THE KERVAIRE INVARIANT OF $8k+2$-MANIFOLDS.

By Edgar H. Brown, Jr. and Franklin P. Peterson.[1]

1. Introduction. The main results of this paper were announced in [6]. Let $\Omega_n(e)$, $\Omega_n(SU)$, and $\Omega_n(Spin)$ denote the n-th framed, SU, and $Spin$ cobordism groups respectively (see [7] and [11].). In [8] Kervaire defined a homomorphism $\Phi\colon \Omega_{4k+2}(e) \to Z_2$, $k \neq 0, 1, 3$, which is the obstruction to a framed $4k+2$-manifold being framed cobordant to a homotopy sphere ([9]). Kervaire showed that $\Phi = 0$ for $k = 2, 4$. In [4] a homomorphism ψ: $\Omega_{8k+2}(Spin) \to Z_2$ was defined such that $\Phi = \psi\rho$ where $\rho\colon \Omega_n(e) \to \Omega_n(Spin)$ is the obvious map. The obvious map of $\Omega_n(SU)$ into $\Omega_n(Spin)$ defines a homomorphism of $\Omega_{8k+2}(SU)$ into Z_2 which we also denote by ψ. This latter map is the main object to the investigated in this paper. In an appendix we briefly discuss $\psi\colon \Omega_{8k+2}(Spin) \to Z_2$.

The main results of this paper are as follows:

Theorem 1.1. *$\Phi\colon \Omega_{8k+2}(e) \to Z_2$ is zero for $k > 0$.*

The following corollaries of (1.1) are implied by the results of [9], [8] and [3].

Corollary 1.2. *$bP_{8k+2} \approx Z_2$, where bP_{8k+2} is the group of homotopy spheres which bound stably parallelizable $8k+2$-manifolds* [9].

Corollary 1.3. *If K is the topological manifold obtained by plumbing two copies of the tangent disc bundle of S^{4k+1} together and then attaching an $8k+2$-disc, then K does not admit a diffedentiable structure.*

(1.3) follows from [8] if one has the result that a C^∞ manifold with underlying topological space K is stably parallelizable. In Appendix 2 we give a proof of this due to John Milnor.

Corollary 1.4. *Every element of $\Omega_{8k+2}(e)$ can be represented by a homotopy sphere, $k \geqq 1$.*

Corollary 1.5. *A finite, 1-connected CW complex has the homotopy type of a stably parallelizable $8k+2$-manifold if and only if there is a stably*

Received June 7, 1965.

[1] The first named author was partially supported by the N. S. F. and the second named author was partially supported by the N. S. F. and by the U. S. Army Research Office.

spherical class $m \in H_{8k+2}(X)$ *such that* $m \cap : H^q(X;Z) \approx H_{8k+2-q}(X;Z)$ *for all* q *and* $\psi(X) = 0$ *(see* § 3 *for a definition of* $\psi(X)$*)* ([3], [12]).

It is known that $\Omega_2(SU) \approx Z_2$ [7]. Let α be the generator. Define $\psi\colon \Omega_2(SU) \to Z_2$ by $\psi(\alpha) = 1$.

THEOREM 1.6. *If* $\beta \in \Omega_{8k+2}(SU)$ *and* $\gamma \in \Omega_{8l}(SU)$, $k \geqq 0$, $l > 0$, *then*

$$\psi(\beta\gamma) = \psi(\beta) I(\gamma)$$

where $I(\gamma)$ *is the index of* γ *reduced* mod 2. ($I(\gamma)$ = *Euler characteristic of* γ mod 2 *also.*)

In [7] it is shown that $\Omega_{16}(SU)$ contain an element γ^{16} with $I(\gamma^{16}) \neq 0$ mod 2.

By (1.6) we have:

COROLLARY 1.7. $\psi(\alpha\gamma^{16}) = 1$.

In § 2 we give some preliminary results about cohomology operations. In § 3 we define $\psi\colon \Omega_{8k+2}(SU) \to Z_2$ and in § 4 we prove Theorems (1.1) and (1.6).

2. Some cohomology operations. Throughout the remainder of the paper all homology and cohomology groups will have Z_2 coefficients unless otherwise specified. m will denote an integer of the form $4k+1$, $k > 0$. Below we show that various cohomology operations are equal. This will always mean equal modulo the largest indeterminacy involved.

In [5] it is shown that the relation

$$Sq^2 Sq^{m-1} + Sq^1(Sq^2 Sq^{m-2}) = 0$$

on m-dimensional cohomology classes gives rise to a secondary operation

$$\phi\colon H^m(X) \cap \operatorname{Ker} Sq^{m-1} \cap \operatorname{Ker} Sq^2 Sq^{m-2} \to H^{2m}(X)/Sq^1 H^{2m-1}(X) + Sq^2 H^{2m-2}(X). \tag{2.1}$$

Furthermore, the following is proved:

2.2) If $\phi(u)$ and $\phi(v)$ are defined, $\phi(u+v)$ is defined and
$$\phi(u+v) = \phi(u) + \phi(v) + uv$$

Let $\hat{\sigma}_m \in H^m(K(Z,m);Z)$ be the generator and let σ_m be $\hat{\sigma}_m$ reduced mod 2. Let $p\colon E \to K(Z,m)$ be the fibration with fibre $K(Z_2, 2m-2)$ and k-invariant $Sq^{m-1}\hat{\sigma}_m$. Then $\phi(p^*\sigma_m)$ is defined since, by the Adem relations,

$$Sq^2 Sq^{m-2} = Sq^m + Sq^{m-1} Sq^1 = Sq^1 Sq^{m-1} + Sq^{m-1} Sq^1$$

is zero on $p^*\sigma_m$. Choose an element $z \in \phi(p^*\sigma_m)$. z defines a cohomology operation

$$\hat{\phi}\colon H^m(X,Z) \cap \operatorname{Ker} Sq^{m-1} \to H^{2m}(X)/Sq^2H^{2m-2}(X)$$

The following is immediate.

(2.3) If $\hat{u} \in H^m(X;Z)$, $Sq^{m-1}\hat{u} = 0$ and u is $\hat{u}$ reduced mod 2, then

$$\hat{\phi}(\hat{u}) = \phi(u).$$

Let $\hat{u} \in H^m(X;Z)$ be viewed as a map $\hat{u}\colon X \to K(Z,m)$. Then the following is proved in [13].

(2.4) If $Sq^{m-1}\hat{u} = 0$,

$$\hat{\phi}(\hat{u}) = Sq^2_{\hat{u}}(Sq^{m-1}\hat{\sigma}_m)$$

Let $f\colon (X,A) \to (Y,B)$ and let $g\colon X \to Y$ be the map defined by f. We need a product formula for functional operations.

(2.5) If $u \in H^q(Y,B)$, $v \in H^p(Y)$ and $g^*v = Sq^2v = Sq^1u = Sq^2u = 0$, then

$$Sq^2{}_f(uv) = (f^*u)(Sq^2{}_g v)$$

(This formula is proved, in the absolute case, in [1].)

Proof. Let $h\colon A \to B$ be the map defined by f. Note that (X,A) is contained in the mapping cylinders (C_g, C_h) and that $A = C_h \cap X$. Hence we may assume f is an inclusion map and $A = X \cap B$. Cup product with u maps the exact sequence of (Y,X) into the exact sequence of the triad (Y,X,B) giving the following commutative diagram:

$$\begin{array}{ccccccc}
H^{p-1}(X) & \longrightarrow & H^p(Y,X) & \longrightarrow & H^p(Y) & \longrightarrow & H^p(X) \\
\downarrow \cup f^*u & & \downarrow \cup u & & \downarrow \cup u & & \downarrow \\
H^{p+q-1}(X,A) & \to & H^{p+q}(Y, X \cup B) & \to & H^{p+q}(Y,B) & \to & H^{p+q}(X,A)
\end{array}$$

Since $Sq^1u = Sq^2u = 0$, $Sq^2uz = uSq^2z$ for any z. Hence Sq^2 maps the above ladder into itself giving a large commutative diagram. (2.5) now follows by chasing around this diagram.

3. Definition of the Kervaire invariant. Throughout this section we view classes $u \in H^q(X)$ as maps $u\colon X \to K(Z_2, q)$. Again, $m = 4k+1$, $k > 0$.

$\Omega_n(X;SU)$ will denote the n-th SU bordism group [7]. Recall, this group is the set of equivalence classes, under an appropriately defined cobordism relation, of triples (M, λ, f) where M is a closed, compact C^∞ n-manifold, λ is an SU reduction of the normal bundle of M embedded in R^{n+k} for large k and $f\colon M \to X$. $\Omega_n(SU) = \Omega_n(pt; SU)$. One may easily show that if X is connected, every element of $\Omega_n(X; SU)$ can be represented by (M, λ, f) where M is connected. Hereafter we assume all spaces are connected. Also we assume all manifolds have an SU structure on their normal bundle, (M, λ, u) will be denoted by (M, u) and $\nu_M\colon M \to BSU_k$ will denote the map defined by this SU structure.

LEMMA 3.1. *If* $\{M, u\} \in \Omega_{2m}(K(Z_2, m); SU)$, *then* $\{M, u\} = \{M', u'\}$ *where* M' *is* 1*-connected. Furthermore, there is a cobordism* (N, v) *between* (M, u) *and* (M', u') *such that if* $i\colon M \to N$ *and* $j\colon M' \to N$ *are the inclusion maps,*

$$i^*\colon H^q(M) \approx H^q(N) \quad \text{for } q > 2$$
$$j^*\colon H^q(M') \approx H^q(N) \quad \text{for } q \neq 2m-1, 2m-2.$$

Proof. We form M' by killing $\pi_1(M)$ by surgery [10]. This process yields a manifold N with an SU reduction such that $\partial N = M - M'$. Furthermore N consists of $M \times I$ with handles $D^2 \times D^{2m-1}$ attached by maps $h\colon S^1 \times D^{2m-1} \to M \times \{0\}$. Up to homotopy type, N is M with 2-cells attached and N is also M' with $(2m-1)$-cells attached. (3.1) now follows from these properties of N.

Let

$$\bar{\phi}\colon \Omega_{2m}(K(Z_2, m); SU) \to Z_2$$

be defined as follows: Let ϕ be the cohomology operation described in (2.1). Let $\{M, u\} \in \Omega_{2m}(K(Z_2, m); SU)$. Since M has an SU reduction, its Stiefel-Whitney classes w_1 and w_2 are zero. Therefore by the Wu formulas,

$$Sq^1 H^{2m-1}(M) = w_1 H^{2m-1}(M) = 0 \text{ and } Sq^2 H^{2m-2}(M) = w_2 H^{2m-2}(M) = 0.$$

Hence, $\phi\colon H^m(M) \cap \operatorname{Ker} Sq^{m-1} \to H^{2m}(M)$. If $Sq^{m-1}u = 0$, we let

$$\bar{\phi}\{M, u\} = \phi(u)([M])$$

where $[M] \in H_{2m}(M)$ denotes the fundamental class. By (3.1) we may always choose M to be 1-connected and hence so that

$$Sq^{m-1} H^m(M) \subset H^{2m-1}(M) \approx H_1(M) = 0.$$

Therefore $\bar{\phi}$ is defined on all of $\Omega_{2m}(K(Z_2, m); SU)$. We show that it is well defined. Let $\beta \in \Omega_{2m}(K(Z_2, m); SU)$. Choose an $(M_1, u_1) \in \beta$ such that

M_1 is 1-connected. Let (M_2, u_2) be any representative of β such that $Sq^{m-1}u_2 = 0$. We show $\phi(u_2)([M_2]) = \phi(u_1)([M_1])$. Let (N, v) be a cobordism between (M_1, u_1) and (M_2, u_2). By surgery we make N 2-connected. Let $j_i\colon M_i \to N$ be the inclusion maps. $H^{2m-1}(N, M_2) \approx H_2(N, M_1) = 0$ since $H_2(N) = H_1(M_1) = 0$. Therefore $j_2{}^*\colon H^{2m-1}(N) \to H^{2m-1}(M_2)$ is an injection. $j_2{}^*Sq^{m-1}v = Sq^{m-1}u_2 = 0$. Hence $Sq^{m-1}v = 0$ is zero. In a similar way one shows that $j_i{}^*\colon H^{2m}(N) \to H^{2m}(M_i)$ is an injection and hence that $Sq^2Sq^{m-2}v = 0$. Therefore $\phi(v)$ is defined and, since $\phi(u_i) = j_i{}^*\phi(v)$, $\phi(u_1) = 0$ if and only if $\phi(u_2) = 0$. Thus $\bar{\phi}$ is well defined.

LEMMA 3.2. *If $\{M, u\} \in \Omega_{2m}(K(Z_2, m); SU)$, $Sq^{m-1}u = 0$ and u is the reduction* mod 2 *of an integer class $\hat{u}$, then*

$$\bar{\phi}\{M, u\} = Sq^2_{\hat{u}}\,(Sq^{m-1}\hat{\sigma}_m)([M])$$

Proof. By (2.3) and (2.4),

$$\phi(u) = \hat{\phi}(\hat{u}) = Sq^2_{\hat{u}}\,(Sq^{m-1}\hat{\sigma}_m).$$

Thus the only thing to check is that the indeterminacy for each of these operations is zero. The indeterminacy of ϕ is $Sq^2H^{2m-2}(M) = w_2H^{2m-2}(M) = 0$. The indeterminacy of $Sq^2{}_u$ is

$$Sq^2H^{2m-2}(M) + \hat{u}^*H^{2m}(K(Z, m))$$

Thus we must show that $v = Sq^{i_1}Sq^{i_2}\cdots Sq^{i_l}u = 0$ if $i_1 + i_2 + \cdots + i_l = m$. By the Wu formulas $v = zu$ where z is a polynomial in the Stiefel-Whitney classes of M. Therefore $z = \nu^*{}_M z'$ for $z' \in H^m(BSU)$. But m is odd and hence $z' \in H^m(BSU) = 0$.

We next define the Kervaire Invariant $\psi\colon \Omega_{2m}(SU) \to Z_2$. Let $\{M\} \in \Omega_{2m}(SU)$. Choose a symplectic basis $\{u_i, v_i \mid i = 1, 2, \cdots, \nu\}$ for $H^m(M)$, that is, $u_1, \cdots, u_\nu, v_1, \cdots, v_\nu$ is a basis for $H^m(M)$, $u_iu_j = v_iv_j = 0$, and $u_iv_j = \delta_{ij}$. Since M is orientable, $u^2 = 0$, $u \in H^m(M)$, and hence such a basis exists. Define

$$\psi\{M\} = \sum_{i=1}^{\nu} \bar{\phi}\{M, u_i\} \cdot \bar{\phi}\{M, v_i\}. \tag{3.3}$$

We show that ψ is a homomorphism and that it is well defined. By (3.1) we may change M by surgery so that it is 1-connected. Furthermore, by (3.1) $\{u_i, v_i\}$ goes over, under this process, to a symplectic basis. Hence we may assume M is 1-connected. By (2.2)

$$\bar{\phi}\{M, \;\}\colon H^m(M) \to Z_2$$

is a quadratic function whose associated bilinear form, namely, cup product, is non-singular. Therefore the right side of 3.3 is independent of the choice of the symplectic basis [2]. One may easily check that the right side of 3.3 is additive with respect to the connected sum operation on manifolds. Thus ψ is a homomorphism and to show that it is well defined it is sufficient to show that the right side of (3.3) is zero if $M=\partial N$. We may make N 2-connected by surgery. Recall, if $j\colon M\to N$ is the inclusion map, $u\in H^m(N)$, $v\in H^m(N)$, then $u\cdot\delta^*v=\delta^*(j^*(u)\cdot v)$, where $\delta^*\colon H^m(M)\to H^{m+1}(N,M)$. From this fact and Poincare duality one may obtain classes $u_i\in H^m(N)$ and $v_i\in H^m(M)$ such that $\{j^*u_i, v_i\}$ is a symplectic basis for $H^m(M)$. $H^{2m-1}(N)\approx H_2(N,M)=0$ and $H^{2m}(N)\approx H_1(N,M)=0$. Therefore $\phi(u_i)$ is defined and equals zero. Therefore $\bar{\phi}(\{M,j^*u_i\})=\phi(j^*u_i)([M])=j^*\phi(u_i)([M])=0$. Thus ψ is well defined.

Remark 3.4. The secondary operation ϕ is not uniquely determined by the relation between primary operations from which it arises, that is, it is only determined up to the addition of a stable primary operation. (3.2) shows that $\psi\colon\Omega_{8k+2}(SU)\to Z_2$ does not depend on the choice of ϕ since for any $\{M\}\in\Omega_{2m}(SU)$, one may choose M 1-connected and with every element of $H^m(M)$ the reduction of an integer class (See § 4.).

Let α be the generator of $\Omega_2(SU)\approx Z_2$. We define $\psi\colon\Omega_2(SU)\to Z_2$ by $\psi(\alpha)=1$.

Finally, to complete the statement of Corollary 1.5, we define $\psi(X)\in Z_2$ when X is a 1-connected, finite CW complex which has a stably spherical class $m\in H_{8k+2}(X;Z)$ such that

$$\cap\, m\colon H^q(X;Z)\approx H_{8k+2-q}(X;Z)$$

for all q. Since $H_{8k+2}(X;Z)$ is generated by a stably spherical class, Sq^i is zero on $H^{8k+2-i}(X)$. Also

$$Sq^{4k}H^{4k+1}(X)\subset H^{8k+1}(X)\approx H_1(X)=0$$

Therefore ϕ defines a quadratic function

$$\phi\colon H^{4k+1}(X)\to H^{8k+2}(X)$$

Let

$$\psi(X)=\sum_{i=1}^{r}\phi(u_i)(m)\cdot\phi(v_i)(m)$$

where $\{u_i, v_i \mid i=1,\cdots,r\}$ is a symplectic basis for $H^{4k+1}(X)$.

4. Proofs of Theorems (1.1) and (1.6). We first prove (1.6). Let

$\beta \in \Omega_{8k+2}(SU)$, $k \geqq 0$ and let $\gamma \in \Omega_{8l}(SU)$, $l > 0$. We wish to show that $\psi(\beta\gamma) = \psi(\beta) I(\gamma)$ where $I(\gamma)$ is the index of $\gamma \bmod 2$.

Let $M \in \beta$ and $N \in \gamma$. Applying surgery to M and N we may choose them so that $\nu_{M*}\colon \pi_i(M) \to \pi_i(BSU)$ is an isomorphism for $i < 4k+1$ and $\nu_{N*}\colon \pi_i(N) \to \pi_i(BSU)$ is an isomorphism for $i < 4l$. Then $H^q(M) = 0$ for q odd and $q \neq 4k+1$ and $H^q(N) = 0$ for q odd. Furthermore the elements of $H^{4k+1}(M)$ and $H^{4l}(N)$ are reduction mod 2 of integer classes because $H^{4l+1}(N;Z) \approx H_{4l-1}(N;Z) = 0$ and $H^{4k+2}(M;Z) \approx H_{4k}(M;Z) \approx H_{4k}(BSU;Z)$ which is free abelian. Note $H^{4(k+l)+1}(M \times N) = H^{4k+1}(M) \otimes H^{4l}(N)$.

LEMMA 4.1. *If $u \in H^{4k+1}(M)$ and $v \in H^{4l}(N)$,*

$$\phi(u \otimes v) = \phi(u) \otimes v^2 \text{ if } k > 0$$
$$= 0 \qquad \text{if } k = 0 \text{ and } v^2 = 0.$$

Proof. Let $\hat{u} \in H^{4k+1}(M;Z)$ and $\hat{v} \in H^{4l}(N;Z)$ be classes which give u and v when reduced mod 2. Below we denote $Sq^2{}_f u$ by $Sq^2(f,u)$.

$$(4.2) \qquad \phi(u \otimes v) = \hat{\phi}(\hat{u} \otimes \hat{v})$$

$$(4.3) \qquad = Sq^2(\hat{u} \otimes \hat{v}, Sq^{4(k+l)}\hat{\sigma}_{4(k+l)+1})$$

$$= Sq^2((\hat{u} \times id)(\hat{\sigma}_{4k+1} \otimes \hat{v}), Sq^{4(k+l)}\hat{\sigma}_{4(k+l)+1})$$

$$(4.4) \qquad = Sq^2(\hat{u} \times id, Sq^{4(k+l)}(\hat{\sigma}_{4k+1} \otimes v))$$

$$(4.5) \qquad = Sq^2(u \times id, Sq^{4k}\hat{\sigma}_{4k+1} \otimes v^2)$$

$$(4.6) \qquad = Sq^2(u, Sq^{4k}\hat{\sigma}_{4k+1}) \otimes v^2$$

$$(4.7) \qquad = \phi(u) \otimes v^2$$

(4.2) follows from (2.3), (4.2) and (4.7) from (3.2), (4.4) from the naturality of $Sq^2{}_f$, (4.5) from the Cartan formula, and (4.6) from (2.5). In the case $k = 0$, one needs $v^2 = 0$, in order that $\phi(u \otimes v)$ be defined and (4.5) yields $\phi(u \otimes v) = 0$.

We continue with the proof of (1.6). Let $v_{4l}(N) \in H^{4l}(N)$ be the class such that $z^2 = z v_{4l}(N)$ for all $z \in H^{4l}(N)$. Recall, $v_{4l}(N)^2([N]) = \text{index}\, N \bmod 2$ (the proof of this is contained in the argument below). Let $u \in H^{4l}(N)$ be a class such that $u = 0$ if $v_{4l}(N) = 0$, $u = v_{4l}(N)$ if $v_{4l}(N)^2 \neq 0$ and $u v_{4l}(N) \neq 0$ if $v_{4l}(N) \neq 0$ and $v_{4l}(N)^2 = 0$. Let $V \subset H^{4l}(N)$ be the subspace spanned by u and $v_{4l}(N)$ and let $U \subset H^{4l}(N)$ be its orthogonal complement, that is, $U = \{z \in H^{4l}(N) \mid zu = z v_{4l}(N) = 0\}$. $H^{4k+1}(M) \otimes U$ is the orthogonal complement of $H^{4k+1}(M) \otimes V$ in $H^{4(k+l)+1}(M \times N)$. Hence a symplectic basis for each of these subspaces will provide a symplectic basis for $H^{4(k+l)+1}(M \times N)$. By (4.1) $H^{4k+1}(M) \otimes U$ makes no contribution to $\psi\{M \times N\}$ as $z^2 = zv_{4l}(N) = 0$ if $z \in U$. Let $\{x_i, y_i\}$ be a symplectic basis for $H^{4k+1}(M)$. We now consider four cases.

Case I. $v_{4l}(N) = 0$. Then $V = 0$ and $\psi\{M \times N\} = 0 = \psi\{M\}v_{4l}^2(N)$.

Case II. $v_{4l}^2(N) = 0$, $v_{4l}(N) \neq 0$. A symplectic basis for $H^{4k+1}(M) \otimes V$ is given by $\{x_i \otimes v_{4l}(N), y_i \otimes v_{4l}(N)\}$ as the first group of terms and $\{x_i \otimes (v_{4l}(N) + u), y_i \otimes u\}$ as the second group. By (4.1) $\phi(x_i \otimes v_{4l}(N)) = \phi(y_i \otimes v_{4l}(N)) = 0$. Hence $\psi\{M \times N\} = 0$.

Case III. $v_{4l}^2(N) \neq 0$, $k > 0$. $\{x_i \otimes v_{4l}(N), y_i \otimes v_{4l}(N)\}$ is a symplectic basis for $H^{4k+1}(M) \otimes V$. Therefore by (4.1),

$$\begin{aligned}\psi\{M \times N\} &= \sum \phi(x_i \otimes v_{4l}(N))([M \times N])\phi(y_i \otimes v_{4l}(N))([M \times N]) \\ &= \sum \phi(x_i)([M])\phi(y_i)([M])v_{4l}^2(N) \\ &= \psi(\{M\})I(\{N\}).\end{aligned}$$

Case IV. $v_{4l}^2(N) \neq 0$, $k = 0$. The generator of $\Omega_2(SU)$ is represented by $M = S^1 \times S^1$ with the non-trivial SU reduction of its normal bundle. Let $x \in H^1(S^1)$ be the generator. $\{x \otimes 1 \otimes v_{4l}(N), 1 \otimes x \otimes v_{4l}(N)\}$ is a symplectic basis for $H^1(M) \otimes V$. Hence

$$\psi\{M \times N\} = \bar{\phi}\{M \times N, x \otimes 1 \otimes v_{4l}(N)\}\,\bar{\phi}\{M \times N, 1 \otimes x \otimes v_{4l}(N)\}.$$

By a symmetry argument this equals $\bar{\phi}\{M \times N, x \otimes 1 \otimes v_{4l}(N)\}$. By Wu formulas $v_{4l}(N)$ is a polynomial in the Stiefel-Whitney classes of N. Therefore $v_{4l}(N) = \nu_N^*(z_{4l})$ where $z_{4l} \in H^{4l}(BSU)$. z_{4l} is the reduction mod 2 of an integer class $\hat{z}_{4l}$. Hence if $\partial P = S^1 \times N - N'$

$$\partial(P, \nu_p^* z_{4l}) = (S^1 \times N, 1 \otimes v_{4l}(N)) - (N', \nu_{N'}^* z_{4l})$$

By (3.1) we may choose N' to be 1-connected. Let $y = \nu_{N'}^* z_{4l}$. $y^2 \in H^{8l}(N') \approx H_1(N') = 0$. Therefore $\phi(x \otimes y)$ is defined. Let $\hat{y} = \nu_{N'}^* \hat{z}_{4l}$ and let $\hat{x} \in H^1(S^1; Z)$ be the generator. By (2.3) and (2.4),

$$\psi\{M \times N\} = \hat{\phi}(\hat{x} \otimes \hat{y})([S^1 \times N'])$$

$$\begin{aligned}\hat{\phi}(\hat{x} \otimes \hat{y}) &= Sq^2{}_{\hat{x} \otimes \hat{y}}(Sq^{4l}\hat{\sigma}_{4l+1}) \\ &= Sq^2{}_{\mathrm{id} \times \hat{y}}(Sq^{4l}(\hat{x} \otimes \hat{\sigma}_{4l})) \\ &= Sq^2{}_{\mathrm{id} \times \hat{y}}(x \otimes \sigma_{4l}^2) \\ &= x \otimes Sq^2{}_{\hat{y}}(\sigma_{4l}^2).\end{aligned}$$

Hence we must show that $Sq^2{}_{\hat{y}}(\sigma_{4l}^2) \neq 0$.

Let ν be the normal bundle of N' embedded in R^{8l+2r} for large r. Let ζ_r be the canonical, real $2r$ bundle over BSU_r. let $\lambda\colon \nu \to \zeta_r$ be the SU reduction of ν, $T(\nu)$ and $T(\zeta_r)$ the Thom spaces, $U \in H^{2r}(T(\zeta_r);Z)$ the Thom class, and let $f\colon S^{8l+2r} \to T(\nu)$ be the map obtained by the Thom construction. Note $T(\lambda)f$ is homotopic to $g\eta$ where η is suspension of the Hopf map and $g\colon S^{8l+2r-1} \to T(\zeta_r)$ is the map obtained from N by the Thom construction. This is because N' and $S^1 \times N$ are SU cobordant.

$$Sq^2{}_{\hat{y}}(\sigma_{4l}{}^2) = Sq^2{}_{\nu_{N'}}(z_{4l}{}^2) \text{ as } \hat{y} = \hat{z}_{4l} \circ \eta_{N'}.$$

Note $Sq^1U = w_1U = 0$ and $Sq^2U = w_2U = 0$.

$$(4.8)\qquad \begin{aligned} f^*(Sq^2{}_{\nu_{N'}}(z_{4l}{}^2) \cdot T(\lambda)^*U) &= f^*Sq^2{}_{T(\lambda)}(z_{4l}{}^2U) \\ &= Sq^2{}_{T(\lambda)f}(z_{4l}{}^2U) \\ &= Sq^2{}_{g\eta}(z_{4l}{}^2U) \\ &= Sq^2{}_{\eta}(g^*z_{4l}{}^2U). \end{aligned}$$

But $Sq^2{}_\eta$ is an isomorphism and $g^*(z_{4l}{}^2U)([S^{8k+2r-1}]) = v_{4l}{}^2([N]) \neq 0$. This completes the proof of (1.6).

We next prove (1.1). Suppose $\gamma = \{M\} \in \Omega_{8k+2}(SU)$ where M is stably parallelizable. We must show that $\psi(\gamma) = 0$. Conner and Floyd [7] and Lashof and Rothenberg (unpublished) show that if $\mu \in \Omega_{2n}(SU)$ has all its Chern numbers zero, then $\mu = \alpha\beta$ where $\alpha \in \Omega_2(SU)$ is the generator and $\beta \in \Omega_{2n-2}(SU)$. Hence $\gamma = \alpha\beta$. By (1.6) $\psi(\gamma) = \psi(\alpha)I(\beta) = I(\beta) = v_{4l}(N)^2([N])$ where $v_{4l}(N)$ and N are as above. Suppose $v_{4l}{}^2(N) \neq 0$. Then by (4.3) in Case IV above, $Sq^2{}_{g\eta}(z_{4l}{}^2U) \neq 0$.

Let θ be the secondary cohomology operation associated with the relation $Sq^2Sq^2 = 0$ on integer classes.

$$\theta\colon H^q(X;Z) \cap \operatorname{Ker} Sq^2 \to H^{q+3}(X)/Sq^2H^{q+1}(X).$$

Let θ_f denote the associated functional operation. In [14] it is shown that

$$(4.9)\qquad \theta_{\eta\eta}x = Sq^2{}_\eta Sq^2{}_\eta x.$$

$Sq^2(z_{4l}{}^2U) = 0$, hence $\theta(z_{4l}{}^2U) \in H^{8l+3+2r}(T(\zeta_r)) = 0$ is defined and is zero.

$$\begin{aligned} Sq^2{}_\eta Sq^2{}_\eta(z_{4l}{}^2U) &= Sq^2{}_\eta\, Sq^2{}_\eta(g^*(z_{4l}{}^2U)) \\ &= \theta_{\eta\eta}(g^*(z_{4l}{}^2U)) \\ &= \theta_{g\eta\eta}(z_{4l}{}^2U). \end{aligned}$$

We show that $\theta_{g\eta\eta}(z_{4l}{}^2U)$ is zero and has zero indeterminacy. Since $Sq^2{}_\eta$ is an isomorphism, this shows that $Sq^2{}_{g\eta}(z_{4l}{}^2U) = 0$, which is the contradiction we seek. Note $g\eta\eta\colon S^{8l+2+2r} \to T(\zeta_r)$ is the map corresponding to $S^1 \times S^1 \times N$

under the Thom construction. By hypothesis, $S^1 \times S^1 \times N$ is SU cobordant to a stably parallelizable manifold. Hence $g\eta\eta$ is homotopic to ih where $i\colon S^{2r} \to T(\zeta_r)$ is the inclusion of a fibre and $h\colon S^{8l+2+2r} \to S^{2r}$. Therefore,

$$\theta_{g\eta\eta}(z_{4l}{}^2 U) = \theta_h(i^* z_{4l}{}^2 U) = 0.$$

The indeterminacy of $\theta_{g\eta\eta}(z_{4l}{}^2 U)$ is

$$(g\eta\eta)^*(H^{8l+2r}(T(\zeta_r))) + \theta(H^{8l+2r-3}(S^{8l+2r})) + Sq^2(H^{8l+2r-2}(S^{8l+2r})) + Sq^2{}_{g\eta\eta}(H^{8l+2r-1}(T(\zeta_r))) = 0.$$

The above argument shows that if $\beta \in \Omega_{8l}(SU)$ and $f\colon S^{8l+2+2r} \to T(\zeta_r)$ is the map associated with $\alpha\beta$, then

$$\psi(\alpha\beta) = \theta_f(z_{4l}{}^2 U)([S^{8l+2+2r}])$$

By examining how the operation θ_f is related to the Thom isomorphism one may prove:

THEOREM 4.10. *If $\alpha \in \Omega_2(SU)$ is the generator and $\beta \in \Omega_{8l}(SU)$, then*

$$\psi(\alpha\beta) = \theta_{\nu_M}(z_{4l}{}^2)([M])$$

where M is a 3-connected manifold representing $\alpha\beta$.

Appendix 1.

We state here, without proof, those parts of our results which go through for $\psi\colon \Omega_{8k+2}(Spin) \to Z_2$.

The first difficulty in generalizing our results to the $Spin$ case is that $\bar{\phi}\colon \Omega_{2m}(K(Z_2, m); Spin) \to Z_2$ depends on the choice of the operation ϕ associated to the given relation among primary operations. One choice would be to choose ϕ so that the third suspension of ϕ is zero on all classes of dimension $m-3$. This is possible and gives rise to two choices for ϕ, each of which give the same $\bar{\phi}$.

The only part of 1.6 that we can prove is the case $k=0$, namely:

THEOREM A1.1. *If $\alpha \in \Omega_2(Spin)$ is the generator and $\beta \in \Omega_{8k}(Spin)$,*

$$\psi(\alpha\beta) = I(\beta)$$

where $I(\beta)$ is the index of β mod 2.

This theorem follows from the arguments used to prove (1.6), except that one must use slightly more complicated cohomology operations. One also sees that $\psi(\alpha\beta)$ is independent of the choice of ϕ.

Recall, quaternionic projective space PQ_n admits a *Spin* structure and has Euler characteristic 1 if n is even. Hence,

COROLLARY A1.2. *If $\alpha \in \Omega_2(Spin)$ is the generator,*

$$\psi(\alpha\{PQ_{2k}\}) = 1$$

The proof of (1.6) breaks down in the *Spin* case for $k > 0$ because $H^{4(k+l)+1}(M \times N)$ is very complicated even if M and N are simplified by surgery. Also, it is not clear whether an analogous theorem to 4.5 goes through in the spin case.

Appendix 2.

In this appendix we give a proof, due to John Milnor, that a differentiable manifold of the same homotopy type as a Kervaire manifold ([8]) is stably parallelizable.

Let n be odd and $n \neq 1, 3, 7$, let $p\colon T \to S^n$ be the tangent disc bundle of S^n, let D^n be the closed n-disc, $h\colon D^n \to S^n$ a homeomorphism into, $k\colon D^n \times D^n \to T$ a bundle map covering h, $\bar{T}$ a copy of T and let $L = T \cup \bar{T}$ with $h(x,y)$ identified to $h(y,x)$ for each $(x,y) \in D^n \times D^n$. L is a manifold with boundary and ∂L is homeomorphic to S^{2n-1}. Let $f\colon S^{2n-1} \to \partial L$ be a homeomorphism and let $K^{2n} = L \bigcup_f D^{2n}$. K^{2n} is the manifold constructed by Kervaire.

THEOREM A2.1. *If M is a differentiable manifold with the same homotopy type as K^{2n}, then M is stably parallelizable.*

Proof. Let S^n, $\bar{S}^n$ and $S_1{}^n \subset K^{2n}$ be the zero cross-section of T, the zero cross-section of $\bar{T}$ and a cross-section of the associated sphere bundle of T, respectively. Clearly K^{2n} has a cell structure $S^n \vee \bar{S}^n \cup e^{2n}$ and hence, since S^n and $S_1{}^n$ are isotopic,

$$K^{2n} = S_1{}^n \vee \bar{S}^n \cup e^{2n}. \qquad K^{2n}/S_1{}^n = \bar{S}^n \cup e^{2n}. \qquad K^{2n}/K^{2n} - \operatorname{Int} T$$

is the Thom space of $\tau(S^n)$ which is $S^n \bigcup_{[\iota,\iota]} e^{2n}$. Consider the quotient map

$$u\colon K^{2n}/S_1{}^n \to K^{2n}/K^{2n} - \operatorname{Int} T$$

u is of degree one on the n and $2n$ cells and hence is a homotopy equivalence. Therefore $K^{2n}/S^n = K^{2n}/S_1{}^n = S^n \bigcup_{[\iota,\iota]} e^{2n}$.

Let $g\colon K^{2n} \to M$ be a homotopy equivalence. We first show that M is almost parallelizable by showing that $g^*\tau(M)$ is trivial on $S^n \vee \bar{S}^n$. Choose g so that $g \mid S^n$ is a smooth embedding and let ν be the normal bundle of $g(S^n)$. Let $T(\nu)$ be the Thom space of ν, $t\colon M \to T(\nu)$ the usual map and

$\bar{t}\colon K^{2n} \to K^{2n}/S^n$ the quotient map. $tg \mid S^n$ is homotopically trivial. Hence, up to homotopy, $tg = \bar{g}\bar{t}$ for some $\bar{g}$. One may easily check that $\bar{g}$ is a homotopy equivalence. Hence $T(\nu) = S^n \bigcup_\alpha e^{2n}$ where $\alpha = [\iota, \iota]$. Recall, $\alpha = J(\beta)$, $\beta \in \pi_{n-1}(0_n)$ where β is the characteristic class of ν. The stable J homomorphism on $\pi_{n-1}(0)$ is a monomorphism and hence β is stably trivial. Hence ν is stably trivial and therefore $g^*\tau(M) \mid S^n$ is trivial. The same argument shows that $g^*\tau(M) \mid \bar{S}^n$ is trivial.

Recall, the obstruction to an almost parallelizable m-manifold being stably parallelizable is in the kernel of J on $\pi_{m-1}(0)$. J is a monomorphism on $\pi_{2n-1}(0)$. Hence M is stably parallelizable.

BRANDEIS UNIVERSITY.
M. I. T.

REFERENCES.

[1] J. Adem, "Un criterio cohomologico para delerminar composiciones esenciales de transformaciones," *Boletin de la Sociedad Matematica Mexicana*, vol. 1 (1956), pp. 38-48.

[2] C. Arf, "Untersuchungen über quadratische formen in körpern der charakteristik 2," *Crelles Mathematical Journal*, vol. 183 (1941), pp. 148-167.

[3] W. Browder, *Homotopy type of differentiable manifolds*, Colloquium on Algebraic Topology, Aarhus University, 1962, pp. 42-46.

[4] E. H. Brown, Jr., "Note on an invariant of Kervaire," *Michigan Mathematical Journal*, vol. 12 (1965), pp. 23-24.

[5] E. H. Brown, Jr. and F. P. Peterson, "Whitehead products and cohomology operations," *Quarterly Journal of Mathematics*, vol. 15 (1964), pp. 116-120.

[6] E. H. Brown, Jr. and F. P. Peterson, "The Kervaire invariant of $(8k+2)$-manifolds," *Bulletin of hhe American Mathematical Society*, vol. 71 (1965), pp. 190-193.

[7] P. E. Conner and E. E. Floyd, "The *SU*-bordism theory," *Bulletin of the American Mathematical Society*, vol. 70 (1964), pp. 670-675.

[8] M. A. Kervaire, "A manifold which does not admit any differentiable structure," *Commentarii Mathematici Helvetici*, vol 34 (1960), pp. 257-270.

[9] M. A. Kervaire and J. W. Milnor, "Groups of homotopy spheres: I," *Annals of Mathematics*, vol. 77 (1963), pp. 504-537.

[10] R. Lashof, *Some theorems of Browder and Novikov on homotopy equivalent manifolds with an application*, Mimeographed Notes, University of Chicago.

[11] J. W. Milnor, "Remarks concerning spin manifolds," to appear.

[12] S. P. Novikov, "Diffeomorphisms of simply connected manifolds," *Soviet Mathematics, American Mathematical Society*, vol. 3 (1962), pp. 540-543.

[13] F. P. Peterson and N. Stein, "Secondary cohomology operations: two formulas," *American Juornal of Mathematics*, vol. 81 (1959), pp. 281-305.

[14] F. P. Peterson, "Functional cohomology operations," *Transactions of the American Mathematical Society*, vol. 86 (1957), pp. 197-211.