

THE 2-PRIMARY HUREWICZ IMAGE OF tmf

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ABSTRACT. We determine the image of the 2-primary tmf -Hurewicz homomorphism, where tmf is the spectrum of topological modular forms. We do this by lifting elements of tmf_* to the homotopy groups of the generalized Moore spectrum $M(8, v_1^8)$ using a modified form of the Adams spectral sequence and the tmf -resolution, and then proving the existence of a v_2^{32} -self map on $M(8, v_1^8)$ to generate 192-periodic families in the stable homotopy groups of spheres.

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1. INTRODUCTION

The Hurewicz theorem implies that the Hurewicz homomorphism

$$h : \pi_*(S^n) \rightarrow \tilde{H}_*(S^n; \mathbb{Z})$$

is an isomorphism for $* = n$, implying the well known result that the 0th stable stem is given by

$$\pi_0^s \cong \mathbb{Z}.$$

In his paper [Ada66], Adams studied the Hurewicz homomorphism for real K-theory

$$h_{\mathrm{KO}} : \pi_*^s \rightarrow \pi_* \mathrm{KO} = \mathrm{KO}^{-*}(pt).$$

The computation of the real K-theory of a point (the homotopy groups of the spectrum KO representing real K-theory) is a consequence of the Bott periodicity theorem [Bot59]: these groups are given by the following 8-fold periodic pattern.

$n \bmod 8$	0	1	2	3	4	5	6	7
$\pi_n \text{KO}$	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	\mathbb{Z}	0	0	0

The map h_{KO} is an isomorphism in degree 0, and Adams showed that h_{KO} is surjective in degrees $* \equiv 1, 2 \pmod{8}$. He did this by constructing what is now known as a v_1 -self map

$$v_1^4 : \Sigma^8 M(2) \rightarrow M(2),$$

where $M(2)$ denotes the mod 2 Moore spectrum, and considering the projections

$$\mu_{8j+1+\epsilon} \in \pi_{8j+1+\epsilon}^s$$

of the elements

$$(1.1) \quad \eta^\epsilon \cdot v_1^{4j} \tilde{\eta} \in \pi_{8j+2+\epsilon} M(2)$$

to the top cell of $M(2)$. Here $\tilde{\eta}$ denotes a lift of $\eta \in \pi_1^s$ to the top cell of $M(2)$ and $\epsilon \in \{0, 1\}$. Because we have

$$\pi_*^s \otimes \mathbb{Q} = 0$$

for $* > 0$, the homomorphism h_{KO} is necessarily trivial in positive degrees $* \equiv 0 \pmod{4}$.

Goerss, Hopkins, and Miller constructed the spectrum tmf of topological modular forms [DFHH14] as a higher analog of the real K-theory spectrum.¹ The homotopy groups of tmf are 576-periodic. The goal of this paper is to determine the image of the 2-local tmf-Hurewicz homomorphism²

$$h_{\text{tmf}} : (\pi_*^s)_{(2)} \rightarrow \pi_* \text{tmf}_{(2)}.$$

Henceforth, everything in this paper is implicitly 2-local.

2-locally, the homotopy groups of tmf are merely 192-periodic. These homotopy groups were originally computed by Hopkins and Mahowald [DFHH14] (see also [Bau08]). These homotopy groups are displayed in Figure 1.1. In this figure:

- A series of i black dots joined by vertical lines corresponds to a factor of $\mathbb{Z}/2^i$ which is annihilated by some power of $c_4 = v_1^4$.
- An open circle corresponds to a factor of $\mathbb{Z}/2$ which is not annihilated by a power of c_4 .
- A box indicates a factor of $\mathbb{Z}_{(2)}$ which is not annihilated by a power of c_4 .
- The non-vertical lines indicate multiplication by η and ν .
- A pattern with a dotted box around it and an arrow emanating from the right face indicates this pattern continues indefinitely to the right by c_4 -multiplication (i.e. tensor the pattern with $\mathbb{Z}_{(2)}[c_4]$).

¹Here, tmf denotes *connective* topological modular forms.

²The 3-primary Hurewicz image has also not been resolved, but would follow from the results in a recent preprint of Shimomura [Shi]. Since $\pi_* \text{tmf}_{(p)}$ has no torsion for $p \geq 5$, the p -primary tmf-Hurewicz image is trivial in positive degrees for these primes.

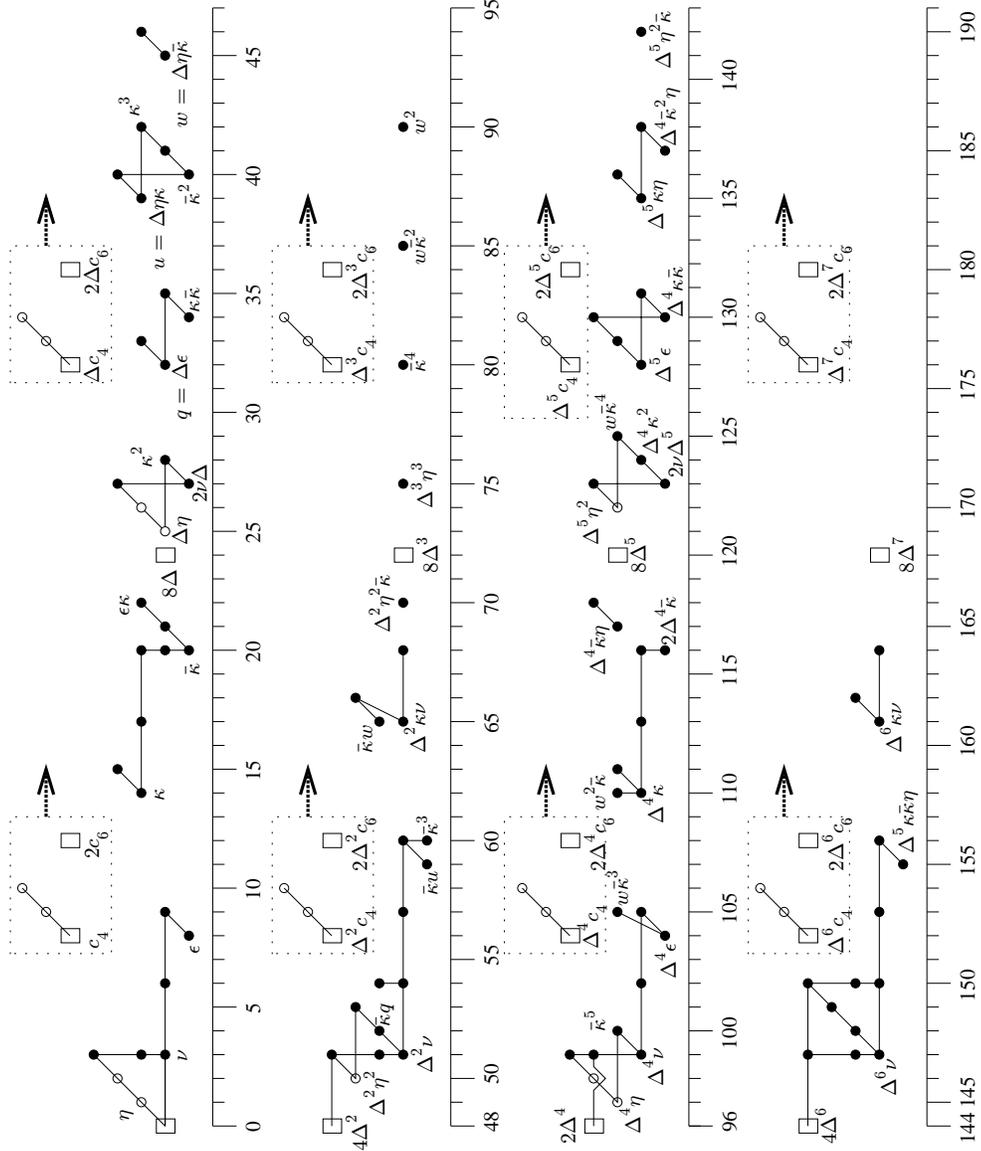


FIGURE 1.1. The homotopy groups of tmf

After localization at the prime 2, the element $\Delta^8 = v_2^{32}$ is a permanent cycle in the descent spectral sequence, and $\pi_*\text{tmf}$ is given by tensoring the pattern depicted in Figure 1.1 with $\mathbb{Z}[\Delta^8]$. Our choice of names for generators in Figure 1.1 is motivated by the fact that the elements

$$\eta, \nu, \epsilon, \kappa, \bar{\kappa}, q, u, w$$

in the stable stems map to the corresponding elements in $\pi_*\text{tmf}$ under the tmf-Hurewicz homomorphism.

The main theorem of this paper is the following.

Theorem 1.2. *The tmf-Hurewicz image is the subgroup of $\pi_*\mathrm{tmf}$ generated by*

- (1) *All the elements of $\pi_{\leq 3}(\mathrm{tmf})$,*
- (2) *The elements $c_4^j \eta^j$ with $j \in \{1, 2\}$,*
- (3) *All the elements of $\pi_*\mathrm{tmf}$ annihilated by a power of $c_4 = v_1^4$, except those non-zero elements of the form*

$$\alpha \Delta^i \nu$$

with $i > 0$ and $\alpha \in \mathbb{Z}_{(2)}$.

Besides representing an advance in our understanding of v_2 -periodic homotopy at the prime 2, Theorem 1.2 also has applications to smooth structures on spheres, as explained in [BHHM20]. Specifically, Hill, Hopkins, and the first two authors consider the following question.

Question 1.3. In which dimensions n do there exist exotic smooth structures on the n -sphere?

Such spheres with exotic smooth structures are called exotic spheres. The work of Kervaire and Milnor [KM63] relates the existence of exotic spheres to the triviality of the Kervaire homomorphism

$$\pi_{4k+2}^s \rightarrow \mathbb{Z}/2$$

and the non-triviality of the cokernel of the J -homomorphism

$$J : \pi_n SO \rightarrow \pi_n^s.$$

Specifically, they prove that exotic spheres exist in dimensions n for which

- $n = 4k$: $n \geq 8$ and there exists a non-trivial element of $\mathrm{coker} J$,
- $n = 4k + 1$: there exists a non-trivial element of $\mathrm{coker} J$, or there does not exist an element of Kervaire invariant 1 in dimension $n + 1$,
- $n = 4k + 2$: there exists a non-trivial element of $\mathrm{coker} J$ with Kervaire invariant 0,
- $n = 4k + 3$: $n \geq 7$.

Combining this with the work of Browder [Bro69], Barratt-Jones-Mahowald [BJM84], Perelman [Per02] [Per03b] [Per03a], Hill-Hopkins-Ravenel [HHR16], and Wang-Xu [WX17], Question 1.3 has been answered completely for n odd:

the only odd dimensions n for which there do not exist exotic spheres are $n = 1, 3, 5$, and 61 .

For n even, the case of $n = 4$ is unresolved. For other even n , by the previous discussion, the question boils down to the existence of non-trivial elements of $\mathrm{coker} J$ (with Kervaire invariant 0). It is shown in [BHHM20]:

the only even dimensions $4 \neq n < 140$ for which there do not exist exotic spheres are $n = 2, 6, 12$, and 56 .

In the case of $n = 8k + 2 \geq 10$, Adams' elements μ_{8k+2} with non-trivial KO-Hurewicz image are not in the image of J and have trivial Kervaire invariant. It thus follows that:

there exist exotic spheres in all dimensions $n = 8k + 2 \geq 10$.

As is explained in [BHHM20], many of the 192-periodic families of elements of Theorem 1.2 also are not in the image of J and have trivial Kervaire invariant. Theorem 1.2 therefore has the following corollary.³

Corollary 1.4. *There exist exotic spheres in the following congruence classes of even dimensions $n \geq 8$ modulo 192:*

2, 6, 8, 10, 14, 18, 20, 22, 26, 28, 32, 34, 40, 42, 46, 50, 52, 54, 58, 60, 66, 68,
70, 74, 80, 82, 90, 98, 100, 102, 104, 106, 110, 114, 116, 118, 122, 124, 128,
130, 136, 138, 142, 146, 148, 150, 154, 156, 162, 164, 170, 178, 186.

(This accounts for over half of the even dimensions.)

We will prove Theorem 1.2 by first showing (Theorem 6.1) that the elements of $\pi_*\mathrm{tmf}$ not in the subgroup described by Theorem 1.2 are not in the Hurewicz image. This will be a relatively straightforward consequence of some v_1 -periodic computations. The elements of Theorem 1.2(1) are already established to be in the Hurewicz image by the preceding discussion, and the elements (2) are in the Hurewicz image because they are the images of the elements μ_{8i+j} . We are left to show that the elements of type (3) lift to π_*^s . This is the main task of this paper.

In [BR], Bruner and Rognes give a systematic and careful study of the Adams spectral sequence for tmf , and in particular they have independently established the Hurewicz image in many low-dimensional cases. Specifically, they prove Theorem 1.2 for degrees $* \leq 101$ and also show that $w\bar{\kappa}^3$, $w^2\bar{\kappa}$, $w\bar{\kappa}^4$, $2\Delta^4\kappa\bar{\kappa}$, and $4\Delta^6\nu^2$ (in dimensions 105, 110, 125, 130, and 150) are in the Hurewicz image. Also, they use a different technique (Anderson duality) to prove that the Hurewicz image is contained in the subgroup of tmf_* described in Theorem 1.2.

Our strategy to lift elements from $\pi_*\mathrm{tmf}$ to π_*^s is to use the methods of [BHHM20]. We summarize that strategy here. We recall the following from [BHHM20, Prop. 6.1].

Proposition 1.5 ([BHHM20]). *Every v_1 -torsion element $x \in \pi_*\mathrm{tmf}$ is 8-torsion and v_1^8 -torsion.*

Let $M(2^i)$ denote the cofiber of 2^i , and let $M(2^i, v_1^j)$ denote the cofiber of a v_1 -self map

$$v_1^j : \Sigma^{2j} M(2^i) \rightarrow M(2^i).$$

³In fact, the v_2^{32} -self map of Theorem 1.7 which is used to construct the periodic families of Theorem 1.2 also immediately implies the existence of some elements not in the image of the J -homomorphism which are in the kernel of the tmf -Hurewicz homomorphism, such as the beta elements $\beta_{32k/8}$. However, we will not concern ourselves here with the few additional dimensions such considerations add to the list of Corollary 1.4.

Corollary 1.6. *Every v_1 -torsion element $x \in \pi_*(\mathrm{tmf})$ lifts to an element*

$$\tilde{x} \in \mathrm{tmf}_*M(8, v_1^8)$$

so that the projection to the top cell maps \tilde{x} to x .

Given a v_1 -torsion element $x \in \pi_{<192}(\mathrm{tmf})$, Proposition 1.5 implies it lifts to an element

$$\tilde{x} \in \mathrm{tmf}_*M(8, v_1^8)$$

so that the projection to the top cell maps \tilde{x} to x . We will then show that \tilde{x} lifts to an element

$$\tilde{y} \in \pi_*M(8, v_1^8).$$

Then the image

$$y \in \pi_*^s$$

given by projecting \tilde{y} to the top cell is an element whose image under the tmf -Hurewicz homomorphism is x .

Every v_1 -torsion element $x' \in \pi_{\geq 192}\mathrm{tmf}$ is of the form $v_2^{32k}x$ for $x \in \pi_{<192}\mathrm{tmf}$. We will prove the following theorem.

Theorem 1.7. *There exists a v_2^{32} -self map*

$$v_2^{32} : \Sigma^{192}M(8, v_1^8) \rightarrow M(8, v_1^8).$$

If $\tilde{x} \in \mathrm{tmf}_*M(8, v_1^8)$ is a lift of x , and $\tilde{y} \in \pi_*M(8, v_1^8)$ is a lift of \tilde{x} , as in the discussion above, then the resulting element

$$v_2^{32k}\tilde{y} \in \pi_*M(8, v_1^8),$$

obtained by composing with the k -fold iterate of the v_2^{32} -self map, projects to an element $y' \in \pi_*^s$ which maps to x' under the tmf -Hurewicz homomorphism.

As in [BHHM20], the analysis above rests on a systematic analysis of the homotopy groups $\pi_*M(8, v_1^8)$. This will be based on computations using the *modified Adams spectral sequence (MASS)*. The E_2 -term of the modified Adams spectral sequence will be analyzed in a region near its vanishing line by means of another spectral sequence, the *algebraic tmf resolution*.

The work of [BHHM20] was hampered by the fact that all of the algebraic tmf resolution computations were performed on the level of the E_1 -term of the algebraic tmf resolution. In this paper, we will show that the weight spectral sequence, used in the context of bo -resolutions by [LM87] and [BBB⁺20], can be used to analyze the E_2 -term of the algebraic tmf resolution, greatly simplifying the computations.

Conventions.

- Homology will be implicitly taken with mod 2 coefficients.
- We let A_* denote the dual Steenrod algebra, $A // A(2)_*$ denote the dual of the Hopf algebra quotient $A // A(2)$, and for an A_* -comodule M (or more generally an object of the stable homotopy category of A_* -comodules [Hov04]) we let

$$\mathrm{Ext}_{A_*}^{s,t}(M)$$

- denote the group $\mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2, M)$.
- Given a Hopf algebroid (B, Γ) , and a comodule M , we will let $C_\Gamma^*(M)$ denote the associated normalized cobar complex.
 - For a spectrum E , we let E_* denote its homotopy groups π_*E .

Outline of paper. In Section 2, we recall the modified Adams spectral sequence (MASS), which takes the form

$${}^{mass}E_2^{*,*} = \mathrm{Ext}_{A_*}(H_*X \otimes H(8, v_1^8)) \Rightarrow \pi_*(X \wedge M(8, v_1^8))$$

for a certain object $H(8, v_1^8)$ in the stable homotopy category of A_* -comodules. We recall how the E_2 -term of the MASS can be studied using the algebraic tmf resolution, which is a spectral sequence that takes the form

$$\mathrm{tmf}_{alg}E_1(M)^{*,*,*} \Rightarrow \mathrm{Ext}_{A_*}^{*,*}(M)$$

for any M in the stable category of A_* -comodules. We then recall how the E_1 -term of the algebraic tmf resolution decomposes as a sum of Ext groups involving tensor powers of bo-Brown-Gitler comodules, and also summarize an inductive method to compute these Ext groups.

In Section 3, we study the d_1 differential in the algebraic tmf resolution for \mathbb{F}_2 , and introduce a tool, the weight spectral sequence (WSS)

$$\mathrm{tmf}_{alg}E_1 = {}^{wss}E_0 \Rightarrow \mathrm{tmf}_{alg}E_2,$$

which serves as an analog of the May spectral sequence, and converges to the E_2 -term of the algebraic tmf resolution. The E_0 -page of the v_0 -localized weight spectral sequence is identified with the cobar complex of a primitively generated Hopf algebra, and this allows us to give “names” to the v_0 -torsion-free classes of $\mathrm{tmf}_{alg}E_1$. We include many charts of summands of $\mathrm{tmf}_{alg}E_1(\mathbb{F}_2)$ corresponding to tensor powers of bo-Brown-Gitler comodules which illustrate this naming convention, and provide the essential data for the rest of the computations in this paper. Finally, we study the g -local WSS⁴ using recent work of Bhattacharya-Bobkova-Thomas [BBT18], and show that many classes are killed in the g -local WSS by d_1 -differentials. This is the key fact we will use to systematically remove obstructions for lifting classes from tmf_*X to π_*X .

In Section 4 we study the structure of the MASS for $M(8, v_1^8)$. We recall the structure of the MASS for $\mathrm{tmf}_*M(8, v_1^8)$, and we explain how to adapt the Ext charts of Section 3 to give the corresponding computations of $\mathrm{tmf}_{alg}E_1(H(8, v_1^8))$. We then explain how to translate the computations of the g -localized algebraic tmf resolution of Section 3 to the case of $H(8, v_1^8)$.

Section 5 is dedicated to the proof of Theorem 1.7. We recall the work of Davis, Mahowald, and Rezk, who discovered topological attaching maps between the first two bo-Brown-Gitler spectra which comprise $\mathrm{tmf} \wedge \mathrm{tmf}$, which give extra differentials in the Adams spectral sequence of $\mathrm{tmf} \wedge \mathrm{tmf}$ that kill some g -torsion-free classes. We then prove a technical lemma (Lemma 5.5) which lifts differentials from the MASS for $\mathrm{tmf}^s \wedge M(8, v_1^8)$ to the MASS for $M(8, v_1^8)$. We prove Theorem 1.7 by listing

⁴Here, $g \in \mathrm{Ext}_{A_*}^{4,24}(\mathbb{F}_2)$ is the element corresponding to the element $h_{2,1}^4$ in the May spectral sequence which detects $\bar{\kappa}$ in the Adams spectral sequence for the sphere.

all elements in $\mathrm{tmf}_{alg} E_1(H(8, v_1^8))$ which could detect a non-trivial differential $d_r(v_2^{32})$ in the MASS for $M(8, v_1^8)$, and then we systematically eliminate these possibilities. Most of these classes are g -torsion-free, and are eliminated in the WSS, or by using Lemma 5.5.

In Section 6, we explain how v_1 -periodic computations give an upper bound on the Hurewicz image.

Section 7 is devoted to showing this upper bound is sharp, by producing lifts of the remaining elements of $\pi_*\mathrm{tmf}$ to the sphere. We begin by identifying multiplicative generators of the Hurewicz image in dimensions less than 192, so that it suffices for us to lift these. We then lift these elements by producing elements in the MASS for $M(8, v_1^8)$ which we show are permanent cycles, and detect elements of $\pi_*M(8, v_1^8)$ which project to the desired elements on the top cell. These elements are then propagated to v_2^{32} -periodic families using the self-map, thus proving Theorem 1.2 in all dimensions.

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2. PRELIMINARIES

The techniques and methods of this paper closely follow those of [BHHM20]. In this section we recall some spectral sequences used in that paper.

The modified Adams spectral sequence. Our computations of $\pi_*M(8, v_1^8)$ and $\mathrm{tmf}_*M(8, v_1^8)$ will be performed using the modified Adams spectral sequence (MASS). We refer the reader to [BHHM20, Sec. 6] for a complete account of the construction of the MASS and summarize the form it takes here.

Let St_{A_*} denote Hovey's stable homotopy category of A_* -comodules [Hov04]. For objects M and N of St_{A_*} , we define groups

$$\mathrm{Ext}_{A_*}^{s,t}(M, N) = \mathrm{St}_{A_*}(\Sigma^t M, N[s])$$

as a group of maps in the stable homotopy category. Here $\Sigma^t M$ denotes the t -fold shift with respect to the internal grading of M , and $N[s]$ denotes the s -fold shift with respect to the triangulated structure of St_{A_*} . This reduces to the usual definition of Ext_{A_*} when M and N are A_* -comodules.

Define $H(8)$ to be the cofiber of the map

$$(2.1) \quad \Sigma^3 \mathbb{F}_2[-3] \xrightarrow{h_0^3} \mathbb{F}_2$$

in the stable homotopy category of A_* -comodules. Define $H(8, v_1^8) \in \text{St}_{A_*}$ to be the cofiber

$$(2.2) \quad \Sigma^{24} H(8)[-8] \xrightarrow{v_1^8} H(8) \rightarrow H(8, v_1^8).$$

For a spectrum X , the MASS takes the form

$${}^{mass} E_2^{s,t}(M(8, v_1^8) \wedge X) = \text{Ext}_{A_*}^{s,t}(H(8, v_1^8) \otimes H_* X) \Rightarrow \pi_{t-s} M(8, v_1^8) \wedge X.$$

Recall the following from [BHHM20, Prop. 7.1].

Proposition 2.3. *$M(8, v_1^8)$ is a weak homotopy ring spectrum.*⁵

It follows that if X is a ring spectrum, the MASS above is a spectral sequence of (non-associative) algebras.

We recall the following key theorem of Mathew.

Theorem 2.4 (Mathew [Mat16]). *We have*

$$H_* \text{tmf} \cong A // A(2)_*$$

as an algebra in A_ -comodules.*

Taking $X = \text{tmf} \wedge Y$ for some Y , and applying a change of rings theorem, the MASS takes the form

$${}^{mass} E_2^{s,t}(\text{tmf} \wedge M(8, v_1^8) \wedge Y) = \text{Ext}_{A(2)_*}^{s,t}(H(8, v_1^8) \otimes H_* Y) \Rightarrow \text{tmf}_{t-s}(M(8, v_1^8) \wedge Y).$$

The algebraic tmf-resolution. The E_2 -page of the MASS for $M(8, v_1^8)$ will be analyzed using an algebraic analog of the tmf-resolution (as in [BHHM20, Sec. 6]).

The (topological) tmf-resolution of a space X is the Adams spectral sequence based on the spectrum tmf:

$$\text{tmf} E_1^{s,t} = \pi_t \text{tmf} \wedge \overline{\text{tmf}}^s \wedge X \Rightarrow \pi_{t-s} X.$$

Here, $\overline{\text{tmf}}$ is the cofiber of the unit:

$$S \rightarrow \text{tmf} \rightarrow \overline{\text{tmf}}.$$

The *algebraic* tmf-resolution is an algebraic analog. Namely, let M be an object of the stable homotopy category of A_* -comodules, and let $\overline{A // A(2)}_*$ denote the cokernel of the unit

$$0 \rightarrow \mathbb{F}_2 \rightarrow A // A(2)_* \rightarrow \overline{A // A(2)}_* \rightarrow 0$$

⁵By this, we mean a spectrum with a possibly non-associative product and a two sided unit in the stable homotopy category.

(note that $H_*\overline{\text{tmf}} = \overline{A//A(2)_*}$). The *algebraic tmf-resolution* of M is a spectral sequence of the form

$$\text{tmf}_{\text{alg}} E_1^{s,t,n}(M) = \text{Ext}_{A(2)_*}^{s,t}(\overline{A//A(2)_*}^{\otimes n} \otimes M) \Rightarrow \text{Ext}_{A_*}^{s+n,t}(M).$$

bo-Brown-Gitler comodules. We recall some material on bo-Brown-Gitler comodules. These are A_* -comodules which are the homology of the bo-Brown-Gitler spectra constructed by [GJM86]. Mahowald used integral Brown-Gitler spectra to analyze the bo resolution [Mah81]. The bo-Brown-Gitler comodules play a similar role in the algebraic tmf resolution [BHHM08], [MR09], [DM10], [BOSS19], [BHHM20].

Endow the mod 2 homology of the connective real K -theory spectrum

$$H_*(\text{bo}) \cong A//A(1)_* = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \dots]$$

with a multiplicative grading by declaring the *weight* of ζ_i to be

$$(2.5) \quad \text{wt}(\zeta_i) = 2^{i-1}.$$

The i th bo-Brown-Gitler comodule is the subcomodule

$$\underline{\text{bo}}_i = F_{4i}A//A(1)_* \subset A//A(1)_*$$

spanned by monomials of weight less than or equal to $4i$. It is isomorphic as an A_* -comodule to the homology of the i th bo-Brown-Gitler spectrum bo_i .

The analysis of the E_1 -page of the algebraic tmf-resolution is simplified via the decomposition of $A(2)_*$ -comodules

$$\overline{A//A(2)_*} \cong \bigoplus_{i>0} \Sigma^{8i} \underline{\text{bo}}_i$$

of [BHHM08, Cor. 5.5]. We therefore have a decomposition of the E_1 -page of the algebraic tmf-resolution for M given by

$$(2.6) \quad \text{tmf}_{\text{alg}} E_1^{s,t,n}(M) \cong \bigoplus_{i_1, \dots, i_n > 0} \text{Ext}_{A(2)_*}^{s,t}(\Sigma^{8(i_1 + \dots + i_n)} \underline{\text{bo}}_{i_1} \otimes \dots \otimes \underline{\text{bo}}_{i_n} \otimes M).$$

For any M , the computation of

$$\text{Ext}_{A(2)_*}^{s,t}(\Sigma^{8(i_1 + \dots + i_n)} \underline{\text{bo}}_{i_1} \otimes \dots \otimes \underline{\text{bo}}_{i_n} \otimes M)$$

can be inductively determined from $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes k} \otimes M)$ by means of a set of exact sequences of $A(2)_*$ -comodules which relate the $\underline{\text{bo}}_i$'s [BHHM08, Sec. 7] (see also [BOSS19]):

$$(2.7) \quad 0 \rightarrow \Sigma^{8j} \underline{\text{bo}}_j \rightarrow \underline{\text{bo}}_{2j} \rightarrow A(2)//A(1)_* \otimes \underline{\text{tmf}}_{j-1} \rightarrow \Sigma^{8j+9} \underline{\text{bo}}_{j-1} \rightarrow 0,$$

$$(2.8) \quad 0 \rightarrow \Sigma^{8j} \underline{\text{bo}}_j \otimes \underline{\text{bo}}_1 \rightarrow \underline{\text{bo}}_{2j+1} \rightarrow A(2)//A(1)_* \otimes \underline{\text{tmf}}_{j-1} \rightarrow 0$$

Here, $\underline{\text{tmf}}_j$ is the j th tmf-Brown-Gitler comodule — it is the subcomodule of

$$H_*(\text{tmf}) \cong A//A(2)_* = \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \dots]$$

spanned by monomials of weight less than or equal to $8j$.⁶

The exact sequences (2.7) and (2.8) can be re-expressed as resolutions in the stable homotopy category of $A(2)_*$ -comodules:

$$\begin{aligned} \underline{\mathrm{bo}}_{2j} &\rightarrow A(2) // A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \rightarrow \Sigma^{8j+9} \underline{\mathrm{bo}}_{j-1} \rightarrow \Sigma^{8j} \underline{\mathrm{bo}}_j[2], \\ \underline{\mathrm{bo}}_{2j+1} &\rightarrow A(2) // A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1} \rightarrow \Sigma^{8j} \underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1[1] \end{aligned}$$

which give rise to spectral sequences

$$(2.9) \quad E_1^{n,s,t} = \begin{cases} \mathrm{Ext}_{A(1)_*}^{s,t}(\underline{\mathrm{tmf}}_{j-1} \otimes M), & n = 0, \\ \mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^{8j+9} \underline{\mathrm{bo}}_{j-1} \otimes M[-1]), & n = 1, \\ \mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^{8j} \underline{\mathrm{bo}}_j \otimes M), & n = 2, \\ 0, & n > 2 \end{cases} \Rightarrow \mathrm{Ext}_{A(2)_*}^{s,t}(\underline{\mathrm{bo}}_{2j} \otimes M),$$

$$E_1^{n,s,t} = \begin{cases} \mathrm{Ext}_{A(1)_*}^{s,t}(\underline{\mathrm{tmf}}_{j-1} \otimes M), & n = 0, \\ \mathrm{Ext}_{A(2)_*}^{s,t}(\Sigma^{8j} \underline{\mathrm{bo}}_j \otimes \underline{\mathrm{bo}}_1 \otimes M), & n = 1, \\ 0, & n > 1 \end{cases} \Rightarrow \mathrm{Ext}_{A(2)_*}^{s,t}(\underline{\mathrm{bo}}_{2j+1} \otimes M).$$

These spectral sequences have been observed to collapse in low degrees (see [BOSS19]) but it is not known if they collapse in general. They inductively build $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_i \otimes M)$ out of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k} \otimes M)$ and $\mathrm{Ext}_{A(1)_*}(\underline{\mathrm{tmf}}_j \otimes M)$.

3. ANALYSIS OF THE ALGEBRAIC TMF RESOLUTION

In this section we will compute the d_1 -differential in the algebraic tmf resolution, and will introduce a tool, the *weight spectral sequence (WSS)*, which is a variant of the May spectral sequence that converges to the E_2 -page of the algebraic tmf resolution.

The d_1 differential in the algebraic tmf resolution. Our approach to understanding the d_1 -differential in the algebraic tmf -resolution will be to compute it on v_0 -torsion-free classes, and then infer its effect on v_0 -torsion classes by means of linearity over $\mathrm{Ext}_{A_*}(\mathbb{F}_2)$.

Consider the algebraic $BP\langle 2 \rangle$ and algebraic BP -resolutions.

$$\begin{aligned} {}^{BP\langle 2 \rangle}_{alg} E^{s,t,n} &= \mathrm{Ext}_{E[2]_*}^{s,t}(\overline{A // E[2]_*}^{\otimes n}) \Rightarrow \mathrm{Ext}_{A_*}^{s+n,t}(\mathbb{F}_2) \\ {}^{BP}_{alg} E^{s,t,n} &= \mathrm{Ext}_{E_*}^{s,t}(\overline{A // E_*}^{\otimes n}) \Rightarrow \mathrm{Ext}_{A_*}^{s+n,t}(\mathbb{F}_2) \end{aligned}$$

Here, $E[2] = E[Q_0, Q_1, Q_2]$ and $E = E[Q_0, Q_1, Q_2, \dots]$ denote subalgebras of the Steenrod algebra, where Q_i are the Milnor generators dual to $\xi_{i+1} \in A_*$.

⁶Technically speaking, as is addressed in [BHHM08, Sec. 7], the comodules $A(2) // A(1)_* \otimes \underline{\mathrm{tmf}}_{j-1}$ in the above exact sequences have to be given a slightly different $A(2)_*$ -comodule structure from the standard one arising from the tensor product. However, this different comodule structure ends up being Ext-isomorphic to the standard one. As we are only interested in Ext groups, the reader can safely ignore this subtlety.

The d_1 -differential in the algebraic tmf-resolution may be studied by means of the zig-zag

$$(3.1) \quad \mathrm{tmf}_{\mathrm{alg}} E_1^{*,*,*} \rightarrow {}^{BP\langle 2 \rangle} E_1^{*,*,*} \leftarrow {}^{BP} E_1^{*,*,*}.$$

Note that

$${}^{BP} E_1^{*,*,n} \cong \mathbb{F}_2[v_0, v_1, v_2, \dots] \otimes \overline{\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots]}^{\otimes n}$$

where $\overline{\mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots]}$ denotes the cokernel of the unit

$$\mathbb{F}_2 \rightarrow \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \dots].$$

The Adams spectral sequences

$${}^{BP} E_1^{n,*,*} = {}^{ass} E_2(BP \wedge \overline{BP}^n) \Rightarrow C_{BP_*BP}^m(BP_*)$$

collapse, where $C_{BP_*BP}^*$ is the normalized cobar complex for BP_*BP , and

$$\zeta_i^2 \in \overline{A//E}_* \text{ detects } t_i \in BP_*BP.$$

We conclude:

Lemma 3.2. *The d_1 differential in the algebraic BP resolution is the associated graded of the differential in the cobar complex for BP_*BP with respect to Adams filtration.*

The weight spectral sequence. Endow the normalized cobar complex

$$C^*(A_*, A_*, \mathbb{F}_2)$$

with a decreasing filtration by weight by defining

$$wt(a_0[a_1 | \dots | a_s]) = wt(a_1) + \dots + wt(a_s).$$

Applying $\mathrm{Ext}_{A_*}(\mathbb{F}_2, -)$ to the resulting filtered A_* -comodule produces a variant of the May spectral sequence which we will call the *modified May spectral sequence* (MMSS)⁷

$$(3.3) \quad {}^{mmss} E_0^{w,s,t} = C_{E^0 A_*}^*(\mathbb{F}_2) \Rightarrow \mathrm{Ext}_{A_*}^{s,t}(\mathbb{F}_2).$$

Since $E^0 A_*$ is primitively generated, we have

$${}^{mmss} E_1^{*,*} = \mathbb{F}_2[h_{i,j} : i \geq 1, j \geq 0].$$

The map $\mathrm{tmf} \rightarrow H$ induces an inclusion

$$\Phi : H_*(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n) \hookrightarrow H_*(H \wedge \overline{H}^n) \cong C^n(A_*, A_*, \mathbb{F}_2).$$

Under this inclusion, the weight filtration restricts to a decreasing filtration on

$$H_*(\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n) \cong A//A(2)_* \otimes \overline{A//A(2)}_*^{\otimes n}$$

by A_* -subcomodules. Because the weights of all of the generators of $A//A(2)_*$ are divisible by 8, we actually work with weights divided by 8. Applying $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2, -)$ and taking cohomology, we get the *weight spectral sequence* (WSS):

$${}^{wss} E_0^{w,n,s,t} = \bigoplus_{i_1 + \dots + i_n = w} \mathrm{Ext}_{A(2)_*}^{s,t}(\underline{\mathrm{bo}}_{i_1} \otimes \dots \otimes \underline{\mathrm{bo}}_{i_n}) \Rightarrow \mathrm{tmf}_{\mathrm{alg}} E_2^{n,s,t}.$$

⁷The authors of [LSWX19] construct a similar modified May spectral sequence, but with a slightly different filtration.

The WSS serves as an analog of the May spectral sequence for the algebraic tmf -resolution.

The map Φ above induces a map of spectral sequences

$$(3.4) \quad \begin{array}{ccc} wss E_0^{w,n,0,t} & \Longrightarrow & \mathrm{tmf}_{\mathrm{alg}} E_0^{n,0,t} \\ \Phi_* \downarrow & & \downarrow \Phi_* \\ mmss E_0^{8w,n,t} & \longrightarrow & \mathrm{Ext}_{A_*}^{n,t}(\mathbb{F}_2) \end{array}$$

The v_0 -localized algebraic tmf resolution. Observe that we have

$$(3.5) \quad v_0^{-1} \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2) = \mathbb{F}_2[v_0^\pm, v_1^4, v_2^2].$$

Note that $c_4, c_6 \in (\mathrm{tmf}_*)_{\mathbb{Q}}$ are detected in the v_0 -localized ASS by v_1^4 and $v_0^3 v_2^2$, respectively.

We recall from [BOSS19] that

$$(3.6) \quad v_0^{-1} \mathrm{Ext}_{A(2)_*}^{*,*,*}(A//A(2)_*) = \mathbb{F}_2[v_0^\pm, v_1^4, v_2^2][\zeta_1^8, \zeta_2^4]$$

and that there is an isomorphism

$$(3.7) \quad v_0^{-1} \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_i) \cong \mathbb{F}_2[v_0^\pm, v_1^4, v_2^2]\{\zeta_1^{8i'} \zeta_2^{4i''}\}_{i=i'+i''}.$$

We will now compute the localized E_1 -page $v_0^{-1} wss E_1$. The following is immediate from the computation of the cobar differential (modulo terms of higher Adams filtration) on the elements ζ_1^8 and ζ_2^4 , using (3.6), (3.7), and (3.1).

Proposition 3.8. *There is an isomorphism of differential graded algebras*

$$v_0^{-1} wss E_0^{*,n,*,*} \cong \mathbb{F}_2[v_0^\pm, v_1^4, v_2^2] \otimes C_{\mathbb{F}_2[\zeta_1^8, \zeta_2^4]}^m$$

where $\mathbb{F}_2[\zeta_1^8, \zeta_2^4]$ is regarded as a primitively generated Hopf algebra.

Corollary 3.9. *There is an isomorphism*

$$v_0^{-1} wss E_1 = \mathbb{F}_2[v_0^\pm, v_1^4, v_2^2] \otimes \mathbb{F}_2[h_{1,3}, h_{1,4}, \dots, h_{2,2}, h_{2,3}, \dots]$$

Charts. For the convenience of the reader we include some charts of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^k)$ for $0 \leq k \leq 3$ as well as $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_2)$.

$\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$: (Figure 3.1)

All of the elements are $c_4 = v_1^4$ -periodic, and v_2^8 -periodic. Exactly one v_1^4 multiple of each element is displayed with the \bullet replaced by a \circ . Observe the wedge pattern beginning in $t - s = 35$. This pattern is infinite, propagated horizontally by $h_{2,1}$ -multiplication and vertically by v_1 -multiplication. Here, $h_{2,1}$ is the name of the generator in the May spectral sequence of bidegree $(t - s, s) = (5, 1)$, and $h_{2,1}^4 = g$.

$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$, for $k = 1, 2, 3$: (Figures 3.2, 3.3, 3.4)

Every element is v_2^8 -periodic. However, unlike $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$, not every element of these Ext groups is v_1^4 -periodic. Rather, it is the case that either an element

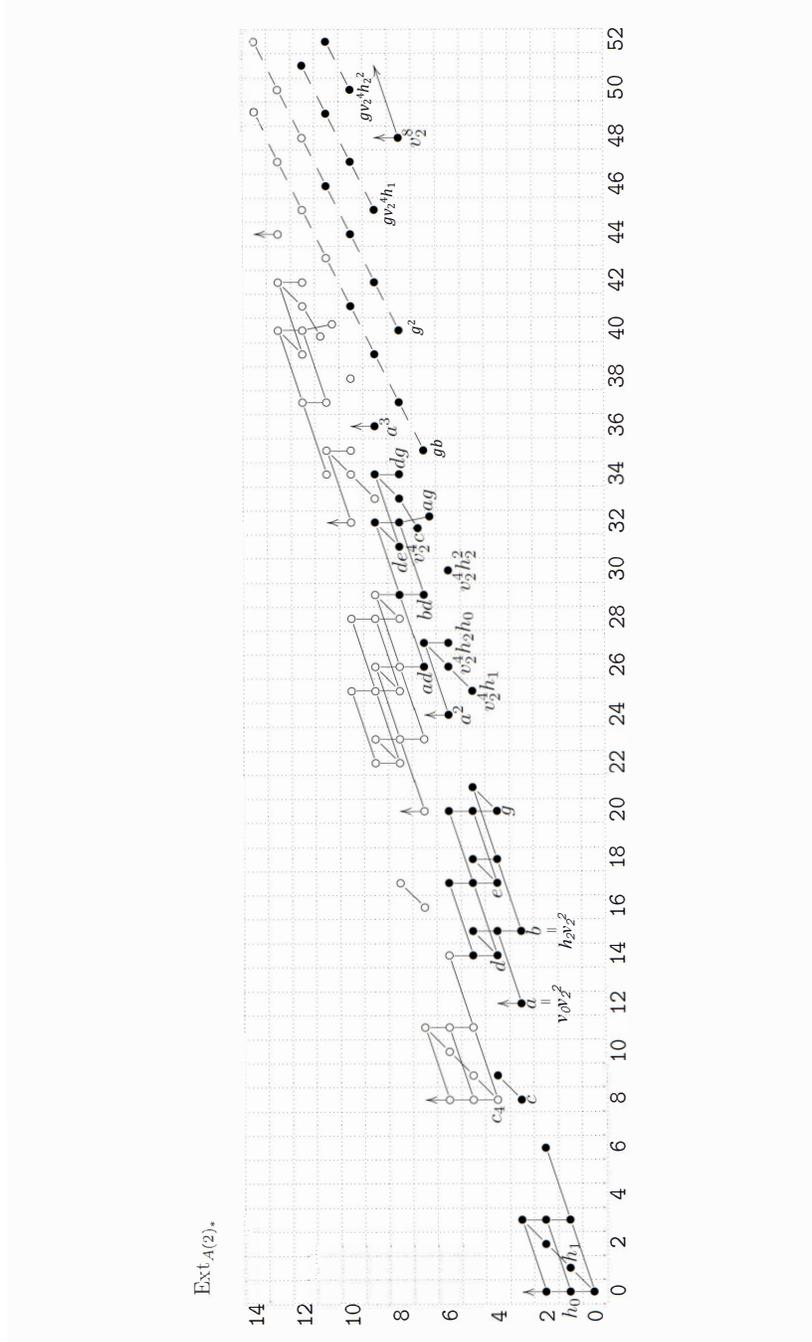
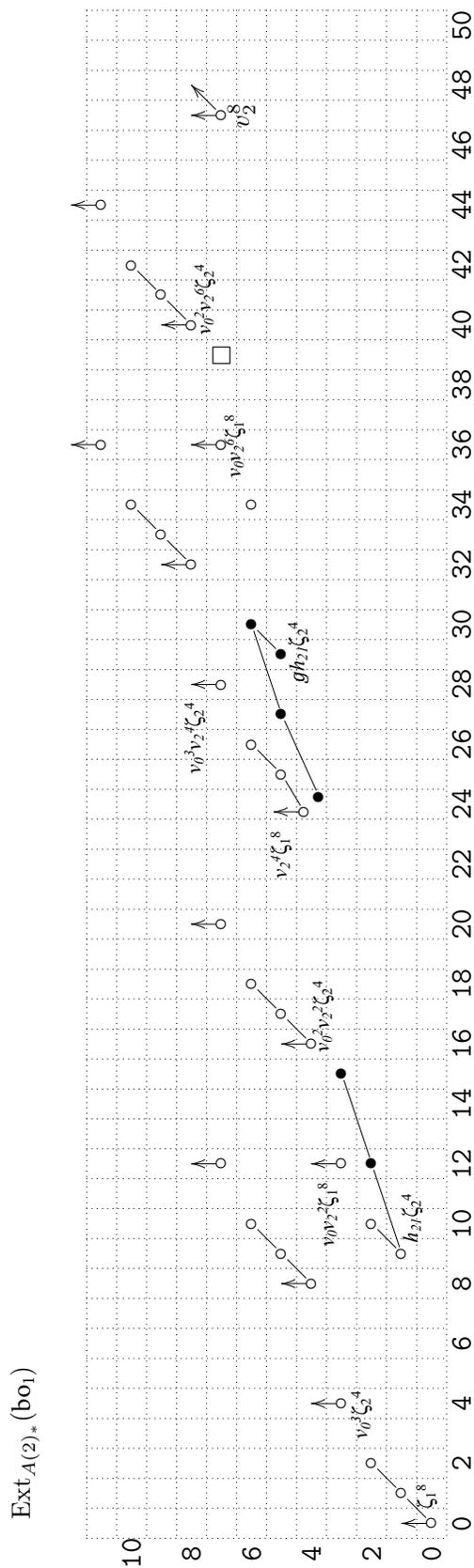


FIGURE 3.1. $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$.



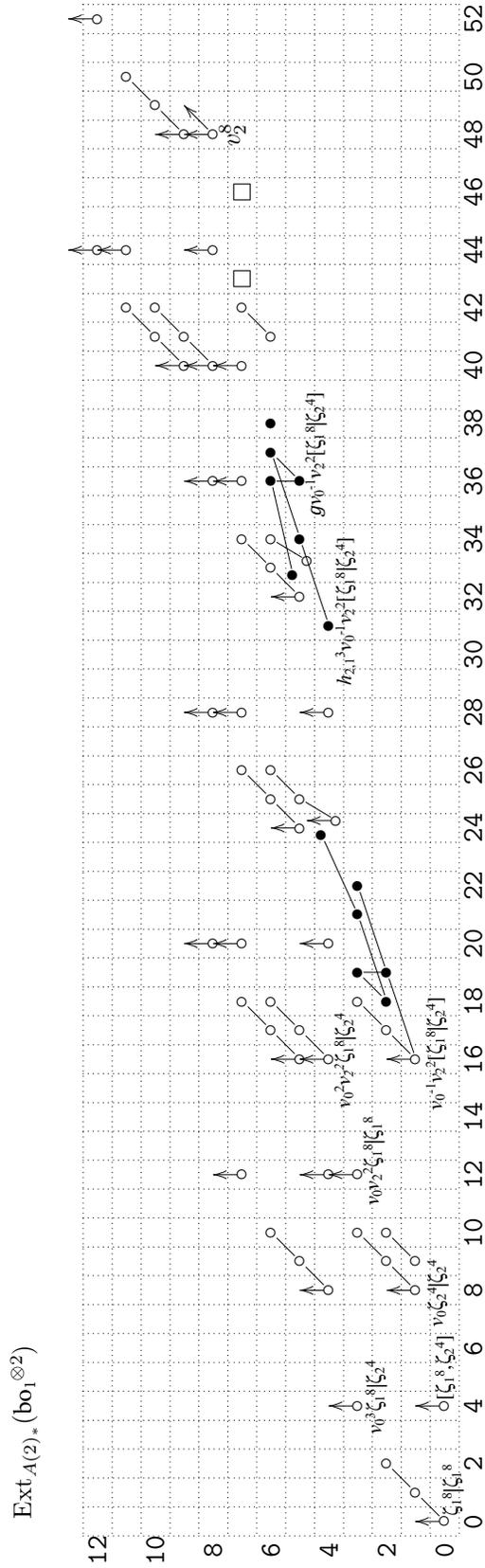


FIGURE 3.3. $\text{Ext}_{A(2)_*}(\text{bo}_1^{\otimes 2})$.

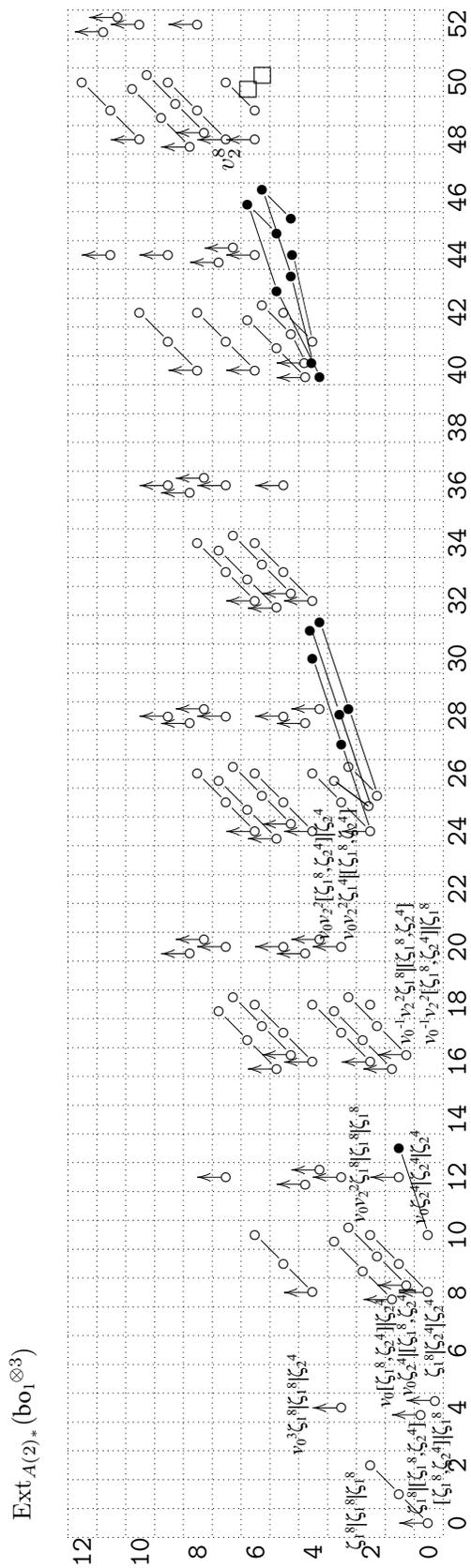


FIGURE 3.4. $\text{Ext}_{A(2)_*}(\text{bo}_1^{\otimes 3})$.

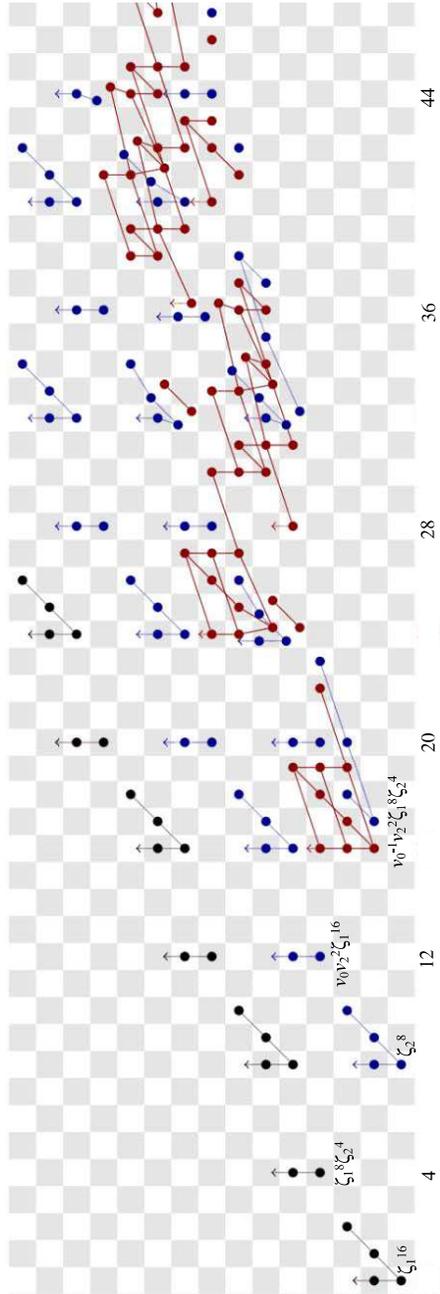


FIGURE 3.5. $\text{Ext}_{A(2)_*}(\underline{bO}_2)$.

$x \in \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$ satisfies $v_1^4 x = 0$, or it is v_1^4 -periodic. Each of the v_1^4 -periodic elements fit into families which look like shifted and truncated copies of $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2)$, and are labeled with a \circ . We have only included the beginning of these v_1^4 -periodic patterns in the chart. The other generators are labeled with a \bullet . A \square indicates a polynomial algebra $\mathbb{F}_2[h_{2,1}]$.

$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_2)$: (Figure 3.5)

Via the spectral sequence (2.9), this Ext chart is assembled out of $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_2)$ (black), $\mathrm{Ext}_{A(2)_*}(\Sigma^8 \underline{\mathrm{bo}}_1)$ (blue), and $\mathrm{Ext}_{A(2)_*}(\Sigma^{17} \mathbb{F}_2[-1])$ (red).

$h_{2,1}$ -towers. Our computations of the MASS for $M(8, v_1^8)$ will rely on a detailed understanding of this spectral sequence near its vanishing line. Since $M(8, v_1^8)$ is a type 2 complex, the Hopkins-Smith Periodicity Theorem [HS98] implies that the E_∞ -page of this MASS has a vanishing line of slope $1/|v_2| = 1/6$. However, $g = h_{2,1}^4$ is not nilpotent in the modified Ext groups $\mathrm{Ext}_{A_*}(H(8, v_1^8))$, and $h_{2,1}$ -multiplication has slope $1/5$. The goal of this subsection is to show that many of the $h_{2,1}$ -towers in the E_1 -page of the algebraic tmf resolution actually kill each other off by the E_2 -page of the algebraic tmf resolution. We will then identify specific $h_{2,1}$ -periodic elements of $\mathrm{Ext}_{A_*}(\mathbb{F}_2)$ that some of these remaining $h_{2,1}$ -towers detect.

Consider the quotient Hopf algebra $C_* := \mathbb{F}_2[\zeta_2]/(\zeta_2^4)$ of $A(2)_*$, with

$$\mathrm{Ext}_{C_*}^{*,*}(\mathbb{F}_2) = \mathbb{F}_2[v_1, h_{2,1}].$$

Lemma 3.10. *Let $C(v_2^8)$ be the cofiber of the map*

$$v_2^8 : \Sigma^{56} \mathbb{F}_2[-8] \rightarrow \mathbb{F}_2$$

in the stable homotopy category $\mathrm{St}_{A(2)_}$. For any $M \in \mathrm{St}_{A(2)_*}$ there is an isomorphism*

$$g^{-1} \mathrm{Ext}_{A(2)_*}(M \otimes C(v_2^8)) \cong h_{2,1}^{-1} \mathrm{Ext}_{C_*}(M).$$

Proof. Since the element $v_2^8 \in \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ maps to zero in $\mathrm{Ext}_{C_*}(\mathbb{F}_2)$, it follows that there is a factorization

$$\begin{array}{ccc} \mathbb{F}_2 & \longrightarrow & A(2) // C_* \\ \downarrow & & \nearrow \\ C(v_2^8) & & \end{array}$$

in $\mathrm{St}_{A(2)_*}$. Explicit computation reveals

$$g^{-1} \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2) = \mathbb{F}_2[v_2^8, v_1, h_{2,1}^\pm]$$

and it follows that the map

$$g^{-1} C(v_2^8) \rightarrow g^{-1} A(2) // C_*$$

induces an isomorphism on $\mathrm{Ext}_{A(2)_*}$, and is hence an equivalence. The result follows. \square

Corollary 3.11. *For any $M \in \mathrm{St}_{A(2)_*}$, there is a v_2^8 -Bockstein spectral sequence*

$$h_{2,1}^{-1} \mathrm{Ext}_{C_*}(M) \otimes \mathbb{F}_2[v_2^8] \Rightarrow g^{-1} \mathrm{Ext}_{A(2)_*}(M).$$

Bhattacharya, Bobkova, and Thomas [BBT18] computed the P_2^1 -Margolis homology of the tmf-resolution, and in the process computed the structure of $A//A(2)_*^{\otimes n}$ as C_* -comodules. From this one can read off the Ext groups

$$h_{2,1}^{-1} \text{Ext}_{C_*}(A//A(2)_*^{\otimes n}),$$

which in turn determines the g -local algebraic tmf-resolution by Corollary 3.11 (the spectral sequence in this corollary will collapse in the cases we consider it).

To state the results of [BBT18] we will need to introduce some notation. The coaction of $\mathbb{F}_2[\zeta_2]/\zeta_2^4$ is encoded in the dual action of the algebra $E[Q_1, P_2^1]$ on $A//A(2)_*^{\otimes n}$. Define elements

$$\begin{aligned} x_{i,j} &= 1 \otimes \cdots \otimes 1 \otimes \underbrace{\zeta_{i+3}}_j \otimes 1 \otimes \cdots \otimes 1, \\ t_{i,j} &= 1 \otimes \cdots \otimes 1 \otimes \underbrace{\zeta_{i+1}^4}_j \otimes 1 \otimes \cdots \otimes 1 \end{aligned}$$

in $A//A(2)_*^{\otimes n}$. The weight filtration on $A//A(2)_*$ induces a *multi-weight filtration* on $A//A(2)_*^{\otimes n}$ indexed by n -tuples of weights. The generators $x_{i,j}$ and $t_{i,j}$ have multi-weight

$$(0, \dots, 0, \underbrace{2^{i+2}}_j, 0, \dots, 0).$$

For sets of multi-indices

$$\begin{aligned} I &= \{(i_1, j_1), \dots, (i_k, j_k)\}, \\ I' &= \{(i'_1, j'_1), \dots, (i'_{k'}, j'_{k'})\} \end{aligned}$$

with $I \cap I' = \emptyset$, let

$$x_I t_{I'} \in A//A(2)_*$$

denote the corresponding monomial. The action of the algebra $E[Q_1, P_2^1]$ on the \mathbb{F}_2 -submodule of $A//A(2)_*^{\otimes n}$ spanned by such monomials is given by

$$\begin{aligned} Q_1(x_I t_{I'}) &= \sum_{\ell} x_{I - \{(i_\ell, j_\ell)\}} t_{I' \cup \{(i_\ell, j_\ell)\}}, \\ P_2^1(x_I t_{I'}) &= \sum_{\ell < \ell'} x_{I - \{(i_\ell, j_\ell), (i_{\ell'}, j_{\ell'})\}} t_{I' \cup \{(i_\ell, j_\ell), (i_{\ell'}, j_{\ell'})\}}. \end{aligned}$$

For an *ordered* set

$$J = ((i_1, j_1), \dots, (i_k, j_k))$$

of multi-indices, let

$$|J| := k$$

denote the number of pairs of indices it contains. Define linearly independent sets of elements

$$\mathcal{T}_J \subset A//A(2)_*^{\otimes n}$$

inductively as follows: for J as above with $|J|$ odd, define

$$\begin{aligned} \mathcal{T}_{J, (i,j)} &= \{z \cdot x_{i,j}\}_{z \in \mathcal{T}_J}, \\ \mathcal{T}_{J, (i,j), (i',j')} &= \{Q_1(z \cdot x_{i,j}) x_{i',j'}\}_{z \in \mathcal{T}_J} \cup \{Q_1(z \cdot x_{i',j'}) x_{i,j}\}_{z \in \mathcal{T}_J}. \end{aligned}$$

Let

$$N_J \subset A//A(2)_*^{\otimes n}$$

denote the \mathbb{F}_2 -subspace with basis

$$Q_1 \mathcal{T}_J := \{Q_1(z)\}_{z \in \mathcal{T}_J}.$$

While the set \mathcal{T}_J depends on the ordering of J , the subspace N_J does not.

The following is the main theorem of [BBT18]⁸

Theorem 3.12 (Bhattacharya-Bobkova-Thomas). *As modules over $\mathbb{F}_2[h_{2,1}^\pm, v_1]$, we have*

$$h_{2,1}^{-1} \mathrm{Ext}_{E[Q_1, P_2^1]}^{*,*}(A // A(2)_*^{\otimes n}) = \mathbb{F}_2[h_{2,1}^\pm][x_{i,j} \cdot t_{i,j}]_{\substack{1 \leq i, \\ 1 \leq j \leq n}} \otimes \left(\mathbb{F}_2[v_1] \oplus \bigoplus_{|J| \text{ odd}} N_J \oplus \bigoplus_{|J| \neq 0 \text{ even}} \mathbb{F}_2[v_1]/v_1^2 \otimes N_J \right)$$

where J ranges over the non-empty subsets of

$$\{(i, j) : 1 \leq i, 1 \leq j \leq n\}$$

and v_1 acts trivially on N_J for $|J|$ odd. The summand

$$h_{2,1}^{-1} \mathrm{Ext}_{E[Q_1, P_2^1]}^{*,*}(\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n})$$

is spanned by those monomials of multiweight $(8i_1, \dots, 8i_n)$.

In light of Lemma 3.10 and Corollary 3.11, we may refer to elements of the g -local algebraic tmf resolution as $v_2^{8j}z$, where z is an element of the $h_{2,1}$ -localized Ext groups described in the theorem above.

Lemma 3.13. *The WSS d_0 -differential on the element*

$$x_{1,1}t_{1,1} \in g^{-1} \mathrm{Ext}_{A(2)_*}^{*,*}(\underline{\mathrm{bo}}_2)$$

is given by

$$d_0^{wss}(x_{1,1}t_{1,1}) = Q_1(x_{1,1}x_{1,2}) \in \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2}).$$

Proof. We use the map of spectral sequences

$${}^{wss}E_0 \rightarrow g^{-1} {}^{wss}E_0.$$

By explicit computation of $g^{-1} \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_2)$, under the map

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_2) \rightarrow g^{-1} \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_2)$$

we have

$$v_0^{-1}v_2^2\zeta_1^8\zeta_2^4 \mapsto h_{2,1}x_{1,1}t_{1,1}.$$

In the WSS we have

$$(3.14) \quad d_0^{wss}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4) = v_0^{-1}v_2^2[\zeta_1^8, \zeta_2^4].$$

Again, by explicit computation of g -local Ext groups, under the map

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2}) \rightarrow g^{-1} \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2})$$

we have

$$v_0^{-1}v_2^2[\zeta_1^8, \zeta_2^4] \mapsto h_{2,1}Q_1(x_{1,1}x_{1,2}).$$

The result follows. \square

⁸The main theorem of [BBT18] is a computation of P_2^1 -Margolis homology, but the actual content of the paper is a decomposition of $A // A(2)_*$ in the stable module category of $E[Q_1, P_2^1]$.

Proposition 3.15. *In $g^{-1} wss E_0$, all of the $h_{2,1}$ -towers coming from $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes k})$, for $k \geq 2$, either support non-trivial d_0 -differentials, or are the target of d_0 -differentials.*

Proof. By Lemma 3.10 and Theorem 3.12, the $h_{2,1}$ -towers coming from

$$\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes k})$$

are supported by the elements $\mathcal{T}_{\{(1,1), \dots, (1,k)\}}$. By Lemma 3.13, the WSS d_0 induces a surjection for $k = 2$

$$d_0^{wss} : \mathbb{F}_2[h_{2,1}^\pm, v_1, v_2^8]\{x_{1,1}t_{1,1}\} \twoheadrightarrow \mathbb{F}_2[h_{2,1}^\pm, v_1, v_2^8]/v_1^2 \otimes N_{\{(1,1), (1,2)\}}.$$

For $k > 2$, observe that

$$\mathcal{T}_{(1,1), \dots, (1,k)} = Q_1(x_{1,1}x_{1,2})\mathcal{T}_{(1,3), \dots, (1,k)} \cup Q_1(x_{1,2}x_{1,3})\mathcal{T}_{(1,1), (1,4), \dots, (1,k)}.$$

For $k > 2$ even the WSS d_0 gives isomorphisms

$$\begin{aligned} d_0^{wss} : \mathbb{F}_2[h_{2,1}^\pm, v_1, v_2^8]/v_1^2 \otimes x_{1,1}t_{1,1}N_{\{(1,2), \dots, (1,k-1)\}} \\ \xrightarrow{\cong} \mathbb{F}_2[h_{2,1}^\pm, v_1, v_2^8]/v_1^2 \otimes Q_1(x_{1,1}x_{1,2})N_{\{(1,3), \dots, (1,k)\}}, \end{aligned}$$

$$\begin{aligned} d_0^{wss} : \mathbb{F}_2[h_{2,1}^\pm, v_1, v_2^8]/v_1^2 \otimes x_{1,2}t_{1,2}N_{\{(1,1), (1,3), \dots, (1,k-1)\}} \\ \xrightarrow{\cong} \mathbb{F}_2[h_{2,1}^\pm, v_1, v_2^8]/v_1^2 \otimes Q_1(x_{1,2}x_{1,3})N_{\{(1,1), (1,4), \dots, (1,k)\}}, \end{aligned}$$

and for $k > 2$ odd the WSS d_0 gives isomorphisms

$$\begin{aligned} d_0^{wss} : \mathbb{F}_2[h_{2,1}^\pm, v_2^8] \otimes x_{1,1}t_{1,1}N_{\{(1,2), \dots, (1,k-1)\}} \\ \xrightarrow{\cong} \mathbb{F}_2[h_{2,1}^\pm, v_2^8] \otimes Q_1(x_{1,1}x_{1,2})N_{\{(1,3), \dots, (1,k)\}}, \end{aligned}$$

$$\begin{aligned} d_0^{wss} : \mathbb{F}_2[h_{2,1}^\pm, v_2^8] \otimes x_{1,2}t_{1,2}N_{\{(1,1), (1,3), \dots, (1,k-1)\}} \\ \xrightarrow{\cong} \mathbb{F}_2[h_{2,1}^\pm, v_2^8] \otimes Q_1(x_{1,2}x_{1,3})N_{\{(1,1), (1,4), \dots, (1,k)\}}. \end{aligned}$$

□

We shall denote the elements of the Mahowald-Tangora wedge [MT68] in $\text{Ext}_{A_*}(\mathbb{F}_2)$ by⁹

$$v_1^i h_{2,1}^j g^2, \quad i \geq 0, j \geq 0.$$

Recall that the Mahowald operator

$$M = \langle g_2, h_0^3, - \rangle$$

leads to an infinite collection of wedges

$$M^k(v_1^i h_{2,1}^j g^2) \in \text{Ext}_{A_*}(\mathbb{F}_2)$$

with non-zero image in

$$\text{Ext}_{B_*}(\mathbb{F}_2) = \text{Ext}_{A(2)_*}(\mathbb{F}_2)[v_3]$$

where B_* is the quotient algebra

$$(3.16) \quad B_* := \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \zeta_4]/(\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2)$$

⁹This notation is slightly misleading, as there are a few wedge elements for which the P operator does not take the element we are denoting $v_1^i x$ to the element we are denoting $v_1^{i+4} x$, but we justify this notation by the fact that the wedge elements map to elements with such names in $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$.

of A_* [MPT70],[Isa20]. The existence of the element $\Delta^2 g^2 \in \text{Ext}_{A_*}(\mathbb{F}_2)$ gives elements

$$\Delta^{2m} M^k (v_1^i h_{2,1}^{j+8m} g^2) \in \text{Ext}_{A_*}(\mathbb{F}_2).$$

These elements are all linearly independent, since they project to linearly independent elements of $\text{Ext}_{B_*}(\mathbb{F}_2)$.

The following proposition gives the elements of $\text{Ext}_{A(2)_*}$ that some of the remaining $h_{2,1}$ towers in $\text{Ext}_{A(2)_*}$ detect in the algebraic tmf resolution.

Proposition 3.17. *The following table lists, for $i \geq 0$, $m \geq 0$, and $j \geq 4$ an $A(2)_*$ -comodule M , an $h_{2,1}$ -tower in $g^{-1} \text{Ext}_{A(2)_*}(M)$, the corresponding $h_{2,1}$ -tower in $\text{Ext}_{A(2)_*}(M)$, and an $h_{2,1}$ -tower in $\text{Ext}_{A_*}(\mathbb{F}_2)$ that it detects in the algebraic tmf resolution (assuming the latter is non-zero).*

M	$g^{-1} \text{Ext}_{A(2)_*}(M)$	$\text{Ext}_{A(2)_*}(M)$	$\text{Ext}_{A_*}(\mathbb{F}_2)$
\mathbb{F}_2	$\Delta^{2m} v_1^i h_{2,1}^{j+8m+8}$	$\Delta^{2m} v_1^i h_{2,1}^{j+8m} g^2$	$\Delta^{2m} v_1^i h_{2,1}^{j+8m} g^2$
bo_1	$\Delta^{2m} h_{2,1}^{j+8m+4} Q_1(x_{1,1})$	$\Delta^{2m} h_{2,1}^{j+8m+4} \zeta_2^4$	$\Delta^{2m} h_{2,1}^{j+8m} n$
bo_2	$\Delta^{2m} h_{2,1}^{j+8m+6} Q_1(x_{2,1})$ $\Delta^{2m} v_1^{i+2} h_{2,1}^{j+8m+11} x_{1,1} t_{1,1}$	$\Delta^{2m} h_{2,1}^{j+8m+1} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})$ $\Delta^{2m} v_1^{i+2} h_{2,1}^{j+8m+2} g^2 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)$	$\Delta^{2m} h_{2,1}^{j+8m} Q_2$ $\Delta^{2m} v_1^i h_{2,1}^{j+8m} M g^2$

(Note that the notation Q_2 in the above table refers to the name of the generator of $\text{Ext}_{A_*}^{7,57+7}(\mathbb{F}_2)$, and not the Milnor generator $Q_2 \in A$.)

Proof. The classes corresponding to $\Delta^{2m} v_1^i h_{2,1}^k$ are clear, because they are in the image of the map

$$\text{Ext}_{A_*}(\mathbb{F}_2) \rightarrow \text{Ext}_{A(2)_*}(\mathbb{F}_2).$$

In the case of the classes corresponding to $\Delta^{2m} h_{2,1}^k n$, $\Delta^{2m} h_{2,1}^k Q_2$, we consider the $h_{2,1}^j$ multiples of $n, Q_2 \in \text{Ext}_{A_*}(\mathbb{F}_2)$ for $j \geq 4$:

$$gn, gt, rn, mn, g^2 n, \dots, \\ gQ_2, gC_0, rQ_2, mQ_2, g^2 Q_2 \dots$$

It suffices to show that

$$n, t, Q_2, C_0$$

are detected in the algebraic tmf resolution by

$$(3.18) \quad h_{2,1}^4 \zeta_2^4 + \alpha_1, h_{2,1}^5 \zeta_2^4 + \alpha_2, h_{2,1}^6 v_0^{-2} v_2^2 \zeta_1^{16} + \alpha_3, h_{2,1}^7 v_0^{-2} v_2^2 \zeta_1^{16} + \alpha_4$$

where $g\alpha_i = r\alpha_i = m\alpha_i = 0$.

Examination of a computer calculation of $\text{Ext}_{A_*}(\overline{A//A(2)_*}^{\otimes 2})$ reveals that none of the elements n, t, Q_2, C_0 are in the image of the map

$$(3.19) \quad \text{Ext}_{A_*}^{*,*}(\overline{A//A(2)_*}^{\otimes 2}) \rightarrow \text{Ext}_{A_*}^{*+2,*}(\mathbb{F}_2).$$

Since the elements n, t, Q_2 , and C_0 map to zero in $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$, they must therefore be detected on the 1-line of the algebraic tmf resolution. Examination of the

relevant Ext charts reveals the only possibility is for the elements to be detected by classes of the form (3.18).

If we consider the class $Mg \in \text{Ext}_{A_*}(\mathbb{F}_2)$, one can both check that it is not in the image of (3.19), and that the only class in $\text{Ext}_{A(2)_*}(\overline{A//A(2)}_*)$ which can detect it is the class

$$e_0^2(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4) \in \text{Ext}_{A(2)_*}(\underline{\text{bo}}_2).$$

It follows from the multiplicative structure of the wedge, and the fact that

$$ge_0^2 = v_1^2h_{2,1}^2g^2,$$

that the elements $v_1^i h_{2,1}^j Mg^2 \in \text{Ext}_{A_*}(\mathbb{F}_2)$ are detected by

$$v_1^{i+2}h_{2,1}^{j+2}g^2(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4) \in \text{Ext}_{A(2)_*}(\underline{\text{bo}}_2)$$

for $i \geq 0$ and $j \geq 4$. □

4. THE MASS FOR $M(8, v_1^8)$

In this and following sections, we shall use the notation

$$x[k]$$

to denote an element of $\text{Ext}_{A(2)_*}(M \otimes H(8, v_1^8))$ detected by an element

$$x \in \text{Ext}_{A(2)_*}(M)$$

on the k -cell of $H(8, v_1^8)$ ($k \in \{0, 1, 17, 18\}$).

The MASS for $\text{tmf}_*M(8, v_1^8)$. The computation of $\text{Ext}_{A(2)_*}(H(8, v_1^8))$ is depicted in Figure 4.1. In this figure, solid dots correspond to classes carried by the “0-cell” of $H(8, v_1^8)$, and open circles correspond to classes carried by the “1-cell” of $H(8, v_1^8)$. The large solid circles correspond to h_0 -torsion free classes of $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$ on the 0-cell of $H(8, v_1^8)$. The classes with solid boxes around them support $h_{2,1}$ towers. Everything is v_2^8 -periodic.

Figure 4.2 depicts the differentials in the MASS for $\text{tmf} \wedge M(8, v_1^8)$ through the same range; the complete computation of this MASS can be similarly accomplished. An explanation of how to determine these differentials can be found in [BHHM20].

The algebraic tmf resolution for $H(8, v_1^8)$. The following lemma explains that, in our $H(8, v_1^8)$ computations, we may disregard terms coming from $\text{Ext}_{A(1)_*}$ in the sequence of spectral sequences (2.9).

Lemma 4.1 (Lemma 8.8 of [BHHM20]). *In the algebraic tmf -resolution for $M = H(8, v_1^8)$, the terms*

$$\text{Ext}_{A(1)_*}(\text{something})$$

in (2.9) do not contribute to $\text{Ext}_{A_}^{s,t}(H(8, v_1^8))$ if*

$$s > \frac{1}{7}(t - s) + \frac{51}{7}.$$

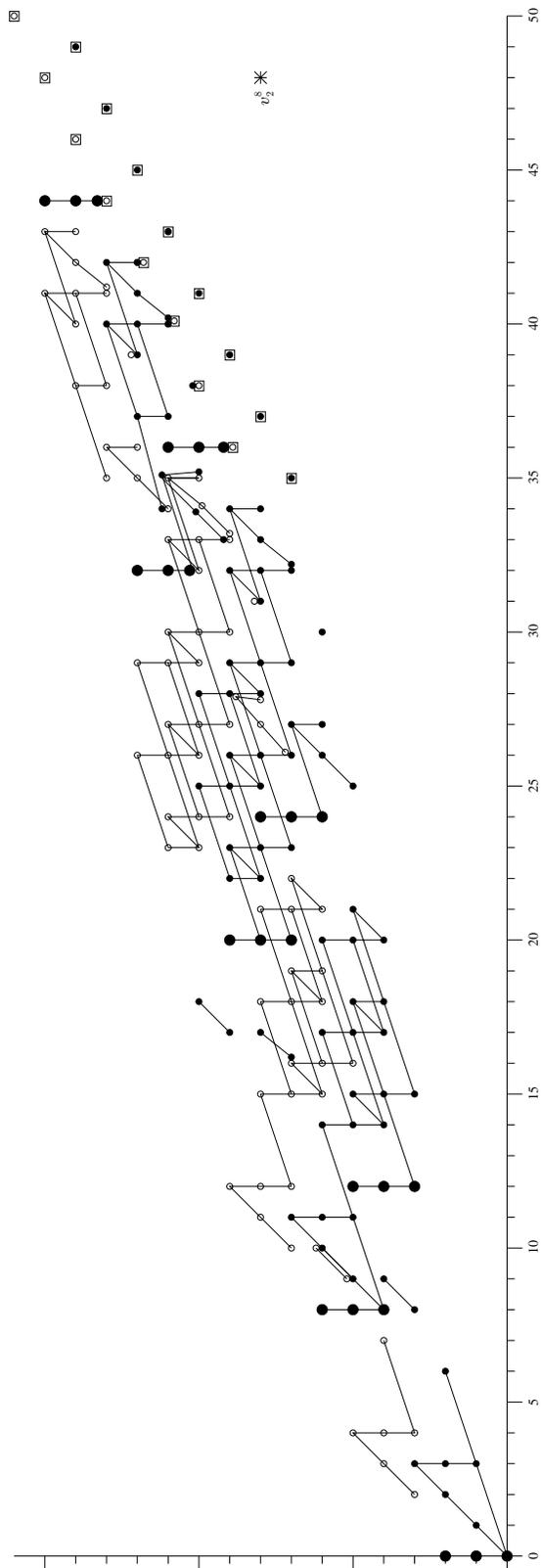


FIGURE 4.1. The groups $\text{Ext}_{A(2)_*}(H(8, v_1^8))$.

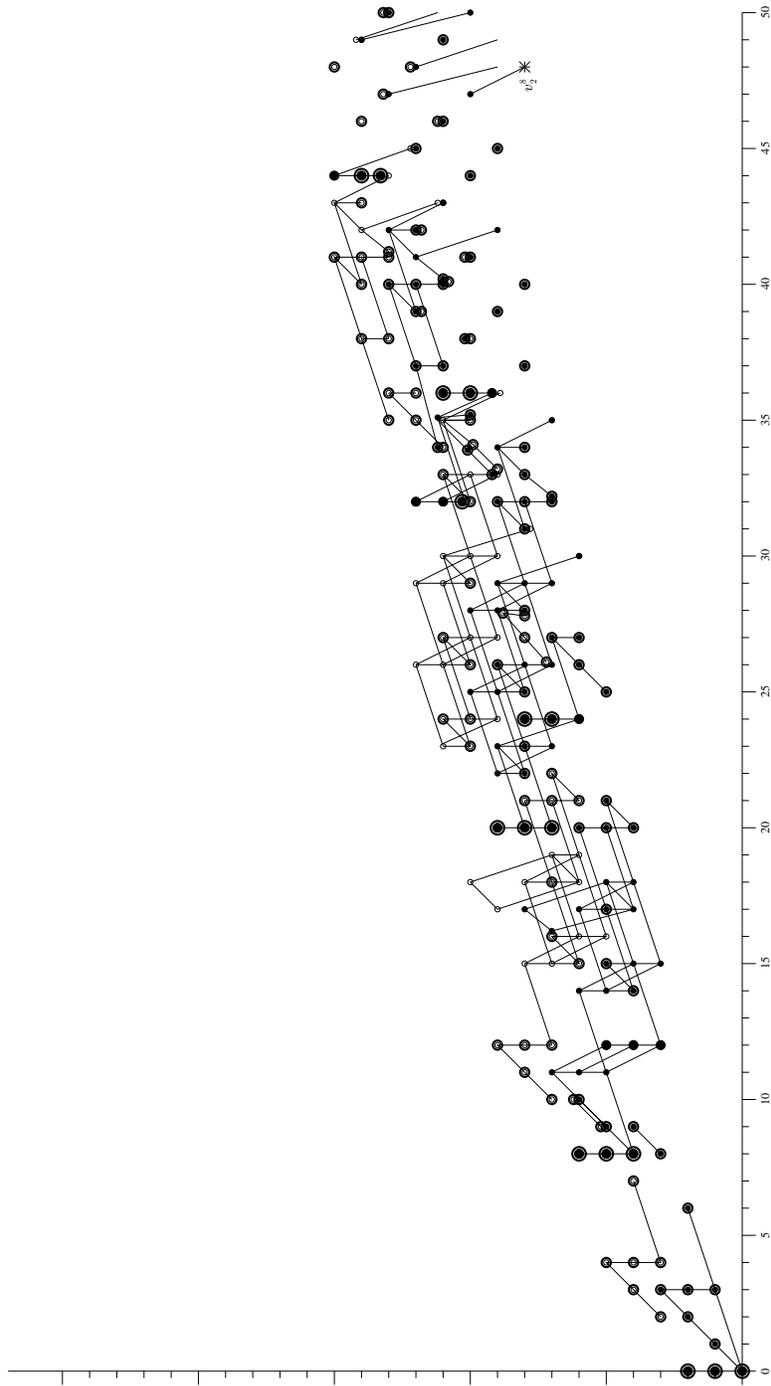


FIGURE 4.2. The MASS for $\text{tmf} \wedge M(8, v_1^8)$.

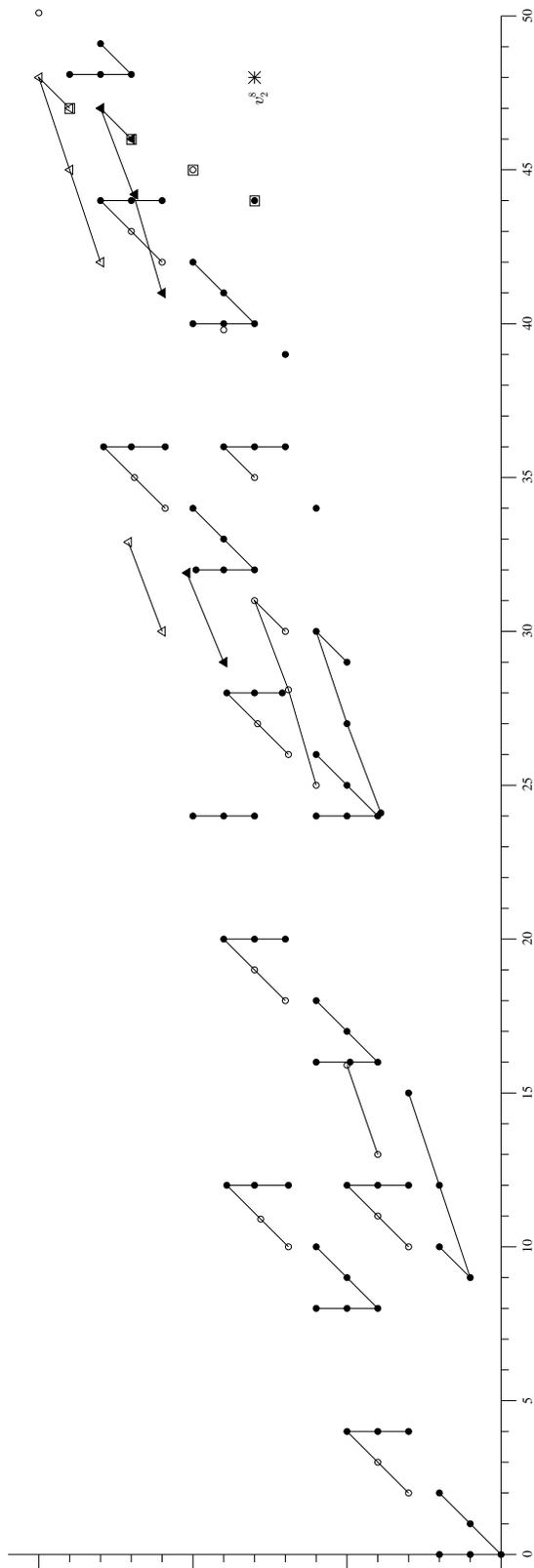


FIGURE 4.3. $\text{Ext}_{A(2)_*}(b\Omega_1 \otimes H(8, v_1^8))$.

For $n > 0$ and $i_1, \dots, i_n > 0$, the terms

$$\mathrm{Ext}_{A(2)_*}^{s,t}(\underline{\mathrm{bo}}_{i_1} \otimes \cdots \otimes \underline{\mathrm{bo}}_{i_n} \otimes H(8, v_1^8))$$

that comprise the terms in the algebraic tmf-resolution for $H(8, v_1^8)$ are in some sense less complicated than $\mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$.

Most of the features of these computations can already be seen in the computation of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$, which is displayed in Figure 4.3. This computation was performed by taking the computation of $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1)$ (see, for example, [BHHM08]) and running the long exact sequences in Ext associated to the cofiber sequences

$$\begin{aligned} \Sigma^3 \underline{\mathrm{bo}}_1[-3] &\xrightarrow{h_0^3} \underline{\mathrm{bo}}_1 \rightarrow \underline{\mathrm{bo}}_1 \otimes H(8), \\ \Sigma^{24} \underline{\mathrm{bo}}_1 \otimes H(8)[-8] &\xrightarrow{v_1^8} \underline{\mathrm{bo}}_1 \otimes H(8) \rightarrow \underline{\mathrm{bo}}_1 \otimes H(8, v_1^8). \end{aligned}$$

In Figure 4.3, as before, solid dots represent generators carried by the 0-cell of $H(8, v_1^8)$ and open circles are carried by the 1-cell. Unlike the case of $\mathrm{Ext}_{A(2)_*}(H(8))$, there is v_1^8 -torsion in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8))$. This results in classes in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$ carried by the 17-cell and the 18-cell of $H(8, v_1^8)$, which are represented by solid triangles and open triangles, respectively. A box around a generator indicates that that generator actually carries a copy of $\mathbb{F}_2[h_{2,1}]$. As before, everything is v_2^8 -periodic.

One can similarly compute

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k} \otimes H(8, v_1^8))$$

for larger values of k by applying the same method to the corresponding computations of

$$\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes k})$$

in [BHHM08]. We do not bother to record the complete results of these computations for small values of k , but will freely use them in what follows. The spectral sequences (2.9) imply these computations control $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_J)$.

$h_{2,1}$ towers in the algebraic tmf resolution for $H(8, v_1^8)$. Theorem 3.12 has the following implication for the g -local algebraic tmf-resolution of $H(8, v_1^8)$:

$$\begin{aligned} h_{2,1}^{-1} \mathrm{Ext}_{E[Q_1, P_2]}^{*,*}(A // A(2)_*^{\otimes n} \otimes H(8, v_1^8)) = \\ \mathbb{F}_2[h_{2,1}^{\pm}, x_{i,j} \cdot t_{i,j}] \otimes \left(\mathbb{F}_2[v_1]/v_1^8 \otimes H(8) \oplus \bigoplus_{|J| \text{ odd}} N_J \otimes H(8, v_1^8) \right) \\ \oplus \bigoplus_{|J| \neq 0 \text{ even}} \mathbb{F}_2[v_1]/v_1^2 \otimes N_J \otimes H(8, v_1^8) \end{aligned}$$

where J ranges over the non-empty subsets of

$$\{(i, j) : 1 \leq i, 1 \leq j \leq n\}$$

This leads to the following twist in the analog of Proposition 3.15.

Proposition 4.2. *In $g^{-1} wss E_0(H(8, v_1^8))$, all of the $h_{2,1}$ -towers coming from*

$$\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes k} \otimes H(8, v_1^8))$$

for $k \geq 3$ are either the source of a non-trivial d_0 -differential, or are the target of a d_0 -differential. For $k = 2$, the $h_{2,1}$ towers

$$v_1^\epsilon h_{2,1}^j Q_1(x_{1,1} x_{1,2})[n]$$

are killed for $\epsilon \in \{0, 1\}$ and $n \in \{0, 1\}$ (but the corresponding towers with $n \in \{17, 18\}$ are not killed).

Proof. Everything is identical to the proof of 3.15, except that the differentials

$$d_0^{wss} : \mathbb{F}_2[v_1, h_{2,1}^\pm]/v_1^8\{x_{1,1} t_{1,1}\} \otimes H(8) \rightarrow \mathbb{F}_2[v_1, h_{2,1}^\pm]/v_1^2\{Q_1(x_{1,1} x_{1,2})\} \otimes H(8, v_1^8)$$

now have non-trivial kernel and cokernel. \square

We now give elements of $\text{Ext}_{A_*}(H(8, v_1^8))$ which these remaining $h_{2,1}$ -towers detect in the algebraic tmf resolution. Note that, as pointed out in [MPT70], the Mahowald operator satisfies

$$h_0^3 M(x) = 0$$

which implies that for any $x \in \text{Ext}_{A_*}(\mathbb{F}_2)$, there exists a lift

$$M(x)[1] \in \text{Ext}_{A_*}(H(8))$$

and thus an element $M(x)[1] \in \text{Ext}_{A_*}(H(8, v_1^8))$. Furthermore, the element $\Delta^2 = v_2^8$ exists in $\text{Ext}_{A_*}(H(8, v_1^8))$ (see Lemma 5.1 below). We conclude that for $0 \leq i \leq 7$, $j, k, l \geq 0$, and $\epsilon \in \{0, 1\}$ the wedge elements

$$v_1^i h_{2,1}^j \Delta^{2k} M^l g^2[\epsilon] \in \text{Ext}_{A_*}(H(8, v_1^8))$$

exist, and we see they are linearly independent by mapping to $\text{Ext}_{B_*}(H(8, v_1^8))$ (where B_* is defined in (3.16)).

Proposition 4.3. *The following table lists, for $m \geq 0$, $0 \leq i \leq 7$, $0 \leq i' \leq 5$, $j \geq 4$, $k \in \{0, 1, 17, 18\}$, and $\epsilon, \epsilon' \in \{0, 1\}$ an $A(2)_*$ -comodule M , an $h_{2,1}$ -tower in $g^{-1} \text{Ext}_{A(2)_*}(M \otimes H(8, v_1^8))$, the corresponding $h_{2,1}$ -tower in $\text{Ext}_{A(2)_*}(M \otimes H(8, v_1^8))$, and an $h_{2,1}$ -tower in $\text{Ext}_{A_*}(H(8, v_1^8))$ that it detects in the algebraic tmf resolution.*

M	$g^{-1} \text{Ext}_{A(2)_*}(M \otimes H(8, v_1^8))$	$\text{Ext}_{A(2)_*}(M \otimes H(8, v_1^8))$	$\text{Ext}_{A_*}(H(8, v_1^8))$
\mathbb{F}_2	$\Delta^{2m} v_1^i h_{2,1}^{j+8}[\epsilon]$	$\Delta^{2m} v_1^i h_{2,1}^j g^2[\epsilon]$	$\Delta^{2m} v_1^i h_{2,1}^j g^2[\epsilon]$
$\underline{\text{bo}}_1$	$\Delta^{2m} h_{2,1}^{j+4} Q_1(x_{1,1})[k]$	$\Delta^{2m} h_{2,1}^{j+4} \zeta_2^4[k]$	$\Delta^{2m} h_{2,1}^j n[k]$
$\underline{\text{bo}}_2$	$\Delta^{2m} h_{2,1}^{j+6} Q_1(x_{2,1})[k]$ $\Delta^{2m} v_1^{i'+2} h_{2,1}^{j+11} x_{1,1} t_{1,1}[\epsilon]$	$\Delta^{2m} h_{2,1}^{j+1} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[k]$ $\Delta^{2m} v_1^{i'+2} h_{2,1}^{j+2} g^2(v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[\epsilon]$	$\Delta^{2m} h_{2,1}^j Q_2[k]$ $\Delta^{2m} v_1^{i'} h_{2,1}^j M g^2[\epsilon]$
$\underline{\text{bo}}_1^{\otimes 2}$	$v_1^{\epsilon'} \Delta^{2m} h_{2,1}^{j+11} Q_1(x_{1,1} x_{1,2})[17 + \epsilon]$	$\Delta^{2m} v_1^{\epsilon'} h_{2,1}^{j+2} g^2(v_0^{-1} v_2^2[\zeta_1^8, \zeta_2^4])[17 + \epsilon]$	$\Delta^{2m} v_1^{6+\epsilon'} h_{2,1}^j M g^2[\epsilon]$

Proof. The cases of

$$\begin{aligned} & \Delta^{2m} v_1^i h_{2,1}^j g^2[\epsilon], \\ & \Delta^{2m} h_{2,1}^j n[\epsilon], \\ & \Delta^{2m} h_{2,1}^j Q_2[\epsilon], \\ & \Delta^{2m} v_1^{i'} h_{2,1}^j M g^2[\epsilon] \end{aligned}$$

follow immediately from Proposition 3.17 since all of these elements are annihilated by v_0^3 .

The elements

$$\begin{aligned} & h_{2,1}^{j+4} \zeta_2^4 \in \text{Ext}_{A(2)_*}(\underline{\mathbf{b}\mathbf{O}}_1), \\ & h_{2,1}^{j+6} \zeta_1^{16} \in \text{Ext}_{A(2)_*}(\underline{\mathbf{b}\mathbf{O}}_2) \end{aligned}$$

lift to elements

$$(4.4) \quad \begin{aligned} & h_{2,1}^{j+4} \zeta_2^4 [17 + \epsilon] \in \text{Ext}_{A(2)_*}(\underline{\mathbf{b}\mathbf{O}}_1 \otimes H(8, v_1^8)), \\ & h_{2,1}^{j+6} \zeta_1^{16} [17 + \epsilon] \in \text{Ext}_{A(2)_*}(\underline{\mathbf{b}\mathbf{O}}_2 \otimes H(8, v_1^8)). \end{aligned}$$

One can explicitly check that the lifts (4.4) are permanent cycles in the algebraic tmf resolution. Therefore they detect the desired elements

$$h_{2,1}^j n[17 + \epsilon], h_{2,1}^j Q_2[17 + \epsilon] \in \text{Ext}_{A_*}(H(8, v_1^8)).$$

Applying Case (5) of the Geometric Boundary Theorem [Beh12, Lem.A.4.1] to the triangle

$$H(8, v_1^8)[-1] \rightarrow \Sigma^{24} H(8)[-8] \xrightarrow{v_1^8} H(8) \rightarrow H(8, v_1^8)$$

and the differential

$$d_1(v_1^{\epsilon'} h_{2,1}^{j+2} g^2(v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)) = v_1^{\epsilon'} h_{2,1}^{j+2} g^2(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])$$

in the algebraic tmf resolution for $\Sigma^{24} H(8)[-8]$ (3.14), we find that the images of the elements

$$v_1^{8+\epsilon'} h_{2,1}^j M(g^2)[\epsilon] \in \text{Ext}_{A_*}(H(8))$$

under the map

$$\text{Ext}_{A_*}(H(8)) \rightarrow \text{Ext}_{A_*}(H(8, v_1^8))$$

are detected by the elements

$$v_1^{\epsilon'} h_{2,1}^{j+2} g^2(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[17 + \epsilon]$$

in the algebraic tmf resolution for $H(8, v_1^8)$. □

5. THE v_2^{32} SELF-MAP ON $M(8, v_1^8)$

We now endeavor to prove Theorem 1.7. We first recall the following lemma.

Lemma 5.1 (Lem. 7.6 of [BHHM20]). *The element*

$$v_2^8 \in \text{Ext}_{A(2)_*}^{8,48+8}(H(8, v_1^8))$$

is a permanent cycle in the algebraic tmf-resolution, and gives rise to an element

$$v_2^8 \in \text{Ext}_{A_*}^{8,48+8}(H(8, v_1^8)).$$

It follows from the Leibniz rule that v_2^{32} persists to the E_4 -page of the MASS for $M(8, v_1^8)$. Our task will then be reduced to showing that $d_r(v_2^{32}) = 0$ for $r \geq 4$. We will do this by identifying the potential targets of such a differential, and show that they either the source or target of shorter differentials. This will necessitate lifting certain differentials from the MASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}^n} \wedge M(8, v_1^8)$ to the MASS for $M(8, v_1^8)$.

As explained in [BOSS19, Sec. 7.4], work of the second author, Davis, and Rezk [MR09],[DM10] implies that the algebraic map

$$\mathrm{Ext}_{A(2)}(\Sigma^8 \underline{\mathrm{bo}}_1 \oplus \Sigma^{16} \underline{\mathrm{bo}}_2) \rightarrow \mathrm{Ext}_{A(2)_*}(\overline{A//A(2)}_*)$$

realizes to a map

$$(5.2) \quad \mathrm{tmf} \wedge \overline{\mathrm{tmf}}_2 \rightarrow \mathrm{tmf} \wedge \overline{\mathrm{tmf}}$$

where $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}_2$ is a spectrum built out of $\mathrm{tmf} \wedge \Sigma^8 \underline{\mathrm{bo}}_1$ and $\mathrm{tmf} \wedge \Sigma^{16} \underline{\mathrm{bo}}_2$. They furthermore show that there is a map

$$(5.3) \quad \Sigma^{32} \mathrm{tmf} \rightarrow \mathrm{tmf} \wedge \overline{\mathrm{tmf}}_2$$

which geometrically realizes the inclusion of the direct summand (2.9)

$$\mathrm{Ext}_{A(2)_*}(\Sigma^{33} \mathbb{F}_2[-1]) \hookrightarrow \mathrm{Ext}_{A(2)_*}(\Sigma^{16} \underline{\mathrm{bo}}_2) \subset \mathrm{Ext}_{A(2)_*}(\Sigma^8 \underline{\mathrm{bo}}_1 \oplus \Sigma^{16} \underline{\mathrm{bo}}_2).$$

The attaching map from $\mathrm{tmf} \wedge \underline{\mathrm{bo}}_2$ to $\mathrm{tmf} \wedge \underline{\mathrm{bo}}_1$ in the spectrum $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}_2$ induces d_3 -differentials from the $h_{2,1}$ -towers in $\underline{\mathrm{bo}}_2$ to the $h_{2,1}$ -towers in $\underline{\mathrm{bo}}_1$ in the ASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}$ under the map (5.2). Furthermore, there are differentials in the ASS's for $\mathrm{tmf} \wedge \underline{\mathrm{bo}}_1$, $\mathrm{tmf} \wedge \underline{\mathrm{bo}}_2$, and tmf , which induce differentials in the ASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}$ under the maps (5.2) and (5.3). We wish to study when these differentials (and more generally differentials in the ASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}^n}$) lift via the tmf resolution to differentials in the ASS for the sphere.

To this end we consider the partial totalizations

$$T^n := \mathrm{Tot}^n(\mathrm{tmf}^{\bullet+1})$$

of the cosimplicial tmf -resolution of the sphere, so that we have

$$S \simeq \varprojlim_n T^n$$

and fiber sequences

$$\Sigma^{-n} \mathrm{tmf} \wedge \overline{\mathrm{tmf}}^n \rightarrow T^n \rightarrow T^{n-1}.$$

The spectrum T^n is a ring spectrum, and in particular has a unit

$$S \rightarrow T^n.$$

We let

$$(5.4) \quad \underline{T}^n = \mathrm{Tot}^n(A//A(2)_*^{\otimes \bullet+1})$$

denote the corresponding construction in the stable homotopy category of A_* -comodules. There is a MASS

$$\mathrm{Ext}_{A_*}^{*,*}(\underline{T}^n \otimes H(8, v_1^8)) \Rightarrow T_*^n M(8, v_1^8)$$

and the algebraic tmf resolution for $H(8, v_1^8)$ truncates to give an algebraic tmf resolution

$$\bigoplus_{i=0}^n \text{Ext}_{A(2)_*}^{*,*} (\overline{A//A(2)_*}^{\otimes i} \otimes H(8, v_1^8)) \Rightarrow \text{Ext}_{A_*}(\underline{T}^n \otimes H(8, v_1^8)).$$

The following lemma will be our key to lifting the desired differentials.

Lemma 5.5. *Suppose x is an element of $\text{Ext}_{A_*}(H(8, v_1^8))$ which is detected in the n -line of the algebraic tmf resolution for $H(8, v_1^8)$ by an element*

$$x' \in \text{Ext}_{A(2)_*}(\overline{A//A(2)_*}^{\otimes n} \otimes H(8, v_1^8)).$$

Furthermore, suppose that in the MASS for $\text{tmf} \wedge \overline{\text{tmf}}^n \wedge M(8, v_1^8)$, there is a differential

$$d_r^{\text{mass}}(x') = y'$$

and that for $2 \leq r' < r$ we have

$$d_{r'}^{\text{mass}}(x) = 0$$

in the MASS for the $M(8, v_1^8)$. Then either of the following is true:

- (1) *The differential*

$$d_r^{\text{mass}}(x)$$

in the ASS for $M(8, v_1^8)$ is detected by y' in the algebraic tmf resolution, or

- (2) *The element y' is the target of a differential in the algebraic tmf resolution for $H(8, v_1^8)$, or in the algebraic tmf resolution for $\underline{T}^n \otimes H(8, v_1^8)$ the element y' detects an element of $\text{Ext}_{A_*}(\underline{T}^n \otimes H(8, v_1^8))$ which is zero in ${}^{\text{mass}}E_r(T^n \wedge M(8, v_1^8))$.*

Proof. Consider the maps of algebraic tmf resolutions and MASS's induced from the zig-zag

$$M(8, v_1^8) \xrightarrow{\alpha} T^n \wedge M(8, v_1^8) \xleftarrow{\beta} \Sigma^{-n} \text{tmf} \wedge \overline{\text{tmf}}^n \wedge M(8, v_1^8).$$

Define

$$\bar{x} := \alpha_*(x) \in \text{Ext}_{A_*}(\underline{T}^n \otimes H(8, v_1^8))$$

Then \bar{x} is detected by x' , regarded as an element of the algebraic tmf resolution for $T^n \wedge M(8, v_1^8)$. In particular, this means that

$$\bar{x} = \beta_*(x')$$

Therefore, the differential

$$d_r^{\text{mass}}(x') = y'$$

in the MASS for $\text{tmf} \wedge \overline{\text{tmf}}^n \wedge M(8, v_1^8)$ maps to a differential

$$d_r^{\text{mass}}(\bar{x}) = \bar{y} := \beta_*(y')$$

in the MASS for $T^n \wedge M(8, v_1^8)$. In particular, either (Case 1) \bar{y} is nonzero in ${}^{\text{mass}}E_r(T^n \wedge M(8, v_1^8))$ and is detected by y' in the algebraic tmf resolution for $\underline{T}^n \otimes H(8, v_1^8)$, or (Case 2) either $\bar{y} = 0$ in ${}^{\text{mass}}E_r(T^n \wedge M(8, v_1^8))$ or y' is killed in the algebraic tmf resolution for $\underline{T}^n \otimes H(8, v_1^8)$. If the latter is true, then y' is killed in the algebraic tmf resolution for $H(8, v_1^8)$, since the algebraic tmf resolution for $\underline{T}^n \otimes H(8, v_1^8)$ is a truncation of the algebraic tmf resolution for $H(8, v_1^8)$.

If we are in Case (2), we are done. If we are in Case (1), consider the differential

$$y := d_r^{mass}(x)$$

in the MASS for $M(8, v_1^8)$ (which is defined by hypothesis). We must have

$$\alpha_*(y) = \bar{y}.$$

Therefore, $d_r^{mass}(x)$ is detected by y' in the algebraic tmf resolution. \square

Remark 5.6. We will primarily be applying Lemma 5.5 to the following two cases:

Case 1: $x = \Delta^{2m} h_{2,1}^j Q_2[k]$. Suppose that we can prove

$$d_2^{ass}(\Delta^{2m} h_{2,1}^j Q_2[k]) = 0$$

in the MASS for $M(8, v_1^8)$. The element $\Delta^{2m} h_{2,1}^j Q_2[k]$ is detected by

$$\Delta^{2m} h_{2,1}^{j+1} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[k] \in \mathrm{Ext}_{A(2)_*}(\mathrm{bo}_2 \otimes H(8, v_1^8))$$

in the algebraic tmf resolution, and it is proven in [BOSS19] that in the ASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}$ there is a differential

$$\begin{aligned} d_3^{ass}(\Delta^{2m} h_{2,1}^{j+1} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})) = \\ \Delta^{2m} h_{2,1}^{j+4} g(h_{2,1} \zeta_2^4) + \epsilon(m) \Delta^{2m-4} h_{2,1}^{j+20} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16}) \end{aligned}$$

where

$$\epsilon(m) = \begin{cases} 1, & m \equiv 2 \pmod{4}, \\ 0, & \text{otherwise.} \end{cases}$$

Lifting this differential to $\mathrm{tmf} \wedge \overline{\mathrm{tmf}} \wedge M(8, v_1^8)$, Lemma 5.5 implies that either the target of the differential $d_3^{ass}(\Delta^{2m} h_{2,1}^j Q_2[k])$ in the MASS for $M(8, v_1^8)$ is detected by

$$\Delta^{2m} h_{2,1}^{j+4} g(h_{2,1} \zeta_2^4)[k] + \epsilon(m) \Delta^{2m-4} h_{2,1}^{j+20} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[k]$$

in the algebraic tmf resolution, or

$$\Delta^{2m} h_{2,1}^{j+4} g(h_{2,1} \zeta_2^4)[k] + \epsilon(m) \Delta^{2m-4} h_{2,1}^{j+20} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[k]$$

is the target of a differential in the algebraic tmf resolution or detects an element of $\mathrm{Ext}_{A_*}(\mathbb{T}^1 \otimes H(8, v_1^8))$ which is zero on the E_3 -page of the MASS for $T^1 \wedge M(8, v_1^8)$.

Case 2: $x = M \Delta^2 v_1^i h_{2,1}^{j+8}[\epsilon]$ for $\epsilon \in \{0, 1\}$ and $0 \leq i \leq 4$. The element $M \Delta^2 v_1^i h_{2,1}^{j+8}[\epsilon]$ is detected by

$$\Delta^2 v_1^{i+2} h_{2,1}^{j+10} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[\epsilon]$$

in the algebraic tmf resolution for $H(8, v_1^8)$, and the map (5.3) implies there is a differential

$$d_2^{mass}(\Delta^2 v_1^{i+2} h_{2,1}^{j+10} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[\epsilon]) = v_1^{i+3} h_{2,1}^{j+19} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[\epsilon]$$

in the MASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}} \wedge M(8, v_1^8)$. Then Lemma 5.5 implies that either $d_2^{mass}(M \Delta^2 v_1^i h_{2,1}^{j+8}[\epsilon])$ is detected by

$$v_1^{i+3} h_{2,1}^{j+19} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[\epsilon]$$

in the algebraic tmf resolution, or $v_1^{i+3}h_{2,1}^{j+19}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4)[\epsilon]$ is killed in the tmf resolution for $H(8, v_1^8)$ or it detects an element which is zero in the E_2 -term of the MASS for $T^1 \wedge M(8, v_1^8)$. However, the element

$$Mv_1^{i+1}h_{2,1}^{j+17}[\epsilon] \in \text{Ext}_{A_*}(H(8, v_1^8))$$

is non-zero, and is detected by $v_1^{i+3}h_{2,1}^{j+19}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4)[\epsilon]$ in the algebraic tmf resolution for $H(8, v_1^8)$. We conclude that $v_1^{i+3}h_{2,1}^{j+19}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4)[\epsilon]$ is not killed in the algebraic tmf resolution for $H(8, v_1^8)$. Since the algebraic tmf resolution for $\underline{T}^1 \otimes H(8, v_1^8)$ is a truncation of the algebraic tmf resolution for $H(8, v_1^8)$, we conclude that $v_1^{i+3}h_{2,1}^{j+19}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4)[\epsilon]$ detects a non-trivial element of the E_2 -page of the MASS for $T^1 \wedge M(8, v_1^8)$. We conclude that

$$d_2^{mass}(M\Delta^2v_1^i h_{2,1}^{j+8}[\epsilon])$$

is non-trivial in the MASS for $M(8, v_1^8)$, and is detected in the algebraic tmf resolution by $v_1^{i+3}h_{2,1}^{j+19}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4)[\epsilon]$.

Proof of Theorem 1.7. By Proposition 2.3, it suffices to prove that

$$v_2^{32} \in \text{Ext}_{A_*}(H(8, v_1^8))$$

is a permanent cycle in the MASS. Furthermore, since $v_2^8 \in {}^{mass}E_2(M(8, v_1^8))$, the Leibniz rule implies that $v_2^{32} \in {}^{mass}E_4(M(8, v_1^8))$. We therefore are left with eliminating possible targets of $d_r^{mass}(v_2^{32})$ for $r \geq 4$.

Suppose that $d_r(v_2^{32})$ is non-trivial for $r \geq 4$. We successively consider terms in the algebraic tmf resolution which could detect $d_r(v_2^{32})$, and then eliminate these possibilities one by one.

The only terms in the algebraic tmf resolution E_1 -page which can contribute to $\text{Ext}_{A_*}^{s, 191+s}(H(8, v_1^8))$ for $s \geq 36$ are

- $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes s})$ for $0 \leq s \leq 6$, and
- $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes s} \otimes \underline{\text{bo}}_2)$ for $0 \leq s \leq 2$.

Furthermore, $\underline{\text{bo}}_1^{\otimes s}$ only contributes $h_{2,1}$ -towers in this range for $s = 5, 6$. We list these contributions below, except we do not list elements in $h_{2,1}$ -towers coming from $\underline{\text{bo}}_1^{\otimes s}$ for $s \geq 2$ which are zero in the WSS E_1 -term (see Proposition 4.2). Also, since v_2^{32} is a permanent cycle in the MASS for $\text{tmf} \wedge M(8, v_1^8)$, we can disregard any terms coming from $\text{Ext}_{A(2)_*}(\mathbb{F}_2)$ (the zero-line of the algebraic tmf resolution). Finally, we do not include any terms which can be eliminated through the application of Case 2 of Remark 5.6.

Table 5.7. List of potential targets of $d_r^{mass}(v_2^{32})$ for $r \geq 4$.

\underline{bo}_1	$h_{2,1}^{31}g(h_{2,1}\zeta_2^4)[0]$ $h_{2,1}^{18}\Delta^2g(h_{2,1}\zeta_2^4)[17]$
\underline{bo}_2	$h_{2,1}^5\Delta^4g(h_{2,1}\zeta_1^{16})[18]$ $v_1^2h_{2,1}^{31}(v_0^{-1}v_2^2\zeta_1^8\zeta_2^4)[1]$
$\underline{bo}_1^{\otimes 2}$	$h_{2,1}^5\Delta^4v_1g(v_0^{-1}v_2^2[\zeta_1^8, \zeta_2^4])[18]$ $h_{2,1}^{15}\Delta^2g(v_0^{-1}v_2^2[\zeta_1^8, \zeta_2^4])[18]$
$\underline{bo}_1 \otimes \underline{bo}_2$	$v_1h_{2,1}^{21}g(v_0^{-1}v_2^2[\zeta_2^4, \zeta_2^8])[18]$ $v_1h_{2,1}^{21}g(v_0^{-1}v_2^2[\zeta_2^8, \zeta_2^4])[18]$
\underline{bo}_1^3	$v_1^4\Delta^6h_1(v_2^2\zeta_1^8 \zeta_1^8 \zeta_2^4)[1]$
$\underline{bo}_1^{\otimes 2} \otimes \underline{bo}_2$	$v_1h_{2,1}^{18}g(v_0^{-2}v_2^4[\zeta_1^8, \zeta_2^4] \zeta_1^8\zeta_2^4)[18]$ $v_1h_{2,1}^{18}g(v_0^{-2}v_2^4(\zeta_1^8 \zeta_1^8\zeta_2^4 \zeta_2^4 + \zeta_2^4 \zeta_1^8\zeta_2^4 \zeta_1^8))[18]$ $v_1h_{2,1}^{18}g(v_0^{-2}v_2^4\zeta_1^8\zeta_2^4 [\zeta_1^8, \zeta_2^4])[18]$
\underline{bo}_1^4	$v_1^4\Delta^6h_1^2\zeta_1^8 \zeta_1^8 \zeta_2^4 \zeta_2^4[1]$

We now eliminate these possibilities one by one. We will consider the terms in the order of *reverse* algebraic tmf filtration.

$\underline{bo}_1^{\otimes 4}$: In the modified May spectral sequence (3.3) there is a differential

$$d_8^{mass}(b_{2,2}h_3^2) = h_3^5$$

which lifts under the map Φ_* of 3.4 to a non-trivial differential

$$d_1^{wss}([\zeta_1^8|\zeta_1^8|\zeta_2^4|\zeta_2^4]) = [\zeta_1^8|\zeta_1^8|\zeta_1^8|\zeta_1^8|\zeta_1^8]$$

in the WSS for \mathbb{F}_2 , and this implies a non-trivial differential

$$d_1^{wss}(v_1^4\Delta^6h_1^2[\zeta_1^8|\zeta_1^8|\zeta_2^4|\zeta_2^4][1]) = v_1^4\Delta^6h_1^2[\zeta_1^8|\zeta_1^8|\zeta_1^8|\zeta_1^8|\zeta_1^8][1]$$

in the WSS for $H(8, v_1^8)$.

$\underline{bo}_1^{\otimes 2} \otimes \underline{bo}_2$: In the cobar complex for $\mathbb{F}_2[\zeta_1^8, \zeta_2^4]$ we find

$$d([\zeta_1^8, \zeta_2^4]|\zeta_1^8\zeta_2^4) \quad \text{and} \quad d(\zeta_1^8|\zeta_1^8\zeta_2^4|\zeta_2^4 + \zeta_2^4|\zeta_1^8\zeta_2^4|\zeta_1^8)$$

are linearly independent, and

$$d([\zeta_1^8, \zeta_2^4]|\zeta_1^8\zeta_2^4 + \zeta_1^8\zeta_2^4|[\zeta_1^8, \zeta_2^4]) = 0$$

However

$$d(\zeta_1^8\zeta_2^4|\zeta_1^8\zeta_2^4) = [\zeta_1^8, \zeta_2^4]|\zeta_1^8\zeta_2^4 + \zeta_1^8\zeta_2^4|[\zeta_1^8, \zeta_2^4]$$

The elements are thus eliminated by multiplying the computations above with $v_1^{-2}v_2^4h_{2,1}^{22}$ and lifting them to the top cell of $H(8, v_1^8)$.

$\underline{\mathbf{bo}}_1^{\otimes 3}$: Note that

$$\mathrm{Ext}_{A_*}^{10,10+48}(\mathbb{F}_2) = 0.$$

We conclude that the class

$$v_1^4 c_0 h_1 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4) \in \mathrm{Ext}_{A(2)_*}(\underline{\mathbf{bo}}_2)$$

must either support or be the target of a differential in the algebraic tmf resolution, for otherwise it would give a non-zero element of $\mathrm{Ext}_{A_*}^{10,10+48}(\mathbb{F}_2)$. However, by examination, there are no classes in $\mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$ which can kill $v_1^4 c_0 h_1 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)$ in the algebraic tmf resolution, so there must be a non-trivial differential

$$d_r(v_1^4 c_0 h_1 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4))$$

in the algebraic tmf resolution for \mathbb{F}_2 . Since the target of this differential must be h_1 -torsion, there is only one possibility:

$$d_2(v_1^4 c_0 h_1 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)) = v_1^4 h_1^2 v_2^2 \zeta_1^8 |\zeta_1^8| \zeta_2^4.$$

It follows that we have

$$d_2(v_1^4 c_0 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)) = v_1^4 h_1 v_2^2 \zeta_1^8 |\zeta_1^8| \zeta_2^4.$$

This differential lifts to a differential

$$d_2(v_1^4 c_0 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]) = v_1^4 h_1 v_2^2 \zeta_1^8 |\zeta_1^8| \zeta_2^4 [1]$$

in the algebraic tmf resolution for $H(8, v_1^8)$. Multiplying by Δ^6 , we have

$$d_2(\Delta^6 v_1^4 c_0 (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]) = \Delta^6 v_1^4 h_1 v_2^2 \zeta_1^8 |\zeta_1^8| \zeta_2^4 [1].$$

$\underline{\mathbf{bo}}_1 \otimes \underline{\mathbf{bo}}_2$: There is a differential

$$d_0^{wss}(\zeta_2^{12}) = [\zeta_2^4, \zeta_2^8]$$

in the WSS for \mathbb{F}_2 which lifts to a differential

$$d_0^{wss}(v_1 h_2^{21} g(v_0^{-1} v_2^2 \zeta_2^{12})) = v_1 h_2^{21} g(v_0^{-1} v_2^2 [\zeta_2^4, \zeta_2^8]).$$

We therefore only have to consider one of the two potential elements. In the modified May spectral sequence (3.3), there is a differential

$$d_8^{mss}(h_{2,3}) = h_{1,3} h_{1,4}$$

which lifts to a differential

$$d_1^{wss}(\zeta_2^8) = \zeta_1^8 |\zeta_1^{16}|.$$

using the map Φ_* of (3.3), and gives a differential

$$d_1^{wss}(\zeta_2^4 |\zeta_2^8) = \zeta_2^4 |\zeta_1^8| \zeta_1^{16}.$$

The elements

$$v_1 g(v_0^{-1} v_2^2 \zeta_2^4 |\zeta_2^8) \in \mathrm{Ext}_{A(2)_*}(\underline{\mathbf{bo}}_1 \otimes \underline{\mathbf{bo}}_2)$$

and

$$v_1 g(v_0^{-1} v_2^2 \zeta_2^4 |\zeta_1^8 |\zeta_1^{16}) \in \mathrm{Ext}_{A(2)_*}(\underline{\mathbf{bo}}_1^{\otimes 2} \otimes \underline{\mathbf{bo}}_2)$$

support $h_{2,1}$ -towers which are non-trivial in $^{wss}E_1$. Therefore we have a non-trivial differential

$$d_1^{wss}(v_1 h_2^{21} g(v_0^{-1} v_2^2 \zeta_2^4 |\zeta_2^8)) = v_1 h_2^{21} g(v_0^{-1} v_2^2 \zeta_2^4 |\zeta_1^8 |\zeta_1^{16}).$$

This differential lifts to the top cell of $H(8, v_1^8)$ to give

$$d_1^{wss}(v_1 h_{2,1}^{21} g(v_0^{-1} v_2^2 \zeta_2^4 | \zeta_2^8)[18]) = v_1 h_{2,1}^{21} g(v_0^{-1} v_2^2 \zeta_2^4 | \zeta_1^8 | \zeta_1^{16})[18]$$

in the WSS for $H(8, v_1^8)$.

$\mathrm{bo}_1^{\otimes 2}$: The element

$$h_{2,1}^5 \Delta^4 v_1 g(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]$$

detects the element

$$\Delta^4 \cdot MP\Delta h_0^2 e_0[18]$$

in the algebraic tmf resolution for $H(8, v_1^8)$. Regarding this element as an element in the MASS for $\mathrm{tmf} \wedge \mathrm{bo}_1^2$, there is a non-trivial differential

$$d_3^{mass}(h_{2,1}^5 \Delta^4 v_1 g(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]) = h_{2,1}^{24} v_1 g(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18].$$

By applying $(-)^{\wedge_{\mathrm{tmf}^2}}$ to the map of tmf modules (5.2), we may consider the composite

$$\mathrm{tmf} \wedge \mathrm{bo}_1^2 \hookrightarrow (\mathrm{tmf} \wedge \overline{\mathrm{tmf}}_2)^{\wedge_{\mathrm{tmf}^2}} \rightarrow \mathrm{tmf} \wedge \overline{\mathrm{tmf}}^2.$$

The differential above maps to a non-trivial differential between elements of the same name in the MASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^2$. We wish to apply Lemma 5.5. We must have

$$d_2^{mass}(\Delta^4 \cdot MP\Delta h_0^2 e_0[18]) = 0$$

in the MASS for $M(8, v_1^8)$, since there are no elements in the algebraic tmf resolution for $H(8, v_1^8)$ which could detect a target for this differential. Thus Lemma 5.5 implies that either

$$d_3^{mass}(\Delta^4 \cdot MP\Delta h_0^2 e_0[18])$$

is non-trivial and detected by $h_{2,1}^{24} v_1 g(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]$, or

$$h_{2,1}^{24} v_1 g(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]$$

is killed in the algebraic tmf resolution for $H(8, v_1^8)$, or detects an element which is killed in the MASS for $T^2 \wedge M(8, v_1^8)$. The only such possibility is for

$$\Delta^2 h_{2,1}^{23} \zeta_2^4[17]$$

to detect the source of a d_2 -differential in the MASS for $T^2 \wedge M(8, v_1^8)$ to do such a killing. Projecting onto the top Moore space of $M(8, v_1^8)$, this would imply

$$\Delta^2 h_{2,1}^{23} \zeta_2^4$$

detects an element in the algebraic tmf resolution for the sphere which supports a non-trivial d_2 -differential in the ASS for the sphere. However, $\Delta^2 h_{2,1}^{23} \zeta_2^4$ detects

$$\Delta^2 g^5 \cdot \Delta h_2 c_1$$

in the ASS for the sphere, and there is a differential

$$\begin{aligned} d_2^{ass}(\Delta^2 g^5 \cdot \Delta h_2 c_1) &= d_2^{ass}(\Delta^2 g^2) \cdot g^3 \cdot \Delta h_2 c_1 \\ &= \Delta^2 h_2^2 g^2 e_0 \cdot g^3 \cdot \Delta h_2 c_1. \end{aligned}$$

However $\Delta^2 h_2^2 e_0 \cdot \Delta h_2 c_1 = 0$ in $\text{Ext}_{A_*}(\mathbb{F}_2)$ [Bru], so this d_2^{ass} is zero.

We now turn our attention to the other potential target coming from $\underline{\mathbf{b}}\mathbf{o}_1^{\otimes 2}$:

$$h_{2,1}^{15} \Delta^2 g(v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18].$$

This element detects

$$\Delta^2 g^2 v_1^6 h_{2,1} M g^3 [0]$$

in the algebraic tmf resolution for $M(8, v_1^8)$. However, in the ASS for the sphere, $v_1^6 h_{2,1} g^3$ is a d_2 -cycle, and so there is a differential

$$\begin{aligned} d_2^{ass}(\Delta^2 g^2 \cdot v_1^6 h_{2,1} g^3) &= d_2^{ass}(\Delta^2 g^2) \cdot v_1^6 h_{2,1} g^3 \\ &= \Delta^2 h_2^2 g^2 e_0 \cdot v_1^6 h_{2,1} g^3 \\ &= v_1^7 h_{2,1}^{22} g^2. \end{aligned}$$

Applying $M(-) = \langle -, 8, g_2 \rangle$, and mapping under the inclusion of the bottom cell of $M(8, v_1^8)$, we get a non-trivial differential

$$d_2^{mass}(\Delta^2 g^2 \cdot v_1^6 h_{2,1} M g^3 [0]) = v_1^7 h_{2,1}^{22} M g^2 [0].$$

bo₁: The element

$$h_{2,1}^{31} g(h_{2,1} \zeta_2^4)$$

detects

$$g^8 n \in \text{Ext}_{A_*}(\mathbb{F}_2)$$

in the algebraic tmf resolution for \mathbb{F}_2 (Prop. 3.17). This element can be eliminated by Case (1) of Remark 5.6, but we can also handle it manually using low dimensional calculations in the ASS for the sphere. There is a differential

$$d_3(mQ_2) = g^3 n$$

in the ASS for the sphere [IWX20b], from which it follows that $g^8 n$ is zero on the E_4 -page of the ASS of the sphere, and hence $g^8 n[0]$ is zero on the E_4 -page of the MASS for $M(8, v_1^8)$.

For the the element

$$h_{2,1}^{18} \Delta^2 g(h_{2,1} \zeta_2^4)[17]$$

we wish to employ Case (1) of Remark 5.6, using the differential

$$d_3^{mass}(h_{2,1}^{15} \Delta^2 g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[17]) = h_{2,1}^{18} \Delta^2 g(h_{2,1} \zeta_2^4)[17]$$

in the MASS for $\text{tmf} \wedge \overline{\text{tmf}} \wedge M(8, v_1^8)$. Note that

$$h_{2,1}^{15} \Delta^2 g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[17]$$

detects the element

$$C'' \cdot \Delta^2 g^2 [17]$$

in the algebraic tmf resolution. Observe that we have [IWX20a], [Bru]

$$\begin{aligned} d_2(C'' \cdot \Delta^2 g^2) &= C'' \cdot d_2(\Delta^2 g^2) \\ &= g^2 \cdot C'' \Delta h_2^2 e_0 \\ &= g^2 \cdot 0 = 0. \end{aligned}$$

It follows that $d_2(C'' \cdot \Delta^2 g^2[17])$ is in the image of the map

$$\mathrm{Ext}_{A_*}(H(8)) \rightarrow \mathrm{Ext}_{A_*}(H(8, v_1^8))$$

but a check of the algebraic tmf resolution for $H(8, v_1^8)$ reveals there are no possible targets in this bidegree. We therefore have

$$d_2(C'' \cdot \Delta^2 g^2[17]) = 0.$$

Therefore the hypotheses of Lemma 5.5 are satisfied. It follows that

$$h_{2,1}^{18} \Delta^2 g(h_{2,1} \zeta_2^4)[17]$$

is either killed in the algebraic tmf resolution for $H(8, v_1^8)$, or detects an element in the MASS which is killed by $d_3(C'' \cdot \Delta^2 g^2[17])$, or detects an element which is killed by a d_2 -differential in the MASS for $T^1 \wedge M(8, v_1^8)$. We just need to eliminate this last possibility.

Any possible source for such a d_2 -differential would necessarily be detected on the 0-line of the algebraic tmf resolution, and would not support a non-trivial d_2 in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. The only such possibility is

$$\Delta^4 h_{2,1}^{19}[1].$$

However, we can express this element as the Hurewicz image of the element

$$gm \cdot \Delta^4 \cdot g^2[1]$$

in the MASS for $M(8, v_1^8)$. This element is therefore necessarily a d_2 -cycle, since it is a product of d_2 -cycles.

bo₂: We begin with the element

$$h_{2,1}^5 \Delta^4 g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})[18]$$

which detects the element

$$\Delta^4 gQ_2[18]$$

in the MASS for $M(8, v_1^8)$. We are in Case (1) of Remark 5.6. An elementary check using the charts of [IWX20a] reveals that the element gQ_2 in the ASS for the sphere lifts to a d_2 -cycle

$$gQ_2[18]$$

supported by the top cell of $H(8, v_1^8)$. Since Δ^4 is a d_2 -cycle in the MASS for $M(8, v_1^8)$, we deduce that

$$\Delta^4 gQ_2[18]$$

is a d_2 -cycle. We therefore deduce that either

$$d_3^{mass}(\Delta^4 gQ_2[18])$$

is detected by

$$\Delta^4 h_{2,1}^8 g(h_{2,1} \zeta_2^4)[18] + h_{2,1}^{24} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})$$

in the algebraic tmf resolution for $H(8, v_1^8)$ or

$$\Delta^4 h_{2,1}^8 g(h_{2,1} \zeta_2^4)[18] + h_{2,1}^{24} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})$$

is killed in the algebraic tmf resolution for $H(8, v_1^8)$, or detects an element which is killed in the MASS for $T^1 \wedge M(8, v_1^8)$. The only possible sources of

such algebraic tmf resolution differentials are wedge elements coming from $\text{Ext}_{A(2)_*}(H(8, v_1^8))$, and we know these all must be permanent cycles in the algebraic tmf resolution because they detect the corresponding wedge elements of $\text{Ext}_{A_*}(H(8, v_1^8))$. The only elements of the algebraic tmf resolution which can detect an element which could support a d_2 -differential killing

$$\Delta^4 h_{2,1}^8 g(h_{2,1} \zeta_2^4)[18] + h_{2,1}^{24} g(h_{2,1} v_0^{-2} v_2^2 \zeta_1^{16})$$

in the MASS for $T^1 \wedge M(8, v_1^8)$ are the elements

$$(5.8) \quad \Delta^2 v_1^6 h_{2,1}^{23}[0] \quad \text{and} \quad \Delta^2 v_1^3 h_{2,1}^{24}[1].$$

However, using the map of spectral sequences

$${}^{mass}E_2^{*,*}(T^1 \wedge M(8, v_1^8)) \rightarrow {}^{mass}E_2^{*,*}(\text{tmf} \wedge M(8, v_1^8))$$

we can eliminate these possibilities on the basis that the elements (5.8) support non-trivial d_2 differentials in the MASS for $M(8, v_1^8)$.

We are left with eliminating

$$v_1^2 h_{2,1}^{31} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]$$

as possibly detecting $d_5^{mass}(v_2^{32})$ in the MASS for $M(8, v_1^8)$. This is the trickiest obstruction to eliminate. In the MASS for $\text{tmf} \wedge \text{tmf} \wedge M(8, v_1^8)$ there is a differential

$$d_2^{mass}(\Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]) = v_1^2 h_{2,1}^{31} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1].$$

The *problem* is that in the WSS for $H(8, v_1^8)$ there is a non-trivial differential

$$d_0^{wss}(\Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]) = \Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[1].$$

Sublemma 5.9. *The element v_2^{32} is a permanent cycle in the MASS for $T^1 \wedge M(8, v_1^8)$.*

Proof of sublemma. The elements of the algebraic tmf resolution which could possibly detect the target of a differential

$$d_r^{mass}(v_2^{32}), \quad r \geq 4,$$

in the MASS for $T^1 \wedge M(8, v_1^8)$ consist of those terms in Table 5.7 coming from $\underline{\text{bo}}_1$ and $\underline{\text{bo}}_2$.

Using (5.3) there is a map

$$\Sigma^{31} \text{tmf} \wedge M(8, v_1^8) \rightarrow \Sigma^{-1} \text{tmf} \wedge \overline{\text{tmf}} \rightarrow T^1$$

and we therefore have a differential

$$d_2^{mass}(\Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]) = v_1^2 h_{2,1}^{31} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]$$

in the MASS for $T^1 \wedge M(8, v_1^8)$. Therefore $v_1^2 h_{2,1}^{31} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]$ cannot be the target of a differential $d_5^{mass}(v_2^{32})$ in the MASS for $T^1 \wedge M(8, v_1^8)$.

Our previous arguments eliminate all the other possibilities. \square

Suppose now for the purpose of generating a contradiction that the differential

$$d_5^{mass}(v_2^{32})$$

in the MASS for $M(8, v_1^8)$ is non-trivial and detected by $v_1^2 h_{2,1}^{31}(v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]$ in the algebraic tmf resolution for $H(8, v_1^8)$. Consider the fiber sequence

$$\Sigma^{-2} \overline{\mathrm{tmf}}^2 \wedge M(8, v_1^8) \rightarrow M(8, v_1^8) \rightarrow T^1 \wedge M(8, v_1^8) \xrightarrow{\partial} \Sigma^{-1} \overline{\mathrm{tmf}}^2.$$

We have proven that v_2^{32} exists in $\pi_{192} T^1 \wedge M(8, v_1^8)$, and because our assumption implies that v_2^{32} does not lift to $\pi_{192} M(8, v_1^8)$, we must have

$$0 \neq \partial(v_2^{32}) \in \pi_{191} \Sigma^{-2} \overline{\mathrm{tmf}}^2 \wedge M(8, v_1^8).$$

Sublemma 5.10. *There exists a choice of $v_2^{32} \in \pi_{192} T^1 \wedge M(8, v_1^8)$ so that $\partial(v_2^{32})$ has modified Adams filtration 34.*

Proof of sublemma. Let $X^{(k)}$ denote the k th modified Adams cover of X - so that the MASS for $X^{(k)}$ is the truncation of the MASS for X obtained by only considering terms in $^{mass}E_2^{s,t}(X)$ for $s \geq k$, and let $X_{(k)}$ denote the cofiber

$$X^{(k+1)} \rightarrow X \rightarrow X_{(k)}$$

Then we have fiber sequences

$$M(8, v_1^8)_{(k)} \rightarrow (T^1 \wedge M(8, v_1^8))_{(k)} \rightarrow (\Sigma^{-1} \overline{\mathrm{tmf}}^2 \wedge M(8, v_1^8))_{(k-2)}.$$

Define $\widetilde{M}_{(k)}$ to be the homotopy pullback

$$\begin{array}{ccc} \widetilde{M}_{(k)} & \longrightarrow & T^1 \wedge M(8, v_1^8) \\ \downarrow & & \downarrow \\ M(8, v_1^8)_{(k)} & \longrightarrow & (T^1 \wedge M(8, v_1^8))_{(k)} \end{array}$$

Then the algebraic tmf resolution for $\widetilde{M}_{(k)}$ is the truncation of the algebraic tmf resolution for $M(8, v_1^8)$ obtained by omitting, for $n \geq 2$ all terms of

$$\mathrm{Ext}_{A(2)_*}(\mathbf{bo}_{i_1} \otimes \cdots \otimes \mathbf{bo}_{i_n} \otimes H(8, v_1^8))$$

of cohomological degree greater than $k - n$. It follows from the map of algebraic tmf resolutions and MASS's associated to the map

$$M(8, v_1^8) \rightarrow \widetilde{M}_{(k)}$$

that there is a differential

$$d_5^{mass}(v_2^{32}) = v_1^2 h_{2,1}^{31}(v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]$$

in the MASS for $\widetilde{M}_{(k)}$. This differential is non-trivial in the MASS for $\widetilde{M}_{(36)}$, because it is non-trivial in the MASS for $M(8, v_1^8)$, and any intervening differentials killing the target in the algebraic tmf resolution or MASS for $\widetilde{M}_{(36)}$ would lift to $M(8, v_1^8)$ because the spectral sequences are isomorphic in the relevant range. The same is not true in the case of $\widetilde{M}_{(35)}$, where

$$d_0^{wss}(\Delta^2 v_1 h_{2,1}^{22}(v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4)[1]) = 0$$

and therefore $\Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4) [1]$ persists to the E_2 -term of the MASS

$$d_2^{mass}(\Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4) [1]) = v_1^2 h_{2,1}^{31} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4) [1].$$

Therefore the proof of Sublemma 5.9 goes through with $T^1 \wedge M(8, v_1^8)$ replaced with $\widetilde{M}_{(35)}$ to show that there exists an element

$$\widetilde{v}_2^{32} \in \pi_{192} \widetilde{M}_{(35)}$$

which is detected by v_2^{32} in the MASS. Consider the diagram

$$\begin{array}{ccccc} & & & \Sigma^{-1} \overline{\text{tmf}}^2 & \\ & & & \downarrow & \\ & & \partial & & \\ \widetilde{M}_{(36)} & \longrightarrow & T^1 \wedge M(8, v_1^8) & \xrightarrow{\partial'} & (\Sigma^{-1} \overline{\text{tmf}}^2 \wedge M(8, v_1^8))_{(34)} \\ & \downarrow & \parallel & & \downarrow \\ \widetilde{M}_{(35)} & \longrightarrow & T^1 \wedge M(8, v_1^8) & \xrightarrow{\partial''} & (\Sigma^{-1} \overline{\text{tmf}}^2 \wedge M(8, v_1^8))_{(33)} \end{array}$$

where the rows are cofiber sequences. The element $\widetilde{v}_2^{32} \in \pi_{192} \widetilde{M}_{(35)}$ maps to an element $v_2^{32} \in T^1 \wedge M(8, v_1^8)$ with

$$\partial''(v_2^{32}) = 0.$$

However, since $d_5^{mass}(v_2^{32})$ is non-trivial in the MASS for $\widetilde{M}_{(36)}$, the element $v_2^{32} \in \pi_{192} T^1 \wedge M(8, v_1^8)$ cannot lift to $\widetilde{M}_{(36)}$, and therefore

$$\partial'(v_2^{32}) \neq 0.$$

It follows that $\partial(v_2^{32})$ has modified Adams filtration 34. \square

However we have

Sublemma 5.11. *There are no elements of $\pi_{191} \Sigma^{-2} \overline{\text{tmf}}^2 \wedge M(8, v_1^8)$ of modified Adams filtration 34.*

Proof of sublemma. The only possible elements in the algebraic tmf resolution for $\overline{\text{tmf}}^2 \wedge M(8, v_1^8)$ which could contribute to modified Adams filtration 34 in this degree are

$$(5.12) \quad \Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4]) [1] \in \text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$$

and the elements of Table 5.7 of algebraic tmf filtration greater than 1 in the appropriate modified Adams filtration. However, the previous arguments eliminate all of the candidates coming from Table 5.7, so we are left with eliminating (5.12). We wish to lift the differential

$$d_3^{mass}(\Delta^6 v_1 h_{2,1}^3 (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4]) [1]) = \Delta^2 v_1 h_{2,1}^{22} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4]) [1]$$

in the MASS for $\text{tmf} \wedge \overline{\text{tmf}}^2 \wedge M(8, v_1^8)$ to a differential in the MASS for $\overline{\text{tmf}}^2 \wedge M(8, v_1^8)$. We therefore must argue that

$$d_2^{mass}(\Delta^6 v_1 h_{2,1}^3 (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4]) [1]) = 0$$

in the MASS for $\overline{\text{tmf}}^2 \wedge M(8, v_1^8)$. We will therefore argue there are no elements in the algebraic tmf resolution for $\overline{\text{tmf}}^2 \wedge M(8, v_1^8)$ which could detect

the target of such a d_2 . Ignoring any possibilities which are eliminated by Proposition 4.2, the only possibilities are

$$\begin{aligned} & \Delta^6 v_1^4 h_1 v_0^{-1} v_2^2 \zeta_1^8 | [\zeta_1^8, \zeta_2^4] [1], \\ & \Delta^6 v_1^4 h_1 v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4] | \zeta_1^8 [1], \\ & \Delta^6 v_1^4 h_0^2 [\zeta_1^8, \zeta_2^4] | \zeta_1^8 | \zeta_1^8 | \zeta_1^8 [0], \\ & \Delta^6 v_1^4 h_0^2 \zeta_1^8 | [\zeta_1^8, \zeta_2^4] | \zeta_1^8 | \zeta_1^8 [0], \\ & \Delta^6 v_1^4 h_0^2 \zeta_1^8 | \zeta_1^8 | [\zeta_1^8, \zeta_2^4] | \zeta_1^8 [0], \\ & \Delta^6 v_1^4 h_0^2 \zeta_1^8 | \zeta_1^8 | \zeta_1^8 | [\zeta_1^8, \zeta_2^4] [0]. \end{aligned}$$

However, these are killed by the respective WSS differentials:

$$\begin{aligned} & d_0^{wss} \Delta^6 v_1^4 h_1 v_0^{-1} v_2^2 \zeta_1^8 | \zeta_1^8 \zeta_2^4 [1], \\ & d_0^{wss} \Delta^6 v_1^4 h_1 v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4 | \zeta_1^8 [1], \\ & d_0^{wss} \Delta^6 v_1^4 h_0^2 \zeta_1^8 \zeta_2^4 | \zeta_1^8 | \zeta_1^8 | \zeta_1^8 [0], \\ & d_0^{wss} \Delta^6 v_1^4 h_0^2 \zeta_1^8 | \zeta_1^8 \zeta_2^4 | \zeta_1^8 | \zeta_1^8 [0], \\ & d_0^{wss} \Delta^6 v_1^4 h_0^2 \zeta_1^8 | \zeta_1^8 | \zeta_1^8 \zeta_2^4 | \zeta_1^8 [0], \\ & d_0^{wss} \Delta^6 v_1^4 h_0^2 \zeta_1^8 | \zeta_1^8 | \zeta_1^8 | \zeta_1^8 \zeta_2^4 [0]. \end{aligned}$$

□

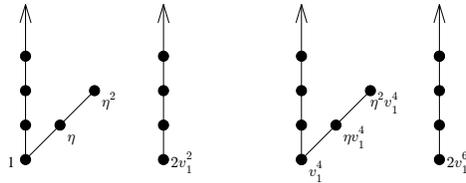
Thus we have arrived at a contradiction, as we have produced an element of modified Adams filtration 34, and subsequently showed no such elements exist. We conclude that our supposition, that the differential $d_5^{mass}(v_2^{32})$ in the MASS for $M(8, v_1^8)$ is non-trivial and detected by $v_1^2 h_{2,1}^{31} (v_0^{-1} v_2^2 \zeta_1^8 \zeta_2^4) [1]$ in the algebraic tmf resolution, is false.

□

6. DETERMINATION OF ELEMENTS NOT IN THE TMF HUREWICZ IMAGE

Theorem 6.1. *The elements of tmf_* not in the subgroup described in Theorem 1.2 are not in the Hurewicz image.*

We first recall some well known K-theory computations. Recall that $\pi_* \mathrm{KO}$ is given by the following v_1^4 -periodic pattern:



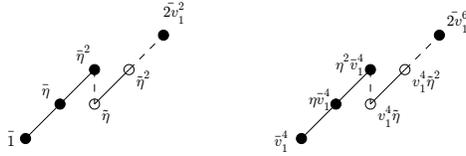
Let

$$M(2^\infty) := \varinjlim_i M(2^i)$$

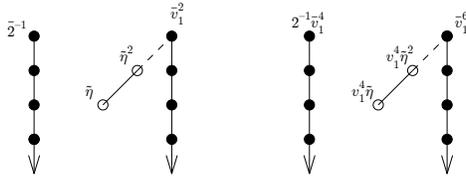
denote the Moore spectrum for $\mathbb{Z}/2^\infty$. Consider the following diagram of cofiber sequences:

$$(6.2) \quad \begin{array}{ccccccc} \Sigma^{-1}\mathrm{KO} \wedge M(2) & \xrightarrow{p} & \mathrm{KO} & \xrightarrow{\cdot 2} & \mathrm{KO} & \xrightarrow{\overline{(\cdot)}} & \mathrm{KO} \wedge M(2) \\ \downarrow \cdot 2^{-1} & & \parallel & & \downarrow \cdot 2^{-1} & & \downarrow \cdot 2^{-1} \\ \Sigma^{-1}\mathrm{KO} \wedge M(2^\infty) & \xrightarrow{p} & \mathrm{KO} & \longrightarrow & \mathrm{KO}_{\mathbb{Q}} & \xrightarrow{\overline{(\cdot)}} & \mathrm{KO} \wedge M(2^\infty) \end{array}$$

The groups $\mathrm{KO}_*M(2)$ are well-known to be given by the following v_1^4 -periodic pattern:



where we denote lifts of elements of KO_* along the map p of Diagram (6.2) with a tilde, and the images of the map $\overline{(\cdot)}$ with a bar. It then follows easily from the map of long exact sequences coming from the above diagram that $\mathrm{KO}_*M(2^\infty)$ is given by the v_1^4 -periodic pattern



where again we denote lifts over the map p with a tilde, and images under the map $\overline{(\cdot)}$ with a bar. The infinite sequences of dots going down represent the elements 2^{-i} in $\mathbb{Z}/2^\infty = \mathbb{Q}/\mathbb{Z}_{(2)}$.

Proof of Theorem 6.1. Recall that we have an equivalence [Lau04, Cor. 3]

$$c_4^{-1}\mathrm{tmf} \simeq \mathrm{KO}[j^{-1}]$$

where $j^{-1} = \Delta/c_4^3$. Applying π_0 to this equivalence, we have a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & \mathrm{KO} \\ \downarrow & & \searrow \\ \mathrm{tmf} & \longrightarrow & c_4^{-1}\mathrm{tmf} \xrightarrow{\simeq} \mathrm{KO}[j^{-1}]. \end{array}$$

Consider the following diagram

$$\begin{array}{ccccc}
\pi_* S & \xleftarrow{p} & \pi_{*+1} M(2^\infty) & \longrightarrow & \mathrm{KO}_{*+1} M(2^\infty) \\
\downarrow h & & \downarrow h & & \searrow i \\
\mathrm{tmf}_* & \xleftarrow{\quad} & \mathrm{tmf}_{*+1} M(2^\infty) & \xrightarrow{L} & c_4^{-1} \mathrm{tmf}_{*+1} M(2^\infty) = \mathrm{KO}_{*+1} M(2^\infty)[j^{-1}] \\
\downarrow L & & & \nearrow p' & \\
c_4^{-1} \mathrm{tmf}_* & & & &
\end{array}$$

Suppose that $x \in \mathrm{tmf}_{>0}$ has non-trivial image in $L(x) \in c_4^{-1} \mathrm{tmf}_*$, and suppose that $x = h(y)$. Since y is torsion, it lifts over p to an element

$$\tilde{y} \in \pi_{*+1} M(2^\infty)$$

The commutativity of the diagram, implies that

$$0 \neq L(x) \in \mathrm{Im}(p' \circ i)$$

and this implies that

$$L(x) \in \{c_4^k \eta^l : k \geq 0, l \in \{1, 2\}\}.$$

Now consider elements of the form

$$x = \alpha \Delta^k \nu \in \mathrm{tmf}_*$$

with $\alpha \not\equiv 0 \pmod{8}$. Suppose that $x = h(y)$. Lift y to an element

$$\tilde{y} \in \pi_{*+1} M(2^\infty).$$

Then we have

$$Lh(\tilde{y}) = \frac{\overline{\alpha \Delta^k v_1^2}}{8} = \frac{\overline{\alpha v_1^{12k+2}}}{4} j^{-k} \neq 0.$$

But the commutativity of the diagram implies that $Lh(\tilde{y})$ is in the image of i , which implies that $k = 0$. \square

7. LIFTING THE REMAINING ELEMENTS OF tmf_* TO π_*^S .

Multiplicative generators of the Hurewicz image below the 192-stem. In this section, we determine a set of elements which multiplicatively generate the tmf -Hurewicz image the below the 192-stem. The results in this section drastically reduce the number of classes which we must lift in the sequel.

Lemma 7.1. *The Hurewicz map $S \rightarrow \mathrm{tmf}$ is a map of E_∞ -ring spectra. In particular, it preserves multiplication and Toda brackets.*

This lemma may be applied as follows. Suppose we wish to lift a class $\alpha \in \pi_*(\mathrm{tmf})$ to a class $\tilde{\alpha} \in \pi_*(S)$.

- (1) Suppose $\alpha = \beta\gamma$ is a product of elements $\beta, \gamma \in \pi_*(\mathrm{tmf})$ with lifts $\tilde{\beta}, \tilde{\gamma} \in \pi_*(S)$. Lemma 7.1 implies that $\tilde{\beta}\tilde{\gamma} \in \pi_*(S)$ must be a lift of α .

- (2) Suppose the Toda bracket $\langle \alpha_1, \dots, \alpha_k \rangle$ is defined, and that $\alpha_1, \dots, \alpha_k \in \pi_*(\text{tmf})$ have lifts $\tilde{\alpha}_1, \dots, \tilde{\alpha}_k \in \pi_*(S)$ so that the Toda bracket $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined. Lemma 7.1 implies that if α is the Hurewicz image of $\tilde{\alpha} \in \langle \tilde{\alpha}_1, \dots, \tilde{\alpha}_k \rangle$, then

$$\alpha \in \langle \alpha_1, \dots, \alpha_k \rangle.$$

With this in mind, it suffices to find a subset of the Hurewicz image which generates the entire Hurewicz image up to the 192-stem under products and Toda brackets. Our desired generating subset is given in Corollary 7.18. We will obtain our generating set by listing generators in lemmas and then recording their products in corollaries, until we have exhausted the tmf-Hurewicz image up to stem 192.

Lemma 7.2. *The classes $2 \in \pi_0(\text{tmf})$, $\eta \in \pi_1(\text{tmf})$, and $\nu \in \pi_3(\text{tmf})$ are in the Hurewicz image.*

Proof. These classes are detected by h_0 , h_1 , and h_2 , respectively, in the ASS for the sphere and for tmf. The Hurewicz map induces a map of spectral sequences which sends $h_i \mapsto h_i$. The map in homotopy $\pi_*(S) \rightarrow \pi_*(\text{tmf})$ then sends $2 \mapsto 2$, $\eta \mapsto \eta$, and $\nu \mapsto \nu$, respectively, since each element survives in the ASS. \square

Corollary 7.3. *The classes $2^i \in \pi_0(\text{tmf})$, $i \geq 1$, $\eta^2 \in \pi_2(\text{tmf})$, $2\nu \in \pi_3(\text{tmf})$, $4\nu = \eta^3 \in \pi_3(\text{tmf})$, $\nu^2 \in \pi_6(\text{tmf})$, and $\nu^3 = \epsilon\eta \in \pi_9(\text{tmf})$ are in the Hurewicz image.*

Lemma 7.4. *The classes $\epsilon \in \pi_8(\text{tmf})$, $\kappa \in \pi_{14}(\text{tmf})$, and $\bar{\kappa} \in \pi_{20}(\text{tmf})$ are in the Hurewicz image.*

Proof. By [Bau08, Table 1], the class $\epsilon \in \pi_8(\text{tmf})$ is in the Toda bracket $\langle \nu, \eta, \nu \rangle$, the class $\kappa \in \pi_{14}(\text{tmf})$ is in the Toda bracket $\langle \nu, 2\nu, \nu, 2\nu \rangle$, and the class $\bar{\kappa} \in \pi_{20}(\text{tmf})$ is in the Toda bracket $\langle \kappa, 2, \eta, \nu \rangle$. The result follows from the fact that these Toda brackets have no indeterminacy, 2 , η , and ν are in the Hurewicz image by Lemma 7.2, and the Toda brackets are defined in $\pi_*(S)$. \square

Corollary 7.5. *The classes $\kappa\eta \in \pi_{15}(\text{tmf})$, $\kappa\nu \in \pi_{17}(\text{tmf})$, $2\bar{\kappa} \in \pi_{20}(\text{tmf})$, $4\bar{\kappa} = \kappa\nu^2 \in \pi_{20}(\text{tmf})$, $\bar{\kappa}\eta \in \pi_{21}(\text{tmf})$, $\bar{\kappa}\eta^2 = \kappa\epsilon \in \pi_{22}(\text{tmf})$, $\bar{\kappa}\epsilon = \kappa^2 \in \pi_{28}(\text{tmf})$, $\bar{\kappa}\kappa \in \pi_{34}(\text{tmf})$, $\bar{\kappa}\kappa\eta \in \pi_{35}(\text{tmf})$, $\bar{\kappa}^2 \in \pi_{40}(\text{tmf})$, $2\bar{\kappa}^2 \in \pi_{40}(\text{tmf})$, $\bar{\kappa}^2\eta \in \pi_{41}(\text{tmf})$, $\bar{\kappa}^2\eta^2 = \kappa^3 \in \pi_{42}(\text{tmf})$, $\bar{\kappa}^2\kappa \in \pi_{54}(\text{tmf})$, $\bar{\kappa}^3 \in \pi_{60}(\text{tmf})$, $2\bar{\kappa}^3 \in \pi_{60}(\text{tmf})$, $\bar{\kappa}^4 \in \pi_{80}(\text{tmf})$, and $\bar{\kappa}^5 \in \pi_{100}(\text{tmf})$ are in the Hurewicz image.*

Lemma 7.6. *The classes $q \in \pi_{32}(\text{tmf})$, $u \in \pi_{39}(\text{tmf})$, and $w \in \pi_{45}(\text{tmf})$ are in the Hurewicz image.*

Proof. By [Isa19, Table 8], the class q is detected by $\Delta h_1 h_3$ in the ASS for S , u is detected by $\Delta h_1 d_0$, and w is detected by $\Delta h_1 g$. The same holds in the ASS for tmf by inspection of [DFHH14, Pg. 215]. The Hurewicz map $S \rightarrow \text{tmf}$ induces a map which sends these elements to the element with the same name. Since there are no elements in higher Adams filtration (except for possibly v_1 -periodic classes), we conclude that the same holds in homotopy. \square

Corollary 7.7. *The classes $q\eta \in \pi_{33}(\text{tmf})$, $w\eta \in \pi_{46}(\text{tmf})$, $\bar{\kappa}q \in \pi_{52}(\text{tmf})$, $\bar{\kappa}q\eta \in \pi_{53}(\text{tmf})$, $\bar{\kappa}u \in \pi_{59}(\text{tmf})$, $\bar{\kappa}w \in \pi_{65}(\text{tmf})$, $\bar{\kappa}w\eta \in \pi_{66}(\text{tmf})$, $\bar{\kappa}^2w \in \pi_{85}(\text{tmf})$, $w^2 \in \pi_{90}(\text{tmf})$, $\bar{\kappa}^3w \in \pi_{105}(\text{tmf})$, $\bar{\kappa}w^2 \in \pi_{110}(\text{tmf})$, $\bar{\kappa}^4w \in \pi_{125}(\text{tmf})$, and $\bar{\kappa}^2w^2 \in \pi_{130}(\text{tmf})$ are in the Hurewicz image.*

Lemma 7.8. *The classes $\{\nu\Delta^2\}\nu \in \pi_{54}(\text{tmf})$, $\{\nu\Delta^2\}\kappa \in \pi_{65}(\text{tmf})$, and $\{\eta^2\Delta^2\}\bar{\kappa} \in \pi_{70}(\text{tmf})$ are in the Hurewicz image.*

Proof. See Lemma 7.23. □

Corollary 7.9. *The classes $\{\nu\Delta^2\}\nu^2 \in \pi_{57}(\text{tmf})$ and $\{\nu\Delta^2\}\kappa\nu$ are in the Hurewicz image.*

Lemma 7.10. *The classes $\{\nu\Delta^4\}\nu \in \pi_{102}(\text{tmf})$, $\{\epsilon\Delta^4\} \in \pi_{104}(\text{tmf})$, $\{\kappa\Delta^4\} \in \pi_{110}(\text{tmf})$, $2\Delta^4\bar{\kappa} \in \pi_{116}(\text{tmf})$, and $\{\eta\Delta^4\}\bar{\kappa} \in \pi_{117}(\text{tmf})$ are in the Hurewicz image.*

Proof. See Lemmas 7.24 and 7.25. □

Corollary 7.11. *The classes $\{\epsilon\Delta^4\}\eta \in \pi_{105}(\text{tmf})$, $\{\kappa\Delta^4\}\eta \in \pi_{111}(\text{tmf})$, $\{\kappa\Delta^4\}\nu \in \pi_{113}(\text{tmf})$, $\{\kappa\Delta^4\}\nu^2 \in \pi_{116}(\text{tmf})$, $\{\eta\Delta^4\}\bar{\kappa}\eta \in \pi_{118}(\text{tmf})$, $\{\kappa\Delta^4\}\kappa \in \pi_{124}(\text{tmf})$, $\{\kappa\Delta^4\}\bar{\kappa} \in \pi_{130}(\text{tmf})$, $\{\kappa\Delta^4\}\bar{\kappa}\eta \in \pi_{131}(\text{tmf})$, $\{\eta\Delta^4\}\bar{\kappa}^2 \in \pi_{137}(\text{tmf})$, and $\{\eta\Delta^4\}\bar{\kappa}^2\eta \in \pi_{138}(\text{tmf})$ are in the Hurewicz image.*

Lemma 7.12. *The class $\{q\Delta^4\} \in \pi_{128}(\text{tmf})$ is in the Hurewicz image.*

Proof. See Lemma 7.26. □

Corollary 7.13. *The classes $\{q\Delta^4\}\eta \in \pi_{129}(\text{tmf})$, $\{q\Delta^4\}\kappa = w\eta\Delta^4 \in \pi_{142}(\text{tmf})$, $\{q\Delta^4\}\bar{\kappa} \in \pi_{148}(\text{tmf})$, $\{q\Delta^4\}\bar{\kappa}\eta \in \pi_{149}(\text{tmf})$, $\{q\Delta^4\}\bar{\kappa}\eta^2 \in \pi_{150}(\text{tmf})$ are in the Hurewicz image.*

Lemma 7.14. *The class $\Delta^4u \in \pi_{135}(\text{tmf})$ is in the Hurewicz image.*

Proof. See Lemma 7.27. □

Corollary 7.15. *The classes $\Delta^4u\eta \in \pi_{136}(\text{tmf})$ and $\Delta^4u\bar{\kappa} \in \pi_{155}(\text{tmf})$ are in the Hurewicz image.*

Lemma 7.16. *The classes $\{\nu\Delta^6\}\nu \in \pi_{150}(\text{tmf})$ and $\{\nu\Delta^6\}\kappa \in \pi_{161}(\text{tmf})$ are in the Hurewicz image.*

Proof. See Lemma 7.28. □

Corollary 7.17. *The classes $\{\nu\Delta^6\}2\nu \in \pi_{150}(\text{tmf})$, $\{\nu\Delta^6\}\nu^2 \in \pi_{153}$, $\{\nu\Delta^6\}\nu^3 \in \pi_{156}$, $\{\nu\Delta^6\}\kappa\eta \in \pi_{162}(\text{tmf})$ and $\{\nu\Delta^6\}\kappa\nu \in \pi_{164}(\text{tmf})$ are in the Hurewicz image.*

Thus our calculation of the Hurewicz image up to dimension 192 has been reduced to showing that the following list of elements is in the Hurewicz image.

Corollary 7.18. *Up to dimension 192, the Hurewicz image is generated under multiplication by*

$$\{2, \eta, \nu, \epsilon, \kappa, \bar{\kappa}, q, u, w, \{\nu\Delta^2\}\nu, \{\nu\Delta^2\}\kappa, \{\eta^2\Delta^2\}\bar{\kappa}, \{\nu\Delta^4\}\nu, \{\epsilon\Delta^4\}, \{\kappa\Delta^4\}, 2\Delta^4\bar{\kappa}, \{\eta\Delta^4\}\bar{\kappa}, \{q\Delta^4\}, \Delta^4u, \{\nu\Delta^6\}\nu, \{\nu\Delta^6\}\kappa\}.$$

Lifting generators. We will now describe our method for lifting generators. Given an element $x \in \mathrm{tmf}_*$, we want to lift it to an element $y \in \pi_*^s$. To this end, we consider the diagram of (M)ASS's:

$$\begin{array}{ccccc}
 & & \mathrm{Ext}_{A(2)_*}(H(8, v_1^8)) & \xlongequal{\quad\quad\quad} & \mathrm{tmf}_* M(8, v_1^8) \\
 & \nearrow & \downarrow & & \downarrow \\
 \mathrm{Ext}_{A_*}(H(8, v_1^8)) & \xlongequal{\quad\quad\quad} & \pi_* M(8, v_1^8) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2) & \xlongequal{\quad\quad\quad} & \mathrm{tmf}_* \\
 \downarrow & \nearrow & \downarrow & & \downarrow \\
 \mathrm{Ext}_{A_*}(\mathbb{F}_2) & \xlongequal{\quad\quad\quad} & \pi_*^s & & \\
 & \nearrow & \downarrow & & \\
 & & \mathrm{tmf}_* & &
 \end{array}$$

First, we identify an element

$$x' \in \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$$

which detects the element x in the ASS for tmf_* , and then we identify an element

$$\tilde{x}' \in \mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$$

which maps to it. This element \tilde{x}' can be regarded as an element of the zero line of the algebraic tmf -resolution for $\mathrm{Ext}_{A_*}(H(8, v_1^8))$. We will show that the element \tilde{x}' is a permanent cycle in the algebraic tmf -resolution, and thus lifts to an element

$$\tilde{y}' \in \mathrm{Ext}_{A_*}(H(8, v_1^8)).$$

We will then show that the element \tilde{y}' is a permanent cycle in the MASS for $M(8, v_1^8)$, and hence detects an element

$$\tilde{y} \in \pi_* M(8, v_1^8).$$

Let $y \in \pi_*^s$ be the projection of \tilde{y} to the top cell. It then follows that the image of y in tmf_* equals x , modulo terms of higher Adams filtration (AF). Furthermore, using the v_2^{32} -self map on $M(8, v_1^8)$, we deduce that the element

$$v_2^{32k} \tilde{y} \in \pi_* M(8, v_1^8)$$

projects on the top cell to an element $v_2^{32k} y \in \pi_*^s$ whose image in tmf_* is $\Delta^{8k} x$ modulo terms of higher Adams filtration. Finally, Theorem 6.1 eliminates the potential ambiguity caused by elements of higher Adams filtration, since the elements of higher Adams filtration are v_1^4 -periodic.

We will show all of the generators of Corollary 7.18 actually come from the top cell of $M(8, v_1^8)$, and thus v_2^{32} periodicity extends our work below dimension 192 to all dimensions.

Lemma 7.19. *The following classes lift to the top cell of $M(8, v_1^8)$:*

- (1) $\kappa \in \pi_{14}(\mathrm{tmf})$,
- (2) $\bar{\kappa} \in \pi_{20}(\mathrm{tmf})$.

Proof. We will check that each element lifts using the AHSS:

- (1) Since κ is 2-torsion (and thus 8-torsion), it lifts to $\kappa[1] \in \pi_{15}(M(8))$. Inspection of [IWX20a, Pg. 3] in stems 31 and 32 and $\text{AF} \geq 12$ reveals that there are no classes which could detect $v_1^8 \kappa[1]$. Therefore $\kappa[1]$ lifts to $\kappa[18] \in \pi_{32}(M(8, v_1^8))$.
- (2) Since $\bar{\kappa}$ is 8-torsion, it lifts to $\bar{\kappa}[1] \in \pi_{21}(M(8))$. Inspection of [IWX20a, Pg. 3] in stems 36 and 37 and $\text{AF} \geq 12$ reveals that there are no classes which could detect $v_1^8 \bar{\kappa}[1]$. Therefore $\bar{\kappa}[1]$ lifts to $\bar{\kappa}[18] \in \pi_{38}(M(8, v_1^8))$.

□

Lemma 7.20. *The following classes lift to the top cell of $M(8, v_1^8)$:*

- (1) $q \in \pi_{32}(\text{tmf})$,
- (2) $u \in \pi_{39}(\text{tmf})$,
- (3) $w \in \pi_{45}(\text{tmf})$.

Proof. We will check that each element lifts using the Atiyah-Hirzebruch spectral sequence (AHSS).

- (1) We begin with $q \in \pi_{32}(\text{tmf})$, which we will define to be the unique non-trivial c_4 -torsion class detected by the element

$$v_2^4 c_0 \in \text{Ext}_{A(2)_*}^{7, 7+32}(\mathbb{F}_2)$$

in the ASS for tmf. The element $v_2^4 c_0$ does *not* lift to Ext_{A_*} . Nevertheless, we claim that there is an element $\tilde{q} \in \pi_{32}^s$ ¹⁰ detected by the element

$$\Delta h_1 h_3 \in \text{Ext}_{A_*}^{6, 6+32}(\mathbb{F}_2)$$

in the ASS for the sphere, which maps to q under the tmf Hurewicz homomorphism. Our strategy will be to argue that \tilde{q} and q lift to

$$\tilde{q}[18] \in \pi_{50} M(8, v_1^8) \quad \text{and} \quad q[18] \in \text{tmf}_{50} M(8, v_1^8)$$

respectively, and that the element which detects $\tilde{q}[18]$ in the MASS for $M(8, v_1^8)$ maps to the element which detects $q[18]$ in the MASS for $\text{tmf} \wedge M(8, v_1^8)$ under the map

$$(7.21) \quad \text{Ext}_{A_*}(H(8, v_1^8)) \rightarrow \text{Ext}_{A(2)_*}(H(8, v_1^8)).$$

Inspection of [IWX20a, Pg. 3] in stem 32 and $\text{AF} \geq 7$ reveals that \tilde{q} is 2-torsion (and thus 8-torsion), so \tilde{q} lifts to $\tilde{q}[1] \in \pi_{33}(M(8))$. Inspection of [IWX20a, Pg. 3] in stems 48 and 49 and $\text{AF} \geq 14$ reveals that there are no classes which could detect $v_1^8 \tilde{q}[1]$. Therefore $\tilde{q}[1]$ lifts to $\tilde{q}[18] \in \pi_{50}(M(8, v_1^8))$. A similar but easier analysis reveals that the lift $q[18]$ exists.

The elements $\Delta h_1 h_3 \in \text{Ext}_{A_*}(\mathbb{F}_2)$ and $v_2^4 c_0 \in \text{Ext}_{A(2)_*}(\mathbb{F}_2)$ are h_0 -torsion, and hence lift to elements

$$\begin{aligned} \Delta h_1 h_3[1] &\in \text{Ext}_{A_*}(H(8)), \\ v_2^4 c_0[1] &\in \text{Ext}_{A(2)_*}(H(8)) \end{aligned}$$

¹⁰The element we are calling $\tilde{q} \in \pi_{32}^s$ is traditionally called q , but we add the tilde to distinguish it from the element we are calling q in $\pi_{32}\text{tmf}$.

which detect $\tilde{q}[1] \in \pi_{33}M(8)$ and $q[1] \in \mathrm{tmf}_{33}M(8)$, respectively, in the MASS. To identify the elements which detect $\tilde{q}[18]$ and $q[18]$ in the MASS, we make use of the Geometric Boundary Theorem [Beh12, Appendix A].¹¹ The differentials

$$\begin{aligned} d_3(v_1^2 h_{2,1} g^2[1]) &= v_1^8 \Delta h_3 h_1[1], \\ d_4(v_1^2 h_{2,1} g^2[1]) &= v_1^8 v_2^4 c_0[1] \end{aligned}$$

in the MASS's for $M(8)$ and $\mathrm{tmf} \wedge M(8)$, respectively, imply that $\tilde{q}[18] \in \pi_{50}M(8, v_1^8)$ and $q[18] \in \mathrm{tmf}_{50}M(8, v_1^8)$ are detected by

$$\begin{aligned} v_1^2 h_{2,1} g^2[1] &\in \mathrm{Ext}_{A_*}(H(8, v_1^8)), \\ v_1^2 h_{2,1} g^2[1] &\in \mathrm{Ext}_{A(2)_*}(H(8, v_1^8)), \end{aligned}$$

in the MASS's for $M(8, v_1^8)$ and $\mathrm{tmf} \wedge M(8, v_1^8)$, respectively, and the former maps to the latter under the map (7.21).

- (2) Since $u \in \pi_{39}\mathrm{tmf}$ is detected by an element of $\mathrm{Ext}_{A(2)_*}$ in the image of the map

$$(7.22) \quad \mathrm{Ext}_{A_*}(\mathbb{F}_2) \rightarrow \mathrm{Ext}_{A(2)_*}(\mathbb{F}_2)$$

we immediately see that the element $u \in \pi_{39}(S)$ maps to it. We are left with lifting $u \in \pi_{39}^s$ to the top cell of $M(8, v_1^8)$. Inspection of [IWX20a, Pg. 3] in stem 39 and $\mathrm{AF} \geq 10$ reveals that u is 2-torsion (and thus 8-torsion), so u lifts to $u[1] \in \pi_{40}(M(8))$. Inspection of [IWX20a, Pg. 3] in stems 55 and 56 and $\mathrm{AF} \geq 17$ reveals that there are no classes which could detect $v_1^8 u[1]$. Therefore $u[1]$ lifts to $u[18] \in \pi_{57}(M(8, v_1^8))$.

- (3) The element $w \in \pi_{45}\mathrm{tmf}$ is detected by an element which is in the image of the map (7.22), and thus we deduce that $w \in \pi_{45}(S)$ maps to it. A similar argument to the case above shows that w lifts to $w[18] \in \pi_{63}(M(8, v_1^8))$.

□

Lemma 7.23. *The following classes lift to the top cell of $M(8, v_1^8)$:*

- (1) $\Delta^2 \nu^2 \in \pi_{54}(\mathrm{tmf})$,
- (2) $\Delta^2 \kappa \nu \in \pi_{65}(\mathrm{tmf})$,
- (3) $\Delta^2 \eta^2 \bar{\kappa} \in \pi_{70}(\mathrm{tmf})$.

Proof. We follow the proof of [BHHM20, Thm. 11.1] (which builds on [BHHM20, Exm. 9.5] and [BHHM20, Prop. 10.1]).

- (1) We begin with $\Delta^2 \nu^2 \in \pi_{54}(\mathrm{tmf})$. This class lifts to an element

$$\Delta^2 \nu^2[1] \in \mathrm{tmf}_{55}(M(8))$$

¹¹We are specifically using case (5) of the Geometric Boundary Theorem since the relevant class (denoted $p_*(y)$ in the theorem statement) is a permanent cycle. We will be using this argument repeatedly in subsequent proofs in this section, and for brevity will simply say “by the Geometric Boundary Theorem...” in these subsequent instances.

which is detected by

$$v_2^8 h_2^2[1] \in \text{Ext}_{A(2)_*}^{12,55+12}(H(8))$$

in the MASS for $\text{tmf} \wedge M(8)$. Let

$$\Delta^2 \nu^2[18] \in \text{tmf}_{72}(M(8, v_1^8)).$$

be a lift of $\Delta^2 \nu^2[1]$. In the MASS for $\text{tmf} \wedge M(8)$, there is a differential

$$d_2(v_2^{10} v_1^4 h_2 h_0[1]) = v_2^8 v_1^8 h_2^2[1].$$

Since $v_2^{10} v_1^4 h_2 h_0[1]$ is a permanent cycle in the MASS for $\text{tmf} \wedge M(8, v_1^8)$, it follows from the Geometric Boundary Theorem that $\Delta^2 \nu^2[18]$ is detected by $v_2^{10} v_1^4 h_2 h_0[1]$ in the MASS for $\text{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^2 \nu^2[18]$ has modified Adams filtration (MAF) 18 and stem 72.

We now check that $v_2^{10} v_1^4 h_2 h_0[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(8, v_1^8)$. Its relative position¹² is $t-s = 65$ and $AF = 17$, its relative position in $\text{Ext}_{A(2)_*}(\mathbf{bo}_1^{\otimes 2} \otimes H(8, v_1^8))$ is $t-s = 58$ and $AF = 16$, and its relative position in $\text{Ext}_{A(2)_*}(\mathbf{bo}_1^{\otimes 3} \otimes H(8, v_1^8))$ is $t-s = 51$ and $AF = 15$, the last of which lies above the vanishing line. Inspection of the relevant charts shows that $v_2^{10} v_1^4 h_2 h_0[1]$ cannot support a nontrivial d_1 -differential since the target bidegrees are zero. Therefore $v_2^{10} v_1^4 h_2 h_0[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(8, v_1^8)$ and therefore it detects an element $\{v_2^{10} v_1^4 h_2 h_0[1]\}$ in $\text{Ext}_{A_*}(H(8, v_1^8))$.

Finally, inspection of the same algebraic tmf resolution charts reveals that there are no possible targets for a nontrivial differential supported by $\{v_2^{10} v_1^4 h_2 h_0[1]\}$ in the MASS for $M(8, v_1^8)$. Therefore $\{v_2^{10} v_1^4 h_2 h_0[1]\}$ is a permanent cycle which detects a lift of $\Delta^2 \nu^2$.

- (2) The class $\Delta^2 \kappa \nu \in \pi_{65}(\text{tmf})$ lifts to an element

$$\Delta^2 \kappa \nu[1] \in \text{tmf}_{66}(M(8))$$

which is detected by

$$v_2^8 h_2 d_0[1] \in \text{Ext}_{A(2)_*}^{15,66+15}(H(8))$$

in the MASS for $\text{tmf} \wedge M(8)$. Lift $\Delta^2 \kappa \nu[1]$ to an element

$$\Delta^2 \kappa \nu[18] \in \text{tmf}_{83}(M(8, v_1^8)).$$

In the MASS for $\text{tmf} \wedge M(8)$, there is a differential

$$d_2(v_2^{10} v_1^4 d_0 h_0[1]) = v_2^8 v_1^8 h_2 d_0[1].$$

It follows from the Geometric Boundary Theorem that $v_2^8 \kappa \nu[18]$ is detected by $v_2^{10} v_1^4 d_0 h_0[1]$ in the MASS for $\text{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^2 \kappa \nu[18]$ has MAF 21 and stem 83.

¹²We will say that an element $x \in \text{Ext}_{A(2)_*}(H(8, v_1^8))$ has *relative position* $(t-s, s)$ in $\text{Ext}_{A(2)_*}(\mathbf{bo}_I \otimes H(8, v_1^8))$ if the image of a differential supported by x in the algebraic tmf resolution lies in $\text{Ext}_{A(2)_*}^{s+1, t}(\mathbf{bo}_I \otimes H(8, v_1^8))$, and the image of a differential supported by x in the MASS could be detected in the algebraic tmf resolution by an element in $\text{Ext}_{A(2)_*}^{s+r, t-r+1}(\mathbf{bo}_1 \otimes H(8, v_1^8))$. In other words, if you were to pretend x were an element in $\text{Ext}_{A(2)_*}^{s, t}(\mathbf{bo}_I \otimes H(8, v_1^8))$, then d_r -differentials in the algebraic tmf resolution “look” like Adams d_1 ’s, and d_r -differentials in the MASS “look” like Adams d_r ’s.

We now check that $v_2^{10}v_1^4d_0h_0[1]$ is a permanent cycle in the algebraic tmf resolution for $H(8, v_1^8)$. Its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1 \otimes H(8, v_1^8))$ is $t-s = 76$ and $AF = 20$, its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$ is $t-s = 69$ and $AF = 19$, and its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes 3} \otimes H(8, v_1^8))$ is $t-s = 62$ and $AF = 18$, the last of which has targets only above the vanishing line. Inspection of the relevant charts shows that $v_2^{10}v_1^4d_0h_0[1]$ cannot support a nontrivial d_1 -differential since the target bidegrees are zero. Therefore $v_2^{10}v_1^4d_0h_0[1]$ is a permanent cycle in the algebraic tmf-resolution for $H(8, v_1^8)$ and detects an element $\{v_2^{10}v_1^4d_0h_0[1]\}$ in $\text{Ext}_{A_*}(H(8, v_1^8))$.

Finally, inspection of the same charts reveals that there are no possible targets for a nontrivial differential supported by $\{v_2^{10}v_1^4d_0h_0[1]\}$ in the MASS for $M(8, v_1^8)$. Therefore $\{v_2^{10}v_1^4d_0h_0[1]\}$ is a permanent cycle.

- (3) The class $\Delta^2\eta^2\bar{\kappa} \in \pi_{70}(\text{tmf})$ lifts to an element

$$\Delta^2\eta^2\bar{\kappa}[1] \in \text{tmf}_{71}(M(8))$$

which is detected by

$$g^2h_{2,1}^6[1] \in \text{Ext}_{A(2)_*}^{16,71+16}(H(8))$$

in the MASS for $\text{tmf} \wedge M(8)$. Lift $\Delta^2\eta^2\bar{\kappa}[1]$ to an element

$$\Delta^2\eta^2\bar{\kappa}[18] \in \text{tmf}_{88}(M(8, v_1^8)).$$

In the MASS for $\text{tmf} \wedge M(8)$, there is a differential

$$d_2(v_2^8v_1^4d_0e_0[1]) = g^2v_1^8h_{2,1}^6[1].$$

It follows from the Geometric Boundary Theorem that $\Delta^2\eta^2\bar{\kappa}[18]$ is detected by $v_2^8v_1^4d_0e_0[1]$ in the MASS for $\text{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^2\eta^2\bar{\kappa}[18]$ has MAF 24 and stem 88.

We now check that $v_2^8v_1^4d_0e_0[1]$ is a permanent cycle in the algebraic tmf-resolution for $H(8, v_1^8)$. Its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1 \otimes H(8, v_1^8))$ is $t-s = 81$ and $AF = 23$ and its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$ is $t-s = 74$ and $AF = 22$, the latter of which lies above the vanishing line. Inspection of the relevant charts shows that $v_2^8v_1^4d_0e_0[1]$ cannot support a nontrivial differential in the algebraic tmf resolution for $H(8, v_1^8)$ since the target bidegrees are zero. Therefore $v_2^8v_1^4d_0e_0[1]$ is a permanent cycle in the algebraic tmf-resolution for $H(8, v_1^8)$ and therefore lifts to an element $\{v_2^8v_1^4d_0e_0[1]\}$ in $\text{Ext}_{A_*}(H(8, v_1^8))$.

Finally, inspection of the same charts reveals that there are no possible targets for a nontrivial differential supported by $\{v_2^8v_1^4d_0e_0[1]\}$ in the MASS for $M(8, v_1^8)$. Therefore $\{v_2^8v_1^4d_0e_0[1]\}$ is a permanent cycle in the MASS for $M(8, v_1^8)$.

□

Lemma 7.24. *The following classes lift to the top cell of $M(8, v_1^8)$:*

- (1) $\Delta^4\nu^2 \in \pi_{102}(\text{tmf})$, $\Delta^4\epsilon \in \pi_{104}(\text{tmf})$, $\Delta^4\kappa \in \pi_{110}(\text{tmf})$,
- (2) $\Delta^42\bar{\kappa} \in \pi_{116}(\text{tmf})$.

Proof.

- (1) These classes were lifted in [BHHM20, Thm. 11.1].
- (2) The class $\Delta^4 2\bar{\kappa} \in \pi_{116}(\mathrm{tmf})$ lifts to an element

$$\Delta^4 2\bar{\kappa}[1] \in \mathrm{tmf}_{117}(M(8))$$

which is detected by

$$v_2^{16} h_0 g[1] \in \mathrm{Ext}_{A(2)_*}^{23, 117+23}(H(8))$$

in the MASS for $\mathrm{tmf} \wedge M(8)$. Lift $\Delta^4 2\bar{\kappa}[1]$ to an element

$$\Delta^4 2\bar{\kappa}[18] \in \mathrm{tmf}_{134}(M(8, v_1^8)).$$

In the MASS for $\mathrm{tmf} \wedge M(8)$, there is a differential

$$d_2(v_2^{18} v_1^4 d_0 h_2[1]) = v_2^{16} v_1^8 h_0 g[1].$$

It follows from the Geometric Boundary Theorem that $\Delta^4 2\bar{\kappa}[18]$ is detected by $v_2^{18} v_1^4 d_0 h_2[1]$ in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^4 2\bar{\kappa}[18]$ has MAF 29 and stem 134.

We now check that $v_2^{18} v_1^4 d_0 h_2[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(8, v_1^8)$. Its relative position in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1 \otimes H(8, v_1^8))$ is $t-s = 127$ and $AF = 28$, its relative position in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$ is $t-s = 120$ and $AF = 27$, and its relative position in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3} \otimes H(8, v_1^8))$ is $t-s = 113$ and $AF = 26$, the last of which lies above the vanishing line. Inspection of the relevant charts shows that $v_2^{16} 2\bar{\kappa}[18]$ cannot support a nontrivial d_1 -differential since the target bidegrees are zero. Therefore $v_2^{16} 2\bar{\kappa}[18]$ is a permanent cycle in the algebraic tmf -resolution for $H(8, v_1^8)$ and lifts to an element $v_2^{16} 2\bar{\kappa}[18]$ in $\mathrm{Ext}_{A_*}(H(8, v_1^8))$.

Finally, inspection of the same charts reveals that there are no possible targets for a nontrivial differential supported by $v_2^{16} 2\bar{\kappa}[18]$ in the MASS for $M(8, v_1^8)$. Therefore $v_2^{16} 2\bar{\kappa}[18]$ is a permanent cycle.

□

Contrary to the previous cases, there are several potential obstructions to lifting $\Delta^4 \bar{\kappa} \eta \in \pi_{117}(\mathrm{tmf})$ to the top cell of $M(8, v_1^8)$ which are tricky to resolve. However, since this element is 2-torsion and v_1^4 -torsion, we may instead attempt to lift it to the top cell of the generalized Moore spectrum $M(2, v_1^4)$ of [BHHM08], where the potential obstructions are much simpler to analyze. It then follows from the fact that the composite

$$\Sigma^8 M(2, v_1^4) \xrightarrow{\cdot 4v_1^4} M(8, v_1^8) \rightarrow S^{18}$$

is projection onto the top cell of $M(2, v_1^4)$ that $\Delta^4 \bar{\kappa} \eta$ does lift to the top cell of $M(8, v_1^8)$.

Lemma 7.25. *The class $\Delta^4 \bar{\kappa} \eta \in \pi_{117}(\mathrm{tmf})$ lifts to the top cell of $M(2, v_1^4)$.*

Proof. The class $\Delta^4 \eta \bar{\kappa} \in \pi_{117}(\mathrm{tmf})$ lifts to an element

$$\Delta^4 \eta \bar{\kappa}[1] \in \mathrm{tmf}_{118}(M(2))$$

which is detected by

$$v_2^{16}h_1g[1] \in \text{Ext}^{21,118+21}(H(2))$$

in the MASS for $\text{tmf} \wedge M(2)$. Lift $\Delta^4\eta\bar{\kappa}[1]$ to an element

$$\Delta^4\eta\bar{\kappa}[10] \in \text{tmf}_{127}(M(2, v_1^4)).$$

In the MASS for $\text{tmf} \wedge M(2)$, there is a differential

$$d_3(v_2^{20}h_2^2[1]) = v_2^{16}v_1^4h_1g[1].$$

It follows from the Geometric Boundary Theorem that $\Delta^4\eta\bar{\kappa}[10]$ is detected by $v_2^{20}h_2^2[1]$ in the MASS for $\text{tmf} \wedge M(2, v_1^4)$. In particular, we see that $\Delta^4\eta\bar{\kappa}[10]$ has MAF 24 and stem 127.

We now check that $v_2^{20}h_2^2[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(2, v_1^4)$. Its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1 \otimes H(2, v_1^4))$ is $t - s = 120$ and $AF = 23$, its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes 2} \otimes H(2, v_1^4))$ is $t - s = 113$ and $AF = 22$, and its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1^{\otimes 3} \otimes H(2, v_1^4))$ is $t - s = 106$ and $AF = 21$. Inspection of the relevant charts [BHHM08, Figs. 6.4-6.5] shows that there is potentially a nontrivial differential

$$d_1(v_2^{20}h_2^2[1]) = x_{119,24},$$

in the algebraic tmf resolution, where

$$x_{119,24} \in \text{Ext}_{A(2)_*}^{24,119+24}(\underline{\text{bo}}_1 \otimes H(2, v_1^4)),$$

but since $v_2^{20}h_2^2[1]$ is v_2^{16} -divisible and $x_{119,24}$ is not, this differential cannot occur (compare with the proof of [BHHM20, Prop. 10.1]). Therefore $v_2^{20}h_2^2[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(2, v_1^4)$ and therefore lifts to an element $\{v_2^{20}h_2^2[1]\}$ in $\text{Ext}_{A_*}(H(2, v_1^4))$.

Finally, inspection of the same charts reveals that there are no possible nontrivial differentials supported by $\{v_2^{20}h_2^2[1]\}$ in the MASS for $M(2, v_1^4)$. Therefore $\{v_2^{20}h_2^2[1]\}$ is a permanent cycle in the MASS for $M(2, v_1^4)$. \square

Lemma 7.26. *The class $\Delta^4q \in \pi_{128}(\text{tmf})$ lifts to the top cell of $M(8, v_1^8)$.*

Proof. The class $\Delta^4q \in \pi_{128}(\text{tmf})$ lifts to an element

$$\Delta^4q[1] \in \text{tmf}_{129}(M(8))$$

which is detected by

$$v_2^{20}c_0[1] \in \text{Ext}_{A(2)_*}^{23,129+23}(H(8))$$

in the MASS for $\text{tmf} \wedge M(8)$. Lift $\Delta^4q[1]$ to an element

$$\Delta^4q[18] \in \text{tmf}_{146}(M(8, v_1^8)).$$

In the MASS for $\text{tmf} \wedge M(8)$, there is a differential

$$d_4(v_2^{16}g^2h_{2,1}v_1^2[1]) = v_2^{20}v_1^8c_0[1].$$

It follows from the Geometric Boundary Theorem that $\Delta^4q[18]$ is detected by $v_2^{16}g^2h_{2,1}v_1^2[1]$ in the MASS for $\text{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^4q[18]$ has MAF 29 and stem 146.

We now check that $v_2^{16}g^2h_{2,1}v_1^2[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(8, v_1^8)$. Its relative position in $\text{Ext}_{A(2)_*}(\underline{\text{bo}}_1 \otimes H(8, v_1^8))$ is $t - s = 139$ and

$AF = 28$, its relative position in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$ is $t - s = 132$ and $AF = 27$, and its relative position in $\mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 3} \otimes H(8, v_1^8))$ is $t - s = 125$ and $AF = 26$.

The proof of Lemma 7.20(1) implies that the element

$$g^2 h_{2,1} v_1^2 [1] \in \mathrm{Ext}_{A(2)_*}(H(8, v_1^8))$$

is a permanent cycle in the algebraic tmf resolution for $H(8, v_1^8)$. It follows from Lemma 5.1 that

$$v_2^{16} g^2 h_{2,1} v_1^2 [1]$$

is a permanent cycle in the algebraic tmf resolution for $H(8, v_1^8)$, and detects an element

$$v_2^{16} \cdot \{g^2 h_{2,1} v_1^2 [1]\} \in \mathrm{Ext}_{A_*}(H(8, v_1^8))$$

which persists to the E_3 -page of the MASS for $M(8, v_1^8)$.

The only possibility for this element to support a non-trivial MASS differential is for it to support a d_3 -differential whose target to be detected by the element

$$v_1 h_{2,1}^{19} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18] \in \mathrm{Ext}_{A(2)_*}(\underline{\mathrm{bo}}_1^{\otimes 2} \otimes H(8, v_1^8))$$

in the algebraic tmf resolution for $H(8, v_1^8)$.

We wish to use Lemma 5.5 to argue that the element $v_1 h_{2,1}^{19} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]$ detects an element in $\mathrm{Ext}_{A_*}(H(8, v_1^8))$ which is zero in the E_3 -page of the MASS. In the MASS for $\mathrm{bo}_1^2 \wedge M(8, v_1^8)$, there is a differential

$$d_2(v_2^8 h_{2,1}^{10} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]) = v_1 h_{2,1}^{19} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18].$$

Using the map

$$\Sigma^{16} \mathrm{tmf} \wedge \mathrm{bo}_1^2 \wedge M(8, v_1^8) \hookrightarrow \mathrm{tmf} \wedge \overline{\mathrm{tmf}}^2 \wedge M(8, v_1^8)$$

we get the same differential in the MASS for $\mathrm{tmf} \wedge \overline{\mathrm{tmf}}^2 \wedge M(8, v_1^8)$. By Proposition 4.3, the element $v_2^8 h_{2,1}^{10} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]$ is a permanent cycle in the algebraic tmf resolution for $H(8, v_1^8)$, detecting the element

$$\Delta^2 v_1^6 M(g^2)[1] \in \mathrm{Ext}_{A_*}(H(8, v_1^8)).$$

Therefore the hypotheses of Lemma 5.5 are satisfied, and we deduce that

$$v_1 h_{2,1}^{19} (v_0^{-1} v_2^2 [\zeta_1^8, \zeta_2^4])[18]$$

detects an element which is zero in the E_3 -page of the MASS, and hence cannot be the target of a non-trivial d_3 -differential in the MASS. \square

Lemma 7.27. *The class $\Delta^4 u \in \pi_{135}(\mathrm{tmf})$ lifts to the top cell of $M(8, v_1^8)$.*

Proof. The class $\Delta^4 u \in \pi_{135}(\mathrm{tmf})$ lifts to an element

$$\Delta^4 u [1] \in \mathrm{tmf}_{136}(M(8))$$

which is detected by

$$v_2^{16} v_1^2 x_{35} [1] \in \mathrm{Ext}_{A(2)_*}^{25, 136+25}(H(8))$$

in the MASS for $\mathrm{tmf} \wedge M(8)$. Lift $\Delta^4 u [1]$ to an element

$$\Delta^4 u [18] \in \mathrm{tmf}_{153}(M(8, v_1^8)).$$

There is a differential in the MASS for $\mathrm{tmf} \wedge M(8)$

$$d_4(v_2^{16}v_1^3h_{2,1}^2g^2[1]) = v_2^{16}v_1^{10}x_{35}[1],$$

so by the Geometric Boundary Theorem, $\Delta^4u[18]$ is detected by $v_2^{16}v_1^3h_{2,1}^2g^2[1]$ in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. In particular, $\Delta^4u[18]$ has MAF 31 and stem 153.

We now check that $v_2^{16}v_1^3h_{2,1}^2g^2[1]$ is a permanent cycle in the algebraic tmf -resolution for $H(8, v_1^8)$. Note that $v_1^3h_{2,1}^2g^2[1]$ detects $u[18]$ in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. In Lemma 7.20, we established that $u[18]$ lifts to $M(8, v_1^8)$, and therefore $v_1^3h_{2,1}^2g^2[1]$ is a permanent cycle in the algebraic tmf resolution, and it detects a permanent cycle in the MASS for $M(8, v_1^8)$. It follows from Lemma 5.1 that

$$v_2^{16}v_1^3h_{2,1}^2g^2[1]$$

is a permanent cycle in the algebraic tmf resolution, and detects an element

$$v_2^{16} \cdot \{v_1^3h_{2,1}^2g^2[1]\} \in \mathrm{Ext}_{A_*}(H(8, v_1^8)).$$

Inspection of the relevant charts shows that the only possible non-trivial MASS differentials supported by this element would be

$$d_2(v_2^{16} \cdot \{v_1^2h_{2,1}^2g^2[1]\}) = \{v_2^8h_{2,1}^{15}\zeta_2^4[18]\}.$$

However, we have

$$d_2(v_2^{16} \cdot \{v_1^3h_{2,1}^2g^2[1]\}) = 0,$$

since it is a product of d_2 -cycles. \square

Lemma 7.28. *The following classes lift to the top cell of $M(8, v_1^8)$:*

- (1) $\Delta^6\nu^2 \in \pi_{150}(\mathrm{tmf})$,
- (2) $\Delta^6\kappa\nu \in \pi_{161}(\mathrm{tmf})$.

Proof.

- (1) The class $\Delta^6\nu^2 \in \pi_{150}(\mathrm{tmf})$ lifts to an element

$$\Delta^6\nu^2[1] \in \mathrm{tmf}_{151}(M(8))$$

which is detected by

$$v_2^{24}h_2^2[1] \in \mathrm{Ext}_{A(2)_*}^{28, 151+28}(H(8))$$

in the MASS for $\mathrm{tmf} \wedge M(8)$. Lift $\Delta^6\nu^2[1]$ to an element

$$\Delta^6\nu^2[18] \in \mathrm{tmf}_{168}(M(8, v_1^8)).$$

In the MASS for $\mathrm{tmf} \wedge M(8)$, there is a differential

$$d_2(v_2^{26}v_1^4h_2h_0[1]) = v_2^{24}v_1^8h_2^2[1].$$

It follows from the Geometric Boundary Theorem that $\Delta^6\nu^2[18]$ is detected by $v_2^{26}v_1^4h_2h_0[1]$ in the MASS for $\mathrm{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^6\nu^2[18]$ has MAF 34 and stem 168.

In Lemma 7.23(1) we showed that $v_2^{10}v_1^4h_2h_0[1]$ is a permanent cycle in the algebraic tmf resolution, detecting an element

$$\{v_2^{10}v_1^4h_2h_0[1]\} \in \mathrm{Ext}_{A_*}(H(8, v_1^8))$$

in the algebraic \mathbf{tmf} resolution for $H(8, v_1^8)$. By Lemma 5.1, this is also true of $v_2^{26}v_1^4h_2h_0[1]$.

Lemma 5.1 implies that $d_2(v_2^{16}) = 0$ in the MASS for $M(8, v_1^8)$. By Lemma 7.23(1), it follows that

$$d_2(v_2^{16} \cdot \{v_2^{10}v_1^4h_2h_0[1]\}) = 0.$$

Inspection of the algebraic \mathbf{tmf} resolution charts reveals that there are no possible targets of a longer MASS differential supported by $v_2^{16} \cdot \{v_2^{10}v_1^4h_2h_0[1]\}$.

- (2) The class $\Delta^6\kappa\nu \in \pi_{161}(\mathbf{tmf})$ lifts to an element

$$\Delta^6\kappa\nu[1] \in \mathbf{tmf}_{162}(M(8))$$

which is detected by

$$v_2^{24}d_0h_2[1] \in \mathrm{Ext}_{A(2)_*}^{31, 161+31}(H(8))$$

in the MASS for $\mathbf{tmf} \wedge M(8)$. Lift $\Delta^6\kappa\nu[1]$ to an element

$$\Delta^6\kappa\nu[18] \in \mathbf{tmf}_{179}(M(8, v_1^8)).$$

In the MASS for $\mathbf{tmf} \wedge M(8)$, there is a differential

$$d_2(v_2^{26}v_1^4h_0d_0[1]) = v_2^{24}v_1^8h_2d_0[1].$$

It follows from the Geometric Boundary Theorem that $\Delta^6\kappa\nu[18]$ is detected by $v_2^{26}v_1^4h_0d_0[1]$ in the MASS for $\mathbf{tmf} \wedge M(8, v_1^8)$. In particular, we see that $\Delta^6\kappa\nu[18]$ has MAF 37 and stem 179.

We showed in Lemma 7.23 that $v_2^{10}v_1^4h_0d_0[1]$ is a permanent cycle in the algebraic \mathbf{tmf} resolution. By Lemma 5.1, it follows that $v_2^{26}v_1^4h_0d_0[1]$ is a permanent cycle in the algebraic \mathbf{tmf} -resolution for $H(8, v_1^8)$ and lifts to an element $\{v_2^{26}v_1^4h_0d_0[1]\}$ in $\mathrm{Ext}_{A_*}(H(8, v_1^8))$.

Finally, inspection of the algebraic \mathbf{tmf} resolution charts reveals that there are no possible nontrivial differentials on $\{v_2^{26}v_1^4h_0d_0[1]\}$ in the MASS for $M(8, v_1^8)$. Therefore $\{v_2^{26}v_1^4h_0d_0[1]\}$ is a permanent cycle.

□

REFERENCES

- [Ada66] J. F. Adams, *On the groups $J(X)$. IV*, *Topology* **5** (1966), 21–71. MR 198470
- [Bau08] Tilman Bauer, *Computation of the homotopy of the spectrum \mathbf{tmf}* , *Groups, homotopy and configuration spaces*, *Geom. Topol. Monogr.*, vol. 13, *Geom. Topol. Publ.*, Coventry, 2008, pp. 11–40. MR 2508200
- [BBB⁺20] A. Beaudry, M. Behrens, P. Bhattacharya, D. Culver, and Z. Xu, *On the E_2 -term of the bo-adams spectral sequence*, *J. Topol.* **13** (2020), no. 1, 356–415.
- [BBT18] Prasit Bhattacharya, Irina Bobkova, and Brian Thomas, *The P_2^1 Margolis homology of connective topological modular forms*, arXiv:1810.05622, 2018.
- [Beh12] Mark Behrens, *The Goodwillie tower and the EHP sequence*, *Mem. Amer. Math. Soc.* **218** (2012), no. 1026, xii+90. MR 2976788
- [BHHM08] M. Behrens, M. Hill, M. J. Hopkins, and M. Mahowald, *On the existence of a v_2^{32} -self map on $M(1, 4)$ at the prime 2*, *Homology Homotopy Appl.* **10** (2008), no. 3, 45–84. MR 2475617
- [BHHM20] ———, *Detecting exotic spheres in low dimensions using coker J* , *J. Lond. Math. Soc. (2)* **101** (2020), no. 3, 1173–1218.

- [BJM84] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*, J. London Math. Soc. (2) **30** (1984), no. 3, 533–550. MR 810962
- [BOSS19] M. Behrens, K. Ormsby, N. Stapleton, and V. Stojanoska, *On the ring of cooperations for 2-primary connective topological modular forms*, J. Topol. **12** (2019), no. 2, 577–657.
- [Bot59] Raoul Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **70** (1959), 313–337. MR 110104
- [BR] Robert R. Bruner and John Rognes, *The Adams spectral sequence for topological modular forms*, preprint.
- [Bro69] William Browder, *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math. (2) **90** (1969), 157–186. MR 251736
- [Bru] Robert R. Bruner, *The cohomology of the mod 2 Steenrod algebra: a computer computation*, <http://www.rrb.wayne.edu/papers/cohom.pdf>.
- [DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014. MR 3223024
- [DM10] Donald M. Davis and Mark Mahowald, *Connective versions of $TMF(3)$* , Int. J. Mod. Math. **5** (2010), no. 3, 223–252. MR 2779050
- [GJM86] Paul G. Goerss, John D. S. Jones, and Mark E. Mahowald, *Some generalized Brown-Gitler spectra*, Trans. Amer. Math. Soc. **294** (1986), no. 1, 113–132. MR 819938
- [HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262. MR 3505179
- [Hov04] Mark Hovey, *Homotopy theory of comodules over a Hopf algebroid*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K -theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 261–304. MR 2066503
- [HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR 1652975
- [Hu99] Po Hu, *Transfinite spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 197–216. MR 1718081
- [Isa19] Daniel C. Isaksen, *Stable stems*, Mem. Amer. Math. Soc. **262** (2019), no. 1269, viii+159. MR 4046815
- [Isa20] ———, *The Mahowald operator in the cohomology of the Steenrod algebra*, arXiv:2001.01758, 2020.
- [IWX20a] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, *Classical and C -motivic Adams charts*, v5.1, <https://s.wayne.edu/isaksen/adams-charts/>, 2020.
- [IWX20b] ———, *More stable stems*, arXiv:2001.04511, 2020.
- [IWX20c] ———, *Stable homotopy groups of spheres*, arXiv:2001.04247, 2020.
- [KM63] Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537. MR 148075
- [Lau04] Gerd Laures, *$K(1)$ -local topological modular forms*, Invent. Math. **157** (2004), no. 2, 371–403. MR 2076927
- [LM87] Wolfgang Lellmann and Mark Mahowald, *The bo -Adams spectral sequence*, Trans. Amer. Math. Soc. **300** (1987), no. 2, 593–623.
- [LSWX19] Guchuan Li, XiaoLin Danny Shi, Guozhen Wang, and Zhouli Xu, *Hurewicz images of real bordism theory and real Johnson-Wilson theories*, Adv. Math. **342** (2019), 67–115. MR 3877362
- [Mah81] Mark Mahowald, *bo -resolutions*, Pacific J. Math. **92** (1981), no. 2, 365–383. MR 618072
- [Mat16] Akhil Mathew, *The homology of tmf* , Homology Homotopy Appl. **18** (2016), no. 2, 1–29. MR 3515195
- [MPT70] Harvey Margolis, Stewart Priddy, and Martin Tangora, *Another systematic phenomenon in the cohomology of the Steenrod algebra*, Topology **10** (1970), 43–46. MR 300272
- [MR09] Mark Mahowald and Charles Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. **5** (2009), no. 2, Special Issue: In honor of Friedrich Hirzebruch. Part 1, 853–872. MR 2508904

- [MT68] Mark Mahowald and Martin Tangora, *An infinite subalgebra of $\mathrm{Ext}_A(Z_2, Z_2)$* , Trans. Amer. Math. Soc. **132** (1968), 263–274. MR 222887
- [Per02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159, 2002.
- [Per03a] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245, 2003.
- [Per03b] ———, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109, 2003.
- [Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR 860042
- [Shi] Katsumi Shimomura, *The existence of β_{9t+3} in stable homotopy of spheres at the prime three*, preprint, <http://www.math.kochi-u.ac.jp/katsumi/>.
- [WX17] Guozhen Wang and Zhouli Xu, *The triviality of the 61-stem in the stable homotopy groups of spheres*, Ann. of Math. (2) **186** (2017), no. 2, 501–580. MR 3702672

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