

Equivariant dendroidal Segal spaces and G - ∞ -operads

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We introduce the analogues of the notions of complete Segal space and of Segal category in the context of equivariant operads with norm maps, and build model categories with these as the fibrant objects. We then show that these model categories are Quillen equivalent to each other and to the model category for G - ∞ -operads built in a previous paper.

Moreover, we establish variants of these results for the Blumberg–Hill indexing systems.

In an appendix, we discuss Reedy categories in the equivariant context.

55U10, 55U35, 55U40; 18G30

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1 Introduction

This paper follows Pereira [21] and Bonventre and Pereira [7] and is the third piece of a larger project aimed at understanding the homotopy theory of *equivariant operads with norm maps*. Here, norm maps are a new piece of structure that must be considered when dealing with equivariant operads (see Remark 3.39 for a brief definition of norm maps or the introductions to [21; 7] for a more extensive discussion). The need to

understand norm maps was made clear by Hill, Hopkins and Ravenel [15], who used them in the context of equivariant ring spectra as part of their solution of the Kervaire invariant one problem.

The starting point of this project was the discovery by the authors, for each finite group G , of a category Ω_G of G -trees whose objects diagrammatically encode compositions of norm maps (in a G -equivariant operad) and whose arrows encode the necessary compatibilities between such compositions. Our categories Ω_G are a somewhat non-obvious generalization of the dendroidal category Ω of Cisinski, Moerdijk and Weiss, and indeed all the key combinatorial concepts in their work, such as faces, degeneracies, boundaries and horns, generalize to G -trees [21, Sections 5–6]. As such, it is natural to attempt to generalize the Cisinski–Moerdijk program [9; 10; 11] to the equivariant context.

Recall that the main result of their program is the existence of a Quillen equivalence

$$W_1: \text{dSet} \rightleftarrows \text{sOp} : N_{\text{hc}},$$

where $\text{dSet} = \text{Set}^{\Omega^{\text{op}}}$ is the category of presheaves on Ω , which are called *dendroidal sets*, and sOp is the category of *simplicial colored operads* (also referred to as simplicial multicategories). Their program was carried out in three main steps:

- (i) they established in [9] the existence of the ∞ -operad model structure on dSet (with some of the key combinatorial analysis based on Moerdijk and Weiss’s previous work in [20]);
- (ii) they established in [10] auxiliary model structures on the categories sdSet and PreOp of dendroidal spaces and preoperads, with fibrant objects the *complete dendroidal Segal spaces* and *Segal operads*, and showed that all three of dSet , sdSet and PreOp are Quillen equivalent;
- (iii) lastly, in [11] they established the existence of the model structure on sOp as well as the Quillen equivalence between sOp and PreOp , finishing the proof of the main result of the program.¹

From the perspective of the Cisinski–Moerdijk program, the previous paper [21] is then the equivariant analogue of the first step [9] (as well as [20]), while the present

¹Recall that by using the inclusions of simplicial categories and simplicial sets into simplicial operads and dendroidal sets (see the introduction to [11]), the Cisinski–Moerdijk program recovers and generalizes the Bergner–Joyal–Lurie–Rezk–Tierney program studying the various models for $(\infty, 1)$ -categories. A survey of these models can be found in Bergner [4].

paper provides the equivariant analogue of the second step [10]; the analogue of the third step spans four papers, culminating in [8]. More explicitly, in [21], and inspired by the category Ω_G of G -trees, the second author equipped the category dSet^G of G -equivariant dendroidal sets with a model structure whose fibrant objects are “equivariant operads with norm maps up to homotopy”, called G - ∞ -operads. Further, it was shown therein [21, Proposition 6.15] that whenever a G -operad $\mathcal{O} \in \text{sOp}^G$ is suitably fibrant, the homotopy coherent nerve $N_{\text{hc}}(\mathcal{O})$ is such a G - ∞ -operad (rather than just an “ ∞ -operad with a G -action”). In the present paper our main results, Theorems 4.30, 4.39 and 4.41, are then the existence of suitable model structures on the categories sdSet^G and PreOp^G of G -dendroidal spaces and G -preoperads, with fibrant objects the *complete equivariant dendroidal Segal spaces* and *equivariant Segal operads*, as well as the existence of Quillen equivalences between all three of dSet^G , sdSet^G and PreOp^G .

Table 1 provides a summary of the parallel narratives for categories, operads and equivariant operads, listing the different model categories as well as the terminology for the fibrant objects.

“categories up to htpy”	“operads up to htpy”	“equivariant operads up to htpy”
simplicial sets sSet Joyal model structure ∞ -categories	dendroidal sets dSet model str. from [9] ∞ -operads	equivariant dendroidal sets dSet^G model structure from [21] G - ∞ -operads
bisimplicial sets ssSet Rezk model structure complete Segal spaces	simp. dend. sets sdSet model str. from [10] compl. dend. Segal spaces	equiv. simp. dend. sets sdSet^G model structure from Section 4.2 compl. equiv. dend. Segal spaces
Segal precategories SeCat Hirschowitz–Simpson Segal categories	Segal preoperads PreOp model str. from [10] Segal operads	equiv. Segal preoperads PreOp^G model structure from Section 4.3 equiv. Segal operads
simplicial categories sCat Bergner model structure	simplicial operads sOp model str. from [11]	equiv. simplicial operads sOp^G model structure forthcoming

Table 1: A summary of models for ∞ -categories, ∞ -operads and G - ∞ -operads.

It is worth noting that, much as was the case of the work in [21], our results are not formal consequences of their nonequivariant analogues, due to the nature of norm maps.² Indeed, in [7], the second piece of our project, the authors introduced the notion

²As a point of contrast, we note that the lack of norms in the categorical case causes the equivariant generalization of this latter program to indeed be formal; see Stephan [26] and Bergner [5].

of *genuine equivariant operads*, which are new algebraic objects motivated by the combinatorics of norm maps as encoded by the category Ω_G of G -trees. And while a priori the work in [7] is largely perpendicular to the Cisinski–Moerdijk program (the main result [7, Theorem III] is what one might call the “operadic Elmendorf–Piacenza theorem”, which is an equivariant phenomenon), some of the new technical hurdles in this paper versus [10] come from the need to work with (colored) genuine equivariant operads, which we repackage in Section 3.3 via an independent (but equivalent) perspective to that of [7].

The organization of the paper is as follows.

Section 2 mostly recalls the necessary notions concerning the category Ω_G of G -trees and the category dSet^G of G -dendroidal sets that were introduced in [21]. However, some new notions and results can be found throughout, most notably the notion of *orbital face* of a G -tree in Definition 2.16 and the associated notion of *orbital horn* in Section 2.3.

The main goal of Section 3 is to establish Proposition 3.22, which roughly states that Segal core inclusions, horn inclusions and orbital horn inclusions can in some circumstances be used interchangeably. The bulk of the work takes place in Section 3.1, where Lemma 3.4, a powerful technical result we call the *characteristic edge lemma*, is established. Section 3.2 then shows Proposition 3.22 via a string of easy applications of Lemma 3.4. Lastly, Section 3.3 recasts the genuine equivariant operads of [7] in a different perspective more suitable for our purposes in Section 5.

Section 4 establishes the desired Quillen equivalences between dSet^G , sdSet^G and PreOp^G via largely abstract methods. Our approach is inspired by [10, Theorem 6.6], which observes that the Rezk/complete model structure on sdSet can be built via two distinct localization procedures. In fact, we will prefer to use the common localization perspective to define the equivariant Rezk/complete model structure on sdSet^G (see Definition 4.22 and Remark 4.23), and then “work backward” (see Remark 4.27) to obtain the analogues of the definitions in [10] and of [10, Theorem 6.6]. As such, in Section 4.1 we first discuss an abstract setting for such common localizations, which is then applied in Section 4.2 to obtain the Quillen equivalence $\text{dSet}^G \rightleftarrows \text{sdSet}^G$ in Theorem 4.30. Section 4.3 then uses purely formal techniques to induce the model structure on PreOp^G from the model structure on sdSet^G and to establish the Quillen equivalence $\text{PreOp}^G \rightleftarrows \text{sdSet}^G$ in Theorem 4.41.

In our last main section, [Section 5](#), motivated by the fact that in our desired model structure on simplicial G -operads sOp^G (to be described in a follow-up paper) the weak equivalences are Dwyer–Kan equivalences (ie characterized by fully faithfulness and essential surjectivity), we establish [Theorem 5.48](#), which gives a Dwyer–Kan-type description of the weak equivalences between the fibrant objects in either of sdSet^G or PreOp^G .

[Section 6](#), which is transversal to the rest of the paper, generalizes all our main results by replacing the category Ω_G of G -trees with certain special subcategories $\Omega_{\mathcal{F}} \subseteq \Omega_G$ which (almost exactly) correspond to the *indexing systems* first identified by Blumberg and Hill [\[6\]](#).

Lastly, the [appendix](#) discusses an equivariant variant of the *generalized Reedy categories* of [\[2\]](#) which plays an essential role in [Section 4.2](#) when describing the model structure on sdSet^G . The key to this [appendix](#) is the *Reedy-admissibility* condition in [Definition A.2\(iv\)](#), which is a fairly nonobvious equivariant generalization of one of the generalized Reedy axioms in [\[2\]](#).

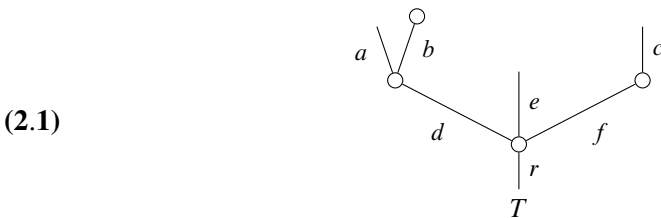
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2 Preliminaries

2.1 The category of trees Ω

We start by recalling the key features of the category Ω of trees that will be used throughout. Our official model for Ω will be Weiss’s algebraic model of *broad posets* as discussed in [\[21, Section 5\]](#), hence we first recall some key notation and terminology. Given a tree diagram T such as



for each edge t of T topped by a vertex \circ , we write t^\uparrow to denote the tuple of edges immediately above t . In our example, $r^\uparrow = def$, $d^\uparrow = ab$, $f^\uparrow = c$ and $b^\uparrow = \epsilon$,

where ϵ is the empty tuple. Edges t for which

- (i) $t^\uparrow \neq \epsilon$, such as r , d and f , are called *nodes*;
- (ii) $t^\uparrow = \epsilon$, such as b , are called *stumps*;
- (iii) t^\uparrow is undefined, such as a , c and e , are called *leaves*.

Each vertex of T is then encoded symbolically as $t^\uparrow \leq t$, which we call a *generating broad relation*. This notation is meant to suggest a form of transitivity: for example, the generating relations $ab \leq d$ and $def \leq r$ generate, via *broad transitivity*, a relation $abef \leq r$ (we note that this is essentially compact notation for the operations and composition in the colored operad $\Omega(T)$ generated by T [19, Section 3; 21, Remark 4.4 and Example 4.6]). The other broad relations obtained by broad transitivity are $aef \leq r$, $dec \leq r$, $abec \leq r$, $aec \leq r$ and $a \leq d$. The set of edges of T together with these broad relations (as well as identity relations $t \leq t$) form the *broad poset* associated to the tree, which is again denoted by T .

Given a broad relation $t_0 \cdots t_n \leq t$, we further write $t_i \leq_d t$. Pictorially, this says that the edge t_i is above t , and it is thus clear that \leq_d defines a partial order on edges of T . Trees always have a single \leq_d -maximal edge, called the *root*. Edges other than the root or the leaves are called *inner edges*. In our example r is the root, a , e and c are leaves, and b , d and f are inner edges.

We denote the sets of edges, inner edges and vertices of T by $E(T)$, $E^i(T)$ and $V(T)$.

The Cisinski–Moerdijk–Weiss category Ω of trees then has as objects the tree diagrams as in (2.1) and as maps $\varphi: T \rightarrow S$ the monotone maps of broad posets (meaning that if $t_1 \cdots t_k \leq t$ then $\varphi(t_1) \cdots \varphi(t_k) \leq \varphi(t)$). In fact, in [27], Weiss characterized those broad posets associated to trees (see³ [21, Definitions 5.1 and 5.9]), so that one is free to work intrinsically with broad posets.

Moreover, our discussion will be somewhat simplified by the assumption that Ω contains exactly one representative of each *planarized* tree. Informally, this means that trees $T \in \Omega$ come with a preferred planar representation, though this can also be formalized in purely algebraic terms; see [7, Section 3.1]. For our purposes, the main consequence is that any map $S \rightarrow T$ in Ω has a (strictly) unique factorization $S \xrightarrow{\cong} S' \rightarrow T$ as an isomorphism followed by a *planar map* [7, Proposition 3.23]. Informally, S' is obtained from S by “pulling back” the planarization of T .

³For clarity, we note that throughout [21] the term *maximum* is used to mean the *greatest element*, ie an element which is the unique maximal element.

We now recall the key classes of maps of Ω . A map $\varphi: S \rightarrow T$ which is injective on edges is called a *face map* while a map that is surjective on edges and preserves leaves is called a *degeneracy map* (the extra requirement ensures that leaves of S do not become stumps of T). Moreover, a face map φ is further called an *inner face map* if $\varphi(r_S) = r_T$ and $\varphi(\underline{l}_S) = \underline{l}_T$ (where $r_{(-)}$ denotes the root edge and $\underline{l}_{(-)}$ the leaf tuple) and called an *outer face map* if it does not admit a factorization as a nonisomorphism inner face map followed by a face map.

The following result is [7, Corollary 3.32], with the additional planar statement following from the unique factorization of maps in Ω as isomorphisms followed by planar maps.

Proposition 2.2 *A map $\varphi: S \rightarrow T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^o} T$$

as a degeneracy followed by an inner face map followed by an outer face map.

Moreover, there is a (strictly) unique factorization with φ^i and φ^o planar.

We next recall an explicit characterization and notation for planar inner/outer faces (planar degeneracies are characterized by edge multiplicities; see [7, Proposition 3.47(ii)]). For any subset $E \subseteq \mathbf{E}^i(T)$, there is a planar inner face $T - E$ which removes the inner edges in E but keeps all broad relations involving edges not in E (this is the hardest class of maps to visualize pictorially, as the vertices adjacent to each $e \in E$ are combined via broad transitivity/composition). For each broad relation $t_1 \cdots t_k = \underline{t} \leq t$ in T , there is a planar outer face $T_{\underline{t} \leq t}$ such that $r_{T_{\underline{t} \leq t}} = t$ and $\underline{l}_{T_{\underline{t} \leq t}} = \underline{t}$ (in fact, by Proposition 2.2 this is the largest such face). Moreover, the edges s of $T_{\underline{t} \leq t}$ are the edges of T such that $s \leq_d t$ and $s \not\leq_d t_i$ for all i while the vertices are the $s^\uparrow \leq s$ such that $s \leq_d t$ and $s \not\leq_d t_i$ for all i (pictorially, $T_{\underline{t} \leq t}$ removes those sections of T not above t and above some t_i).

Remark 2.3 Inner faces $T - E \hookrightarrow T$ are always full, ie $T - E$ contains all broad relations of T between those edges in $\mathbf{E}(T - E) = \mathbf{E}(T) \setminus E$. By contrast, whenever T has stumps, some of its outer faces $T_{\underline{t} \leq t}$ are not full, the main example being given by the maximal outer faces that “remove stumps” [21, Notation 5.41].

Remark 2.4 Following [2, Example 2.8], one has a degree function $|-|: \Omega \rightarrow \mathbb{N}$ given by $|T| = |\mathbf{V}(T)|$ such that nonisomorphism face maps (resp. degeneracies) strictly increase (decrease) $|-|$. As such, the subcategory of face maps is denoted by Ω^+ while that of degeneracies is denoted by Ω^- .

We now collect a couple of useful lemmas concerning faces.

Lemma 2.5 Consider a diagram of planar faces in Ω (implicitly regarded as inclusion maps)

$$\begin{array}{ccc} V & \xleftarrow{\text{out}} & U \\ \text{inn} \downarrow & & \downarrow \\ \bar{V} & \xleftarrow{\text{out}} & \bar{U} \end{array}$$

such that the horizontal maps are outer face maps and the left vertical map is an inner face map.

Then $E^i(V) = E^i(U) \cap E^i(\bar{V})$.

Proof Write r and $\underline{l} = l_1 \cdots l_n$ for the root and leaf tuple of V or, equivalently, of \bar{V} . Since the horizontal maps are outer, an edge $e \in E^i(U)$ (resp. $e \in E^i(\bar{U})$) is also in $E^i(V)$ (resp. in $E^i(\bar{V})$) if and only if $e <_d r$ and $e \not\leq_d l_i$ for all i . But then $E^i(V) = E^i(U) \cap E^i(V) = E^i(U) \cap E^i(\bar{V})$. \square

Lemma 2.6 Let $\{U_i \hookrightarrow T\}$ be a collection of planar outer faces of T with a common root t . Then there are planar outer faces $U^\cup \hookrightarrow T$ and $U^\cap \hookrightarrow T$, also with root t , such that

$$(2.7) \quad \begin{aligned} E(U^\cup) &= \bigcup_i E(U_i), & V(U^\cup) &= \bigcup_i V(U_i), \\ E(U^\cap) &= \bigcap_i E(U_i), & V(U^\cap) &= \bigcap_i V(U_i). \end{aligned}$$

Moreover, these are the smallest (resp. largest) outer faces containing (contained in) all U_i .

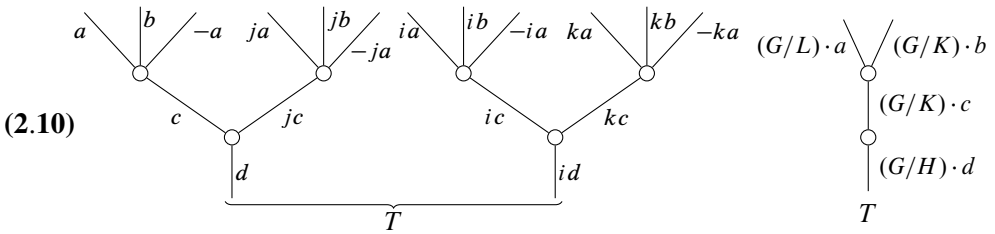
Remark 2.8 More generally, U^\cup and U^\cap can be defined whenever the U_i have a common edge.

Proof Since edges and vertices are simply elements and generating broad relations of the broad poset of a tree, (2.7) generates prebroad posets (see [21, Remark 5.2]) U^\cup and U^\cap . It now suffices to check that U^\cup and U^\cap are trees, ie that they satisfy the axioms in [21, Definitions 5.1, 5.3 and 5.9]. Antisymmetry and simplicity are inherited from T ; the root axiom follows since the U_i have a common root (in the U^\cap case note that if s is in U^\cap , then so is any s' such that $s \leq_d s' \leq_d t$); and the nodal axiom is obvious from (2.7) (which, a posteriori, is correct as an identity on sets of edges and vertex sets). \square

Notation 2.9 We will write $\iota: \Delta \rightarrow \Omega$ for the standard inclusion, which sends $[n]$ to the linear tree with $n + 1$ edges (recall that all notions in this section generalize the usual notions for Δ). Additionally, we will as usual write $\eta = \iota([0])$ for the “stick tree” containing a single edge and no vertices, and note that ι induces an identification between Δ and the overcategory $\Omega \downarrow \eta$.

2.2 The category of G -trees Ω_G

We next recall the category Ω_G of G -trees introduced in [21, Section 5.3]. We start with an explicit and representative example of a G -tree (for more examples, see [21, Section 4.3]). Letting $G = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the group of quaternionic units and $G \geq H \geq K \geq L$ denote the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$ and $L = \{1\}$, there is a G -tree T with *expanded representation* given by the two trees on the left below and *orbital representation* given by the (single) tree on the right:



Note that the edge labels on the expanded representation encode the action of G so that the edges a, b, c and d have isotropy L, K, K and H .

Formally, the definition of Ω_G [21, Definition 5.44] is given as follows. Given a nonequivariant forest diagram F (ie a finite collection of tree diagrams side by side), there is an associated broad poset just as before, and one thus obtains a category Φ of forests and broad monotone maps. Letting Φ^G denote the category of G -objects in Φ , referred to as G -forests, the category $\Omega_G \subset \Phi^G$ of G -trees is defined as the full subcategory of those G -forests such that the G -action is transitive on tree components.

We note that any G -tree T can be written as an induction $T \simeq G \cdot_H T_*$, where T_* is some fixed tree component, $H \leq G$ is the subgroup sending T_* to itself, and we regard $T_* \in \Omega^H$, ie as a tree with an H -action (where we caution that $\Omega^G \subsetneq \Omega_G$). For example, in (2.10) we have $T \simeq G \cdot_H T_d$ for $H \leq G$, $T \in \Omega_G$ as defined therein and $T_d \in \Omega^H$ the tree component containing d .

We similarly assume that G -trees (and forests in general) are planarized, meaning that they come with a total order of the tree components, each of which is planarized.

If $T \in \Omega_G$ has tree components T_1, \dots, T_k , we write $E(T) = \coprod_i E(T_i)$, $E^i(T) = \coprod_i E^i(T_i)$ and $V(T) = \coprod_i V(T_i)$ for its sets of edges, inner edges and vertices, as well as $E_G(T) = E(T)/G$, $E_G^i(T) = E^i(T)/G$ and $V_G(T) = V(T)/G$ for its sets of edge orbits, inner edge orbits and G -vertices.

Before discussing face maps in the equivariant context, it is worth commenting on the complementary roles of the expanded and orbital representations. On the one hand, the G -broad posets associated to G -trees are diagrammatically represented by the expanded representation, so that the arrows of Ω_G are best understood from that perspective. On the other hand, the diagrams encoding compositions of norm maps of an equivariant operad \mathcal{O} are given by the orbital representations of G -trees (see Example 3.37 and Remark 3.39, or alternatively [21, Example 4.9; 7, (1.10)]). As a result, different aspects of our discussion are guided by different representations, and this will require us to discuss the different notions of face/boundary/horn suggested by the two representations. We start by recalling the notion of face discussed in [21], which is motivated by the expanded representation.

Definition 2.11 Let $T \in \Omega_G$ be a G -tree with nonequivariant tree components T_1, T_2, \dots, T_k .

A *face* of T is an underlying face map $U \hookrightarrow T_i$ in Ω for some $1 \leq i \leq k$. Further, we abbreviate faces of T as $U \hookrightarrow T$, and call them *planar/outer* faces whenever so is the map $U \hookrightarrow T_i$.

Notation 2.12 Given $T \in \Omega_G$, we write $\text{Face}(T)$ for the G -poset of *planar faces* $U \hookrightarrow T$. We note that the G -action is given by the unique factorization of the composite $U \hookrightarrow T \xrightarrow{g} T$ as $U \simeq gU \hookrightarrow T$ such that $gU \hookrightarrow T$ is planar,

$$(2.13) \quad \begin{array}{ccc} U & \hookrightarrow & T \\ \simeq \downarrow & & \downarrow g \\ gU & \hookrightarrow & T \end{array}$$

Alternatively, planar faces $U \hookrightarrow T$ can be viewed as subbroad posets $U \subseteq T$, identifying this G -action with the natural action on subsets. However, we prefer the planar face framework since it is more readily related to the presheaves $\Omega[T]$ discussed in the next section (see Remark 2.30).

Notation 2.14 Given $T \in \Omega_G$ and a face $\varphi: U \hookrightarrow T$ we write \bar{U}^T , or just \bar{U} when no confusion should arise, for the planar face in the unique factorization $U \xrightarrow{\varphi^i} \bar{U}^T \xrightarrow{\varphi^o} T$ (see Proposition 2.2) with φ^o planar. We will call \bar{U}^T the *outer closure* of U , due

to it being the smallest planar outer face of T such that $U \hookrightarrow \bar{U}^T$ (to see this, let $V \hookrightarrow T$ be a planar outer face such that $U \hookrightarrow V$; then Proposition 2.2 applied to $U \hookrightarrow \bar{U}^V \hookrightarrow V \hookrightarrow T$ implies it must be $\bar{U}^V = \bar{U}^T$, showing $\bar{U}^T \hookrightarrow V$).

Remark 2.15 Recalling the notation $\Omega^+ \subset \Omega$ for the subcategory of face maps, we write $\Omega^+ \downarrow T$ for the category of all faces of $T \in \Omega_G$. By pulling back the planarization of T one then obtains a *planarization functor*

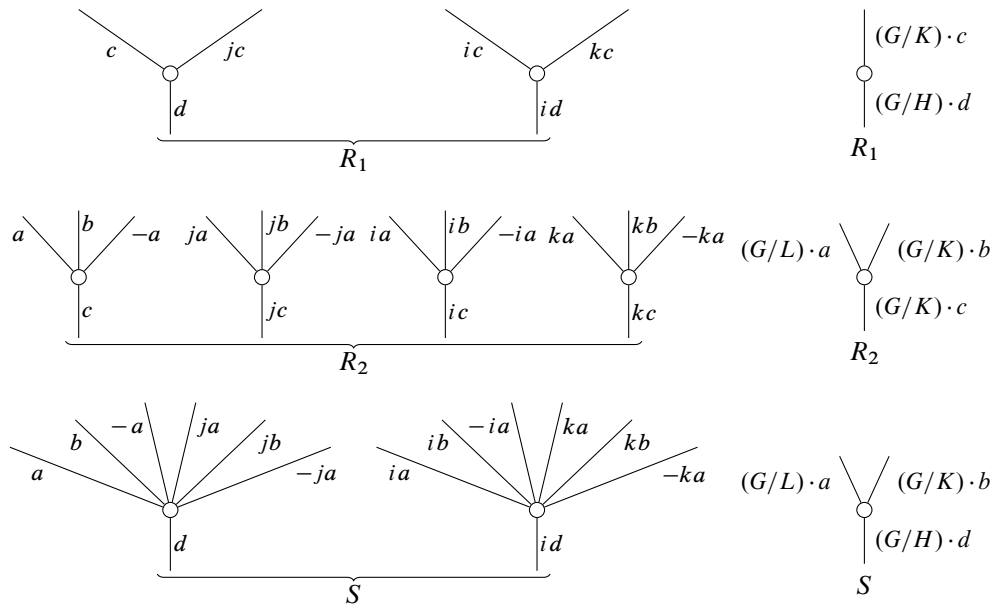
$$\Omega^+ \downarrow T \xrightarrow{\text{pl}} \text{Face}(T)$$

which respects the G -actions on the two categories. Note, however, that the inclusion $\text{Face}(T) \subset \Omega^+ \downarrow T$ (which is a section of pl) does not respect the G -actions, as displayed in (2.13).

We now introduce the notion of face of a G -tree suggested by the orbital representation.

Definition 2.16 Let $T \in \Omega_G$ be a G -tree. An *orbital face* of T is a map $S \hookrightarrow T$ in Ω_G which is injective on edges. Further, an orbital face is called *inner/outer* if any of its component maps is (and thus all are) and *planar* if it is a planar map of forests [7, Definition 3.21].

Example 2.17 The following are three planar orbital faces of the G -tree T in (2.10), with $R_1 \hookrightarrow T$, $R_2 \hookrightarrow T$ orbital outer faces and $S \hookrightarrow T$ an orbital inner face:



These examples illustrate our motivation for the term “orbital face”: the tree diagrams in the orbital representations of R_1 , R_2 and S look like faces of the tree in the orbital representation of T .

Adapting the notation for (nonequivariant) inner faces, we write $S = T - Gc = T - \{c, jc, ic, kc\}$ and analogously throughout the paper. We will need no analogous notation for orbital outer faces.

Notation 2.18 In the remainder of the paper we sometimes need to consider (nonequivariant) faces and orbital faces simultaneously. As such, we reserve the letters U , V and W for trees in Ω and the letters R , S and T for G -trees in Ω_G .

One has the following orbital face analogue of Proposition 2.2:

Proposition 2.19 Any orbital face $\varphi: S \hookrightarrow T$ in Ω_G has a factorization $S \xrightarrow{\varphi^i} R \xrightarrow{\varphi^o} T$, unique up to unique isomorphism, as an orbital inner face followed by an orbital outer face.

Moreover, there is a (strictly) unique factorization with φ^o planar.

Proof It suffices to prove the planar statement. Writing $T = \coprod_{i \in I} T_i$ and $S = \coprod_{j \in J} S_j$ for the tree component decompositions, $\varphi: S \hookrightarrow T$ is described by a map $\varphi: J \rightarrow I$ together with planar face maps $S_j \hookrightarrow T_{\varphi(j)}$. Abbreviating $\bar{S}_j = \bar{S}_j^{T_{\varphi(j)}}$ (see Notation 2.14), the uniqueness in Proposition 2.2 shows that the maps $S_j \hookrightarrow \bar{S}_j \hookrightarrow T_{\varphi(j)}$ must be the components of the desired factorization $S \hookrightarrow R \hookrightarrow T$, so that it remains only to show that $R = \coprod_j \bar{S}_j$ admits a unique compatible G -action and that the natural map $R \rightarrow T$ is injective. To obtain the G -action on R , we again apply Proposition 2.2 to obtain unique dashed arrows as in the diagram

$$\begin{array}{ccccc}
 S_j & \hookrightarrow & \bar{S}_j & \hookrightarrow & T_{\varphi(j)} \\
 \downarrow g & & \downarrow g & & \downarrow g \\
 S_{gj} & \hookrightarrow & \bar{S}_{gj} & \hookrightarrow & T_{g\varphi(j)}
 \end{array}$$

That the G -action is associative and unital — meaning that the composite $\bar{S}_j \xrightarrow{g} \bar{S}_{gj} \xrightarrow{\bar{g}} \bar{S}_{\bar{g}gj}$ equals $\bar{S}_j \xrightarrow{\bar{g}\bar{g}} \bar{S}_{\bar{g}\bar{g}j}$ and that $\bar{S}_j \xrightarrow{e} \bar{S}_{ej}$ is the identity $\bar{S}_j = \bar{S}_{ej}$ — follows from the analogous properties for S and T and the uniqueness in Proposition 2.2. Lastly, the remaining claim that $R \rightarrow T$ is injective follows since each map $\bar{S}_j \hookrightarrow T$ is injective together with the fact that the roots $r_{\bar{S}_j} = r_{S_j}$ are \leq_d -incomparable, so that edges of T in different \bar{S}_j are also \leq_d -incomparable [21, Corollary 5.25]. \square

The argument at the end of the previous proof has the following two consequences:

Remark 2.20 If $U \in \text{Face}(T)$ has isotropy H , the induced map $G \cdot_H U \rightarrow T$ is injective on edges if and only if H is also the isotropy of the root r_U .

Remark 2.21 Orbital inner faces $S \hookrightarrow T$ are full (see Remark 2.3), ie all broad relations in T between edges of S are also in S .

We next discuss the interactions between (nonequivariant) faces and orbital faces.

Proposition 2.22 Let $T \in \Omega_G$. For any (nonequivariant) face $U \hookrightarrow T$ there is a smallest planar orbital face $GU \hookrightarrow T$ such that $U \hookrightarrow GU$.

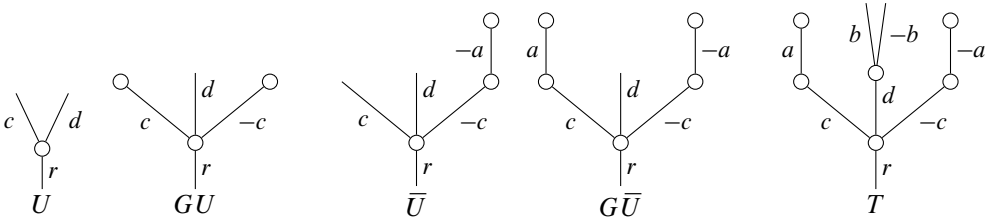
Proof Without loss of generality, we assume U is planar throughout.

Suppose first that $U = \bar{U}^T$ is outer and write $H \leq G$ for the isotropy of its root r_U . By Lemma 2.6 there exists a smallest planar outer face containing all $hU \hookrightarrow T$ for $h \in H$, which we denote by HU . Moreover, HU inherits the H -action from T (by either its construction or its characterization). The natural map $G \cdot_H HU \rightarrow T$ is then injective on edges (Remark 2.20) and we thus let GU be $G \cdot_H HU$ with the planar structure induced from T . The claim that GU is the smallest planar orbital face such that $U \hookrightarrow GU$ is clear.

Before tackling the general case, we collect some key observations. Firstly, if U is outer then so is the (nonequivariant) face HU and the orbital face GU . Secondly, the root tuple of GU is $G \cdot_H r_U$. Lastly, we need to characterize the leaf tuple of GU . We call a leaf l of U *orbital* if all the edges in $Hl \cap E(U)$ are leaves of U , and claim that the leaves of GU are the tuple \underline{l} formed by the G -orbits of the orbital leaves of U . Indeed, a leaf l of U is also a leaf of HU if and only if $l \in E(hU)$ implies that l is a leaf of hU for all $h \in H$ if and only if $h^{-1}l \in E(U)$ implies that l is a leaf of U for all $h \in H$.

In the general case, we define GU as the orbital inner face of $G\bar{U}$ that removes all edge orbits not represented in U (that all such edge orbits are inner follows from the description of the roots and leaves of $G\bar{U}$ in the previous paragraph). For the remaining claim that GU is the smallest planar orbital face with $U \hookrightarrow GU$, let $U \hookrightarrow S$ be any such face, and write $S \hookrightarrow \bar{S} \hookrightarrow T$ for the planar orbital factorization given by Proposition 2.19. Then, by the outer case established before, we have $GU \hookrightarrow G\bar{U} \hookrightarrow \bar{S}$ and, since all edges of GU (which are the orbits of the edges of U) are in S , it follows that $GU \hookrightarrow S \hookrightarrow \bar{S}$ due to the inner face inclusion $S \hookrightarrow \bar{S}$ being full (Remark 2.21). \square

Example 2.23 Much of the complexity in the previous proof is needed to handle the scenario of nonouter faces $U \hookrightarrow T$ of G -trees T which have stumps, which is easily the subtlest case, as illustrated by the following example (where $G = \mathbb{Z}/2 = \{\pm 1\}$):



Remark 2.24 If $U \hookrightarrow T$ is outer, the characterization of the roots and leaves of GU in the previous proof shows that the inner edges of GU are the G -orbits of the inner edges of U .

Remark 2.25 For any inner face $V - e$ of a face $V \hookrightarrow T$ one has that $G(V - e)$ is either $GV - Ge$ or GV . Indeed, the latter holds if and only if $V - e$ contains either an inner edge or a leaf of the form ge .

Remark 2.26 Writing $\text{Face}_o(T)$ for the poset of planar orbital faces, Proposition 2.22 gives a G -equivariant functor (note that G acts trivially on $\text{Face}_o(T)$)

$$\text{Face}(T) \xrightarrow{G(-)} \text{Face}_o(T), \quad U \mapsto GU.$$

Moreover, there is a natural inclusion $\text{Face}_o(T) \subseteq \text{Face}(T)/G$ (sending an orbital face S to the class of components $[S_*]$) whose left adjoint is the induced functor $\text{Face}(T)/G \rightarrow \text{Face}_o(T)$.

Remark 2.27 Following the intuition in Example 2.17, there is an isomorphism of posets

$$\text{Face}_o(T) \xrightarrow{\cong} \text{Face}(T/G), \quad S \mapsto S/G,$$

where T/G , which is formally defined below, can be informally thought of as the underlying tree in the orbital representation of T .

However, we should first caution that though this claim is intuitive, some care is needed. For example, the broad poset of T/G is in general *not* the quotient of the broad poset of T , as that may fail the simplicity axiom in [21, Definition 5.9]. In fact, the assignment $T \mapsto T/G$ is *not* a functor $\Omega_G \rightarrow \Omega$, as shown by the following (for

$G = \mathbb{Z}/2 = \{\pm 1\}$), since no dashed arrow exists:

(2.28)

We now outline the formal construction of T/G , starting with some preliminary notation.

Given \underline{e} and \underline{f} , tuples of edges of T , write $\underline{f} \leq \underline{e}$ if $\underline{e} = e_1 e_2 \cdots e_k$ and there is a tuple decomposition $\underline{f} = \underline{f}_1 \underline{f}_2 \cdots \underline{f}_k$ such that $\underline{f}_i \leq e_i$. When the e_i are \leq_d -incomparable, Proposition 5.30 of [21] says that such decomposition is unique, so that \underline{e} and \underline{f} consist of distinct edges and we can regard \underline{e} and \underline{f} as subsets $\underline{e}, \underline{f} \subseteq \mathbf{E}(T)$.

We now say that a relation $\underline{f} \leq \underline{e}$ is an *orbital relation* if $\underline{e} \subseteq \mathbf{E}(T)$ is a transitive G -subset and $\underline{f} \subseteq \mathbf{E}(T)$ is a G -subset. Reinterpreting the orbital relations of T as broad relations on the set $\mathbf{E}_G(T) = \mathbf{E}(T)/G$ of edge orbits, one readily checks that this defines a dendroidally ordered set [21, Definition 5.9], ie a tree, which we denote by T/G . Note that one hence has a functor $(-)/G: \Omega_G^+ \rightarrow \Omega$, where Ω_G^+ is the subcategory of orbital face maps, and planarizations of the T/G are chosen arbitrarily.

Lastly, we observe that, in analogy to the nonequivariant case, the orbital outer faces of T are indexed by orbital relations.

2.3 Equivariant dendroidal sets

Recall [21, Section 5.4] that the category of G -equivariant dendroidal sets is the presheaf category $\text{dSet}^G = \text{Set}^{\Omega^{\text{op}} \times G}$. Given $T \in \Omega_G$ with nonequivariant tree components T_1, \dots, T_k , we extend the usual notation for representable functors to obtain $\Omega[T] \in \text{dSet}^G$ via

$$\Omega[T] = \Omega[T_1] \amalg \cdots \amalg \Omega[T_k]$$

regarded as a G -object in dSet . One further defines *boundaries* (in the union formula we regard the injections $\Omega[U] \rightarrow \Omega[T]$ as inclusions; the equivalence between the colimit and union formulas follows from Proposition 2.2)

$$\partial\Omega[T] = \text{colim}_{U \in \text{Face}(T), U \neq T_i} \Omega[U] = \bigcup_{U \in \text{Face}(T), U \neq T_i} \Omega[U]$$

and, for $\emptyset \neq E \subseteq E^i(T)$ a nonempty G -subset of inner edges (we abbreviate $E_i = E \cap E^i(T_i)$), G -inner horns

$$\Lambda^E[T] = \operatorname{colim}_{U \in \operatorname{Face}(T), (T_i - E_i) \not\rightarrow U} \Omega[U] = \bigcup_{U \in \operatorname{Face}(T), (T_i - E_i) \not\rightarrow U} \Omega[U],$$

which, informally, are the subcomplexes of $\Omega[T]$ that remove the inner faces $T_i - D$ for $D \subseteq E_i$.

Lastly, letting $\operatorname{Face}_{\text{sc}}(T)$ denote those outer faces of T with no inner edges (these are either single edges t or generated by single vertices $t^\uparrow \leq t$), we define the *Segal core of T* ,

$$\operatorname{Sc}[T] = \operatorname{colim}_{U \in \operatorname{Face}_{\text{sc}}(T)} \Omega[U] = \bigcup_{U \in \operatorname{Face}_{\text{sc}}(T)} \Omega[U].$$

Note that if $T \simeq G \cdot_H T_*$ for some $T_* \in \Omega^H$ then

$$(2.29) \quad \begin{aligned} \Omega[T] &\simeq G \cdot_H \Omega[T_*], & \partial\Omega[T] &\simeq G \cdot_H \partial\Omega[T_*], \\ \Lambda^E[T] &\simeq G \cdot_H \Lambda^{E_*}[T_*], & \operatorname{Sc}[T] &\simeq G \cdot_H \operatorname{Sc}[T_*]. \end{aligned}$$

As a cautionary note, we point out that though representable functors $\Omega[T]$ are defined for $T \in \Omega_G$, evaluations $X(U)$ of $X \in \operatorname{dSet}^G$ are defined only for $U \in \Omega$ (see [Notation 2.18](#)).

Remark 2.30 For $T \in \Omega_G$, a planar face $\varphi_U: U \hookrightarrow T$ can also be regarded as a dendrex $\varphi_U \in \Omega[T](U)$. However, the G -isotropy H of $U \in \operatorname{Face}(T)$ must not be confused with the G -isotropy of φ_U . Instead, $\Omega[T](U)$ has a larger $G \times \operatorname{Aut}(U)^{\text{op}}$ -action, where $\operatorname{Aut}(U)^{\text{op}}$ acts by precomposition, and the $G \times \operatorname{Aut}(U)^{\text{op}}$ -isotropy of φ_U is a subgroup $\Gamma \leq G \times \operatorname{Aut}(U)$ consisting of elements $(h, \phi^{-1}(h))$ where $h \in H$ and $\phi: H \rightarrow \operatorname{Aut}(U)$ is a homomorphism. Indeed, noting that (2.13) implies there is an identity $\Omega[gU] = g\Omega[U]$ as subpresheaves of $\Omega[T]$, it follows that the G -isotropy H of $U \in \operatorname{Face}(T)$ coincides with the G -isotropy of the subpresheaf $\Omega[U] \subseteq \Omega[T]$, so that, by Yoneda, $U \in \Omega$ has a canonical H -action $\phi: H \rightarrow \operatorname{Aut}(U)$ (more explicitly, $\phi(h)$ is the left isomorphism in (2.13)). We abuse notation by writing $U \in \Omega^H \subseteq \Omega_H$ to denote this.

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts.

The saturation of the boundary inclusions $\partial\Omega[T] \rightarrow \Omega[T]$ is the class of G -normal monomorphisms, ie those monomorphisms $X \rightarrow Y$ in dSet^G such that $Y(U) \setminus X(U)$

has an $\text{Aut}(U)$ -free action for all $U \in \Omega$ (nonequivariantly, this is [9, Proposition 1.5], and also holds equivariantly by [21, Remark 6.7]; alternatively, it can be shown using [21, Propositions 6.5(ii) and 5.62]). Moreover, since one can forget the G -action when verifying this condition, we will usually call these simply *normal monomorphisms*.

The saturation of the G -inner horn inclusions $\Lambda^E[T] \rightarrow \Omega[T]$ is called the class of G -inner anodyne maps, while those $X \in \text{dSet}^G$ with the right lifting property against all G -inner horn inclusions are called G - ∞ -operads.

We can now recall the statement of [21, Theorem 2.1], which was the main result therein.

Theorem 2.31 *There is a model structure on dSet^G such that the cofibrations are the normal monomorphisms and the fibrant objects are the G - ∞ -operads.*

Remark 2.32 The definition of G - ∞ -operads just given is a priori distinct from the original definition [21, Definition 6.12], which used only *generating G -inner horn inclusions*, ie those inclusions $\Lambda^{Ge}[T] \rightarrow \Omega[T]$ with $E = Ge$ an inner edge orbit. The definition herein has the technical advantages of being naturally compatible with restricting the G -action and of allowing for a simpler proof of Lemma 3.4, which is our main tool for showing that maps are G -inner anodyne. The equivalence between the two definitions follows from [21, Proposition 6.17], although we will also independently recover this fact from Lemma 3.4 as Corollary 3.20.

In addition to the G -inner horns defined above, we now introduce a new kind of horn that, much like orbital faces, is naturally suggested by the orbital representation of G -trees. Given $\emptyset \neq E \subseteq \mathbf{E}^i(T)$ a G -equivariant set of inner edges, we define the associated *orbital G -inner horn* by

$$\Lambda_o^E[T] = \text{colim}_{S \in \text{Face}_o(T), (T-E) \not\rightarrow S} \Omega[S] = \bigcup_{S \in \text{Face}_o(T), (T-E) \not\rightarrow S} \Omega[S],$$

where we note that the equivalence between the colimit and union formulas now follows from Proposition 2.22.

Remark 2.33 One can strengthen the identification $\text{Face}_o(T) \simeq \text{Face}(T/G)$ in Remark 2.27:

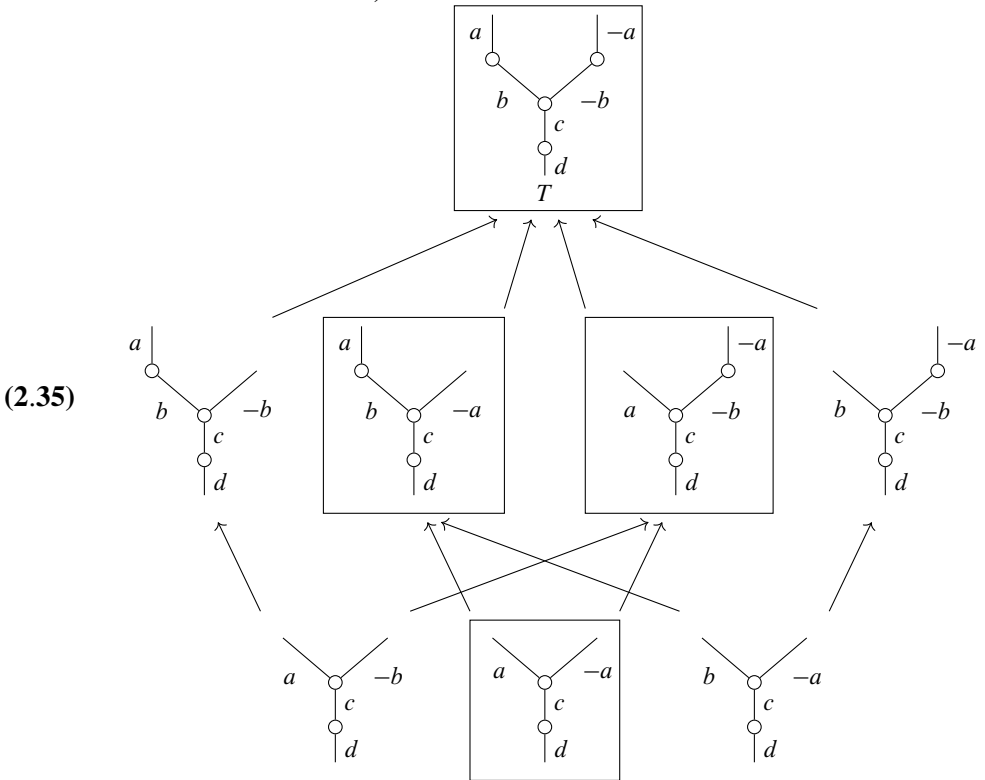
Say a subcomplex $A \subseteq \Omega[T]$ is *orbital* if it is the union of orbital faces $\Omega[S]$ for $S \in \text{Face}_o(T)$. Equivalently, by Proposition 2.22 this means that for $U \in \text{Face}(T)$ one has $\Omega[U] \subseteq A$ if and only if $\Omega[GU] \subseteq A$. There is then a natural bijection of posets

(under inclusion)

$$\left\{ \text{orbital subcomplexes } \bigcup_i \Omega[S_i] \text{ of } \Omega[T] \right\} \leftrightarrow \left\{ \text{subcomplexes } \bigcup_i \Omega[S_i/G] \text{ of } \Omega[T/G] \right\}.$$

In particular, $\Lambda_o^{Ge}[T]$ corresponds to $\Lambda^{[e]}[T/G]$ and $\text{Sc}[T]$ corresponds to $\text{Sc}[T/G]$.

Example 2.34 Let $G = \mathbb{Z}/2 = \{\pm 1\}$, and consider the tree $T \in \Omega^G \subset \Omega_G$ at the top below. The following depicts the poset of planar faces of T *not in* $\Lambda_o^{Gb}[T]$; by contrast, $\Lambda^{Gb}[T]$ lacks only the boxed faces (which are precisely those faces pictured below that are *inner* faces of T):



3 Equivariant inner anodyne maps

Much as in [10, Section 2], we need to show that the inclusions $\text{Sc}[T] \rightarrow \Omega[T]$ for $T \in \Omega_G$ are G -inner anodyne. In addition, parts of the equivariant dendroidal story

are naturally described in terms of orbital G -inner horns $\Lambda_o^E[T]$ (rather than G -inner horns $\Lambda^E[T]$), and one must hence also show that the inclusions $\Lambda_o^E[T] \rightarrow \Omega[T]$ are G -inner anodyne.

In practice, the proofs of such results are long as well as somewhat repetitive, since they share many technical arguments. In fact, dealing with the case of orbital horn inclusions requires using many of the arguments in the long proof of [21, Theorem 7.1].

As such, we split our technical analysis into two parts. In Section 3.1 we prove Lemma 3.4, which we call the *characteristic edge lemma* and which abstractly identifies sufficient conditions for a map to be G -inner anodyne (see Remark 3.8 for a comparison with previous results in the literature). Then, in Section 3.2, we deduce that the desired maps are G -inner anodyne by applying Lemma 3.4, and further establish Proposition 3.22, which, informally, says that Segal core inclusions, G -inner horn inclusions and orbital G -inner horn inclusions can be used interchangeably in some contexts.

Lastly, Section 3.3 briefly discusses *colored genuine equivariant operads* (which in the single color case were first introduced in [7]), which play an important role in Section 5.1.

3.1 The characteristic edge lemma

Definition 3.1 Let $T \in \Omega_G$, $A \subseteq \Omega[T]$ a subdendroidal set, and $\{U_i\}_{i \in I} \subseteq \text{Face}(T)$ a subset.

Given a set Ξ^i of inner edges of U_i and a subface $V \hookrightarrow U_i$, write $\Xi_V^i = \Xi^i \cap E^i(V)$. Suppose further that the indexing set I is a finite G -poset. For each $i \in I$, write

$$A_{<i} = A \cup \bigcup_{j < i} \Omega[U_j]$$

We say that $\{\Xi^i \subseteq E^i(U_i)\}$ is a *characteristic inner edge collection of $\{U_i\}$ with respect to A* if

- (Ch0) A , $\{U_i\}$ and $\{\Xi^i\}$ are all G -equivariant, ie $gA = A$, $gU_i = U_{gi}$ and $g\Xi^i = \Xi^{gi}$ as appropriate;
- (Ch1) for all i , any outer face $V \simeq \bar{V}^{U_i}$ of U_i such that $\Xi_V^i = \emptyset$ is contained in $A_{<i}$;
- (Ch2) for all i , any face $V \hookrightarrow U_i$ such that $(V - \Xi_V^i) \in A$ is contained in $A_{<i}$;
- (Ch3) for all $j \not\prec i$, all faces $V \hookrightarrow U_i$ such that $(V - \Xi_V^i) \hookrightarrow U_j$ are contained in $A_{<i}$.

Remark 3.2 If $gi \neq i$, then i and gi are incomparable in I . Indeed, if $i < gi$ then $i < gi < g^2i < \dots$ would violate antisymmetry, and likewise if $i > gi$. Hence, (Ch3) applies when $j = gi$ for $gi \neq i$.

In particular, we assume throughout that if $gi \neq i$ then $U_{gi} \neq U_i$, or else U_i would be in $A_{<i}$.

Remark 3.3 In some of the main examples (see Propositions 3.13 and 3.16), there exists a G -equivariant set Ξ of inner edges of T such that $\Xi^i = \Xi \cap E^i(U_i)$.

We caution that, for fixed A and $\{U_i\}$, our characteristic conditions are *not* monotone on such Ξ since increasing Ξ makes (Ch1) more permissive while making (Ch2)–(Ch3) more restrictive.

Lemma 3.4 If $\{\Xi^i\}_{i \in I}$ is a characteristic inner edge collection of $\{U_i\}_{i \in I}$ with respect to A , then the map

$$(3.5) \quad A \rightarrow A \cup \bigcup_{i \in I} \Omega[U_i]$$

is G -inner anodyne. In fact, (3.5) is built cellularly⁴ from G -inner horn inclusions $\Lambda^E[S] \rightarrow \Omega[S]$ for $S \in \Omega_G$.

Recall that a subset $S \subseteq \mathcal{P}$ of a poset \mathcal{P} is called a *lower set*⁵ if $s \in S$ and $p < s$ implies $p \in S$.

Proof We start with the case of $I \simeq G/H$ transitive so that, abbreviating $U = U_{[e]}$, $\{U_i\}$ is the set of conjugates gU . We likewise abbreviate $\Xi = \Xi^{[e]}$ and $\Xi_V = \Xi_V^{[e]}$ for $V \hookrightarrow U$. Moreover, in this case one has $A_{<[g]} = A$ in (Ch1)–(Ch3) and that H is also the isotropy of U in $\text{Face}(T)$.

We write $\text{Face}_{\Xi}^{\text{lex}}(U)$ for the H -poset of planar faces $V \hookrightarrow U$ such that $\Xi_V \neq \emptyset$ and $\Xi_V = \Xi_{\bar{V}}$ ordered as follows: $V \leq V'$ if either

- (i) $\bar{V} \hookrightarrow \bar{V}'$ and $\bar{V} \neq \bar{V}'$, or
- (ii) $\bar{V} = \bar{V}'$ and $V \hookrightarrow V'$.

⁴Recall that a map f is built cellularly from a class of maps I if it can be obtained as a (possibly transfinite) composition of pushouts of maps in I (but without retracts). See [17, Definition 2.1.9; 24, Example 12.6.12].

⁵In [21, Sections 7–8] such sets were called *convex*, following the terminology in [12, Section 0].

(Alternatively, this is the lexicographic order of pairs (\bar{V}, V) . We note that here and in the remainder of the proof all outer closures are implicitly taken in U (rather than T), ie $\bar{V} = \bar{V}^U$.

For any H -equivariant lower set C of $\text{Face}_{\Xi}^{\text{lex}}(U)$ we write (for the identity $\Omega[gV] = g\Omega[V]$, see Remark 2.30)

$$A_C = A \cup \bigcup_{g \in G, V \in C} \Omega[gV] = A \cup \bigcup_{g \in G, V \in C} g\Omega[V].$$

It now suffices to show that whenever $C \subseteq C'$ the map $A_C \rightarrow A_{C'}$ is built cellularly from G -inner horn inclusions (indeed, setting $C = \emptyset$ and $C' = \text{Face}_{\Xi}^{\text{lex}}(U)$ recovers (3.5) when $I \simeq G/H$).

Without loss of generality we can assume that C' is obtained from C by adding the H -orbit of a single $W \hookrightarrow U$. Further, we may assume $W \notin A_C$ or else $A_C = A_{C'}$. Letting $K \leq H$ denote the isotropy of W in $\text{Face}_{\Xi}^{\text{lex}}(U)$ and regarding $W \in \Omega^K \subseteq \Omega_K$, we claim there is a pushout diagram

$$(3.6) \quad \begin{array}{ccc} G \cdot_K \Lambda^{\Xi_W}[W] & \longrightarrow & A_C \\ \downarrow \iota & \lrcorner & \downarrow \\ G \cdot_K \Omega[W] & \longrightarrow & A_{C'} \end{array}$$

where we note that the inner edge set Ξ_W is K -equivariant since $\Xi_W = \Xi \cap E^i(W)$ and Ξ is H -equivariant by (Ch0). The pushout (3.6) will follow once we establish the following claims:

- (a) all proper outer faces V of W are in A_C ;
- (b) an inner face $W - D$ of W is in A_C if and only if $D \not\subseteq \Xi_W$;
- (c) the G -isotropy (ie the isotropy in $\text{Face}(T)$) of faces $W - D$ for $D \subseteq \Xi_W$ is contained in K .

To see why these claims suffice, write P for the pushout of (3.6) with $A_{C'}$ removed. Surjectivity of $P \rightarrow A_{C'}$ is immediate from the assumptions on C and C' . For injectivity, since $A_C \rightarrow A_{C'}$ is certainly injective, it suffices to check that the restriction of the bottom horizontal map in (3.6) to the complement of the image of the left vertical map ι

- (i) is injective, and
- (ii) has image that does not overlap with the image of A_C .

Condition (ii) follows from (a)–(b). As for (i), injectivity is automatic when further restricting to the intersection with an individual conjugate $\Omega[gW] = g\Omega[W]$, with (c) ensuring injectivity across different conjugate intersections.

To check (a), writing $\bar{V} = \bar{V}^U$ for the corresponding outer face of U , one has

$$\Xi_V = \Xi \cap E^i(V) = \Xi \cap E^i(W) \cap E^i(\bar{V}) = \Xi \cap E^i(\bar{W}) \cap E^i(\bar{V}) = \Xi \cap E^i(\bar{V}) = \Xi_{\bar{V}},$$

where the second step follows from Lemma 2.5 (applied to $V \hookrightarrow W \hookrightarrow U$ and $V \hookrightarrow \bar{V} \hookrightarrow U$) and the third since by definition of $\text{Face}_{\Xi}^{\text{lex}}(U)$ we have $\Xi_W = \Xi_{\bar{W}}$. Thus either $\Xi_V = \Xi_{\bar{V}} = \emptyset$, so that $\bar{V} \in A$ by (Ch1), or $\Xi_V = \Xi_{\bar{V}} \neq \emptyset$, so that $V \in \text{Face}_{\Xi}^{\text{lex}}(U)$ with $V < W$, and thus $V \in C$. In either case one has $V \in A_C$.

We now check the “if” direction of (b). If $D \not\subseteq \Xi_W$ then $W' = W - (D \setminus \Xi_W)$ is in $\text{Face}_{\Xi}^{\text{lex}}(U)$ (since $\bar{W}' = \bar{W}$ and thus $\Xi_{W'} = \Xi_W = \Xi_{\bar{W}} = \Xi_{\bar{W}'}$) and $W' < W$, and thus $W' \in A_C$.

For the “only if” direction of (b), note first that it suffices to consider $D = \Xi_W$. The assumption $W \notin A_C$ together with (Ch2) imply that $W' = W - \Xi_W$ is not in A , and thus it remains to show that W' is not a face of any gV with $g \in G$ and $V \in C$. Suppose otherwise, ie $W' \hookrightarrow gV$. If it were $g \notin H$, then it would be $W' \hookrightarrow gV \hookrightarrow gU \neq U$, and (Ch3) would imply $W \in A$. Thus we need only consider $g \in H$, and since C is H -equivariant, we can set $g = e$. It now suffices to show that if $W' \hookrightarrow V$ then it must be $W \leq V$ in $\text{Face}_{\Xi}^{\text{lex}}(U)$, as this would then contradict $W \notin C$ due to C being a lower set. Since $W' \hookrightarrow V$ implies $\bar{W} = \bar{W}' \hookrightarrow \bar{V}$, the condition $W \leq V$ is automatic from the definition of \leq unless $\bar{W} = \bar{V}$. In this latter case, by definition of $\text{Face}_{\Xi}^{\text{lex}}(U)$ the face V must contain as inner edges all edges in $\Xi_V = \Xi_{\bar{V}} = \Xi_{\bar{W}} = \Xi_W$, so that not only $W - \Xi_W = W' \hookrightarrow V$ but also $W \hookrightarrow V$. But then we have $W \leq V$ in either case, establishing the desired contradiction.

We now show (c). If $g(W - D) = W - D$ then $g(W - \Xi_W) \hookrightarrow U$, and thus $W - \Xi_W \hookrightarrow g^{-1}U$, so that by (Ch3) it must be $g \in H$ or else it would be $W \in A$. Now suppose $h(W - D) = W - D$ with $h \in H$. Since Ξ is H -equivariant (by (Ch0)) and $\Xi_{W-D} = \Xi_W \setminus D$ (due to $D \subseteq \Xi_W$) it follows that $h(W - \Xi_W) = W - \Xi_W$, so that we may assume $D = \Xi_W$. Now note that $hW, h(W - \Xi_W) = W - \Xi_W, W$ are all faces of U with a common outer closure \bar{W} . Hence $h\Xi_W = \Xi_{hW} \subseteq \Xi_{\bar{W}} = \Xi_W$, where the last step follows since $W \in \text{Face}_{\Xi}^{\text{lex}}(U)$, and for cardinality reasons it must in fact be $h\Xi_W = \Xi_W$. But then hW and W have the same outer closure and the same inner edges, and thus $hW = W$, establishing (c).

Lastly, we address the case of general I . For each G -equivariant lower set J of I , set

$$A_J = A \cup \bigcup_{j \in J} \Omega[U_j].$$

As before, it suffices to check that for all lower sets $J \subseteq J'$ the map $A_J \rightarrow A_{J'}$ is built cellularly from G -inner horn inclusions, and again we can assume that J' is obtained from J by adding a single G -orbit Gj of I . By the I transitive case, it now suffices to check that $\{\Xi^{g_j}\}_{g_j \in G_j}$ is also a characteristic inner edge collection of $\{U_{g_j}\}_{g_j \in G_j}$ with respect to A_J . (Ch0) is clear, and since by G -equivariance and the lower set property we have $A_{<g_j} \subseteq A_J$, the new (Ch1)–(Ch3) conditions follow from the original conditions (for (Ch2), note that if $(V - \Xi_V^i) \in A_J$ but $(V - \Xi_V^i) \notin A$, then the fact that J is a lower set implies $(V - \Xi_V^i) \hookrightarrow U_j$ for some $j \not\leq i$, so that in this case the new (Ch2) follows from the original (Ch3)). \square

Remark 3.7 The requirement $A \subseteq \Omega[T]$ in Definition 3.1 can be relaxed. Consider an inclusion $A \subseteq B$ with $B \in \text{dSet}^G$, a set of nondegenerate dendrices $\{b_i \in B(U_i)\}_{i \in I}$ and a collection of edges $\{\Xi^i \subseteq E^i(U_i)\}_{i \in I}$, with I a finite G -poset. Letting $A_{<i} = A \cup \bigcup_{j < i} b_j(\Omega[U_j])$, suppose then that

(Ch0.1) the maps $b_i: \Omega[U_i] \rightarrow B$ are injective away from $b_i^{-1}(A_{<i})$;

(Ch0.2) A , $\{U_i\}$, $\{b_i\}$ and $\{\Xi^i\}$ are all G -equivariant in the sense that

- (i) $gA = A$;
- (ii) there are isomorphisms $U_i \xrightarrow{g} U_{gi}$ such that the composite $U_i \xrightarrow{g} U_{gi} \xrightarrow{\bar{g}} U_{\bar{g}gi}$ is $U_i \xrightarrow{\bar{g}g} U_{\bar{g}gi}$ and $U_i \xrightarrow{e} U_{ei}$ is the identity $U_i = U_{ei}$;
- (iii) the composites $\Omega[U_i] \xrightarrow{b_i} B \xrightarrow{g} B$ and $\Omega[U_i] \xrightarrow{g} \Omega[U_{gi}] \xrightarrow{b_{gi}} B$ coincide;
- (iv) $g(\Xi^i) = \Xi^{gi}$.

The original conditions (Ch1)–(Ch3) can now be reinterpreted by, for each face $\varphi_V: V \hookrightarrow U_i$, replacing expressions such as “ V contained in $A_{<i}$ ” and $(V - \Xi_V^i) \hookrightarrow U_j$ by $b_i(\varphi_V) \in A_{<i}$ and $b_i(\varphi_{V - \Xi_V^i}) \in b_j(\Omega[U_j])$.

The proof of Lemma 3.4 now carries through mostly unchanged to show that the inclusion

$$A \rightarrow A \cup \bigcup_{i \in I} b_i(\Omega[U_i])$$

is G -inner anodyne (again built cellularly from G -inner horn inclusions). Indeed, writing $A_C = A \cup \bigcup_{g \in G, V \in C} gb_{[e]}(\Omega[V])$, the obvious analogues of (a)–(b) in the proof show that one can form the analogous diagram (3.6) and that $A_{C'} \setminus A_C$ is generated

by the dendrices $gb_i(\varphi_{W-D})$ for $D \subseteq \Xi_W$. That (3.6) is indeed a pushout then follows from (Ch0.1), which ensures that all dendrices attached by each conjugate inclusion $g(\Lambda^{\Xi_W}[W] \rightarrow \Omega[W])$ are distinct, together with the analogue of (c), which ensures that all such conjugate inclusions attach different dendrices. As a technical note, precisely reformulating (c) requires accounting for the isotropy issue discussed in Remark 2.30, and is thus slightly cumbersome. However, just as in the first line of the proof of (c), it is immediate from (Ch3) that $\Lambda^{\Xi_W}[W] \rightarrow \Omega[W]$ and $g(\Lambda^{\Xi_W}[W] \rightarrow \Omega[W])$ could only possibly attach common dendrices if $g \in H$, so that the bulk of the argument in the proof of (c) concerns the H -isotropy in $\text{Face}(W)$, and thus carries through with no noteworthy changes.

Remark 3.8 Lemma 3.4 readily recovers several arguments in the literature:

(i) In [22, Section 10] (also [23, Section 6.2]), Rezk introduces the notion of *covers*, which in our language are the simplicial subsets $\text{Sc}[n] \subseteq A \subseteq \Delta[n]$ such that if $V \hookrightarrow [n]$ is in A then so is the outer closure $\bar{V}^{[n]}$ (in words, A is generated by outer faces). Similarly, in the proof of [10, Propostion 2.4] Cisinski and Moerdijk use subcomplexes that can be regarded as *dendroidal covers*, ie subcomplexes $\text{Sc}[T] \subseteq A \subseteq \Omega[T]$ such that if V is in A then so is \bar{V}^T (note that this definition extends unchanged to the equivariant context). Lastly, the subcomplexes $\Omega[T] \cup_I \Omega[S] \subseteq \Omega[T \circ_I S]$ in the grafting result [20, Lemma 5.2], and likewise for the equivariant analogue [21, Proposition 6.19], are also dendroidal covers.

Lemma 3.4 implies that any inclusion $A \rightarrow A'$ of G -equivariant (dendroidal) covers for $T \in \Omega_G$ is G -inner anodyne. Indeed, let $I = \text{Face}_{A'}^{\text{out}}(T)$ be the G -poset of planar outer faces $V \hookrightarrow T$ contained in A' , ordered by inclusion, $\Xi = \mathbf{E}^i(T)$ and $U_V = V$. (Ch0) is clear, (Ch1) follows since $\text{Sc}(T) \subseteq A$, (Ch2) follows since A is a cover and (Ch3) follows since the U_i are outer.

Alternatively, one can also use $I = \text{Face}_{A',o}^{\text{out}}(T)$ for the G -trivial set of orbital outer faces $GV \hookrightarrow T$, together with an *arbitrary* total order (see Remark 3.17 for a similar example).

Lastly, note that in the special case $\{U_i\} = \{T\}$ and $\Xi = \mathbf{E}^i(T)$, (Ch1) says precisely $\text{Sc}[T] \subseteq A$.

(ii) In [20, Lemma 9.7], Moerdijk and Weiss introduced a *characteristic edge* condition that can be regarded as a special case of our characteristic edge collection condition as generalized in Remark 3.7, and which served as one of our main inspirations.

Therein, they work in the case of $B = \Omega[T] \otimes \Omega[S]$ a tensor product of (nonequivariant) representable dendroidal sets, in which case (Ch0.1) is easily verified (and (Ch0.2) is automatic). In our notation, they then require that $I \simeq *$ (so that (Ch3) is also automatic), the dendrex $b_* \in (\Omega[T] \otimes \Omega[S])(U_*)$ encodes a special type of subtree U_* of $\Omega[T] \otimes \Omega[S]$, which they call an *initial segment*, and they further require that $\Xi^* = \{\xi\}$ is a singleton, called the *characteristic edge*. Moreover, they then demand that A should contain all outer faces of the subtree U_* , from which (Ch1) follows, as well as the key characteristic condition [20, Lemma 9.7(ii)], which coincides with (Ch2) in this specific setting.

Similarly, in [21, Lemma 7.39] the second author introduced a *characteristic edge orbit* condition that generalizes that in [20] to the equivariant context by letting $I \simeq G/H$ and the $\Xi^{[g]} = \Xi \cap E^i(U_{[g]})$ be determined by a G -edge orbit $\Xi \simeq G\xi$ (see Remark 3.3).

However, both of the lemmas in [20] and [21] have the drawback of needing to be used iteratively (so that much effort therein is spent showing that this can be done) while Lemma 3.4 is designed so that a single use suffices for the natural applications. Indeed, conditions (Ch1) and (Ch3), the first of which relaxes the requirement in [20; 21] that A should contain all outer faces of the U_i , essentially provide abstract conditions under which the original characteristic edge arguments of [20; 21] can be iterated.

Example 3.9 As indicated above, Lemma 3.4 can be used to reorganize and streamline the rather long proofs of [21, Theorems 7.1 and 7.2]. We illustrate this in the hardest case, that of [21, Theorem 7.1(i)], which states that if $S, T \in \Omega_G$ are open (ie have no stumps) and $G\xi$ is an inner edge orbit of T , the maps

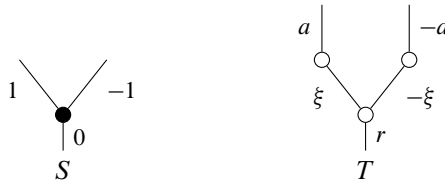
$$(3.10) \quad \partial\Omega[S] \otimes \Omega[T] \amalg_{\partial\Omega[S] \otimes \Lambda^{G\xi}[T]} \Omega[S] \otimes \Lambda^{G\xi}[T] \rightarrow \Omega[S] \otimes \Omega[T]$$

are G -inner anodyne.

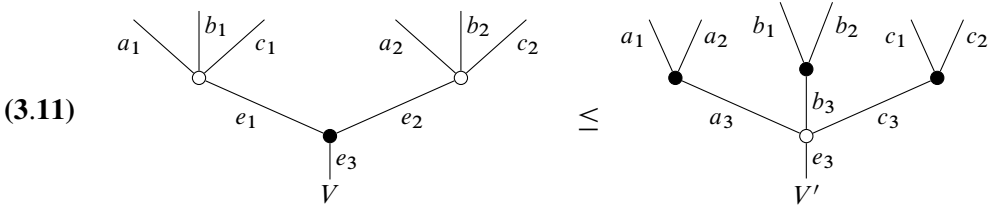
Given $S, T \in \Omega_G$, it is possible [21, Section 7.1] to define a G -equivariant broad poset $S \otimes T$ so that $(\Omega[S] \otimes \Omega[T])(V) = \text{hom}(V, S \otimes T)$ where the hom set is taken in broad posets. Intuitively, $S \otimes T$ is an object with edge set $E(S) \times E(T)$ and where each edge (s, t) of $S \otimes T$ may, depending on whether $s \in S$ and $t \in T$ are leaves or not, admit two *distinct* vertices: an S -vertex $(s, t)^{\uparrow S} = s^{\uparrow} \times t \leq (s, t)$ and a T -vertex $(s, t)^{\uparrow T} = s \times t^{\uparrow} \leq (s, t)$.

To recover [21, Theorem 7.1(i)] from Lemma 3.4, we first let $I = \text{Max}(S \otimes T)$ be the G -poset of maximal subtrees $U \hookrightarrow S \otimes T$ (these are called *percolation schemes* in [20, Section 9]), ordered lexicographically [21, Definition 7.29]. As an example, let

$G = \mathbb{Z}/_2 = \{\pm 1\}$ and consider the $\mathbb{Z}/_2$ -trees



We depict the $\mathbb{Z}/_2$ -poset $\text{Max}(S \otimes T)$ in Figure 1 (note that (s, t) is abbreviated as t_s). In words, the maximal subtrees are built by starting with the “double root” r_0 and iteratively choosing between the available S and T vertices (along all upward paths) until the “double leaves” $a_1, a_{-1}, -a_1$ and $-a_{-1}$ are reached. The generating relations $U \leq U'$ in a generic $\text{Max}(S \otimes T)$ occur whenever U contains an outer face V shaped as on the left below and, by “replacing” V with V' as on the right, one obtains U' :



The claim that \leq is indeed a partial order (at least if one of S and T is open) is [21, Proposition 7.31]. As an aside, we note that V and V' above have a common inner face $V - \{e_1, e_2\} = V' - \{a_3, b_3, c_3\}$, which encodes a (universal!) example of a Boardman–Vogt relation (see [19, Section 5.1]).

Returning to the task of proving that (3.10) is G -inner anodyne, we define Ξ^U for each maximal subtree $U \hookrightarrow S \otimes T$ to be the set of inner edges of U of the form $(g\xi)_s$ such that the vertex $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$ in U is a T -vertex (see Figure 1). We now verify (Ch1)–(Ch3). We recall that, since S and T are assumed open, Lemma 7.19 of [21] guarantees that, for faces $S' \hookrightarrow S$ and $T' \hookrightarrow T$, a factorization $V \hookrightarrow S' \otimes T' \hookrightarrow S \otimes T$ exists if and only if the edges of V are in $E(S') \times E(T')$.

For (Ch1), note that there is an equivariant grafting decomposition $T = T_{\not\leq G\xi} \amalg_{G\xi} T^{\leq G\xi}$, where $T_{\not\leq G\xi}$ contains the edges $t \in T$ such that $t \not\leq g\xi$ for all $g \in G$ (pictorially, this is a lower equivariant outer face of T) while $T^{\leq G\xi}$ contains the edges $t \in T$ such that $t \leq g\xi$ for some $g \in G$ (an upper equivariant outer face of T). But one now readily checks that if $V \hookrightarrow U$ is an outer face such that $\Xi_V^U = \emptyset$, then either $V \hookrightarrow S \otimes T_{\not\leq G\xi}$ or $V \hookrightarrow S \otimes T^{\leq G\xi}$, and thus $V \in A$.

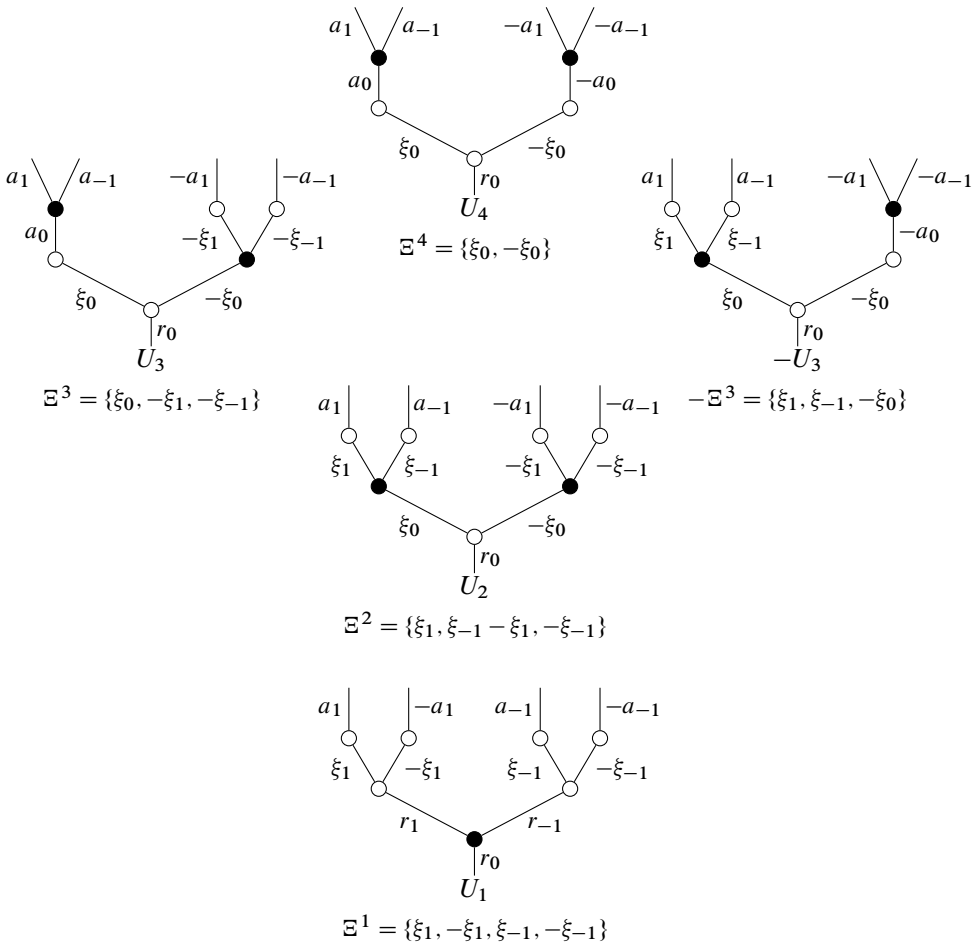


Figure 1: The $\mathbb{Z}/2$ -poset $\text{Max}(S \otimes T)$ and characteristic edges Ξ^i .

For (Ch3), suppose $U_j \not\leq U_i$, $V \hookrightarrow U_i$ and $(V - \Xi_V^{U_i}) \hookrightarrow U_j$. Then it follows from [21, Lemma 7.37] that there exists a generating relation $U_k < U_i$ such that $(V - \Xi_V^{U_i}) \hookrightarrow U_k$ (indeed, Lemma 7.37 of [21] makes the claim that such a relation can be performed on an “elementary subtree” [21, Definition 7.16] or, in our notation, on the outer closure \bar{V}^{U_i} ; and, via grafting, it is straightforward to extend such a $<$ relation for \bar{V}^{U_i} to one for U_i [21, Remark 7.30]). But then, as one sees from (3.11), all edges e of U_i that are not in U_k are topped by the S -vertex $e^{\uparrow S} \leq e$, and thus we have $e \notin \Xi^{U_i}$. Therefore $V \hookrightarrow U_k$, as desired.

Lastly, for (Ch2), suppose $V \hookrightarrow U$ and $(V - \Xi_V^U) \in A$. If it were $(V - \Xi_V^U) \hookrightarrow S \otimes \Lambda^{G\xi}[T]$, then it would also be $V \hookrightarrow S \otimes \Lambda^{G\xi}[T]$ since all edges of Ξ_V^U have

T -coordinate in $G\xi$. Now consider the more interesting case $(V - \Xi_V^U) \hookrightarrow S' \otimes T$ for some face $S' \hookrightarrow S$. Then it will also be $V \hookrightarrow S' \otimes T$ unless there is at least one edge $(g\xi)_s \in \Xi_V^U$ such that $s \notin S'$. But then, since the outer closure \bar{V}^U can have no leaf with S -coordinate s (this would contradict $s \notin S'$), by Lemma 2.6 there exists some minimal outer face $U_{(g\xi)_s}^{<s}$ of U with root $(g\xi)_s$ and such that its leaves have S -coordinate $<_d s$. By minimality, one has that $U_{(g\xi)_s}^{<s} \hookrightarrow \bar{V}^U$ and that all inner edges of $U_{(g\xi)_s}^{<s}$ have S -coordinate s . Further, note that $U_{(g\xi)_s}^{<s}$ has at least one inner edge (since by definition of Ξ^U the vertex $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$ is a T -vertex) and that V contains none of those inner edges (or else it would be $s \in S'$). Thus, by applying [21, Lemma 7.34] to $U_{(g\xi)_s}^{<s}$, one obtains a maximal subtree $U' < U$ containing all edges of U that are not inner edges of $U_{(g\xi)_s}^{<s}$. But then $V \hookrightarrow U'$ and (Ch2) follows.

Remark 3.12 We briefly outline how to modify the example above to prove [21, Theorem 7.1(ii)], in which case some notable subtleties arise. The result again states that (3.10) is G -inner anodyne, but now with one of S and T allowed to have stumps while the other is required to be linear, ie of the form $G/H \cdot [n]$.

One again sets $I = \text{Max}(S \otimes T)$, with maximal trees defined just as before, but some caution is needed. To see why, note that if the black nodes \bullet in (3.11) are replaced with stumps then V' becomes a subtree of V , so that not all maximal trees are maximal with regard to inclusion.

When S has stumps and T is linear this causes no issues and the proof above holds (notably, it can now be $\Xi^U = \emptyset$, in which case (Ch1) demands $U \in A$, as indeed follows from the argument).

However, when S is linear and T has stumps the proof above breaks down (more specifically, the tree $U_{(g\xi)_s}^{<s}$ that appears when arguing (Ch2) may now fail to have inner edges). The solution is then to reverse the poset structure on $\text{Max}(S \otimes T)$ and to modify the Ξ^U to be those inner edges $(g\xi)_s$ such that $(g\xi)_s \in t_s^{\uparrow T}$ for some t_s (pictorially, this says that these are the lowermost edges with T -coordinate in $G\xi$, whereas before they were the uppermost ones). The arguments for (Ch1)–(Ch3) then hold. For (Ch2), only the argument for the interesting case of $V - \Xi_V^U \hookrightarrow S' \otimes T$ with $s \notin S'$ changes. In this case, there is then a \leq_d -largest edge t'_s such that $(g\xi)_s <_d t'_s$, where s cannot be the root of S (or else it would be $s \in S'$). Pictorially, t'_s looks like the edge $e_1 \in V$ in (3.11) in the case where the \bullet node is unary (since S is assumed linear). But then, since V cannot contain t'_s , there exists a maximal subtree $U' > U$ such that $V \hookrightarrow U'$, and (Ch2) follows.

Lastly, we note that [21, Theorem 7.2] follows from a minor variant of the argument for [21, Theorem 7.1(ii)] when S is linear.

3.2 Segal core, horn and orbital horn inclusions

Proposition 3.13 *Let $T \in \Omega_G$. For G -subsets $\emptyset \neq F \subseteq E \subseteq E^i(T)$ the inclusions*

$$(3.14) \quad \Lambda_o^E[T] \rightarrow \Omega[T], \quad \Lambda_o^E[T] \rightarrow \Lambda_o^F[T]$$

are G -inner anodyne.

Proof We are free to assume that $T \in \Omega^G \subseteq \Omega_G$. Indeed, otherwise, writing $T = G \cdot_H T_*$, where $T_* \in \Omega^H$ is a fixed component, $E_* = E \cap E^i(T_*)$ and $F_* = F \cap E^i(T_*)$, the maps in (3.14) are $G \cdot_H (\Lambda_o^{E_*}[T_*] \rightarrow \Omega[T_*])$ and $G \cdot_H (\Lambda_o^{E_*}[T_*] \rightarrow \Lambda_o^{F_*}[T_*])$.

In the $\Lambda_o^E[T] \rightarrow \Omega[T]$ case we apply Lemma 3.4 with $I = \{*\}$ a singleton and

$$\Xi^* = E, \quad U_* = T, \quad A = \Lambda_o^E[T].$$

It remains to check the characteristic conditions in Definition 3.1. (Ch0) and (Ch3) are clear.

Note that for $V \hookrightarrow T$ we have $V \notin A$ if and only if $GV = T - E'$ for some G -subset $E' \subseteq E$.

For (Ch1), the condition $\Xi_V = \emptyset$ says that none of the inner edges of V are in E and thus, by Remark 2.24, that the orbital outer face GV contains none of the edge orbits in E as inner edge orbits. Since $E \neq \emptyset$, the orbital outer face GV is not T itself, and hence $A = \Lambda_o^E[T]$ contains V .

For (Ch2), note that if $V \notin A$, ie $GV = T - E'$ for $E' \subseteq E$, then Remark 2.25 implies that $G(V - \Xi_V) = T - E''$ for $E' \subseteq E'' \subseteq E$, and thus also $(V - \Xi_V) \notin A$.

In the $\Lambda_o^E[T] \rightarrow \Lambda_o^F[T]$ case we instead apply Lemma 3.4 with $I = (E \setminus F)/G$, with an arbitrary choice of total order, and (writing elements of $(E \setminus F)/G$ as orbits $Ge \subseteq E \setminus F$)

$$\Xi^{Ge} = F, \quad U_{Ge} = T - Ge, \quad A = \Lambda_o^E[T].$$

Note that the U_{Ge} are the orbital inner faces $T - Ge$ for $Ge \subseteq E \setminus F$, and thus the map (3.5) in Lemma 3.4 is indeed $\Lambda_o^E[T] \rightarrow \Lambda_o^F[T]$. Further, we are free to abbreviate $\Xi = \Xi^{Ge}$ and $\Xi_V = \Xi_V^{Ge}$, since Ξ^{Ge} is independent of Ge . We again check the characteristic conditions. (Ch0) is clear.

For (Ch1), note that for an outer face $V \hookrightarrow U_i$, and writing $\bar{V} = \bar{V}^T$, Lemma 2.5 implies $E^i(V) = E^i(U_i) \cap E^i(\bar{V})$ and hence, since $\Xi_{U_i} = F = \Xi$, the hypothesis $\Xi_V = \emptyset$ in (Ch1) implies that also $\Xi_{\bar{V}} = \emptyset$. Hence, just as before, $G\bar{V}$ is an orbital outer face other than T , hence V is in $A = \Lambda_o^E[T]$. The argument for (Ch2) is identical to the one in the $\Lambda_o^E[T] \rightarrow \Omega[T]$ case. Lastly, (Ch3) follows since if $V \notin A$, so that $GV = T - E'$ and $G(V - \Xi_V) = T - E' - F'$ with $E' \subseteq E$, $F' \subseteq F$, then $GV \hookrightarrow T - Ge$ if and only if $G(V - \Xi_V) \hookrightarrow T - Ge$ (by fullness of $T - Ge$; see Remark 2.21) and thus $V \hookrightarrow T - Ge$ if and only if $V - \Xi_V \hookrightarrow T - Ge$. \square

Example 3.15 Continuing Example 2.34, we consider the inclusion $\Lambda_o^{Gb}[T] \rightarrow \Omega[T]$. Denoting the leftmost tree in the middle row of (2.35) by S , the intersection of the G -poset in (2.35) with the G -poset $\text{Face}_{\Xi}^{\text{lex}}(U) = \text{Face}_{Gb}^{\text{lex}}(T)$ specified by the proofs of Proposition 3.13 and Lemma 3.4 is the G -poset $S \rightarrow T \leftarrow -S$. The argument in those proofs then shows that $\Lambda_o^{Gb}[T] \rightarrow \Omega[T]$ is built cellularly by attaching $G \cdot (\Lambda^b[S] \rightarrow \Omega[S])$ followed by $\Lambda^{Gb}[T] \rightarrow \Omega[T]$.

Proposition 3.16 Let $T \in \Omega_G$. For G -subsets $\emptyset \neq F \subseteq E \subseteq E^i(T)$ the inclusions

$$\Lambda^E[T] \rightarrow \Lambda^F[T]$$

are G -inner anodyne.

Proof Arguing just as at the beginning of Proposition 3.13, we may assume $T \in \Omega^G \subseteq \Omega_G$.

We now apply Lemma 3.4 with $I = \mathcal{P}_0(E \setminus F)$ the poset of nonempty subsets $\emptyset \neq E' \subseteq (E \setminus F)$, ordered by reverse inclusion, and

$$\Xi^{E'} = F, \quad U_{E'} = T - E', \quad A = \Lambda^E[T].$$

We again need to verify the characteristic conditions, and as in the previous result we abbreviate $\Xi = \Xi^{E'}$ and $\Xi_V = \Xi_V^{E'}$. (Ch0) is clear. (Ch1) follows as in the previous proof, except noting that \bar{V} (rather than $G\bar{V}$) is not T . (Ch2) follows since $V \in A$ if and only if $V - \Xi_V \in A$. Similarly, (Ch3) follows since $V \hookrightarrow T - E'$ if and only if $(V - \Xi_V) \hookrightarrow T - E'$ and since, if $V \hookrightarrow T - E'$ and $V \hookrightarrow T - E''$, then $V \hookrightarrow T - (E' \cup E'')$. \square

Remark 3.17 By specifying to the nonequivariant case $G = *$, the previous results yield two distinct proofs that inclusions of nonequivariant horns $\Lambda^E[T] \rightarrow \Lambda^F[T]$ are

inner anodyne, with the first proof using $I = E \setminus F$ (with an arbitrary total order) and the second using $I = \mathcal{P}_0(E \setminus F)$.

The discrepancy is explained as follows: when T , E and F are G -equivariant, showing that $\Lambda^E[T] \rightarrow \Lambda^F[T]$ is G -inner anodyne requires a control of isotropies not needed when showing that the underlying map is nonequivariant inner anodyne, and since this control is given by (Ch3), it is necessary to include in the $\{U_i\}$ the “intersections” of $T - e$ and $T - ge$ for $e \in E \setminus F$.

The following repeats Remark 3.8(i), but the alternative proof will simplify the proof of Proposition 3.22.

Proposition 3.18 *Let $T \in \Omega_G$. The inclusions $\text{Sc}[T] \rightarrow \Omega[T]$ are G -inner anodyne.*

Proof We may yet again assume $T \in \Omega^G \subseteq \Omega_G$. Now apply Lemma 3.4 with $I = *$, $U_* = T$ and $\Xi^* = E^i(T)$. The conditions (Ch0)–(Ch3) follow as in Remark 3.8(i). \square

Remark 3.19 All G -inner horn inclusions attached in the proof of the characteristic edge lemma, Lemma 3.4, correspond to G -trees whose nonequivariant components are faces of the U_i . Moreover, when $I \simeq G/H$ has a transitive G -action, the last horn inclusion attached (corresponding to the maximum of $\text{Face}_{\Xi}^{\text{lex}}(U)$) is $G \cdot_H (\Lambda^{\Xi}[U] \rightarrow \Omega[U])$, while all other horn inclusions used are of the form $G \cdot_K (\Lambda^{\Xi^w}[W] \rightarrow \Omega[W])$ for some W with $|W| < |U|$.

Corollary 3.20 *G -inner horn inclusions $\Lambda^E[T] \rightarrow \Omega[T]$ are built cellularly from generating horn inclusions $\Lambda^{Ge}[S] \rightarrow \Omega[S]$.*

Proof The proof is by induction on $|T_*|$ for $T_* \in \Omega$ a tree component (see Remark 2.4). As in the proofs of Propositions 3.13 and 3.16 one is free to assume $T \in \Omega^G$. A choice of edge orbit Ge in E yields a factorization $\Lambda^E[T] \rightarrow \Lambda^{Ge}[T] \rightarrow \Omega[T]$, hence we need only show that $\Lambda^E[T] \rightarrow \Lambda^{Ge}[T]$ is built cellularly from generating horns. But this is immediate from the induction hypothesis, Remark 3.19 and the proof of Proposition 3.16, since all U_i therein satisfy $|U_i| < |T|$. \square

Following the discussion preceding [14, Proposition 3.6.8], a class of normal monomorphisms of dSet^G (or, more generally, a subclass of the cofibrations in a model category) is called *hypersaturated* if it is saturated (ie it is closed under pushouts, transfinite

composition and retracts) and satisfies the following additional cancellation property: if f and g are normal monomorphisms (or, more generally, cofibrations)

$$(3.21) \quad A \xrightarrow{f} B \xrightarrow{g} C$$

such that both f and gf are in the class, then so is g .

The following is an equivariant generalization of [10, Propositions 2.4 and 2.5]:

Proposition 3.22 *The following sets of maps generate the same hypersaturated class:*

- the generating G -inner horn inclusions $\Lambda^{Ge}[T] \rightarrow \Omega[T]$ for $T \in \Omega_G$ and $e \in E^i(T)$;
- the G -inner horn inclusions $\Lambda^E[T] \rightarrow \Omega[T]$ for $T \in \Omega_G$ and G -subset $\emptyset \neq E \subseteq E^i(T)$;
- the orbital G -inner horn inclusions $\Lambda^E_o[T] \rightarrow \Omega[T]$ for $T \in \Omega_G$ and G -subset $\emptyset \neq E \subseteq E^i(T)$;
- the G -Segal core inclusions $\text{Sc}[T] \rightarrow \Omega[T]$ for $T \in \Omega_G$.

In the following proof we refer to the hypersaturation of the orbital horn (resp. Segal core) inclusions as the orbital (resp. Segal) hypersaturation.

Proof That the first two hypersaturations coincide is clear from Corollary 3.20.

The fact that G -inner horn inclusions generate the maps has been established in Proposition 3.13 and Remark 3.8(i) for the orbital case and Proposition 3.18 for the Segal case.

To see that the G -inner horn inclusions are in the orbital hypersaturation, we again argue by induction on $|T_*|$, with the base cases those where $\Lambda^E[T] = \Lambda^E_o[T]$. Recalling that in the proof of Proposition 3.13 one sets $I = *$, $U_* = T$ and $\Xi^* = E$, Remark 3.19 implies that in the factorization $\Lambda^E_o[T] \rightarrow \Lambda^E[T] \rightarrow \Omega[T]$ the first map $\Lambda^E_o[T] \rightarrow \Lambda^E[T]$ is built cellularly out of G -horns with $|S_*| < |T_*|$. But then the induction hypothesis says that $\Lambda^E_o[T] \rightarrow \Lambda^E[T]$ is in the orbital hypersaturation, and by the cancellation property (3.21) so is $\Lambda^E[T] \rightarrow \Omega[T]$.

For the claim that the G -inner horn inclusions are in the Segal hypersaturation, note that the proof of Proposition 3.18 sets $I = *$, $U_* = T$ and $\Xi^* = E^i(T)$. Therefore, arguing exactly as above for the factorization $\text{Sc}[T] \rightarrow \Lambda^{E^i(T)}[T] \rightarrow \Omega[T]$, one

obtains by induction on $|T_*|$ that $\Lambda^{E^i(T)}[T] \rightarrow \Omega[T]$ is in the Segal hypersaturation. But now, letting $\emptyset \neq E \subseteq E^i(T)$ be any G -subset and considering the factorization $\Lambda^{E^i(T)}[T] \rightarrow \Lambda^E[T] \rightarrow \Omega[T]$, the induction hypothesis applies to the cells of $\Lambda^{E^i(T)}[T] \rightarrow \Lambda^E[T]$ (just as in [Corollary 3.20](#)), which is thus also in the Segal hypersaturation. But, by the cancellation property, so is $\Lambda^E[T] \rightarrow \Omega[T]$, finishing the proof. \square

Remark 3.23 The identification between orbital subcomplexes $\bigcup_i \Omega[S_i] \subseteq \Omega[T]$ and subcomplexes of $\bigcup_i \Omega[S_i/G] \subseteq \Omega[T/G]$ described in [Remark 2.33](#) is compatible with attaching horn inclusions. As such, nonequivariant results concerning horns in Ω imply the analogous results for orbital horns in Ω_G . For example, mimicking [Lemma 5.1](#) of [\[20\]](#), one has pushouts

$$(3.24) \quad \begin{array}{ccc} \Lambda_o^{E-Ge}[T - Ge] & \longrightarrow & \Lambda_o^E[T] \\ \downarrow & \lrcorner & \downarrow \\ \Omega[T - Ge] & \longrightarrow & \Lambda_o^{E-Ge}[T] \end{array}$$

which imply the orbital horn analogue of [Corollary 3.20](#). It is worth noting that while setting $G = *$ in [Corollary 3.20](#) does recover [\[20, Lemma 5.1\]](#), the analogue of the pushouts (3.24) does not hold for (nonorbital) G -inner horns, so that the proof of [Corollary 3.20](#) (see also the original proof in [\[21, Proposition 6.17\]](#)) is intrinsically harder when $G \neq *$, due to isotropy concerns (this is closely related to the two proofs discussed in [Remark 3.17](#)).

Similarly, [Propositions 2.4 and 2.5](#) of [\[10\]](#) (or our [Remark 3.8\(i\)](#) and [Proposition 3.22](#) when $G = *$) imply that the Segal core inclusions $\text{Sc}[T] \rightarrow \Omega[T]$ are built cellularly from the orbital horn inclusions $\Lambda_o^E[S] \rightarrow \Omega[S]$, and that the two classes have the same hypersaturation.

This last observation indicates an alternative route for proving [Proposition 3.22](#) (which the authors considered in early versions of this work) without making direct use of the characteristic edge lemma machinery. Namely, following the considerations above, the main missing claim is the first part of [Proposition 3.13](#), stating that the inclusions $\Lambda_o^E[T] \rightarrow \Omega[T]$ are G -inner anodyne, and this latter claim is not too hard to prove directly. Indeed, while the proof does require some of the ideas in the proof of [Lemma 3.4](#), many of the subtler arguments in that proof become trivial when $I = *$ is a singleton, as is the case in [Proposition 3.13](#).

Remark 3.25 Since for each $T \in \Omega_G$ the cellular decomposition of $\partial\Omega[T]$ is built by attaching boundary inclusions $\partial\Omega[S] \rightarrow \Omega[S]$ with $|S_*| < |T_*|$, it readily follows that the sets

$$\{\partial\Omega[T] \rightarrow \Omega[T] : T \in \Omega_G\}, \quad \{\emptyset \rightarrow \Omega[T] : T \in \Omega_G\}$$

have the same hypersaturation. Similarly, the following sets also have the same hypersaturation:

$$\{\partial\Omega[T] \rightarrow \Omega[T] : T \in \Omega_G, T \neq G/H \cdot \eta\}, \quad \left\{ \coprod_{e \in \mathbf{E}(T)} \Omega[\eta] \rightarrow \Omega[T] : T \in \Omega_G \right\}.$$

We end this section with some necessary remarks about hypersaturations of *simplicial horns*.

Remark 3.26 Setting $G = e$ and restricting to the overcategory $\text{dSet}/_\eta \simeq \text{sSet}$ (where as usual we abbreviate $\Omega[\eta]$ as just η ; see [Notation 2.9](#)), [Proposition 3.22](#) recovers the well-known claim that the hypersaturation of the simplicial inner horn inclusions $\{\Lambda^i[m] \rightarrow \Delta[m] : 0 < i < m\}$ coincides with the hypersaturation of the simplicial Segal core inclusions $\{\text{Sc}[m] \rightarrow \Delta[m] : m \geq 0\}$.

Remark 3.27 We will use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns $\{\Lambda^i[m] \rightarrow \Delta[m] : 0 \leq i \leq m, 0 < m\}$ matches the hypersaturation of all vertex inclusion maps $\{\Delta[0] \rightarrow \Delta[m]\}$.

Call the latter hypersaturation S . One easily checks that the maps $\{0\} \rightarrow \text{Sc}[m]$ are in S , so that by cancellation so are the maps $\text{Sc}[m] \rightarrow \Delta[m]$ and hence by [Remark 3.26](#) so are all inner horn inclusions. Moreover, for left horns $\Lambda^0[m]$ the maps $\{0\} \rightarrow \Lambda^0[m]$ are built cellularly from left horn inclusions $\Lambda^0[k] \rightarrow \Delta[k]$ with $k < m$ (in join notation — see [\[18, Section 1.2.8\]](#) or [\[21, Section 7.4\]](#) — $\{0\} \rightarrow \Lambda^0[m]$ is $\Delta[0] \star (\emptyset \rightarrow \partial\Delta[m-1])$), and the filtration follows from the cellular filtration of $\partial\Delta[m-1]$. But hence by induction and the cancellation property all left horn inclusions $\Lambda^0[m] \rightarrow \Delta[m]$ are in S . The case of right horn inclusions $\Lambda^m[m] \rightarrow \Delta[m]$ is dual.

Remark 3.28 The smallest hypersaturated class containing the inner horn inclusions and the left horn inclusion $\Lambda^0[2] \rightarrow \Delta[2]$ in fact contains all left horn inclusions $\Lambda^0[m] \rightarrow \Delta[m]$ for $m \geq 2$. Indeed, this follows inductively from the left diagram below since the bottom map is inner while the top and left maps are given by the center and

right pushout diagrams:

$$\begin{array}{ccccc}
 \Lambda^{0,1}[m] & \longrightarrow & \Lambda^0[m] & & \Lambda^0[m-1] & \longrightarrow & \Lambda^{0,1}[m] & & \Lambda^0[m-1] & \longrightarrow & \Lambda^{0,1}[m] \\
 \downarrow & & \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow & \lrcorner & \downarrow \\
 \Lambda^1[m] & \longrightarrow & \Delta[m] & & \Delta[m-1] & \xrightarrow{d^1} & \Lambda^0[m] & & \Delta[m-1] & \xrightarrow{d^0} & \Lambda^1[m]
 \end{array}$$

The case of right horn inclusions is dual.

Remark 3.29 Write $[\widetilde{m}] = (0 \rightrightarrows 1 \rightrightarrows \dots \rightrightarrows m)$ for the contractible groupoid on objects $0, 1, \dots, m$. The k -simplices of $[\widetilde{m}]$ are encoded as strings $a_0 a_1 \dots a_k$ with $a_i \in \{0, 1, \dots, m\}$, and one has that a simplex is nondegenerate if and only if $a_{i-1} \neq a_i$ for $1 \leq i \leq k$. We claim that the maps

$$(3.30) \quad \Delta[m] = N[m] \xrightarrow{012\dots m} N[\widetilde{m}], \quad m \geq 1,$$

are built cellularly out of left horn inclusions $\Lambda^0[k] \rightarrow \Delta[k]$ with $k \geq 2$.

Indeed, we show a little more. We say a subcomplex $A \subseteq N[\widetilde{m}]$ is *0-stable* if a k -simplex \underline{a} is in A if and only if the $(k+1)$ -simplex $0\underline{a}$ is. We claim that any inclusion $A \rightarrow A'$ of 0-stable subcomplexes is built cellularly from left horn inclusions $\Lambda^0[k] \rightarrow \Delta[k]$ with $k \geq 1$. Indeed, it suffices to check this when A' attaches as little as possible to A , and 0-stability guarantees that in that case the only two nondegenerate simplices in $A \setminus A'$ have the form \underline{a} and $0\underline{a}$ (note that \underline{a} cannot start with a 0). But then $A \rightarrow A'$ is a pushout of $\Lambda^0[k+1] \rightarrow \Delta[k+1]$, where k is the dimension of \underline{a} .

The desired claim follows by noting that both the domain and codomain of (3.30) are 0-stable and that the inclusions $\Lambda^0[1] \rightarrow \Delta[1]$ are unneeded since (3.30) is an isomorphism on 0-simplices.

3.3 Genuine equivariant operads

Recall that categories can be identified with their nerves, since the *nerve functor* $N: \text{Cat} \rightarrow \text{sSet}$, given by $NC(n) = \text{Cat}([n], \mathcal{C})$, is fully faithful. Moreover, the essential image of the nerve is characterized as those simplicial sets with the strict right lifting property against the inner horn inclusions [18, Proposition 1.1.2.2] (here *strict* means that the usual lifts are unique).

More generally, one has a similar operadic story. Any tree $U \in \Omega$ has a naturally associated colored operad $\Omega(U) \in \text{Op}$ [19, Section 3], and [20, Proposition 5.3 and

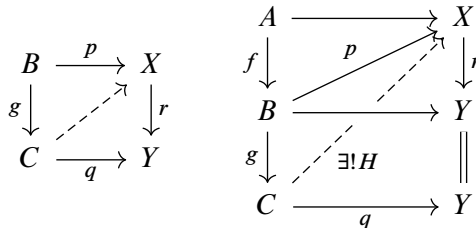
Theorem 6.1] show that the operadic nerve $N: \text{Op} \rightarrow \text{dSet}$, given by $NO(U) = \text{Op}(\Omega(U), \mathcal{O})$, is again fully faithful with essential image the dendroidal sets with the strict right lifting property against dendroidal inner horn inclusions. Moreover, Corollary 2.6 of [10] provides an alternative characterization via strict lifts against Segal core inclusions. The equivalence between these two characterizations is an observation concerning the notion of hypersaturation discussed in the previous section, as follows.

In the next result, note that we need not assume that the maps in (3.21) are cofibrations. More precisely, we slightly modify the notion of ‘‘hypersaturation’’ by using the cancellation property ‘‘if f and gf are in the class, then so is g ’’ without further requirements on f and g . Alternatively, this means that normal monomorphisms/cofibrations are replaced with the class of all maps.

Proposition 3.31 *If two classes \mathcal{C} and \mathcal{D} of maps in a category have the same hypersaturation, then the two classes of maps with the strict right lifting property against \mathcal{C} and \mathcal{D} coincide.*

Proof It suffices to check that the hypersaturation closure conditions are compatible with strict right lifting properties. The claims concerning pushouts, transfinite compositions and retracts follow from the easy observation that the proofs of the analogous claims for the usual right lifting property [24, Lemma 11.1.4] are compatible with the uniqueness requirement.

We thus address only the cancellation property (3.21). Suppose that r has the strict right lifting property against f and gf , and consider a lifting problem as on the left below:



By assumption, there is a unique lift H for the outer square on the right, and we claim that H is also the unique lift for the left square. Noting that $pf = Hgf$ and $rp = qg = rHg$, it follows that both p and Hg are lifts for the top square in the right diagram, so that by the uniqueness assumption we have $p = Hg$. This shows that H is also in fact a lift for the left square. Uniqueness follows since any lift of the left square induces a lift of the outer right square. □

Roughly speaking, our goal in this section is that of describing those presheaves with the strict right lifting property against any of the classes of maps in Proposition 3.22, which we call genuine equivariant operads. However, some care is needed. Namely, it is essential to work with the category $\text{dSet}_G = \text{Set}^{\Omega_G^{\text{op}}}$ of *genuine G -dendroidal sets* rather than with the category $\text{dSet}^G = \text{Set}^{\Omega^{\text{op}} \times G}$ of G -dendroidal sets, ie it is essential to work with presheaves that are evaluated on G -trees $T \in \Omega_G$ rather than nonequivariant trees $U \in \Omega$ (the motivation for this is given in Remark 3.42 below). To relate these presheaf categories, note that the fully faithful inclusion $\nu: \Omega \times G^{\text{op}} \rightarrow \Omega_G$ given by $U \mapsto G \cdot U$ induces an adjunction (note that ν_* is itself fully faithful, being a right Kan extension along a fully faithful functor [24, Corollary 1.4.5])

$$(3.32) \quad \nu^*: \text{dSet}_G \rightleftarrows \text{dSet}^G : \nu_*$$

Explicitly, one has $\nu_* X(T) \simeq X(T_*)^H \simeq \text{dSet}^G(\Omega[T], X)$, where $T \simeq G \cdot_H T_*$ for $T_* \in \Omega^H$, and the H -action on $X(T_*)$ is defined diagonally, ie by the composites $X(T_*) \xrightarrow{X(h^{-1})} X(T_*) \xrightarrow{h} X(T_*)$.

Remark 3.33 The functor ν_* does not preserve arbitrary colimits, due to the presence of fixed points in its formula. Nonetheless, ν_* does preserve pushouts where one leg is a monomorphism.

Remark 3.34 Mimicking the notation in Section 2.3, we write $\Omega_G[-]: \Omega_G \rightarrow \text{dSet}_G$ for the Yoneda embedding. On the other hand, in Section 2.3 we extended the notation $\Omega[-]$ to obtain a functor $\Omega[-]: \Omega_G \rightarrow \text{dSet}^G$. These two “representable functors” are related by $\Omega_G[T] \simeq \nu_* \Omega[T]$.

The following definition is then the main purpose of this section:

Definition 3.35 $Z \in \text{dSet}_G$ is called a *genuine equivariant operad* if Z has the strict right lifting property against the images under ν_* of the Segal core inclusions, ie against the maps

$$(3.36) \quad \nu_*(\text{Sc}[T] \rightarrow \Omega[T]), \quad T \in \Omega_G.$$

Equivalently, by Propositions 3.22 and 3.31, one may replace Segal core inclusions with either orbital G -inner horn inclusions or G -inner horn inclusions.

Example 3.37 To illustrate the role of the strict lifting condition against the maps in (3.36), consider the G -tree T in (2.10), along with the subgroup $K = \langle -1 \rangle$ therein and the orbital faces R_1, R_2 and S in Example 2.17. The strict lifting condition then

says that the left map in

$$(3.38) \quad Z(R_1) \times_{Z(G/K \cdot \eta)} Z(R_2) \xleftarrow{\cong} Z(T) \rightarrow Z(S)$$

is an isomorphism, so T induces a *composition map* $Z(R_1) \times_{Z(G/K \cdot \eta)} Z(R_2) \rightarrow Z(S)$. Here we note that R_1, R_2 and S are G -corollas (ie G -trees with a single G -vertex). Informally, one then thinks of the $Z(C)$, where C ranges over the G -corollas, as the mapping sets of the genuine equivariant operad Z , so that the strict lifting conditions equip these mapping sets with associative and unital composition maps.

We caution, however, that this is not quite the whole story, since the composition maps need also be compatible with the presheaf structure, which is more complex in the equivariant context. More explicitly, nonequivariantly one needs only compatibility with the symmetric group actions, reflecting the fact that all (nondegenerate) maps between corollas are symmetry isomorphisms. But in the equivariant context G -corollas are also related via *quotient maps* (such as the map in (2.28)), which induce subtler compatibility conditions. Nonetheless, our intended application in Section 5 will not require an explicit discussion of these additional compatibilities.

Remark 3.39 Consider a single colored G -operad \mathcal{O} on the category of sets (ie an operad with a G -action commuting with all structure) and a finite H -set A for some subgroup $H \leq G$. Write $\Gamma_A \leq G \times \Sigma_{|A|}$ for the graph of the homomorphism $H \rightarrow \Sigma_{|A|}$ encoding A . We then abbreviate $\mathcal{O}(A)^H = \mathcal{O}(|A|)^{\Gamma_A}$, and call this the *set of A -norm maps of \mathcal{O}* . This name comes from the fact that, for each \mathcal{O} -algebra R , $\mathcal{O}(A)^H$ indexes operations $N^A R \rightarrow R$, where the A -norm object $N^A R$ denotes $R^{\times |A|}$ together with the twisted H -action given by the graph subgroup $\Gamma_A \leq G \times \Sigma_{|A|}$.

Letting $T, R_1, R_2, S \in \Omega_G$ and $H, K, L \leq G$ again be as in (2.10) and Example 2.17, the diagram of hom sets (recall that $\Omega(T)$ denotes the colored operad generated by T [19, Section 3; 21, Remark 4.4 and Example 4.6])

$$(3.40) \quad \text{Op}^G(\Omega(R_1), \mathcal{O}) \times \text{Op}^G(\Omega(R_2), \mathcal{O}) \xleftarrow{\cong} \text{Op}^G(\Omega(T), \mathcal{O}) \rightarrow \text{Op}^G(\Omega(S), \mathcal{O})$$

can be interpreted, after unpacking notation (see [21, Section 4.3]), as a composition of norm maps

$$(3.41) \quad \mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K \rightarrow \mathcal{O}(H/L \amalg H/K)^H$$

The diagrams (3.38) for genuine operads Z can then be regarded as abstracting the diagrams (3.40) for G -operads \mathcal{O} , though with two key differences. The more obvious difference is the fact that (3.40) features no analogue of the $Z(G/K \cdot \eta)$ term, though

this is simply since we chose \mathcal{O} to be single-colored. The subtler, and more crucial, difference is the fact that the terms in (3.38) need not be described by fixed-point sets as in (3.41).

Therefore, one can regard genuine equivariant operads as objects that mimic the composition combinatorics of the norm maps in a (regular) equivariant operad, while relaxing the fixed-point conditions. In fact, the reader of [7] may recognize this as the informal description of genuine equivariant operads given in the introduction to that work, though our current formal setting is rather different. The connection between the two settings is as follows. There is a nerve functor

$$N_G: \text{Op}_G \rightarrow \text{dSet}_G, \quad N_G \mathcal{P}(T) = \text{Op}_G(\Omega_G(T), \mathcal{P}),$$

where Op_G denotes a colored generalization of the genuine equivariant operads of [7] and $\Omega_G(T)$ denotes $\Omega(T)$ upon the standard inclusion $\nu_*: \text{Op}^G \rightarrow \text{Op}_G$. Moreover, N_G is fully faithful and its essential image are the genuine equivariant operads in the sense of Definition 3.35. However, we do not presently require these facts, and thus delay their proof to a sequel [7, Appendix B].

We end this section by explaining why it is that genuine G -dendroidal sets dSet_G , rather than G -dendroidal sets dSet^G , must be used in Definition 3.35.

Remark 3.42 Suppose $X \in \text{dSet}^G$ has the strict right lifting property against all Segal core inclusions $\text{Sc}[T] \rightarrow \Omega[T]$ for $T \in \Omega_G$. By specifying to the cases of $T \simeq G \cdot T_*$ the free G -trees, (2.29) implies that, after forgetting the G -action so as to regard X as an object in dSet , X has the strict lifting property against the inclusions $\text{Sc}[T_*] \rightarrow \Omega[T_*]$ for $T_* \in \Omega$. But the strict lifting properties with respect to all other G -trees $T \in \Omega_G$ are now automatic. Indeed, writing $T \simeq G \cdot_H T_*$ for some $T_* \in \Omega^H$, one has that, by (2.29), G -equivariant lifts against $G \cdot_H (\text{Sc}[T_*] \rightarrow \Omega[T_*])$ are the same as H -equivariant lifts against $\text{Sc}[T_*] \rightarrow \Omega[T_*]$. Consider now the following diagram, where ϕ is an H -equivariant map and Φ is the unique nonequivariant lift:

$$\begin{array}{ccccc} \text{Sc}[T_*] & \xrightarrow{h} & \text{Sc}[T_*] & \xrightarrow{\phi} & X & \xrightarrow{h^{-1}} & X \\ \downarrow & & \downarrow & \nearrow \text{dashed } \Phi & & & \\ \Omega[T_*] & \xrightarrow{h} & \Omega[T_*] & & & & \end{array}$$

Then $h^{-1}\Phi h$ is a lift for the composite lifting problem, but since $h^{-1}\phi h = \phi$, that composite lifting problem in fact coincides with the middle lifting problem, so that

strictness implies that also $h^{-1}\Phi h = \Phi$. In other words, Φ is in fact also the unique H -equivariant lift.

In summary, we have shown that if we had instead used dSet^G in Definition 3.35, then nonfree G -trees would be superfluous, so that by [20, Theorem 6.1] the $X \in \text{dSet}^G$ with such a lifting property would be simply the nerves of G -operads. To see why this is an unsatisfactory situation we recall a fundamental basic example. The category Top^G of G -spaces admits two main equivariant notions of weak equivalence: the fine/genuine equivalences, which care about all fixed-point spaces, and the coarse/naïve equivalences, which care only about the total spaces. However, this distinction vanishes when working in the discrete setting of G -sets Set^G , unless one instead works with G -coefficient systems $\text{Set}^{\text{Op}G}$ (recall that $\text{Op}G$ is the orbit category formed by the G -sets G/H for $H \leq G$ and G -equivariant maps between them). Similarly, the category sOp^G of G -simplicial operads admits two natural notions of weak equivalence, one which cares about the spaces of norm maps for all H -sets A and one which cares only about the spaces of norm maps for trivial H -sets (which are simply the usual multiplication). However, this distinction vanishes when working in the discrete setting of G -dendroidal sets dSet^G , unless one works instead with genuine G -dendroidal sets dSet_G .

4 Quillen equivalences

Our main goal in this section is to prove Theorems 4.30 and 4.41, which jointly establish the Quillen equivalence of three model categories: the category of equivariant dendroidal sets dSet^G with the “ G - ∞ -operad” model structure of [21, Theorem 2.1]; the category of equivariant dendroidal spaces sdSet^G with the “complete equivariant dendroidal Segal space” model structure in Section 4.2; and the category of equivariant preoperads PreOp^G with the “equivariant Segal operad” model structure in Section 4.3.

Our perspective will be that these Quillen equivalences are best understood in light of the observation from [10, Theorem 6.6], which says that the complete dendroidal Segal space model structure on $\text{sdSet} = \text{dSet}^{\Delta^{\text{op}}}$ can be obtained via two distinct left Bousfield localization procedures. As such, we will find it helpful to first discuss an abstract framework for such “joint left Bousfield localizations”, and then *define* the model structure on sdSet^G within that framework (Definition 4.22), with the analogues of the original definitions in [10] recovered a posteriori (Remark 4.27).

4.1 Joint left Bousfield localizations

Throughout we assume familiarity with the theory of left Bousfield localizations as in [16].

Proposition 4.1 *Suppose a category \mathcal{C} admits two model structures $\mathcal{C}_1 = (C, W_1, F_1)$ and $\mathcal{C}_2 = (C, W_2, F_2)$ with a common class of cofibrations C , and assume further that both model structures are cofibrantly generated and admit left Bousfield localizations with respect to any set of maps.*

Then \mathcal{C}_1 and \mathcal{C}_2 have a smallest joint left Bousfield localization $\mathcal{C}_{1,2} = (C, W, F)$ and

- (i) *$c \in \mathcal{C}$ is $\mathcal{C}_{1,2}$ -fibrant if and only if it is simultaneously \mathcal{C}_1 -fibrant and \mathcal{C}_2 -fibrant;*
- (ii) *for $\mathcal{C}_{1,2}$ -fibrant $c, d \in \mathcal{C}$ one has that $c \rightarrow d$ is in W if and only if it is in W_1 if and only if it is in W_2 .*

Proof The joint localized model structure $\mathcal{C}_{1,2}$ can be obtained by either left Bousfield localizing \mathcal{C}_1 with regard to the generating trivial cofibrations of \mathcal{C}_2 or vice versa. Indeed, denoting the first localization in the previous sentence by $L_2\mathcal{C}_1$ and the vice versa localization by $L_1\mathcal{C}_2$, the identity functors $\mathcal{C}_1 \rightarrow L_1\mathcal{C}_2$ and $\mathcal{C}_2 \rightarrow L_2\mathcal{C}_1$ are left Quillen and thus, by the universal property of left Bousfield localizations [16, Proposition 3.3.18] (cofibrant approximations [16, Definition 8.1.2] cause no issue since \mathcal{C}_1 and \mathcal{C}_2 have the same cofibrations), so are the identity functors $L_2\mathcal{C}_1 \rightarrow L_1\mathcal{C}_2$ and $L_1\mathcal{C}_2 \rightarrow L_2\mathcal{C}_1$. This implies that $L_2\mathcal{C}_1$ and $L_1\mathcal{C}_2$ indeed coincide.

For (i), the claim that joint fibrant objects are fibrant in both of the original model structures follows since $C \cap W$ contains both $C \cap W_1$ and $C \cap W_2$ (in fact, this shows that $F \subseteq F_1 \cap F_2$). The converse claim follows from the observation that fibrant objects in any model structure are already local with respect to the weak equivalences in that same model structure.

Lastly, (ii) follows from the local Whitehead theorem [16, Theorem 3.2.13], stating that the local equivalences between local objects match the initial weak equivalences. \square

The prototypical example of Proposition 4.1 is given by the category $\text{ssSet} \simeq \text{Set}^{\Delta^{\text{op}} \times \Delta^{\text{op}}}$ of bisimplicial sets together with the two possible Reedy structures over the Kan model structure on sSet . Explicitly, writing the levels of $X \in \text{ssSet}$ as $X_n(m)$, one can either form a Reedy model structure with respect to the *horizontal index* m or with respect to the *vertical index* n .

In either case, the generating cofibrations are then given by the maps

$$(\partial\Delta[n] \rightarrow \Delta[n]) \square (\partial\Delta[m] \rightarrow \Delta[m]), \quad n, m \geq 0,$$

where \square denotes the pushout product (see for example [24, Construction 11.1.7]).

Further, in the horizontal Reedy model structure the generating trivial cofibrations are the maps

$$(4.2) \quad (\Lambda^i[n] \rightarrow \Delta[n]) \square (\partial\Delta[m] \rightarrow \Delta[m]), \quad n \geq 1, n \geq i \geq 0, m \geq 0,$$

while for the vertical Reedy model structure the generating trivial cofibrations are the maps

$$(4.3) \quad (\partial\Delta[n] \rightarrow \Delta[n]) \square (\Lambda^j[m] \rightarrow \Delta[m]), \quad n \geq 0, m \geq 1, m \geq j \geq 0.$$

We caution the reader about a possible hiccup with the terminology: the weak equivalences for the horizontal Reedy structure are the *vertical equivalences*, ie maps inducing Kan equivalences of simplicial sets $X_\bullet(m) \rightarrow Y_\bullet(m)$ for each $m \geq 0$, and dually for the vertical Reedy structure.

Notation 4.4 Given a fixed $X \in \text{ssSet}$ we will also write $X_{(-)}: \text{sSet}^{\text{op}} \rightarrow \text{sSet}$ for the unique limit-preserving functor such that $X_{\Delta[n]} = X_n$. Explicitly, $X_K(m) = \text{sSet}(K, X(m))$.

Note that for maps $K \rightarrow L$ and $A \rightarrow B$ in sSet , X has the right lifting property against $(K \rightarrow L) \square (A \rightarrow B)$ if and only if $X_L \rightarrow X_K$ has the right lifting property against $A \rightarrow B$ (see Remark 4.13).

In the next result we refer to the localized model structure given by Proposition 4.1 as the *joint Reedy model structure* and we write $\delta^*: \text{ssSet} \rightarrow \text{sSet}$ for the diagonal functor.

Proposition 4.5 *Suppose that $X, Y \in \text{ssSet}$ are horizontal Reedy fibrant. Then:*

- (i) *for each fixed n all vertex maps $X_n \rightarrow X_0$ are trivial Kan fibrations in sSet ;*
- (ii) *any vertical Reedy fibrant replacement \tilde{X} of X is fibrant in the joint Reedy model structure;*
- (iii) *a map $X \rightarrow Y$ is a joint weak equivalence if and only if it is a horizontal weak equivalence if and only if $X_0 \rightarrow Y_0$ is a Kan equivalence in sSet ;*
- (iv) *the canonical map $X_0 \rightarrow \delta^*(X)$ (with levels $X_0(n) \rightarrow X_n(n)$ induced by degeneracies) is a Kan equivalence in sSet .*

Proof (i) follows since the trivial cofibrations for the horizontal Reedy structure include all the maps of the form $(\Delta[0] \rightarrow \Delta[n]) \square (\partial\Delta[m] \rightarrow \Delta[m])$.

For (ii), the fact that \tilde{X} is vertical fibrant implies that for any monomorphism $K \rightarrow L$ in \mathbf{sSet} the induced map $\tilde{X}_L \rightarrow \tilde{X}_K$ is a Kan fibration. Since $X \rightarrow \tilde{X}$ is a horizontal equivalence, (i) implies that all vertex maps $\tilde{X}_n \rightarrow \tilde{X}_0$ are trivial Kan fibrations, so that by [Remark 3.27](#) one has that $\tilde{X}_L \rightarrow \tilde{X}_K$ is a trivial Kan fibration whenever $K \rightarrow L$ is anodyne (since the $K \rightarrow L$ with this property are hypersaturated; see [Remark 4.14](#) for a similar argument). Therefore, \tilde{X} also has the right lifting property against (4.2). That is, \tilde{X} is also horizontal fibrant, as desired.

The first “if and only if” in (iii) follows from (ii) since the localizing maps $X \rightarrow \tilde{X}$ and $Y \rightarrow \tilde{Y}$ are horizontal equivalences while the second “if and only if” in (iii) follows from (i).

For (iv), note first that $\delta^*: \mathbf{ssSet} \rightarrow \mathbf{sSet}$ is left Quillen for either the horizontal or vertical Reedy structures (and thus also for the joint Reedy structure). But, noting that all objects in \mathbf{ssSet} are cofibrant, and regarding X_0 as a bisimplicial set that is vertically constant, the claim follows by noting that by (i) the map $X_0 \rightarrow X$ is a horizontal weak equivalence in \mathbf{ssSet} . □

Corollary 4.6 *A map $f: X \rightarrow Y$ in \mathbf{ssSet} is a joint equivalence if and only if it induces a Kan equivalence on diagonals $\delta^*(X) \rightarrow \delta^*(Y)$ in \mathbf{sSet} .*

Proof Since horizontal Reedy fibrant replacement maps $X \rightarrow \tilde{X}$ are vertical equivalences, they are also diagonal equivalences (since δ^* is left Quillen), so one reduces to the case of X and Y horizontal Reedy fibrant. The result now follows by combining parts (iii) and (iv) of [Proposition 4.5](#). □

Corollary 4.7 *The adjunction (where \mathbf{ssSet} has the joint Reedy model structure)*

$$\delta_! : \mathbf{sSet} \rightleftarrows \mathbf{ssSet} : \delta^*$$

is a Quillen equivalence. Moreover, given $f: X \rightarrow Y$ in \mathbf{ssSet} ,

- $\delta^*(f)$ is a Kan fibration in \mathbf{sSet} if f has the right lifting property against both sets of maps in (4.2) and (4.3);
- $\delta^*(X)$ is a Kan complex in \mathbf{sSet} if X has the right lifting property against all maps in (4.2) as well as the maps in (4.3) with $m \geq 2$ (and dually for (4.2) with $n \geq 2$ and all maps in (4.3)).

Note that the first “moreover” claim is not quite formal, since the maps in (4.2)–(4.3) are not known to be generating trivial cofibrations for the joint model structure on ssSet .

Proof Recall that $\delta_!$ is the unique colimit-preserving functor such that $\delta_!(\Delta[n]) = \Delta[n] \times \Delta[n]$.

Throughout the proof we write F for both a nonempty subset $\emptyset \neq F \subseteq \{0, 1, \dots, n\}$ and the corresponding face $F \rightarrow [n]$ in Δ , and let $\Delta F \subseteq \Delta[n]$ denote the associated subsheaf.

Moreover, note that for any simplex $x \in \Delta[n]^{\times 2}$ there is a smallest pair of faces F_1^x and F_2^x such that $x \in \Delta F_1^x \times \Delta F_2^x$, and thus also a smallest F^x such that $x \in (\Delta F^x)^{\times 2}$, namely $F^x = F_1^x \cup F_2^x$.

To see that $\delta_!$ preserves cofibrations it is enough to show that $\delta_!(\partial\Delta[n] \rightarrow \Delta[n])$ is a monomorphism for all $n \geq 0$. Since $\partial\Delta[n] = \text{colim}_{\text{faces } F \neq [n]} \Delta F$, this map is $(\text{colim}_{\text{faces } F \neq [n]} \Delta F^{\times 2}) \rightarrow \Delta[n]^{\times 2}$, which is a monomorphism since

- (i) for any face F the map $\Delta F^{\times 2} \rightarrow \Delta[n]^{\times 2}$ is a monomorphism;
- (ii) in the colimit defining $\delta_!(\partial\Delta[n])$, any simplex $x \in \Delta F^{\times 2}$ is identified with the simplex $x \in (\Delta F^x)^{\times 2}$.

The claim that $\delta_!$ preserves trivial cofibrations follows easily from Remark 3.27 together with Corollary 4.6, but here we give a harder argument needed to establish the stronger “moreover” claims. Namely, we will argue that the maps $\delta_!(\Lambda^i[n] \rightarrow \Delta[n])$ are built cellularly out of the maps in (4.2)–(4.3). One has a factorization

$$\delta_! \Lambda^i[n] \rightarrow \Lambda^i[n] \times \Delta[n] \rightarrow \Delta[n]^{\times 2},$$

where the second map is clearly built cellularly out of the maps in (4.2), and we claim that the first map is likewise built cellularly out of the maps in (4.3). Writing $\text{Face}_{\supseteq \{i\}}$ for the poset of faces of $[n]$ strictly containing $\{i\}$, this first map can be factored as a sequence of maps of the form

$$(4.8) \quad \delta_! \Lambda^i[n] \cup \bigcup_{G \in C} \Lambda^i[n] \times \Delta G \rightarrow \delta_! \Lambda^i[n] \cup \bigcup_{G \in C'} \Lambda^i[n] \times \Delta G$$

for lower sets $C \subseteq C' \subseteq \text{Face}_{\supseteq \{i\}}$ such that $C' = C \amalg \{F\}$ for some face $F \supsetneq \{i\}$. Note that a simplex $x \in \Lambda^i[n] \times \Delta F$ will be in $\bigcup_{G \in C} \Lambda^i[n] \times \Delta G$ if and only if $F_2^x \not\supseteq F - \{i\}$, and that if $F_2^x \supseteq F - \{i\}$ then $x \in \delta_! \Lambda^i[n]$ if and only if $F^x = F_1^x \cup F_2^x \not\supseteq [n] - \{i\}$ if and only if $F_1^x \not\supseteq [n] - F$; it follows that (4.8) is a pushout of the map

$$(4.9) \quad (\Lambda^F[n] \rightarrow \Lambda^i[n]) \square (\Lambda^i F \rightarrow \Delta F),$$

where $\Lambda^F[n]$ is the union of those ΔH with $H \not\supseteq [n] - F$ (this is consistent with the horn notation in Section 2.3) and $\Lambda^i F$ the union of the ΔH with $F \supseteq H \not\supseteq F - \{i\}$. Most maps in (4.9) can in fact be built from either (4.2) or (4.3), but there is one crucial exception: for $F = [n]$ we have $\Lambda^F[n] = \emptyset$, so only (4.3) can be used. The first “moreover” claim now follows. For the second “moreover” claim, note that any $X \in \text{ssSet}$ admits (degenerate) liftings against $\delta_!(\Lambda^i[1] \rightarrow \Delta[1]) = (\{i\}^{\times 2} \rightarrow \Delta[1]^{\times 2})$, and that the only step in the filtration of $\delta_!(\Lambda^i[n] \rightarrow \Delta[n])$ for $n \geq 2$ requiring (4.3) (ie the case $F = [n]$ in (4.9)) uses the map $\Lambda^i[n] \times (\Lambda^i[n] \rightarrow \Delta[n])$.

Lastly, the Quillen equivalence condition is that for all $X \in \text{sSet}$ and joint fibrant $Y \in \text{ssSet}$, a map $X \rightarrow \delta^* Y$ is a weak equivalence if and only if $\delta_! X \rightarrow Y$ is. Factoring the former map as $X \rightarrow \delta^* \delta_! X \rightarrow \delta^* Y$, by Corollary 4.6 this reduces to showing that the unit maps $X \rightarrow \delta^* \delta_! X$ are weak equivalences. This latter claim follows by cellular induction on X , since those pushouts attaching cells are homotopy pushouts (due to sSet being left proper). □

Remark 4.10 Just as in the proof of Proposition 4.5, hypersaturations simplify the lifting conditions in the previous result.

Indeed, $X \rightarrow Y$ is a vertical fibration (ie it has the lifting property against (4.3)) if and only if, for each monomorphism $K \rightarrow L$ in sSet , $X_L \rightarrow X_K \times_{Y_K} Y_L$ is a Kan fibration. The lifting property against (4.2) then requires that $X_L \rightarrow X_K \times_{Y_K} Y_L$ is a trivial Kan fibration when $K \rightarrow L$ is a horn inclusion. But a straightforward argument (see Remark 4.14) shows that the $K \rightarrow L$ with this property are hypersaturated, so that by Remark 3.27 it suffices to check that the maps $X_n \rightarrow X_0 \times_{Y_0} Y_n$, induced by maps $[0] \rightarrow [n]$, are trivial Kan fibrations.

Similarly, if X is vertical fibrant (ie X has the lifting property against (4.3)) then, by Remarks 3.26 and 3.28, to check that X has the lifting property against (4.2) for $n \geq 2$ it suffices to check that the maps $X_n \rightarrow X_{\text{Sc}[n]}$ for $n \geq 0$, $X_2 \rightarrow X_{\Lambda^0[2]}$ and $X_2 \rightarrow X_{\Lambda^2[2]}$ are trivial Kan fibrations.

Remark 4.11 The adjunction $\delta^*: \text{ssSet} \rightleftarrows \text{sSet} : \delta_*$ can also be shown to be a Quillen equivalence.

4.2 Complete equivariant dendroidal Segal spaces

We now turn to our main application of Proposition 4.1, the category $\text{sdSet}^G = \text{Set}^{\Delta^{\text{op}} \times \Omega^{\text{op}} \times G}$ of G -equivariant simplicial dendroidal sets.

Since Δ is a (usual) Reedy category the model structure on dSet^G in [21, Theorem 2.1] induces a model structure on sdSet^G that we will refer to as the *simplicial Reedy model structure*.

On the other hand, in the context of Definition A.2, $\Omega^{\text{op}} \times G$ is a generalized Reedy category such that the families $\{\mathcal{F}_U^\Gamma\}_{U \in \Omega}$ of G -graph subgroups are Reedy-admissible (see Example A.6) and hence, using the underlying Kan model structure on sSet , Theorem A.8 yields a model structure on sdSet^G that we will refer to as the *equivariant dendroidal Reedy model structure*, or simply as the *dendroidal Reedy model structure* for the sake of brevity.

Throughout, we will write the levels of $X \in \text{sdSet}^G$ as $X_n(U)$ for $n \geq 0$ and $U \in \Omega$. We now extend Notation 4.4. Note that the representable functor of $U \in \Omega \times G^{\text{op}}$ is given by $\Omega[G \cdot U] = G \cdot \Omega[U]$.

Notation 4.12 Given a fixed $X \in \text{sdSet}^G$ we will also write

$$X(-): (\text{dSet}^G)^{\text{op}} \rightarrow \text{sSet}, \quad X_{(-)}: \text{sSet}^{\text{op}} \rightarrow \text{dSet}^G$$

for the unique limit-preserving functors such that $X(\Omega[G \cdot U]) = X(U)$ and $X_{\Delta[n]} = X_n$. Explicitly, $(X(A))_n = \text{dSet}^G(A, X_n)$ and $X_K(U) = \text{sSet}(K, X(U))$.

Moreover, for fixed $J \in \text{dSet}^G$ we define $X^J \in \text{sdSet}^G$ by $X^J(U) = X(\Omega[G \cdot U] \otimes J)$, where \otimes is the tensor product of dendroidal sets (see Example 3.9 for an informal definition or [20, Section 9; 21, Section 7] for an in-depth discussion).

Remark 4.13 The notation just defined fits into a two-variable adjunction (see [24, Section 10.1])

$$\begin{aligned} \text{sSet} \times \text{dSet}^G &\overset{\times}{\rightarrow} \text{sdSet}^G, & \text{sdSet}^G \times (\text{dSet}^G)^{\text{op}} &\overset{(-)(-)}{\rightarrow} \text{sSet}, \\ & & \text{sdSet}^G \times (\text{sSet})^{\text{op}} &\overset{(-)(-)}{\rightarrow} \text{dSet}^G. \end{aligned}$$

For our purposes, the relevance of this is that for maps $K \rightarrow L$ in sSet , $A \rightarrow B$ in dSet^G and $X \rightarrow Y$ in sdSet^G the three induced lifting problems shown below are all equivalent:

$$\begin{array}{ccc} \begin{array}{ccc} K \times B \amalg_{K \times A} L \times A & \longrightarrow & X \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ L \times B & \longrightarrow & Y \end{array} & \begin{array}{ccc} K & \longrightarrow & X(B) \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ L & \longrightarrow & X(A) \times_{Y(A)} Y(B) \end{array} & \begin{array}{ccc} A & \longrightarrow & X_L \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ B & \longrightarrow & X_K \times_{Y_K} Y_L \end{array} \end{array}$$

Remark 4.14 Fix $X \in \text{sdSet}^G$ and suppose that, for all normal monomorphisms $A \rightarrow B$ in dSet^G , $X(B) \rightarrow X(A)$ is a Kan fibration in sSet . Then the class of normal monomorphisms $A \rightarrow B$ such that $X(B) \rightarrow X(A)$ is a *trivial* Kan fibration is hypersaturated. Indeed, saturation follows from properties of trivial fibrations while (3.21) follows from 2-out-of-3 for Kan equivalences. A similar remark holds if $X_L \rightarrow X_K$ is a fibration in dSet^G for all monomorphisms $K \rightarrow L$ in sSet .

Notation 4.15 Writing $[\widetilde{m}] = (0 \rightrightarrows 1 \rightrightarrows \cdots \rightrightarrows m)$ for the contractible groupoid with objects $0, 1, \dots, m$ (see Remark 3.29), we write $J^m \in \text{sSet}$ for the nerve

$$J^m = N[\widetilde{m}]$$

and abbreviate $J = J^1$. Moreover, we slightly abuse notation by also writing $J^m \in \text{dSet}^G$ for the corresponding dendroidal set (under the standard inclusion $\iota_! : \text{sSet} \rightarrow \text{dSet}$; see Notation 2.9) with the trivial G -action. In fact, we use $J^m \in \text{dSet}^G$ by default and the few exceptions — Proposition 5.26, the end of Proposition 5.35 and Theorem 5.48 — should be clear from context.

Lastly, we make a notational remark that may avoid confusion. The term J^m will always feature in expressions $X(J^m)$ and X^{J^m} (or slight variants) as defined in Notation 4.12. As such, J^m as m changes should be thought of as “varying in the dendroidal direction” (which corresponds to the horizontal simplicial direction m), never in the (vertical) simplicial direction n .

Remark 4.16 $J^\bullet \in (\text{sSet})^\Delta$ (and thus $J^\bullet \in (\text{dSet}^G)^\Delta$) is a Reedy cofibrant cosimplicial object.

To see this, we mimic the notation (and argument) in the proof of Corollary 4.7 by writing F for both a subset $\emptyset \neq F \subseteq \{0, 1, \dots, m\}$ and face $F \rightarrow [m]$, as well as $\widetilde{F} \rightarrow [\widetilde{m}]$ for the natural subgroupoid. Further, note that for any simplex $x \in N[\widetilde{m}]$, which can be identified with a nonempty string on $\{0, 1, \dots, n\}$, there is a smallest face $F^x \rightarrow [m]$ such that $x \in N\widetilde{F}^x$, namely F^x is the set of “letters” of x . The latching map $L_m J^\bullet \rightarrow J^m$ is then $(\text{colim}_{\text{faces } F \neq [m]} N\widetilde{F}) \rightarrow N[\widetilde{m}]$, which is a monomorphism since

- (i) for any face F the map $N\widetilde{F} \rightarrow N[\widetilde{m}]$ is a monomorphism;
- (ii) any simplex $x \in N\widetilde{F}$ is identified with the simplex $x \in N\widetilde{F}^x$ in the colimit.

Proposition 4.17 *Both the simplicial and dendroidal Reedy model structures on sdSet^G have generating cofibrations given by the maps*

$$(4.18) \quad (\partial\Delta[n] \rightarrow \Delta[n]) \square (\partial\Omega[T] \rightarrow \Omega[T]), \quad n \geq 0, T \in \Omega_G.$$

Further, the dendroidal Reedy structure has as generating trivial cofibrations the maps

$$(4.19) \quad (\Lambda^i[n] \rightarrow \Delta[n]) \square (\partial\Omega[T] \rightarrow \Omega[T]), \quad n \geq 1, n \geq i \geq 0, T \in \Omega_G,$$

while the simplicial Reedy structure has as generating trivial cofibrations the maps

$$(4.20) \quad (\partial\Delta[n] \rightarrow \Delta[n]) \square (A \rightarrow B), \quad n \geq 0,$$

for $\{A \rightarrow B\}$ a set of generating trivial cofibrations of dSet^G .

Proof The claims concerning the simplicial Reedy structure are standard. For the dendroidal Reedy model structure, the result follows by [Proposition A.33](#) and [Example A.34](#). □

We call the saturation of the maps in (4.18) the class of *normal monomorphisms* of sdSet^G .

Remark 4.21 The category sdSet^G is simplicially tensored and cotensored via the pointwise formulas $(K \times X)(U) = K \times X(U)$ and $(X^K)(U) = X(U)^K$ for $K \in \text{sSet}$ and $X \in \text{sdSet}^G$, and hence becomes a simplicial category with mapping spaces $(\text{sdSet}^G(A, X))_\bullet = \text{sdSet}^G(\Delta[\bullet] \times A, X)$.

The generating sets (4.18)–(4.19) then show that these mapping spaces make the dendroidal Reedy model structure into a simplicial model structure, though the same does not hold for the simplicial Reedy model structure (since the pushout product of an anodyne map in sSet with a map in (4.18) will be in the saturation of (4.19) rather than in that of (4.20)).

Definition 4.22 The *joint Reedy model structure* on sdSet^G is the smallest joint left Bousfield localization (see [Proposition 4.1](#)) of the dendroidal and simplicial Reedy model structures.

Remark 4.23 Historically, the joint Reedy model structure has been called the *Rezk model structure* and its fibrant objects have been called *complete (dendroidal) Segal spaces*. This is because most discussions in the literature [22; 10] prefer to first introduce

the Segal space model structure on $\text{ssSet}/\text{sdSet}$, which is an intermediate localization of the horizontal/dendroidal Reedy model structure. In this work we first focus on the properties of the Rezk model structure that are consequences of the joint perspective, postponing the Segal space perspective to [Section 5](#).

Proposition 4.24 *The joint fibrant objects $X \in \text{sdSet}^G$ have the following equivalent characterizations:*

- (i) X is both simplicial Reedy fibrant and dendroidal Reedy fibrant;
- (ii) X is simplicial Reedy fibrant and all the simplicial structure maps $X_0 \rightarrow X_n$ are weak equivalences in dSet^G ;
- (iii) X is dendroidal Reedy fibrant and all the natural maps

$$(4.25) \quad X(\Omega[T]) \rightarrow X(\text{Sc}[T]) \quad \text{and} \quad X(\Omega[T]) \rightarrow X(\Omega[T] \otimes J)$$

for $T \in \Omega_G$ are Kan equivalences in sSet .

Proof (i) simply repeats [Proposition 4.1\(i\)](#). In the remainder we write $K \rightarrow L$ for a generic monomorphism in sSet and $A \rightarrow B$ for a generic normal monomorphism in dSet^G .

For (ii), note that X is simplicial fibrant if and only if $X_L \rightarrow X_K$ is always a fibration in dSet^G . And X will also have the right lifting property against [\(4.19\)](#) if and only if $X_L \rightarrow X_K$ is a trivial fibration whenever $K \rightarrow L$ is anodyne. But, by [Remark 3.27](#), it suffices to consider the vertex inclusions $\Delta[0] \rightarrow \Delta[n]$. The claim now follows from 2-out-of-3 applied to the composites $X_0 \rightarrow X_n \rightarrow X_0$.

For (iii), note first that X is dendroidal fibrant if and only if $X(B) \rightarrow X(A)$ is always a Kan fibration. Further, X will have the right lifting property against [\(4.20\)](#) if and only if $X(B) \rightarrow X(A)$ is a trivial Kan fibration whenever $A \rightarrow B$ is a generating trivial cofibration of dSet^G . By adjunction, this is equivalent to $X_L \rightarrow X_K$ always being a fibration in dSet^G . Further, since for $K = \emptyset$ we have $X_K = *$, this requires X_L to always be fibrant. In other words, X has the right lifting property against [\(4.20\)](#) if and only if $X_L \rightarrow X_K$ is always a fibration between fibrant objects in dSet^G . By the characterizations of fibrant objects and of fibrations between fibrant objects in dSet^G (see [\[21, Proposition 8.8\]](#) and the beginning of [\[21, Section 8.1\]](#)), it suffices to check that the maps $X_L \rightarrow X_K$ have the right lifting property against the maps (\square^\otimes denotes

the pushout product with respect to \otimes)

$$(4.26) \quad \begin{aligned} \Lambda^{Ge}[T] &\rightarrow \Omega[T], & T \in \Omega_G, e \in E^i(T), \\ (\partial\Omega[T] \rightarrow \Omega[T]) \square^{\otimes} (\{i\} \rightarrow J), & T \in \Omega_G, i = \{0, 1\}. \end{aligned}$$

(Note that the case $K = \emptyset$ shows that all the X_L are fibrant.)

It now suffices to check that $X(B) \rightarrow X(A)$ is a trivial Kan fibration when $A \rightarrow B$ is in (4.26). To finish the proof, we use Remark 4.14 combined with Proposition 3.22 and Remark 3.25. □

Remark 4.27 The characterizations of joint fibrant objects in Proposition 4.24 show that, to obtain the joint Reedy structure, one needs not localize the dendroidal Reedy structure with respect to *all generating trivial cofibrations* of the simplicial Reedy structure, and vice versa.

Indeed, since all objects appearing in (4.25) can be regarded as mapping spaces (see Remark 4.21), one needs only localize the dendroidal Reedy model structure with respect to the maps

$$(4.28) \quad \text{Sc}[T] \rightarrow \Omega[T], \quad \Omega[T] \otimes J \rightarrow \Omega[T], \quad T \in \Omega_G,$$

recovering the definition of the *dendroidal Rezk model category* from [10, Definitions 5.4 and 6.2].

Similarly, Proposition 4.24(ii) shows that one needs only localize the simplicial Reedy model structure with respect to the maps $(\Delta[0] \rightarrow \Delta[n]) \square (\partial\Omega[T] \rightarrow \Omega[T])$ (since then for X locally fibrant the maps $X_n \rightarrow X_0$ must be trivial fibrations in dSet^G), and by Remark 3.25 it is also enough to localize the vertex maps $\Omega[T] \rightarrow \Delta[n] \times \Omega[T]$, which is the same as localizing with respect to the projection maps

$$\Delta[n] \times \Omega[T] \rightarrow \Omega[T], \quad n \geq 0, T \in \Omega_G,$$

recovering the definition of the *locally constant model category* of [10, Definition 4.6].

Lastly, the fact that the two localizations just described coincide recovers Theorem 6.6 of [10].

We now obtain the following partial analogue of Proposition 4.5. Note that the equivalences in the simplicial Reedy model structure are the dendroidal equivalences and vice versa.

Corollary 4.29 Suppose that $X, Y \in \text{sdSet}^G$ are dendroidal Reedy fibrant. Then

- (i) for all n , the vertex maps $X_n \rightarrow X_0$ are trivial fibrations in dSet^G ;
- (ii) any simplicial Reedy fibrant replacement \tilde{X} of X is fibrant in the joint Reedy model structure;
- (iii) a map $X \rightarrow Y$ is a joint weak equivalence if and only if it is a dendroidal weak equivalence if and only if $X_0 \rightarrow Y_0$ is an equivalence in dSet^G ;
- (iv) regarding X_0 as a simplicially constant object in sdSet^G , the map $X_0 \rightarrow X$ is a dendroidal equivalence, and thus a joint equivalence.

Proof The proof adapts that of Proposition 4.5. (i) follows since X then has the right lifting property with respect to all maps $(\Delta[0] \rightarrow \Delta[m]) \square (\partial\Omega[T] \rightarrow \Omega[T])$. (ii) follows from (i) and the characterization in Proposition 4.24 (ii). The first “if and only if” in (iii) follows from (ii) since the simplicial fibrant replacement maps $X \rightarrow \tilde{X}$ are dendroidal equivalences and the second “if and only if” in (iii) follows from (i). (iv) follows from (i). □

Theorem 4.30 The constant/ 0^{th} level adjunction

$$c_1: \text{dSet}^G \rightleftarrows \text{sdSet}^G : (-)_0$$

where sdSet^G is given the Rezk/joint Reedy model structure, is a Quillen equivalence.

Proof It is clear that the constant functor c_1 preserves normal monomorphisms and all weak equivalences, hence the adjunction is Quillen. Consider any map $c_1(A) \rightarrow X$ with X joint fibrant and perform a “trivial cofibration followed by fibration” factorization as on the left,

$$c_1(A) \xrightarrow{\sim} c_1(\widetilde{A}) \twoheadrightarrow X, \quad A \xrightarrow{\sim} c_1(\widetilde{A})_0 \rightarrow X_0,$$

for the simplicial Reedy model structure. Proposition 4.24(ii) now implies that $c_1(\widetilde{A})$ is in fact joint fibrant and thus that the leftmost composite is a joint equivalence if and only if $c_1(\widetilde{A}) \rightarrow X$ is a dendroidal equivalence in sdSet^G if and only if $c_1(\widetilde{A})_0 \rightarrow X_0$ is an equivalence in dSet^G if and only if the rightmost composite is an equivalence in dSet^G . □

Remark 4.31 Given a G - ∞ -operad $X \in \text{dSet}^G$, one can obtain an explicit model for $c_1(\widetilde{X})$ as the object $X^{J^\bullet} \in \text{sdSet}^G$, where J^m was defined in Notation 4.15 and $X^{J^m} \in \text{dSet}^G$ is defined as in Notation 4.12. Indeed, since J^\bullet is a Reedy cofibrant cosimplicial object in dSet^G (see Remark 4.16), one has that $X^{J^\bullet} \in \text{sdSet}^G$ is simplicial

fibrant. Hence, by Proposition 4.24(ii) $c_!(X) \rightarrow X^{J^\bullet}$ will be a joint fibrant replacement provided that it is a dendroidal equivalence. But this follows from [21, Corollary 8.21], which implies that the maps $X^{J^m} \rightarrow X^{J^0} = X$ are trivial fibrations in dSet^G (formally, loc. cit. says that $v_*(X^{J^m}) = X^{(J^m)} \rightarrow v_*(X)$ is a trivial fibration in dSet_G , which is an equivalent statement, as noted at the end of the proof of [21, Theorem 8.22]).

4.3 Equivariant Segal operads

Recall that the category PreOp of *preoperads* is the full subcategory $\text{PreOp} \subset \text{sdSet}$ of those X such that $X(\eta)$ is a discrete simplicial set. Writing γ^* for the inclusion, one has left and right adjoints $\gamma_!$ and γ_* ,

$$\begin{array}{ccc} & \xleftarrow{\gamma_!} & \\ \text{PreOp}^G & \xrightarrow{\gamma^*} & \text{sdSet}^G \\ & \xleftarrow{\gamma_*} & \end{array}$$

described as follows [10, Section 7]: $\gamma_!X(U) = X(U)$ if $U \notin \Delta$ while $\gamma_!X([n])$ for $[n] \in \Delta$ is given by the pushout on the left below; $\gamma_*X(U)$ is given by the pullback on the right below:

$$(4.32) \quad \begin{array}{ccc} X(\eta) & \xrightarrow{\quad} & \pi_0 X(\eta) \\ \downarrow & \lrcorner & \downarrow \\ X([n]) & \xrightarrow{\quad} & \gamma_!X([n]) \end{array} \quad \begin{array}{ccc} \gamma_*X(U) & \xrightarrow{\quad} & X(U) \\ \downarrow & \lrcorner & \downarrow \\ \prod_{E(U)} X_0(\eta) & \xrightarrow{\quad} & \prod_{E(U)} X(\eta) \end{array}$$

Remark 4.33 Any monomorphism $A \rightarrow B$ in sdSet^G such that $A(\eta) \rightarrow B(\eta)$ is an isomorphism induces a pushout square

$$(4.34) \quad \begin{array}{ccc} A & \xrightarrow{\quad} & \gamma^*\gamma_!A \\ \downarrow & \lrcorner & \downarrow \\ B & \xrightarrow{\quad} & \gamma^*\gamma_!B \end{array}$$

The assignment $U \mapsto \prod_{E(U)} Y(\eta)$ is the 0-coskeleton of Y in the dendroidal Reedy direction (see (A.9)). To avoid confusion with the coskeleton in the simplicial direction, and since η is the only tree of degree 0, we denote this dendroidal coskeleton by $\text{csk}_\eta Y$. We then have the following:

Proposition 4.35 *Let $X \in \text{sdSet}^G$. Then*

- (i) *if $X \in \text{sdSet}^G$ is dendroidal Reedy fibrant then so is $\gamma^*\gamma_*X$;*

- (ii) regarding X_0 as a simplicially constant object of sdSet^G , the left square below is a pullback;
- (iii) if $A \rightarrow A'$ is a map in dSet^G such that $A(\eta) \simeq A'(\eta)$, the right square below is a pullback:

$$\begin{array}{ccc}
 \gamma^* \gamma_* X & \longrightarrow & X \\
 \downarrow & \lrcorner & \downarrow \\
 \text{csk}_\eta X_0 & \longrightarrow & \text{csk}_\eta X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \gamma^* \gamma_* X(A') & \longrightarrow & X(A') \\
 \downarrow & \lrcorner & \downarrow \\
 \gamma^* \gamma_* X(A) & \longrightarrow & X(A)
 \end{array}$$

Proof (ii) is immediate from the observation that $(\text{csk}_\eta Y)(U) = \prod_{\mathbf{E}(U)} Y(\eta)$. Moreover, it readily follows that for $B \in \text{dSet}^G$ we have $(\text{csk}_\eta Y)(B) = (\prod_{B(\eta)} Y(\eta))^G$, where $(-)^G$ denotes fixed points for the conjugation action. Hence, since $(-)(B)$ preserves pullbacks, (iii) follows from (ii).

For (i), formal considerations imply that if X is dendroidal (Reedy) fibrant then the map $X \rightarrow \text{csk}_\eta X$ is a dendroidal fibration (and $\text{csk}_\eta X$ is dendroidal fibrant). Hence, the result will follow provided that $\text{csk}_\eta X_0$ is also dendroidal fibrant. But since $\text{csk}_\eta X_0$ is η -coskeletal, it suffices to check that the η -matching map $(\text{csk}_\eta X_0)(\eta) \rightarrow M_\eta(\text{csk}_\eta X_0)$ is a G -fibration in sSet^G . But this is simply $X_0(\eta) \rightarrow *$ regarded as a map of constant simplicial sets, and the result follows. \square

Notation 4.36 In the remainder of the section we write \mathcal{I}' for the set of maps

$$(\partial\Delta[n] \rightarrow \Delta[n]) \square (\partial\Omega[T] \rightarrow \Omega[T]), \quad n \geq 0, T \in \Omega_G, T \neq G/H \cdot \eta.$$

Further, we note that [Remark 4.33](#) applies to these maps.

In what follows, we say a map in PreOp^G is a normal monomorphism if it is one in sdSet^G .

Lemma 4.37 *The normal monomorphisms in PreOp^G are the saturation of the set of maps $\{\emptyset \rightarrow G/H \cdot \eta : H \leq G\} \cup \gamma_!(\mathcal{I}')$.*

Proof If $A \rightarrow B$ in PreOp^G is a normal monomorphism, then using the cellular filtration in sdSet^G one can write $\gamma^*(A \rightarrow B)$ as a transfinite composition of pushouts of maps in $\{\emptyset \rightarrow G/H \cdot \eta\} \cup \mathcal{I}'$, and hence $\gamma_! \gamma^*(A \rightarrow B) \simeq (A \rightarrow B)$ can similarly be written as a transfinite composition of pushouts of maps in $\{\emptyset \rightarrow G/H \cdot \eta\} \cup \gamma_!(\mathcal{I}')$. The remaining claim that the maps in $\{\emptyset \rightarrow G/H \cdot \eta\} \cup \gamma_!(\mathcal{I}')$ are normal monomorphisms follows from the pushouts [\(4.34\)](#). \square

Lemma 4.38 Any map in PreOp^G which has the right lifting property against all normal monomorphisms in PreOp^G is a joint equivalence in sdSet^G .

Proof We simply adapt the proof of [10, Lemma 8.12] mutatis mutandis.

Choose a normalization E_∞ of $*$ in dSet^G , ie a normal object such that $E_\infty \rightarrow *$ is a trivial fibration. Regarding E_∞ as a simplicially constant object in sdSet^G , a map $X \rightarrow Y$ in PreOp^G will have the right lifting property against all normal monomorphisms if and only if so does $E_\infty \times (X \rightarrow Y)$, so that one is free to assume that X and Y are normal.

One is thus free to pick a section $s: Y \rightarrow X$ of $p: X \rightarrow Y$ and, regarding $J \in \text{dSet}^G$ as a simplicially constant object of sdSet^G , our assumption yields the lift

$$\begin{array}{ccc}
 X \amalg X & \xrightarrow{(\text{id}_X, sp)} & X \\
 \downarrow & \nearrow \text{---} & \downarrow p \\
 X \otimes J & \longrightarrow & Y
 \end{array}$$

showing that p is a homotopy equivalence (the implicit claim that $X \otimes J$ is a cylinder object follows from [21, Proposition 7.25], which implies $X \amalg X \rightarrow X \otimes J$ is a normal monomorphism, and [21, Theorem 7.1], which implies the inclusions $X \rightarrow X \otimes J$ are G -inner anodyne on each simplicial level, thus dendroidal equivalences and hence joint equivalences). □

Theorem 4.39 The category PreOp^G of G -preoperads has a model structure such that

- the cofibrations are the normal monomorphisms;
- the weak equivalences are the maps that become Rezk/joint equivalences when regarded as maps in sdSet^G .

Proof We repeat the proof of the nonequivariant analogue [10, Theorem 8.13], applying J Smith’s theorem [1, Theorem 1.7] with the required set of generating cofibrations the set $\{\emptyset \rightarrow G/H \cdot \eta \mid H \leq G\} \cup \gamma_!(\mathcal{I}')$ given by Lemma 4.37. Indeed, conditions c0 and c2 in [1, Theorem 1.7] are inherited from sdSet^G and c1 follows from Lemma 4.38. The technical “solution set” condition c3 follows from [1, Proposition 1.15] since weak equivalences are accessible, being the preimage by γ^* of the weak equivalences in sdSet^G (see [18, Corollary A.2.6.5; 18, Corollary A.2.6.6]). □

Definition 4.40 Following [11, Definition 5.5] (compare with [10, Definition 8.1]), an equivariant preoperad $X \in \text{PreOp}^G$ is called an *equivariant Segal operad* if the maps

$$\gamma^* X(\Omega[T]) \rightarrow \gamma^* X(\text{Sc}(T))$$

are Kan equivalences in sSet for every $T \in \Omega_G$.

The work in Section 5 will allow us to show that the fibrant objects in PreOp^G are the equivariant Segal operads X such that $\gamma^* X$ is dendroidal Reedy fibrant. See Corollary 5.53.

Theorem 4.41 *The adjunction*

$$\gamma^*: \text{PreOp}^G \rightleftarrows \text{sdSet}^G : \gamma_*$$

is a Quillen equivalence.

Proof It is tautological that the left adjoint γ^* preserves and detects cofibrations and weak equivalences, so it suffices to show that for all fibrant $X \in \text{sdSet}^G$ the counit map $\gamma^* \gamma_* X \rightarrow X$ is a weak equivalence. But, by Proposition 4.35(i), both $\gamma^* \gamma_* X$ and X are dendroidal fibrant, so that the result follows from Corollary 4.29(iii) together with the observation that $(\gamma^* \gamma_* X)_0 = X_0$. \square

5 Equivariant dendroidal Segal spaces

As outlined in the introduction, one of the main aims of our overall project is to show that the model structures on sdSet^G and PreOp^G defined in Sections 4.2 and 4.3 are Quillen equivalent to a suitable model structure on the category sOp^G of (colored) G -operads. However, our present description of the weak equivalences in sdSet^G and PreOp^G is rather different from the description of the desired weak equivalences in sOp^G , which are the Dwyer–Kan equivalences, characterized by fully faithfulness and essential surjectivity requirements.

As such, our goal in this final main section is to prove Theorem 5.48, which states that weak equivalences between fibrant objects in either of sdSet^G or PreOp^G do indeed admit a Dwyer–Kan-type description. Moreover, in Corollary 5.51 we also characterize the fibrant objects of PreOp^G (this independently extends a result that first appeared in Bergner’s work [3]).

To do so, it is useful to consider yet another model structure on the category sdSet^G , whose fibrant objects are the so called *equivariant dendroidal Segal spaces*, and which “interpolate” between the fibrant objects in the categories sdSet^G and PreOp^G (see [Remark 5.54](#) for a precise statement).

5.1 The homotopy genuine operad and Dwyer–Kan equivalences

Definition 5.1 The *equivariant Segal space model structure* on the category sdSet^G , which we denote by sdSet_S^G , is the left Bousfield localization of the dendroidal Reedy model structure with respect to the equivariant Segal core inclusions (see [Remark 4.27](#), more specifically [\(4.28\)](#))

$$\text{Sc}[T] \rightarrow \Omega[T], \quad T \in \Omega_G.$$

Notation 5.2 We will refer to the fibrant objects of sdSet_S^G as *equivariant dendroidal Segal spaces*, or just *dendroidal Segal spaces*. Further, a preoperad $X \in \text{PreOp}^G$ is called *fibrant* if γ^*X is a dendroidal Segal space (for now this is just terminology, foreshadowing [Corollary 5.51](#)).

The following is the equivariant analogue of [\[10, Corollary 5.6\]](#), and is an immediate consequence of [Proposition 3.22](#) and [Remark 4.21](#).

Proposition 5.3 Suppose $X \in \text{sdSet}^G$ is dendroidal Reedy fibrant. The following are equivalent:

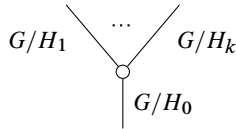
- (i) X is an equivariant dendroidal Segal space;
- (ii) $X(\Omega[T]) \rightarrow X(\text{Sc}[T])$ is a trivial Kan fibration for all $T \in \Omega_G$;
- (iii) $X(\Omega[T]) \rightarrow X(\Lambda^{Ge}[T])$ is a trivial Kan fibration for all $T \in \Omega_G$ and $e \in E^i(T)$;
- (iv) $X(\Omega[T]) \rightarrow X(\Lambda^E[T])$ is a trivial Kan fibration for all $T \in \Omega_G$ and G -subsets $\emptyset \neq E \subseteq E^i(T)$;
- (v) $X(\Omega[T]) \rightarrow X(\Lambda_o^E[T])$ is a trivial Kan fibration for all $T \in \Omega_G$ and G -subsets $\emptyset \neq E \subseteq E^i(T)$.

Proposition 5.4 If $X \in \text{sdSet}^G$ is a dendroidal Segal space, then $\gamma_*X \in \text{PreOp}^G$ is fibrant.

Proof By [Proposition 4.35\(i\)](#), γ_*X is dendroidal fibrant. And, since $\text{Sc}[T](\eta) = \Omega[T](\eta)$, [Proposition 4.35\(iii\)](#) shows that $(\gamma^*\gamma_*X)(\Omega[T]) \rightarrow (\gamma^*\gamma_*X)(\text{Sc}[T])$ is a trivial Kan fibration. □

To define fully faithfulness for dendroidal Segal spaces, we now discuss mapping spaces. Nonequivariantly, these are indexed by tuples of objects and described using corollas [10, 3.6]. Equivariantly, these are indexed by “equivariant tuples” of objects, as suggested by G -corollas.

Notation 5.5 Given subgroups $H_i \leq G$ for $0 \leq i \leq k$ such that $H_0 \geq H_i$ for $1 \leq i \leq k$, we write $C_{\coprod_i H_0/H_i}$ for the G -corolla (well defined up to isomorphism) whose orbital representation is



Writing C_n for the nonequivariant corolla with n leaves, we note that $C_{\coprod_i H_0/H_i} \simeq G \cdot_{H_0} C_{\Sigma_i |H_0/H_i|}$, where $C_{\Sigma_i |H_0/H_i|}$ is regarded as a (nonequivariant) corolla together with the obvious H_0 -action.

Definition 5.6 Let $X \in \text{dSet}^G$ be a G - ∞ -operad. A G -profile on X is a map

$$\partial\Omega[C] \rightarrow X$$

for some G -corolla C . More explicitly, a G -profile is described by the following data:

- subgroups $H_i \leq G$, $0 \leq i \leq k$ such that $H_0 \geq H_i$ for $1 \leq i \leq k$;
- objects $x_i \in X(\eta)^{H_i}$ for $0 \leq i \leq k$.

To simplify notation, we write a G -profile as $(x_1, \dots, x_k; x_0)$, and refer to it as a C -profile on X . Further, for $X \in \text{sdSet}_S^G$ a dendroidal Segal space we define a C -profile on X as a C -profile on X_0 .

Definition 5.7 Given a dendroidal Segal space $X \in \text{sdSet}_S^G$ and associated C -profile $(x_1, \dots, x_k; x_0)$ on X we define the space of maps $X(x_1, \dots, x_k; x_0) \in \text{sSet}$ via the pullback square

$$\begin{array}{ccc} X(x_1, \dots, x_k; x_0) & \longrightarrow & X(\Omega[C]) \\ \downarrow & \lrcorner & \downarrow \\ \Delta[0] & \xrightarrow{(x_1, \dots, x_k; x_0)} & \prod_{0 \leq i \leq k} X(\eta)^{H_i} \end{array}$$

To discuss essential surjectivity, and adapting [10, 8.8], we associate to each equivariant dendroidal Segal space X a discretized algebraic structure $\text{ho}(X)$, which in the equivariant setting is a genuine equivariant operad (see Definition 3.35).

Definition 5.8 Let $X \in \text{sdSet}^G$ be a dendroidal Segal space. The *homotopy genuine operad* $\text{ho}(X) \in \text{dSet}_G$ is defined by

$$\text{ho}(X) = \pi_0(v_*(\gamma_* X)),$$

where v_* and γ_* are defined in (3.32) and (4.32).

Proposition 5.9 For any dendroidal Segal space $X \in \text{sdSet}^G$ one has that $\text{ho}(X) \in \text{dSet}_G$ is a genuine equivariant operad.

Proof By Proposition 5.4 we are free to assume X is in $\text{PreOp}^G \subset \text{sdSet}^G$, so we suppress γ_* from the notation throughout.

The required strict lifting condition is equivalent to the maps $\text{dSet}_G(v_*\Omega[T], \text{ho}(X)) \rightarrow \text{dSet}_G(v_*\text{Sc}[T], \text{ho}(X))$ being isomorphisms. On the other hand, the Segal condition for X says that the maps $X(\Omega[T]) \rightarrow X(\text{Sc}[T])$ are trivial Kan fibrations, so that the maps $\pi_0(X(\Omega[T])) \rightarrow \pi_0(X(\text{Sc}[T]))$ are isomorphisms. Hence, noting that for $A \in \text{dSet}^G$ there is a natural transformation (letting \bullet highlight the index with regards to which the π_0 are computed, and using that v_* is fully faithful)

$$\begin{aligned} (5.10) \quad \pi_0(X_\bullet(A)) &= \pi_0(\text{dSet}^G(A, X_\bullet)) = \pi_0(\text{dSet}_G(v_*A, v_*X_\bullet)) \\ &\rightarrow \text{dSet}_G(v_*A, \pi_0 v_* X_\bullet) = \text{dSet}_G(v_*A, \text{ho}(X_\bullet)), \end{aligned}$$

it is enough to show that (5.10) is an isomorphism when A is of the form $\Omega[T]$ or $\text{Sc}[T]$. The case of $A = \Omega[T]$ is tautological since, by Remark 3.34, $\text{dSet}_G(v_*\Omega[T], Y) = Y(T)$ for any $Y \in \text{dSet}_G$.

It remains to tackle the case $A = \text{Sc}[T]$. We argue by induction on the number of G -vertices of T . The base cases of T either a G -corolla or a single G -edge orbit are automatic since then $\text{Sc}[T] = \Omega[T]$. Otherwise, choosing an equivariant grafting decomposition $T = R \amalg_{Ge} S$ (see [21, (5.18) and Proposition 6.19]) one has a pushout decomposition $\text{Sc}[T] \simeq \text{Sc}[R] \amalg_{\Omega[Ge]} \text{Sc}[S]$, so that, by induction, it suffices to check that both sides of (5.10) turn this pushout into a pullback. For the right side this follows since, by Remark 3.33, one has the analogous decomposition $v_*\text{Sc}[T] \simeq v_*\text{Sc}[R] \amalg_{v_*\Omega[Ge]} v_*\text{Sc}[S]$. For the left side this follows since π_0 preserves the pullback decomposition $X(\text{Sc}[T]) \simeq X(\text{Sc}[R]) \times_{X(\Omega[Ge])} X(\text{Sc}[S])$ due to $X(\Omega[Ge])$ being discrete. □

Recall that O_G denotes the *orbit category* with objects the G -sets G/H for $H \leq G$ and arrows the G -equivariant maps between them. In the following, we use the natural inclusion $\Omega \times O_G \rightarrow \Omega_G$ given by $(T, G/H) \mapsto G/H \cdot T$.

Remark 5.11 Writing $\iota: \Delta \rightarrow \Omega$ and $\iota_G: \Delta \times O_G \rightarrow \Omega \times O_G \rightarrow \Omega_G$ for the (composite) inclusions, unpacking definitions shows that $\iota_G^* \text{ho}(X)$ is the G -coefficient system of nerves of categories $(\iota_G^* \text{ho}(X))(G/H)$ for $H \leq G$, which are the simplicial sets with m -simplices

$$(5.12) \quad \begin{aligned} \text{ho}(X)(G/H \cdot [m]) &= \pi_0((\gamma_* X([m]))^H) = \pi_0(\iota^* \gamma_*(X^H))(m) \\ &= \text{ho}(\iota^*(X^H))(m), \end{aligned}$$

where the second identity follows from the discussion following (3.32), and the second ho is the analogue of Definition 5.8 for simplicial Segal spaces [22, Section 5.5].

Definition 5.13 A map $f: X \rightarrow Y$ of equivariant dendroidal Segal spaces is called

- *fully faithful* if for all G -corollas C and C -profiles $(x_1, \dots, x_n; x_0)$ on X the maps

$$X(x_1, \dots, x_k; x_0) \rightarrow Y(f(x_1), \dots, f(x_k); f(x_0))$$

are Kan equivalences in sSet ;

- *essentially surjective* if the map $\iota_G^* \text{ho}(X) \rightarrow \iota_G^* \text{ho}(Y)$ of G -coefficient systems of categories is levelwise essentially surjective;
- a *DK equivalence* if it is both fully faithful and essentially surjective.

Remark 5.14 Definitions 5.7, 5.8 and 5.13 depend only on the fibrant preoperads $\gamma_* X$ and $\gamma_* Y$, since $X(x_1, \dots, x_k; x_0) = \gamma_* X(x_1, \dots, x_k; x_0)$. In fact, for each G -corolla C one has a decomposition

$$\text{ho}(X)(C) = \coprod_{C\text{-profiles } (x_1, \dots, x_k; x_0)} \pi_0(X(x_1, \dots, x_k; x_0)),$$

so that, given $\varphi \in X_0(x_1, \dots, x_k; x_0)$ we will write $[\varphi] \in \text{ho}(X)(C)$ for the corresponding class.

Remark 5.15 One can extend the previous definitions to G - ∞ -operads $X, Y \in \text{dSet}^G$ by applying them to the dendroidal Segal spaces $X^{J^\bullet}, Y^{J^\bullet} \in \text{sdSet}^G$ (see Remark 4.31).

Remark 5.16 Given a map $X \rightarrow Y$ between dendroidal Segal spaces and $T \in \Omega_G$, consider the diagram

$$(5.17) \quad \begin{array}{ccc} X(\Omega[T]) & \longrightarrow & Y(\Omega[T]) \\ \sim \downarrow & & \downarrow \sim \\ X(\text{Sc}[T]) & \longrightarrow & Y(\text{Sc}[T]) \\ \downarrow & & \downarrow \\ \prod_{[e_i] \in \mathbf{E}_G(T)} (X(\eta))^{H_i} & \longrightarrow & \prod_{[e_i] \in \mathbf{E}_G(T)} (Y(\eta))^{H_i} \end{array}$$

where $H_i \leq G$ is the isotropy of $e_i \in \mathbf{E}(T)$. Since the fibers of the bottom vertical maps are products of spaces of maps for G -profiles and the top vertical maps are trivial fibrations by the Segal conditions, $X \rightarrow Y$ is fully faithful if and only if (5.17) induces Kan equivalences between the fibers of the vertical composite maps (for the “if” claim, let T be a G -corolla, see [10, Proposition 5.7]).

This remark readily implies the following, which is the analogue of [10, Corollary 5.10]:

Corollary 5.18 *Let $f: X \rightarrow Y$ in sdSet_G be a map between dendroidal Segal spaces. Then*

- (i) *if f is a simplicial equivalence then f is fully faithful;*
- (ii) *if f is fully faithful, then f is also a simplicial equivalence if and only if the maps $X(\eta)^H \rightarrow Y(\eta)^H$ for $H \leq G$ are Kan equivalences.*

Remark 5.19 In what follows, we will repeatedly use the observation that, for $X \rightarrow Y$ a trivial Kan fibration in sSet , any two lifts of the form below are homotopic:

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \sim \\ A & \longrightarrow & Y \end{array}$$

Definition 5.20 Let $X \in \text{sdSet}^G$ be a dendroidal Segal space.

For $H \leq G$, we call $f \in X_0(\Omega[C_H/H]) = X_0([1])^H$ an H -equivalence if $[f]$ is an isomorphism in the category $\iota_G^* \text{ho}(X)(G/H) = \text{ho}(\iota^*(X^H))$.

In what follows, and in analogy to [22, Section 11.2], we will need to understand the interaction between the homotopy genuine operad $\text{ho}(X)$ and the mapping spaces $X(x_1, \dots, x_n; x_0)$.

Suppose C and D are G -corollas that can be grafted, ie that C has a leaf orbit and D a root orbit both isomorphic to G/H . Write this orbit as Ge and write $T = C \amalg_{Ge} D$ for the grafted G -tree [21, (5.18) and Proposition 6.19]. For any dendroidal Segal space X one then has $X(\text{Sc}[T]) \simeq X(\Omega[C]) \times_{X(\eta)H} X(\Omega[D])$ and one can hence choose a section in the middle row of

$$\begin{array}{ccc}
 \{\varphi\} \times X(z_1, \dots, z_l; e) & \xrightarrow{\varphi \circ_{Ge} (-)} & X(z_1, \dots, z_l, y_2, \dots, y_k; x) \\
 \downarrow & \dashrightarrow & \downarrow \\
 (5.21) \quad X(\Omega[C]) \times_{X(\eta)H} X(\Omega[D]) & \xleftarrow{\sim} X(\Omega[T]) \xrightarrow{\sim} & X(\Omega[T - Ge]) \\
 \uparrow & & \uparrow \\
 X(e, y_2, \dots, y_k; x) \times \{\psi\} & \xrightarrow{(-) \circ_{Ge} \psi} & X(z_1, \dots, z_l, y_2, \dots, y_k; x)
 \end{array}$$

thus defining maps $\varphi \circ_{Ge} (-)$ and $(-) \circ_{Ge} \psi$ for any choice of $\varphi \in X_0(e, y_2, \dots, y_k; x)$ and $\psi \in X_0(z_1, \dots, z_l; e)$, respectively.

Proposition 5.22 (i) *The maps $\varphi \circ_{Ge} (-)$ and $(-) \circ_{Ge} \psi$ are well defined up to homotopy.*

(ii) *If $[\varphi] = [\bar{\varphi}]$ then the maps $\varphi \circ_{Ge} (-)$ and $\bar{\varphi} \circ_{Ge} (-)$ are homotopic, and likewise for $[\psi] = [\bar{\psi}]$.*

(iii) *$[\varphi \circ_{Ge} \psi]$ depends only on $[\varphi]$ and $[\psi]$.*

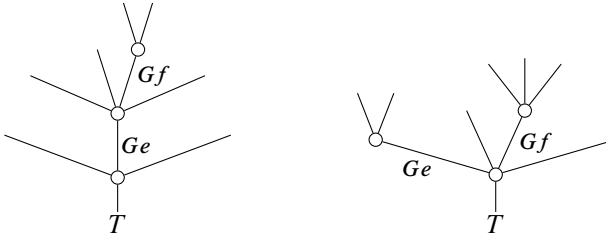
(iv) *The homotopy classes of the maps $\varphi \circ_{Ge} (-)$ and $(-) \circ_{Ge} \psi$ are natural with respect to maps $f: X \rightarrow Y$ between dendroidal Segal spaces.*

Proof Noting that all possible middle row sections in (5.21) (and homotopies between them) are necessarily compatible with the projections to $X(\partial\Omega[T - Ge])$, (i) follows from Remark 5.19. The middle row in (5.21) gives the necessary homotopies for (ii). (iii) is immediate from (ii). Lastly, (iv) follows from Remark 5.19 applied to the two diagonal \nearrow paths in

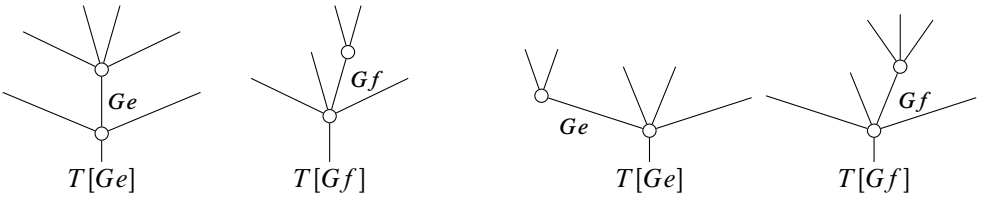
$$\begin{array}{ccc}
 X(\Omega[T]) & \xrightarrow{\quad\quad\quad} & Y(\Omega[T]) \\
 \downarrow \sim & & \downarrow \sim \\
 X(\Omega[C]) \times_{X(\eta)H} X(\Omega[D]) & \rightarrow & Y(\Omega[C]) \times_{Y(\eta)H} Y(\Omega[D])
 \end{array} \quad \square$$

We will now show that the operations $\varphi \circ_{Ge} (-)$ and $(-) \circ_{Ge} \psi$ satisfy the obvious compatibilities one expects, but we will find it convenient to first package these compatibilities into a common format. In the categorical case (corresponding to linear trees),

there are three types of “associativity” compatibilities, corresponding to homotopies $\varphi \circ (\psi \circ (-)) \sim (\varphi \circ \psi) \circ (-)$, $\varphi \circ ((-) \circ \psi) \sim (\varphi \circ (-)) \circ \psi$, $((-) \circ \varphi) \circ \psi \sim (-) \circ (\varphi \circ \psi)$, but in the operadic case there are instead five cases, corresponding to the different possible roles of the nodes in G -trees T with exactly three G -vertices, whose orbital representation falls into one of the two cases illustrated below:



Since all these compatibilities can be simultaneously encoded in terms of such trees, we will refer to all types of compatibility simply as *associativity*. As noted pictorially above, such a G -tree T has exactly two inner edge orbits Ge and Gf . In the next result, we write $T[Ge]$ (resp. $T[Gf]$) for the orbital outer face of T with Ge (resp. Gf) as its single inner edge orbit:



Proposition 5.23 *The operations $\varphi \circ_{Ge} (-)$ and $(-) \circ_{Ge} \psi$ satisfy all associativity conditions with respect to G -trees with three G -vertices. Further, if $C = C_{H/H}$ and $\varphi = s(e)$ is the degeneracy on e , then $\varphi \circ_{Ge} (-)$ is homotopic to the identity, and similarly for $D = C_{H/H}$ and $\psi = s(e)$.*

Proof We abbreviate

$$Sc_{T[Ge]}[T] = Sc[T] \amalg_{Sc[T[Ge]]} \Omega[T[Ge]] = Sc[T] \amalg_{\Lambda_{\sigma}^{Ge}[T[Ge]]} \Omega[T[Ge]],$$

which can be regarded as the union $Sc[T] \cup \Omega[T[Ge]]$ of subcomplexes of $\Omega[T]$. We now consider the following diagram, where all solid maps are Kan fibrations, and the maps labeled \sim are trivial Kan fibrations ($Sc_{T[Ge]}[T]$ is a cover in the sense of

Remark 3.8(i), hence both maps $\text{Sc}[T] \rightarrow \text{Sc}_{T[Ge]}[T] \rightarrow \Omega[T]$ are G -inner anodyne), so that one can choose the indicated sections:

$$\begin{array}{ccccc}
 X(\text{Sc}[T]) & \xleftarrow{\sim} & X(\text{Sc}_{T[Ge]}[T]) & \xrightarrow{\sim} & X(\text{Sc}[T - Ge]) \\
 \uparrow \sim & & \uparrow \sim & & \uparrow \sim \\
 X(\text{Sc}_{T[Gf]}[T]) & \xleftarrow{\sim} & X(\Omega[T]) & \xrightarrow{\sim} & X(\Omega[T - Ge]) \\
 \downarrow & & \downarrow & & \downarrow \\
 X(\text{Sc}[T - Gf]) & \xleftarrow{\sim} & X(\Omega[T - Gf]) & \xrightarrow{\sim} & X(\Omega[T - Ge - Gf])
 \end{array}$$

Since the desired associativity conditions now amount to the claim that the top right and left bottom composites $X(\text{Sc}[T]) \rightarrow X(\Omega[T - Ge - Gf])$ are homotopic, the associativity result follows from **Remark 5.19**. For the “further” claim, note that by **Remark 5.19** one is free to modify (5.21) so as to use any lift of the form below. But then since the G -tree T is degenerate on the G -corolla D , such a lift is given by the degeneracy operator and the result follows:

$$\{s(e)\} \times X(z_1, \dots, z_l; e) \xrightarrow{\quad} X(\Omega[T]) \xrightarrow{\sim} X(\text{Sc}[T]) \quad \square$$

Remark 5.24 In the nonequivariant case the associativity and unit conditions in the previous result capture all the key compatibilities of the $\varphi_{\circ_e}(-)$ and $(-)\circ_e \psi$ operations. However, in the equivariant case there are further “compatibilities with quotients of G -trees”, which reflect the remarks in **Example 3.37**. Nonetheless, describing these extra compatibilities would require using G -trees with more than three G -vertices, and since such compatibilities are not needed for our present goals, we omit their discussion.

Corollary 5.25 *DK equivalences between dendroidal Segal spaces satisfy 2-out-of-6, ie when in $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ the maps gf and hg are DK equivalences then so are f, g, h and hgf .*

Proof Applying the 2-out-of-6 properties in sSet and Cat to mapping spaces and homotopy categories $i_G^* \text{ho}(-)$, the only nonobvious conditions are the fully faithfulness of g and h for C -profiles not in the image of f . But, since by **Proposition 5.23** the

maps $j \circ_{Ge} (-)$, $(-) \circ_{Ge} j$ are weak equivalences when j is an H -equivalence, this last claim follows from essential surjectivity. \square

Recall that by replacing sdSet^G with the simpler category ssSet in Definition 5.1 one recovers the Segal spaces of [22]. The following roughly summarizes (and slightly refines) [22, Lemma 5.8, Theorem 6.2, Proposition 11.1 and Lemma 11.10] in our setup.

Adapting Notation 4.12, for fixed $X \in \text{ssSet}$ we write $X(-): \text{sSet}^{\text{op}} \rightarrow \text{sSet}$ for the limit-preserving functor such that $X(\Delta[m]) = X(m)$.

Proposition 5.26 *Let $X \in \text{ssSet}$ be a Segal space. Then*

- (i) *equivalences of X define a subset of connected components $X^h(1) \subseteq X(1)$;*
- (ii) *the pullbacks*

$$(5.27) \quad \begin{array}{ccc} X^h(m) & \xrightarrow{\quad\quad\quad} & X(m) \\ \downarrow & & \downarrow \\ X^h(1) \times_{X(0)} \cdots \times_{X(0)} X^h(1) & \xrightarrow{\quad\quad\quad} & X(1) \times_{X(0)} \cdots \times_{X(0)} X(1) \end{array}$$

define a Segal space $X^h \subseteq X$, consisting of a union of connected components at each level;

- (iii) *the maps $X^h(2) \xrightarrow{(d_2, d_1)} X^h(\Lambda^0[2])$ and $X^h(2) \xrightarrow{(d_0, d_1)} X^h(\Lambda^2[2])$ are trivial fibrations;*
- (iv) *each map $X(J^m) \rightarrow X(\Delta[m]) = X(m)$ factors through a weak equivalence $X(J^m) \xrightarrow{\sim} X^h(m)$.*

Proof For (i), given $f: x \rightarrow y$ in $X_0(1)$ one has that $[f]$ has a left inverse if and only if there exists p as in the left diagram below, but, for any path H between f and f' in $X(1)$, there is a lift in the right diagram:

$$(5.28) \quad \begin{array}{ccc} & X(2) & \\ & \downarrow (d_2, d_1) & \\ \{0\} & \xrightarrow{p} & X(2) \\ & \uparrow & \downarrow (d_2, d_1) \\ \{0\} & \xrightarrow{(f, s_0(x))} & X(1) \times_{X(0)} X(1) \end{array} \quad \begin{array}{ccc} \{0\} & \xrightarrow{p} & X(2) \\ \sim \downarrow & \nearrow & \downarrow (d_2, d_1) \\ \Delta[1] & \xrightarrow{(H, s_0 d_1(H))} & X(1) \times_{X(0)} X(1) \end{array}$$

showing that f' is also left-invertible. The situation for right inverses is identical, thus (i) follows.

For (ii), that X^h is closed under the simplicial operators follows since equivalences are closed under composition. Moreover, this closure under composition further gives pullbacks

$$\begin{array}{ccc}
 X^h(\text{sk}_1 \Delta[m]) & \longrightarrow & X(\text{sk}_1 \Delta[m]) \\
 \downarrow & & \downarrow \\
 X^h(1) \times_{X(0)} \cdots \times_{X(0)} X^h(1) & \longrightarrow & X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)
 \end{array}$$

so that the pullbacks in (5.27) are equivalent to the left pullbacks below, and cellular induction yields the more general right pullbacks, for all $K \in \text{sSet}$,

$$(5.29) \quad \begin{array}{ccc}
 X^h(\Delta[m]) & \longrightarrow & X(\Delta[m]) \\
 \downarrow & \lrcorner & \downarrow \\
 X^h(\text{sk}_1 \Delta[m]) & \longrightarrow & X(\text{sk}_1 \Delta[m])
 \end{array} \quad \begin{array}{ccc}
 X^h(K) & \longrightarrow & X(K) \\
 \downarrow & \lrcorner & \downarrow \\
 X^h(\text{sk}_1 K) & \longrightarrow & X(\text{sk}_1 K)
 \end{array}$$

Since $\text{sk}_1(\partial\Delta[m]) = \text{sk}_1 \Delta[m]$ if $m \geq 2$, it follows that the maps $X^h(m) \rightarrow X^h(\partial\Delta[m])$ for $m \geq 2$ are Kan fibrations, and since the composite $X^h(1) \rightarrow X(1) \rightarrow X(0) \times X(0)$ is clearly a Kan fibration, X^h is indeed Reedy fibrant. The Segal condition for X^h is obvious from the pullback (5.27).

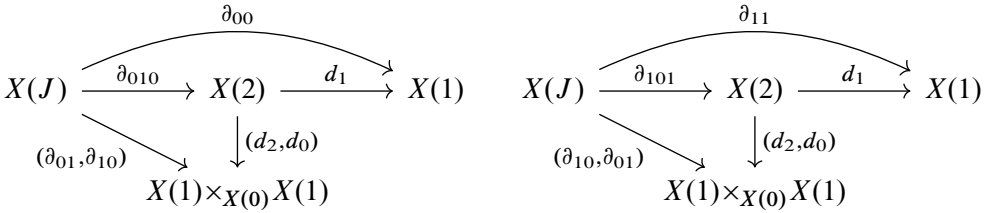
For (iii), it suffices by symmetry to establish the first claim. It is then enough to show that for any choice of section in the following diagram the top composite is a Kan equivalence:

$$\begin{array}{ccccc}
 & & \overset{\sim}{\curvearrowright} & & \\
 X^h(1) \times_{X_0} X^h(1) & \xleftarrow{(d_2, d_0)} & X^h(2) & \xrightarrow{(d_2, d_1)} & X^h(1) \times_{X_0} X^h(1) \\
 & \searrow (id, d_0) & & \swarrow (id, d_0) & \\
 & & X^h(1) \times X(0) & &
 \end{array}$$

But this composite is a map of Kan fibrations over $X^h(1) \times X(0)$ with the map between the fibers over $(f: x \rightarrow y, z)$ computing the map $(-)\circ f: X^h(y; z) \rightarrow X^h(x; z)$, which is a Kan equivalence since $f \in X_0^h(1)$ is an equivalence. Thus the composite is a Kan equivalence, establishing (iii).

Lastly, for (iv) note first that (iii) says that X^h is local with respect to the outer horn inclusions $\Lambda^0[2] \rightarrow \Delta[2]$ and $\Lambda^2[2] \rightarrow \Delta[2]$, and hence by Remarks 3.28 and 3.29 the maps $X^h(J^m) \rightarrow X^h(m)$ are Kan equivalences. It remains to show $X^h(J^m) = X(J^m)$. We first focus on the case of $J = J^1$, for which we consider the following diagrams,

where maps out of $X(J)$ are labeled as $\partial_{\underline{a}}$ for \underline{a} the associated simplex of J (which is a string on $\{0, 1\}$):



Since ∂_{00} and ∂_{11} are degenerate, for any $x \in X_0(J)$ one has that $\partial_{01}(x)$ and $\partial_{10}(x)$ are inverse equivalences, as exhibited by $\partial_{010}(x)$ and $\partial_{101}(x)$ (see (5.28)). This shows that ∂_{01} and ∂_{10} factor through $X^h(1)$, so that $X(J) \rightarrow X(\text{sk}_1 J)$ factors through $X^h(\text{sk}_1 J)$. But, since all 1–simplices of J^m are in the image of some map $J \rightarrow J^m$, the map $X(J^m) \rightarrow X(\text{sk}_1 J^m)$ likewise factors through $X^h(\text{sk}_1 J^m)$ for any m . The pullback (5.29) now finishes the proof. \square

Remark 5.30 The proof of (ii) shows that the inclusion $X^h \rightarrow X$ is a Reedy fibration.

Remark 5.31 Writing $J \xrightarrow{ij} J^m$ for the map sending 0 to i and 1 to j , Proposition 5.26(iv) combined with the Segal condition for X^h and Proposition 5.26(iii) shows that the maps

$$X(J^m) \xrightarrow{(01,12,\dots,(m-1)m)} X(J) \times_{X(0)} X(J) \times_{X(0)} \cdots \times_{X(0)} X(J),$$

$$X(J^2) \xrightarrow{(01,02)} X(J) \times_{X(0)} X(J), \quad X(J^2) \xrightarrow{(12,02)} X(J) \times_{X(0)} X(J),$$

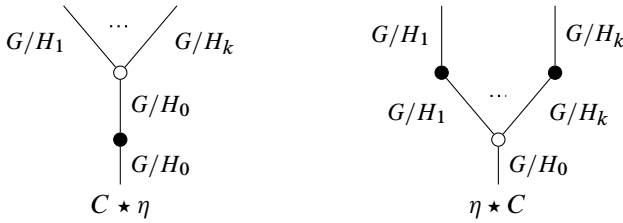
are trivial Kan fibrations.

5.2 Rezk completion and fibrant Segal operads

To prove the characterization of complete/joint equivalences in Theorem 5.48 we will need to establish some technical properties of the completion $X \rightarrow \tilde{X}$ of a dendroidal Segal space $X \in \text{sdSet}^G$, which are given by Propositions 5.35 and 5.41.

We first need to discuss some preliminary constructions. We will make use of a decomposition of the tensor product $[1] \otimes C$, where $[1]$ is the 1–simplex regarded as a G –trivial G –tree and C is a G –corolla (see Notation 5.5). Adapting the discussion in Example 3.9, $[1] \otimes C$ is the union of two maximal G –subtrees $C \star \eta$ and $\eta \star C$, whose orbital representations are depicted below. Explicitly, and noting that the edges (ie η –dendrices) of $\Omega[1] \otimes \Omega[C]$ are $\{0, 1\} \times E(C)$, the tree $C \star \eta$ (resp. $\eta \star C$) has

the edges (i, e) such that $i = 0$ or $e \in E(C)$ is a root (resp. $i = 1$ or $e \in E(C)$ is a leaf), where we recall that in our convention 0 is the leaf of $[1]$ while 1 is the root:



Moreover, just as in (3.11), $C \star \eta$ and $\eta \star C$ have a common orbital *inner* face, which we denote simply by C (since this face is canonically isomorphic to the original C), leading to a decomposition

$$(5.32) \quad \Omega[1] \otimes \Omega[C] \simeq \Omega[C \star \eta] \amalg_{\Omega[C]} \Omega[\eta \star C].$$

Note that this holds even if $k = 0$, which is an exceptional case since then $[1] \otimes C = C \star \eta$.

Remark 5.33 Equation (5.32) is the (equivariant) dendroidal generalization of the familiar decomposition

$$\Delta[1] \times \Delta[1] \simeq \Delta[2] \amalg_{\Delta[1]} \Delta[2].$$

Proposition 5.35 will make use of the cube diagram below, where $\partial^l \Omega[C] = \partial^l \Omega[C] \amalg \partial^r \Omega[C]$ is the decomposition of the edges of C into leaves and roots. All maps are the inclusions determined by the decomposition (5.32), but it seems worthwhile to be explicit regarding how $\partial^l \Omega[C]$ and $\partial^r \Omega[C]$ include into $\Omega[1] \otimes \Omega[C]$. An edge l in $\partial^l \Omega[C]$ includes as $(0, l)$ while an edge r in $\partial^r \Omega[C]$ includes as $(1, r)$:

$$(5.34) \quad \begin{array}{ccc} \partial \Omega[C] & \xrightarrow{\quad} & \Omega[1] \otimes \partial^l \Omega[C] \amalg \partial^r \Omega[C] \\ \downarrow & \searrow & \downarrow \quad \searrow \\ \partial^l \Omega[C] \amalg \Omega[1] \otimes \partial^r \Omega[C] & \xrightarrow{\quad} & \Omega[1] \otimes \partial \Omega[C] \\ \downarrow & \downarrow & \downarrow \\ \Omega[C] & \xrightarrow{\quad} & \Omega[\eta \star C] \\ \downarrow & \searrow & \downarrow \quad \searrow \\ \Omega[C \star \eta] & \xrightarrow{\quad} & \Omega[1] \otimes \Omega[C] \end{array}$$

Moreover, note that this is a projective cofibrant cube (recall that $\chi: (0 \rightarrow 1)^{\times n} \rightarrow \text{dSet}^G$ is projective cofibrant if the maps $\text{colim}_{\underline{j} < \underline{i}} \chi_{\underline{j}} \rightarrow \chi_{\underline{i}}$ are normal monomorphisms for each $\underline{i} \in (0 \rightarrow 1)^{\times n}$). Indeed, since all maps in (5.34) are monomorphisms of presheaves, we can regard all objects as subpresheaves of $\Omega[1] \otimes \Omega[C]$, so that projective

cofibrancy is the same as (5.34) being strongly cartesian, ie the four objects $\partial\Omega[C]$, $\Omega[C]$, $\Omega[1] \otimes \partial^l \Omega[C] \amalg \partial^r \Omega[C]$ and $\partial^l \Omega[C] \amalg \Omega[1] \otimes \partial^r \Omega[C]$ being the intersection of the objects they map to.

Lastly, note that both of the horizontal faces of (5.34) are pushout squares.

Proposition 5.35 *Let $X \in \text{sdSet}^G$ be a dendroidal Segal space. Then the map $X \rightarrow X^J$ is a DK equivalence.*

Proof Note first that for any $T \in \Omega_G$ the map $X^J(\Omega[T]) \rightarrow X^{\Omega[1]}(\Omega[T])$ can be rewritten as $(X^{\Omega[T]})(J) \rightarrow (X^{\Omega[T]})(\Delta[1])$, where $X^{\Omega[T]} \in \text{ssSet}$ is defined by $X^{\Omega[T]}(m) = X(\Omega[m] \otimes \Omega[T])$, with $[m]$ given the G -trivial action (ie we are restricting Notation 4.12). Since $X^{\Omega[T]}$ is a (simplicial) Segal space (by adjunction together with [21, Proposition 7.25 and Theorem 7.1]), Proposition 5.26(iv) says that this map is a weak equivalence onto a subset of components, ie a *homotopy monomorphism*. Hence, for any G -corolla $C \simeq C_{\amalg_i} H_0/H_i$, the horizontal maps in the right square below are homotopy monomorphisms:

$$(5.36) \quad \begin{array}{ccccc} X(\Omega[C]) & \longrightarrow & X^J(\Omega[C]) & \longrightarrow & X^{\Omega[1]}(\Omega[C]) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{0 \leq i \leq k} X(\eta)^{H_i} & \longrightarrow & \prod_{0 \leq i \leq k} (X^J(\eta))^{H_i} & \longrightarrow & \prod_{0 \leq i \leq k} (X^{\Omega[1]}(\eta))^{H_i} \end{array}$$

Since fully faithfulness of $X \rightarrow X^J$ is the statement that the leftmost square in (5.36) induces weak equivalences on fibers, it suffices to show that so does the composite square.

Now note that $X^{\Omega[1]}(\Omega[C]) = X(\Omega[1] \otimes \Omega[C])$, so that (5.34) induces the cube (5.37) below with dual properties: this is an injective fibrant cube whose horizontal faces are pullback squares. Moreover, we will find it convenient to slightly repackage these properties. Regarding the top and bottom faces of (5.37) as objects in the category of square diagrams, the vertical maps in (5.37) can then be collectively regarded as a projective fibration between projective fibrant pullback squares:

$$(5.37) \quad \begin{array}{ccccc} X(\Omega[1] \otimes \Omega[C]) & \longrightarrow & & \twoheadrightarrow & X(\Omega[\eta \star C]) \\ \downarrow & \searrow & & & \downarrow \\ & X(\Omega[C \star \eta]) & \longrightarrow & & X(\Omega[C]) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i X([1])^{H_i} & \longrightarrow & & \twoheadrightarrow & (\prod_{i \neq 0} X([1])^{H_i}) \times X(\eta)^{H_0} \\ \downarrow & \searrow & & & \downarrow \\ & (\prod_{i \neq 0} X(\eta)^{H_i}) \times X([1])^{H_0} & \longrightarrow & & \prod_i X(\eta)^{H_i} \end{array}$$

Noting next that fibers (in the category of square diagrams) of a fibration between pullback squares are fibrant pullback squares, it follows that the fibers of the top left vertical map in (5.37) are homotopy pullbacks of the fibers of the remaining vertical maps. And since this top left vertical map is the right vertical map in (5.36), the desired claim that the total diagram in (5.36) induces equivalences on fibers will follow provided that the same holds for the following total diagrams (which are also the composites of (5.36) with the left and back faces of (5.37); compare with [22, Lemma 12.4]), where the horizontal maps are the obvious degeneracies:

$$\begin{array}{ccc}
 X(\Omega[C]) & \xrightarrow{s} & X(\Omega[C \star \eta]) \\
 \parallel & & \downarrow \sim \\
 X(\Omega[C]) & \xrightarrow{s} & X(\Omega[C]) \times_{X(\eta)^{H_0}} X([1])^{H_0} \\
 \downarrow & & \downarrow \\
 \prod_i X(\eta)^{H_i} & \xrightarrow{s} & (\prod_{i \neq 0} X(\eta)^{H_i}) \times X([1])^{H_0} \\
 \\
 X(\Omega[C]) & \xrightarrow{s} & X(\Omega[\eta \star C]) \\
 \parallel & & \downarrow \sim \\
 X(\Omega[C]) & \xrightarrow{s} & \prod_{i \neq 0} X([1])^{H_i} \times_{\prod_{i \neq 0} X(\eta)^{H_i}} X(\Omega[C]) \\
 \downarrow & & \downarrow \\
 \prod_i X(\eta)^{H_i} & \xrightarrow{s} & (\prod_{i \neq 0} X([1])^{H_i}) \times X(\eta)^{H_0}
 \end{array}$$

But this is clear from the fact that the top right vertical maps in these diagrams are trivial Kan fibrations (by the Segal condition for X) and the fact that the bottom squares are pullback squares.

Lastly, to check essential surjectivity, since G acts trivially on J , equation (5.12) yields $\iota_G^* \text{ho}(X^J)(G/H) = \text{ho}(\iota^*(X^J)^H) = \text{ho}(\iota^*(X^H)^J)$, where $(-)^H$ denotes fixed points, so we reduce to the case of $X \in \text{ssSet}$ a (simplicial) Segal space. Since J is a contractible Kan complex, one has a map $F: J \times J \rightarrow \{0\} \times J$ such that $F|_{\{0\} \times J} = \text{id}_{\{0\} \times J}$ and $F|_{\{1\} \times J} = (0, 0)$. But noting that $X(\{0\} \times J) \rightarrow X(J \times J)$ can be rewritten as $X^J(0) \rightarrow X^J(J)$, one has that one of the composites below is the identity on $X^J(0)$ while the other factors through $X(0) \rightarrow X^J(0)$

$$X^J(0) \xrightarrow{X(F)} X^J(J) \rightarrow X^J(1) \rightrightarrows X^J(0),$$

showing that any object in $(X^J)_0(\eta)$ is equivalent to one in the image of $X_0(\eta) \rightarrow (X^J)_0(\eta)$. \square

Definition 5.38 Two maps $f, f': X \rightrightarrows Y$ between dendroidal Segal spaces are called J -homotopic, written $f \sim_J f'$, if there is a homotopy H such that the two composites $X \xrightarrow{H} Y^J \rightrightarrows Y$ are f and f' .

Further, a map $f: X \rightarrow Y$ of dendroidal Segal spaces is called a J -homotopy equivalence if there exists $g: Y \rightarrow X$ such that $gf \sim_J \text{id}_X$ and $fg \sim_J \text{id}_Y$.

Remark 5.39 For $f \sim_J f'$, Proposition 5.35 and 2-out-of-3 applied to $X \xrightarrow{H} Y^J \rightrightarrows Y$ imply that f is a DK equivalence if and only if f' is. Thus, by 2-out-of-6 for DK equivalences (see Corollary 5.25), J -homotopy equivalences are DK equivalences.

Remark 5.40 Let X be a dendroidal Segal space. All simplicial operators $X^{J^m} \rightarrow X^{J^{m'}}$ (see Notation 4.15) are induced by equivalences of groupoids $[\widetilde{m'}] \rightarrow [\widetilde{m}]$, implying that the operators $X^{J^m} \rightarrow X^{J^{m'}}$ are J -homotopy equivalences and hence also DK equivalences.

Proposition 5.41 Let $X \in \text{sdSet}^G$ be a dendroidal Segal space. Then there is a complete dendroidal Segal space \widetilde{X} and complete/joint equivalence $X \rightarrow \widetilde{X}$ such that

- (i) $X \rightarrow \widetilde{X}$ is a DK equivalence;
- (ii) $X_0(\eta) \rightarrow \widetilde{X}_0(\eta)$ is an isomorphism.

Our proof will adapt the construction of the completion functor in [22, Section 14].

Firstly, let $X^{J^\bullet} \in (\text{sdSet}^G)^{\Delta^{\text{op}}} = \text{ssdSet}^G$ be the object whose m^{th} level is X^{J^m} . Then, writing $\delta^*: \text{ssdSet}^G \rightarrow \text{sdSet}^G$ for the diagonal functor in the two simplicial directions (adapting the functor in Section 4.1), we set $\widetilde{X} = \delta^*(X^{J^\bullet})$ with the natural map $X \rightarrow \widetilde{X}$ induced by degeneracies (see Proposition 4.5)(iv).

Remark 5.42 For fixed n and $K \in \text{sSet}$, and regarding $X_n^{J^\bullet} \in \text{sdSet}^G$, one can translate between the $(-)_(-)$ and $(-)^{(-)}$ notations in Notation 4.12 via

$$(X_n^{J^\bullet})_K = (X_n^{J^\bullet})_{(\text{colim}_{\Delta[k] \rightarrow K} \Delta[k])} = \lim_{\Delta[k] \rightarrow K} X_n^{J^k} = X_n^{(\text{colim}_{\Delta[k] \rightarrow K} J^k)},$$

where $K \simeq \text{colim}_{\Delta[k] \rightarrow K} \Delta[k]$ is the standard decomposition of $K \in \text{sSet}$ as a colimit of simplices. In the special cases of K being either $\partial\Delta[m]$ or $\text{Sc}[m]$, one has (recall

that L_m is the latching object)

$$\left(\operatorname{colim}_{\Delta[k] \rightarrow \partial \Delta[m]} J^k \right) \simeq L_m J^\bullet, \quad \left(\operatorname{colim}_{\Delta[k] \rightarrow \operatorname{Sc}[m]} J^k \right) \simeq J \amalg_{\Delta[0]} \cdots \amalg_{\Delta[0]} J.$$

Proof of Proposition 5.41 Most of the proof will be spent showing that $\tilde{X} = \delta^*(X^{J^\bullet}) \in \operatorname{sdSet}^G$ is dendroidal Reedy fibrant by establishing the lifting conditions needed to apply Corollary 4.7.

Since J^\bullet is a Reedy cofibrant cosimplicial object (see Remark 4.16), it follows from [21, Proposition 7.25] that the maps (where \square^\otimes denotes the pushout product with regard to \otimes)

$$X((L_m J^\bullet \rightarrow J^m) \square^\otimes (\partial \Omega[T] \rightarrow \Omega[T]))$$

are Kan fibrations. Remark 5.42 then implies that $X^{J^\bullet} \in \operatorname{ssdSet}^G$ has the right lifting property against all maps

$$(5.43) \quad (\Lambda^i[n] \rightarrow \Delta[n]) \square (\partial \Delta[m] \rightarrow \Delta[m]) \square (\partial \Omega[T] \rightarrow \Omega[T]),$$

$$n \geq 1, n \geq i \geq 0, m \geq 0, T \in \Omega_G.$$

We next claim that $X^{J^\bullet} \in \operatorname{ssdSet}^G$ also has the lifting property against the two sets of maps

$$(5.44) \quad (\partial \Delta[n] \rightarrow \Delta[n]) \square (\Lambda^j[m] \rightarrow \Delta[m]) \times \Omega[G/H \cdot \eta],$$

$$n \geq 0, m \geq 2, m \geq j \geq 0, H \leq G$$

and

$$(5.45) \quad (\partial \Delta[n] \rightarrow \Delta[n]) \square (\Lambda^j[m] \rightarrow \Delta[m]) \square (\partial \Omega[T] \rightarrow \Omega[T]),$$

$$n \geq 0, m \geq 1, m \geq j \geq 0, T \in \Omega_G \setminus \{G/H \cdot \eta\}.$$

Note that, just as in Remark 4.14, one can replace the set of maps $\Lambda^j[m] \rightarrow \Delta[m]$, $\partial \Omega[T] \rightarrow \Omega[T]$ appearing in (5.44) and (5.45) with any set which has the same hypersaturation.

For (5.44), by Remarks 3.26 and 3.28 one needs only consider the inclusions $\operatorname{Sc}[m] \rightarrow \Delta[m]$ and $\Lambda^0[2] \rightarrow \Delta[2]$ and $\Lambda^2[2] \rightarrow \Delta[2]$. But the claimed lifting condition against (5.44) then amounts to Remark 5.31 applied to each of the simplicial Segal spaces $\iota^*(X^H)$, $H \leq G$ (also, see Remark 5.42). For (5.45), by Remarks 3.25 and 3.27 one needs only show that there are lifts against the maps

$$(\partial \Delta[n] \rightarrow \Delta[n]) \square (\Delta[0] \rightarrow \Delta[m]) \square \left(\prod_{e \in \mathbf{E}(T)} \Omega[\eta] \rightarrow \Omega[T] \right).$$

But lifts against these maps are equivalent to the claim that in the square

$$(5.46) \quad \begin{array}{ccc} X^{J^m}(\Omega[T]) & \longrightarrow & X(\Omega[T]) \\ \downarrow & & \downarrow \\ \prod_{[e_i] \in \mathbf{E}_G(T)} (X^{J^m}(\eta))^{H_i} & \longrightarrow & \prod_{[e_i] \in \mathbf{E}_G(T)} X(\eta)^{H_i} \end{array}$$

the map from $X^{J^m}(\Omega[T])$ to the pullback of the other terms is a trivial Kan fibration. That it is a Kan fibration follows from the lifting condition against (5.43), and that it is a Kan equivalence follows since Remarks 5.40 and 5.16 imply (5.46) induces Kan equivalences between fibers.

We finally show that $\tilde{X} = \delta^*(X^{J^\bullet})$ is dendroidal fibrant, ie that

$$(\tilde{X}(\Omega[T]) \rightarrow \tilde{X}(\partial\Omega[T])) = \delta^*(X^{J^\bullet}(\Omega[T]) \rightarrow X^{J^\bullet}(\partial\Omega[T]))$$

is a Kan fibration for any $T \in \Omega_G$. When $T = G/H \cdot \eta$ the target is the terminal object so this follows from the lifting conditions against (5.43)–(5.44) and the second “moreover” condition in Corollary 4.7. For $T \neq G/H \cdot \eta$ this follows from the lifting conditions against (5.43) and (5.45) and the first “moreover” condition in Corollary 4.7.

To see that \tilde{X} is a complete Segal space, note that by Proposition 4.5(iv) the natural map $X_0^{J^\bullet} \rightarrow \delta^*(X^{J^\bullet}) = \tilde{X}$ is a simplicial equivalence, so that this map is an equivalence between fibrant objects in the dendroidal Reedy model structure. Thus, since $X_0^{J^\bullet}$ is a complete Segal space by Remark 4.31, Lemma 3.2.1 of [16] implies that so is \tilde{X} (alternatively, by 2-out-of-3 the completion conditions in (4.25) applied to $X_0^{J^\bullet}$ readily imply those same conditions for \tilde{X}).

We now address conditions (i)–(ii) in the statement. The natural map $X = X^{J^0} \rightarrow \delta^*(X^{J^\bullet}) = \tilde{X}$ clearly satisfies (ii). For the remaining claim in (i) that this is a DK equivalence, fully faithfulness is the claim that this map induces equivalences on the fibers over $\prod_{[e_i]} (X(\eta))^{H_i}$ for each $T \in \Omega_G$. But the fibers of $\tilde{X} = \delta^*(X^{J^\bullet})$ are diagonals of the fibers of X^{J^\bullet} over $\prod_{[e_i]} (X^{J^\bullet}(\eta))^{H_i}$ for each $T \in \Omega_G$, and the lifting conditions against (5.43) and (5.45) imply that these fibers are joint Reedy fibrant in \mathbf{ssSet} , so fully faithfulness now follows from Proposition 4.5(iv) applied to these fibers. Essential surjectivity is trivial since $X \rightarrow \tilde{X}$ is the identity on objects. \square

Remark 5.47 A notable difference between the proof of Proposition 5.41 and the arguments in [22, Section 14] being adapted is that [22] did not establish the analogue of our main fibrancy claim when $X \in \mathbf{ssSet}$ is a Segal space, namely that $\delta^*(X^{J^\bullet})$

is horizontal Reedy fibrant in ssSet . Instead, Rezk [22] defines \tilde{X} as the horizontal fibrant replacement of $\delta^*(X^{J^\bullet})$ and the analogue of (5.46) is used to conclude that the fibers of $\delta^*(X^{J^\bullet})$ are homotopy fibers, and thus preserved by the replacement. As such, the lifting conditions against (5.44) are technically not essential for the proof, serving only to obtain the neat observation that $\delta^*(X^{J^\bullet})$ need not be replaced.

Theorem 5.48 *A map $f: X \rightarrow Y$ of dendroidal Segal spaces is a complete/joint equivalence if and only if it is a DK equivalence.*

Proof By Proposition 5.41(i) we can assume that X and Y are complete/joint fibrant, so that by Proposition 4.1(ii) complete/joint equivalences coincide with simplicial equivalences. But then by Corollary 5.18 it remains only to show that if f is a DK equivalence then the maps $X(\eta)^H \rightarrow Y(\eta)^H$ for $H \leq G$ are Kan equivalences. Since the induced maps $\iota^*(X^H) \rightarrow \iota^*(Y^H)$ in ssSet are DK equivalences of simplicial Segal spaces, this last claim is immediate from the simplicial analogue [22, Theorem 7.7], but we nonetheless include a full argument for the sake of completeness.

As such, assuming that $f: X \rightarrow Y$ in ssSet is a DK equivalence of complete simplicial Segal spaces, it remains to show $X(0) \rightarrow Y(0)$ is a Kan equivalence. The completion condition states that $Z(J) \xrightarrow{\sim} Z(0)$ for $Z = X, Y$ are Kan equivalences, so that the fibers of the left diagram below are weakly equivalent to the homotopy fibers of the right diagram, ie the loop space of $Z(0)$ at z :

$$(5.49) \quad \begin{array}{ccc} & Z(J) & Z(0) \\ & \downarrow & \downarrow \\ \Delta[0] & \xrightarrow{(z,z)} Z(0) \times X(0) & \Delta[0] \xrightarrow{(z,z)} Z(0) \times Z(0) \end{array}$$

Therefore, since $Z(J) \rightarrow Z(1)$ is a homotopy monomorphism (see Proposition 5.26(i), (iv)), fully faithfulness implies that $X(0) \rightarrow Y(0)$ induces isomorphisms on homotopy groups. Injectivity of $X(0) \rightarrow Y(0)$ on components is similar ($z, z' \in Z_0(0)$ are in the same component if and only if the (z, z') fiber in (5.49) is nonempty) while essential surjectivity implies surjectivity on components (the equivalences $Z(0) \xrightarrow{\sim} Z(J) \xrightarrow{\sim} Z^h(1)$ imply both vertex maps $Z^h(1) \rightrightarrows Z(0)$ are equivalences, and hence $z, z' \in Z_0(0)$ are isomorphic in $\text{ho}(Z)$ if and only if they are connected in $Z(0)$).

It follows that $X(0) \rightarrow Y(0)$ is indeed a Kan equivalence, as required. □

Remark 5.50 It is clear from the construction of the completion \tilde{X} in Proposition 5.41 that the map $X \rightarrow \tilde{X}$ is a monomorphism. However, it seems unlikely that this is a

normal monomorphism (unless X is assumed normal). To address this while preserving the properties in [Proposition 5.41](#), note that the identification $X_0(\eta) = \tilde{X}_0(\eta)$ means that one can perform a “cofibration followed by trivial fibration” factorization

$$X \twoheadrightarrow \hat{X} \twoheadrightarrow \tilde{X}$$

in the dendroidal Reedy model structure in such a way that $X_0(\eta) = \hat{X}_0(\eta) = \tilde{X}_0(\eta)$. Indeed, this follows by performing the small object argument against [\(4.18\)](#) with the maps $\emptyset \rightarrow \Delta[0] \times \Omega[G/H \cdot \eta]$, $H \leq G$ omitted (note that $\hat{X} \rightarrow \tilde{X}$ will still have the lifting property against the omitted maps). The claims that \hat{X} is complete and $X \rightarrow \hat{X}$ is a DK equivalence are inherited from the analogous properties of \tilde{X} .

Corollary 5.51 *A preoperad $X \in \text{PreOp}^G$ is fibrant (in the model structure from [Theorem 4.39](#)) if and only if γ^*X is fibrant in the dendroidal Segal space model structure on sdSet^G .*

Proof We start with the “only if” direction. Recall that γ^*X is a dendroidal Segal space if and only if it has the right lifting property against the maps of the form

$$(5.52) \quad (\Lambda^i[n] \rightarrow \Delta[n]) \square (\partial\Omega[T] \rightarrow \Omega[T]), \quad (\partial\Delta[n] \rightarrow \Delta[n]) \square (\text{Sc}[T] \rightarrow \Omega[T]).$$

With the exception of the first type of maps when $T = G \cdot_H \eta$, in which case the lifting condition against γ^*X is automatic since $\gamma^*X(\eta)$ is discrete, all other maps induce isomorphisms at the η -level, so that by [Remark 4.33](#) applying $\gamma_!$ to these maps yields trivial cofibrations in PreOp^G . Thus, if $X \in \text{PreOp}^G$ is fibrant, an adjunction argument shows that $\gamma^*(X)$ indeed has the lifting property against all maps [\(5.52\)](#), ie that $\gamma^*(X)$ is a dendroidal Segal space.

For the “if” direction, we form the completion $\gamma^*X \rightarrow \hat{X}$ described in [Remark 5.50](#). Then $\gamma_*\hat{X} \in \text{PreOp}^G$ is fibrant by [Theorem 4.41](#) and the adjoint map $X \rightarrow \gamma_*\hat{X}$ has the following properties:

- (i) it is a normal monomorphism (this is inherited from $\gamma^*X \rightarrow \hat{X}$; see the characterization of normal monomorphisms given before [Theorem 2.31](#));
- (ii) it is an isomorphism at the η -level;
- (iii) it is a DK equivalence when regarded as a map in sdSet^G (since, by [Remark 5.14](#), $\gamma^*\gamma_*\hat{X} \rightarrow \hat{X}$ is tautologically a DK equivalence);
- (iv) by [Corollary 5.18\(ii\)](#) it is hence a simplicial equivalence and thus a trivial dendroidal Reedy cofibration when regarded as a map in sdSet^G .

But then the hypothesis that $\gamma^* X$ is a dendroidal Segal space yields a lift

$$\begin{array}{ccc} \gamma^* X & \xlongequal{\quad} & \gamma^* X \\ \downarrow & \nearrow \text{---} & \\ \gamma^* \gamma_* \widehat{X} & & \end{array}$$

showing that X is a retract of $\gamma_* \widehat{X}$ and finishing the proof. □

Corollary 5.53 *The fibrant objects of PreOp^G are the equivariant Segal operads X (see Definition 4.40) such that $\gamma^* X$ is dendroidal Reedy fibrant in sdSet^G .*

Proof This simply combines Corollary 5.51 with the characterization in Proposition 5.3. □

Remark 5.54 For any dendroidal Segal space $X \in \text{sdSet}^G$ one hence has complete equivalences

$$\gamma_* X \rightarrow X \rightarrow \widetilde{X},$$

where $\gamma_* X$ is a fibrant preoperad and \widetilde{X} is a complete dendroidal Segal space.

6 Analogous results for indexing systems

Just as in [21, Section 9], we dedicate our final section to outlining the generalizations of our results parametrized by the *indexing systems* of Blumberg and Hill [6]. Or, more precisely, we will work with the *weak indexing systems* of [21, Section 9; 7, Section 4.4], which are a slight generalization of indexing systems, and were also independently identified by Gutierrez and White [13].

We begin by recalling the key notion of sieve.

Definition 6.1 A *sieve* of a category \mathcal{C} is a full subcategory $\mathcal{S} \subseteq \mathcal{C}$ such that for any arrow $c \rightarrow s$ in \mathcal{C} such that $s \in \mathcal{S}$, we have also $c \in \mathcal{S}$.

Note that a sieve $\mathcal{S} \subseteq \mathcal{C}$ determines a presheaf $\delta_{\mathcal{S}} \in \text{Set}^{\mathcal{C}^{\text{op}}}$ via $\delta_{\mathcal{S}}(c) = *$ if $c \in \mathcal{S}$ and $\delta_{\mathcal{S}}(c) = \emptyset$ if $c \notin \mathcal{S}$. In fact, there is a clear bijection between sieves and such *characteristic presheaves*, ie presheaves taking only the values $*$ and \emptyset , and we will hence blur the distinction between the two concepts.

Sieves are prevalent in equivariant homotopy theory. Indeed, families \mathcal{F} of subgroups of G are effectively the same as sieves $O_{\mathcal{F}} \subseteq O_G$ of the orbit category O_G (formed by the G -sets G/H).

Weak indexing systems can then be thought of as the operadic analogue of families. In particular, they are described by certain sieves $\Omega_{\mathcal{F}} \subseteq \Omega_G$, though additional conditions are needed to ensure compatibility with the operadic composition and unit. In the following, we abbreviate $\delta_{\mathcal{F}} = \delta_{\Omega_{\mathcal{F}}}$ and, for each G -vertex v of $T \in \Omega_G$, we write $T_v \hookrightarrow T$ for the orbital outer face whose only G -vertex is v .

Definition 6.2 A weak indexing system is a full subcategory $\Omega_{\mathcal{F}} \subseteq \Omega_G$ such that

- (i) $\Omega_{\mathcal{F}}$ is a sieve of Ω_G ;
- (ii) for each $T \in \Omega_G$ we have $T \in \Omega_{\mathcal{F}}$ if and only if $T_v \in \Omega_{\mathcal{F}}$ for all $v \in V_G(T)$ or, equivalently, if

$$(6.3) \quad \delta_{\mathcal{F}}(T) = \prod_{v \in V_G(T)} \delta_{\mathcal{F}}(T_v).$$

Remark 6.4 Given (i), condition (ii) can be reinterpreted as combining the following:

- (ii') the characteristic presheaf $\delta_{\mathcal{F}}$ is Segal, ie $\delta_{\mathcal{F}}(T) = \delta_{\mathcal{F}}(\text{Sc}[T])$ for all $T \in \Omega_G$;
- (ii'') $(G/G \cdot \eta) \in \Omega_{\mathcal{F}}$.

Here, (ii'') reflects the existence of units in G -operads, which are encoded by the G -trivial 1-corolla $G/G \cdot [1]$ (note that by the sieve condition (i) we have $(G/G \cdot [1]) \in \Omega_{\mathcal{F}}$ if and only if $(G/G \cdot \eta) \in \Omega_{\mathcal{F}}$).

Similarly, (ii') reflects the composition in G -operads. Indeed, (i) and (ii'') imply that all stick G -trees $G/H \cdot \eta$ are in $\Omega_{\mathcal{F}}$, so that the right-hand side of (6.3) can be reinterpreted as $\delta_{\mathcal{F}}(\text{Sc}[T])$ (more formally, $\delta_{\mathcal{F}}(\text{Sc}[T])$ is defined via an analogue of Notation 4.12, so as to obtain a functor $\delta_{\mathcal{F}}(-): (\text{dSet}_G)^{\text{op}} \rightarrow \text{sSet}$ and by reinterpreting $\text{Sc}[T]$ as an object in dSet_G via applying v_*).

Remark 6.5 The original notion of indexing system in [6, Definition 3.22] is recovered by demanding that all G -trivial n -corollas $G/G \cdot C_n$ are in $\Omega_{\mathcal{F}}$.

Remark 6.6 The \mathcal{F} in the notation $\Omega_{\mathcal{F}}$ is meant to suggest an alternative description of (weak) indexing systems in terms of families of subgroups.

Namely, given a weak indexing system $\Omega_{\mathcal{F}}$ and $n \geq 0$, we let \mathcal{F}_n denote the family of those subgroups of $\Gamma \leq G \times \Sigma_n = G \times \text{Aut}(C_n)$ which are graphs of partial

homomorphisms $G \geq H \rightarrow \Sigma_n$ such that the associated G -corolla $G \cdot_H C_n$ is in $\Omega_{\mathcal{F}}$. \mathcal{F} then stands for the collection $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$.

More generally, for each $U \in \Omega$, we similarly write \mathcal{F}_U for the family of graph subgroups of $G \times \text{Aut}(U)$ encoding partial homomorphisms $G \geq H \rightarrow \text{Aut}(U)$ such that $G \cdot_H U \in \Omega_{\mathcal{F}}$.

The fact that each \mathcal{F}_U is a family is a consequence of the sieve condition (i). On the other hand, (ii) imposes more complex conditions on $\{\mathcal{F}_n\}_{n \geq 0}$ which [6, Definition 3.22] makes explicit.

All results in the paper now extend to the context of a general weak indexing system $\Omega_{\mathcal{F}}$ by essentially replacing Ω_G with $\Omega_{\mathcal{F}}$ throughout. The following are some notable modifications:

- notions in dSet^G discussed in Section 2 such as “ G -normal monomorphism”, “ G -inner horn”, “ G -inner anodyne” and “ G - ∞ -operad” are replaced (by restricting $T \in \Omega_G$ to $T \in \Omega_{\mathcal{F}}$) with “ \mathcal{F} -normal monomorphism”, “ \mathcal{F} -inner horn”, “ \mathcal{F} -inner anodyne” and “ \mathcal{F} - ∞ -operad”;
- the model structure on dSet^G from [21, Theorem 2.1] is replaced with the model structure $\text{dSet}_{\mathcal{F}}^G$ (on the *same* underlying category) from [21, Theorem 2.2], whose cofibrations are the \mathcal{F} -normal monomorphisms and whose fibrant objects are the \mathcal{F} - ∞ -operads;
- genuine dendroidal sets $\text{dSet}_G = \text{Set}^{\Omega_G^{\text{op}}}$ are replaced with \mathcal{F} -dendroidal sets $\text{dSet}_{\mathcal{F}} = \text{Set}^{\Omega_{\mathcal{F}}^{\text{op}}}$.

We briefly outline the main reasons why these substitutions do not affect our proofs.

Firstly, the characteristic edge lemma, Lemma 3.4, extends automatically. Indeed, if $T \in \Omega_{\mathcal{F}}$ the sieve condition for $\Omega_{\mathcal{F}}$ implies that the filtrations produced by the original Lemma 3.4 must necessarily use only \mathcal{F} -inner horn inclusions. Therefore, all results in Section 3.2, most notably Proposition 3.22 concerning hypersaturations, extend to a general weak indexing system $\Omega_{\mathcal{F}}$ via the same proof.

Next, for Section 4.2, one can again consider two different Reedy model structures on sdSet^G . Firstly, using the fact that Δ is Reedy and the model structure $\text{dSet}_{\mathcal{F}}^G$, one obtains an \mathcal{F} -simplicial Reedy model structure on sdSet^G . Secondly, using the fact that $\Omega^{\text{op}} \times G$ is generalized Reedy such that the families $\{\mathcal{F}_U\}$ in Remark 6.6 are Reedy-admissible (see Example A.7) together with the Kan model structure on sSet ,

Theorem A.8 yields a \mathcal{F} -dendroidal Reedy model structure on sdSet^G . Thus, by applying **Proposition 4.1** to combine the two structures, one obtains an \mathcal{F} -joint/ \mathcal{F} -Rezk model structure, which we denote by $\text{sdSet}_{\mathcal{F}}^G$. The remaining discussion in **Section 4.2** then follows through to yield the analogue of **Theorem 4.30**.

Theorem 6.7 *The constant/ 0^{th} level adjunction*

$$c_! : \text{dSet}_{\mathcal{F}}^G \rightleftarrows \text{sdSet}_{\mathcal{F}}^G : (-)_0$$

is a Quillen equivalence.

The modifications for **Section 4.3** are entirely straightforward, with the model structure $\text{sdSet}_{\mathcal{F}}^G$ inducing a model structure $\text{PreOp}_{\mathcal{F}}^G$ via the obvious analogue of **Theorem 4.39**, and yielding the analogue of **Theorem 4.41**.

Theorem 6.8 *The adjunction*

$$\gamma^* : \text{PreOp}_{\mathcal{F}}^G \rightleftarrows \text{sdSet}_{\mathcal{F}}^G : \gamma_*$$

is a Quillen equivalence.

For **Section 5.1**, \mathcal{F} -dendroidal Segal spaces $X \in \text{sdSet}^G$ are defined in the natural way by localizing the \mathcal{F} -dendroidal Reedy model structure against the Segal core inclusions $\text{Sc}[T] \rightarrow \Omega[T]$, $T \in \Omega_{\mathcal{F}}$. The most notable difference is then that in **Notation 5.5** and afterwards one works only with \mathcal{F} -corollas, ie G -corollas $C \in \Omega_{\mathcal{F}}$, and thus only with \mathcal{F} -profiles, thus obtaining a notion of \mathcal{F} -fully faithfulness and of \mathcal{F} -DK equivalence (essential surjectivity needs not be changed due to condition (ii'') in **Remark 6.4** implying that all the stick G -trees $G/H \cdot \eta$ are in $\Omega_{\mathcal{F}}$). Thus, noting that the Segal condition (ii') in **Remark 6.4** ensures that the grafted G -trees $T = C \sqcup_{Ge} D$ in (5.21) are in $\Omega_{\mathcal{F}}$ whenever C and D are \mathcal{F} -corollas, the remaining discussion in **Sections 5.1** and **5.2** generalizes to yield the analogue of **Theorem 5.48**.

Theorem 6.9 *A map of $X \rightarrow Y$ of \mathcal{F} -dendroidal Segal spaces is an \mathcal{F} -complete equivalence if and only if it is an \mathcal{F} -DK equivalence.*

Appendix Equivariant Reedy model structures

In [2], Berger and Moerdijk extend the notion of Reedy category so as to allow for categories \mathbb{R} with nontrivial automorphism groups $\text{Aut}(r)$ for $r \in \mathbb{R}$. For such \mathbb{R} and suitable model category \mathcal{C} they then show that there is a *Reedy model structure* on $\mathcal{C}^{\mathbb{R}}$

defined by modifying the usual characterizations of Reedy cofibrations, weak equivalences and fibrations (see [2, Theorem 1.6] or [Theorem A.8](#) below) to be determined by the $\text{Aut}(r)$ -projective model structures on $\mathcal{C}^{\text{Aut}(r)}$ for each $r \in \mathbb{R}$.

The purpose of this appendix is to show that, under suitable conditions, this can also be done by replacing the $\text{Aut}(r)$ -projective model structures on $\mathcal{C}^{\text{Aut}(r)}$ with the more general $\mathcal{C}_{\mathcal{F}_r}^{\text{Aut}(r)}$ model structures for $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ a nice collection of families of subgroups of each $\text{Aut}(r)$.

To do so, we first need some key notation. For each map $r \rightarrow r'$ in the category \mathbb{R} we will write $\text{Aut}(r \rightarrow r')$ for its automorphism group in the arrow category and write

$$(A.1) \quad \text{Aut}(r) \xleftarrow{\pi_r} \text{Aut}(r \rightarrow r') \xrightarrow{\pi_{r'}} \text{Aut}(r')$$

for the obvious projections. We now introduce our equivariant generalization of the “generalized Reedy categories” of [2, Definition 1.1], the novelty of which is in axiom (iv).

Definition A.2 A *generalized Reedy category structure* on a small category \mathbb{R} consists of wide subcategories \mathbb{R}^+ and \mathbb{R}^- and a degree function $|-|: \text{ob}(\mathbb{R}) \rightarrow \mathbb{N}$ such that

- (i) noninvertible maps in \mathbb{R}^+ (resp. \mathbb{R}^-) raise (lower) degree; isomorphisms preserve degree;
- (ii) $\mathbb{R}^+ \cap \mathbb{R}^- = \text{Iso}(\mathbb{R})$;
- (iii) every map f in \mathbb{R} factors as $f = f^+ \circ f^-$ with $f^+ \in \mathbb{R}^+$ and $f^- \in \mathbb{R}^-$, and this factorization is unique up to isomorphism (meaning that if $f = \tilde{f}^+ \circ \tilde{f}^-$ is another such factorization, then $f^+ = \tilde{f}^+ \circ \phi$ and $\phi \circ f^- = \tilde{f}^-$ for some isomorphism ϕ).

Let $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ be a collection of families of subgroups of the groups $\text{Aut}(r)$. The collection $\{\mathcal{F}_r\}$ is called *Reedy-admissible* if

- (iv) for all maps $r \rightarrow r'$ in \mathbb{R}^- one has $\pi_{r'}(\pi_r^{-1}(H)) \in \mathcal{F}_{r'}$ for all $H \in \mathcal{F}_r$.

We note that condition (iv) above should be thought of as a constraint on the pair $(\mathbb{R}, \{\mathcal{F}_r\})$. The original setup of [2] then deals with the case where $\{\mathcal{F}_r\} = \{\{e\}\}$ is the collection of trivial families. Indeed, our setup recovers the setup in [2], as follows.

Example A.3 When $\{\mathcal{F}_r\} = \{\{e\}\}$, Reedy-admissibility coincides with axiom (iv) in [2, Definition 1.1], stating that if $\theta \circ f^- = f^-$ for some $f^- \in \mathbb{R}^-$ and $\theta \in \text{Iso}(\mathbb{R})$, then θ is an identity.

Example A.4 For any generalized Reedy category \mathbb{R} (ie if \mathbb{R} satisfies (i)–(iii)), the collection $\{\mathcal{F}_{\text{all}}\}$ of the families of all subgroups of $\text{Aut}(r)$ is Reedy-admissible.

Example A.5 Let G be a group and set $\mathbb{R} = G \times (0 \rightarrow 1)$ with $\mathbb{R} = \mathbb{R}^+$ (and thus necessarily $\mathbb{R}^- = \text{Iso}(\mathbb{R})$). Then any pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reedy-admissible.

Similarly, set $\mathbb{R} = G \times (0 \leftarrow 1)$ with $\mathbb{R} = \mathbb{R}^-$. Then a pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reedy-admissible if and only if $\mathcal{F}_0 \supseteq \mathcal{F}_1$.

Example A.6 Letting \mathbb{S} denote any generalized Reedy category in the sense of [2, Definition 1.1] (see Example A.3) and G a group, we set $\mathbb{R} = G \times \mathbb{S}$ with $\mathbb{R}^+ = G \times \mathbb{S}^+$ and $\mathbb{R}^- = G \times \mathbb{S}^-$. Further, for each $s \in \mathbb{S}$ we write \mathcal{F}_s^Γ for the family of G -graph subgroups of $G \times \text{Aut}_{\mathbb{S}}(s)$, ie those subgroups $\Gamma \leq G \times \text{Aut}_{\mathbb{S}}(s)$ which are graphs of partial homomorphisms $G \supseteq H \rightarrow \text{Aut}_{\mathbb{S}}(s)$. We note that G -graph subgroups are also characterized by the condition $\Gamma \cap \text{Aut}_{\mathbb{S}}(s) = \{e\}$.

Reedy-admissibility of the collection $\{\mathcal{F}_s^\Gamma\}$ then follows since, for every map $s \rightarrow s'$ in \mathbb{S}^- , one has that the homomorphism $\pi_s: \text{Aut}_{\mathbb{S}}(s \rightarrow s') \rightarrow \text{Aut}_{\mathbb{S}}(s)$ is injective (we note that this is a restatement of axiom (iv) in [2, Definition 1.1] for \mathbb{S} ; see Example A.3).

Our primary example of interest is obtained by setting $\mathbb{S} = \Omega^{\text{op}}$ in the previous example. Moreover, in this case we are also interested in certain subfamilies $\{\mathcal{F}_U\}_{U \in \Omega} \subseteq \{\mathcal{F}_U^\Gamma\}_{U \in \Omega}$.

Example A.7 Let $\mathbb{R} = G \times \Omega^{\text{op}}$ and let $\{\mathcal{F}_U\}_{U \in \Omega}$ be the family of graph subgroups determined by a weak indexing system \mathcal{F} (see Remark 6.6). Then $\{\mathcal{F}_U\}$ is Reedy-admissible. To see this, recall first that each $\Gamma \in \mathcal{F}_U$ encodes an H -action on $U \in \Omega$ for some $H \leq G$ such that $G \cdot_H U$ is an \mathcal{F} -tree. Given a face map $\varphi: U' \hookrightarrow U$, the subgroup $\pi_U^{-1}(\Gamma)$ is then determined by the largest subgroup $\bar{H} \leq H$ such that U' inherits the \bar{H} -action from U along φ (thus making φ a \bar{H} -map), so that $\pi_{U'}(\pi_U^{-1}(\Gamma))$ is the graph subgroup encoding the \bar{H} -action on U' . Thus, we see that Reedy-admissibility is simply the sieve condition for the induced map of G -trees $G \cdot_{\bar{H}} U' \rightarrow G \cdot_H U$.

We now state the main result. We will assume throughout that \mathcal{C} is a model category such that for any group G and family of subgroups \mathcal{F} , the category \mathcal{C}^G admits the \mathcal{F} -model structure with weak equivalences/fibrations detected by the fixed points X^H for $H \in \mathcal{F}$ (for example, this is the case whenever \mathcal{C} is a cofibrantly generated cellular model category in the sense of [26]).

The definitions of L_r and M_r are recalled below the result.

Theorem A.8 *Let \mathbb{R} be generalized Reedy and $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ a Reedy-admissible collection of families. Then there is an $\{\mathcal{F}_r\}$ -Reedy model structure on $\mathcal{C}^{\mathbb{R}}$ such that a map $A \rightarrow B$ is*

- a (trivial) cofibration if $A_r \amalg_{L_r A} L_r B \rightarrow B_r$ is a (trivial) \mathcal{F}_r -cofibration in $\mathcal{C}^{\text{Aut}(r)}$ for all $r \in \mathbb{R}$;
- a weak equivalence if $A_r \rightarrow B_r$ is an \mathcal{F}_r -weak equivalence in $\mathcal{C}^{\text{Aut}(r)}$ for all $r \in \mathbb{R}$;
- a (trivial) fibration if $A_r \rightarrow B_r \times_{M_r B} M_r A$ is a (trivial) \mathcal{F}_r -fibration in $\mathcal{C}^{\text{Aut}(r)}$ for all $r \in \mathbb{R}$.

The proof of **Theorem A.8** is given near the end of the appendix after establishing some routine generalizations of the key lemmas in [2]. We note that the work in [2] has two main components: a formal analysis of the *latching* and *matching objects* $L_r A$ and $M_r A$, which depends only on axioms (i), (ii) and (iii) of [2, Definition 1.1] and a model category analysis, which depends on the extra axiom (iv) of [2, Definition 1.1].

Since axioms (i)–(iii) in **Definition A.2** repeat (i)–(iii) of [2, Definition 1.1], we will only briefly recall the definitions of latching and matching objects. Writing $\iota_n: \mathbb{R}_{\leq n} \rightarrow \mathbb{R}$ for the inclusion of the full subcategory of those $r \in \mathbb{R}$ with $|r| \leq n$, we have adjunctions

$$(A.9) \quad \begin{array}{ccc} \mathcal{C}^{\mathbb{R}} & \begin{array}{c} \xleftarrow{\iota_{n,!}} \\ \xrightarrow{\quad} \\ \xleftarrow{\iota_n^*} \\ \xrightarrow{\quad} \\ \xleftarrow{\iota_{n,*}} \end{array} & \mathcal{C}^{\mathbb{R}_{\leq n}} \end{array}$$

One then defines n -skeleta by $\text{sk}_n A = \iota_{n,!} \iota_n^* A$ and n -coskeleta by $\text{csk}_n A = \iota_{n,*} \iota_n^* A$ as well as r -latching objects by $L_r A = (\text{sk}_{|r|-1} A)_r$ and r -matching objects by $M_r A = (\text{csk}_{|r|-1} A)_r$. Axioms (i)–(iii) then imply that $\text{sk}_n A$ (resp. $\text{csk}_n A$) depends only on the restriction to $\mathbb{R}_{\leq n}^+$ (resp. $\mathbb{R}_{\leq n}^-$). We refer the reader to [2, Sections 4 and 6] for a detailed discussion.

We now turn to the model categorical analysis, which depends on the Reedy-admissibility condition (iv) in **Definition A.2**, and is the actual novelty of this appendix. We first recall the following; see [7, Propositions 6.5 and 6.6].

Proposition A.10 *Let $\phi: G \rightarrow \bar{G}$ be a homomorphism and \mathcal{F} and $\bar{\mathcal{F}}$ families of subgroups of G and \bar{G} . Then the leftmost (resp. rightmost) adjunction below is a*

Quillen adjunction:

$$\bar{G} \cdot_G (-): \mathcal{C}_{\mathcal{F}}^{\bar{G}} \rightleftarrows \mathcal{C}_{\bar{\mathcal{F}}}^{\bar{G}} : \text{res}_{\bar{G}}, \quad \text{res}_{\bar{G}}: \mathcal{C}_{\bar{\mathcal{F}}}^{\bar{G}} \rightleftarrows \mathcal{C}_{\mathcal{F}}^{\bar{G}} : \text{Hom}_G(\bar{G}, -)$$

provided that for $H \in \mathcal{F}$ we have $\phi(H) \in \bar{\mathcal{F}}$ (resp. for $\bar{H} \in \bar{\mathcal{F}}$ we have $\phi^{-1}(\bar{H}) \in \mathcal{F}$).

Proof For $H \leq G$ and $\bar{H} \leq \bar{G}$, the fixed-point formulas $(\text{res}_{\bar{G}}^{\bar{G}} Y)^H = Y^{\phi(H)}$ and $(\text{Hom}_G(\bar{G}, X))^{\bar{H}} = \prod_{[\bar{g}] \in \phi(G) \setminus \bar{G}/\bar{H}} X^{\phi^{-1}(\bar{g}\bar{H}\bar{g}^{-1})}$ show that the right adjoints preserve (trivial) fibrations. □

For \mathcal{F}_{all} the family of all subgroups of G the model structure $\mathcal{C}_{\mathcal{F}_{\text{all}}}^G$ is called the *fine/genuine model structure*. We regard this as the default model structure, and hence write it simply as \mathcal{C}^G .

Corollary A.11 For any homomorphism $\phi: G \rightarrow \bar{G}$, the functor $\text{res}_{\bar{G}}^{\bar{G}}: \mathcal{C}^{\bar{G}} \rightarrow \mathcal{C}^G$ preserves all four genuine classes of cofibrations, trivial cofibrations, fibrations and trivial fibrations.

Corollary A.12 For any \mathcal{F} , an \mathcal{F} –(trivial) cofibration is also a genuine G –(trivial) cofibration.

Remark A.13 It is clear from the definitions that $X \rightarrow Y$ is an \mathcal{F} –weak equivalence (resp. \mathcal{F} –fibration) if and only if it is a genuine H –weak equivalence (resp. H –fibration) for each $H \in \mathcal{F}$. However, while an \mathcal{F} –cofibration is necessarily a genuine H –cofibration for each $H \in \mathcal{F}$, the converse is rarely true.

For example, consider $\mathcal{C} = \text{sSet}$ and $\mathcal{F} = \{\{e\}\}$ the family consisting only of the trivial subgroup. Then the \mathcal{F} –cofibrant objects turn out to be the G –free objects of sSet^G , but all objects are genuine $\{e\}$ –cofibrant, since this just means that their restriction to $\text{sSet}^{\{e\}} = \text{sSet}$ is cofibrant.

Lemmas A.16 and A.18 below formalize straightforward arguments implicit in the proofs of [2, Lemma 5.2; 2, Theorem 1.6].

Definition A.14 Consider a commutative diagram

$$(A.15) \quad \begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in $\mathcal{C}^{\mathbb{R}}$. A collection of maps $f_r: B_r \rightarrow X_r$ for $|r| \leq n$ that induce a lift of the restriction to $\mathcal{C}^{\mathbb{R}_{\leq n}}$ is called an n -partial lift.

Similarly, given a map $f: X \rightarrow Y$ in $\mathcal{C}^{\mathbb{R}}$, a factorization $\iota_n^* X \rightarrow A \rightarrow \iota_n^* Y$ of $\iota_n^* f$ in $\mathcal{C}^{\mathbb{R}_{\leq n}}$ is called an n -partial factorization.

Lemma A.16 *Let \mathcal{C} be any bicomplete category, and consider a commutative diagram as in (A.15). Then any $(n-1)$ -partial lift uniquely induces commutative diagrams*

$$(A.17) \quad \begin{array}{ccc} A_r \amalg_{L_r A} L_r B & \longrightarrow & X_r \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ B_r & \longrightarrow & Y_r \times_{M_r Y} M_r X \end{array}$$

in $\mathcal{C}^{\text{Aut}(r)}$ for each $r \in \mathbb{R}$ such that $|r| = n$. Furthermore, extensions of the $(n-1)$ -partial lift to an n -partial lift are in bijection with choices of $\text{Aut}(r)$ -equivariant lifts in the diagrams (A.17) for r ranging over representatives of the isomorphism classes of $r \in \mathbb{R}$ with $|r| = n$.

Lemma A.18 *Let \mathcal{C} be any bicomplete category. Then extensions*

$$\begin{array}{ccc} \mathbb{R}_{\leq n-1} & \xrightarrow{A} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed } \tilde{A} & \\ \mathbb{R}_{\leq n} & & \end{array}$$

are determined uniquely up to unique isomorphism (in $\mathcal{C}^{\mathbb{R}_{\leq n}}$) by choices of $\text{Aut}(r)$ -equivariant factorizations

$$(\iota_{n-1, !} A)_r \dashrightarrow \tilde{A}_r \dashrightarrow (\iota_{n-1, * } A)_r$$

for r ranging over representatives of the isomorphism classes of $r \in \mathbb{R}$ with $|r| = n$.

More generally, given a map $X \rightarrow Y$ in $\mathcal{C}^{\mathbb{R}}$, extensions of an $(n-1)$ -partial factorization $\iota_{\leq n-1}^* X \rightarrow A \rightarrow \iota_{\leq n-1}^* Y$ in $\mathcal{C}^{\mathbb{R}_{\leq n-1}}$ to an n -partial factorization $\iota_{\leq n}^* X \rightarrow \tilde{A} \rightarrow \iota_{\leq n}^* Y$ in $\mathcal{C}^{\mathbb{R}_{\leq n}}$ are determined uniquely up to unique isomorphism by choices of $\text{Aut}(r)$ -equivariant factorizations

$$X_r \amalg_{L_r X} (\iota_{n-1, !} A)_r \dashrightarrow \tilde{A}_r \dashrightarrow Y_r \times_{M_r Y} (\iota_{n-1, * } A)_r$$

for r ranging over representatives of the isomorphism classes of $r \in \mathbb{R}$ with $|r| = n$.

In the next result, by $\{\mathcal{F}_r\}$ -cofibration/trivial cofibration/fibration/trivial fibration we mean a map as described in [Theorem A.8](#), regardless of whether such a model structure exists.

Corollary A.19 *Let \mathbb{R} be generalized Reedy and $\{\mathcal{F}_r\}$ an arbitrary family of subgroups of $\text{Aut}(r)$ for $r \in \mathbb{R}$. Then a map in $\mathcal{C}^{\mathbb{R}}$ is an $\{\mathcal{F}_r\}$ -cofibration (resp. trivial cofibration) if and only if it has the left lifting property with respect to all $\{\mathcal{F}_r\}$ -trivial fibrations (resp. fibrations), and vice versa for the right lifting property.*

Lemma A.20 *Let \mathbb{S} be generalized Reedy with $\mathbb{S} = \mathbb{S}^+$, K a group and $\pi: \mathbb{S} \rightarrow K$ a functor.*

Consider a map $A \rightarrow B$ in $\mathcal{C}^{\mathbb{S}}$ such that for all $s \in \mathbb{S}$ the maps $A_s \amalg_{L_s A} L_s B \rightarrow B_s$ are $\text{Aut}(s)$ -genuine (resp. trivial) cofibrations. Then $\text{Lan}_{\pi: \mathbb{S} \rightarrow K}(A \rightarrow B)$ is a K -genuine (trivial) cofibration.

Proof By adjunction, one needs only show that for any K -fibration $X \rightarrow Y$ in \mathcal{C}^K , the map $\pi^*(X \rightarrow Y)$ has the right lifting property against all maps $A \rightarrow B$ in $\mathcal{C}^{\mathbb{S}}$ as in the statement. By [Corollary A.19](#), it thus suffices to check that the maps

$$(\pi^* X)_s \rightarrow (\pi^* Y)_s \times_{M_s \pi^* Y} M_s \pi^* X$$

are $\text{Aut}(s)$ -fibrations. But since $M_s Z = *$ (recall $\mathbb{S} = \mathbb{S}^+$) this map is just $X \rightarrow Y$ with the $\text{Aut}(s)$ -action induced by $\pi: \text{Aut}(s) \rightarrow K$, hence [Corollary A.11](#) finishes the proof. □

Lemma A.21 *Let \mathbb{S} be generalized Reedy with $\mathbb{S} = \mathbb{S}^-$, K a group and $\pi: \mathbb{S} \rightarrow K$ a functor.*

Consider a map $X \rightarrow Y$ in $\mathcal{C}^{\mathbb{S}}$ such that for all $s \in \mathbb{S}$ the maps $X_s \rightarrow Y_s \times_{M_s Y} M_s X$ are $\text{Aut}(s)$ -genuine (resp. trivial) fibrations. Then $\text{Ran}_{\pi: \mathbb{S} \rightarrow K}(A \rightarrow B)$ is a K -genuine (trivial) fibration.

Proof This follows dually to the previous proof. □

Remark A.22 Lemmas [A.20](#) and [A.21](#) generalize key parts of the proofs of Lemmas 5.3 and 5.5 of [\[2\]](#). The duality of their proofs reflects the duality in [Corollary A.11](#).

Notation A.23 Suppose a group K acts on a category \mathbb{R} , ie that one has a homomorphism $K \rightarrow \text{Aut}_{\text{Cat}}(\mathbb{R})$, where the target is the group of automorphisms of \mathbb{R} as a category. We write $K \times \mathbb{R} \rightarrow K$ for the category obtained from \mathbb{R} by formally adding action arrows $r \rightarrow kr$ for $r \in \mathbb{R}$ and $k \in K$, which satisfy three natural relations: the maps $r \xrightarrow{e} er$ are the identity id_r , and the two diagrams below commute for all $k, \bar{k} \in K$ and maps $f: r \rightarrow r'$ in \mathbb{R} :

$$\begin{array}{ccc}
 r & \xrightarrow{k} & kr \\
 & \searrow \bar{k} & \swarrow \bar{k} \\
 & \bar{k}k & \bar{k}kr
 \end{array}
 \qquad
 \begin{array}{ccc}
 r & \xrightarrow{k} & kr \\
 f \downarrow & & \downarrow kf \\
 r' & \xrightarrow{k} & kr'
 \end{array}$$

Note that if $\mathbb{R} = G$ is a group (regarding groups as one-object categories) this recovers the usual $K \times G$ construction and, if K acts trivially on \mathbb{R} , this recovers the product of categories $K \times \mathbb{R}$.

Alternatively, $K \times \mathbb{R}$ admits a succinct formal definition: if one regards \mathbb{R} with its K -action as a functor $K \xrightarrow{\mathbb{R}} \text{Cat}$, the associated Grothendieck fibration is the natural functor $K \times \mathbb{R} \rightarrow K$.

We note that for a functor $A: K \times \mathbb{R} \rightarrow \mathcal{C}$ the Kan extension $\text{Ran}_{K \times \mathbb{R} \rightarrow K} A$ (resp. $\text{Lan}_{K \times \mathbb{R} \rightarrow K} A$) is simply the (co)limit $\lim_{\mathbb{R}} A$ (resp. $\text{colim}_{\mathbb{R}} A$) together with the naturally induced K -action.

Notation A.24 We borrow the $\mathbb{R}^+(r)$ notation introduced after [2, Remark 1.5] for the full subcategory of the overcategory $\mathbb{R} \downarrow r$ with objects the arrows $r' \xrightarrow{+} r$ in \mathbb{R}^+ with $|r'| < |r|$. One has a *domain* functor $d: \mathbb{R}^+(r) \rightarrow \mathbb{R}$ given by $(r' \rightarrow r) \mapsto r'$ such that (see the discussion after [2, Remark 1.5] and [2, Lemma 4.4(i)])

(i) by Definition A.2(iii) arrows in $\mathbb{R}^+(r)$ forget under d to arrows in \mathbb{R}^+ , so that $\mathbb{R}^+(r)$ becomes generalized Reedy with $|r' \rightarrow r| = |r'|$, $(\mathbb{R}^+(r))^+ = \mathbb{R}^+(r)$ and $(\mathbb{R}^+(r))^- = \text{Iso}(\mathbb{R}^+(r))$;

(ii) for $(r' \rightarrow r) \in \mathbb{R}^+(r)$ the functor d induces isomorphisms

$$(\mathbb{R}^+(r))^+(r' \rightarrow r) \xrightarrow{\cong} \mathbb{R}^+(r');$$

(iii) for $A \in \mathcal{C}^{\mathbb{R}}$, $(r' \rightarrow r) \in \mathbb{R}^+(r)$ there are natural isomorphisms $L_{r' \rightarrow r}(d^*A) \simeq L_{r'}A$;

(iv) for $A \in \mathcal{C}^{\mathbb{R}}$ there are natural isomorphisms $L_r A \simeq \text{colim}_{\mathbb{R}^+(r)}(d^*A)$.

Moreover, writing $\mathbb{R}^-(r)$ for the full subcategory of $r \downarrow \mathbb{R}$ with objects the $r \twoheadrightarrow r'$ in \mathbb{R}^- with $|r'| < |r|$, one has a *target* functor $t: \mathbb{R}^-(r) \rightarrow \mathbb{R}$ satisfying the dual properties of (i)–(iv).

Remark A.25 Lemma A.20 will be applied when $K \leq \text{Aut}_{\mathbb{R}}(r)$ and $\mathbb{S} = K \rtimes \mathbb{R}^+(r)$ for \mathbb{R} a given generalized Reedy category and $r \in \mathbb{R}$. Similarly, Lemma A.21 will be applied when $\mathbb{S} = K \rtimes \mathbb{R}^-(r)$. It is straightforward to check that in the \mathbb{R}^+ (resp. \mathbb{R}^-) case maps in \mathbb{S} are identified with squares as on the left (right):

$$(A.26) \quad \begin{array}{ccc} r'' & \xrightarrow{+} & r \\ +\downarrow & & \downarrow \simeq \\ r' & \xrightarrow{+} & r \end{array} \qquad \begin{array}{ccc} r & \xrightarrow{-} & r' \\ \simeq\downarrow & & \downarrow - \\ r & \xrightarrow{-} & r'' \end{array}$$

where maps labeled $+$ are in \mathbb{R}^+ , maps labeled $-$ are in \mathbb{R}^- , the horizontal maps are noninvertible and the maps labeled \simeq are automorphisms in K . Using the projection $\pi: \mathbb{S} \rightarrow K$ and the domain (resp. target) functor $d: \mathbb{S} \rightarrow \mathbb{R}$ ($t: \mathbb{S} \rightarrow \mathbb{R}$), Lemmas A.20 and A.21 will be applied to maps $d^*A \rightarrow d^*B$ (resp. $t^*A \rightarrow t^*B$) in $\mathcal{C}^{\mathbb{S}}$ induced from maps $A \rightarrow B$ in $\mathcal{C}^{\mathbb{R}}$ so that Notations A.23 and A.24(iv) give the natural identifications below, compatibly with the K -actions.

$$\text{Lan}_{\pi} d^*(A \rightarrow B) \simeq (L_r A \rightarrow L_r B), \quad \text{Ran}_{\pi} t^*(A \rightarrow B) \simeq (M_r A \rightarrow M_r B).$$

Lastly, note that (in the \mathbb{R}^+ case) the natural inclusion $(\mathbb{R}^+(r))^+(r' \rightarrow r) \xrightarrow{\iota} \mathbb{S}^+(r \rightarrow r')$ is an equivalence of categories (indeed, by adding a \searrow arrow to the left square in (A.26), the bottom left triangle becomes an object in the source of ι and the top right triangle an isomorphism in the target of ι). Hence Notation A.24(iii) (and its dual) also apply to \mathbb{S} .

We can now prove the following lemmas, which are the key to Theorem A.8 (to simplify our proofs, the lemmas are stated as dually as possible, but care is needed; see Remarks A.13 and A.30).

Lemma A.27 *Let \mathbb{R} be generalized Reedy and $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ a Reedy-admissible family. Suppose further that $A \rightarrow B$ is an $\{\mathcal{F}_r\}$ -Reedy cofibration in $\mathcal{C}^{\mathbb{R}}$. Then:*

- (i) *for all $r \in \mathbb{R}$ and $H \in \mathcal{F}_r$ the maps $L_r A \rightarrow L_r B$ and $A_r \rightarrow B_r$ are H -genuine cofibrations;*
- (ii) *the maps $A_r \rightarrow B_r$ are all \mathcal{F}_r -weak equivalences if and only if so are the maps $A_r \amalg_{L_r A} L_r B \rightarrow B_r$.*

Proof We start with (ii), arguing by induction on $|r|$ that the analogous claim restricted to $|r| \leq n$ holds. The $n = 0$ case is obvious. Otherwise, it suffices to show that, for each r with $|r| = n$ and $H \in \mathcal{F}_r$, the map $A_r \rightarrow B_r$ is an H -genuine weak equivalence if and only if so is $A_r \amalg_{L_r A} L_r B \rightarrow B_r$.

We apply Lemma A.20 with $K = H$ and $\mathbb{S} = H \rtimes \mathbb{R}^+(r)$ to the map $d^*A \rightarrow d^*B$. By Notation A.24(iii), the hypothesis to be checked is that, for each $(r' \rightarrow r) \in \mathbb{S}$, the map $A_{r'} \amalg_{L_{r',A}} L_{r'} B \rightarrow B_{r'}$ is a genuine trivial cofibration for the group $\text{Aut}_{\mathbb{S}}(r' \rightarrow r)$. But by induction this map is an $\mathcal{F}_{r'}$ -trivial cofibration, so this holds by Corollaries A.12 and A.11. It now follows that the maps labeled \sim in

$$(A.28) \quad \begin{array}{ccccc} L_r A & \xrightarrow{\sim} & L_r B & & \\ \downarrow & & \downarrow & & \\ A_r & \xrightarrow{\sim} & A_r \amalg_{L_r A} L_r B & \xrightarrow{\sim} & B_r \end{array}$$

are H -genuine trivial cofibrations, finishing the proof of (ii).

Lastly, (i) follows from a streamlined version of the argument above (with no induction needed), with the same instance of Lemma A.20 showing that the maps $L_r A \rightarrow L_r B$ are H -genuine cofibrations and thus, by again considering (A.28), that so are the maps $A_r \rightarrow B_r$. \square

Lemma A.29 Let \mathbb{R} be generalized Reedy and $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ a Reedy-admissible family. Suppose further that $X \rightarrow Y$ is an $\{\mathcal{F}_r\}$ -Reedy fibration in $\mathcal{C}^{\mathbb{R}}$. Then:

- (i) for all $r \in \mathbb{R}$ and $H \in \mathcal{F}_r$ the maps $M_r X \rightarrow M_r Y$ and $X_r \rightarrow Y_r$ are H -genuine fibrations;
- (ii) the maps $X_r \rightarrow Y_r$ are all $\{\mathcal{F}_r\}$ -weak equivalences if and only if so are the maps $X_r \rightarrow Y_r \times_{M_r Y} M_r X$.

Proof For (ii) we again argue by induction on $|r|$. We need to show that, for r with $|r| = n$ and $H \in \mathcal{F}_r$, the map $X_r \rightarrow Y_r$ is an H -genuine weak equivalence if and only if so is $X_r \rightarrow Y_r \times_{M_r Y} M_r X$.

We apply Lemma A.21 with $K = H$ and $\mathbb{S} = H \rtimes \mathbb{R}^-(r)$ to the map $t^*A \rightarrow t^*B$. By the dual of Notation A.24(iii), the hypothesis to be checked is that, for each $(r \rightarrow r') \in \mathbb{S}$, the map $X_{r'} \rightarrow Y_{r'} \times_{M_{r',Y}} M_{r'} X$ is a genuine trivial fibration for the group $\text{Aut}_{\mathbb{S}}(r \rightarrow r') \simeq \pi_r^{-1}(H)$, where π_r is as in (A.1). But since $\{\mathcal{F}_r\}$ is Reedy-admissible (see Definition A.2(iv)), we know by induction that this map is a $\pi_{r'}(\pi_r^{-1}(H))$ -trivial fibration, and this suffices by Corollary A.11. It now follows that

the maps labeled \sim in

$$\begin{array}{ccc}
 X_r & \twoheadrightarrow & Y_r \times_{M_r Y} M_r X & \xrightarrow{\sim} & Y_r \\
 & & \downarrow & & \downarrow \\
 & & M_r X & \xrightarrow{\sim} & M_r Y
 \end{array}$$

are H -genuine trivial fibrations, finishing the proof of (ii). Part (i) again follows similarly. □

Remark A.30 The proofs of Lemmas A.27 and A.29 are similar, but not dual, since Lemma A.29 uses Reedy-admissibility while Lemma A.27 does not. This reflects the difference in the proofs of [2, Lemmas 5.3 and 5.5] as discussed in [2, Remark 5.6], albeit with a caveat.

Setting $K = \{e\}$ in Lemma A.20 yields that $\lim_{\mathbb{S}}(A \rightarrow B)$ is a cofibration provided that $A \rightarrow B$ is a genuine Reedy cofibration, ie a Reedy cofibration for $\{\mathcal{F}_{\text{all}}\}$ the families of all subgroups. On the other hand, the proof of [2, Lemma 5.3] argues that $\lim_{\mathbb{S}}(A \rightarrow B)$ is a cofibration provided that $A \rightarrow B$ is a projective Reedy cofibration, ie a Reedy cofibration for $\{\{e\}\}$ the collection of trivial families (note that all projective cofibrations are genuine cofibrations, so that our claim is more general). Since the cofibration half of the analogue of Corollary A.11 for projective model structures only holds if $\phi: G \rightarrow \bar{G}$ is injective, the proof of [2, Lemma 5.3] also requires an injectivity check that is not needed in our proof of Lemma A.27.

Proof of Theorem A.8 Lemmas A.27(ii) and A.29(ii) say that the characterizations of trivial cofibrations (resp. trivial fibrations) in the statement of Theorem A.8 are correct, ie that they describe the maps that are both cofibrations (resp. fibrations) and weak equivalences.

We refer to the model category axioms in [17, Definition 1.1.3]. Both 2-out-of-3 and the retract axioms are immediate (note that retracts commute with Kan extensions). The lifting axiom follows from Corollary A.19 while the task of building factorizations $X \rightarrow A \rightarrow Y$ of a given map $X \rightarrow Y$ follows by a similar standard argument from Lemma A.18 by iteratively factoring the maps

$$X_r \amalg_{L_r X} L_r A \rightarrow Y_r \times_{M_r Y} M_r A$$

in $\mathcal{C}^{\text{Aut}(r)}$, thus building both A and the factorization inductively (recall that $L_r A$ and $M_r A$ depend only on the restriction $\iota_{|r-1}^* A$, and are thus well defined in each inductive step). □

Remark A.31 If \mathbb{R} and \mathbb{S} are generalized Reedy categories, then $\mathbb{R} \times \mathbb{S}$ is again generalized Reedy with $|(r, s)| = |r| + |s|$, $(\mathbb{R} \times \mathbb{S})^+ = \mathbb{R}^+ \times \mathbb{S}^+$ and $(\mathbb{R} \times \mathbb{S})^- = \mathbb{R}^- \times \mathbb{S}^-$. Moreover, given corresponding Reedy-admissible families $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ and $\{\mathcal{F}_s\}_{s \in \mathbb{S}}$, one has induced admissible families $\{\mathcal{F}_{(r,s)}\}_{(r,s) \in \mathbb{R} \times \mathbb{S}}$, where

$$\mathcal{F}_{(r,s)} = \{H \times \bar{H} : H \in \mathcal{F}_r, \bar{H} \in \mathcal{F}_s\}.$$

It is then straightforward to check that the $\{\mathcal{F}_{(r,s)}\}$ -Reedy model structure on $\mathcal{C}^{\mathbb{R} \times \mathbb{S}} \simeq (\mathcal{C}^{\mathbb{R}})^{\mathbb{S}}$ can also be obtained iteratively as the $\{\mathcal{F}_s\}$ -Reedy model structure over the $\{\mathcal{F}_r\}$ -Reedy model structure over the model structure on \mathcal{C} (and vice versa).

We note that, given a map $f: A \rightarrow B$ in $\mathcal{C}^{\mathbb{R} \times \mathbb{S}}$ and writing $l_{(-)}f$ for the relative latching map $A_{(-)} \amalg_{L_{(-)}A} L_{(-)}B \rightarrow B_{(-)}$, one has $l_{(r,s)}f \simeq l_r l_s f \simeq l_s l_r f$ and dually for relative matching maps.

Lastly, we briefly discuss the cofibrant generation of equivariant Reedy model structures.

For $K \in \text{Set}$ and $X \in \mathcal{C}$, we write $K \cdot X = \coprod_K X$ and $\{K, X\} = X^{\times K}$ for the standard tensoring and cotensoring of \mathcal{C} over Set . One then has a two-variable adjunction

$$\text{Set}^{\mathbb{R}} \times \mathcal{C} \xrightarrow{(-) \cdot (-)} \mathcal{C}^{\mathbb{R}}, \quad \mathcal{C}^{\text{op}} \times \mathcal{C}^{\mathbb{R}} \xrightarrow{\mathcal{C}(-, -)} \text{Set}^{\mathbb{R}}, \quad (\text{Set}^{\mathbb{R}})^{\text{op}} \times \mathcal{C}^{\mathbb{R}} \xrightarrow{\{-, -\}_{\mathbb{R}}} \mathcal{C},$$

where $\{K_{\bullet}, X_{\bullet}\}_{\mathbb{R}} = \int_{r \in \mathbb{R}} \{K_r, X_r\}$ is the end (sometimes called the weighted limit [24, Section 7.1]).

The following is essentially [25, Lemma 3.5], rewritten in our notation:

Lemma A.32 For $X \in \mathcal{C}^{\mathbb{R}}$ and $r \in \mathbb{R}$, there is a natural identification $M_r X \simeq \{\text{sk}_{|r|-1} \mathbb{R}(r, -), X\}_{\mathbb{R}}$.

Proof Abbreviating $\mathbb{R}_{\leq n-1}$ as $\mathbb{R}_{< n}$, this follows from the calculation

$$\begin{aligned} M_r X &\simeq \int_{\bar{r} \in \mathbb{R}_{< |r|}} \{\mathbb{R}(r, \bar{r}), X_{\bar{r}}\} \simeq \int_{\bar{r} \in \mathbb{R}_{< |r|}} \left\{ \mathbb{R}(r, \bar{r}), \int_{\hat{r} \in \mathbb{R}} \{\mathbb{R}(\bar{r}, \hat{r}), X_{\hat{r}}\} \right\} \\ &\simeq \int_{\bar{r} \in \mathbb{R}_{< |r|}} \int_{\hat{r} \in \mathbb{R}} \{\mathbb{R}(r, \bar{r}), \{\mathbb{R}(\bar{r}, \hat{r}), X_{\hat{r}}\}\} \simeq \int_{\hat{r} \in \mathbb{R}} \int_{\bar{r} \in \mathbb{R}_{< |r|}} \{\mathbb{R}(r, \bar{r}) \times \mathbb{R}(\bar{r}, \hat{r}), X_{\hat{r}}\} \\ &\simeq \int_{\hat{r} \in \mathbb{R}} \left\{ \int_{\bar{r} \in \mathbb{R}_{< |r|}} \mathbb{R}(r, \bar{r}) \times \mathbb{R}(\bar{r}, \hat{r}), X_{\hat{r}} \right\} \simeq \int_{\hat{r} \in \mathbb{R}} \{(\text{sk}_{|r|-1} \mathbb{R}(r, -))(\hat{r}), X_{\hat{r}}\} \\ &= \{\text{sk}_{|r|-1} \mathbb{R}(r, -), X\}_{\mathbb{R}} \end{aligned}$$

where the second step is the Yoneda lemma, the fourth step is Fubini together with adjointness for $\{-, -\}$, and the sixth step is the co-Yoneda lemma [24, Example 1.4.6]. □

Combining this lemma with the observation that, if a group H acts on $K \in \text{Set}^{\mathbb{R}}$, then $\{K/H, X\}_{\mathbb{R}} = (\{K, X\}_{\mathbb{R}})^H$ now yields the following:

Proposition A.33 *Suppose the model category \mathcal{C} is cofibrantly generated, \mathbb{R} is generalized Reedy and $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ is a Reedy-admissible collection of families.*

Then the $\{\mathcal{F}_r\}$ -Reedy model structure on $\mathcal{C}^{\mathbb{R}}$ is also cofibrantly generated.

More explicitly, letting \mathcal{I} (resp. \mathcal{J}) denote a set of generating (resp. trivial) cofibrations of \mathcal{C} , the generating cofibrations for $\mathcal{C}^{\mathbb{R}}$ are the maps of the form

$$((\text{sk}_{|r|-1} \mathbb{R}(r, -) \rightarrow \mathbb{R}(r, -))/H) \square i, \quad r \in \mathbb{R}, H \in \mathcal{F}_r, i \in \mathcal{I},$$

while the generating trivial cofibrations are the maps of the form

$$((\text{sk}_{|r|-1} \mathbb{R}(r, -) \rightarrow \mathbb{R}(r, -))/H) \square j, \quad r \in \mathbb{R}, H \in \mathcal{F}_r, j \in \mathcal{J}.$$

Example A.34 For $\mathbb{R} = \Delta^{\text{op}}$, it is well known that $\text{sk}_{n-1} \Delta^{\text{op}}(n, -) = \text{sk}_{n-1} \Delta[n] = \partial \Delta[n]$.

Similarly, letting $\mathbb{R} = G \times \Omega^{\text{op}}$ as in Example A.7 and letting $\Gamma \in \mathcal{F}_U$ be a graph subgroup encoding an H -action on U , then

$$\begin{aligned} (\text{sk}_{|U|-1}(G \times \Omega^{\text{op}})(U, -))/\Gamma &= (\text{sk}_{|U|-1}(G \cdot \Omega[U]))/\Gamma = (G \cdot \text{sk}_{|U|-1} \Omega[U])/ \Gamma \\ &= (G \cdot \partial \Omega[U])/ \Gamma = G \cdot_H \partial \Omega[U] = \partial \Omega[G \cdot_H U], \end{aligned}$$

where the third equality is [2] (see also [21, Corollary 5.63]) and the last equality is (2.29).

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