

OPERADIC MULTIPLICATIONS IN EQUIVARIANT SPECTRA, NORMS, AND TRANSFERS

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ABSTRACT. We study homotopy-coherent commutative multiplicative structures on equivariant spaces and spectra. We define N_∞ operads, equivariant generalizations of E_∞ operads. Algebras in equivariant spectra over an N_∞ operad model homotopically commutative equivariant ring spectra that only admit certain collections of Hill-Hopkins-Ravenel norms, determined by the operad. Analogously, algebras in equivariant spaces over an N_∞ operad provide explicit constructions of certain transfers. This characterization yields a conceptual explanation of the structure of equivariant infinite loop spaces.

To explain the relationship between norms, transfers, and N_∞ operads, we discuss the general features of these operads, linking their properties to families of finite sets with group actions and analyzing their behavior under norms and geometric fixed points. A surprising consequence of our study is that in stark contrast to the classical setting, equivariantly the little disks and linear isometries operads for a general incomplete universe U need not determine the same algebras.

Our work is motivated by the need to provide a framework to describe the flavors of commutativity seen in recent work of the second author and Hopkins on localization of equivariant commutative ring spectra.

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1. INTRODUCTION

One of the most important ideas in modern stable homotopy theory is the notion of a structured ring spectrum, an enhancement of the representing object for a multiplicative cohomology theory. A structured ring spectrum is a spectrum equipped

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with a homotopy-coherent multiplication; classically the coherence data is packaged up in an operad. When the multiplication is coherently commutative (as in the familiar examples of $H\mathbb{Z}$, ku , and MU), the classical operadic description of the multiplication involves an E_∞ operad.

May originally observed that all E_∞ operads are equivalent up to a zig-zag of maps of operads [19] and showed that equivalent E_∞ operads have equivalent homotopical categories of algebras. As an elaboration of this basic insight it is now well-understood that all possible notions of commutative ring spectrum agree. For instance, in the symmetric monoidal categories of EKMM S -modules [6] and of diagram spectra [18] (i.e., symmetric spectra and orthogonal spectra), the associated categories of commutative monoids are homotopically equivalent to the classical category of E_∞ -ring spectra [15]. Moreover, the homotopy theories of the categories of commutative monoids are equivalent to the homotopy theories of the category of algebras over any reasonable E_∞ operad.

Our focus in this paper is on equivariant generalizations of E_∞ ring spectra. At first blush, it might seem that we can give an analogous account of the situation. After all, for any compact Lie group G and universe U of finite dimensional G -representations, there is the classical notion of an equivariant E_∞ ring spectrum structured by the equivariant linear isometries operad on U [15]. For each U , there are equivariant analogues of the modern categories of spectra (i.e., equivariant orthogonal spectra and equivariant S -modules) that are symmetric monoidal categories [17, 11]. Moreover, once again commutative monoids in these categories are equivalent to classical equivariant E_∞ ring spectra (see [17, §4-5]).

However, this is not the whole story. Fix a symmetric monoidal category $\mathcal{S}p_G$ of equivariant spectra that is tensored over G -spaces and is a model of the equivariant stable homotopy category specified by a complete universe U . For any operad \mathcal{O} of G -spaces, we can form the category of \mathcal{O} -algebras in $\mathcal{S}p_G$. There are many different G -operads \mathcal{O} such that the underlying non-equivariant operad is E_∞ ; for instance, for any universe U' , the equivariant linear isometries operad over U' provides an example. Any operad with that property might be entitled to be thought of as a G - E_∞ operad. However, operadic algebras in $\mathcal{S}p_G$ over different such operads can look very different, as the following example illustrates.

Motivating Example. Let \mathcal{E} be an E_∞ operad in spaces, and view it as an operad in G -spaces by giving it the trivial G -action. Thus the n^{th} space is equivalent to $E\Sigma_n$ with a trivial G -action. Let \mathcal{E}_G denote any E_∞ operad in G -spaces for which the n^{th} space $(\mathcal{E}_G)_n$ is a universal space for $(G \times \Sigma_n)$ -bundles in G -spaces (e.g., the G -linear isometries operad for a complete universe U). Then algebras over \mathcal{E} and algebras over \mathcal{E}_G in orthogonal G -spectra are different. In fact, for almost all positive cofibrant orthogonal G -spectra E ,

$$\mathcal{E}_{n+} \wedge_{\Sigma_n} E^{\wedge n} \not\cong (\mathcal{E}_G)_{n+} \wedge_{\Sigma_n} E^{\wedge n}.$$

The easiest way to see this generic inequality is by computing the G -geometric fixed points. If $E = \Sigma^\infty G_+$, then for all n , $E^{\wedge n}$ is a free G -spectrum. This means, in particular, that the geometric fixed points of the free \mathcal{E} -algebra on E are S^0 . However, if $n = |G|$, then $(\mathcal{E}_G)_n$ has cells of the form $G \times \Sigma_n/\Gamma$, where Γ is the graph of the homomorphism $G \rightarrow \Sigma_n$ describing the left action of G on itself. The G -spectrum

$$(G \times \Sigma_n/\Gamma)_+ \wedge_{\Sigma_n} E^{\wedge n}$$

is the Hill-Hopkins-Ravenel norm $N_e^G(E)$, and in particular, the geometric fixed points are non-trivial.

Moreover, it turns out there are many intermediate classes of G -operads that structure equivariant commutative ring spectra that are richer than \mathcal{E} -algebras but are not \mathcal{E}_G -algebras. Our interest in these different notions of equivariant commutative ring spectra was motivated by recent work of Hopkins and the second author which showed that equivariantly, Bousfield localization does not necessarily take \mathcal{E}_G -algebras to \mathcal{E} -algebras. For formal reasons, the Bousfield localization of any equivariant commutative ring spectrum must have a multiplication that is an \mathcal{E} -algebra, but that is all that is guaranteed. An antecedent of this general result appears in work of McClure [22] which shows that the Tate spectrum of an \mathcal{E}_G -algebra only necessarily has a multiplication that is structured by \mathcal{E} and is usually not itself an \mathcal{E}_G -algebra.

Our goal in this paper is to provide conceptual descriptions of these intermediate multiplications on equivariant spaces and spectra in terms of the Hill-Hopkins-Ravenel norm. We do this via a careful study of the G -operads that structure intermediate multiplications, which we characterize in terms of the allowable norms on algebras over them, as suggested by the example above. For this reason, we refer to such operads as N_∞ operads.

Fix a finite group G . A G -operad \mathcal{O} consists of a sequence of $G \times \Sigma_n$ spaces \mathcal{O}_n , $n \geq 0$, equipped with a G -fixed identity element $1 \in \mathcal{O}_1$ and a composition map satisfying equivariant analogues of the usual axioms (see Definition 3.1 for details).

Definition 1.1. An N_∞ operad is a G -operad such that

- (i) The space \mathcal{O}_0 is G -contractible,
- (ii) The action of Σ_n on \mathcal{O}_n is free, and
- (iii) \mathcal{O}_n is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$.

In particular, the space \mathcal{O}_1 is also G -contractible.

Forgetting the G -action, an N_∞ operad yields a non-equivariant E_∞ operad. Examples include the equivariant little isometries operads and equivariant little disks operads; see Definition 3.6 for details.

Our main theorem is a classification of N_∞ operads in terms of the relationship between the universal spaces \mathcal{O}_n forced by the operadic structure maps. Associated to an N_∞ operad, there is a naturally defined collection (indexed by the subgroups of G) of categories of finite sets, called admissible sets. We can organize the admissible sets as follows. Define a symmetric monoidal coefficient system to be a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of symmetric monoidal categories and lax functors that assigns isomorphic categories to conjugate subgroups.

There is a canonical coefficient system that assigns to the orbit G/H the category of finite H -sets and H -maps, with symmetric monoidal product given by disjoint union. We have a poset \mathcal{I} of certain sub-coefficient systems of the canonical coefficient system, ordered by inclusion (i.e., the ones closed under Cartesian product and induction, see Definition 3.17). Let $\mathcal{N}_\infty\text{-Op}$ denote the category of N_∞ operads, regarded as a full subcategory of G -operads and G -operad maps.

Theorem 1.2. *There is a functor*

$$\underline{\mathcal{C}}: \mathcal{N}_\infty\text{-Op} \longrightarrow \mathcal{I}$$

which descends to a fully-faithful embedding

$$\underline{\mathcal{C}}: \text{Ho}(\mathcal{N}_\infty\text{-Op}) \longrightarrow \mathcal{I},$$

where the homotopy category is formed with respect to the maps of G -operads which are levelwise $G \times \Sigma_n$ -equivalences.

We conjecture that in fact this embedding is an equivalence of categories; as we explain in Section 5.2, there are natural candidate N_∞ operads to represent each object in \mathcal{I} . An interesting question is to determine if all homotopy types are realized by equivariant little disks or linear isometries operads.

Remark 1.3. The proof of the preceding theorem involves a calculation of the derived mapping space between N_∞ operads (see Proposition 5.5); in particular, we show that the space of endomorphisms of an N_∞ operad is contractible.

The import of this classification theorem is that it allows us to give a characterization of structures on algebras over N_∞ operads. The indexed product construction that underlies the norm makes sense in the symmetric monoidal category of G -spaces with the Cartesian product, where the resulting functor is simply coinduction. In this situation, we show in Theorem 7.1 that an algebra over an N_∞ operad has precisely those transfers $H \rightarrow G$ such that G/H is an admissible G -set. Specifically, we have the following result.

Theorem 1.4. *For an algebra X over a suitable N_∞ operad, the abelian group valued coefficient system*

$$\underline{\pi}_k(X): \underline{\mathcal{S}et} \longrightarrow \mathcal{A}b$$

defined by

$$(T \in \mathcal{S}et^H) \mapsto \pi_k(F(T, X)^H)$$

has transfer maps

$$f_*: \pi_k(F(T, i_H^* X)^H) \longrightarrow \pi_k(F(S, i_H^* X)^H)$$

for any H -map $f: T \rightarrow S$ of admissible H -sets and all $k \geq 0$.

These are therefore incomplete Mackey functors, studied by Lewis during his analysis of incomplete universes [12, 13].

Remark 1.5. In the result above, “suitable” refers to a certain technical property of N_∞ operads that we prove for the equivariant Steiner and linear isometries operads in Section 6.2.

In orthogonal G -spectra, we show in Theorem 6.14 that an algebra over a suitable N_∞ operad is characterized as a G -spectrum equipped with maps

$$G_+ \wedge_H N^T R \longrightarrow R$$

for the admissible H -sets T . This gives rise to the following characterization:

Theorem 1.6. *If R is an algebra over an N_∞ operad \mathcal{O} , then*

$$\underline{\pi}_0(R)$$

is a commutative Green functor.

If the \mathcal{O} action interchanges with itself, then for any admissible H -set H/K we have a “norm map”

$$\underline{\pi_0(R)}(G/K) \xrightarrow{n_K^H} \underline{\pi_0(R)}(G/H)$$

which is a homomorphism of commutative multiplicative monoids.

The maps n_K^H satisfy the multiplicative version of the Mackey double-coset formula.

Thus just as the homotopy groups of algebras in spaces over the Steiner operad on an incomplete universe gave incomplete Mackey functors with only some transfers, the zeroth homotopy group of an algebra in spectra over the linear isometries operad on an incomplete universe gives incomplete Tambara functors with only some norms.

The paper is organized as follows. In Section 2, we explain our assumptions and conventions about the kinds of operadic actions and categories of equivariant spectra we are working with. We introduce the notion of N_∞ operads in Section 3. We use this to explain in Section 4 that associated to an N_∞ operad, there is a naturally defined collection (indexed by the subgroups of G) of categories of finite sets, called admissible sets, and that if two operads have the same admissible sets, then they are equivalent. In Section 4.3, we perform a surprising computation: we show that for a generic incomplete universe, the little disks operad and the linear isometries operad are different. In Section 5 we discuss the connection between the homotopy category of N_∞ operads and the poset \mathcal{I} . In Section 6, we then show that the admissible sets correspond to indexed products that an algebra over the operad must have. In Section 7, we work out this characterization in equivariant spaces and spectra. In the case of equivariant spectra, this structure reflects the “norm-restriction” adjunction for commutative equivariant ring spectra. In the case of algebras in G -spaces over N_∞ operads, this perspective explains the transfers that arise in G -equivariant infinite loop space theory. Finally, in the appendix we collect some miscellaneous technical results: in Section A we show that weakly equivalent N_∞ operads have equivalent homotopical categories of algebras and we explain the comparison to rigid realizations of N_∞ operadic algebras in terms of equivariant EKMM spectra, and finally in Section B we describe geometric fixed points of algebras.

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2. CONVENTIONS ON OPERADIC ALGEBRAS IN EQUIVARIANT SPECTRA

Fix a finite group G . Let $\mathcal{S}p_G$ denote the category of orthogonal G -spectra [17]. Fix a complete universe U of G -representations. We will always regard $\mathcal{S}p_G$ as equipped with the homotopy theory specified by the weak equivalences detected by the equivariant stable homotopy groups indexed by U [17, III.3.2]; $\mathcal{S}p_G$ is a model of the equivariant stable category and all representation spheres are invertible [17, III.3.8]. However, the multiplicative structures we study are often described by linear isometries operads over other universes and in general the language of incomplete universes is very useful in describing N_∞ operads. The key point we

want to emphasize is that although the multiplicative structure varies, the additive structure does not.

We now want to be clear about what we mean by an operadic algebra in $\mathcal{S}p_G$. Since $\mathcal{S}p_G$ is tensored over G -spaces (with the tensor of a G -space A and an orthogonal G -spectrum E computed as $A_+ \wedge E$), we can define the category $\mathcal{S}p_G[\mathcal{O}]$ of \mathcal{O} -algebras for any operad \mathcal{O} in G -spaces. This is the notion of operadic algebra we study in this paper. (See Section A for more detail.) However, there is the potential for terminological confusion: even when \mathcal{O} is a classical G - E_∞ operad, for instance the G -linear isometries operad, the category $\mathcal{S}p_G[\mathcal{O}]$ is *not* equivalent to the classical category of G - E_∞ ring spectra [15]. The latter is defined using the category of “coordinate-free” G -spectra and the twisted half-smash product, and requires of necessity operads augmented over the G -linear isometries operad. (This terminological point is clearly explained in [21, §13].)

We could also have worked with the equivariant analogues of EKMM S -modules (e.g., see [7] for a discussion of this category) based on U . However, since we rely at various points on the homotopical analysis of the norm from [11], it is convenient for our purposes to work with orthogonal G -spectra. We have no doubt that our theorems are independent of the specific model of the equivariant stable category, however.

Finally, we note that our results have analogues in the situation when the (additive) homotopy theory on $\mathcal{S}p_G$ is indexed on an incomplete universe. However, in this situation some care must be taken. The underlying analysis was begun by Lewis [14], who analyzed the homotopy theory and groups for G -spectra on incomplete universes, and various subtleties about the connections between the additive and multiplicative structures are known to experts. In this draft, we leave the elaboration in this setting to the interested reader. However, we note that our analysis in Section 4.3 of the linear isometries operads also provides a criterion for the special case when both the additive and multiplicative universes are the same (albeit potentially incomplete).

3. EQUIVARIANT OPERADS AND INDEXING SYSTEMS

In this section, we define N_∞ operads and give a number of examples. We then move on to introduce definitions and notations for *indexing systems*, which allows us to precisely state our main result describing the homotopy category of N_∞ operads in terms of a certain poset.

3.1. Equivariant N_∞ operads. In this section we review the definitions and standard examples of G -operads that we will work with.

Definition 3.1. A G -operad \mathcal{O} consists of a sequence of $G \times \Sigma_n$ spaces \mathcal{O}_n , $n \geq 0$, such that

- (i) There is a G -fixed identity element $1 \in \mathcal{O}_1$,
- (ii) and we have G -equivariant compositions maps

$$\mathcal{O}_k \times \mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k} \longrightarrow \mathcal{O}_{n_1 + \cdots + n_k}$$

which satisfy the usual compatibility conditions with each other and with the action of the symmetric groups (see [4, 2.1]). In particular, if $n_1 = \cdots = n_k = n$, then the map is actually $(G \times \Sigma_k \wr \Sigma_n)$ -equivariant.

When $\mathcal{O}_0 = *$, we say that \mathcal{O} is a reduced operad.

Remark 3.2. Note that in contrast to the usual convention, we will treat G -operads as having left actions of symmetric groups via the inversion, as this makes certain formulas easier to understand. It also allows a simultaneous equivariant treatment of the G and Σ_n -actions.

We will primarily be interested in the equivariant analogues of E_∞ operads.

Definition 3.3. An N_∞ operad is a G -operad such that

- (i) The space \mathcal{O}_0 is G -contractible,
- (ii) The action of Σ_n on \mathcal{O}_n is free,
- (iii) and \mathcal{O}_n is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$.

In particular, the space \mathcal{O}_1 is also G -contractible.

Historically, most sources have focused on the situation where \mathcal{O}_n is a universal principle (G, Σ_n) -bundle; i.e., \mathcal{O}_n^A is nonempty and contractible for every

$$\Lambda = \{(h, \rho(h)) \mid h \in H\},$$

for $H \subset G$ and $\rho: H \rightarrow \Sigma_n$ a homomorphism (e.g., see [4]). As we shall recall, this is the analogue of restricting attention to a complete universe. We will refer to such an N_∞ operad as “complete” and follow the literature in calling these E_∞ G -operads. For any $H \subset G$, there is a forgetful functor from N_∞ operads on G to N_∞ operads on H . When $G = e$, it is clear from the definition that an N_∞ operad is an ordinary E_∞ operad.

Lemma 3.4. *The underlying non-equivariant operad for any N_∞ operad is an E_∞ operad.*

The category $\mathcal{N}_\infty\text{-Op}$ of N_∞ operads, regarded as a full subcategory of the category of G -operads and G -operad maps, is a category with weak equivalences. The weak equivalences are ultimately lifted from the homotopy theory on G -spaces where a map $f: X \rightarrow Y$ of G -spaces is a G -equivalence if the induced maps $f^H: X^H \rightarrow Y^H$ on H -fixed points are nonequivariant weak equivalences for each (closed) subgroup $H \subset G$.

Definition 3.5. A map $\mathcal{O} \rightarrow \mathcal{O}'$ of N_∞ operads is a weak equivalence if each map $\mathcal{O}(n) \rightarrow \mathcal{O}'(n)$ is an equivalence of $G \times \Sigma_n$ -spaces.

Note that this definition of weak equivalence does not generalize the usual weak equivalences on operads (i.e., the maps of operads which are underlying equivalences of spaces for each n) when $G = e$; rather, this is a generalization of Rezk’s notion of weak equivalence of operads [23, §3.2.10]. The generalization of the usual notion would lead to a weak equivalence of N_∞ operads being a levelwise G -equivalence of spaces, and under this definition the linear isometries operad on a genuine universe and any G -trivial E_∞ operad would be equivalent via a zig-zag.

We now turn to examples. The N_∞ operads which arise most frequently in equivariant algebraic topology are the linear isometries operad on a universe U and variants of the little disks operad on a universe U . To be precise, let U denote a countably infinite-dimensional real G -inner product space which contains each finite dimensional sub-representation infinitely often and for which the G -fixed points are non-empty. We emphasize that U is not assumed to be complete. Our presentation is heavily based on the excellent treatment of [8, §10]; we refer the interested reader to that paper for more discussion.

Definition 3.6.

- (i) The *linear isometries* operad $\mathcal{L}(U)$ has n^{th} space $\mathcal{L}(U^n, U)$ of (nonequivariant) linear isometries from U^n to U . The $G \times \Sigma_n$ -action is by conjugation and the diagonal action. The distinguished element $1 \in \mathcal{L}(U, U)$ is the identity map, and the structure maps are induced from composition.
- (ii) The *little disks* operad $\mathcal{D}(U)$ has n^{th} space $\mathcal{D}(U)_n$ given as the colimit of embeddings of n copies of the disk in the unit disk of a finite subrepresentation V in U . Precisely, let $D(V)$ denote the unit disk in V . A little disk is a (nonequivariant) affine map $D(V) \rightarrow D(V)$. We define $\mathcal{D}_V(U)_n$ as the space of n -tuples of nonoverlapping little disks, where G acts by conjugation on each disk and Σ_n in the obvious way. The distinguished element $1 \in \mathcal{D}_V(U)_1$ is the identity map and the structure maps are induced from composition. For $V \subseteq W$, there is a map induced by taking the disk $v \mapsto av + b$ to the disk $w \mapsto aw + b$. We define $\mathcal{D}(U) = \text{colim}_V \mathcal{D}_V(U)$.
- (iii) The *embeddings* operad can be defined as follows. Fix a real representation $V \subset U$ with G -invariant inner product, and let $\mathcal{E}(V)_n$ be the G -space of n -tuples of topological embeddings $V \rightarrow V$ with disjoint image (topologized as a G -subspace of the space of all embeddings with G acting by conjugation). The distinguished element $1 \in \mathcal{E}(V)_1$ is the identity map and the structure maps are induced by composition and disjoint union. As above, we can pass to the colimit over V .
- (iv) The *Steiner* operad $\mathcal{K}(U)$ is a (superior) variant of the little disks operad $\mathcal{D}(U)$. Fix a real representation $V \subset U$ with G -invariant inner product. Define $R_V \subset E(V)_1$ to be the G -subspace of distance reducing embeddings $f: V \rightarrow V$. A Steiner path is a map $h: I \rightarrow R_V$ with $h(1) = \text{id}$. Let P_V denote the G -space of Steiner paths (with G -action coming from the action on R_V). There is a natural projection map $\pi: P_V \rightarrow R_V$ given by evaluation at 0. Define $\mathcal{K}(V)_n$ to be the G -space of n -tuples of Steiner paths $\{h_i\}$ such that the projections $\pi(h_i)$ have disjoint images. The Steiner operad is defined to be $\mathcal{K}(U) = \text{colim}_V \mathcal{K}(V)$.

Remark 3.7. The equivariant little disks operad is unfortunately extremely poorly behaved; products of disks are not necessarily disks, and as observed in [21, §3], the colimit over inclusions $V \subseteq W$ that defines $\mathcal{D}(U)$ is not compatible with the colimit of $\Omega^V \Sigma^V$. These problems are fixed by the Steiner operad, and for these reasons the equivariant Steiner operad is preferable in most circumstances. Moreover, the Steiner operad is necessary for capturing multiplicative structures (i.e., E_∞ ring spaces) via operad pairings — there are equivariant operad pairs $(\mathcal{K}(V), \mathcal{L}(V))$ for each $V \subset U$ and $(\mathcal{K}_U, \mathcal{L}_U)$. In contrast, it does not seem possible to have an operad pairing involving the little disks operad. See [8, 10.2] for further discussion of this point.

We have the following result about the G -homotopy type of the little disks and Steiner operads [8, 9.7, 10.1].

Proposition 3.8. *Let $V \subset U$ be a real representation with G -invariant inner product. Then the n th spaces $\mathcal{D}(V)_n$ and $\mathcal{K}(V)_n$ are $G \times \Sigma_n$ -equivalent to the equivariant configuration space $F(V, n)$.*

Passing to colimits, this has the following corollary:

Corollary 3.9. *The G -operads $\mathcal{D}(U)$ and $\mathcal{K}(U)$ are N_∞ operads for any universe U .*

We have a similar result for $\mathcal{L}(U)$ when U is a complete universe. Let $\Lambda \subset G \times \Sigma_n$ be isomorphic to $H \in G$ such that H acts on $\mathcal{L}(U)$ via this isomorphism. Then the fixed points $(\mathcal{L}(U)_n)^\Lambda$ are isomorphic to the space of H -linear isometries, and (as U is a complete H -universe) this is contractible.

One of our original motivations for this paper was to understand the relationship between $\mathcal{D}(U)$ or $\mathcal{K}(U)$ and $\mathcal{L}(U)$ in the case of a general universe U . We give an answer in the spirit of Lewis' beautiful work relating dualizability of an orbit G/H to whether it embeds in the universe U [14]. The surprising conclusion of our study will be just how far apart $\mathcal{K}(U)$ and $\mathcal{L}(U)$ can be for an incomplete universe U ; see Section 4.3.

3.2. Indexing systems. There is a close connection between our N_∞ operads and certain subcategories of the categories of finite G -sets. However, as is often the case in equivariant homotopy, we never want to consider just the group G ; instead we should consider all subgroups on equal footing. This motivates the following replacement for a category.

Definition 3.10. A *categorical coefficient system* is a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of small categories.

As we will almost never be talking about abelian group valued coefficient systems in this paper, we will often abusively drop the prefix “categorical”.

Definition 3.11. A *symmetric monoidal coefficient system* is a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of symmetric monoidal categories and lax functors that assigns isomorphic categories to conjugate subgroups. If $\underline{\mathcal{C}}$ is a symmetric monoidal coefficient system, then the *value at H* is $\underline{\mathcal{C}}(G/H)$, and will often abusively be denoted $\underline{\mathcal{C}}(H)$.

For a symmetric monoidal coefficient system $\underline{\mathcal{C}}$, let

$$i_H^*: \underline{\mathcal{C}}(G) \longrightarrow \underline{\mathcal{C}}(H)$$

denote the restriction map associated to the natural map $G/H \rightarrow G/G$.

Notice that because of the requirement that the symmetric monoidal coefficient systems that we consider take equivalent values on conjugate subgroups, we need only consider maps associated to subgroup inclusions $H_1 \subset H_2 \subset G$.

We can also consider “enriched” coefficient systems that take values in enriched categories. Most of the naturally arising categories in equivariant homotopy actually sit in enriched symmetric monoidal coefficient systems.

Definition 3.12. Let $\underline{\mathcal{T}op}_{(-)}$ be the enriched coefficient system of spaces. The value at H is $\mathcal{T}op_H$, the category of H -spaces and all (not just equivariant) maps. Similarly, let $\underline{\mathcal{T}op}^{(-)}$ be the associated “level-wise fixed points”, the value at H is $\mathcal{T}op^H$, the category of H -spaces and H -maps. There are two compatible symmetric monoidal structures: disjoint union and Cartesian product.

Let $\underline{\mathcal{S}p}_{(-)}$ be the enriched coefficient system of spectra. The value at H is $\mathcal{S}p_H$, the category of H -spectra and all maps. Let $\underline{\mathcal{S}p}^{(-)}$ be the associated coefficient system whose value at H is the category of H -spectra and H -maps. We again

have two symmetric monoidal structures we can consider: wedge sum and smash product.

The most important category for our study of N_∞ operads is the coefficient system of finite G -sets.

Definition 3.13. Let $\underline{\text{Set}}$ be the symmetric monoidal coefficient system of finite sets. The value at H is Set^H , the category of finite H -sets and H -maps. The symmetric monoidal operation is disjoint union.

We will associate to every N_∞ operad a subcoefficient system of $\underline{\text{Set}}$. The operadic structure gives rise to additional structure on the coefficient system.

Definition 3.14. We say that a full sub symmetric monoidal coefficient system \mathcal{F} of $\underline{\text{Set}}$ is *closed under self-induction* if whenever $H/K \in \mathcal{F}(H)$ and $T \in \mathcal{F}(K)$, $H \times_K T \in \mathcal{F}(H)$.

Definition 3.15. Let $\mathcal{C} \subset \mathcal{D}$ be a full subcategory. We say that \mathcal{C} is a *truncation subcategory* of \mathcal{D} if whenever $X \rightarrow Y$ is monic in \mathcal{D} and Y is in \mathcal{C} , then X is also in \mathcal{C} .

A truncation sub coefficient system of a symmetric monoidal coefficient system $\underline{\mathcal{D}}$ is a sub coefficient system that is levelwise a truncation category.

In particular, for finite G -sets, truncation subcategories are those which are closed under passage to subobjects.

Definition 3.16. An *indexing system* is a truncation sub symmetric monoidal coefficient system $\underline{\mathcal{F}}$ of $\underline{\text{Set}}$ that is closed under self induction and Cartesian product.

Definition 3.17. Let $\text{Coef}(\text{Set})$ be the poset of all subcoefficient systems of $\underline{\text{Set}}$, ordered by inclusion. Let \mathcal{I} be the poset of all indexing systems.

With this, we can state our main result describing the homotopy category of N_∞ operads.

Theorem 3.18. *There is a functor*

$$\underline{\mathcal{C}}: \mathcal{N}_\infty\text{-Op} \longrightarrow \mathcal{I}$$

which descends to a fully-faithful embedding of categories

$$\underline{\mathcal{C}}: \text{Ho}(\mathcal{N}_\infty\text{-Op}) \longrightarrow \mathcal{I}.$$

In the next section, we give a construction of the functor $\underline{\mathcal{C}}$ appearing in the statement of the theorem.

4. ADMISSIBLE SETS AND N_∞ OPERADS

We now describe the functor $\underline{\mathcal{C}}$ from the category of N_∞ operads to the poset \mathcal{I} described above. This proceeds in two steps. We first define a functor, also called $\underline{\mathcal{C}}$, from symmetric sequences with an analogous universal property for their constituent spaces to the poset $\text{Coef}(\text{Set})$. We then show that if a symmetric sequence arises from an operad, then the resulting value of $\underline{\mathcal{C}}$ actually lands in \mathcal{I} .

4.1. Symmetric sequences and the functor $\underline{\mathcal{C}}$. We begin looking very generally at what sorts of families of subgroups can arise, looking only at the universal space property of the spaces in an N_∞ operad and the freeness of the Σ_n -action.

Definition 4.1. An N_∞ symmetric sequence is a symmetric sequence \mathcal{O} in G -spaces such that for each n ,

- (i) \mathcal{O}_n is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ and
- (ii) Σ_n acts freely.

In particular, the underlying symmetric sequence for an N_∞ operad is always of this form.

Our entire analysis hinges on a standard observation about the structure of subgroups of $G \times \Sigma_n$ which intersect Σ_n trivially.

Proposition 4.2. *If $\Gamma \subset G \times \Sigma_n$ is such that $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$, then there is a subgroup H of G and a homomorphism $f: H \rightarrow \Sigma_n$ such that Γ is the graph of f .*

Thus the subgroup Γ is equivalent to an H -set structure on $\underline{n} = \{1, \dots, n\}$. It will be essential to our future analysis to recast the whole story in terms of H -sets.

Definition 4.3. For an H -set T , let Γ_T denote the graph of the homomorphism $H \rightarrow \Sigma_{|T|}$ defining the H -set structure. We write that an H -set T is *admissible* for \mathcal{O} if $\Gamma_T \in \mathcal{F}_{|T|}(\mathcal{O})$.

The requirements associated to the stipulation that $\mathcal{F}_*(\mathcal{O})$ forms a family (closure under subgroups and conjugacy) translates to the following observation in terms of admissibility:

Proposition 4.4. *If an H -set T of cardinality n is admissible, then*

- (i) *for all subgroups $K \subset H$, $i_K^*(T)$ is admissible, and*
- (ii) *the gHg^{-1} -set $g \cdot T$ is admissible.*
- (iii) *every H -set isomorphic to T (as an H -set) is admissible*

Proposition 4.4 actually shows that the admissible sets assemble into a sub-coefficient system of $\underline{\mathcal{S}et}$. This allows us to define the functor $\underline{\mathcal{C}}$.

Definition 4.5. Let $\underline{\mathcal{C}}(\mathcal{O})$ denote the full subcoefficient system of $\underline{\mathcal{S}et}$ whose value at H is the full subcategory of $\mathcal{S}et_H$ spanned by the admissible H -sets.

Proposition 4.6. *If $\mathcal{O} \rightarrow \mathcal{O}'$ is a map of symmetric sequences, then*

$$\underline{\mathcal{C}}(\mathcal{O}) \subset \underline{\mathcal{C}}(\mathcal{O}').$$

Proof. Let T be an admissible set for \mathcal{O} . By definition, this means that $\mathcal{O}_{|T|}^{\Gamma_T} \neq \emptyset$. Since we have a $G \times \Sigma_{|T|}$ -equivariant map $\mathcal{O}_{|T|}$ to $\mathcal{O}'_{|T|}$, we know that the Γ_T fixed points of $\mathcal{O}'_{|T|}$ cannot be empty. \square

To refine our map, we recall the relevant notion of weak equivalence for G -symmetric sequences.

Definition 4.7. A map $f: \mathcal{O} \rightarrow \mathcal{O}'$ between G -symmetric sequences is a weak equivalence if for each n it induces a weak equivalence of $G \times \Sigma_n$ spaces.

Notice that a weak equivalence of N_∞ operads give rise to an underlying equivalence of N_∞ -symmetric sequences. Unpacking the definition immediately gives the following proposition.

Proposition 4.8. *If $f: \mathcal{O} \rightarrow \mathcal{O}'$ is a weak equivalence between N_∞ -symmetric sequences, then $\underline{\mathcal{C}}(\mathcal{O}) = \underline{\mathcal{C}}(\mathcal{O}')$.*

4.2. Symmetric monoidal structure of $\underline{\mathcal{C}}(\mathcal{O})$ and the operadic structure.
For an N_∞ operad \mathcal{O} , the spaces \mathcal{O}_n do not exist in isolation, and the structure maps on \mathcal{O} assemble to show that $\underline{\mathcal{C}}(\mathcal{O})$ has extra structure. We first show that $\underline{\mathcal{C}}(\mathcal{O})$ is never empty.

Proposition 4.9. *For all subgroups H and for all finite sets T of cardinality n , the trivial H -set T is admissible.*

Proof. This follows from condition (3) of Definition 3.3. \square

Next we see that $\underline{\mathcal{C}}(\mathcal{O})$ is a symmetric monoidal coefficient system.

Lemma 4.10. *The coefficient system $\underline{\mathcal{C}}(\mathcal{O})$ is closed under (levelwise) coproducts, and is thus a symmetric monoidal subcoefficient system of $\underline{\mathcal{S}et}$.*

Proof. We give the proof for the case of $S \amalg T$; other cases are analogous. Let $m_1 = |S|$ and $m_2 = |T|$. By definition, the fact that S and T are admissible H -sets means that there exist subgroups $\Gamma_1 \subset G \times \Sigma_{m_1}$ and $\Gamma_2 \subset G \times \Sigma_{m_2}$ which are the graphs of homomorphisms

$$f_1: H \longrightarrow \Sigma_{m_1} \quad \text{and} \quad f_2: H \longrightarrow \Sigma_{m_2}$$

respectively.

Since \mathcal{O} is an operad, we know there exists a composition map

$$\gamma: \mathcal{O}_2 \times \mathcal{O}_{m_1} \times \mathcal{O}_{m_2} \longrightarrow \mathcal{O}_{m_1+m_2}$$

which is at least $G \times (\{e\} \times \Sigma_{m_1} \times \Sigma_{m_2})$ equivariant. Let $\Gamma \subset G \times \Sigma_{m_1+m_2}$ be the subgroup specified by the graph

$$\Gamma = \{(h, f_1(h) \amalg f_2(h)) \mid h \in H\}.$$

Consider the map γ^Γ induced by passage to fixed points. On the left hand side, by hypothesis we know that the fixed points are contractible — this is true for \mathcal{O}_{m_1} and \mathcal{O}_{m_2} by admissibility, and for \mathcal{O}_2 by Proposition 4.9. Therefore, $\mathcal{O}_{m_1+m_2}^\Gamma$ cannot be empty and is therefore contractible. Translating, this means precisely that $S \amalg T$ is an admissible H -set. \square

Already we have neglected structure on the category of finite G -sets. In addition to the disjoint union, there is a Cartesian product. This is a form of the disjoint union, however, as $G/K \times G/H$ is the “disjoint union” of G/H indexed by the G -set G/K :

$$G/K \times G/H \cong \coprod_{G/K} G/H,$$

where G acts on both the indexing set and the constituents. Induction has a similar formulation as an indexed coproduct, and our admissible sets are closed under some forms of each operation.

Lemma 4.11. *For each H , the category $\mathcal{C}_H(\mathcal{O})$ is closed under Cartesian product, and thus $\underline{\mathcal{C}}(\mathcal{O})$ inherits the structure of a symmetric bimonoidal category levelwise.*

Proof. Without loss of generality, we may assume that $H = G$, and let S be an admissible G -set of cardinality m and T one of cardinality n . Associated to S is a subgroup Γ_S which is the graph of $f: G \rightarrow \Sigma_m$, and associated to T , we have a similar subgroup Γ_T and function $h: G \rightarrow \Sigma_n$. Now there is an embedding

$$\Delta: \Sigma_m \times \Sigma_n \longrightarrow \Sigma_m \wr \Sigma_n$$

which is just the diagonal on the Σ_n factor, and we let $F: G \rightarrow \Sigma_m \wr \Sigma_n$ be $\Delta \circ (f \times h)$. Finally, let $\Gamma_{S \times T}$ be the graph of F .

We need now to show two things:

- (i) that $(\mathcal{O}_m \times \mathcal{O}_n^m)^{\Gamma_{S \times T}}$ is non-empty (which in turn forces the $\Gamma_{S \times T}$ -fixed points of \mathcal{O}_{mn} to be non-empty) and
- (ii) that the function F classifies the G -set $S \times T$.

For the first part, we observe that $\Gamma_{S \times T}$ acts on $\mathcal{O}_m \times \mathcal{O}_n^m$ via its natural action on the two named factors. Thus

$$(\mathcal{O}_m \times \mathcal{O}_n^m)^{\Gamma_{S \times T}} = \mathcal{O}_m^{\Gamma_{S \times T}} \times (\mathcal{O}_n^m)^{\Gamma_{S \times T}}.$$

The action on the \mathcal{O}_m factor factors through the canonical quotient map

$$G \times \Sigma_m \wr \Sigma_n \longrightarrow G \times \Sigma_m,$$

and the image of $\Gamma_{S \times T}$ under this quotient map is Γ_S . By assumption, $\mathcal{O}_m^{\Gamma_S}$ is contractible, and hence so is $\mathcal{O}_m^{\Gamma_{S \times T}}$.

The action on the second factor is slightly more complicated. We make the following observation: the diagonal map $\mathcal{O}_n \rightarrow \mathcal{O}_n^m$ is $(G \times \Sigma_m \times \Sigma_n)$ -equivariant, where Σ_m acts trivially on the first factor and where we have identified $\Sigma_m \times \Sigma_n$ with its image under Δ . The group $\Gamma_{S \times T}$ is contained in the subgroup $G \times \text{Im}(\Delta)$, and so the diagonal map is $\Gamma_{S \times T}$ -equivariant. By construction, the action of $\Gamma_{S \times T}$ on \mathcal{O}_n is via Γ_T , and we therefore have fixed points. This implies that \mathcal{O}_n^m has $\Gamma_{S \times T}$ -fixed points as well.

For the second part, we make a simple observation: in the arrow category of finite sets, the automorphism group of the canonical projection $S \times T \rightarrow S$ is isomorphic to $\Sigma_m \wr \Sigma_n$. The Σ_m acts by permuting the base, and then the Σ_n^m acts as the automorphisms of the fibers. By our construction of F , the resulting G -set is the one in which the base is the G -set S , and where all of the fibers are the G -set T . \square

Lemma 4.12. *The symmetric monoidal coefficient system $\underline{\mathcal{C}}(\mathcal{O})$ is closed under self-induction.*

Proof. Without loss of generality, we may assume $H = G$, as for the proof given, we may simply replace all instances of G with H . Now assume that G/K is in $\mathcal{C}_G(\mathcal{O})$, and let T be in $\mathcal{C}_K(\mathcal{O})$. Let n be the cardinality of T , and let m be the index of K in G .

Associated to T is a homomorphism $\pi: K \rightarrow \Sigma_n$, and by assumption, $\mathcal{O}_n^{\Gamma_T} \simeq *$. Finally, let $g_1, \dots, g_m \in G$ be a complete set of coset representatives for G/K , and let $\sigma: G \rightarrow \Sigma_m$ be the homomorphism induced by the left action of G on G/K . Again, by assumption, $\mathcal{O}_m^{\Gamma_{G/K}} \simeq *$.

To prove the result, we must explicitly describe the induced set $G \times_K T$. The argument is standard. Since $\{g_1, \dots, g_m\}$ is a complete set of coset representatives of G , for $1 \leq i \leq n$, we have a homomorphism

$$(\sigma, (k_1, \dots, k_n)): G \longrightarrow \Sigma_m \wr K,$$

where σ and each of the functions k_i are defined by

$$g \cdot g_i = g_{\sigma(i)} k_i(g).$$

The homomorphism $G \rightarrow \Sigma_{nm}$ describing the induced set $G \times_K T$ arises from this homomorphism via the map π :

$$\text{Ind}(g) = \left(\sigma(g), (\pi(k_1(g)), \dots, \pi(k_m(g))) \right) \in \Sigma_m \wr \Sigma_n.$$

We need to now analyze the fixed points of Γ , the graph of Ind , on $\mathcal{O}_m \times (\mathcal{O}_n)^m$. The group $G \times \Sigma_m \wr \Sigma_n$ acts independently on \mathcal{O}_m and on \mathcal{O}_n^m . On \mathcal{O}_m , it acts via the canonical quotient to $G \times \Sigma_m$, and on \mathcal{O}_n^m , G acts diagonally while $\Sigma_m \wr \Sigma_n$ has the obvious action. Thus

$$\left(\mathcal{O}_m \times (\mathcal{O}_n)^m \right)^\Gamma = \mathcal{O}_m^{\Gamma_{G/K}} \times (\mathcal{O}_n^m)^\Gamma.$$

It will suffice to show that these fixed points are non-empty. The first factor is actually contractible, by assumption, so we need only produce a fixed point for the second factor. Since the Γ_T -fixed points of \mathcal{O}_n are non-empty, we can find a point $x \in \mathcal{O}_n$ such that

$$(k, \pi k) \cdot x = x$$

for all $k \in K$. Then we quickly show that

$$y = ((g_1, 1) \cdot x, \dots, (g_m, 1) \cdot x)$$

is a Γ -fixed point. To streamline the typesetting, let $\sigma = \sigma(g)$, and $k_i = k_i(g)$, and let

$$\gamma = \left(g, \sigma, (\pi(k_1), \dots, \pi(k_m)) \right).$$

Then we have a chain of equalities

$$\begin{aligned} \gamma \cdot y &= (g, 1) \cdot \left((g_{\sigma^{-1}(1)}, \pi k_{\sigma^{-1}(1)}) \cdot x, \dots, (g_{\sigma^{-1}(m)}, \pi k_{\sigma^{-1}(m)}) \cdot x \right) \\ &= \left((g \cdot g_{\sigma^{-1}(1)}, \pi k_{\sigma^{-1}(1)}) \cdot x, \dots, (g \cdot g_{\sigma^{-1}(m)}, \pi k_{\sigma^{-1}(m)}) \cdot x \right) \\ &= \left((g_1, 1)(k_{\sigma^{-1}(1)}, \pi k_{\sigma^{-1}(1)}) \cdot x, \dots, (g_m, 1)(k_{\sigma^{-1}(m)}, \pi k_{\sigma^{-1}(m)}) \cdot x \right) \\ &= ((g_1, 1) \cdot x, \dots, (g_m, 1) \cdot x) = y. \end{aligned}$$

Thus we conclude that $(\mathcal{O}_m \times (\mathcal{O}_n)^m)^\Gamma$ is non-empty, and therefore so is \mathcal{O}_{nm}^Γ . \square

One way to package Lemmata 4.10, 4.11, and 4.12 is via the G -symmetric monoidal structure on the category of finite G -sets. Induction is actually a special kind of disjoint union: we simply allow the group G to act on the indexing set (in this case G/H) for the disjoint union. Working more generally, we see that we can easily make sense of a disjoint union of $(-)$ -sets S_t indexed by a G -set T provided

- (i) S_t is a $\text{Stab}(t)$ -set and
- (ii) $S_{g \cdot t}$ is in bijective correspondence with S_t and the action of g intertwines the $\text{Stab}(t)$ and $g\text{Stab}(t)g^{-1}$ actions.

Our lemmas can then be repackaged in this language.

Corollary 4.13. *If $T \in \underline{\mathcal{C}}(\mathcal{O})(G/G)$ and if for all $t \in T$, we have an admissible $\text{Stab}(t)$ -set S_t satisfying the compatibility condition above, then*

$$\coprod_{t \in T} S_t \in \underline{\mathcal{C}}(\mathcal{O})(G/G).$$

Warning 4.14. While it is true that $\underline{\mathcal{C}}(\mathcal{O})$ forms a coefficient system and is closed under some indexed coproducts, it is not true that $\underline{\mathcal{C}}(\mathcal{O})$ is always closed under *arbitrary* induction (making it a kind of category-valued Mackey functor). The norm machinery described in Section 6.2.2 can be used to produce operads which close up $\underline{\mathcal{C}}(\mathcal{O})$ under certain inductions.

Thus far we have used only the composition structure of the operad (and hence, all of this would work in a non-unital context). For the last piece of structure, we must have a unital algebra.

Lemma 4.15. *The coefficient system $\underline{\mathcal{C}}(\mathcal{O})$ is a truncation subcoefficient system of $\underline{\mathcal{S}et}$: if $Z = S \amalg T$ is an admissible G -set, then both S and T are admissible.*

Proof. We use the unit map to show this. The admissibility of Z shows that there is a map $f: G \rightarrow \Sigma_{|Z|}$ and $\mathcal{O}_{|Z|}^{\Gamma_Z} \simeq *$. The disjoint union decomposition of Z into $S \amalg T$ shows that we can choose this map to factor through the inclusion $\Sigma_{|S|} \times \Sigma_{|T|} \subset \Sigma_{|Z|}$ (in fact, the subgroup Γ_Z corresponding to Z probably does not have this property; however, a conjugate of Γ_Z will). In this case, the projection of Γ_Z onto $G \times \Sigma_{|S|}$ realizes the subgroup Γ_S corresponding to S , and similarly for T .

We now use the composition and the identity to deduce the result. Consider the composition:

$$\mathcal{O}_{|Z|} \times \mathcal{O}_1^{|S|} \times \mathcal{O}_0^{|T|} \longrightarrow \mathcal{O}_{|S|}.$$

This map is $(G \times \Sigma_{|S|} \times \Sigma_{|T|})$ -equivariant, where on the first factor, the action is via the obvious inclusion and where the action on the target is via the quotient to $G \times \Sigma_{|S|}$. Since the map defining the G -action on Z factors through $\Sigma_{|S|} \times \Sigma_{|T|}$, the group Γ_Z is actually a subgroup of $G \times \Sigma_{|S|} \times \Sigma_{|T|}$. The Γ_Z -action on $\mathcal{O}_{|S|}$ is via the quotient $\Gamma_{|S|}$, so

$$\mathcal{O}_{|S|}^{\Gamma_Z} = \mathcal{O}_{|S|}^{\Gamma_S}.$$

Since the spaces in the operad are universal spaces for a family, it will again suffice to show that

$$(\mathcal{O}_{|Z|} \times \mathcal{O}_1^{|S|} \times \mathcal{O}_0^{|T|})^{\Gamma_Z} = \mathcal{O}_{|Z|}^{\Gamma_Z} \times (\mathcal{O}_1^{|S|} \times \mathcal{O}_0^{|T|})^{\Gamma_Z} \neq \emptyset.$$

By assumption, the first factor is non-empty. For the second, the diagonal map

$$\mathcal{O}_1 \times \mathcal{O}_0 \longrightarrow \mathcal{O}_1^{|S|} \times \mathcal{O}_0^{|T|}$$

is $\Sigma_{|S|} \times \Sigma_{|T|}$ -equivariant, with the image being the fixed points. The space $\mathcal{O}_1 \times \mathcal{O}_0$ is G -equivariantly contractible, so we know that in fact

$$\emptyset \neq (\mathcal{O}_1^{|S|} \times \mathcal{O}_0^{|T|})^{G \times \Sigma_{|S|} \times \Sigma_{|T|}} \subset (\mathcal{O}_1^{|S|} \times \mathcal{O}_0^{|T|})^{\Gamma_Z}. \quad \square$$

Corollary 4.16. *The coefficient system $\underline{\mathcal{C}}(\mathcal{O})$ is closed under finite limits.*

Proof. Equalizers are subobjects in $\underline{\mathcal{S}et}$, and Lemma 4.11 shows that each category is also closed under finite products. \square

Putting together all of these lemmas, we deduce the following theorem.

Theorem 4.17. *The functor $\underline{\mathcal{C}}$ is a functor from the homotopy category of N_∞ operads to the poset \mathcal{I} .*

4.3. Application: Linear isometries and little disks. We pause here to provide a surprising application: for all but three finite groups G , there are universes U such that the linear isometries and little disks (or Steiner) operads associated to U are inequivalent. To show this, we need only apply our functor $\underline{\mathcal{C}}$.

Theorem 4.18. *For the equivariant linear isometries operad on U , the admissible H -sets are those T such that there is an H -equivariant embedding*

$$\mathbb{Z}[T] \otimes U \longrightarrow U.$$

Proof. In fact, the statement of the theorem is a restatement of definition of the linear isometries operad. If T is an admissible H -set, then by definition

$$\mathcal{L}(U^{\oplus n}, U)^{\Gamma_T} = \mathcal{L}_{\Gamma_T}(U^{\oplus n}, U) \neq \emptyset.$$

The group Γ_T acts on U via the quotient H . The only question is how it acts on

$$U^{\oplus n} = \mathbb{Z}\{1, \dots, n\} \otimes U.$$

On the tensor factor U , the Γ_T -action is again via the quotient H . On the other tensor factor, by the definition of T , the Γ_T -action is the H -action on $\mathbb{Z}[T]$. This gives the result. \square

The truncation and disjoint union conditions on our indexing sets shows that admissibility is completely determined by the admissibility of orbits H/K . The condition for admissibility for $\mathcal{L}(U)$ then is that there is an H -equivariant embedding

$$\mathrm{Ind}_K^H i_K^* U \longrightarrow i_H^* U.$$

This requirement is actually a ‘‘cofamily’’ condition in H : if K is subconjugate to some K' in H , then $\mathbb{Z}[H/K'] \otimes U$ H -embeds into U whenever $\mathbb{Z}[H/K] \otimes U$ does.

Theorem 4.19. *For the equivariant little disks operad on U , the admissible H -sets are those T such that there is an H -equivariant embedding*

$$T \longrightarrow U.$$

Proof. This is essentially due to Lewis. An embedding of T into U can be fattened into a tiny equivariant neighborhood of T embedded into U . This is an embedding of $T \times D$ into U which is H -equivariant, and this is exactly what an element of the Γ_T -fixed points of

$$\mathcal{D} \left(\prod_1^n D, D \right) = \mathcal{D}(\{1, \dots, n\} \times D, D)$$

looks like. Just as in the linear isometries case, the existence of a single embedding is sufficient to have a contractible space. \square

The condition on the category $\underline{\mathcal{C}}(\mathcal{L}(U))$ described in Theorem 4.18 is much more stringent than the one for the category $\underline{\mathcal{C}}(\mathcal{D}(U))$ described in Theorem 4.19.

Theorem 4.20. *If G is a finite group of order bigger than 3, then there is a universe U such that $\mathcal{L}(U)$ and $\mathcal{D}(U)$ are not equivalent.*

Proof. The conditions on the group guarantee that there is a faithful representation of G with one dimensional fixed points which properly embeds in the regular representation¹. Call this representation V , and let U be the infinite direct sum of V with itself. Then by assumption, G embeds into U , so $G/\{e\}$ is an admissible G -set for $\mathcal{D}(U)$. However, U is not the infinite regular representation, since V was not the regular representation, and so $G/\{e\}$ is not an admissible G -set for $\mathcal{L}(U)$. \square

If G has order 2 or 3, then this will fail: there are only two irreducible real representations: the trivial one and multiplication by the corresponding root of unity. Thus in these cases there are only two universes: the trivial universe and the complete universe.

There is an important converse to Theorem 4.20.

Theorem 4.21. *If N is a normal subgroup in G and if U_N is the universe generated by $\mathbb{R}[G/N]$, then $\mathcal{L}(U_N)$ and $\mathcal{D}(U_N)$ are equivalent.*

This universe is the N -fixed points of the complete universe, and this statement should be viewed as an analogue of the symmetric monoidal embedding of G/N -spectra in G -spectra.

Proof. We just have to show that the admissible sets are the same in both cases, and these are the sets with stabilizer containing N . Since N is normal in G , there is no difference between restricting to H and restricting to HN , and in this case, U_N restricts to U_N but with G replaced by HN . It therefore suffices to look at those G sets which are admissible.

The admissible G -sets for $\mathcal{D}(U_N)$ are those with stabilizer H such that G/H embeds in U_N . Since G is finite,

$$\text{Emb}_G(G/H, U) = U^H - \bigcup_{H < K} U^K,$$

where $H < K$ means H is properly subconjugate to K . For all subgroups H , the H fixed points are equal to the HN -fixed points, and so if H does not contain N , there are no embeddings of G/H into U . On the other hand, if H does contain N , then the transfer shows that the H -fixed points of U_N is the universe generated by $\mathbb{R}[G/H]$. This visibly contains G/H . Thus the admissible G -sets for $\mathcal{D}(U_N)$ are those with stabilizers containing N .

For $\mathcal{L}(U)$, we need only to determine those H such that $\mathbb{Z}[G/H] \otimes U_N$ embeds in U_N . The universe U_N has the defining feature that all of N fixes U_N and such that larger subgroups of G move points. If H contains N , then the desired condition obviously holds. If H does not contain N , then N does not fix $\mathbb{Z}[G/H]$, and therefore, there are no embeddings.

Thus in both cases, the admissible G -sets are precisely those whose stabilizers contain N , and $\mathcal{D}_n(U_N)$ and $\mathcal{L}_n(U_N)$ are equivalent. \square

5. THE HOMOTOPY CATEGORY OF N_∞ OPERADS

In this section, we show that the functor $\underline{\mathcal{C}}$ is a fully-faithful embedding and explain why we believe that it is fact an equivalence.

¹If $|G|$ has prime factors bigger than 3, simply induce up the sum of a trivial representation with one of the irreducible 2-dimensional representations of C_p for $p|G|$ bigger than 3. If the only prime factors of $|G|$ are 2 and 3, then it is easy to build the desired representations by hand.

5.1. Faithfulness. We begin by recording some easy results about the relationships of coefficient systems that correspond to natural constructions on operads.

Proposition 5.1. *If \mathcal{O} and \mathcal{O}' are N_∞ operads, then $\mathcal{O} \times \mathcal{O}'$ is an N_∞ operad, and*

$$\underline{\mathcal{C}}(\mathcal{O} \times \mathcal{O}') = \underline{\mathcal{C}}(\mathcal{O}) \cap \underline{\mathcal{C}}(\mathcal{O}').$$

Proof. The only part that requires any proof is the second part; the operadic properties are straightforward. The second part is actually a standard observation in equivariant homotopy theory: if $E\mathcal{F}$ and $E\mathcal{F}'$ are universal spaces for families \mathcal{F} and \mathcal{F}' respectively, then $E \times E'$ is a universal space for $\mathcal{F} \cap \mathcal{F}'$. This follows immediately from consideration of the fixed points. The translation to the categorical version is then as above. \square

Corollary 5.2. *If $\underline{\mathcal{C}}(\mathcal{O}) \subset \underline{\mathcal{C}}(\mathcal{O}')$, then the natural projection*

$$\mathcal{O} \times \mathcal{O}' \longrightarrow \mathcal{O}$$

is a weak equivalence.

Proof. For all n , both $(\mathcal{O} \times \mathcal{O}')_n$ and \mathcal{O}_n are universal spaces for the same family of subgroups. \square

Corollary 5.3. *If $\underline{\mathcal{C}}(\mathcal{O}) = \underline{\mathcal{C}}(\mathcal{O}')$, then in the homotopy category, \mathcal{O} and \mathcal{O}' are isomorphic.*

Proof. Apply Corollary 5.2 twice. \square

Corollary 5.4. *If $\underline{\mathcal{C}}(\mathcal{O}) \subset \underline{\mathcal{C}}(\mathcal{O}')$, then in the homotopy category, we have a map*

$$\mathcal{O} \longrightarrow \mathcal{O}'.$$

In order to go further, we calculate the derived space of maps between two operads \mathcal{O} and \mathcal{O}' .

Proposition 5.5. *The derived mapping space from any G -operad \mathcal{O} to an N_∞ operad \mathcal{O}' is either empty or contractible.*

Proof. We perform the calculation in the category of G -operads in simplicial sets. Since G is discrete, there is a model structure on G -simplicial sets where the weak equivalences and fibrations are detected on passage to fixed point spaces (and the cofibrations are the monomorphisms) [23, 3.1.9]. Let $\text{SymSeq}_{G\text{Set}^{\Delta^{\text{op}}}}$ denote the category of symmetric sequences of G -simplicial sets. Since this is equivalent to the product (over $n \geq 0$) of the categories of $G \times \Sigma_n$ -simplicial sets, there is a levelwise model structure on $\text{SymSeq}_{G\text{Set}^{\Delta^{\text{op}}}}$ in which the weak equivalences and fibrations are detected pointwise. The forgetful functor from the category of G -operads in simplicial sets to $\text{SymSeq}_{G\text{Set}^{\Delta^{\text{op}}}}$ has a left adjoint free functor, and the transfer argument of [23, 3.2.10] applies to lift the model structure on $\text{SymSeq}_{G\text{Set}^{\Delta^{\text{op}}}}$ to one on G -operads in simplicial sets. Note that these model structures are simplicial and cofibrantly-generated.

Let $G\text{-Op}(\mathcal{T})$ denote the category of G -operads in topological spaces and let $G\text{-Op}(\text{Set}^{\Delta^{\text{op}}})$ denote the category of G -operads in simplicial sets. The geometric realization and singular complex functors preserve products and so induce an adjoint pair

$$\text{Sing}: G\text{-Op}(\mathcal{T}) \rightleftarrows G\text{-Op}(\text{Set}^{\Delta^{\text{op}}}): |-|.$$

Furthermore, since both of these functors preserve weak equivalences [23, 3.1.10], we can compute the derived mapping space in either category. More precisely, the fact that $|-|$ and Sing preserve equivalences and are such that the unit and co-unit of the adjunction are natural weak equivalences implies that there is a weak equivalence

$$L^H G\text{-Op}(\mathcal{T})(\mathcal{O}, \mathcal{O}') \simeq L^H G\text{-Op}(\text{Set}^{\Delta^{\text{op}}})(\text{Sing } \mathcal{O}, \text{Sing } \mathcal{O}'),$$

where L^H denotes the Dwyer-Kan simplicial mapping space.

This latter can be computed as the internal mapping space in the model category of operads in G -simplicial sets after replacing the source with a cofibrant object and the target with a fibrant object. In this model structure a cofibrant replacement of a G -operad can be computed as a retract of a cell operad. Moreover, fibrant objects are levelwise fibrant and so in particular $\text{Sing } \mathcal{O}'$ is fibrant.

Thus, we can compute the mapping space by resolving the G -operad \mathcal{O} as a cell object. That is, $\mathcal{O} = \text{colim}_n X_n$, where each stage X_n can be described as the (homotopy) pushout

$$\begin{array}{ccc} \mathbb{F}A_n & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \mathbb{F}B_n & \longrightarrow & X_{n+1}. \end{array}$$

Here \mathbb{F} is the free functor from $\text{SymSeq}_{G\text{Set}^{\Delta^{\text{op}}}}$ to $G\text{-Op}(\text{Set}^{\Delta^{\text{op}}})$. Therefore, there is an equivalence

$$\text{Map}(\mathcal{O}, -) \simeq \text{holim}_n \text{Map}(X_n, -).$$

Since the diagram \mathbb{N} is contractible, it suffices to show that $\text{Map}(X_n, -)$ is contractible. Inductively, we can use the pushout description of X_n above to reduce to the case of free G -operads. Finally, observe that maps from a free operad into any N_∞ operad are contractible or empty: by adjunction, they are computed on the level of symmetric sequences, and any N_∞ operad is made up of universal spaces. \square

We can now immediately deduce the following result.

Corollary 5.6. *The functor $\underline{\mathcal{C}}$ is a faithful embedding of $\text{Ho}(\mathcal{N}_\infty\text{-Op})$ into \mathcal{I} .*

5.2. Towards fullness. We now explain why we believe that in fact $\underline{\mathcal{C}}$ is an equivalence of categories. We will use the categorical Barratt-Eccles operad of Guillou-May [8, 2.3]. To produce operads in spaces, we simply take the geometric realization of the nerve.

Definition 5.7. The categorical Barratt-Eccles operad is defined by

$$\mathbb{O}_n = i^* \text{Map}(G, \Sigma_n),$$

where $i^*: \mathcal{E}ns \rightarrow \text{Cat}$ is the right-adjoint to the ‘‘object’’ functor.

The operadic structure maps are simply induced by the embeddings of products of symmetric groups into bigger ones.

The functor i^* assigns to each set the category whose objects are the set and for which there is a unique morphism between any pair of objects.

Remark 5.8. The operad \mathbb{O} is the norm from trivial categories to G -categories of the Barratt-Eccles operad $\underline{\Sigma}$, defined by

$$\underline{\Sigma}_n = i^* \Sigma_n.$$

From this perspective, it is immediate that \mathbb{O}_n has fixed points for all subgroups H of $G \times \Sigma_n$ for which $H \cap \Sigma_n = \{e\}$.

Associated to an element of $\mathit{Coe}f(\mathit{Set})$ is a collection of families \mathcal{F}_n of subgroups of $G \times \Sigma_n$: T is an H -set in our coefficient system if and only if Γ_T is in $\mathcal{F}_{|T|}$. Using this, we can build a sub-symmetric sequence in categories of \mathbb{O} .

Definition 5.9. If \mathcal{F}_* is a sequence of families of subgroups of $G \times \Sigma_*$, then let

$$\mathbb{O}_n^{\mathcal{F}} = i^* \{f \in \mathbb{O}_n \mid \mathit{Stab}(f) \in \mathcal{F}_n\}.$$

Since the family is closed under conjugation, for each n , $\mathbb{O}_n^{\mathcal{F}}$ is a $G \times \Sigma_n$ -subcategory of \mathbb{O} . By construction, the geometric realization of $\mathbb{O}^{\mathcal{F}}$ is an N_∞ symmetric sequence, and similarly, we immediately have the following.

Proposition 5.10. *Let \mathcal{F}_* be the sequence of families of subgroups associated to an N_∞ symmetric sequence \mathcal{O} . Then we have*

$$\underline{\mathcal{L}}(|\mathbb{O}^{\mathcal{F}}|) = \underline{\mathcal{L}}(\mathcal{O}).$$

We make the following conjecture, which would establish an equivalence of categories between $\mathit{Ho}(\mathcal{N}_\infty\text{-Op})$ and \mathcal{I} .

Conjecture 5.11. *If \mathcal{O} is an N_∞ operad and if \mathcal{F} is the associated sequence of families of subgroups, then $\mathbb{O}^{\mathcal{F}}$ is a sub-operad of \mathbb{O} .*

An interesting question (about which we do not have a conjectural answer) is whether or not all homotopy types in $\mathcal{N}_\infty\text{-Op}$ are realized by the operads that “arise in nature”, i.e., the equivariant Steiner and linear isometries operads.

6. THE STRUCTURE OF N_∞ -ALGEBRAS

Although we can consider algebras over an N_∞ operad \mathcal{O} in any symmetric monoidal category enriched over G -spaces, we are most interested in the examples of orthogonal G -spectra with the smash product and G -spaces with the Cartesian product. In both of these examples, the notion of weak equivalence of operads given in Definition 3.5 is validated by the fact that a weak equivalence of N_∞ operads induces a Quillen equivalence of the associated categories of algebras. (See Appendix A for details.) Therefore, the associated data of the coefficient system captures all of the relevant structure. We now turn to describing this structure in geometric terms.

Specifically, the name N_∞ refers to the additional structure encoded by an N_∞ operad: norms, or more precisely indexed products. In spectra with the smash product, these arise as the Hill-Hopkins-Ravenel norm, and the operadic structure encodes the analogue of the counit of the adjunction between the norms and the forgetful functors for commutative ring spectra. In spaces with the Cartesian product, these arise as coinduction, and the operadic structure maps encode the transfer in algebras over the little disks operad.

In the following definition, we use the technical device of exploiting the equivalence of categories between orthogonal G -spectra on the complete universe and

orthogonal G -spectra on a trivial universe [17, §VI.1], as pioneered in the Hill-Hopkins-Ravenel construction of the norm. Specifically, given an orthogonal G -spectrum X on a complete universe, we forget to the trivial universe, perform the construction indicated in the formula, and then left Kan extend back to the complete universe.

Definition 6.1. Let T be an G -set.

- (i) If E is an orthogonal G -spectrum, then let

$$N^T E = (G \times \Sigma_{|T|}/\Gamma_{T+}) \wedge_{\Sigma_{|T|}} E^{\wedge|T|}.$$

- (ii) If X is a G -space, then let

$$N^T X = (G \times \Sigma_{|T|}/\Gamma_T) \times_{\Sigma_{|T|}} X^{\times|T|}.$$

As stated, there is a potential conflict of notation — $N^T E$ could refer to the preceding definition or to the Hill-Hopkins-Ravenel norm. This ambiguity is resolved by the following proposition, which uses the fact that G -spaces and orthogonal G -spectra are tensored over G -spaces. If X and Y are G -spaces, we write $F(X, Y)$ to denote the space of all continuous maps from X to Y , given the conjugation G -action.

Proposition 6.2. Let T be an H -set.

- (i) Let E be an orthogonal G -spectrum. Then a decomposition $T = \coprod_i H/K_i$ gives a homeomorphism

$$(G \times \Sigma_{|T|}/\Gamma_{T+}) \wedge_{\Sigma_{|T|}} E^{|T|} \cong G_+ \wedge_H \bigwedge_i N_{K_i}^H i_{K_i}^* E,$$

where $N_{K_i}^H$ is the Hill-Hopkins-Ravenel norm.

- (ii) Let X be a G -space. Then we have a homeomorphism

$$(G \times \Sigma_{|T|}/\Gamma_T) \times_{\Sigma_{|T|}} X^{\times|T|} \cong G \times_H F(T, X).$$

Proof. The first statement is essentially the definition of the norm and is proved in [11]. The second follows immediately from the Cartesian product endowing G -spaces with a symmetric monoidal structure. \square

Proposition 6.3. The assignments

$$(T, E) \mapsto N^T(E) \quad \text{and} \quad (T, X) \mapsto N^T(X)$$

specify strong symmetric monoidal functors in both factors, and moreover we have natural homeomorphisms

$$N^{S \times T}(E) \cong N^S N^T(E) \quad \text{and} \quad N^{S \times T}(X) \cong N^S N^T(X).$$

Proof. The first part is immediate from the definition. For the second, unpacking Lemma 4.11 makes the above isomorphisms very clear. The identification of the subgroup of $\Sigma_{|S \times T|}$ associated to $\Gamma_{S \times T}$ shows that the two sides are the same. \square

6.1. The structure of \mathcal{O} -algebras. We focus on the general structure of \mathcal{O} -algebras in G -spaces and orthogonal G -spectra. For brevity of exposition, we will describe all of our maps and structure for orthogonal G -spectra herein, using the smash product. Everything we say holds *mutatis mutandis* for G -spaces using the Cartesian product.

We start with the most basic structure: an algebra over an N_∞ operad looks like an ordinary, classical algebra over a non-equivariant E_∞ operad.

Proposition 6.4. *If R is an \mathcal{O} -algebra in spectra, then R is a naive E_∞ ring spectrum in the sense that R has a multiplication that is unital, associative, and commutative up to all higher homotopy.*

Proof. Choose an ordinary, non-equivariant E_∞ operad \mathcal{E} and endow it with a trivial G -action. Since $\underline{\mathcal{C}}(\mathcal{E})$ is the initial object in \mathcal{I} , we know that we have a map from (an operad equivalent to) \mathcal{E} to \mathcal{O} . Thus any \mathcal{O} -algebra is by restriction an \mathcal{E} -algebra. \square

The other admissible sets appear as extra structure. To explain this, recall that the category of algebras over an N_∞ operad admits coproducts, and using this, we can easily deduce that the forgetful functor i_H^* has a left adjoint $i_{H!}$. We therefore have units and counits of the adjunction

$$i_{H!} \circ i_H^* \xrightarrow{\epsilon} Id \quad \text{and} \quad Id \xrightarrow{\eta} i_H^* \circ i_{H!}.$$

If we are looking at commutative ring orthogonal G -spectra, the coproduct is the smash product, and then we have a natural isomorphism of functors

$$(6.5) \quad i_{H!}(-) \cong N_H^G(-),$$

expressing the norm as the left adjoint to the forgetful functor (in fact, both functors are “induction” using the symmetric monoidal structure given by the coproduct) [11]. In orthogonal G -spectra, we have a natural isomorphism

$$i_{H!}(-) \cong G_+ \wedge_H (-),$$

and the counit map is the canonical action map.

For a general N_∞ operad, the smash product is not usually the coproduct in the category of algebras (though the coproduct is underlain by the smash product, since the restriction of any N_∞ operad is an ordinary E_∞ operad). In fact, if the coproduct were equivalent to the smash product, then for formal reasons Equation 6.5 would always hold. But the simple calculation in our motivating example shows that this is not the case when our operad is a trivial N_∞ operad. Thus in general, we will not have the “norm-restriction” adjunction given by Equation 6.5 for all $H \subset G$. We will return to this perspective several times in the remainder of the paper.

If T is an admissible H -set for \mathcal{O} with associated subgroup Γ_T , then by definition, there is a $(G \times \Sigma_{|T|})$ -contractible space of maps

$$(G \times \Sigma_{|T|})/\Gamma_T \longrightarrow \mathcal{O}_{|T|}.$$

If E is an orthogonal G -spectrum, then smashing this over $\Sigma_{|T|}$ with $E^{\wedge|T|}$ gives the following.

Proposition 6.6. *For an orthogonal G -spectrum E , we have a contractible space of maps*

$$G_+ \wedge_H N^T i_H^* E \longrightarrow \mathcal{O}_{|T|+} \wedge_{\Sigma_{|T|}} E^{\wedge|T|}.$$

This contractible space of maps gives us extra structure for an \mathcal{O} -algebra.

Lemma 6.7. *If R is an \mathcal{O} -algebra and T is an admissible H -set, then there is a contractible space of choices for a map*

$$G_+ \wedge_H N^T i_H^* R \longrightarrow R.$$

Proof. The maps in question are the composite

$$G_+ \wedge_H N^T i_H^* R \longrightarrow \mathcal{O}_{|T|+} \wedge_{\Sigma_{|T|}} R^{\wedge |T|} \longrightarrow R,$$

where the first map is any of the maps in Proposition 6.6 arising from the contractible space

$$F_{G \times \Sigma_{|T|}}(G \times \Sigma_{|T|}/\Gamma_T, \mathcal{O}_{|T|}) = \mathcal{O}_{|T|}^{\Gamma_T}. \quad \square$$

Remark 6.8. By convention, we assume that the empty set is always admissible. In this case, there is again a contractible space of choices for a map

$$G_+ \wedge_H N^{\emptyset} i_H^* R \longrightarrow R,$$

since by assumption, $N^{\emptyset} i_H^* R$ is the symmetric monoidal unit.

We can strengthen these results. Recall that the category of algebras over an E_∞ operad is homotopically tensored over finite sets in the sense that given an algebra R and a map $T \rightarrow S$ of finite sets, we have a contractible space of maps $R^{|T|} \rightarrow R^{|S|}$. A very similar result holds in this context, where here the algebras over an N_∞ operad \mathcal{O} are homotopically tensored over $\mathcal{C}_G(\mathcal{O})$.

Theorem 6.9. *If T and S are admissible G -sets and $f: T \rightarrow S$ is a G -map, then for any \mathcal{O} -algebra R , we have a contractible space of maps $N^T R \rightarrow N^S R$.*

Proof. For S a trivial G -set, this is the content of Lemma 6.7. For the general case, we observe that a general map between G sets can be written as a disjoint union of surjective maps onto orbits inside S . Disjoint unions correspond to external smash products, and hence, it suffices to consider S a single orbit and $T \rightarrow S$ surjective. This, however, can be rewritten as

$$T \longrightarrow \coprod_{|T/G|} S \longrightarrow S,$$

where the first map is the disjoint union of the surjection restricted to each orbit of T and the second is just the fold map. It will therefore suffice to show two things:

- (i) That associated to the fold map we have a contractible space of maps, and
- (ii) associated to a surjective map $G/H \rightarrow G/K$, we have a contractible space of maps.

The fold map in turn is just S times the fold map sending $|T/G|$ points with trivial G action to a single point. We have a contractible space of maps

$$R^{\wedge |T/G|} \longrightarrow R$$

by Lemma 6.7 again, applied to the trivial G -set. Taking the norm $N^S(-)$ of these produces the required contractible space of maps for the fold.

Now consider $T = G/H$ and $S = G/K$ and let the map be the canonical quotient map. By possibly composing with an automorphism of T , we may assume that H is a subgroup of K . In this case, the map is

$$G \times_K (K/H \longrightarrow K/K).$$

Since K/H is a summand of $i_K^*(G/H)$, we know that K/H is an admissible K -set. Lemma 6.7 gives us a contractible space of maps

$$N^{K/H}(i_K^*R) \longrightarrow i_K^*R.$$

Applying the functor N_K^G produces a contractible space of maps

$$N^{G/H}(R) \longrightarrow N^{G/K}(R),$$

as required. \square

Remark 6.10. One way of interpreting Theorem 6.9 is that equivariant operads should really be indexed on finite G -sets, not just (a skeleton of) finite sets. Such a definition is very natural using the perspective on ∞ -operads developed in Lurie [9] — instead of working with fibrations over Segal's category Γ , equivariant ∞ -operads should be defined as fibrations over the equivariant analogue Γ_G . We intend to return to explore this perspective in future work.

Corollary 6.11. *If S, S' , and S'' are finite admissible G -sets, and*

$$S \xrightarrow{f} S' \xrightarrow{f'} S''$$

are maps of G -sets and if R is an \mathcal{O} -algebra, then for any choice of maps coming from Theorem 6.9, the following diagram commutes up to homotopy

$$\begin{array}{ccc} N^S R & \xrightarrow{f_{\sharp}} & N^{S'} R \\ & \searrow (f' \circ f)_{\sharp} & \downarrow f'_{\sharp} \\ & & N^{S''} R. \end{array}$$

With this, we can show that the norm maps from Lemma 6.7 provide half of the data needed to show that the norm is left adjoint to the forgetful functor on the homotopy category.

Definition 6.12. Let G/H be an admissible G -set for an N_{∞} operad \mathcal{O} . If R is an \mathcal{O} -algebra, then let

$$\epsilon: N_H^G i_H^* R \longrightarrow R$$

be the composite of the canonical identification

$$N_H^G i_H^* R \cong N^{G/H} R$$

and map induced by $G/H \rightarrow G/G$.

Similarly, if Q is an $i_H^* \mathcal{O}$ -algebra, then let

$$\eta: Q \longrightarrow i_H^* N_H^G Q$$

be the composite of the map induced by $H/H \rightarrow i_H^* G/H$ and the natural identification

$$N^{i_H^* G/H} Q \cong i_H^* N_H^G Q.$$

These maps play the role that the counit and unit play for a putative adjunction of homotopy categories

$$N_H^G: \mathrm{Ho}(i_H^* \mathcal{O}\text{-Alg}) \rightleftarrows \mathrm{Ho}(\mathcal{O}\text{-Alg}): i_H^*.$$

Following Mac Lane [16, §IV.1], we need only show that the two composites

$$i_H^* R \xrightarrow{\eta_{i_H^*}} i_H^* N_H^G i_H^* R \xrightarrow{i_H^* \epsilon} i_H^* R$$

and

$$N_H^G Q \xrightarrow{N_H^G \eta} N_H^G i_H^* N_H^G Q \xrightarrow{\epsilon N_H^G} N_H^G Q$$

are homotopic to the identity. The first always holds and we prove it now. We believe that the second always holds, but we show it only for spaces in § 7.1.

Theorem 6.13. *If G/H is an admissible G -set for \mathcal{O} , then the composite*

$$i_H^* R \xrightarrow{\eta^*} i_H^* N_H^G i_H^* R \xrightarrow{i_H^* \epsilon} i_H^* R$$

is homotopic to the identity.

Proof. The map η is the map from Lemma 6.7 induced by the inclusion of H/H into $i_H^* G/H$. The map ϵ is induced by the canonical map $G/H \rightarrow G/G$, and hence restricts to the map induced by the canonical map $i_H^* G/H \rightarrow H/H$. The composite is therefore the identity on H/H , and by Corollary 6.11, the composite of the induced maps is homotopic to the identity. \square

We close by summarizing some of the compatibilities between the structure maps for an algebra over an N_∞ operad.

Theorem 6.14. *An \mathcal{O} -algebra R is an orthogonal G -spectrum with maps*

$$G_+ \wedge_H N^T i_H^* R \longrightarrow R$$

for all admissible H -sets T such that the following conditions hold.

- (i) *For all admissible G -sets S and T , the following diagram homotopy commutes*

$$\begin{array}{ccc} N^{S \amalg T} R \simeq N^S R \wedge N^T R & \longrightarrow & R \wedge R \\ & \searrow & \downarrow \\ & & R. \end{array}$$

- (ii) *For all admissible G -sets S and T , the following diagram homotopy commutes*

$$\begin{array}{ccc} N^{S \times T} R \simeq N^S N^T R & \longrightarrow & N^S R \\ & \searrow & \downarrow \\ & & R. \end{array}$$

- (iii) *For all admissible sets S and T such that for some $K \subset G$, $i_K^*(S) \cong i_K^*(T)$, the following diagram homotopy commutes*

$$\begin{array}{ccc} i_K^* N^S R \simeq N^{i_K^* S} i_K^* R & \longleftarrow & N^{i_K^* T} i_K^* R \simeq i_K^* N^T R \\ & \searrow & \swarrow \\ & i_K^* R. & \end{array}$$

In fact, all of these diagrams commute up to coherent homotopy; this coherence data is precisely the information encoded by the operad.

The first two conditions express compatibility with the multiplication and with the other norms. The third part shows that the structure is well-behaved upon passage to fixed points. We spell out a short, illuminating, example.

Example 6.15. Let $G = C_2$ (although any finite group will work here), and let \mathcal{O} denote an N_∞ operad weakly equivalent to the Steiner operad on the complete universe. By assumption, \mathcal{O}_2 is the universal space $E_{C_2}\Sigma_2$ for Σ_2 -bundles in C_2 -spaces. If we let ρ_2 denote the regular representation of C_2 and τ denote the sign representation of Σ_2 , then a cofibrant model for \mathcal{O}_2 is given by

$$S(\infty(\rho_2 \otimes \tau)) = \lim_{\substack{\longrightarrow \\ \downarrow}} S(n\rho_2 \otimes \tau).$$

Inside of this is of course $S(\rho_2 \otimes \tau)$. This has a cell structure given by

$$((C_2 \times \Sigma_2)/C_2 \amalg (C_2 \times \Sigma_2)/\Delta) \cup_f (C_2 \times \Sigma_2) \times e^1,$$

where Δ is the diagonal copy of $C_2 = \Sigma_2$, and f is the canonical quotient

$$f: (C_2 \times \Sigma_2) \times S^0 = (C_2 \times \Sigma_2) \amalg (C_2 \times \Sigma_2) \longrightarrow (C_2 \times \Sigma_2)/C_2 \amalg (C_2 \times \Sigma_2)/\Delta.$$

Thus if we have an \mathcal{O} -algebra R , then the zero cells together give a map

$$R^{\wedge 2} \wedge N^{C_2} i_e^* R \longrightarrow R,$$

while the attaching map for the one-cell identifies the restriction of the map on the first factor with the restriction of the map on the second factor.

6.2. Multiplicative action maps. Based on the example of algebras over the commutative operad, one expects that the operations parametrized by N_∞ operads are multiplicative in the sense that for any point $o \in \mathcal{O}(n)$, the induced map

$$\mu_o: X^{\wedge n} \longrightarrow X$$

is itself a map of \mathcal{O} -algebras, where the domain is given the diagonal action of \mathcal{O} . More generally, we would expect this also to hold equivariantly, where now the maps described in Lemma 6.7 and Theorem 6.9 are maps of appropriate algebras.

Classically, this situation is described via the formalism of interchange of operads [5, §1], which we review below. To study the case of Theorem 6.9, wherein we consider the norm of a map of \mathcal{O} -algebras, we need to also address the connection between algebras over the norm of an operad and the norm of algebras over an operad.

6.2.1. Interchange of operads. Recall that given an object X which is simultaneously an \mathcal{O} -algebra and an \mathcal{O}' -algebra, we say that the two actions interchange if for each point $x \in \mathcal{O}_n$, the map $X^n \rightarrow X$ is a map of \mathcal{O}' -algebras and vice-versa. We can express this relationship by requiring that the diagram

$$\begin{array}{ccc} (X^n)^m & \xrightarrow{\cong} & (X^m)^n & \xrightarrow{\beta^n} & X^n \\ \downarrow \alpha^m & & & & \downarrow \alpha \\ X^m & \xrightarrow{\beta} & & & X \end{array}$$

commute for each $\alpha \in \mathcal{O}(n)$ and $\beta \in \mathcal{O}'(m)$, where the homeomorphism is given by the permutation that takes lexicographic order to other lexicographic order.

Interchange of operads is described by the tensor product of operads; by construction, X is an \mathcal{O} -algebra and an \mathcal{O}' -algebra such that the actions interchange if and only if X is an $\mathcal{O} \otimes \mathcal{O}'$ -algebra [5, §1]. The universal property of the tensor product of operads can also be described in terms of the theory of pairings of operads [20] (see [8, §6.1] for a discussion in the equivariant setting); a pairing

$$(\mathcal{O}, \mathcal{O}') \longrightarrow \mathcal{O}''$$

is a collection of suitable coherent maps $\mathcal{O}_n \times \mathcal{O}'_m \rightarrow \mathcal{O}''_{nm}$. In this language, the tensor product is the universal recipient for pairings.

The N_∞ -condition is a homotopical one, parameterizing (as we saw above) the ways to coherently multiply elements where we allow the group to act on both the elements and on the coordinates. We therefore expect that the tensor product of N_∞ operads will always be N_∞ :

Conjecture 6.16. *If \mathcal{O} and \mathcal{O}' are N_∞ operads, then (subject to suitably cofibrancy conditions) $\mathcal{O} \otimes \mathcal{O}'$ is an N_∞ operad and moreover*

$$\underline{\mathcal{L}}(\mathcal{O} \otimes \mathcal{O}') = \underline{\mathcal{L}}(\mathcal{O}) \vee \underline{\mathcal{L}}(\mathcal{O}'),$$

where \vee denotes the least upper bound in the poset \mathcal{I} .

In particular, for any algebra over an N_∞ operad \mathcal{O} , the operad action interchanges with itself.

An immediate corollary of the definition of interchange is that when the operadic action interchanges with itself, the maps in Lemma 6.7 are maps of \mathcal{O} -algebras:

Proposition 6.17. *Let R be an algebra over an N_∞ operad \mathcal{O} , and assume that the \mathcal{O} -action interchanges with itself. Then for any admissible H -set T , the structure maps in Lemma 6.7*

$$N^T i_H^* R \longrightarrow i_H^* R$$

are maps of \mathcal{O} -algebras.

We intend to return to a general analysis of the theory of the tensor product of G -operads elsewhere. However, for the cases of most interest in applications, namely the equivariant Steiner and linear isometries operads, it is possible to verify the necessary interchange relations directly.

In [8, §10], it is shown that there is a pairing of operads

$$(\mathcal{K}(V), \mathcal{K}(W)) \longrightarrow \mathcal{K}(V \oplus W),$$

relying on an interchange map

$$\theta: \mathcal{K}_n(U) \times \mathcal{K}_m(U) \longrightarrow \mathcal{K}_{nm}(U \oplus U)$$

that takes n Steiner paths $\{k_1, \dots, k_n\}$ and m Steiner paths $\{k'_1, \dots, k'_m\}$ to the collection of the nm product paths

$$k_i \times k'_j: I \longrightarrow R_U \times R_U \subset R_{U \oplus U}.$$

ordered lexicographically. Choosing a homeomorphism $U \oplus U \rightarrow U$, we deduce the following consequence:

Proposition 6.18. *Let X be an algebra over the equivariant Steiner operad on U . Then the operad action satisfies interchange with itself.*

Corollary 6.19. *If X is an algebra over $\mathcal{K}(U)$, then for any admissible H -set T , the structure maps*

$$N^T i_H^* X \longrightarrow i_H^* X$$

are maps of $\mathcal{K}(U)$ -algebras.

Essentially the same construction works for the linear isometries operad. To be precise, given $f \in \mathcal{L}_n(U)$ and $g \in \mathcal{L}_m(U)$, we can decompose these into their components — $f: U^n \rightarrow U$ gives rise to $f_1, f_2, \dots, f_n: U \rightarrow U$ and $g: U^m \rightarrow U$

gives rise to $g_1, g_2, \dots, g_m: U \rightarrow U$. The interchange map here takes $\{f_i\}, \{g_i\}$ to the map

$$(U \oplus U)^{mn} \longrightarrow U \oplus U$$

by the lexicographic pairings $\{f_i \oplus g_j\}$. Therefore, using again a chosen homeomorphism $U \oplus U \rightarrow U$, we have the following result.

Proposition 6.20. *Let R be an algebra over the equivariant linear isometries operad on U . Then the operad action satisfies interchange with itself.*

Corollary 6.21. *If R is an algebra over $\mathcal{L}(U)$, then for any admissible H -set T , the structure maps*

$$N^T i_H^* R \longrightarrow i_H^* R$$

are maps of $\mathcal{L}(U)$ -algebras.

6.2.2. *Norms of N_∞ operads.* The maps in Theorem 6.9 are all norms of structure maps given by Lemma 6.7. We now show that they are also maps of algebras in an appropriate sense, provided our operad action interchanges with itself.

Definition 6.22. If T is a G -set and \mathcal{O} is an N_∞ operad, then let $N^T(\mathcal{O})$ be the N_∞ operad given by

$$(N^T \mathcal{O})_n = F(T, \mathcal{O}_n).$$

Naturality of the function object immediately gives the following Proposition.

Proposition 6.23. *If $f: S \rightarrow T$ is a surjective map of finite G -sets, then f^* is a map of G -operads*

$$N^T \mathcal{O} \longrightarrow N^S \mathcal{O},$$

and hence any $N^S \mathcal{O}$ -algebra is naturally a $N^T \mathcal{O}$ -algebra.

We now tie this to the algebras over an N_∞ operad \mathcal{O} , using the norm construction. One of the defining features of the norm in spectra is a homeomorphism

$$N_H^G \Sigma^\infty(X_+) \cong \Sigma^\infty(N_H^G(X)_+),$$

which follows immediately from the fact that Σ_+^∞ is a symmetric monoidal functor from spaces with Cartesian product to spectra with the smash product. Thus we expect a close connection between algebras in spaces or spectra over an N_∞ operad and those over its norm. The following proposition covers both cases immediately.

Proposition 6.24. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax symmetric monoidal functor between symmetric monoidal categories \mathcal{C} and \mathcal{D} . Given an operad \mathcal{O} in \mathcal{C} , then $F\mathcal{O}$ is an operad in \mathcal{D} , and F induces a functor*

$$\mathcal{C}[\mathcal{O}] \longrightarrow \mathcal{D}[F\mathcal{O}]$$

connecting the categories of \mathcal{O} -algebras and $F\mathcal{O}$ -algebras.

Proof. The fact that $F\mathcal{O}$ forms an operad in \mathcal{D} is a standard consequence of regarding operads as monoids in symmetric sequences; e.g., see [24, 3.3] for a more detailed discussion. To see that F induces a functor on algebras, it suffices to exhibit a natural map

$$(F\mathcal{O})FX \longrightarrow F(\mathcal{O}X)$$

in \mathcal{D} , where $(F\mathcal{O})X$ denotes the free $F\mathcal{O}$ -algebra on X . Writing this out, we want a natural map

$$\prod_{n=0}^{\infty} F\mathcal{O}(n) \otimes_{\Sigma_n} (FX)^n \longrightarrow F \left(\prod_{n=0}^{\infty} \mathcal{O}(n) \otimes_{\Sigma_n} X^n \right).$$

The lax symmetric monoidal structure of F induces a composite

$$F\mathcal{O}(n) \otimes (FX)^n \longrightarrow F\mathcal{O}(n) \otimes F(X^n) \longrightarrow F(\mathcal{O}(n) \otimes X^n),$$

and now we map this into the orbits and then the coproduct. By the universal property of the coproduct, as n varies these maps assemble into the desired map. \square

Corollary 6.25. *If R is an \mathcal{O} -algebra in spaces or spectra for an N_∞ H -operad \mathcal{O} , then $N_H^G(R)$ is naturally a $N_H^G\mathcal{O}$ -algebra.*

When the N_∞ H -operad is the restriction of a N_∞ G -operad, then we can combine Corollaries 6.23 and 6.25.

Corollary 6.26. *If R is an \mathcal{O} -algebra in spaces or spectra for a N_∞ G -operad \mathcal{O} , then $N_H^G i_H^*(R)$ is again naturally an \mathcal{O} -algebra.*

More generally, if T is a finite G -set, then $N^T(R)$ is naturally an \mathcal{O} -algebra.

This gives us the language to state the refinement of Theorem 6.9.

Corollary 6.27. *Let R be an algebra over an N_∞ operad \mathcal{O} , and assume that the \mathcal{O} -action interchanges with itself. Then for any surjective maps $S \rightarrow T$ of admissible H -sets, the structure maps in Theorem 6.9*

$$N^S i_H^* R \longrightarrow N^T i_H^* R$$

are maps of $N^T i_H^ \mathcal{O}$ -algebras.*

7. N_∞ -SPACES AND N_∞ -RING SPECTRA: TRANSFERS AND NORMS

In this section, we interpret the structure on algebras over N_∞ operads in the two cases of most interest: G -spaces and orthogonal G -spectra. In the former, the admissible sets control which transfer maps exist; this provides a conceptual interpretation of the way in which N_∞ operads controls the structure of equivariant infinite loop spaces. In the latter, the admissible sets control which norms exist; this provides a conceptual interpretation of the way in which N_∞ operads controls the structure of equivariant commutative ring spectra.

7.1. N_∞ algebras in spaces and the transfer. We begin by applying the machinery developed above to produce the transfer in algebras over an N_∞ operad in spaces. The most important examples of N_∞ operads from the point of view of spaces are the equivariant Steiner operads $\mathcal{K}(U)$, which model equivariant infinite loop spaces. The goal of this section is to describe how the transfer naturally arises from the operadic structure maps.

In this section, we state our results in terms of an operad \mathcal{O} such that the action of \mathcal{O} on any \mathcal{O} -algebra X interchanges with itself. (Recall that Proposition 6.18 tells us this is true for $\mathcal{K}(U)$.) The following is a restatement of Theorem 6.9 in the context of G -spaces.

Theorem 7.1. *If \mathcal{O} is an N_∞ operad, S and T are admissible H -sets, and $f: T \rightarrow S$ is an H -map, then for any \mathcal{O} -algebra X in G -spaces, we have a contractible space of maps*

$$F(T, i_H^* X) \longrightarrow F(S, i_H^* X),$$

and if the map f is surjective, then any choice is homotopic to a map of $N^S(\mathcal{O})$ -algebras.

Applying fixed points and passing to homotopy groups, we produce interesting maps:

Theorem 7.2. *If S and T are admissible H -sets, if $f: T \rightarrow S$ is an H -map, and if X is an \mathcal{O} -algebra in G -spaces, then there is unique, canonical (in X) map of abelian groups*

$$f_*: \pi_k(F(T, i_H^* X)^H) \longrightarrow \pi_k(F(S, i_H^* X)^H)$$

for all $k \geq 0$.

Proof. Without loss of generality, we may assume that $H = G$. If the map f is not surjective, then we may use the splitting

$$S = \text{Im}(f) \amalg S'$$

to produce a decomposition

$$F(S, X) \cong F(\text{Im}(f), X) \times F(S', X).$$

Proposition 6.4 guarantees that for all $k \geq 0$, the induced decomposition on homotopy groups of fixed points is a splitting of abelian monoids. Our map f_* is the composite of the map induced by $f: T \rightarrow \text{Im}(f)$ with the inclusion of the summand associated to S' . We therefore may assume that f is surjective.

Since the spaces in our operad are contractible, there is a unique homotopy class for the structure map given by Theorem 7.1

$$f_\#^G: F(T, X)^G \longrightarrow F(S, X)^G,$$

which gives rise to a unique map of homotopy groups:

$$f_*: \pi_k(F(T, X)^G) \longrightarrow \pi_k(F(S, X)^G).$$

Proposition 6.4 guarantees that for all $k \geq 0$, the homotopy groups of all fixed points of $N^T(X)$ are abelian monoids. It is obvious that f_* is a map of abelian groups for $k \geq 1$. Since we may assume that the $f_\#$ comes from a surjective map, our interchange assumption guarantees that the map $f_\#$ is a map of $N^S(\mathcal{O})$ -algebras. Thus f_* is a map of abelian monoids for all k . \square

Corollary 7.3. *If H/K is an admissible H -set, then associated to the canonical projection map*

$$\pi_K^H: H/K \longrightarrow H/H$$

we have a natural map of abelian monoids

$$\text{tr}_K^H = \pi_{K*}^H: \pi_k X^K \longrightarrow \pi_k X^H.$$

This map has the feel of the transfer map: on homotopy groups, we have a map that goes from the fixed points for a subgroup back to the fixed points for a larger group. We shall shortly verify that upon passage to spectra that this does give the usual transfer. Before doing so, we deduce some very nice structural corollaries from Theorem 6.14.

Proposition 7.4. *If H/K is an admissible H -set, then the double coset formula determining the restriction of tr_K^H to any subgroup K' of H holds:*

$$\text{res}_{K'}^H \text{tr}_K^H = \bigoplus_{g \in K' \backslash H/K} \text{tr}_{K' \cap gKg^{-1}}^{K'} \text{res}_{K' \cap gKg^{-1}}^K.$$

This proposition, often called the ‘‘Mackey double coset formula’’ really has a simpler interpretation: the restriction to a subgroup K' of the transfer associated to an H -set T is the transfer associated to the K' -set $i_{K'}^* T$. As such, this is an immediate consequence of Theorem 6.14 (iii).

Corollary 7.5. *For an \mathcal{O} -algebra X for which the \mathcal{O} -action interchanges with itself, the abelian group valued coefficient system*

$$\underline{\pi}_k(X): \underline{\mathcal{S}et} \longrightarrow \mathcal{A}b$$

defined by

$$(T \in \mathcal{S}et^H) \mapsto \pi_k(F(T, X)^H)$$

has transfers for any admissible sets.

These are therefore incomplete Mackey functors, studied by Lewis during his analysis of incomplete universes [12, 13].

Remark 7.6. The forgetful functor on abelian group valued coefficient systems has a right adjoint: coinduction. By the universal property of the product, we have a natural isomorphism

$$\underline{\pi}_k F(G/H, X) \cong \text{CoInd}_H^G i_H^* \underline{\pi}_k(X).$$

This can be further simplified, using the construction of coinduction:

$$\text{CoInd}_H^G i_H^* \underline{M}(T) = \underline{M}(G/H \times T).$$

This final formulation has an obvious extension to more general G -sets than orbits, and we follow Lewis’s notation

$$\underline{M}_S(T) = \underline{M}(S \times T)$$

for a fixed G -set S .

The \mathcal{O} -algebra structure that interchanges with itself endows the homotopy coefficient system of an \mathcal{O} -algebra X with natural transformations

$$\underline{\pi}_k(X)_T \longrightarrow \underline{\pi}_k(X)$$

for all admissible sets T and which commute with restriction. If all sets are admissible, then this is equivalent to a Mackey functor structure on $\underline{\pi}_k(X)$ [10].

Remark 7.7. One of the classical ways to package the data of a Mackey functor is via additive functors from the Burnside category of spans of finite G -sets into some other category. There is an ‘‘incomplete’’ version of these that can be used in our context. The appropriate notion of a ‘‘span’’ for our incomplete Mackey functors is an isomorphism class of a pair of maps $S \leftarrow U \rightarrow T$, where $U \rightarrow T$ is a pull-back of a map between admissible sets. These objects forms a subcategory of the Burnside category. A full treatment of this approach also engages with the issues from Remark 6.10 of indexing our operads on finite G -sets rather than on natural numbers. We intend to return to this issue in a subsequent paper.

Having seen that the homotopy groups of an \mathcal{O} -algebra in G -spaces have transfers analogous to those possessed by the homotopy groups of genuine spectra, we restrict attention to $\mathcal{O} = \mathcal{K}(U)$ for a universe U and show that we are in fact constructing the usual transfer. Recall that an equivariant $\mathcal{K}(U)$ -algebra X is “group-like” if $\pi_0(X^H)$ is an abelian group for all $H \subset G$. We have the following delooping result:

Proposition 7.8 ([4]). *If X is a group-like $\mathcal{K}(U)$ -algebra then there is an equivariant spectrum \mathfrak{X} indexed on U for which X is the zero space. Similarly, a map of $\mathcal{K}(U)$ -algebras $X \rightarrow Y$ deloops to a map $\mathfrak{X} \rightarrow \mathfrak{Y}$ of spectra indexed on U .*

We can now deloop any of our structure maps since, as Corollary 6.19 implies that they are infinite loop maps.

Corollary 7.9. *Fix some universe U , and let H/K be an admissible H -set for $\mathcal{K}(U)$. If X is a grouplike $\mathcal{K}(U)$ -algebra, then we have a map of spectra indexed by U :*

$$F_K(H, \mathfrak{X}) \longrightarrow i_H^* \mathfrak{X},$$

where \mathfrak{X} is the spectrum whose zero space is X , and where $F_K(H, \mathfrak{X})$ is the coinduced spectrum. Moreover, the homotopy class is unique.

In this context, we see another interpretation of Theorem 6.14 (iii). The relevant spaces in the operad \mathcal{O} parameterize the homotopies making the diagrams

$$\begin{array}{ccc} i_K^* N^S \mathfrak{X} \simeq N^{i_K^* S} i_K^* \mathfrak{X} & \longleftarrow & N^{i_K^* T} i_K^* \mathfrak{X} \simeq i_K^* N^T \mathfrak{X} \\ & \searrow & \swarrow \\ & i_K^* \mathfrak{X} & \end{array}$$

commute. This is again an incarnation of the double-coset formula.

Thus we have a collection of maps of spectra

$$F(H/K, i_H^* \mathfrak{X}) \longrightarrow i_H^* \mathfrak{X}$$

for every admissible H -set H/K , and these satisfy the Mackey double-coset formula.

At this point, we have maps of infinite loop spaces that look like the transfer. To show that these are the transfer (up to a unit), we prove a universal property analogous to that enjoyed by the transfer. Recall that in the equivariant Spanier-Whitehead category (i.e., the full subcategory of the equivariant stable category spanned by the suspension spectra of finite G -CW complexes), the transfer is a map

$$X \longrightarrow G_+ \wedge_H X$$

which realizes one of the natural maps needed to establish the adjunction

$$[i_H^* Y, X]_H \cong [Y, G_+ \wedge_H X]_G$$

in the Wirthmuller isomorphism [15, §II.5] (see also [1, 5.2] for a discussion).

Since we are working with infinite loop spaces, rather than the actual spectra, we have the dual result (arising from the fact that the natural map from the coproduct to the Cartesian product is a weak equivalence of infinite loop spaces).

Theorem 7.10. *If G/H is an admissible G -set, then we have a natural weak equivalence of functors*

$$F_H(G, -) \simeq i_{H!}(-)$$

on \mathcal{O} -algebras, arising via a natural isomorphism:

$$[F_H(G, X), Y]_G \cong [X, i_H^* Y]_H.$$

Proof. Recall the maps ϵ and η from Definition 6.12. Theorem 6.13 shows that for any \mathcal{O} -algebra Y , the composite

$$i_H^* Y \longrightarrow i_H^* F_H(G, i_H^* Y) \cong i_H^* F(G/H, Y) \longrightarrow i_H^* Y$$

is homotopic to the identity. We need only show this is so for the other composite:

$$F: F_H(G, X) \xrightarrow{F_H(G, \eta)} F_H(G, i_H^* F_H(G, X)) \xrightarrow{\epsilon} F_H(G, X).$$

This is greatly simplified by the fact that $F_H(G, -)$ is also the right adjoint to $i_H^*(-)$ in spaces (in fact, it is also so in \mathcal{O} -algebras, but we will not need or use this). We need only show that the map F is homotopic to the adjoint of the identity map. The adjoint to F is

$$i_H^* F_H(G, X) \xrightarrow{i_H^* F} i_H^* F_H(G, X) \xrightarrow{ev_H} X.$$

The map $i_H^*(F)$ live entirely in the category of $i_H^*\mathcal{O}$ -algebras, as here it is actually a composite of structure maps.

When we restrict to H , we have an H -equivariant homeomorphism

$$i_H^* F_H(G, X) \cong F(i_H^* G/H, X).$$

Under this identification, the first map in $i_H^* F$ is simply the map induced by the inclusion $H/H \rightarrow i_H^* G/H$, by definition. In a similar way, we have an equivariant homeomorphism

$$i_H^* F_H(G, i_H^* F_H(G, X)) \cong F(i_H^*(G \times_H G/H), X),$$

and hence the composite to this point is the map induced by

$$i_H^* G \times_H H/H \longrightarrow i_H^* G \times_H G/H.$$

The composite of this maps of H -sets with the restriction of the G -equivariant homeomorphism

$$G \times_H G/H \cong G/H \times G/H$$

is simply the diagonal map

$$i_H^* G/H \xrightarrow{\Delta} i_H^* G/H \times G/H.$$

Thus upon restricting, we see that the map $i_H^* F_H(G, \eta)$ is the map induced by the diagonal map of $i_H^* G/H$.

The map ϵ is also induced by a map of G -sets, in this case the projection $G/H \rightarrow G/G$. Under our identifications (and by Corollary 6.11), this makes the map $i_H^* F$ homotopic to the map induced by the composite

$$i_H^* G/H \xrightarrow{\Delta} i_H^* G/H \times i_H^* G/H \longrightarrow i_H^* G/H,$$

where the last map is the projection onto the second factor. This composite is the identity, finishing the proof. \square

The argument has a very simple algebraic interpretation. If we have an $i_H^*\mathcal{O}$ -algebra X , then the \mathcal{O} -algebra $N_H^G X$ is “generated” (up to homotopy) by those functions of the form

$$x\delta_e(g) = \begin{cases} x & g \in eH \\ * & g \notin eH, \end{cases}$$

where $*$ is any unit for the \mathcal{O} -algebra structure. The action of an element $g' \in G$ takes $x\delta_e$ to a function $x\delta_{g'}$ (where in the definition, we replace all instance of e with g'). Now since, up to homotopy, multiplication by $*$ is the identity, we have a decomposition

$$f = \prod_{g \in G/H} g \cdot f(g)\delta_e.$$

A map of \mathcal{O} -algebras is then completely and uniquely determined by what it does to the generating space X , which is the content of the theorem.

7.2. N_∞ -ring spectra and the norm. We now study the case of N_∞ algebras in orthogonal G -spectra. The arguments are essentially the same as in the preceding subsection, but the interpretation is different. Lemma 6.7 and Theorem 6.14 show that algebras over an N_∞ operad \mathcal{O} have the counit and unit maps for admissible H -sets reminiscent of those arising from the adjunction between the norm and the forgetful functor on commutative ring spectra [11]. We derive some easy computational consequences of this. The proof of the following is identical to the proof of Theorem 7.2 and Proposition 7.4, so we omit it.

Theorem 7.11. *If R is an algebra over an N_∞ operad \mathcal{O} , then*

$$\underline{\pi_0(R)}$$

is a commutative Green functor.

If the \mathcal{O} action interchanges with itself, then for any admissible H -set H/K we have a “norm map”

$$\underline{\pi_0(R)}(G/K) \xrightarrow{n_K^H} \underline{\pi_0(R)}(G/H)$$

which is a homomorphism of commutative multiplicative monoids.

The maps n_K^H satisfy the multiplicative version of the Mackey double-coset formula.

Thus just as the homotopy groups of algebras in spaces over the Steiner operad on an incomplete universe gave incomplete Mackey functors with only some transfers, the zeroth homotopy group of an algebra in spectra over the linear isometries operad on an incomplete universe gives incomplete Tambara functors with only some norms.

APPENDIX A. THE HOMOTOPY THEORY OF ALGEBRAS OVER N_∞ OPERADS IN $\mathcal{S}p_G$

In this section, we quickly present some technical results about the abstract homotopy theory of categories of algebras over N_∞ operads.

A.1. Model structure and comparison results. Given an N_∞ operad \mathcal{O} , there is an associated monad \mathbb{O} on $\mathcal{S}p_G$ formed in the usual fashion: for an object X in $\mathcal{S}p_G$, the free \mathcal{O} -algebra can be described as

$$\mathbb{O}X = \bigvee_n \mathcal{O}(n) \wedge_{\Sigma_n} X^{\wedge n}.$$

The category of \mathcal{O} -algebras is the category $\mathcal{S}p_G[\mathbb{O}]$ of algebras over the monad \mathbb{O} . For our model category results, we require a mild hypothesis on the spaces in the operad (we could equivalently assume that the operads arise as the geometric

realization of simplicial operads). We believe that in fact this sort of hypothesis is unnecessary, but we do not study that issue here.

Proposition A.1. *Let \mathcal{O} be an N_∞ operad for which each space \mathcal{O}_n is of the homotopy type of a $G \times \Sigma_n$ -CW complex. Then the category of \mathcal{O} -algebras has a model structure in which the weak equivalences and fibrations are lifted from Sp_G .*

Proof. We use the criteria of [18, 5.13], which gives conditions for a monad on a topological model category to generate lifted model structures on the associated category of algebras. These conditions then follow from the analogous arguments in [17, §III.8] for the commutative operad; the hypothesis on \mathcal{O} implies that the arguments apply with trivial modification. \square

Associated to a map $f: \mathcal{O} \rightarrow \mathcal{O}'$ of operads is an adjoint pair

$$f_! : Sp_G[\mathcal{O}] \rightleftarrows Sp_G[\mathcal{O}'] : f^*$$

where f^* is the pullback and $f_!$ is the coequalizer

$$\mathcal{O}' \circ X \rightrightarrows \mathcal{O}' X,$$

where one map is the action map on X and the other is the composite of f and the multiplication on \mathcal{O}' . In the standard model structures on $Sp_G[\mathcal{O}]$ and $Sp_G[\mathcal{O}']$, it is clear that this pair forms a Quillen adjunction since f^* clearly preserves fibrations and weak equivalences.

The following result justifies the notion of weak equivalence of N_∞ operad. The argument is a standard cellular induction starting from the easy comparison on free algebras; e.g., see [2, 3.14].

Theorem A.2. *Let $f: \mathcal{O} \rightarrow \mathcal{O}'$ be a weak equivalence of N_∞ operads. Assume that \mathcal{O}_1 and \mathcal{O}'_1 have nondegenerate G -fixed basepoints and each \mathcal{O}_n and \mathcal{O}'_n are of the homotopy type of $G \times \Sigma_n$ CW-complexes. Then the adjoint pair $(f_!, f^*)$ is a Quillen equivalence.*

A.2. Comparison to rigid commutative monoids. In the category of orthogonal spectra, the symmetric monoidal structure given by the smash product is constructed so that a commutative monoid encodes the classical homotopy-coherence data of an E_∞ ring spectrum [18]. The key technical underpinning of this comparison is the equivalence

$$(A.3) \quad (E\Sigma_i)_+ \wedge_{\Sigma_i} X^{\wedge i} \longrightarrow X^{\wedge i} / \Sigma_i,$$

for a positive cofibrant orthogonal spectrum X [18, 15.5]. Furthermore, since the category of orthogonal spectra is enriched in spaces, we can consider E_∞ objects in orthogonal spectra; these have a homotopy theory equivalent to that of commutative monoids and hence classical E_∞ ring spectra [21, 13.2].

The category of orthogonal G -spectra is also symmetric monoidal, and we have the following analogue of equation (A.3)

$$(A.4) \quad (E_G \Sigma_i)_+ \wedge_{\Sigma_i} X^{\wedge i} \longrightarrow X^{\wedge i} / \Sigma_i,$$

for a positive cofibrant orthogonal G -spectrum [17, III.8.4] (see also [11, B.49]). Once again, this implies that the homotopy theory of commutative monoids is equivalent to the homotopy theory of classical E_∞ ring spectrum (over the linear isometries operad). Moreover, we have the following comparison between algebras over complete N_∞ operads and commutative monoids in the category of orthogonal

G -spectra, which we follows from the same kind of inductive argument as Theorem A.2, using the equivalence of equation (A.3) to start the induction (i.e., to do the comparison on the free algebras).

Theorem A.5. *Let X be an algebra in orthogonal G -spectra over a complete N_∞ operad \mathcal{O} . Assume that \mathcal{O} has a nondegenerate G -fixed basepoint and each \mathcal{O}_n has the homotopy type of a $G \times \Sigma_n$ CW-complex. Then there exists a commutative monoid \tilde{X} in orthogonal G -spectra such that $X \simeq \tilde{X}$ as algebras over \mathcal{O} . (Here we are using the pullback along the terminal map from \mathcal{O} to the commutative operad to give \tilde{X} the structure of an \mathcal{O} -algebra). This correspondence is functorial, and therefore there is a Quillen equivalence between the model category $Sp_G[\mathbb{P}]$ of commutative monoids in Sp_G and the model category $Sp_G[\mathbb{O}]$ of \mathcal{O} -algebras in Sp_G .*

In fact, using the same argument we can obtain a more general comparison result on the category of orthogonal G -spectra indexed on an incomplete universe U . Specifically, there is a Quillen equivalence between algebras over the commutative operad and algebras over any N_∞ operad weakly equivalent to the G -linear isometries operad indexed on U .

APPENDIX B. ALGEBRAS AND GEOMETRIC FIXED POINTS

One of the most important constructions in equivariant stable homotopy theory is that of geometric fixed points for a normal subgroup N (e.g., see [17, §V.4]). We finish our general analysis of \mathcal{O} -algebras by describing the structure carried by their N -geometric fixed points. We let $\Phi^N(-)$ denote the point-set N -geometric fixed point functor [17, §V.4].

We first address the effect of fixed points on the operad and the admissible sets.

Lemma B.1. *Let N be a normal subgroup of G , and let \mathcal{O} be an N_∞ operad. Then*

- (i) \mathcal{O}^N is a N_∞ G/N -operad and
- (ii) the admissible H/N -sets for \mathcal{O}^N are the admissible H sets for \mathcal{O} which are fixed by N .

Proof. It is obvious that \mathcal{O}^N still forms a G -operad, and it is also clear that there are no fixed points for the symmetric factor. Both parts of the lemma then rely on understanding the way families behave upon passage to fixed points by a normal subgroup. Let Γ be a subgroup of $G/N \times \Sigma_n$. Then

$$(\mathcal{O}_n^N)^\Gamma = \mathcal{O}_n^{\pi^{-1}(\Gamma)},$$

where $\pi: G \rightarrow G/N$ is the canonical projection, is either empty or contractible. Thus \mathcal{O}_n is in fact a universal space, making \mathcal{O}^N and N_∞ G/N -operad.

For the second part, we again use the above equality of fixed points. If Γ_T corresponds to an admissible H/N -set T for \mathcal{O}^N , then the above equality shows that $\pi^{-1}(\Gamma_T)$ corresponds to an admissible H -set for \mathcal{O} . Since this contains $N \times \{1\}$, we see that this admissible H -set is simply T again, now viewed as an H -set. Thus the admissible sets for \mathcal{O}^N are precisely the admissible sets for \mathcal{O} which are fixed by N . \square

The N_∞ G/N -operad \mathcal{O}^N is also an N_∞ operad via the quotient $G \rightarrow G/N$. Thus it is a sub N_∞ operad of \mathcal{O} , and by restriction of structure, any \mathcal{O} -algebra R is also a \mathcal{O}^N -algebra. This is the heart of the following theorem.

Theorem B.2. *If R is an \mathcal{O} -algebra, then $\Phi^N(R)$ is an \mathcal{O}^N -algebra.*

Proof. Since N acts trivially on \mathcal{O}^N , the fact that Φ^N is lax symmetric monoidal gives rise to a canonical composite

$$\mathcal{O}_{n+}^N \wedge_{\Sigma_n} \Phi^N(R)^n \longrightarrow \mathcal{O}_{n+}^N \wedge_{\Sigma_n} \Phi^N(R^n) \longrightarrow \Phi^N(\mathcal{O}_{n+}^N \wedge_{\Sigma_n} R^n).$$

All of our structure maps are then induced by Φ^N applied to the structure maps for the \mathcal{O}^N -algebra R . \square

Corollary B.3. *If \mathcal{O}' is any sub N_∞ operad of \mathcal{O} on which N acts trivially and R is an \mathcal{O} -algebra, then $\Phi^N(R)$ is a \mathcal{O}' -algebra.*

In particular, in the absolute worst case possible, we choose \mathcal{O}' the G -fixed subspace. The only admissible sets are those with trivial action (and this becomes an operad modeling a “coherently homotopy commutative multiplication” with no other structure). Then Corollary B.3 shows that for any \mathcal{O} -algebra R and for any normal subgroup N , $\Phi^N(R)$ is a \mathcal{O}' -algebra and in particular, has a coherently homotopy commutative multiplication.

Remark B.4. The same statements are true for the actual fixed points, rather than the geometric fixed points. The proofs also largely carry through *mutatis mutandis*. The only change is in the proof of Theorem B.2, in which the homeomorphism comparing $\Phi^N(R)^{\wedge n}$ and $\Phi^N(R^{\wedge n})$ is replaced by a map

$$(R^N)^{\wedge n} \longrightarrow (R^{\wedge n})^N.$$

See [24] for analysis of operads obtained in this fashion.

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