of Chap. I, §4, this defines a cochain

$$c_\varphi \in C^{n+1}(K,L;\tilde{\omega}_n(Y)),$$

Now $c_\varphi(g\sigma)$ is represented by $\varphi\circ f_{g\sigma}: S^n \to Y^{G_{g\sigma}} = Y^{gG_\sigma g^{-1}}$ and $\varphi\circ f_{g\sigma} = \varphi\circ g\circ f_\sigma = g\circ\varphi\circ f_\sigma$ so that $c_\varphi(g\sigma) = g_\#(c_\varphi(\sigma))$. This means that c_φ is an equivariant cochain (by the defintion of $\tilde{\omega}_n(Y)$), that is

$$c_\varphi \in C_G^{n+1}(K,L;\tilde{\omega}_n(Y)).$$

It is called the obstruction cochain.

(1.1) Proposition. $\delta c_\varphi = 0$.

Proof. Let τ be an $(n+2)$-cell and consider the computation of $(\delta c_\varphi)(\tau)$. To calculate this, one "pushes" the coefficients to those on τ; that is to $\pi_n(Y^{G_\tau})$, and calculates the classical coboundary. But c_φ restricted to $K(\tau)$ and with coefficients pushed to $\pi_n(Y^{G_\tau})$ is just the obstruction cochain, in the classical sense, to extending $\varphi | K^n \cap K(\tau)$ to $K^{n+1} \cap K(\tau)$. Thus $(\delta c_\varphi)(\tau) = 0$ is a fact from the classical theory.

(1.2) Proposition. $c_\varphi = 0$ iff φ can be extended equivariantly to $K^{n+1} \cup L$.

Proof. If $c_\varphi(\sigma) = 0$ then clearly we may extend φ to $K^n \cup L \cup \sigma$ in such a way that $\varphi(\sigma) \subset Y^{G_\sigma}$. Define, for $g \in G$ and $x \in \sigma$,

$$\varphi(gx) = g\,\varphi(x) \in g(Y^{G_\sigma}) = Y^{gG_\sigma g^{-1}} = Y^{G_{g\sigma}}.$$

If $gx = g'x$ then $g' = gh$ for some $h \in G_\sigma$ so that $g'\varphi(x) = g\varphi(x)$ (since $\varphi(x) \in Y^{G_\sigma}$), which shows that this definition is valid. The proof is completed by taking an $(n+1)$-cell from each orbit of G on the $(n+1)$-cells and following the procedure above.

Now suppose that φ and θ are equivariant maps $K^n \cup L \to Y$ and let $F: (K^{n-1} \cup L) \times I \to Y$ be an equivariant homotopy between $\varphi | K^{n-1} \cup L$ and $\theta | K^{n-1} \cup L$. Define an equivariant map $\varphi \#_F \theta: (K \times I)^n \cup (L \times I) \to Y$ by

$$\begin{cases} (\varphi \#_F \theta)(x,0) = \varphi(x) \\ (\varphi \#_F \theta)(x,1) = \theta(x) \\ (\varphi \#_F \theta)(x,t) = F(x,t). \end{cases}$$

If $\varphi | K^{n-1} \cup L = \theta | K^{n-1} \cup L$ and F is the constant homotopy $\#$ will denote $\#_F$.

The deformation cochain $d_{\varphi,F,\theta} \in C^n_G(K,L;\tilde{\omega}_n(Y))$ is defined by

$$d_{\varphi,F,\theta}(\sigma) = c_{\varphi \#_F \theta}(\sigma \times I).$$

It is clear that

$$\delta d_{\varphi,F,\theta} = c_\theta - c_\varphi . \tag{1.3}$$

If $\#_F = \#$, that is if F is constant, then we put $d_{\varphi,\theta} = d_{\varphi,F,\theta}$.

(1.4) Proposition. *Let $\varphi: K^n \cup L \to Y$ be equivariant and let $d \in C^n_G(K,L;\tilde{\omega}_n(Y))$. Then there is an equivariant map $\theta: K^n \cup L \to Y$, coinciding with φ on $K^{n-1} \cup L$, such that $d_{\varphi,\theta} = d$.*

Proof. Let σ be an n-cell of K, not in L, and choose a characteristic map $f_\sigma: (B^n, S^{n-1}) \to (K^n, K^{n-1})$ for σ. Let $J^n = B^n \times \{0\} \cup S^{n-1} \times I \subset B^n \times I$ and define $\Psi: J^n \to Y^{G_\sigma}$ by $\Psi(x,t) = \varphi(f_\sigma(x))$. As shown in non-equivariant obstruction theory, Ψ may be extended to a map $\Psi': \partial(B^n \times I) \to Y^{G_\sigma}$ representing the element (or any element) $d(\sigma) \in \pi_n(Y^{G_\sigma})$. It is clear that such extensions may be chosen *equivariantly*, since d is an equivariant cochain.

Now θ can be defined by $\theta|K^{n-1} \cup L = \varphi|K^{n-1} \cup L$ and, for an n-cell σ and $x \in \sigma$,

$$\theta(x) = \Psi'(f_\sigma^{-1}(x),1).$$

It is clear that $d_{\varphi,\theta} = d$.

The cocycle $c_\varphi \in C_G^{n+1}(K,L;\tilde{\omega}_n(Y))$ represents a cohomology class

$$[c_\varphi] \in H_G^{n+1}(K,L;\tilde{\omega}_n(Y))$$

which depends, by (1,3), only on the <u>equivariant homotopy class</u> of $\varphi|K^{n-1} \cup L$. Moreover, if $[c_\varphi] = 0$, then by (1.4) $\varphi|K^{n-1} \cup L$ extends to θ: $K^n \cup L \to Y$ such that $c_\theta = 0$ (choose d with $\delta d = -c_\varphi$). Hence, by (1.2), we have the following result:

(1.5) <u>Theorem</u>. <u>Let</u> φ: $K^n \cup L \to Y$ <u>be equivariant. Then</u> $\varphi|K^{n-1} \cup L$ <u>can be extended to an equivariant map</u> $K^{n+1} \cup L \to Y$ <u>iff</u> $[c_\varphi] = 0$.

<u>Remark</u>. Suppose that φ,θ: $K \to Y$ are equivariant and that F: $(K^{n-1} \cup L)\times I \to Y$ is an equivariant homotopy between the restrictions of φ and θ to $K^{n-1} \cup L$. As above we obtain an equivariant map $\varphi \#_F \theta = (K^{n-1}\times I) \cup Q \to Y$ where $Q = (L\times I) \cup (K\times \partial I)$. Then the obstruction to extending $\varphi \#_F \theta$ to $(K^n\times I) \cup Q$ is

$$c_{\varphi \#_F \theta} \in C_G^{n+1}(K\times I, L\times I \cup K\times \partial I;\tilde{\omega}_n(Y)),$$

This group is isomorphic to $C_G^n(K,L;\tilde{\omega}_n(Y))$ and this isomorphism takes $c_{\varphi \#_F \theta}$ into $d_{\varphi,F,\theta}$ (now a cocycle).

2. Primary obstructions

At various points in this section we shall make one or more of the following assumptions:

(1) Y^H is r-simple, non-empty and arcwise connected for all r and $H \subset G$ (e.g. $\tilde{\omega}_0(Y) = 0 = \tilde{\omega}_1(Y)$).

(2) $H_G^{r+1}(K,L;\tilde{\omega}_r(Y)) = 0$ for all $r < n$.

(3) $H_G^{r}(K,L;\tilde{\omega}_r(Y)) = 0$ for all $r < n$.

(4) $H_G^{r-1}(K,L;\tilde{\omega}_r(Y)) = 0$ for all $r < n$.

Numbers appearing in each statement indicate which of these assumptions are used. The results in this section are all easy applications of §1 to the study of extensions of equivariant maps and homotopies. The proofs will be omitted since they offer no difficulties.

Suppose first that we are given an equivariant map $f: L \to Y$.

(2.1) Lemma. (1,2) *There exists an equivariant extension f_n of f to $K^n \cup L$.*

(2.2) Lemma. (1,3) *If f_n and g_n are equivariant extensions of f to $K^n \cup L$ then $[c_{f_n}] = [c_{g_n}]$.*

(Hint: Use (2.1) to find a homotopy $f_{n-1} \sim g_{n-1}$ relative to L.)

(2.3) Definition. (1,2,3) *Let $\gamma^{n+1}(f) \in H_G^{n+1}(K,L;\tilde{\omega}_n(Y))$ be the (unique) cohomology class $[c_{f_n}]$ for any equivariant extension f_n of f to $K^n \cup L$. $\gamma^{n+1}(f)$ is called the primary obstruction to extending f and is an invariant of the equivariant homotopy class of f.*

(2.4) Proposition. If $k\colon K' \to K$ is cellular and equivariant then $\gamma^{n+1}(f \cdot k) = k^*(\gamma^{n+1}(f))$ when this is defined.

(This is also true without cellularity but we have not yet defined k^* in the general case.)

(2.5) Theorem (Extension). (1,2,3) If we also have that $H_G^{r+1}(K,L;\tilde{\omega}_r(Y)) = 0$ for $n < r < \dim(K-L)$ then an equivariant map $f\colon L \to Y$ has an equivariant extension to K iff $\gamma^{n+1}(f) = 0$.

Now suppose that we are given two equivariant maps $f,g\colon K \to Y$ such that $f|L = g|L$. These induce an equivariant map $f\#g\colon Q \to Y$ where $Q = (K\times\partial I) \cup (L\times I)$.

There is a natural isomorphism

$$(2.6) \qquad \lambda\colon H_G^n(K,L;\tilde{\omega}_n(Y)) \xrightarrow{\approx} H_G^{n+1}(K\times I,Q;\tilde{\omega}_n(Y))$$

(induced by the obvious isomorphism on the cochain level). We define, under conditions (1,3,4):

$$(2.7) \qquad \omega^n(f,g) = \lambda^{-1}(\gamma^{n+1}(f\#g))$$

and note that

$$(2.8) \qquad \omega^n(f,g) + \omega^n(g,h) = \omega^n(f,h)$$

and

$$(2.9) \qquad \omega^n(f\circ k,g\circ k) = k^*(\omega^n(f,g))$$

(where $k\colon (K',L') \to (K,L)$ is cellular and equivariant) when this is defined.

An application of (2.5) to this situation yields:

(2.10) Theorem (Homotopy). (1,3,4) If we also have that $H^r_G(K,L;\tilde{\omega}_r(Y)) = 0$ for $n < r \leq \dim(K-L)$ and if $f,g\colon K \to Y$ are equivariant with $f|L = g|L$, then f and g are equivariantly homotopic (relative to L) iff $\omega^n(f,g) = 0$.

A standard argument now proves the following result:

(2.11) Theorem (Classification). Assume that (1) holds and also that

$$\begin{cases} H^r_G(K,L;\tilde{\omega}_r(Y)) = 0 = H^{r-1}_G(K,L;\tilde{\omega}_r(Y)) & \text{for } r < n \\ H^r_G(K,L;\tilde{\omega}_r(Y)) = 0 = H^{r+1}_G(K,L;\tilde{\omega}_r(Y)) & \text{for } r > n, \end{cases}$$

Let $f\colon K \to Y$ be an equivariant map. Then the equivariant homotopy classes (relative to L) of maps $g\colon K \to Y$ (with $g|L = f|L$) are in one-one correspondence with the elements of

$$H^n_G(K,L;\tilde{\omega}_n(Y))$$

and $g \to \omega^n(g,f)$ is such a correspondence.

As a matter of notation, we shall use double brackets: $[[X;Y]]$, where X and Y are G-spaces, to denote the equivariant homotopy classes of (equivariant) maps $X \to Y$. Thus, for $L = \emptyset$, the conclusion of (2.11) states that $[[g]] \leftrightarrow \omega^n(g,f)$ is a one-one correspondence

$$[[K;Y]] \approx H^n_G(K;\tilde{\omega}_n(Y)).$$

3. The characteristic class of a map

In this section we assume that Y is a G-space with base point $y_0 \in Y^G$ such that

$$\tilde{\omega}_q(Y,y_0) = 0 \text{ for } q < n,$$

for a given integer $n \geq 1$. If $n = 1$, we assume that $\tilde{\omega}_1(Y,y_0)$ (that is, each $\pi_1(Y^H,y_0)$) is abelian.

Let K be a G-complex and let 0 denote the constant (equivariant) map $0: K \to y_0 \in Y$. For any equivariant map $f: K \to Y$ we define the characteristic class of f to be

$$(3.1) \qquad \chi^n(f) = \omega^n(f,0) \in H^n_G(K;\tilde{\omega}_n(Y)).$$

If $k: K' \to K$ is cellular and equivariant then by (2.9)

$$\chi^n(f \circ k) = k^*(\chi^n(f)).$$

(The cellularity condition is unnecessary as will follow from later facts.)

The following four results are standard and immediate consequences of the definitions and of §2. We shall omit their proofs:

(3.2) Proposition. *If* $H^r_G(K;\tilde{\omega}_r(Y)) = 0$ for $r > n$ *then two maps* $f,g: K \to Y$ *are homotopic iff* $\chi^n(f) = \chi^n(g)$.

(3.3) Theorem. *If* (K,L) *is a* G-*complex pair and* $f: L \to Y$ *is given with characteristic class* $\chi^n(f) \in H^n_G(L;\tilde{\omega}_n(Y))$, *then the primary obstruction to extending* f *to* K *equivariantly is*

$$\gamma^{n+1}(f) = \delta^*(\chi^n(f))$$

where $\delta^*: H^n_G(L;\tilde{\omega}_n(Y)) \to H^{n+1}_G(K,L;\tilde{\omega}_n(Y))$ *is the coboundary*.

(3.4) <u>Corollary</u>. <u>If</u> $H_G^{r+1}(K,L;\tilde{\omega}_r(Y)) = 0$ <u>for</u> $r > n$ <u>then an equivariant map</u> $f: L \to Y$ <u>has an equivariant extension to</u> K <u>iff</u>

$$\chi^n(f) \in \mathrm{Im}[i^*: H_G^n(K;\tilde{\omega}_n(Y)) \to H_G^n(L;\tilde{\omega}_n(Y))].$$

(3.5) <u>Theorem</u>. <u>If</u> $f,g: K \to Y$ <u>are equivariant and if</u> $f|L = g|L$, <u>then</u>

$$\chi^n(f) - \chi^n(g) = j^*(\omega^n(f,g)).$$

(Here $\chi^n(f)$ and $\chi^n(g)$ are in $H_G^n(K;\tilde{\omega}_n(Y))$, $\omega^n(f,g)$ is in $H_G^n(K,L;\tilde{\omega}_n(Y))$ and j^* is induced by $(K,\phi) \to (K,L)$.)

We conclude this section with some remarks on the case in which Y is, itself, a G-<u>complex</u>. These remarks will not be used in any essential way elsewhere in these notes. The identity $1: Y \to Y$ yields a class

$$\chi^n(Y) = \chi^n(1) = \omega^n(1,0) \in H_G^n(Y;\tilde{\omega}_n(Y)),$$

which is the primary obstruction to equivariantly contracting Y and is called the characteristic class of Y.

For any $f: K \to Y$ we obviously have

$$\chi^n(f) = f^*(\chi^n(Y)). \qquad (3.6)$$

By Chap. I, (10.5) we have that

$$H_G^n(Y;\tilde{\omega}_n(Y)) \approx \mathrm{Hom}(\tilde{\omega}_n(Y),\tilde{\omega}_n(Y))$$

and it can be shown that under this isomorphism $\chi^n(Y)$ corresponds to the identity homomorphism. (Perhaps the easiest way to prove this is to note that Y has the equivariant homotopy type of a G-complex which has no cells in dimensions between 0 and n, and then to prove the result in this case. See §7.) This is, of course, an important result since it allows the computation of the characteristic class.

4. Hopf G-spaces

Let Y be a G-space with base point y_0. Let G act diagonally on Y×Y, that is, $g(y,y') = (gy,gy')$. Such a space Y together with a base point preserving equivariant map $\theta: Y\times Y \to Y$ is said to be a Hopf G-space if the restriction $Y \vee Y \to Y$ of θ is equivariantly homotopic to $1 \vee 1$. This obviously implies that Y^H, for $H \subset G$, is a Hopf-space.

For example, if Y is any G-space with base point, then the loop space ΩY is a Hopf G-space, where the action of G on a loop, or generally on a path, $f: I \to Y$, is defined by $g(f)(t) = g(f(t))$.

Let us denote the product $\theta(y,y')$ by $y \sqcup y'$ in a given Hopf G-space Y. Let (K,L) be a pair of G-complexes and let $\varphi,\psi: K^n \cup L \to Y$ be equivariant, where Y is (also) as in §1. We have the map

$$\varphi \sqcup \psi: K^n \cup L \to Y$$

defined by $(\varphi \sqcup \psi)(x) = \varphi(x) \sqcup \psi(x)$. Since addition in the homotopy groups of a Hopf-space is induced by the Hopf-space operation, as is well-known, and since each Y^{G_σ} is a Hopf-space, it follows immediately that

$$c_{\varphi \sqcup \psi} = c_\varphi + c_\psi$$

in $C_G^{n+1}(K,L;\tilde{\omega}_n(Y))$.

It follows immediately that in the situation of (3.1), with Y a Hopf G-space and $f,f': K \to Y$ equivariant, we have

(4.1) $$\chi^n(f \sqcup f') = \chi^n(f) + \chi^n(f').$$

5.* Equivariant deformations and homotopy type

In this section we shall prove some elementary facts concerning equivariant deformation. These results could be encompassed in an obstruction theory of deformation (which contains the obstruction theory of extensions) but we have chosen not to do so.

Let $Y \supset B$ be a pair of G-spaces and assume that

(5.1) $\tilde{\omega}_q(Y,B) = 0$ for all $0 \leq q \leq n$,

in the sense that, for every subgroup $H \subset G$, every map $(B^q, S^{q-1}) \to (Y^H, B^H)$ is deformable, through such maps, to a map into B^H. (We allow the case $n = \infty$.)

(5.2) Lemma. *Let* (K,L) *be a pair of* G-*complexes with* $\dim(K-L) \leq n$ *and let* $\varphi\colon K,L \to Y,B$ *be an equivariant map. Then* φ *is equivariantly homotopic relative to* L *to a map into* B.

Proof. Consider $K\times I$. We wish to extend the map $\varphi|L\times I \cup \varphi\times\{0\}$ on $L\times I \cup K\times\{0\}$ to $K\times I$ such that $K\times\{1\}$ goes into B. The extension is defined inductively on the $K^n\times I$ and proceeds much as in the proof of (1.2). The details are omitted.

As above, double brackets $[[X;Y]]$ denote the set of equivariant homotopy classes of equivariant maps $X \to Y$, where X and Y are G-spaces.

(5.3) Corollary. *Inclusion* $i\colon B \to Y$ *induces a one-one correspondence*

$$i_\#\colon [[K;B]] \xrightarrow{\approx} [[K;Y]]$$

for every G-*complex* K *with* $\dim K < n$.

Proof. $i_\#$ is onto by (5.2). If f: K → B can be equivariantly deformed, through Y, to g: K → B, then by (5.2) the homotopy may be deformed, relative to the ends, into B. This shows that $i_\#$ is one-one.

(5.4) Theorem. Let f: Y → Y' be an equivariant map of G-spaces such that $f_\#: \tilde{\omega}_q(Y) \approx \tilde{\omega}_q(Y')$ for all $q \geq 0$. Then

$$f_\#: [[K;Y]] \to [[K;Y']]$$

is a one-one correspondence for every G-complex K.

Proof. Let M_f be the mapping cylinder of f, with the natural G-action. M_f and Y' have the same equivariant homotopy type so that f may be replaced by the inclusion i: $Y \to M_f$. The hypothesis implies easily that $\tilde{\omega}_q(M_f,Y) = 0$ for all $q \geq 0$. Thus the result follows from (5.3).

(5.5) Corollary. If φ: K → K' is an equivariant map between two G-complexes such that $\varphi_\#: \tilde{\omega}_q(K) \approx \tilde{\omega}_q(K')$ for all $q \geq 0$ then φ is an equivariant homotopy equivalence.

Proof. $\varphi_\#: [[K';K]] \xrightarrow{\approx} [[K';K']]$ by (5.4). Let $\psi: K' \to K$ represent $\varphi_\#^{-1}(1)$. That is, $\varphi\psi: K' \to K'$ is equivariantly homotopic to the identity. Clearly $\psi_\# = \varphi_\#^{-1}$ is bijective so that there is (similarly) a $\theta: K \to K'$ with $\psi\theta \sim 1$ (equivariantly). Then $\theta \sim \varphi\psi\theta \sim \varphi$ so that $\psi\varphi \sim \psi\theta \sim 1$ as was to be shown.

(5.6) Proposition. Every equivariant map $f: K_1 \to K_2$ between two G-complexes is equivariantly homotopic to a cellular map. An equivariant homotopy between cellular maps may be deformed equivariantly, relative to the ends, into a cellular homotopy.

Proof. This is an easy consequence of (5.2) using $(Y,B) = (K_2, K_2^n)$.

This result can be used to extend the definition of the induced cohomology homomorphism of an equivariant map $K_1 \to K_2$ to arbitrary (non-cellular) maps. Another method of doing this is given in §6.

6. Eilenberg-MacLane G-complexes

Let $\tilde{\omega}$ be any element of the abelian category $\mathcal{C}_G$. A G-space of type $(\tilde{\omega},n)$ is defined to be a G-space Y with

$$\tilde{\omega}_q(Y,y_0) = \begin{cases} 0 & \text{for } q \neq n \\ \tilde{\omega} & \text{for } q = n, \end{cases}$$

where $y_0 \in Y^G \neq \emptyset$.

For such a space (2.11) provides a one-one correspondence

(6.1) $$[[K;Y]] \approx H_G^n(K;\tilde{\omega})$$

for all G-complexes K, given by

$$[[f]] \leftrightarrow \omega^n(f,0) = \chi^n(f)$$

(where 0 denotes the constant map $K \to y_0$ and the notation on the right is from §3). Moreover, if φ: $K \to K'$ is cellular and equivariant then, by (2.9),

(6.2) $$\begin{array}{ccc} [[K';Y]] & \xrightarrow{\approx} & H_G^n(K';\tilde{\omega}) \\ \downarrow{\scriptstyle \varphi^\#} & & \downarrow{\scriptstyle \varphi^*} \\ [[K;Y]] & \xrightarrow{\approx} & H_G^n(K;\tilde{\omega}) \end{array}$$

commutes, where $\varphi^{\#}([[f]]) = [[f \circ \varphi]]$. Thus (6.1) is a natural equivalence of functors.

Note that if Y is a G-space of type $(\tilde{\omega},n)$ then the loop space ΩY (see §4) has type $(\tilde{\omega},n-1)$. This is an immediate consequence of the obvious fact that $(\Omega Y)^H = \Omega(Y^H)$.

If Y is a Hopf G-space then we can define an addition in [[K,Y]] by

(6.3) $$[[f]] + [[f']] = [[f \,\square\, f']].$$

Then, by (4.1), the correspondence (6.1) preserves addition. Thus, in this case, if we are given an equivariant map $\varphi: K \to K'$ (not necessarily cellular) we can define $\varphi^*: H^n_G(K';\tilde{\omega}) \to H^n_G(K;\tilde{\omega})$ by commutativity of (6.2), since $\varphi^{\#}$ is always defined. The obvious additivity of $\varphi^{\#}$ implies that φ^* is a homomorphism. Thus, in this way, we can dispense with the definitions of φ^* in Chap. I, §7 as well as Proposition (5.6) (used to extend the definition of φ^* to non-cellular maps).

We shall now show how to construct a G-<u>complex</u> K of type $(\tilde{\omega},n)$ for any $\tilde{\omega} \in \mathcal{C}_G$ and $n \geq 1$. We shall restrict our attention, for convenience only, to the case $n > 1$. This is not much loss of generality since ΩK has type $(\tilde{\omega},1)$ when K has type $(\tilde{\omega},2)$. The construction is based on the following two lemmas which use the notation of Chap. I, §9, 10.

First we shall introduce some further notation. If T is a G-set, T^+ is T together with a disjoint base point, S^qT^+ is the q^{th} reduced suspension of T^+ (that is, the one point union of q-spheres, one for each member of T), and CS^qT^+ is the reduced cone of this (that is, the one point union of (q+1)-cells, one for each member of T). Note that there are natural isomorphisms (for q > 1)

(6.4) $$F_T \approx \tilde{\omega}_q(S^qT^+) \approx \underline{C}_q(S^qT^+;Z) \approx \underline{H}_q(S^qT^+;Z)$$

of elements of $\mathcal{C}_G$.

(6.5) Lemma. Let $q > 1$ and let Y be a G-space with base point y_0 and with $\tilde{\omega}_0(Y,y_0) = 0 = \tilde{\omega}_1(Y,y_0)$. Then for any G-set T, the assignment to an equivariant homotopy class [[f]] (of a map $f\colon S^qT^+ \to Y$) of the induced morphism $f_\#\colon F_T \approx \tilde{\omega}_q(S^qT^+) \to \tilde{\omega}_q(Y)$ in $\mathcal{C}_G$ is a one-one correspondence

$$[[S^qT^+,Y]] \xrightarrow{\approx} \mathrm{Hom}\,(F_T,\tilde{\omega}_q(Y)).$$

In particular, every morphism $\alpha\colon F_T \to \tilde{\omega}_q(Y)$ in $\mathcal{C}_G$ is represented by an equivariant map $f\colon S^qT^+ \to Y$ and f is equivariantly extendible to $CS^qT^+ \to Y$ iff $\alpha = f_\#$ is trivial.

Proof. A direct proof of this should be fairly obvious. However, we note that it is, in fact, a special case of the equivariant homotopy classification theorem (2.11). That is, take $K = S^qT^+$, let L be the base point, and let $0\colon K \to Y$ be the constant map into y_0. The conditions of (2.11) are satisfied for $n = q$ since, in fact, K has no cells in dimensions other than 0 and q. The classification assigns to an equivariant map $f\colon K \to Y$ the class $\omega^q(f,0)$ in

$$H^q_G(S^qT^+;\tilde{\omega}_q(Y)) \approx \mathrm{Hom}(\underline{H}_q(S^qT^+;Z),\tilde{\omega}_q(Y))$$

$$\approx \mathrm{Hom}(\tilde{\omega}_q(S^qT^+),\tilde{\omega}_q(Y))$$

(see (9.5) of Chap. I). It is obvious from the definition of $\omega^n(f,0)$ that the corresponding homomorphism $\tilde{\omega}_q(S^qT^+) \to \tilde{\omega}_q(Y)$ is precisely the induced map $f_\#$.

(6.6) Lemma. *Let $q > 1$ and let Y be a G-space with base point y_0 and with $\tilde{\omega}_0(Y,y_0) = 0 = \tilde{\omega}_1(Y,y_0)$. Let $f\colon S^qT^+ \to Y$ be an equivariant base point preserving map and let $Y' = Y \cup_f CS^qT^+$ be the (reduced) mapping cone of f with the obvious G action. Let $i\colon Y \to Y'$ denote the inclusion. Then we have the following facts:*

(1) $i_\#\colon \tilde{\omega}_r(Y) \to \tilde{\omega}_r(Y')$ *is an isomorphism for* $r < q$.

(2) $i_\#\colon\tilde{\omega}_q(Y) \to \tilde{\omega}_q(Y')$ *is an epimorphism with* Kernel $i_\# = \operatorname{Im}\{f_\#\colon F_T \approx \tilde{\omega}_q(S^qT^+) \to \tilde{\omega}_q(Y)\}$.

Proof. For $H \subset G$ it is clear that Y'^H is just the mapping cone of $(S^qT^+)^H \to Y^H$. But $(S^qT^+)^H = S^q(T^H)^+$. Thus $i_\#(G/H)\colon \pi_r(Y^H) \to \pi_r(Y'^H)$ is induced by the inclusion of Y^H in the mapping cone of the restriction of $f\colon S^q(T^H)^+ \to Y^H$. Similarly $f_\#(G/H)\colon \pi_r((S^qT^+)^H) \to \pi_r(Y^H)$ is induced by the restriction of f. Since (1) and (2) are true iff the corresponding statements for the values on each $G/H \in \mathcal{C}_G$ are true, and since these corresponding statements are known results (see Hu, *Homotopy Theory*, p. 168) concerning (non-equivariant) attaching of cells, the lemma follows.

Using these two lemmas, the construction of $K(\tilde{\omega},n)$ complexes is now quite straightforward. Thus let T and R be G-sets such that there is an exact sequence

$$F_R \xrightarrow{\alpha} F_T \xrightarrow{\beta} \tilde{\omega} \to 0$$

in $\mathcal{C}_G$ (see Chap. I, §10). Let $n > 1$ and put $K^n = S^nT^+$. Let

$$f: S^nR^+ \to S^nT^+$$

be an equivariant map inducing α (via $F_T \approx \tilde{\omega}_n(S^nT^+)$, etc.). This exists by (6.5). Let $K^{n+1} = K^n \cup_f CS^nR^+$. By (6.6) we have

$$\begin{cases} \tilde{\omega}_n(K^{n+1}) \approx \tilde{\omega} \\ \tilde{\omega}_r(K^{n+1}) = 0 \text{ for } r < n. \end{cases}$$

If K^q has been constructed to be a G-complex of dimension q ($q \geq n + 1$) such that

$$\text{(6.7)} \qquad \begin{cases} \tilde{\omega}_n(K^q) \approx \tilde{\omega} \\ \tilde{\omega}_r(K^q) = 0 \text{ for } r < n \text{ and } n < r < q \end{cases}$$

let V be a G-set such that there is an epimorphism

$$F_V \xrightarrow{\gamma} \tilde{\omega}_q(K^q).$$

Let $v: S^qV^+ \to K^q$ be an equivariant map inducing γ and let $K^{q+1} = K^q \cup_v S^qV^+$. Then, by (6.6), K^{q+1} satisfies (6.7) with q replaced by q + 1. Let $K = \bigcup_q K^q$. This is clearly a G-complex of type $(\tilde{\omega}, n)$.

7.* n-connected G-complexes

The method of killing the groups $\tilde{\omega}_q$ used in the construction of $K(\tilde{\omega},n)$ in the last section is, of course, an important tool. We shall use it here in a rather straightforward way to prove the following result:

(7.1) Proposition. *Let* K *be a* G-*complex with* $\tilde{\omega}_q(K) = 0$ *for all* $0 \leq q < n$. *Then* K *has the same equivariant homotopy type as a* G-*complex with no cells in dimensions* q *for* $0 < q < n$.

<u>Proof</u>. Let $L = K^{n-1}$. Then the inclusion $L \to K$ is equivariantly homotopic to a constant map, by (2.10). That is, K is an equivariant retract of $K \cup C_L$. But $K \cup C_L$ has the same equivariant homotopy type as K/L. Thus there exist equivariant maps

$$K \xrightarrow{\varphi} K/L \xrightarrow{\psi} K$$

with $\psi\varphi$ equivariantly homotopic to 1. Clearly K/L has no q-cells for $0 < q < n$ so that $\tilde{\omega}_q(K/L) = 0$ for $q < n$.

Suppose that for some $q \geq n$ we have constructed a G-complex $K_q \supset K/L$ and an equivariant map $\psi_q \colon K_q \to K$ with $\psi_q\varphi = \psi\varphi$ such that $(\psi_q)_\# \colon \tilde{\omega}_r(K_q) \to \tilde{\omega}_r(K)$ is a monomorphism for $r < q$. Let T be a G-set and

$$\alpha \colon F_T \to \mathrm{Ker}\{(\psi_q)_\# \colon \tilde{\omega}_q(K_q) \to \tilde{\omega}_q(K)\}$$

an epimorphism in $\mathcal{C}_G$. Let

$$f \colon S^qT^+ \to K_q$$

be an equivariant map inducing α. f may be assumed to be cellular by (5.6) (or merely because $\tilde{\omega}_q(K_q^q) \to \tilde{\omega}_q(K_q)$ is an epimorphism). Let $K_{q+1} = K_q \cup_f CS^qT^+$. By (6.5), ψ_q extends to $\psi_{q+1} \colon K_{q+1} \to K$ and, by (6.6), $(\psi_{q+1})_\# \colon \tilde{\omega}_r(K_{q+1}) \to \tilde{\omega}_r(K)$ is a monomorphism for $r \leq q$. Let (K', ψ') be the union of the (K_q, ψ_q).

Thus we obtain a G-complex $K' \supset K/L$ with no q-cells for $0 < q < n$ and equivariant maps

$$K \xrightarrow{\varphi} K' \xrightarrow{\psi'} K$$

with $\psi'\varphi = \psi\varphi \sim 1$. Also

$$\psi'_\# \colon \tilde{\omega}_*(K') \to \tilde{\omega}_*(K),$$

being a monomorphism with $\psi'_\# \varphi_\# = 1$, must be an isomorphism and it follows from (5.5) that K and K' have the same equivariant homotopy type.