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The homotopy groups of the algebraic K–theory of the sphere spectrum

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We calculate $\pi_* K(\mathbb{S}) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$, the homotopy groups of $K(\mathbb{S})$ away from 2, in terms of the homotopy groups of $K(\mathbb{Z})$, the homotopy groups of $\mathbb{C}P^{\infty}_{-1}$ and the homotopy groups of $\mathbb{S}$. This builds on work of Waldhausen, who computed the rational homotopy groups (building on work of Quillen and Borel) and Rognes, who calculated the groups at odd regular primes in terms of the homotopy groups of $\mathbb{C}P^{\infty}_{-1}$ and the homotopy groups of $\mathbb{S}$.

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1 Introduction

The algebraic K–theory of the sphere spectrum $K(\mathbb{S})$ is Waldhausen's $A(*)$, the algebraic K–theory of the one-point space. The underlying infinite loop space of $K(\mathbb{S})$ splits as a copy of the underlying infinite loop space of $\mathbb{S}$ and the smooth Whitehead space of a point $Wh^{\mathrm{Diff}}(*)$. For a highly connected compact manifold M, the second loop space of $Wh^{\mathrm{Diff}}(*)$ approximates the stable concordance space of M, and the loop space of $Wh^{\mathrm{Diff}}(*)$ parametrizes stable h–cobordisms; see Waldhausen, Jahren and Rognes [34]. As a consequence, computation of the algebraic K–theory of the sphere spectrum is a fundamental problem in algebraic and differential topology.

Early efforts in this direction were carried out by Waldhausen in the 1980s using the "linearization" map $K(\mathbb{S}) \to K(\mathbb{Z})$ from the K–theory of the sphere spectrum to the K–theory of the integers. This is induced by the map of (highly structured) ring spectra $\mathbb{S} \to \mathbb{Z}$. Waldhausen showed that because the map $\mathbb{S} \to \mathbb{Z}$ is an isomorphism on homotopy groups in degree zero (and below) and a rational equivalence on higher homotopy groups, the map $K(\mathbb{S}) \to K(\mathbb{Z})$ is a rational equivalence. Borel's computation of the rational homotopy groups of $K(\mathbb{Z})$ then applies to calculate the rational homotopy groups of $K(\mathbb{S})$: $\pi_q K(\mathbb{Z}) \otimes \mathbb{Q}$ is rank 1 when $q = 0$ or $q = 4k+1$ for $k > 1$ and is zero for all other q.

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In the 1990s, work of Bökstedt, Carlsson, Cohen, Goodwillie, Hsiang and Madsen revolutionized the computation of algebraic K–theory with the introduction and study of *topological cyclic homology* TC, an analogue of negative cyclic homology that can be computed using the methods of equivariant stable homotopy theory. TC is the target of the *cyclotomic trace*, a natural transformation $K \to \mathrm{TC}$. For a map of highly structured ring spectra such as $\mathbb{S} \to \mathbb{Z}$, naturality gives a diagram

$$\begin{array}{ccc} K(\mathbb{S}) & \longrightarrow & \mathrm{TC}(\mathbb{S}) \\ \big\downarrow & & \big\downarrow \\ K(\mathbb{Z}) & \longrightarrow & \mathrm{TC}(\mathbb{Z}) \end{array}$$

the *linearization/cyclotomic trace square*. A foundational theorem of Dundas [8] (building on work of McCarthy [21] and Goodwillie [11]) states that the square above becomes homotopy cartesian after p–completion, which means that the maps of homotopy fibers become weak equivalences after p–completion.

In the 2000s, Rognes [29] used the linearization/cyclotomic trace square to compute the homotopy groups of $K(\mathbb{S})$ at odd regular primes in terms of the homotopy groups of $\mathbb{S}$ and the homotopy groups of $\mathbb{C}P^{\infty}_{-1}$ (assuming the now-affirmed Quillen–Lichtenbaum conjecture). The answer is easiest to express in terms of the torsion subgroups: Because $\pi_n K(\mathbb{S})$ is finitely generated, it is the direct sum of a free part and a torsion part, the free part being $\mathbb{Z}$ when $n = 0$ or $n \equiv 1 \bmod 4$, $n > 1$, and 0 otherwise. The main theorem of Rognes [29] is that for p an odd regular prime the p–torsion of $\pi_* K(\mathbb{S})$ is

$$\mathrm{tor}_p(\pi_* K(\mathbb{S})) \cong \mathrm{tor}_p(\pi_* \mathbb{S} \oplus \pi_{*-1} c \oplus \pi_{*-1} \overline{\mathbb{C}P}{}^{\infty}_{-1})$$

(which can be made canonical, as discussed below). Here c denotes the additive p–complete cokernel of J spectrum (the connected cover of the homotopy fiber of the map $\mathbb{S}^{\wedge}_p \to L_{K(1)}\mathbb{S}$); its homotopy groups are all torsion and are direct summands of $\pi_*\mathbb{S}$. The spectrum $\overline{\mathbb{C}P}{}^{\infty}_{-1}$ is a wedge summand of $(\mathbb{C}P^{\infty}_{-1})^{\wedge}_p$,

$$(\mathbb{C}P^{\infty}_{-1})^{\wedge}_p \simeq \overline{\mathbb{C}P}{}^{\infty}_{-1} \vee \mathbb{S}^{\wedge}_p;$$

see Madsen and Schlichtkrull [19, (1.3)] and Rognes [29, page 166]. Using unpublished work of Knapp, Rognes [29, 4.7] calculates the order of these torsion groups in degrees $\leq 2(2p+1)(p-1)-4$.

The description of $\mathrm{tor}_p(\pi_* K(\mathbb{S}))$ above uses an identification of the homotopy type of $\mathrm{TC}(\mathbb{S})^{\wedge}_p$ as $\mathbb{S}^{\wedge}_p \vee \Sigma\mathbb{C}P^{\infty}_{-1}$ due to Bökstedt, Hsiang and Madsen [4, 5.15, 5.17]. Unfortunately, there does not seem to be a canonical map realizing this weak equivalence, and so the isomorphism above is not canonical. We do have a canonical splitting $\mathrm{TC}(\mathbb{S}) \simeq \mathbb{S} \vee Cu$, where Cu is the homotopy cofiber of the unit map $\mathbb{S} \to \mathrm{TC}(\mathbb{S})$, or equivalently, the homotopy fiber of the map $\mathrm{TC}(\mathbb{S}) \to \mathrm{THH}(\mathbb{S})$. As we discuss below, the splitting of the p–completion $Cu^{\wedge}_p$,

$$Cu^{\wedge}_p \simeq \overline{Cu} \vee \Sigma\mathbb{S}^{\wedge}_p$$

corresponding to the splitting $(\mathbb{C}P^{\infty}_{-1})^{\wedge}_p \simeq \overline{\mathbb{C}P}^{\infty}_{-1} \vee \mathbb{S}^{\wedge}_p$, turns out to be canonical. Rognes' canonical identification of the p–torsion of $\pi_* K(\mathbb{S})$ is then

$$\mathrm{tor}_p(\pi_* K(\mathbb{S})) \cong \mathrm{tor}_p(\pi_*\mathbb{S} \oplus \pi_{*-1} c \oplus \pi_* \overline{Cu}).$$

This paper computes $\pi_* K(\mathbb{S})^{\wedge}_p$ in the case of irregular primes, thereby completing the computation of the homotopy groups of the algebraic K–theory of the sphere spectrum away from the prime 2. As a first step, we prove the following splitting theorem in Section 4:

Theorem 1.1 *Let p be an odd prime. The long exact sequence on homotopy groups induced by the p–completed linearization/cyclotomic trace square breaks up into noncanonically split short exact sequences*

$$0 \to \pi_* K(\mathbb{S})^{\wedge}_p \to \pi_*\mathrm{TC}(\mathbb{S})^{\wedge}_p \oplus \pi_* K(\mathbb{Z})^{\wedge}_p \to \pi_*\mathrm{TC}(\mathbb{Z})^{\wedge}_p \to 0.$$

Choosing appropriate splittings in the previous theorem, we can identify the p–torsion groups. The identification is again in terms of $\pi_*\mathbb{S}$ and $\pi_* Cu$ (or, noncanonically, $\pi_*\Sigma\mathbb{C}P^{\infty}_{-1}$) but now involves also $\pi_* K(\mathbb{Z})$, which is not fully understood at irregular primes. In the statement, $K^{\mathrm{red}}(\mathbb{Z})$ denotes the wedge summand of $K(\mathbb{Z})^{\wedge}_p$ complementary to j,

$$K(\mathbb{Z})^{\wedge}_p \simeq j \vee K^{\mathrm{red}}(\mathbb{Z}),$$

where j is the p–complete additive image of J spectrum, the connective cover of $L_{K(1)}\mathbb{S}$; see Dwyer and Mitchell [10, 2.1, 9.7]. (As discussed below, this splitting is canonical.)

Theorem 1.2 *Let p be an odd prime. The p–torsion in $\pi_* K(\mathbb{S})$ admits canonical isomorphisms*

(a) $$\mathrm{tor}_p(\pi_* K(\mathbb{S})) \cong \mathrm{tor}_p(\pi_* c \oplus \pi_{*-1} c \oplus \pi_* \overline{Cu} \oplus \pi_* K(\mathbb{Z}))$$

(b) $$\cong \mathrm{tor}_p(\pi_*\mathbb{S} \oplus \pi_{*-1} c \oplus \pi_* \overline{Cu} \oplus \pi_* K^{\mathrm{red}}(\mathbb{Z})).$$

In formula (a), the map $\mathrm{tor}_p(\pi_* K(\mathbb{S})) \to \mathrm{tor}_p(\pi_*(K(\mathbb{Z})))$ is induced by the linearization map and the map

$$\mathrm{tor}_p(\pi_* K(\mathbb{S})) \to \mathrm{tor}_p(\pi_* c \oplus \pi_{*-1} c \oplus \pi_* \overline{Cu})$$

is induced by the composite of the cyclotomic trace map $K(\mathbb{S}) \to \mathrm{TC}(\mathbb{S})$ and a canonical splitting (2.3) of the homotopy groups $\pi_* \mathrm{TC}(\mathbb{S})^{\wedge}_p$ as

$$\pi_* \mathrm{TC}(\mathbb{S})^{\wedge}_p \cong \pi_*(j) \oplus \pi_*(c) \oplus \pi_*(\Sigma j) \oplus \pi_*(\Sigma c) \oplus \pi_*(\overline{Cu}),$$

explained in the first part of Section 2, followed by the projection onto the non-j summands.

In formula (b), the map $\mathrm{tor}_p(\pi_* K(\mathbb{S})) \to \mathrm{tor}_p(\pi_* K^{\mathrm{red}}(\mathbb{Z}))$ is induced by the linearization map and the canonical map $\pi_* K(\mathbb{Z}) \to \pi_* K^{\mathrm{red}}(\mathbb{Z})$ which is the quotient by the image of the unit map $\pi_* \mathbb{S} \to \pi_* K(\mathbb{Z})$. The map

$$\mathrm{tor}_p(\pi_* K(\mathbb{S})) \to \mathrm{tor}_p(\pi_* \mathbb{S} \oplus \pi_{*-1} c \oplus \pi_* \overline{Cu})$$

is induced by the composite of the cyclotomic trace map $K(\mathbb{S}) \to \mathrm{TC}(\mathbb{S})$ and the canonical splitting of homotopy groups

$$\pi_* \mathrm{TC}(\mathbb{S})^{\wedge}_p \cong \pi_*(\mathbb{S}^{\wedge}_p) \oplus \pi_*(\Sigma j) \oplus \pi_*(\Sigma c) \oplus \pi_*(\overline{Cu})$$

followed by projection away from the Σj summand. (The splitting of $\pi_* \mathrm{TC}(\mathbb{S})^{\wedge}_p$ here is related to the splitting above by the canonical splitting on homotopy groups $\pi_* \mathbb{S}^{\wedge}_p \cong \pi_*(j) \oplus \pi_*(c)$.)

Formula (b) generalizes the computation of Rognes [29] at odd regular primes because $K^{\mathrm{red}}(\mathbb{Z})$ is torsion-free if (and only if) p is regular (see for example Weibel [36, Section VI.10]). Part of the argument for the theorems above involves making certain splittings in prior K–theory and TC computations canonical and canonically identifying certain maps (or at least their effect on homotopy groups). Although we construct the splittings and prove their essential uniqueness calculationally, we offer in Section 5 a theoretical explanation in terms of a conjectural extension of Adams operations on algebraic K–theory to an action of the p–adic units and a conjecture on the consistency of Adams operations on K–theory and Adams operations on TC. This perspective leads to a splitting of $K(\mathbb{S})^{\wedge}_p$ and the linearization/cyclotomic trace square into $p-1$ summands (which is independent of the conjectures), expanding on certain splittings discovered by Rognes [29, Section 3] (in the case of regular primes); see Theorem 5.1.

The identification of the maps in Section 3 allows us to prove the following theorem, which slightly sharpens Theorem 1.2:

Theorem 1.3 *Let p be an odd prime. Let $\alpha\colon \pi_*\mathrm{TC}(\mathbb{S}) \to \pi_* j$ be the induced map on homotopy groups given by the composite of the canonical maps $\mathrm{TC}(\mathbb{S}) \to \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}$ and $\mathbb{S} \to j$; let $\beta\colon \pi_* K(\mathbb{Z}) \to \pi_* j$ be the canonical splitting; and let $\gamma\colon \pi_*\mathrm{TC}(\mathbb{S}) \to \pi_*\Sigma\mathbb{S}^1 \to \pi_*\Sigma j$ be the map induced by the splitting of (2.3) below. Then the p–torsion subgroup of $\pi_* K(\mathbb{S})$ maps isomorphically to the subgroup of the p–torsion subgroup of $\pi_*\mathrm{TC}(\mathbb{S}) \oplus \pi_* K(\mathbb{Z})$, where the appropriate projections composed with α and β agree and the appropriate projection composed with γ is zero.*

Theorems 1.2 and 1.3 provide a good understanding of what the linearization/cyclotomic trace square does on the odd torsion part of the homotopy groups. In contrast, the maps on mod torsion homotopy groups are not fully understood. We have that $\pi_n K(\mathbb{S})$ and $\pi_n K(\mathbb{Z})$ mod torsion are rank one for $n = 0$ and $n \equiv 1 \bmod 4$, $n > 1$; the map $K(\mathbb{S}) \to K(\mathbb{Z})$ is a rational equivalence and an isomorphism in degree zero, but is not an isomorphism on mod torsion homotopy groups in degrees congruent to 1 mod $2(p-1)$ by the work of Klein and Rognes (see the proof of [14, 6.3(i)]). The mod torsion homotopy groups of $\mathrm{TC}(\mathbb{S})^\wedge_p$ are rank one in degree zero and odd degrees ≥ -1; the map $K(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{S})^\wedge_p$ on mod torsion homotopy groups is an isomorphism in degree zero, by necessity zero in degrees not congruent to 1 mod 4, and for odd regular primes an isomorphism in degrees congruent to 1 mod 4. For irregular primes, the map is not fully understood.

In principle, we can use Theorem 1.2 to calculate $\pi_* K(\mathbb{S})$ in low degrees. In practice, we are limited by a lack of understanding of $\pi_*\mathbb{C}P^\infty_{-1}$ and $\pi_* K(\mathbb{Z})$; we know $\pi_*\mathbb{S}$ and $\pi_* c$ in a comparatively larger range. The calculations of $\pi_*\overline{\mathbb{C}P}{}^\infty_{-1}$ for $* < |\beta_2| - 1 = (2p+1)(2p-2)-3$ in [29, 4.7] work for irregular primes as well as odd regular primes; although they only give the order of the torsion rather than the torsion group, in many cases the order is either 1 or p, and the group structure is also determined. At primes that satisfy the Kummer–Vandiver conjecture, we know $\pi_* K(\mathbb{Z})$ in terms of Bernoulli numbers; at other primes we know $\pi_* K(\mathbb{Z})$ in odd degrees, but do not know $K_{4n}\mathbb{Z}$ at all (except $n = 0, 1$) and only know the order of $\pi_{4n+2} K(\mathbb{Z})$ and again only in terms of Bernoulli numbers. As the formula [29, 4.7] for $\pi_*\overline{\mathbb{C}P}{}^\infty_{-1}$ is somewhat messy, we do not summarize the answer here, but for convenience, we have included it in Section 6.

As a consequence of the work of Rognes [29, 3.6, 3.8], the cyclotomic trace

$$trc_p\colon K(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{S})^\wedge_p$$

is injective on homotopy groups at odd regular primes. In Theorem 1.2 above, the contribution in $\mathrm{tor}_p(\pi_* K(\mathbb{S}))$ of p–torsion from $\mathrm{tor}_p(\pi_* K^{\mathrm{red}}(\mathbb{Z}))$ maps to zero in $\pi_* \mathrm{TC}(\mathbb{S})$ under the cyclotomic trace. This then gives the following complete answer to the question the authors posed in [3]:

Corollary 1.4 *For an odd prime p, the cyclotomic trace trc_p: $K(\mathbb{S})^{\wedge}_p \to \mathrm{TC}(\mathbb{S})^{\wedge}_p$ is injective on homotopy groups if and only if p is regular.*

Theorem 1.1 contains the following more general injectivity result:

Corollary 1.5 *For an odd prime p, the map of ring spectra*

$$K(\mathbb{S})^{\wedge}_p \to \mathrm{TC}(\mathbb{S})^{\wedge}_p \times K(\mathbb{Z})^{\wedge}_p$$

is injective on homotopy groups.

Conventions

Throughout this paper p denotes an odd prime. For a ring R, we write $R^{\times}$ for its group of units.

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2 The spectra in the linearization/cyclotomic trace square

We begin by reviewing the descriptions of the spectra $\mathrm{TC}(\mathbb{S})^{\wedge}_p$, $K(\mathbb{Z})^{\wedge}_p$, and $\mathrm{TC}(\mathbb{Z})^{\wedge}_p$ in the p–completed linearization/cyclotomic trace square

$$\begin{array}{ccc} K(\mathbb{S})^{\wedge}_p & \longrightarrow & \mathrm{TC}(\mathbb{S})^{\wedge}_p \\ \downarrow & & \downarrow \\ K(\mathbb{Z})^{\wedge}_p & \longrightarrow & \mathrm{TC}(\mathbb{Z})^{\wedge}_p \end{array}$$

These three spectra have been identified in more familiar terms up to weak equivalence. We discuss how canonical these weak equivalences are. Often this will involve studying splittings of the form $X \simeq Y \vee Z$ (or with additional summands). We will say that the *splitting is canonical* when we have a canonical isomorphism in the stable category $X \simeq X' \vee X''$ with X' and X'' (possibly noncanonically) isomorphic in the stable category to Y and Z; we will say that *the identification of the summand Y is canonical* when further the isomorphism in the stable category $X' \simeq Y$ is canonical.

To illustrate the above terminology, and justify its utility, consider the example when X is noncanonically weakly equivalent to $Y \vee Z$ and $[Y, Z] = 0 = [Z, Y]$ (where $[-,-]$ denotes maps in the stable category). In the terminology above, this gives an example of a canonical splitting $X \simeq Y \vee Z$ without canonical identification of the summands. As another example, if we have a canonical map $Y \to X$ in the stable category and a canonical map $X \to Y$ in the stable category giving a retraction, then we have a canonical splitting with canonical identification of summands $X \simeq Y \vee F$, where F is the homotopy fiber of the retraction map $X \to Y$. In this second example, if we also have a noncanonical weak equivalence $Z \to F$, we then have a canonical splitting $X \simeq Y \vee Z$ with a canonical identification of the summand Y (but not the summand Z).

The splitting of $\mathrm{TC}(\mathbb{S})^\wedge_p$

Historically, $\mathrm{TC}(\mathbb{S})^\wedge_p$ was the first of the terms in the linearization/cyclotomic trace square to be understood. Work of Bökstedt, Hsiang and Madsen [4, 5.15, 5.17] identifies the homotopy type of $\mathrm{TC}(\mathbb{S})^\wedge_p$ as

$$\mathrm{TC}(\mathbb{S})^\wedge_p \simeq \mathbb{S}^\wedge_p \vee \Sigma(\mathbb{C}P^\infty_{-1})^\wedge_p.$$

The inclusion of the $\mathbb{S}$ summand is the unit of the ring spectrum structure and is split by the canonical map $\mathrm{TC}(\mathbb{S})^\wedge_p \to \mathrm{THH}(\mathbb{S})^\wedge_p$ and canonical identification $\mathbb{S}^\wedge_p \simeq \mathrm{THH}(\mathbb{S})^\wedge_p$ (also induced by the ring spectrum structure). We therefore get a canonical isomorphism in the stable category between $\mathrm{TC}(\mathbb{S})^\wedge_p$ and $\mathbb{S}^\wedge_p \vee Cu^\wedge_p$, where $Cu^\wedge_p$ is the homotopy cofiber of the map $\mathbb{S}^\wedge_p \to \mathrm{TC}(\mathbb{S})^\wedge_p$; in the terminology at the beginning of the section, this is a canonical splitting with canonical identification

(2.1) $$\mathrm{TC}(\mathbb{S})^\wedge_p \simeq \mathbb{S}^\wedge_p \vee Cu^\wedge_p.$$

We have a canonical splitting with noncanonical identification

$$\mathrm{TC}(\mathbb{S})^\wedge_p \simeq \mathbb{S}^\wedge_p \vee \Sigma(\mathbb{C}P^\infty_{-1})^\wedge_p.$$

As already indicated, we make use of a further splitting from [29, Section 3] (see also the remarks preceding [19, (1.3)]),

$$(\mathbb{C}P^{\infty}_{-1})^{\wedge}_{p} \simeq \mathbb{S}^{\wedge}_{p} \vee \overline{\mathbb{C}P}^{\infty}_{-1},$$

induced by the splitting $\Sigma^{\infty}_{+}\mathbb{C}P^{\infty} \simeq \Sigma^{\infty}\mathbb{C}P^{\infty} \vee \mathbb{S}$. The splitting exists after inverting 2, but for notational convenience, we use it only after p–completion. It follows that there exists an analogous splitting

$$Cu^{\wedge}_{p} \simeq \Sigma\mathbb{S}^{\wedge}_{p} \vee \overline{Cu}. \tag{2.2}$$

There is a canonical map $\Sigma\mathbb{S}^{\wedge}_{p} \to \mathrm{TC}(\mathbb{S})^{\wedge}_{p}$ in the stable category arising from [4, 5.15–5.17], which in particular gives a canonical isomorphism

$$\pi_1(\mathrm{TC}(\mathbb{S})^{\wedge}_{p}) \cong \lim \mathbb{Z}/p^n.$$

Ravenel's calculation [22, 1.11(iii)] reveals that $[\mathbb{C}P^{\infty}_{-1}, \mathbb{S}^{\wedge}_{p}] \cong \mathbb{Z}^{\wedge}_{p}$, so there is a unique map $Cu^{\wedge}_{p} \to \Sigma\mathbb{S}^{\wedge}_{p}$ in the stable category such that the composite self-map of $\Sigma\mathbb{S}^{\wedge}_{p}$ is the identity.

Finally, we have a canonical isomorphism of homotopy groups $\pi_*\mathbb{S}^{\wedge}_{p} \cong \pi_*(j) \oplus \pi_*(c)$ from classical work in homotopy theory on Whitehead's J–homomorphism and Bousfield's work on localization of spectra [7, Section 4]. As above, j denotes the connective cover of the $K(1)$–localization of the sphere spectrum and c denotes the homotopy fiber of the map $\mathbb{S}^{\wedge}_{p} \to j$. The map $\mathbb{S}^{\wedge}_{p} \to j$ induces an isomorphism from the p–Sylow subgroup of the image-of- J subgroup of $\pi_q\mathbb{S}^{\wedge}_{p}$ to $\pi_q j$.

Putting this all together, we have a canonical isomorphism

$$\pi_*\mathrm{TC}(\mathbb{S})^{\wedge}_{p} \cong \pi_*(j) \oplus \pi_*(c) \oplus \pi_*(\Sigma j) \oplus \pi_*(\Sigma c) \oplus \pi_*(\overline{Cu}). \tag{2.3}$$

The splitting of $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$

Next up historically is $\mathrm{TC}(\mathbb{Z})$, which was first identified by Bökstedt and Madsen [5; 6] and Rognes [26]. They expressed the answer on the infinite-loop space level and (equivalently) described the connective cover spectrum $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty)$ as having the homotopy type of

$$j \vee \Sigma j \vee \Sigma^3 ku^{\wedge}_{p} \simeq j \vee \Sigma j \vee \Sigma^3\ell \vee \Sigma^5\ell \vee \cdots \vee \Sigma^{3+2(p-2)}\ell$$

(noncanonical isomorphism in the stable category). Here ku denotes connective complex topological K–theory (the connective cover of periodic complex topological

K–theory KU) and ℓ denotes the Adams summand of $ku^\wedge_p$ (the connective cover of the Adams summand L of $KU^\wedge_p$). A standard calculation (eg see [18, 2.5.7]) shows that before taking the connective cover $\pi_{-1}(\mathrm{TC}(\mathbb{Z})^\wedge_p)$ is free of rank one over $\mathbb{Z}^\wedge_p$ and, as we explain in the subsection on the homotopy type of $\mathrm{TC}(\mathbb{Z})^\wedge_p$ below, the argument for [29, 3.3] extends to show that summand $\Sigma^{3+2(p-3)}\ell$ above becomes $\Sigma^{-1}\ell$ in $\mathrm{TC}(\mathbb{Z})^\wedge_p$.

The noncanonical splitting above rigidifies into a canonical splitting and canonical identification

(2.4) $\mathrm{TC}(\mathbb{Z})^\wedge_p \simeq$
$$j \vee \Sigma j' \vee \Sigma^{-1}\ell_{\mathrm{TC}}(0) \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p) \vee \Sigma^{-1}\ell_{\mathrm{TC}}(2) \vee \cdots \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p-2).$$

Here the numbering replaces 1 with p but otherwise numbers sequentially $0, \ldots, p-2$. Each $\Sigma^{-1}\ell_{\mathrm{TC}}(i)$ is a spectrum that is noncanonically weakly equivalent to $\Sigma^{2i-1}\ell \cong \Sigma^{2i}(\Sigma^{-1}\ell)$ but that admits a canonical description by work of Hesselholt and Madsen [12, Theorem D] and Dwyer and Mitchell [10, Section 13] in terms of units of cyclotomic extensions of $\mathbb{Q}^\wedge_p$; we omit a detailed description of the identification as we do not use it. The spectrum j' is the connective cover of the $K(1)$–localization of the $(\mathbb{Z}^\wedge_p)^\times$–Moore spectrum $M_{(\mathbb{Z}^\wedge_p)^\times}$ or, equivalently, the p–completion of $j \wedge M_{(\mathbb{Z}^\wedge_p)^\times}$. Since the p–completion of $(\mathbb{Z}^\wedge_p)^\times$ is noncanonically isomorphic to $\mathbb{Z}^\wedge_p$, we have that j' is noncanonically weakly equivalent to j but with $\pi_0 j'$ canonically isomorphic to $((\mathbb{Z}^\wedge_p)^\times)^\wedge_p$ (and isomorphisms in the stable category from j to j' are in canonical bijection with isomorphisms $\mathbb{Z}^\wedge_p \to ((\mathbb{Z}^\wedge_p)^\times)^\wedge_p$). Much of the canonical splitting in (2.4) follows by a calculation of maps in the stable category. Temporarily writing $x(0) = j \vee \Sigma^{-1}\ell$, $x(1) = \Sigma j \vee \Sigma^{2p-1}\ell$ and $x(i) = \Sigma^{2i-1}\ell$ for $i = 2, \ldots, p-2$, the results above give us a canonical splitting without canonical identification (see the start of this section for an explanation of this terminology)

$$\mathrm{TC}(\mathbb{Z})^\wedge_p \simeq x(0) \vee \cdots \vee x(p-2)$$

because, for $i \neq i'$, we have $[x(i), x(i')] = 0$; we provide a detailed computation as Proposition 2.14 at the end of the section. The canonical map $\mathbb{S} \to j$ induces an isomorphism $[j, \mathrm{TC}(\mathbb{Z})^\wedge_p] \cong \pi_0(\mathrm{TC}(\mathbb{Z})^\wedge_p)$ and so we have a canonical map $\eta\colon j \to \mathrm{TC}(\mathbb{Z})^\wedge_p$ coming from the unit of the ring spectrum structure. Likewise, the canonical map $M_{(\mathbb{Z}^\wedge_p)^\times} \to j'$ induces an isomorphism

$$[\Sigma j', \mathrm{TC}(\mathbb{Z})^\wedge_p] \cong [\Sigma M_{(\mathbb{Z}^\wedge_p)^\times}, \mathrm{TC}(\mathbb{Z})^\wedge_p] \cong \mathrm{Hom}\big(((\mathbb{Z}^\wedge_p)^\times)^\wedge_p, \pi_1 \mathrm{TC}(\mathbb{Z})^\wedge_p\big).$$

The canonical isomorphism $\pi_1 \mathrm{TC}(\mathbb{Z})^\wedge_p \cong ((\mathbb{Z}^\wedge_p)^\times)^\wedge_p$ [12, Theorem D] then gives a canonical map $u\colon \Sigma j' \to \mathrm{TC}(\mathbb{Z})^\wedge_p$. The restriction along η and u induce bijections

$$[\mathrm{TC}(\mathbb{Z})^\wedge_p, j] \cong [j, j] \quad \text{and} \quad [\mathrm{TC}(\mathbb{Z})^\wedge_p, \Sigma j'] \cong [\Sigma j', \Sigma j']$$

(see Proposition 2.15 below), giving retractions to η and u that are unique in the stable category. This gives a canonical splitting and identification of the j and $\Sigma j'$ summands in (2.4).

The splitting of $K(\mathbb{Z})^\wedge_p$

The homotopy type of $K(\mathbb{Z})^\wedge_p$ is still not fully understood at irregular primes even in light of the confirmation of the Quillen–Lichtenbaum conjecture. The Quillen–Lichtenbaum conjecture implies that $K(\mathbb{Z})^\wedge_p$ can be understood in terms of its $K(1)$–localization $L_{K(1)}K(\mathbb{Z})$, and work of Dwyer and Friedlander [9] or Dwyer and Mitchell [10, 12.2] identifies the homotopy type of $K(\mathbb{Z})^\wedge_p$ at regular odd primes as $j \vee \Sigma^5 ko^\wedge_p$ (noncanonically). At any prime, Quillen's Brauer induction and reduction mod r (for r prime and a generator of $(\mathbb{Z}/p^2)^\times$) induce a splitting

$$K(\mathbb{Z})^\wedge_p \simeq j \vee K^{\mathrm{red}}(\mathbb{Z}) \tag{2.5}$$

for some p–complete spectrum we denote as $K^{\mathrm{red}}(\mathbb{Z})$ (see for example [10, 2.1, 5.4 and 9.7]). We argue in Proposition 2.17 at the end of this section (once we have reviewed more about $K^{\mathrm{red}}(\mathbb{Z})$) that $[j, K^{\mathrm{red}}(\mathbb{Z})] = 0$ and $[K^{\mathrm{red}}(\mathbb{Z}), j] = 0$, and it follows that the splitting in (2.5) is canonical; the identification of the summand j is also canonical, as the map $j \to K(\mathbb{Z})$ described is then the unique one taking the canonical generator of $\pi_0 j$ to the unit element of $\pi_0 K(\mathbb{Z})$ in the ring spectrum structure.

The splitting $K(\mathbb{Z})^\wedge_p \simeq j \vee K^{\mathrm{red}}(\mathbb{Z})$ corresponds to a splitting

$$L_{K(1)}K(\mathbb{Z}) \simeq J \vee L_{K(1)}K^{\mathrm{red}}(\mathbb{Z}),$$

where $J = L_{K(1)}\mathbb{S}$ is the $K(1)$–localization of $\mathbb{S}$. We have that $K^{\mathrm{red}}(\mathbb{Z})$ is 4–connected [16; 17; 27] and the Quillen–Lichtenbaum conjecture (as reformulated by Waldhausen [33, Section 4]) implies that $K(\mathbb{Z})^\wedge_p \to L_{K(1)}K(\mathbb{Z})$ induces an isomorphism on homotopy groups in degrees 2 and above. Thus, $K^{\mathrm{red}}(\mathbb{Z})$ is the 4–connected cover of $L_{K(1)}K^{\mathrm{red}}(\mathbb{Z})$. This makes it straightforward to convert statements about the homotopy type of $L_{K(1)}K^{\mathrm{red}}(\mathbb{Z})$ into statements about the homotopy type of $K^{\mathrm{red}}(\mathbb{Z})^\wedge_p$.

A lot about $L_{K(1)}K(\mathbb{Z})$ is known by work of Dwyer and Mitchell [10], which studies $L_{K(1)}K(R_F)$, where $R_F = \mathcal{O}_F[1/p]$ is the ring of p–integers in a number field; in the case $F=\mathbb{Q}$, $R_F=\mathbb{Z}[1/p]$. By Quillen's localization theorem (and the triviality of $L_{K(1)}K(\mathbb{F}_p)\simeq L_{K(1)}H\mathbb{Z}^{\wedge}_p$),

$$L_{K(1)}K(\mathbb{Z})\simeq L_{K(1)}K(\mathbb{Z}[1/p])$$

and [10, 1.7] in particular computes $(KU^{\wedge}_p)^*(K^{\mathrm{red}}(\mathbb{Z}))$ together with the action of $(KU^{\wedge}_p)^0(KU^{\wedge}_p)$ in number-theoretic terms in terms of the *Iwasawa module* M. Let μ_{p^n} denote the $(p^n)^{\mathrm{th}}$ roots of unity and let $\mu_{p^\infty}=\bigcup\mu_{p^n}$. The Iwasawa module M is then the Galois group of the maximal abelian p–extension of $\mathbb{Q}(\mu_{p^\infty})$ unramified except at p. It comes with an action of $\Gamma'=\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty}/\mathbb{Q}))$ and its completed group ring, typically denoted as Λ' in Iwasawa theory. The canonical isomorphism $\Gamma'\cong(\mathbb{Z}^{\wedge}_p)^{\times}$ (induced by the canonical isomorphisms $\mathrm{Aut}(\mu_{p^n})\cong(\mathbb{Z}/p^n)^{\times}$) and the interpolation of Adams operations on $KU^{\wedge}_p$ to p–adic units gives an isomorphism between Γ' and this group of p–adically interpolated Adams operations, and induces an isomorphism of $\mathbb{Z}^{\wedge}_p$–algebras $\Lambda'\cong(KU^{\wedge}_p)^0(KU^{\wedge}_p)$. From here on, we identify these two algebras via this isomorphism, the utility of which is explained in [10, Sections 4 and 6]. The main result of Dwyer and Mitchell [10, 1.7] proves

$$(KU^{\wedge}_p)^0(K^{\mathrm{red}}(\mathbb{Z}))=0,$$
$$(KU^{\wedge}_p)^{-1}(K^{\mathrm{red}}(\mathbb{Z}))\cong M,$$

as Λ'–algebras. As discussed in [10, Section 8], M is finitely generated projective dimension one as a Λ'–module. Working in terms of L, the Adams summand of $KU^{\wedge}_p$, it follows that $L^*(K^{\mathrm{red}}(\mathbb{Z}))$ is concentrated in odd degrees, where it is a finitely generated projective dimension one L^0L–module in each degree. In particular, $L_{K(1)}K^{\mathrm{red}}(\mathbb{Z})$ splits as

$$L_{K(1)}K^{\mathrm{red}}(\mathbb{Z})\simeq Y_0\vee\cdots\vee Y_{p-2},$$

where Y_i is the fiber of a map from a finite wedge of copies of $\Sigma^{2i-1}L$ to a finite wedge of copies of $\Sigma^{2i-1}L$, giving an L^0L–resolution of the module $L^{2i-1}Y_i$. We note that $[Y_i,Y_{i'}]=0$ unless $i=i'$.

Letting y_i be the 4–connected cover of Y_i, we show below in Proposition 2.18 that $[y_i,y_{i'}]=0$ for $i\neq i'$ and so obtain a canonical splitting and canonical identification of summands

(2.6) $$K(\mathbb{Z})^{\wedge}_p\simeq j\vee y_0\vee\cdots\vee y_{p-2}.$$

Dwyer and Mitchell [10, 8.10] relate the L^0L–modules $L^{2i-1}Y_i$ to class groups of cyclotomic fields. Write A_m for the p–Sylow subgroup of the ideal class group of the cyclotomic field $\mathbb{Q}(\mu_{p^{m+1}})$ and let A_∞ denote the inverse limit of A_m over the norm maps, with its natural structure of a Λ'–module. Usual notation is to write Γ for the subgroup of Γ' given by $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$ or, equivalently, the subgroup of $(\mathbb{Z}^\wedge_p)^\times$ of p–adic units that are congruent to 1 mod p. Writing Δ for $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ or, equivalently, $(\mathbb{Z}/p)^\times$, the Teichmüller character $\omega\colon \Delta \to (\mathbb{Z}^\wedge_p)^\times$ then induces a splitting $\Gamma' \cong \Gamma \times \Delta$, which in turn induces an isomorphism $\Lambda' = \Lambda[\Delta]$, for a certain subring Λ of Λ'. In these terms [10, 8.10] gives an exact sequence

$$\textbf{(2.7)}\quad 0 \to \mathrm{Ext}^1_\Lambda(A_\infty, \Lambda) \to (KU^\wedge_p)^1(K^{\mathrm{red}}(\mathbb{Z})) \to \mathrm{Hom}_\Lambda(E'_\infty(\mathrm{red}), \Lambda) \to \mathrm{Ext}^2_\Lambda(A_\infty, \Lambda) \to 0.$$

(Note that in the case under consideration here, A_∞ is also isomorphic to the Λ'–modules denoted by L_∞ and A'_∞ in [10, Section 12].) Here $E'_\infty(\mathrm{red})$ is a certain Λ'–module defined in terms of cyclotomic units, the details of which will not come into play here, except to note that by [10, 9.10], $E'_\infty(\mathrm{red})$ is noncanonically isomorphic to $(KU^\wedge_p)^0(KO^\wedge_p)$ as a Λ'–module.

To decompose (2.7) in terms of the Y_i, we employ the "Adams splitting" or "eigensplitting" of Δ–equivariant $\mathbb{Z}^\wedge_p$–modules. Any $\mathbb{Z}^\wedge_p[\Delta]$–module X decomposes as a direct sum of pieces corresponding to the powers ω^j of the Teichmüller character $\omega\colon \Delta \to (\mathbb{Z}^\wedge_p)^\times$. The ω^j–character piece $\epsilon_j X$ is the submodule where the element α of Δ acts by multiplication by $\omega^j(\alpha) \in \mathbb{Z}^\wedge_p$ (for all $\alpha \in \Delta$). (For $\Lambda' = \Lambda[\Delta]$–modules, the ω^j–character piece is the summand where $\psi^{\omega(\alpha)}$ acts by $\omega^j(\alpha)$ for all $\alpha \in \Delta$.) This relates to the Adams decomposition of $KU^\wedge_p$ as

$$\epsilon_j((KU^\wedge_p)^q(Z)) \cong L^{2j+q}(Z)$$

for any spectrum Z, and

$$(\Sigma^{2j}L)^0(\Sigma^{2j}L) \xrightarrow{\cong} (KU^\wedge_p)^0(\Sigma^{2j}L) \to (KU^\wedge_p)^0(KU^\wedge_p) = \Lambda'$$

is reasonably interpreted as the inclusion of $\epsilon_j\Lambda'$. (The projection $(KU^\wedge_p)^0(KU^\wedge_p) \to (\Sigma^{2j}L)^0(\Sigma^{2j}L)$ induces an isomorphism of rings from Λ to $(\Sigma^{2j}L)^0(\Sigma^{2j}L)$.) In Δ–equivariant terms, the exact sequence (2.7) is better written as

$$0 \to \mathrm{Ext}^1_\Lambda(A_\infty, L^0L) \to (KU^\wedge_p)^1(K^{\mathrm{red}}(\mathbb{Z})) \to \mathrm{Hom}_\Lambda(E'_\infty(\mathrm{red}), L^0L) \to \mathrm{Ext}^2_\Lambda(A_\infty, L^0L) \to 0$$

(see [10, 8.11]) with the canonical action of Δ on Hom and Ext^i. Taking the ϵ_j piece of the Δ–eigensplitting of this sequence, we get exact sequences

$$0 \to \mathrm{Ext}^1_\Lambda(\epsilon_{-j} A_\infty, L^0 L) \to L^{2j+1} Y_{j+1} \to \mathrm{Hom}_\Lambda(\epsilon_{-j} E'_\infty(\mathrm{red}), L^0 L) \\ \to \mathrm{Ext}^2_\Lambda(\epsilon_{-j} A_\infty, L^0 L) \to 0,$$

since $L^{2j+1} Y_k = 0$ for $k \neq j+1$ (where we understand $Y_{p-1} = Y_0$). Now $\mathrm{Hom}_\Lambda(\epsilon_{-j} E'_\infty(\mathrm{red}), L^0 L)$ is zero when j is odd and a free Λ–module of rank 1 when j is even. Specifically, Y_i is closely related to $\epsilon_j A_\infty$ for $i + j \equiv 1 \bmod (p-1)$.

For fixed p, several of the ω^j–character pieces of A_∞ are always zero. In particular, $\epsilon_0 A_0 = 0$ (because it is canonically isomorphic to the p–Sylow subgroup of the projective class group of $\mathbb{Z}$) and [35, 10.7] then implies that $\epsilon_0 A_\infty = 0$. From the exact sequence above, $L^1 Y_1 \cong L^0 L$ (noncanonically) and so Y_1 is noncanonically weakly equivalent to ΣL. It follows that y_1 is noncanonically weakly equivalent to $\Sigma^{1+2(p-1)} \ell$. In terms of $K(\mathbb{Z})^\wedge_p$, we obtain a further canonical splitting (without canonical identification) $K^{\mathrm{red}}(\mathbb{Z}) \simeq \Sigma^{2p-1} \ell \vee K^{\mathrm{red}\#}(\mathbb{Z})$ for some p–complete spectrum $K^{\mathrm{red}\#}(\mathbb{Z})$. We use the identification of y_1 as a key step in the proof of Theorem 1.1 in Section 4.

Another useful vanishing result is $\epsilon_1 A_0 = 0$ [35, 6.16]. As a consequence, we see that

(2.8) $$Y_0 \simeq *.$$

This simplifies some formulas and arguments.

Although these are the only results we use, other vanishing results for $\epsilon_j A$ give other vanishing results for the summands Y_i. Herbrand's theorem [35, 6.17] and Ribet's converse [24; 35, 15.8] state that for $3 \le j \le p-2$ odd, $\epsilon_j A_0 \neq 0$ if and only if $p \mid B_{p-j}$, where B_n denotes the Bernoulli number, numbered by the convention $t/(e^t - 1) = \sum B_n t^n / n!$. We see that for $p - 3 \ge i \ge 2$ even, $Y_i \simeq *$ when p does not divide B_{i+1}. In particular, Y_2, Y_4, Y_6, Y_8 and Y_{10} are trivial, Y_{12} is trivial for $p \neq 691$, and for every even i, Y_i is only nontrivial for finitely many primes.

A prime p is regular precisely when p does not divide the class number of $\mathbb{Q}(\zeta_p)$ or, in other words, when $A_0 = 0$ and therefore $A_\infty = 0$. Then, for an odd regular prime, we have that Y_{2k} is trivial for all k and Y_{2k+1} is noncanonically weakly equivalent to $\Sigma^{4k+1} L$. It follows that $L_{K(1)} K^{\mathrm{red}}(\mathbb{Z})$ is noncanonically weakly equivalent to $\Sigma KO^\wedge_p$ and $K^{\mathrm{red}}(\mathbb{Z})$ is noncanonically weakly equivalent to $\Sigma^5 ko^\wedge_p$, since $K^{\mathrm{red}}(\mathbb{Z})$

is the 4–connected cover of $L_{K(1)}K^{\mathrm{red}}(\mathbb{Z})$. This leads precisely to the description of $K(\mathbb{Z})^{\wedge}_p$ as noncanonically weakly equivalent to $j \vee \Sigma^5 ko^{\wedge}_p$, as indicated above.

Now consider the case when p satisfies the Kummer–Vandiver condition. This means that p does not divide the class number of $\mathbb{Q}(\zeta_p+\zeta_p^{-1})$ (the fixed field of $\mathbb{Q}(\zeta_p)$ under complex conjugation). The p–Sylow subgroup is precisely the subgroup of A_0 fixed by complex conjugation, which is the internal direct sum of $\epsilon_j A_0$ for $0 \le j < p-1$ even. It follows that $\epsilon_j A_0 = 0$ for j even, and so again Y_{2k+1} is noncanonically weakly equivalent to $\Sigma^{4k+1}L$; this splits a copy of $\Sigma KO^{\wedge}_p$ (with noncanonical identification) off $L_{K(1)}K^{\mathrm{red}}$ as in [10, 12.2]. Now the even summands Y_{2k} may be nonzero, but the Λ–modules $\epsilon_j A_\infty$ are cyclic for j odd (see for example [35, 10.16]) and Y_{2k} is (noncanonically) weakly equivalent to the homotopy fiber of a map $\Sigma^{4k-1}L \to \Sigma^{4k-1}L$ determined by the p–adic L–function $L_p(s;\omega^{2k})$ [10, 12.2]. As above, $Y_0 \simeq *$ and in the other cases, for $n>0$, $n \equiv 2k-1 \bmod (p-1)$,

$$\pi_{2n}Y_{2k} \cong \mathbb{Z}^{\wedge}_p/L_p(-n,\omega^{2k}) = \mathbb{Z}^{\wedge}_p/\left(\tfrac{B_{n+1}}{n+1}\right),$$
$$\pi_{2n+1}Y_{2k} = 0$$

(noncanonical isomorphisms). The groups are of course zero for $n \not\equiv 2k-1 \bmod (p-1)$. (For $n<0$, $n \equiv 2k-1 \bmod (p-1)$, the L–function formula for $\pi_{2n}Y_{2k}$ still holds, and $\pi_{2n+1}Y_{2k}=0$ still holds provided the value of the L–function is nonzero. If the value of the L–function is zero, then $\pi_{2n+1}Y_{2k} \cong \mathbb{Z}^{\wedge}_p$, though it is conjectured [30; 15] that this case never occurs.)

For p not satisfying the Kummer–Vandiver condition, the odd summands satisfy

$$\pi_{2n}Y_{2k+1} = \text{finite},$$
$$\pi_{2n+1}Y_{2k+1} \cong \mathbb{Z}^{\wedge}_p$$

(noncanonical isomorphism) for $n \equiv 2k+1 \bmod (p-1)$ (and zero otherwise) with the finite group unknown. As always $Y_0 \simeq *$, and the Mazur–Wiles theorem [20; 35, 15.14] implies that in the even summands,

$$\#(\pi_{2n}Y_{2k}) = \#\left(\mathbb{Z}^{\wedge}_p/\left(\tfrac{B_{n+1}}{n+1}\right)\right),$$
$$\pi_{2n+1}Y_{2k} = 0$$

for $n>0$, $n \equiv 2k-1 \bmod (p-1)$ (and $\pi_{2n}Y_{2k}=0$ for $n \not\equiv 2k-1 \bmod (p-1)$), although the precise group in the first case is unknown. (In this case, for $n<0$, $n \equiv 2k-1 \bmod (p-1)$, it is known that $\#(\pi_{2n}Y_{2k}) = \#(\mathbb{Z}^{\wedge}_p/L_p(-n,\omega^{2k}))$ and

$\pi_{2n+1}Y_{2k}=0$ provided $L_p(-n,\omega^{2k})$ is nonzero. If $L_p(-n,\omega^{2k})=0$, then $\pi_{2n}Y_{2k}\cong \mathbb{Z}_p^\wedge\oplus$ finite and $\pi_{2n+1}Y_{2k}\cong\mathbb{Z}_p^\wedge$, noncanonically.) For more on the homotopy groups of $K(\mathbb{Z})$, see for example [36, Section VI.10].

Supporting calculations

In several places above, we claimed (implicitly or explicitly) that certain Hom sets in the stable category were zero. Here we review some calculations and justify these claims. All of these computations follow from well-known facts about the spectrum L together with standard facts about maps in the stable category. In particular, in several places, we make use of the fact that for a $K(1)$–local spectrum Z, the localization map $X\to L_{K(1)}X$ induces an isomorphism $[L_{K(1)}X, Z]\to[X,Z]$; also, several times we make use of the fact that if X is $(n-1)$–connected, then the $(n-1)$–connected cover map $Z[n,\infty)\to Z$ induces an isomorphism $[X, Z[n,\infty)]\to[X,Z]$. We begin with results on $[\ell,\Sigma^q\ell]$.

Proposition 2.9 *The map $[\ell,\Sigma^q\ell]\to[\ell,\Sigma^qL]\cong[L,\Sigma^qL]$ is an injection for $q\le 2(2p-2)$. In particular, $[\ell,\Sigma^q\ell]=0$ if $q\not\equiv 0 \bmod (2p-2)$ and $q<2(2p-2)$.*

Proof We have a cofiber sequence

$$\Sigma^{q-1-(2p-2)}H\mathbb{Z}_p^\wedge\to\Sigma^q\ell\to\Sigma^{q-(2p-2)}\ell\to\Sigma^{q-(2p-2)}H\mathbb{Z}_p^\wedge$$

and a corresponding long exact sequence

$$\cdots\to[\ell,\Sigma^{q-1-(2p-2)}H\mathbb{Z}_p^\wedge]\to[\ell,\Sigma^q\ell]\to[\ell,\Sigma^{q-(2p-2)}\ell]\to\cdots.$$

First we note that the map $[\ell,\Sigma^q\ell]\to[\ell,\Sigma^{q-(2p-2)}\ell]$ is injective for $q\le 2(2p-2)$: When $q\ne(2p-2)+1$ this follows from the fact that

$$[\ell,\Sigma^{q-1-(2p-2)}H\mathbb{Z}_p^\wedge]=H^{q-1-(2p-2)}(\ell;\mathbb{Z}_p^\wedge)=0$$

for $q-1-(2p-2)<(2p-2)$ unless $q-1-(2p-2)=0$. In the case $q=(2p-2)+1$, the image of $[\ell,\Sigma^0H\mathbb{Z}_p^\wedge]$ in $[\ell,\Sigma^q\ell]$ in the long exact sequence is still zero because the map $[\ell,\ell]\to[\ell,H\mathbb{Z}_p^\wedge]\cong\mathbb{Z}_p^\wedge$ is surjective. Now, when $q-(2p-2)<2p-2$,

$$[\ell,\Sigma^{q-(2p-2)}\ell]\cong[\ell,\Sigma^{q-(2p-2)}L]\cong[L,\Sigma^{q-(2p-2)}L],$$

since then $\Sigma^{q-(2p-2)}\ell\to\Sigma^{q-(2p-2)}L$ is a weak equivalence on connective covers. For the remaining case $q=2(2p-2)$, we have seen that the map $[\ell,\Sigma^q\ell]\to[\ell,\Sigma^{q-(2p-2)}\ell]$ is an injection and the map $[\ell,\Sigma^{q-(2p-2)}\ell]\to[\ell,\Sigma^{q-(2p-2)}L]$ is an injection. □

Next, using the cofiber sequence

$$\Sigma^{-1}\ell \to \Sigma^{(2p-2)-1}\ell \to j \to \ell$$

and applying the previous result, we obtain the following calculation:

Proposition 2.10 $[j, \Sigma^q \ell] = 0$ *if* $q \not\equiv 0 \bmod (2p-2)$ *and* $q < 2(2p-2)$.

Proof Looking at the long exact sequence

$$\cdots \to [\ell, \Sigma^q \ell] \to [j, \Sigma^q \ell] \to [\Sigma^{(2p-2)-1}\ell, \Sigma^q \ell] \to [\Sigma^{-1}\ell, \Sigma^q \ell] \to \cdots$$

and using the isomorphism $[\Sigma^{(2p-2)-1}\ell, \Sigma^q \ell] \cong [\ell, \Sigma^{q+1-(2p-2)}\ell]$, we have that both $[\ell, \Sigma^q \ell]$ and $[\Sigma^{(2p-2)-1}\ell, \Sigma^q \ell]$ are 0 when $q \not\equiv 0, -1 \bmod (2p-2)$ and $q \leq 2(2p-2)$. In the case when $q \equiv -1 \bmod (2p-2)$, using also the isomorphism $[\Sigma^{-1}\ell, \Sigma^q \ell] \cong [\ell, \Sigma^{q+1}\ell]$, we have a commutative diagram

$$\begin{array}{ccc} [\ell, \Sigma^{q+1-(2p-2)}\ell] & \longrightarrow & [\ell, \Sigma^{q+1}\ell] \\ \downarrow & & \downarrow \\ [L, \Sigma^{q+1-(2p-2)}L] & \rightarrowtail & [L, \Sigma^{q+1}L] \end{array}$$

where the feathered arrows are known to be injections. The statement now follows in this case as well. □

For maps the other way, we have the following result. The proof is similar to the proof of the previous proposition.

Proposition 2.11 $[\Sigma^q \ell, j] = 0$ *if* $q \not\equiv -1 \bmod (2p-2)$ *and* $q \geq -(2p-2)$.

We also have the following result for $q = -1$:

Proposition 2.12 $$[\Sigma^{-1}\ell, j] = 0.$$

Proof Let $j_{-1} = J[-1, \infty)$, where $J = L_{K(1)}\mathbb{S} \simeq L_{K(1)}j$. Then we have a cofiber sequence $\Sigma^{-2}H\pi_{-1}J \to j \to j_{-1} \to \Sigma^{-1}H\pi_{-1}J$ and a long exact sequence

$$\cdots \to [\Sigma^{-1}\ell, \Sigma^{-2}H\pi_{-1}J] \to [\Sigma^{-1}\ell, j] \to [\Sigma^{-1}\ell, j_{-1}] \to [\Sigma^{-1}\ell, \Sigma^{-1}H\pi_{-1}J] \to \cdots.$$

Since $\Sigma^{-1}\ell$ is (–2)–connected, the inclusion of j_{-1} in J induces a bijection

$$[\Sigma^{-1}\ell, j_{-1}] \to [\Sigma^{-1}\ell, J] \cong [\Sigma^{-1}L, J].$$

It follows that a map $\Sigma^{-1}\ell \to j_{-1}$ is determined by the map on π_{-1}, and therefore that the image of $[\Sigma^{-1}\ell, j]$ in $[\Sigma^{-1}\ell, j_{-1}]$ is zero. But $H^{-2}(\Sigma^{-1}\ell; \pi_{-1}J) = 0$, so $[\Sigma^{-1}\ell, j] = 0$. □

In the case of maps between suspensions of j, we only need to consider two cases:

Proposition 2.13 *$[j, \Sigma j] = 0$ and $[\Sigma j, j] = 0$.*

Proof As in the previous proof, we let $j_{-1} = J[-1, \infty)$, and we use the cofiber sequence $\Sigma^{-1} H\pi_{-1}J \to \Sigma j \to \Sigma j_{-1} \to H\pi_{-1}J$ and the induced long exact sequence

$$\cdots \to [j, \Sigma^{-1} H\pi_{-1}J] \to [j, \Sigma j] \to [j, \Sigma j_{-1}] \to [j, H\pi_{-1}J] \to \cdots .$$

Since the connective cover of ΣJ is Σj_{-1}, we have that the map $[j, \Sigma j_{-1}] \to [j, \Sigma J]$ is a bijection, and the maps

$$[J, \Sigma J] \to [j, \Sigma J] \to [\mathbb{S}, \Sigma J] \cong \pi_{-1}J$$

are isomorphisms. It follows that the map $[j, \Sigma j_{-1}] \to [j, H\pi_{-1}J]$ is an isomorphism. Since $[j, \Sigma^{-1} H\pi_{-1}J] = 0$, this proves $[j, \Sigma j] = 0$. For the other calculation, the map $j \to J$ induces a weak equivalence of 1–connected covers, and the induced map

$$[\Sigma j, j] \to [\Sigma j, J] \cong [\Sigma J, J] \cong [\Sigma \mathbb{S}, J] = \pi_1 J = 0$$

is a bijection. □

The following propositions are now clear:

Proposition 2.14 *In the notation above, the summands $x(i)$ of $\mathrm{TC}(\mathbb{Z})^\wedge_p$ satisfy $[x(i), x(i')] = 0$ for $i \neq i'$.*

Proposition 2.15 *Let $k = 0, 1$. In the notation above, $[x(i), \Sigma^k j] = 0$ for $i \neq k$ and the inclusion of $\Sigma^k j$ in $x(k)$ induces a bijection $[x(k), \Sigma^k j] \to [\Sigma^k j, \Sigma^k j]$.*

Eliminating the summands where maps out of j or Σj are trivial, and looking at the connective and 0–connected covers of $K(1)$–localizations, we get the following proposition:

Proposition 2.16 *The map $\mathbb{S} \to j$ induces isomorphisms $[j, \mathrm{TC}(\mathbb{Z})^\wedge_p] \cong \pi_0 \mathrm{TC}(\mathbb{Z})^\wedge_p$ and $[\Sigma j, \mathrm{TC}(\mathbb{Z})^\wedge_p] \cong \pi_1 \mathrm{TC}(\mathbb{Z})^\wedge_p$.*

For the summands of $K(\mathbb{Z})^\wedge_p$, we first consider the splitting of j.

Proposition 2.17 *$[j, K^{\mathrm{red}}(\mathbb{Z})] = 0$ and $[K^{\mathrm{red}}(\mathbb{Z}), j] = 0$.*

Proof As indicated above, $K^{\mathrm{red}}(\mathbb{Z}) \simeq y_0 \vee \cdots \vee y_{p-2}$. We have that y_1 is (noncanonically) weakly equivalent to $\Sigma^{2p-1}\ell$, and applying Propositions 2.10 and 2.11, we see that $[j, y_1] = 0$ and $[y_1, j] = 0$. In addition, $y_2 \simeq *$ and $y_0 \simeq *$. For $2 < i \leq p-2$, y_i is the fiber of a map from a finite wedge of copies of $\Sigma^{2i-1}\ell$ to a finite wedge of copies of $\Sigma^{2i-1}\ell$. Looking at the long exact sequences

$$\cdots \to \bigoplus [j, \Sigma^{2i-2}\ell] \to [j, y_i] \to \bigoplus [j, \Sigma^{2i-1}\ell] \to \cdots,$$
$$\cdots \to \prod [\Sigma^{2i-1}\ell, j] \to [y_i, j] \to \prod [\Sigma^{2i-2}\ell, j] \to \cdots,$$

we again see from Propositions 2.10 and 2.11 that $[j, y_i] = 0$ and $[y_i, j] = 0$. □

Finally, the spectra Y_i clearly satisfy $[Y_i, Y_{i'}] = 0$ for $i \neq i'$; we now verify that the same holds for the covers y_i.

Proposition 2.18 *With the notation above, the summands y_i of $K^{\mathrm{red}}(\mathbb{Z})$ satisfy $[y_i, y_{i'}] = 0$ for $i \neq i'$.*

Proof Each y_i is the fiber of a map from a finite wedge of copies of $\Sigma^{2i-1}\ell$ to a finite wedge of copies of $\Sigma^{2i-1}\ell$ except that $y_1 \simeq \Sigma^{2p-1}\ell$ (noncanonically), $y_0 \simeq *$ and $y_2 \simeq *$. First, for $i > 2$, looking at the long exact sequence

$$\cdots \to \prod [\Sigma^{2i-1}\ell, \Sigma^q \ell] \to [y_i, \Sigma^q \ell] \to \prod [\Sigma^{2i-2}\ell, \Sigma^q \ell] \to \cdots,$$

we see from Proposition 2.9 that $[y_i, \Sigma^q \ell] = 0$ when $q \not\equiv 2i-1, 2i-2 \bmod (2p-2)$ and $q \leq 2(2p-2) + 2i - 3$. In particular, $[y_i, y_1] = 0$ for $i > 2$. For $i' > 2$, looking at the long exact sequence

$$\cdots \to \bigoplus [y_i, \Sigma^{2i'-2}\ell] \to [y_i, y_{i'}] \to \bigoplus [y_i, \Sigma^{2i'-1}\ell] \to \cdots,$$

we see that $[y_i, y_{i'}] = 0$ for $i \neq i'$ in the remaining cases. □

The homotopy type of $\mathrm{TC}(\mathbb{Z})^{\wedge}_p$

To determine the homotopy type of $\mathrm{TC}(\mathbb{Z})^{\wedge}_p$ from that of $\mathrm{TC}(\mathbb{Z})^{\wedge}_p[0,\infty)$, we need to study the cofiber sequence

$$\Sigma^{-2} H\mathbb{Z}^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p[0,\infty) \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p \to \Sigma^{-1} H\mathbb{Z}^{\wedge}_p.$$

The argument of [29, 3.3] studies the induced cofiber sequence on the cofiber of the inclusion of j,

$$\Sigma^{-2} H\mathbb{Z}^{\wedge}_p \to C\big(j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p[0,\infty)\big) \to C(j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p) \to \Sigma^{-1} H\mathbb{Z}^{\wedge}_p,$$

and shows that $C(j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p})$ has the homotopy type of

$$\Sigma j \vee \Sigma^{3}\ell \vee \cdots \vee \Sigma^{2p-5}\ell \vee \Sigma^{-1}\ell \vee \Sigma^{2p-1}\ell.$$

In terms of the noncanonical weak equivalence

$$C\big(j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty)\big) \simeq \Sigma j \vee \Sigma^{3} ku^{\wedge}_{p},$$

the map $\Sigma^{-2} H\mathbb{Z}^{\wedge}_{p} \to C\big(j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty)\big)$ factors through a map

$$\Sigma^{-2} H\mathbb{Z}^{\wedge}_{p} \xrightarrow{\lambda} \Sigma^{2p-3}\ell \xrightarrow{\alpha} \Sigma^{3} ku^{\wedge}_{p},$$

where $\alpha\colon \Sigma^{2p-3}\ell \to \Sigma^{3} ku^{\wedge}_{p}$ is the inclusion of an Adams summand and the map $\lambda\colon \Sigma^{-2} H\mathbb{Z}^{\wedge}_{p} \to \Sigma^{2p-3}\ell$ is a generator of $[\Sigma^{-2} H\mathbb{Z}^{\wedge}_{p}, \Sigma^{2p-3}\ell] \cong \mathbb{Z}^{\wedge}_{p}$. It follows in particular that pulling back along λ induces an isomorphism $[\Sigma^{2p-3}\ell, j] \to [\Sigma^{-2} H\mathbb{Z}^{\wedge}_{p}, j]$. The splitting

$$\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty) \simeq j \vee C\big(j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty)\big)$$

is noncanonical, determined by a choice of map $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty) \to j$ (in the stable category) such that the composite map $j \to j$ is the identity; choosing an arbitrary such map, we can alter it by a map $\Sigma^{2p-3}\ell \to j$ to get a splitting where the composite map

$$\Sigma^{-2} H\mathbb{Z}^{\wedge}_{p} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}[0,\infty) \to j$$

is the zero map. It follows that $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$ is noncanonically weakly equivalent to

$$j \vee \Sigma j \vee \Sigma^{3}\ell \vee \cdots \vee \Sigma^{2p-5}\ell \vee \Sigma^{-1}\ell \vee \Sigma^{2p-1}\ell.$$

3 The maps in the linearization/cyclotomic trace square

The previous section discussed the corners of the linearization/cyclotomic trace square; in this section, we discuss the edges. The main observation is that with respect to the canonical splittings of the previous section, the cyclotomic trace is diagonal and the linearization map is diagonal on the p–torsion part of the homotopy groups.

Theorem 3.1 *In terms of the splittings (2.4) and (2.6) of the previous section, the cyclotomic trace* $K(\mathbb{Z})^{\wedge}_{p} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$ *splits as the wedge of the identity map* $j \to j$

and maps

$$\begin{aligned} y_0 &\to \Sigma^{-1}\ell_{\mathrm{TC}}(0),\\ y_1 &\to \Sigma^{-1}\ell_{\mathrm{TC}}(p),\\ y_2 &\to \Sigma^{-1}\ell_{\mathrm{TC}}(2),\\ &\vdots\\ y_{p-2} &\to \Sigma^{-1}\ell_{\mathrm{TC}}(p-2). \end{aligned}$$

Proof We have that each y_i fits into a fiber sequence of the form

$$y_i \to \bigvee \Sigma^{2i-1}\ell \to \bigvee \Sigma^{2i-1}\ell$$

except in the case $i = 1$, where the suspension is $\Sigma^{2p-1}\ell$ rather than $\Sigma^{1}\ell$ (and the cases $i = 0$ and $i = 2$, where $y_i = *$ anyway). Choosing a noncanonical weak equivalence $\Sigma^{2q-1}\ell \simeq \Sigma^{-1}\ell_{\mathrm{TC}}(q)$, and looking at the long exact sequences of maps into $\Sigma^{2q-1}\ell$, Proposition 2.9 implies $[y_i, \Sigma^{-1}\ell_{\mathrm{TC}}(q)] = 0$ unless $q \equiv i \bmod (p-1)$. Likewise, $[j, \Sigma^{-1}\ell_{\mathrm{TC}}(q)] = 0$ for all q by Proposition 2.10. □

Next, we turn to the linearization map.

Theorem 3.2 *In terms of the splittings (2.1), (2.2) and (2.4) of the previous section, the linearization map* $\mathrm{TC}(\mathbb{S})^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ *admits factorizations as follows:*

(i) *The map* $\mathbb{S}^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ *factors through the canonical map* $\mathbb{S}^{\wedge}_p \to j$*, which is a canonically split surjection on homotopy groups in all degrees.*

(ii) *The map* $\Sigma\mathbb{S}^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ *factors through a map* $\Sigma\mathbb{S}^{\wedge}_p \to \Sigma j'$ *that is an isomorphism on* π_1 *and is a split surjection on homotopy groups in all degrees.*

(iii) *The map* $\overline{Cu} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ *factors through* $j \vee \bigvee \Sigma^{-1}\ell_{\mathrm{TC}}(i)$ *for* $i = 0, p, 2, \dots, p-2$*. On* p*–torsion, the map* $\mathrm{tor}_p(\pi_*(\overline{Cu})) \to \mathrm{tor}_p(\pi_*(\mathrm{TC}(\mathbb{Z})^{\wedge}_p))$ *is zero.*

Proof The statement (i) is clear from the construction of the map $j \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ since the linearization map is a map of ring spectra. For (ii), the composite map $\Sigma\mathbb{S}^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ is determined by where the generator goes in $\pi_1\mathrm{TC}(\mathbb{Z})^{\wedge}_p$, but the inclusion of $\Sigma j'$ in $\mathrm{TC}(\mathbb{Z})^{\wedge}_p$ induces an isomorphism on π_1, and so $\Sigma\mathbb{S}^{\wedge}_p$ factors through $\Sigma j'$. The map $\mathrm{TC}(\mathbb{S})^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ is a $(2p{-}3)$–equivalence [5, 9.10]; the map $\Sigma\mathbb{S} \to \Sigma j'$ is therefore an isomorphism on π_1. Using the image of the generator of $\pi_1\Sigma\mathbb{S}$ as a generator for $\pi_1\Sigma j'$ gives a weak equivalence $\Sigma j \to \Sigma j'$ such that the map $\Sigma\mathbb{S} \to \Sigma j' \to \Sigma j$ obtained by composing with the inverse is the suspension of the canonical map $\mathbb{S} \to j$. This completes the proof of (ii).

To show the factorization of $\overline{Cu} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$ for (iii), it suffices to check that the composite map

$$\overline{Cu} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p} \to \Sigma j'$$

is trivial. Using the noncanonical weak equivalence $\overline{Cu} \simeq \Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1}$, the Atiyah–Hirzebruch spectral sequence to calculate $(\Sigma j)^*(\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1})$ with

$$E_2^{s,t} = H^{s+t}(\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1}; \pi_{-t}(\Sigma j')) \cong H^{s+t}(\overline{\mathbb{C}P}{}^{\infty}_{-1}; \pi_{-t}(j))$$

is zero along the line of total degree zero, and so $[\overline{Cu}, \Sigma j'] = 0$.

Finally, to see that the map $\overline{Cu} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$ is zero on the torsion subgroup of $\pi_*\overline{Cu}$, it suffices to note that the composite map

$$\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1} \simeq \overline{Cu} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p} \to j$$

is zero on the torsion subgroup of $\pi_*\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1}$, or equivalently that the composite map to J is zero on the torsion subgroup of $\pi_*\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1}$. Postcomposing with the (canonical) map $J \to L$ in the fiber sequence

$$\Omega L \to J \to L \to L,$$

the map $\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1} \to L$ is trivial (since $L^*(\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1})$ is concentrated in odd degrees). It follows that the map $\Sigma\overline{\mathbb{C}P}{}^{\infty}_{-1} \to J$ factors through the map $\Omega L \to J$, and therefore is zero on torsion. □

It would be reasonable to expect that the augmentations $\mathrm{TC}(\mathbb{S})^{\wedge}_{p} \to \mathbb{S}^{\wedge}_{p}$ and $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p} \to j$ are compatible, although we see no K–theoretic, THH–theoretic, or calculational reasons why this should hold. Such a compatibility would imply that the map $\overline{Cu} \to \mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$ factors through $\bigvee \Sigma^{-1}\ell_{\mathrm{TC}}(i)$ and would then (combined with the observations in Section 5) say that the linearization map is fully diagonal with respect to the splittings of the previous section.

4 Proof of the main results

We now apply the work of the previous two sections to prove the theorems stated in the introduction. We begin with Theorem 1.1, which is an immediate consequence of the following theorem:

Theorem 4.1 *The map*

$$\pi_n \mathrm{TC}(\mathbb{S})^\wedge_p \oplus \pi_n K(\mathbb{Z})^\wedge_p \to \pi_n \mathrm{TC}(\mathbb{Z})^\wedge_p$$

is (noncanonically) split surjective.

We apply the splittings of $\mathrm{TC}(\mathbb{S})^\wedge_p$, $\mathrm{TC}(\mathbb{Z})^\wedge_p$ and $K(\mathbb{Z})^\wedge_p$ and the maps on homotopy groups to prove the previous theorem by breaking it into pieces and showing that different pieces in the splitting induce surjections on homotopy groups. Theorem 3.2(i)–(ii) provide two surjection results; we write two more in Lemmas 4.2 and 4.4. The first of these is essentially due to Klein and Rognes [14].

Lemma 4.2 *Under the splittings (2.1), (2.2) and (2.4), the composite map*

$$\overline{Cu} \to \mathrm{TC}(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{Z})^\wedge_p \to \Sigma^{-1}\ell_{\mathrm{TC}}(0) \vee \cdots \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p-2)$$

induces a split surjection on π_{2q+1} for $q \not\equiv 0 \bmod (p-1)$

Proof Klein and Rognes [14, 5.8, (17)] (and independently Madsen and Schlichtkrull [19, 1.1]) construct a space-level map

$$\mathrm{SU}^\wedge_p \to \Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})^\wedge_p.$$

They study the composite map

$$\mathrm{SU}^\wedge_p \to \Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})^\wedge_p \to \Omega^\infty(\mathrm{TC}(\mathbb{Z})^\wedge_p) \to \mathrm{SU}^\wedge_p \tag{4.3}$$

induced by the linearization map $\mathrm{TC}(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{Z})^\wedge_p$, the projection map

$$\mathrm{TC}(\mathbb{Z})^\wedge_p[0,\infty) \to \Sigma^3 ku^\wedge_p$$

and the Bott periodicity isomorphism $\Omega^\infty\Sigma^3 ku \simeq \mathrm{SU}$. In [14, 6.3(i)], Klein and Rognes show that their map (4.3) induces an isomorphism of homotopy groups in all degrees except those congruent to $1 \bmod 2(p-1)$. This proves the statement except in degree -1, where it follows from the fact that the linearization map $\mathrm{TC}(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{Z})^\wedge_p$ is a $(2p-3)$–equivalence [5, 9.10]. □

The other lemma constructs a split surjection onto $\pi_*\mathrm{TC}(\mathbb{Z})^\wedge_p$ in degrees congruent to $1 \bmod 2(p-1)$. We give a direct argument for the following lemma, but it can also be proved using the vanishing results in [3].

Lemma 4.4 *Under the splittings of (2.4) and (2.6), the composite map*

$$y_1 \to K(\mathbb{Z})^{\wedge}_p \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p \to \Sigma^{-1}\ell_{\mathrm{TC}}(p)$$

is a weak equivalence.

Proof Since y_1 and $\Sigma^{-1}\ell_{\mathrm{TC}}(p)$ are both (noncanonically) weakly equivalent to $\Sigma^{2p-1}\ell$, it suffices to show that the map becomes a weak equivalence after $K(1)$–localization. Indeed, by v_1 periodicity, it suffices to show that the map on $K(1)$–localizations is an isomorphism on any homotopy group in degree congruent to $1 \bmod (2p-2)$. We check this on π_1. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & L_{K(1)}K(\mathbb{Z}[1/p]) & \longrightarrow & & & L_{K(1)}K(\mathbb{Q}^{\wedge}_p) \\
 & \nearrow\simeq & & & & \nearrow\simeq & \\
L_{K(1)}K(\mathbb{Z}) & & \longrightarrow & & L_{K(1)}K(\mathbb{Z}^{\wedge}_p) & & \\
\downarrow & & & & \downarrow\simeq & & \\
L_{K(1)}\mathrm{TC}(\mathbb{Z}) & & \xrightarrow{\ \simeq\ } & & L_{K(1)}\mathrm{TC}(\mathbb{Z}^{\wedge}_p) & &
\end{array}$$

where the vertical maps are induced by the cyclotomic trace and the indicated maps are weak equivalences, for the diagonal maps by Quillen's localization sequence, for the indicated horizontal map by [12, Addendum 6.2], and for the indicated vertical map by [12, Theorem D]. On π_1, the canonical maps

$$\mathbb{Z}[1/p]^{\times} \to \pi_1(K(\mathbb{Z}[1/p])) \to \pi_1(L_{K(1)}K(\mathbb{Z}[1/p])),$$
$$(\mathbb{Q}^{\wedge}_p)^{\times} \to \pi_1(K(\mathbb{Q}^{\wedge}_p)) \to \pi_1(L_{K(1)}K(\mathbb{Q}^{\wedge}_p))$$

induce isomorphisms

$$(\mathbb{Z}[1/p]^{\times})^{\wedge}_p \cong \pi_1(L_{K(1)}K(\mathbb{Z})) \cong \pi_1(Y_1),$$
$$((\mathbb{Q}^{\wedge}_p)^{\times})^{\wedge}_p \cong \pi_1(L_{K(1)}\mathrm{TC}(\mathbb{Z})) \cong \pi_1(L_{K(1)}\Sigma j') \oplus \pi_1(L_{K(1)}\Sigma^{-1}\ell_{\mathrm{TC}}(p)),$$

under which the map $\pi_1(Y_1) \to \pi_1(L_{K(1)}\mathrm{TC}(\mathbb{Z}^{\wedge}_p))$ corresponds to the map induced by the inclusion of $\mathbb{Z}[1/p]$ in $\mathbb{Q}^{\wedge}_p$. By construction, the map $\Sigma j' \to \mathrm{TC}(\mathbb{Z})^{\wedge}_p$ sends $\pi_1(j')$ isomorphically onto the subgroup $((\mathbb{Z}^{\wedge}_p)^{\times})^{\wedge}_p$ of $((\mathbb{Q}^{\wedge}_p)^{\times})^{\wedge}_p$ under the isomorphism above. The map

$$(\mathbb{Z}[1/p]^{\times})^{\wedge}_p \to ((\mathbb{Q}^{\wedge}_p)^{\times})^{\wedge}_p/((\mathbb{Z}^{\wedge}_p)^{\times})^{\wedge}_p$$

is an isomorphism of free $\mathbb{Z}^{\wedge}_p$–modules of rank 1. □

We now have everything we need for the proof of Theorem 4.1.

Proof of Theorem 4.1 Combining previous results, we have two families of (noncanonical) splittings

$$\pi_*(\mathbb{S} \vee \Sigma\mathbb{S} \vee \overline{Cu} \vee j \vee y_0 \vee \cdots \vee y_{p-2}) \\ \to \pi_*(j \vee \Sigma j' \vee \Sigma^{-1}\ell_{\mathrm{TC}}(0) \vee \cdots \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p-2)).$$

For both splittings we use Lemma 4.4 to split the $\Sigma^{-1}\ell_{\mathrm{TC}}(p)$ summand in the codomain (canonically) using the y_1 summand of the domain, we use Theorem 3.2(ii) to split the $\pi_*\Sigma j'$ summand in the codomain (canonically) using the $\Sigma\mathbb{S}^1$ summand in the domain, and we use Lemma 4.2 to split the

$$\pi_*(\Sigma^{-1}\ell_{\mathrm{TC}}(0) \vee \Sigma^{-1}\ell_{\mathrm{TC}}(2) \vee \cdots \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p-2))$$

summands in the codomain (noncanonically) using the $\pi_*\overline{Cu}$ summand in the domain. We then have a choice on the remaining summand of the codomain, $\pi_* j$. We can use Theorem 3.1 to split this (canonically) using the $\pi_* j$ summand in the domain or use Theorem 3.2 to split this (canonically) using the $\pi_*\mathbb{S}$ summand in the domain. □

This completes the proof of Theorem 1.1. We now prove the remaining theorems from the introduction.

Proof of Theorems 1.2 and 1.3 Since the long exact sequence of the homotopy cartesian linearization/cyclotomic trace square breaks into split short exact sequences, we get split short exact sequences on p–torsion subgroups

$$0 \to \mathrm{tor}_p(\pi_n K(\mathbb{S})) \to \mathrm{tor}_p(\pi_n \mathrm{TC}(\mathbb{S})^{\wedge}_p \oplus \pi_n K(\mathbb{Z})) \to \mathrm{tor}_p(\pi_n \mathrm{TC}(\mathbb{Z})^{\wedge}_p) \to 0.$$

Using the splittings of (2.1), (2.4) and (2.5), leaving out the nontorsion summands, we can identify $\mathrm{tor}_p(\pi_n K(\mathbb{S}))$ as the kernel of a map

$$\mathrm{tor}_p(\pi_n\mathbb{S} \oplus \pi_{n-1}\mathbb{S} \oplus \pi_n\overline{Cu} \oplus \pi_n j \oplus \pi_n K^{\mathrm{red}}(\mathbb{Z})) \to \mathrm{tor}_p(\pi_n j \oplus \pi_{n-1} j'),$$

which by Theorems 3.1 and 3.2 is mostly diagonal: it is the direct sum of the canonical maps

$$\mathrm{tor}_p(\pi_n\mathbb{S}) \oplus \mathrm{tor}_p(\pi_n j) \to \mathrm{tor}_p(\pi_n j), \\ \mathrm{tor}_p(\pi_{n-1}\mathbb{S}) \to \mathrm{tor}_p(\pi_{n-1} j'),$$

and the zero maps on $\mathrm{tor}_p(\pi_n\overline{Cu})$ and $\mathrm{tor}_p(\pi_n K^{\mathrm{red}}(\mathbb{Z}))$. The isomorphism (a) uses the canonical splitting $\pi_n j \to \pi_n\mathbb{S}$ on the $\pi_n\mathbb{S}$ summand, while the isomorphism (b) uses the identity of $\pi_n j$ on the $\pi_n j$ summand. □

5 Conjecture on Adams operations

In Section 2 we produced canonical splittings on $K(\mathbb{Z})^\wedge_p$ and $\mathrm{TC}(\mathbb{Z})^\wedge_p$ and in Section 3 we showed that the cyclotomic trace is diagonal with respect to these splittings. The purpose of this section is to prove the following splitting of the linearization/cyclotomic trace square and relate it to conjectures on Adams operations.

Theorem 5.1 *The spectrum $K(\mathbb{S})^\wedge_p$ splits into $p-1$ summands,*

$$K(\mathbb{S})^\wedge_p \simeq \mathcal{K}_0 \vee \cdots \vee \mathcal{K}_{p-2},$$

and the linearization/cyclotomic trace square splits into the wedge sum of $p-1$ homotopy cartesian squares

$$\begin{array}{ccc} \mathcal{K}_0 & \longrightarrow & j \\ \downarrow & & \downarrow \\ \mathbb{S}^\wedge_p \vee \Sigma\mathbb{C}P^\infty_{-1}[-1] & \longrightarrow & j \vee \Sigma^{-1}\ell_{\mathrm{TC}}(0) \end{array} \qquad \begin{array}{ccc} \mathcal{K}_1 & \longrightarrow & y_1 \\ \downarrow & & \downarrow \\ \Sigma\mathbb{C}P^\infty_{-1}[0] & \longrightarrow & \Sigma j \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p) \end{array}$$

for $i=0,1$, and

$$\begin{array}{ccc} \mathcal{K}_i & \longrightarrow & y_i \\ \downarrow & & \downarrow \\ \Sigma\mathbb{C}P^\infty_{-1}[i-1] & \longrightarrow & \Sigma^{-1}\ell_{\mathrm{TC}}(i) \end{array}$$

for $i=2,\dots p-2$.

We are using the notation from Section 2 for the summands y_i and $\Sigma^{-1}\ell_{\mathrm{TC}}(i)$ of $K(\mathbb{Z})^\wedge_p$ and $\mathrm{TC}(\mathbb{Z})^\wedge_p$. The spectra $\mathbb{C}P^\infty_{-1}[i]$ are the wedge summands of the "Adams splitting" previously used by Rognes [29, Section 5],

(5.2) $$\mathbb{C}P^\infty_{-1} \simeq \mathbb{C}P^\infty_{-1}[-1] \vee \mathbb{C}P^\infty_{-1}[0] \vee \cdots \vee \mathbb{C}P^\infty_{-1}[p-3].$$

Here we are following the numbering of Rognes [29, page 169], which has its rationale in that the $[i]$ piece has its ordinary cohomology concentrated in degrees $2i \mod 2p-2$ and starts in degree $2i$. Note that the splitting of the theorem fails to be canonical because of problems with the identification of $\mathrm{TC}(\mathbb{S})^\wedge_p$ as $\mathbb{S}^\wedge_p \vee \Sigma\mathbb{C}P^\infty_{-1}$.

In the theorem, for the $i=0$ square, we have used that $y_0 = *$ as noted in Section 2. In using these squares to study the homotopy type of $\mathcal{K}_i$, we can simplify the $i=0$

square to a cofiber sequence

$$\mathcal{K}_0 \to \mathbb{S}^\wedge_p \vee \Sigma\mathbb{C}P^\infty_{-1}[-1] \to \Sigma^{-1}\ell_{\mathrm{TC}}(0) \to \Sigma\mathcal{K}_0$$

since Theorem 3.1 indicates that the map $j \to j \vee \Sigma^{-1}\ell_{\mathrm{TC}}(0)$ factors through the identity map $j \to j$. For the $i = 1$ square, the splitting of $\mathbb{C}P^\infty_{-1}$ fits into a fiber sequence with the splitting of

$$(\Sigma^\infty\mathbb{C}P^\infty)^\wedge_p \simeq \Sigma^\infty K(\mathbb{Z}^\wedge_p, 2)) \simeq \mathbb{C}P^\infty[1] \vee \cdots \vee \mathbb{C}P^\infty[p-1],$$

which identifies $\mathbb{C}P^\infty_{-1}[0]$ as $\mathbb{S}^\wedge_p \vee \mathbb{C}P^\infty[p-1]$. Theorem 3.1 and Lemma 4.4 indicate that the map $y_1 \to \Sigma j \vee \Sigma^{-1}\ell_{\mathrm{TC}}(p)$ factors through a weak equivalence $y_1 \to \Sigma^{-1}\ell_{\mathrm{TC}}(p)$, and we get a weak equivalence

$$\mathcal{K}_1 \simeq \Sigma c \vee \mathbb{C}P^\infty[p-1],$$

where by definition c is the homotopy fiber of the canonical map $\mathbb{S} \to j$.

Proof of Theorem 5.1 Let $\epsilon_i\mathrm{TC}(\mathbb{S})^\wedge_p$, $\epsilon_i\mathrm{TC}(\mathbb{Z})^\wedge_p$ and $\epsilon_i K(\mathbb{Z})^\wedge_p$ be the summands of $\mathrm{TC}(\mathbb{S})^\wedge_p$, $\mathrm{TC}(\mathbb{Z})^\wedge_p$ and $K(\mathbb{Z})^\wedge_p$, respectively, specified in the i^{th} square in the statement of the theorem. Our work in Section 3 shows that the map $K(\mathbb{Z})^\wedge_p \to \mathrm{TC}(\mathbb{Z})^\wedge_p$ restricts to a sum of maps $\epsilon_i K(\mathbb{Z})^\wedge_p \to \epsilon_i\mathrm{TC}(\mathbb{Z})^\wedge_p$. The Atiyah–Hirzebruch spectral sequence implies that the $\mathbb{C}P^\infty_{-1}[i-1]$ summand of $\epsilon_i\mathrm{TC}(\mathbb{S})^\wedge_p$ factors uniquely through $\epsilon_i\mathrm{TC}(\mathbb{Z})^\wedge_p$ as there are no essential maps to the other summands; Theorem 3.2 then implies that the map $\mathrm{TC}(\mathbb{S})^\wedge_p \to \mathrm{TC}(\mathbb{Z})^\wedge_p$ decomposes as a wedge sum of maps $\epsilon_i\mathrm{TC}(\mathbb{S})^\wedge_p \to \epsilon_i\mathrm{TC}(\mathbb{Z})^\wedge_p$. □

In the proof above, we argued calculationally using the paucity of maps in the stable category between the summands; however, there is a conceptual reason to expect much of this behavior based on p–adically interpolated of Adams operations. In [1, 10.7], we showed that the splitting of $\overline{\mathbb{C}P}^\infty_{-1}$ arises from a p–adic interpolation of the Adams operations on $\mathrm{TC}(\mathbb{S})^\wedge_p$ (constructed there). Such a p–adic interpolation is an extension to $(\mathbb{Z}^\wedge_p)^\times$ of the action of the monoid $\mathbb{Z}^\times_{(p)} \cap \mathbb{Z}$ on $\mathrm{TC}(-;p)$ (in the p–complete stable category) acting by Adams operations. Using the Teichmüller character ω to embed $(\mathbb{Z}/p)^\times$ in $(\mathbb{Z}^\wedge_p)^\times$, we then get an action of the ring $\mathbb{Z}^\wedge_p[(\mathbb{Z}/p)^\times]$, which then produces an eigensplitting of $\mathrm{TC}(\mathbb{S})^\wedge_p$ into the $p-1$ summands corresponding to the powers of the Teichmüller character. The wedge summand $\epsilon_i\mathrm{TC}(\mathbb{S})^\wedge_p$ in the proof of Theorem 5.1 corresponds to the character ω^i in the sense that the p–adically interpolated Adams operation $\psi^{\omega(\alpha)}$ acts on it by multiplication by $\omega^i(\alpha) \in \mathbb{Z}^\wedge_p$ for all $\alpha \in (\mathbb{Z}/p)^\times$.

We can imagine that p–adically interpolated Adams operations act on the whole linearization/cyclotomic trace square; we would then obtain an eigensplitting into $p-1$ squares exactly as in the statement of Theorem 5.1. We regard this as providing evidence for the following pair of conjectures, which together would give a conceptual (as opposed to calculational) proof of Theorem 5.1.

Conjecture 5.3 *Let R be an E_∞ ring spectrum. There exists a homomorphism from $(\mathbb{Z}_p^\wedge)^\times$ to the composition monoid $[K(R)_p^\wedge, K(R)_p^\wedge]$, which is natural in the obvious sense and satisfies the following properties.*

(i) *When R is a ring, the restriction to $\mathbb{Z}\cap(\mathbb{Z}_p^\wedge)^\times$ gives Quillen's Adams operations on the zeroth space.*

(ii) *The induced map $(\mathbb{Z}_p^\wedge)^\times \to \operatorname{Hom}(\pi_* K(R)_p^\wedge, \pi_* K(R)_p^\wedge)$ is continuous, where the target is given the p–adic topology.*

Conjecture 5.4 *For R a connective E_∞ ring spectrum, the cyclotomic trace $K(R)_p^\wedge \to \mathrm{TC}(R)_p^\wedge$ commutes with the (conjectural) p–adically interpolated Adams operations.*

The preceding conjecture on p–adic interpolation of the Adams operations in p–completed algebraic K–theory is weaker than the (known) results in the case of topological K–theory in that we are only asking for continuity on homotopy groups rather than continuity for (some topology on) endomorphisms. Nevertheless, it implies a natural action of $\mathbb{Z}_p^\wedge[(\mathbb{Z}/p)^\times]$ (in the stable category) on p–completed algebraic K–theory spectra (of connective ring spectra), which is all that is needed for the eigensplitting. To compare to the splitting used in Theorem 5.1, note that for an algebraically closed field of characteristic prime to p or a strict Henselian ring A with p invertible, the Adams operation ψ^k acts on $\pi_{2s} L_{K(1)} K(A)$ by multiplication by k^s. As a consequence, for any scheme satisfying the hypotheses for Thomason's spectral sequence [31, 4.1], the eigensplitting on homotopy groups is compatible with the filtration from E_∞ in the sense that the subquotient of $H^s(R;\mathbb{Z}/p^n(i))$ comes from the ω^i summand. For $R=\mathbb{Z}[1/p]$, we see that the ω^i summand of $L_{K(1)}K(R)$ has homotopy groups only in degrees congruent to $2i-1$ and $2i-2$ mod $2(p-1)$ except when $i=0$, where the unit of $\pi_0 L_{K(1)}(R)$ is also in the trivial character summand. As a consequence, we see that this splitting agrees with the splitting described above for $L_{K(1)}K(\mathbb{Z}[1/p]) \simeq L_{K(1)}K(\mathbb{Z})$. Specifically, the summand corresponding to the trivial character is J and the summand corresponding to ω^i is Y_i for $i=1,\dots,p-2$.

The preceding conjectures also leads to the same splitting of $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$. Hesselholt and Madsen [12, Theorem D, Addendum 6.2] show that the completion map and cyclotomic trace

$$\mathrm{TC}(\mathbb{Z})^{\wedge}_{p} \to \mathrm{TC}(\mathbb{Z}^{\wedge}_{p})^{\wedge}_{p} \leftarrow K(\mathbb{Z}^{\wedge}_{p})^{\wedge}_{p}$$

are weak equivalences after taking the connective cover and so in particular induce weak equivalences after $K(1)$–localization. The Quillen localization sequence identifies the homotopy fiber of the map $K(\mathbb{Z}^{\wedge}_{p}) \to K(\mathbb{Q}^{\wedge}_{p})$ as $K(\mathbb{F}_p)$. Since the p–completion of $K(\mathbb{F}_p)$ is weakly equivalent to $H\mathbb{Z}^{\wedge}_{p}$, its $K(1)$–localization is trivial. Combining these maps, we obtain a canonical isomorphism in the stable category from $L_{K(1)}\mathrm{TC}(\mathbb{Z})$ to $L_{K(1)}K(\mathbb{Q}^{\wedge}_{p})$. The Hesselholt–Madsen proof of the Quillen–Lichtenbaum conjecture for certain local fields [13, Theorem A] (or in this case, inspection from the calculation of $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$), shows that the map

$$\mathrm{TC}(\mathbb{Z})^{\wedge}_{p} \to L_{K(1)}\mathrm{TC}(\mathbb{Z}) \simeq L_{K(1)}K(\mathbb{Q}^{\wedge}_{p})$$

becomes a weak equivalence after taking 1–connected covers. Again looking at Thomason's spectral sequence, we see that the conjectural Adams operations would then split $L_{K(1)}K(\mathbb{Q}^{\wedge}_{p})$ into summands as follows. The summand corresponding to the trivial character is $J \vee \Sigma^{-1}L_{\mathrm{TC}}(0)$, the summand corresponding to ω is $\Sigma J \vee \Sigma^{-1}L_{\mathrm{TC}}(1)$ and the summand corresponding to ω^i is $\Sigma^{-1}L_{\mathrm{TC}}(i)$ for $i = 2, \dots, p-2$. A short argument now shows that the eigensplitting gives the splitting of $\mathrm{TC}(\mathbb{Z})^{\wedge}_{p}$ used in Theorem 5.1.

We have proposed relatively strong conjectures in Conjectures 5.3 and 5.4; for a proof of Theorem 5.1, it would be enough for the operations to exist and be compatible just for regular rings. The work of Riou [25] makes the existence of such operations plausible. But given the work of Dundas [8], the more general conjectures above (at least for connective ring spectra R with $\pi_0 R$ regular) are not far removed from the corresponding conjectures for rings. We do also note that work on nonexistence of determinants [2, 2.3; 32, Proof of 3.7] is often cited as evidence against the existence of Adams operations on the algebraic K–theory of ring spectra.

6 Low-degree computations

Theorem 1.2 describes the p–torsion in $K(\mathbb{S})$ in terms of the p–torsion in various pieces. For convenience, we review in Proposition 6.1 below what is known about the homotopy groups of these pieces at least up to the range in which [29] describes the homotopy groups of $\overline{\mathbb{C}P}^{\infty}_{-1} \simeq \Sigma^{-1}\overline{Cu}$. As a consequence of Theorem 1.2, irregular primes potentially contribute in degrees divisible by 4 but otherwise make no contribution to the torsion of $K(\mathbb{S})$ until degree 22. Thus, $\pi_* K(\mathbb{S})$ in degrees ≤ 21 not divisible by 4 is fully computed (up to some 2–torsion extensions) by the work of Rognes [28; 29]. For convenience, we assemble the computation of $\pi_* K(\mathbb{S})$ for $* \leq 22$ in Table 1 on page 131.

The table compiles the results reviewed in Proposition 6.1 (qv for sources) and [28, 5.8] into the computation of $\pi_n K(\mathbb{S})$ for $n \leq 22$. The description of $\pi_n K(\mathbb{S})$ is divided into columns:

1. The nontorsion part.
2. The contribution from the torsion of $\mathbb{S}$.
3. The remaining 2–torsion (from [28]).
4. The contribution from Σc for odd primes.
5. The torsion contribution from $\overline{Cu}$ for odd primes.
6. The torsion contribution from $K^{\mathrm{red}}(\mathbb{Z})$ for odd primes.

Presently $K_{4n}(\mathbb{Z})$ is unknown for $n > 1$, conjectured to be 0 (the Kummer–Vandiver conjecture) and if nonzero is a finite group with order a product of irregular primes, each of which is $> 10^8$.

Summands denoted by $[m]$ are finite groups of order m whose isomorphism class is not known.

Proposition 6.1 *The p–torsion groups $\mathrm{tor}_p(\pi_*\mathbb{S})$, $\mathrm{tor}_p(\pi_* c)$, $\mathrm{tor}_p(\pi_* K^{\mathrm{red}}(\mathbb{Z}))$ and $\mathrm{tor}_p(\pi_*\overline{Cu})$ are known in at least the following ranges, as follows:*

$\boldsymbol{\pi_*\mathbb{S}}$, $\boldsymbol{\pi_* j}$ (see for example [23, 1.1.13]) *$\pi_*\mathbb{S}$ splits as*

$$\pi_*\mathbb{S} = \pi_* j \oplus \pi_* c.$$

Also, $\mathrm{tor}_p(\pi_k j)$ is zero unless $2(p-1)$ divides $k+1$, in which case it is cyclic of order p^{s+1}, where $k+1 = 2(p-1)p^s m$ for m relatively prime to p. See below for $\pi_* c$.

$\boldsymbol{\pi_* c}$ (see for example [23, 1.1.14]) *In degrees $\leq 6p(p-1)-6$, $\mathrm{tor}_p(\pi_* c)$ is $\mathbb{Z}/p$ in the following degrees and zero in all others* (in the table, $\alpha_1 \in \pi_{2p-3} j$):

Generator	Degree
β_1	$2p(p-1)-2$
$\alpha_1\beta_1$	$2(p+1)(p-1)-3$
β_1^2	$4p(p-1)-4$
$\alpha_1\beta_1^2$	$2(2p+1)(p-1)-5$
β_2	$2(2p+1)(p-1)-2$
$\alpha_1\beta_2$	$4(p+1)(p-1)-3$
β_1^3	$6p(p-1)-6$

$\boldsymbol{\pi_* K^{\mathrm{red}}(\mathbb{Z})}$ (see for example [36, Section VI.10] or Section 2) $\operatorname{tor}_p(\pi_* K^{\mathrm{red}}(\mathbb{Z}))$ *is zero in odd degrees. If* p *is regular, then* $\operatorname{tor}_p(\pi_{2k} K^{\mathrm{red}}(\mathbb{Z}))=0$. *If* p *satisfies the Kummer–Vandiver condition, then*

$$\operatorname{tor}_p(\pi_{4k} K^{\mathrm{red}}(\mathbb{Z}))=0 \quad \text{and} \quad \operatorname{tor}_p(\pi_{4k+2} K^{\mathrm{red}}(\mathbb{Z}))=\mathbb{Z}^\wedge_p/(B_{2k+2}/(2k+2)),$$

where B_n *denotes the Bernoulli number, numbered by the convention* $t/(e^t-1)=\sum B_n t^n/n!$. *If* p *does not satisfy the Kummer–Vandiver condition then*

$$\operatorname{tor}_p(\pi_{4k} K^{\mathrm{red}}(\mathbb{Z}))=0$$

for $k=1$ *and is an unknown finite group for* $k>1$, *while* $\operatorname{tor}_p(\pi_{4k+2} K^{\mathrm{red}}(\mathbb{Z}))$ *is an unknown group of order* $\#\big(\mathbb{Z}^\wedge_p/(B_{2k+2}/(2k+2))\big)$ *for all* k.

$\boldsymbol{\pi_* \overline{Cu}}$ (see [29, 4.7])

- *In odd degrees* $\le |\beta_2|-2=2(2p+1)(p-1)-4$,

$$\operatorname{tor}_p(\pi_{2n+1}\overline{Cu})=\mathbb{Z}/p$$

in degrees $n=p^2-p-1+m$ *or* $n=2p^2-2p-2+m$ *for* $1\le m\le p-3$ *and* $\operatorname{tor}_p(\pi_{2n+1}\overline{Cu})$ *is zero otherwise.*

- *In even degrees* $\le 2p(p-1)$,

$$\operatorname{tor}_p(\pi_{2n}\overline{Cu})=\mathbb{Z}/p$$

for $m(p-1)<n<mp$ *for* $2\le m\le p-1$ *and* $\operatorname{tor}_p(\pi_{2n}\overline{Cu})$ *is zero otherwise, except that*

$$\operatorname{tor}_p(\pi_{2(p(p-1)-1)}\overline{Cu})=0.$$

n	$\pi_n K(\mathbb{S})$					
0	$\mathbb{Z}$					
1		$\mathbb{Z}/2$				
2		$\mathbb{Z}/2$				
3		$\mathbb{Z}/8\times\mathbb{Z}/3$	$\oplus\ \mathbb{Z}/2$			
4	0					
5	$\mathbb{Z}$					
6		$\mathbb{Z}/2$				
7		$\mathbb{Z}/16\times\mathbb{Z}/3\times\mathbb{Z}/5$	$\oplus\ \mathbb{Z}/2$			
8		$(\mathbb{Z}/2)^2$				$\oplus\ K_8(\mathbb{Z})$
9	$\mathbb{Z}$	$\oplus\ (\mathbb{Z}/2)^3$	$\oplus\ \mathbb{Z}/2$			
10		$\mathbb{Z}/2\times\mathbb{Z}/3$	$\oplus\ \mathbb{Z}/8\times(\mathbb{Z}/2)^2$			
11		$\mathbb{Z}/8\times\mathbb{Z}/9\times\mathbb{Z}/7$	$\oplus\ \mathbb{Z}/2$	$\oplus\ \mathbb{Z}/3$		
12			$\oplus\ \mathbb{Z}/4$			$\oplus\ K_{12}(\mathbb{Z})$
13	$\mathbb{Z}$	$\oplus\ \mathbb{Z}/3$				
14		$(\mathbb{Z}/2)^2$	$\oplus\ \mathbb{Z}/4$	$\oplus\ \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/9$	
15		$\mathbb{Z}/32\times\mathbb{Z}/2\times\mathbb{Z}/3\times\mathbb{Z}/5$	$\oplus\ (\mathbb{Z}/2)^2$			
16		$(\mathbb{Z}/2)^2$	$\oplus\ \mathbb{Z}/8\times\mathbb{Z}/2$		$\oplus\ \mathbb{Z}/3$	$\oplus\ K_{16}(\mathbb{Z})$
17	$\mathbb{Z}$	$\oplus\ (\mathbb{Z}/2)^4$	$\oplus\ (\mathbb{Z}/2)^2$			
18		$\mathbb{Z}/8\times\mathbb{Z}/2$	$\oplus\ \mathbb{Z}/32\times(\mathbb{Z}/2)^3$		$\oplus\ \mathbb{Z}/3\times\mathbb{Z}/5$	
19		$\mathbb{Z}/8\times\mathbb{Z}/2\times\mathbb{Z}/3\times\mathbb{Z}/11$	$\oplus\ [64]$			
20		$\mathbb{Z}/8\times\mathbb{Z}/3$	$\oplus\ [128]$		$\oplus\ \mathbb{Z}/3$	$\oplus\ K_{20}(\mathbb{Z})$
21	$\mathbb{Z}$	$\oplus\ (\mathbb{Z}/2)^2$	$\oplus\ [16]$	$\oplus\ \mathbb{Z}/3$		
22		$(\mathbb{Z}/2)^2$	$\oplus\ [2^?]$		$\oplus\ \mathbb{Z}/3$	$\oplus\ \mathbb{Z}/691$
	①	②	③	④	⑤	⑥

Table 1: The homotopy groups of $K(\mathbb{S})$ in low degrees

Beyond this range, [29, 4.7] *only describes the size of the group. In all even degrees* $\leq |\beta_2|-2 = 2(2p+1)(p-1)-4$, *the size of* $\operatorname{tor}_p(\pi_{2n}\overline{Cu})$ *is*

$$\#(\operatorname{tor}_p(\pi_{2n}\overline{Cu})) = p^{a(n)+b(n)-c(n)-d(n)+e(n)},$$

where

$$a(n) = \lfloor (n-1)/(p-1) \rfloor,$$
$$b(n) = \lfloor (n-1)/(p(p-1)) \rfloor,$$
$$c(n) = \lfloor n/p \rfloor,$$
$$d(n) = \lfloor n/p^2 \rfloor,$$

and

$$e(n) = \begin{cases} +1 & \text{if } n = p^2 - 2 + mp \text{ for } 1 \le m \le p-3, \\ -1 & \text{if } n = p - 1 + mp \text{ for } m \ge p-2, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\lfloor x \rfloor$ denotes the greatest integer $\le x$. The formulas a, b, c and d above count the number of positive integers $< n$ (in a and b) or $\le n$ (in c and d) that are divisible by the denominator.

The formula above shows that for $p = 3$, $\pi_{14}(\overline{Cu})$ has order 9. Rognes [29, 4.9(b)] shows that this group is $\mathbb{Z}/9$.

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