*Journal of Topology* 3 (2010) 535–560 
doi:10.1112/jtopol/jtq017

# Topological Hochschild homology of Thom spectra which are $E_\infty$-ring spectra

Andrew J. Blumberg

Abstract

We identify the topological Hochschild homology (THH) of the Thom spectrum associated to an $E_\infty$ classifying map $X \to BG$ for $G$ an appropriate group or monoid (e.g. $U$, $O$, and $F$). We deduce the comparison from the observation of McClure, Schwanzl, and Vogt that THH of a cofibrant commutative $S$-algebra ($E_\infty$-ring spectrum) $R$ can be described as an indexed colimit together with a verification that the Lewis–May operadic Thom spectrum functor preserves indexed colimits and is in fact a left adjoint. We prove a splitting result $\mathrm{THH}(Mf) \simeq Mf \wedge BX_+$, which yields a convenient description of $\mathrm{THH}(MU)$. This splitting holds even when the classifying map $f : X \to BG$ is only a homotopy commutative $A_\infty$ map, provided that the induced multiplication on $Mf$ extends to an $E_\infty$-ring structure; this permits us to recover Bokstedt's calculation of $\mathrm{THH}(H\mathbb{Z})$.

## 1. *Introduction*

The algebraic $K$-theory of ring spectra encodes subtle and interesting invariants. It has long been known that the $K$-theory of ordinary rings contains a great deal of arithmetic information. On the other hand, Waldhausen showed that there is a deep connection between the $K$-theory of the sphere spectrum and the geometry of high-dimensional manifolds (as seen by pseudo-isotopy theory) [**36**]. Waldhausen's 'chromatic' program for analyzing $K(S)$ in terms of a chromatic tower of $K$-theory spectra suggests a connection between these seemingly disparate bodies of work, as such a tower can be regarded as interpolating from arithmetic to geometry [**35**]. Recently, Rognes' development of a Galois theory of $S$-algebras [**32**] and attendant generalizations of classical $K$-theoretic descent [**1**] along with Lurie's work on derived algebraic geometry [**18**] have raised the prospect of an arithmetic theory of ring spectra, which would provide a unified viewpoint on these phenomena. To gain insight into the situation, examples provided by computations of the $K$-theory of ring spectra that do not come from ordinary rings are essential.

Of course, computation of algebraic $K$-theory tends to be extremely difficult. However, for connective ring spectra, algebraic $K$-theory is in principle tractable via 'trace methods', which relates $K$-theory to the more computable topological Hochschild homology (THH) and topological cyclic homology (TC). Specifically, there is a topological lifting of the Dennis trace to a 'cyclotomic trace' map [**7**], and the fiber of this map is well understood [**11**, **28**]. Moreover, $\mathrm{TC}(R)$ is built as a certain homotopy limit of the fixed-point spectra of $\mathrm{THH}(R)$ with regard to the action of subgroups of the circle, and so is relatively computable via the methods of equivariant stable homotopy theory. One of the major early successes of this methodology was the resolution of the '$K$-theory Novikov conjecture' by Bokstedt, Hsiang, and Madsen [**7**]. Central to their results was a computation of the TC and THH of the 'group ring' $\Sigma^\infty(\Omega X)_+$ for a space $X$; these theories receive the trace map from Waldhausen's $A(X)$.

Received 15 March 2009; revised 1 March 2010; published online 14 July 2010.

2000 *Mathematics Subject Classification* 19D55, 18G55, 55P43.

The author was supported in part by an NSF postdoctoral fellowship.

Thom spectra associated to multiplicative classifying maps provide a natural generalization of the suspension spectra of monoids. Moreover, many interesting ring spectra arise naturally as Thom spectra. The purpose of this paper is to provide an explicit and conceptual description of the THH of Thom spectra which are $E_\infty$-ring spectra. As the starting point for the calculation of TC is the determination of THH, this description provides necessary input to ongoing work to understand the TC and $K$-theory of such spectra. This paper is a companion to a joint paper with Cohen and Schlichtkrull [4], which uses somewhat different methods to study the THH of Thom spectra which are $A_\infty$-ring spectra.

The operadic approach to Thom spectra of Lewis and May (see [17, 7.3; 27]) provides a Thom spectrum functor $M$ that yields structured ring spectra when given suitable input. Specifically, for suitable topological groups and monoids $G$, Lewis constructs a Thom spectrum functor

$$M : \mathcal{T}/BG \longrightarrow \mathcal{S} \setminus S$$

from the category of based spaces over $BG$ to the category $\mathcal{S} \setminus S$ of unital spectra. Furthermore, he shows that if $f : X \to BG$ is an $E_n$ map, then $Mf$ is an $E_n$-ring spectrum, where $E_n$ denotes an operad that is augmented over the linear isometries operad $\mathscr{L}$ and weakly equivalent to the little $n$-cubes operad. In particular, $M$ takes $E_\infty$ maps to $E_\infty$-ring spectra. Since $E_\infty$-ring spectra can be functorially replaced by commutative $S$-algebras, we can regard $M$ as restricting to a functor

$$M : \mathcal{T}[\mathscr{L}]/BG \longrightarrow \mathcal{CA}_S.$$

Thus, $M$ produces output that is suitable for the construction of THH.

The development of symmetric monoidal categories of spectra has made possible direct constructions of THH that mimic the classical algebraic descriptions of Hochschild homology, replacing the tensor product with the smash product. Thus, for a cofibrant $S$-algebra $R$, THH$(R)$ can be computed as the realization of the cyclic bar construction $N^{\mathrm{cyc}}R$ with respect to the smash product, where $N^{\mathrm{cyc}}R$ is the simplicial spectrum

$$[k] \to \underbrace{R \wedge R \wedge \ldots \wedge R}_{k+1}$$

with the usual Hochschild structure maps [12, 9.2.1].

Recall that the category of commutative $S$-algebras is enriched and tensored over unbased spaces, and more generally has all indexed colimits [12, 7.2.9]. When $R$ is commutative, McClure, Schwanzl, and Vogt [29] made precise an insight of Bokstedt's that there should be a homeomorphism

$$|N^{\mathrm{cyc}}R| \cong R \otimes S^1.$$

Here $R \otimes S^1$ denotes the tensor of the commutative $S$-algebra $R$ with the unbased space $S^1$. Thus, we can describe THH$(Mf)$ by studying $Mf \otimes S^1$.

The category of $\mathscr{L}$-spaces is also tensored over unbased spaces, and this induces a tensored structure on the category of $\mathscr{L}$-maps $f : X \to BG$. Our first main theorem, proved in Section 5, states that the Thom spectrum functor is compatible with the topologically tensored structures on its domain and range categories.

Theorem 1.1. *The Thom spectrum functor*

$$M : \mathcal{T}[\mathscr{L}]/BG \longrightarrow \mathcal{CA}_S$$

*preserves indexed colimits and in fact is a continuous left adjoint. In particular, for an unbased space $A$ and an $\mathscr{L}$-map $X \to BG$, there is a homeomorphism*

$$M(f \otimes A) \cong Mf \otimes A.$$

*When $G$ is a group, $M$ is a Quillen left adjoint with respect to the standard model structures on $\mathcal{T}[\mathscr{L}]/BG$ and $\mathcal{CA}_S$.*

This theorem follows from an appropriate categorical viewpoint on the Thom spectrum functor. The category of $\mathscr{L}$-spaces can be regarded as the category $\mathcal{T}[\mathbb{K}]$ of algebras over a certain monad $\mathbb{K}$ on the category $\mathcal{T}$ of based spaces. We can utilize this description to describe the category of $\mathscr{L}$-maps $X \to BG$ as the category $(\mathcal{T}/BG)[\mathbb{K}_{BG}]$ of algebras over a closely related monad $\mathbb{K}_{BG}$. Similarly, the category of $E_\infty$-ring spectra can be regarded as the category $(\mathcal{S} \setminus S)[\tilde{\mathbb{C}}]$ of algebras over a monad $\tilde{\mathbb{C}}$ on the category $\mathcal{S} \setminus S$ of unital spectra. Each of these categories admits the structure of a topological model category, by which we mean a model category structure compatible with an enrichment in spaces [**12**, 7.2–7.4]. In particular, each of these categories has tensors with unbased spaces.

Furthermore, the work of Lewis [**17**, 7] describes the interaction of $M$ with these monads. Specifically, Lewis [**17**, 7.7.1] show that

$$M\mathbb{K}_{BG}f \cong \tilde{\mathbb{C}}Mf$$

and, moreover, that in fact $M$ takes the monad $\mathbb{K}_{BG}$ to the monad $\tilde{\mathbb{C}}$ (that is, the indicated isomorphism is suitably compatible with the monad structure maps). In Section 2, we study this situation more generally and prove the following result about the preservation of indexed colimits by induced functors on categories of monadic algebras; Theorem 1.1 is then a straightforward consequence.

Theorem 1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories tensored over unbased spaces and let $\mathbb{M}_A$ be a continuous monad on $A$ and $\mathbb{M}_B$ be a continuous monad on $B$, such that $\mathbb{M}_A$ and $\mathbb{M}_B$ preserve reflexive coequalizers. Let $F : \mathcal{A} \to \mathcal{B}$ be a continuous functor such that:*
1. *$F$ preserves colimits and tensors;*
2. *there is an isomorphism $F\mathbb{M}_A X \cong \mathbb{M}_B F X$ which is compatible with the monad structure maps.*

*Then $F$ restricts to a functor*

$$F_{\mathbb{M}} : \mathcal{A}[\mathbb{M}_A] \longrightarrow \mathcal{B}[\mathbb{M}_B],$$

*which preserves colimits and tensors. If $F$ is a left adjoint, then $F_{\mathbb{M}}$ is also a left adjoint.*

Remark 1.3. Note that a consequence of Theorem 1.1 is that the Thom spectrum functor from $\mathscr{L}$-spaces over $BG$ to commutative $S$-algebras commutes with homotopy colimits. Essentially the same result has also been obtained in the ‘symmetric Thom spectrum’ setup of Schlichtkrull [**34**], where it is used to analyze the interaction of the Thom spectrum construction with ‘higher’ topological Hochschild homology.

In order to use the formula $M(f \otimes S^1) \cong Mf \otimes S^1$ provided by Theorem 1.1 to compute $\mathrm{THH}(Mf)$, we must first ensure that we have homotopical control over $Mf$. Two technical issues arise. First, the cyclic bar construction description of $\mathrm{THH}(R)$ only has the correct homotopy type when the point-set smash product $R \wedge R$ represents the derived smash product (for instance, if $R$ is cofibrant as a commutative $S$-algebra). Second, when working over $BF$, Lewis’ construction of the Thom spectrum functor preserves weak equivalences only for certain classifying maps (‘good’ maps), notably Hurewicz fibrations.

We show in Section 6 that, by appropriate cofibrant replacement of $f : X \to BG$, we can ensure that $Mf$ is suitable for computing the derived smash product. The second problem can be handled by the classical device of functorial replacement by a Hurewicz fibration. Unfortunately, it turns out to be complicated to analyze the interaction of these two replacements. In the companion paper [4] we discuss the technical details of the interaction between these processes. In the present context, we are able to obtain our main applications without confronting this issue; although with the tools described herein the next result is only practically applicable when $G$ is a group, in which case all maps are good, the splitting in Theorem 1.6 holds more generally.

COROLLARY 1.4. *Let $f : X \to BG$ be a good map of $\mathscr{L}$-spaces such that $X$ is a cofibrant $\mathscr{L}$-space. Then $\mathrm{THH}(Mf)$ and $M(f \otimes S^1)$ are isomorphic in the derived category.*

Just as $R \otimes S^1$ is the cyclic bar construction in the category of commutative $S$-algebras, for an $\mathscr{L}$-space $X$ we can similarly regard $X \otimes S^1$ as a cyclic bar construction [3, 6.7]. Unlike commutative $S$-algebras, $\mathscr{L}$-spaces are tensored over based spaces and the tensor with an unbased space is constructed by adjoining a disjoint basepoint. Thus, for an $\mathscr{L}$-space $X$ it is preferable to think of the unbased tensor $X \otimes S^1$ as the based tensor $X \otimes S^1_+$. This description allows us to construct a natural map to the free loop space

$$X \otimes S^1_+ \longrightarrow L(X \otimes S^1),$$

which is a weak equivalence when $X$ is group-like. Note that the based tensor $X \otimes S^1$ is a model of the classifying space of $X$, so that we have recovered the familiar relationship between $N^{\mathrm{cyc}}X$ and $L(BX)$ (see [7]). Furthermore, in Section 7 we use the stable splitting of $S^1_+$ to provide an extremely useful splitting of $\mathrm{THH}(Mf)$.

THEOREM 1.5. *Let $f : X \to BG$ be a good map of $\mathscr{L}$-spaces such that $X$ is a cofibrant and group-like $\mathscr{L}$-space. Then there is a weak equivalence of commutative $S$-algebras*

$$\mathrm{THH}(Mf) \simeq Mf \wedge BX_+.$$

This theorem provides convenient formulas describing THH for various bordism spectra, notably

$$\mathrm{THH}(MU) \simeq MU \wedge BBU_+.$$

Furthermore, we show that this splitting theorem holds when $f : X \to BG$ is only an $E_2$ map, provided that the induced multiplicative structure on $Mf$ 'extends to' an $E_\infty$-structure. In this context, the result follows from a separate analysis that exploits the multiplicative equivalence

$$Mf \wedge Mf \simeq Mf \wedge X_+$$

induced by the Thom isomorphism. Note that in the statement of the following theorem we do not require $X$ to be cofibrant, and so we can always arrange for $f$ to be a good map.

THEOREM 1.6. *Let $\mathcal{C}_2$ denote an $E_2$-operad augmented over the linear isometries operad and let $f : X \to BG$ be a good $\mathcal{C}_2$ map such that $X$ is group-like. Assume that there is a map $\gamma : Mf \to M'$ that is a weak equivalence of homotopy commutative $S$-algebras such that $M'$ is a commutative $S$-algebra. Then there is a weak equivalence of $S$-modules*

$$\mathrm{THH}(Mf) \simeq Mf \wedge BX_+.$$

Although the hypotheses of this theorem may seem strange, in fact this situation arises in nature. It has long been known that $H\mathbb{Z}/2$ is the Thom spectrum of an $E_2$ map $f : \Omega^2 S^3 \to BO$

(see [**10**, **19**]). There is a similar construction of $H\mathbb{Z}/p$ for odd primes due to Hopkins, which is described in [**20**]. Constructions of $H\mathbb{Z}$ as a Thom spectrum over $\Omega^2 S^3\langle 3\rangle$ are also well known [**10**, **19**], but these descriptions only yield an $H$-space structure on $H\mathbb{Z}$.

In Section 9, we provide a new construction of $H\mathbb{Z}$ as the Thom spectrum associated to an $E_2$ map. Then Theorem 1.6 allows us to recover the classical computations of Bokstedt of $\mathrm{THH}(\mathbb{Z}/2)$, $\mathrm{THH}(\mathbb{Z}/p)$, and $\mathrm{THH}(\mathbb{Z})$.

## 2. *Colimit-preserving functors in categories of monadic algebras*

In this section, we prove Theorem 1.2. The theorem is essentially a straightforward consequence of categorical results due to Kelly describing the construction of colimits and tensors in enriched categories of monadic algebras. We begin by briefly reviewing the relevant background material, largely following the exposition of [**12**].

Let $\mathcal{V}$ denote a symmetric monoidal category and let $\mathcal{C}$ be a category enriched over $\mathcal{V}$. In such a context we can define tensors and cotensors (and more generally enriched or indexed colimits and limits). The tensors and cotensors are particularly important in the setting of topological categories, as a consequence of the following result of Kelly [**12**, 7.2.6].

Theorem 2.1. *A topological category has all indexed colimits, provided that it is cocomplete and tensored. Dually, a topological category has all indexed limits, provided that it is complete and cotensored.*

For our application, we need to understand the tensor in the category of commutative $S$-algebras and the tensor in the category of $E_\infty$ spaces. Unlike in the case of spectra, where the tensor of an unbased space $A$ and a spectrum $X$ is the smash product $X \wedge A_+$, there is not a familiar construction that yields the tensor. For that matter, construction of colimits in these categories is not obvious either. The key observation of McClure and Hopkins [**13**], further developed in [**12**], is that since these categories admit descriptions as algebras over monads, we can apply general constructions for lifting colimits and tensors from a category $\mathcal{C}$ to the category $\mathcal{C}[\mathbb{A}]$ of algebras for a monad $\mathbb{A}$ on $\mathcal{C}$; that is, colimits and tensors in $\mathcal{C}[\mathbb{A}]$ can be constructed in terms of certain colimits and tensors in $\mathcal{C}$.

The category of commutative $S$-algebras is precisely the category of algebras over a certain monad in $S$-modules, and the category of $\mathscr{L}$-spaces is the category of algebras over a certain monad in based spaces. Moreover, these monads preserve reflexive coequalizers. (This latter technical condition is needed in order to apply the lifting results.)

Now, let $F : \mathcal{C} \to \mathcal{D}$ be a functor between topological categories, let $\mathbb{A} : \mathcal{C} \to \mathcal{C}$ be a monad on $\mathcal{C}$, and let $\mathbb{B} : \mathcal{D} \to \mathcal{D}$ be a monad on $\mathcal{D}$. The following easy lemma provides a simple condition for $F$ to yield a functor on the associated categories of algebras, $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$.

Lemma 2.2. *Let $\phi : \mathbb{B}F(X) \cong F(\mathbb{A}X)$ be a natural isomorphism such that the following diagrams commute for any object $X$ of $\mathcal{C}$:*

$$\begin{array}{ccc} \mathbb{B}F(X) & \xrightarrow{\phi} & F(\mathbb{A}X) \\ & \nwarrow^{\eta_B} & \uparrow{\scriptstyle F(\eta_A)} \\ & & F(X) \end{array} \qquad \begin{array}{ccc} \mathbb{B}\mathbb{B}F(X) & \xrightarrow{\mu_B} & \mathbb{B}F(X) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ F(\mathbb{A}\mathbb{A}X) & \xrightarrow{F(\mu_A)} & F(\mathbb{A}X) \end{array}$$

*Then, if $X$ is a $\mathbb{A}$-algebra in $\mathcal{C}$ with an action map $\psi : \mathbb{A}X \to X$, we have that $F(X)$ is a $\mathbb{B}$-algebra in $\mathcal{D}$ with the action map*

$$\mathbb{B}F(X) \cong F(\mathbb{A}X) \xrightarrow{F(\psi)} F(X)\ .$$

*Therefore $F$ yields a functor from $\mathcal{C}[\mathbb{A}]$ to $\mathcal{C}[\mathbb{B}]$.*

We now prove Theorem 1.2. We assume that we are in the situation described in the preceding lemma, with the additional hypothesis that $\mathcal{C}$ and $\mathcal{D}$ are topological categories.

Theorem 2.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be cocomplete topological categories, and $\mathbb{A} : \mathcal{C} \to \mathcal{C}$ and $\mathbb{B} : \mathcal{D} \to \mathcal{D}$ be continuous monads. Further, suppose that there is a continuous functor $F : \mathcal{C} \to \mathcal{D}$ that satisfies the hypothesis of the preceding lemma and therefore yields a functor $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$. Then the following conditions hold.*

(i) *If $F : \mathcal{C} \to \mathcal{D}$ preserves colimits and tensors, and the monads $\mathbb{A}$ and $\mathbb{B}$ preserve reflexive coequalizers, then $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$ preserves colimits and tensors in $\mathcal{C}[\mathbb{A}]$. Therefore $F$ preserves all indexed colimits in $\mathcal{C}$.*

(ii) *Furthermore, if $F$ is a left adjoint as a functor from $\mathcal{C}$ to $\mathcal{D}$, then $F$ induces a left adjoint from $\mathcal{C}[\mathbb{A}]$ to $\mathcal{D}[\mathbb{B}]$.*

*Proof.* First, we handle the issue of colimits. We can apply [**12**, 2.7.4] to describe colimits in the category $\mathcal{C}[\mathbb{A}]$ of $\mathbb{A}$-algebras. Given a diagram of $\{R_i\}$ of $\mathbb{A}$-algebras, we can describe $F(\operatorname{colim} R_i)$ as $F$ applied to the reflexive coequalizer that creates colimits in the category $\mathcal{C}[\mathbb{A}]$.

$$F\left(\mathbb{A}(\operatorname{colim} \mathbb{A}R_i) \underset{\mu\circ\mathbb{A}\alpha}{\overset{\mathbb{A}(\operatorname{colim}\xi_i)}{\rightrightarrows}} \mathbb{A}(\operatorname{colim} R_i)\right).$$

Since $F$ commutes with colimits in $\mathbb{A}$, this is homeomorphic to the reflexive coequalizer

$$\mathbb{B}(\operatorname{colim} \mathbb{B}FR_i) \underset{\mu\circ\mathbb{B}F(\alpha)}{\overset{\mathbb{B}(\operatorname{colim} F(\xi_i))}{\rightrightarrows}} \mathbb{B}(\operatorname{colim} FR_i)\ .$$

This is precisely the colimit of the diagram $\{FR_i\}$ in the category of $\mathbb{B}$-algebras by [**12**, 2.7.4] once again.

Next, we consider tensors. We can express $F(X \otimes A)$ as $F$ applied to the reflexive coequalizer that creates the tensors in the category $\mathcal{C}[\mathbb{A}]$ as follows:

$$F\left(\mathbb{A}(\mathbb{A}X \otimes A) \underset{\mu\circ\mathbb{A}\nu}{\overset{\mathbb{A}(\xi\otimes\mathrm{id})}{\rightrightarrows}} \mathbb{A}(X \otimes A)\right).$$

We can rewrite this expression using the fact that $F$ commutes with colimits in $\mathcal{A}$, as follows:

$$\mathbb{B}F(\mathbb{A}X \otimes A) \underset{\mu\circ\mathbb{B}\nu}{\overset{\mathbb{B}(\xi\otimes\mathrm{id})}{\rightrightarrows}} \mathbb{B}F(X \otimes A)\ .$$

As $F$ commutes with tensors in $\mathcal{A}$, this becomes

$$\mathbb{B}(\mathbb{B}FX \otimes A) \underset{\mu\circ\mathbb{B}\nu}{\overset{\mathbb{B}(\mathbb{B}(\xi\otimes\mathrm{id})}{\rightrightarrows}} \mathbb{B}(FX \otimes A)\ .$$

This is precisely the diagram expressing the tensor $FX \otimes A$ in the category $\mathcal{C}[\mathbb{B}]$. It is now a consequence of Theorem 2.1 that $M$ preserves all indexed colimits.

Finally, assume that $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint. There is a diagram of categories

$$\begin{array}{ccc} \mathcal{C}[\mathbb{A}] & \xrightarrow{F} & \mathcal{D}[\mathbb{B}] \\ {\scriptstyle U}\downarrow\uparrow{\scriptstyle G} & & {\scriptstyle V}\downarrow\uparrow{\scriptstyle G} \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D}. \end{array}$$

Here $U$ and $V$ denote forgetful functors and $G$ denotes the free algebra functors. The square commutes in the sense that $F \circ G = G \circ F$ and $F \circ U = V \circ F$. To show that $F : \mathcal{C}[\mathbb{A}] \to \mathcal{D}[\mathbb{B}]$ is a continuous left adjoint, it suffices to show that $F$ preserves tensors and $F$ is a left adjoint when the enrichment is ignored [**8**, 6.7.6]. We know that the former holds, and since $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint by hypothesis and $\mathcal{C}[\mathbb{A}]$ has coequalizers, we can apply the adjoint lifting theorem [**8**, 4.5.6] and conclude the latter. $\square$

## 3. *Parameterized spaces and operadic algebras*

In this section, we review the definitions of the domain and range categories of the Lewis–May operadic Thom spectrum functor. We begin by discussing operadic algebras.

### 3.1. *Review of operadic algebras*

Let $\mathscr{I}$ be the (unbased) category of finite-dimensional or countably infinite dimensional real inner product spaces and linear isometries. This is a symmetric monoidal category under the direct sum.

Definition 3.1. Let $U^j$ be the direct sum of $j$ copies of $U$ (an infinite-dimensional real inner product space), and let $\mathscr{L}(j)$ be the mapping space $\mathscr{I}(U^j, U)$. The action of $\Sigma_j$ on $U^j$ by permutation induces an action of $\Sigma_j$ on $\mathscr{L}(j)$. There are the maps

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \ldots \times \mathscr{L}(j_k) \longrightarrow \mathscr{L}(j_1 + \ldots + j_k)$$

given by $\gamma(g; f_1, \ldots, f_k) = g \circ (f_1 \oplus \ldots \oplus f_k)$. The spaces $\mathscr{L}(j)$ form an operad, which we will refer to as the linear isometries operad.

The properties of the linear isometries operad have been explored at length, notably in [**12**, Section XI]. Recall that $\mathscr{L}$ is an $E_\infty$-operad, as $\mathscr{L}(j)$ is contractible, $\mathscr{L}(1)$ contains the identity, $\mathscr{L}(0)$ is a point, and $\Sigma_n$ acts freely on $\mathscr{L}(n)$. We can consider both based spaces and spectra that admit actions of $\mathscr{L}$. We will make frequent use of the fact that, for any operad $\mathcal{O}$, there is an associated monad $\mathbb{O}$ such that objects $X$ with actions by $\mathcal{O}$ are precisely algebras over $\mathbb{O}$ (see [**23**]).

A space $X$ with an action of the operad $\mathscr{L}$ is the same as an algebra over a certain monad $\mathbb{K}$ on the category of based spaces. Since the monad $\mathbb{K}$ preserves reflexive coequalizers, standard lifting techniques suffice to show the following theorem [**3**, 6.2; **13**].

Theorem 3.2. *The category $\mathcal{T}[\mathbb{K}]$ of $\mathscr{L}$-spaces admits the structure of a topological model category. Fibrations and weak equivalences are created in the category $\mathcal{T}$, and cofibrations are defined as having the left-lifting property with respect to acyclic fibrations.*

Since $\mathscr{L}$ is an $E_\infty$ operad, we can functorially associate a spectrum $Z$ to an $\mathscr{L}$-space $X$ such that the map $X \to \Omega^\infty Z$ is a group completion. When $\pi_0(X)$ is a group and not just a monoid, this map is a weak equivalence. Such $\mathscr{L}$-spaces $X$ for which $\pi_0(X)$ is a group are said to be group-like.

Similarly, the category of $E_\infty$-ring spectra can be described as a category of algebras over monads, following [**12**, 2.4]. Let $\mathcal{S}$ denote the category of coordinate-free spectra [**17**]. For clarity, we emphasize that $\mathcal{S}$ is not a symmetric monoidal category of spectra prior to passage to the homotopy category. An $E_\infty$-ring spectrum structured by the operad $\mathscr{L}$ is an algebra over a certain monad $\mathbb{C}$ in $\mathcal{S}$.

Since the Thom spectrum associated to an object $f$ of $\mathcal{T}/BG$ will have a natural unit $S \to Mf$ induced by the inclusion of the basepoint, we also consider the category $\mathcal{S} \setminus S$ of unital spectra. In this setting, an $E_\infty$-ring spectrum $X$ over the operad $\mathscr{L}$ is the same as an algebra over the monad $\tilde{\mathbb{C}}$, where $\tilde{\mathbb{C}}X$ is a 'reduced' version of $\mathbb{C}$ quotiented to ensure that the unit provided by the algebra structure coincides with the existing unit.

There is a close relationship between the category of algebras over $\mathbb{C}$ and algebras over $\tilde{\mathbb{C}}$ (see [**12**, 2.4.9]). The category $\mathcal{S} \setminus S$ is itself a category of algebras over $\mathcal{S}$ for the monad $\mathbb{U}$ which takes $X$ to $X \vee S$. The monad $\mathbb{C}$ is precisely the composite monad $\tilde{\mathbb{C}}\mathbb{U}$, and in this situation the categories of algebras are equivalent [**12**, 2.6.1]. Therefore the two notions of $E_\infty$-ring spectrum we have described are equivalent. In the language of [**12**], $\tilde{\mathbb{C}}$ is the 'reduced' monad associated to the monad $\mathbb{C}$. Both of these monads preserve reflexive coequalizers.

Finally, given an $E_\infty$-ring spectrum, the functor $S \wedge_{\mathscr{L}} -$ converts it to a weakly equivalent commutative $S$-algebra (see [**12**, 2.3.6, 2.4.2]). Moreover, $S \wedge_{\mathscr{L}} -$ is a continuous left adjoint.

### 3.2. *Parameterized operadic algebras*

Now we move on to consider the category of spaces over a fixed base space $B$. The category $\mathcal{U}/B$ has objects maps $p : X \to B$, where $X$ and $B$ are objects of $\mathcal{U}$. A morphism $(p_1 : X \to B) \to (p_2 : Y \to B)$ is a map $f : X \to Y$ such that $p_2 f = p_1$. The properties of this category have been investigated in a variety of places [**14**, **16**; **17**, 7.1]. In particular, this is a topological category where the tensor of $p : X \to B$ and an unbased space $A$ is given by the composite

$$X \times A \xrightarrow{\pi_1} X \xrightarrow{p} B$$

(where $\pi_1$ is the projection onto the first factor).

Since we shall be interested in spaces that admit operad actions, we also consider the related category of based spaces over $B$. This is the category $\mathcal{T}/B$, defined in the same fashion as $\mathcal{U}/B$, replacing spaces with based spaces and requiring that the maps be based. The category $\mathcal{T}/B$ inherits the structure of a category tensored over unbased spaces from $\mathcal{U}/B$, where the tensor of $X \to B$ and an unbased space $A$ is given by $X \wedge A_+ \to B$.

Colimits in $\mathcal{T}/B$ are formed as follows. Given a diagram $D \to \mathcal{T}/B$, via the forgetful functor we obtain a diagram $D \to \mathcal{T}/B \to \mathcal{T}$. The colimit over $D \to \mathcal{T}/B$ is computed by taking the colimit of this diagram in $\mathcal{T}$ and using the induced map to $B$ given by the universal property of the colimit.

When $B$ is an $\mathscr{L}$-space, there is a category where the objects are $\mathscr{L}$-maps $X \to B$ and the morphisms are $\mathscr{L}$-maps over $B$. We will sometimes refer to this category as $\mathscr{L}$-spaces over $B$. We can regard this category as algebras over a monad on $\mathcal{T}/B$. Given a map $f : Y \to B$, where $B$ is an $\mathscr{L}$-space, the space $\mathbb{K}Y$ admits an $\mathscr{L}$-map to $B$ given by the unique extension of $f$ (see [**17**, 7.7]). This specifies a monad on $\mathcal{T}/B$, with structure maps inherited from those of $\mathbb{K}$, which we refer to as $\mathbb{K}_B$. Denote by $(\mathcal{T}/B)[\mathbb{K}_B]$ the category of $\mathbb{K}_B$-algebras.

There is a model structure on this category defined in analogy with the standard model structure on $\mathcal{T}/B$. We need to verify the existence of tensors and colimits in $(\mathcal{T}/B)[\mathbb{K}_B]$. In order to show that $(\mathcal{T}/B)[\mathbb{K}_B]$ is topologically cocomplete, it will suffice to show that the monad $\mathbb{K}_B$ preserves reflexive coequalizers. This follows immediately from the fact that $\mathbb{K}$ preserves reflexive coequalizers, since colimits in $\mathcal{T}/B$ are constructed by taking the colimit in $\mathcal{T}$ and using the natural map to $B$.

Proposition 3.3. *The category* $(\mathcal{T}/B)[\mathbb{K}_B]$ *is topologically cocomplete* (*and in particular has all colimits and tensors with based spaces*).

It will be useful later on to write out an explicit description of the tensor in $(\mathcal{T}/B)[\mathbb{K}_B]$. We regard the category of $\mathscr{L}$-spaces as tensored over unbased spaces via the tensor over based spaces: for an unbased space $A$ the tensor with an $\mathscr{L}$-space $X$ is the based tensor $X \otimes A_+$.

Lemma 3.4. *The tensor of an unbased space* $A$ *and* $(X \to B)$ *is given by*

$$X \otimes A_+ \longrightarrow X \otimes S_0 \cong X \longrightarrow B,$$

*where the first map is the collapse map that takes* $A$ *to the nonbasepoint of* $S^0$.

## 4. *The operadic Thom spectrum functor*

In this section we review the operadic theory of Thom spectra developed by Lewis [**17**, 7.3] and May [**27**]. Our discussion is updated slightly to take account of more recent developments in the theory of diagram spectra [**21**, **22**]. In particular, our terminology regarding $\mathscr{I}$-spaces reflects the modern usage and is at variance with the definitions in the original articles.

### 4.1. *The definition of M*

Recall that $\mathscr{I}$ denotes the category of finite-dimensional or countably infinite-dimensional real inner product spaces and linear isometries.

Definition 4.1. An $\mathscr{I}$-space is a continuous functor $X$ from $\mathscr{I}$ to the category of based topological spaces.

We restrict attention to $\mathscr{I}$-spaces with the property that $X(V)$ is the colimit of $X(W)$ for the finite-dimensional subspaces $W \subset V$. This constraint implies that it is sufficient to consider the restriction of $X$ to the full subcategory of $\mathscr{I}$ consisting of the finite-dimensional real inner product spaces [**27**, 1.1.8, 1.1.9].

The idea of using $\mathscr{I}$ to capture structure about infinite loop spaces and operad actions dates back to Boardman and Vogt's original treatment [**5**]. In the context of Thom spectra, $\mathscr{I}$-spaces first arose in [**27**]. More recently, May has introduced the terminology of 'functors with cartesian product' (FCP) to highlight the connection to diagram spectra [**26**], in analogy with Bokstedt's 'functors with smash product' (FSPs).

Definition 4.2. An FCP over $\mathscr{I}$ ($\mathscr{I}$-FCP) is a $\mathscr{I}$-space equipped with a unital and associative 'Whitney sum' natural transformation $\omega$ from $X \times X$ to $X \circ \oplus$.

A commutative $\mathscr{I}$-FCP is a $\mathscr{I}$-FCP for which the natural transformation $X \times X$ to $X \circ \oplus$ is commutative. We assume in the following that by default the $\mathscr{I}$-FCP are commutative. The commutative $\mathscr{I}$-FCP encode an $E_\infty$-structure [**27**, 1.1.6]; specifically, a commutative $\mathscr{I}$-FCP $X$ yields an $\mathscr{L}$-space structure on $X(\mathbb{R}^\infty)$. The essential observation is that we can use the

Whitney sum to obtain a natural map $\mathscr{L}(j) \times X(R^{\infty})^j \to X(R^{\infty})$ specified by

$$(f, x_1, x_2, \ldots, x_j) \longmapsto Xf(x_1 \oplus x_2 \oplus \ldots \oplus x_j).$$

Similarly, a noncommutative $\mathscr{I}$-FCP yields a non-$\Sigma$ $\mathscr{L}$-space structure on $X(\mathbb{R}^{\infty})$.

There is an obvious product structure on the category of $\mathscr{I}$-spaces specified by the levelwise cartesian product: if $X$ is an $\mathscr{I}$-space, then $X \times X$ is specified by $V \mapsto X(V) \times X(V)$. The product of (commutative) $\mathscr{I}$-FCP is itself a (commutative) $\mathscr{I}$-FCP.

Definition 4.3. A (commutative) monoid $\mathscr{I}$-FCP is a (commutative) $\mathscr{I}$-FCP such that each $X(V)$ is itself a topological monoid and the levelwise monoid maps assemble into a morphism of (commutative) $\mathscr{I}$-FCP $X \times X \to X$. A (commutative) monoid $\mathscr{I}$-FCP is group-like if each $X(V)$ is group-like. A (commutative) group $\mathscr{I}$-FCP is a (commutative) monoid $\mathscr{I}$-FCP in which each $X(V)$ is a topological group.

The classical groups furnish examples; for instance, there are commutative group $\mathscr{I}$-FCP given by the functors specified by $V \mapsto O(V)$ and $V \mapsto U(V)$. The most important example for our purposes is the commutative $\mathscr{I}$-FCP assembled from the classifying spaces for spherical fibrations.

Definition 4.4. Let $F$ be the commutative monoid $\mathscr{I}$-FCP given by taking $F(V)$ to be the space of based homotopy equivalences of $S^V$.

For any monoid $\mathscr{I}$-FCP $X$, we can construct a related $\mathscr{I}$-FCP $BX$ via the two-sided bar construction. Specifically, define $BX$ as the functor specified by

$$BX(V) = B(*, X(V), *),$$

where $B(-,-,-)$ denotes the geometric realization of the two-sided bar construction. When $X$ is equipped with an augmentation to $F$ which is a map of monoid $\mathscr{I}$-FCP, we can construct $EX$ as

$$EX(V) = B(*, X(V), S^V),$$

where $X(V)$ acts on $S^V$ via the augmentation. There is a projection map $\pi : EX \to BX$ and a section defined by the basepoint inclusion $* \hookrightarrow S^V$. This section is a Hurewicz cofibration, when $X$ is group-like, $\pi$ is a quasifibration, and $\pi$ has fiber $S^V$ (see [**17**, 7.2]). If $X$ actually takes values in groups, then $\pi$ is a bundle.

When $X = F$, this construction provides a model of the universal quasifibration with spherical fibers [**24**]. More generally, we obtain universal quasifibrations and fibrations with spherical fibers and prescribed structure groups. Note that we are following Lewis in letting $EG(V)$ denote the total space of the universal spherical quasifibration rather than the associated principal quasifibration.

Moving on, we now describe the Thom spectrum construction. Let $G$ be a group-like commutative monoid $\mathscr{I}$-FCP that is augmented over $F$. Abusing notation, we write $BG$ to denote both the $\mathscr{I}$-FCP $BG$ and the space $\operatorname{colim}_V BG(V)$. We assume that we are given a map of spaces $f : Y \to BG$. The filtration of $BG$ by inner product spaces $V$ induces a filtration on $Y$ defined as $Y(V) = f^{-1}(BG(V))$.

Associated to the inclusion $V \subset W$ is an inclusion $Y(V) \subset Y(W)$, and this induces a map of pullbacks as follows:

$$\begin{CD} Z_W @>>> EG(W) \\ @VVV @VVV \\ Y(W) @>>> BG(W) \end{CD} \qquad\qquad \begin{CD} Q_V @>>> EG(V) @>>> EG(W) \\ @VVV @VVV @VVV \\ Y(V) @>>> BG(V) @>>> BG(W). \end{CD}$$

Upon passage to Thom spaces, we can identify the Thom space of $Q_V$ as the fiberwise suspension $\Sigma^{W-V}$ of the Thom space of $Z_V$ (see [**17**, 7.2.2]), and so the map $Q_V \to Z_W$ is the suspension map $\Sigma^{W-V} Z_V \to Z_W$. One checks that these suspension maps are appropriately coherent [**17**, 7.2.1], and this permits the following definition.

Definition 4.5. The Thom prespectrum associated to $f : Y \to BG$ is specified as follows. Set $Tf(V)$ to be the Thom space of the pullback $Z_V$ in the following diagram:

$$\begin{CD} Z_V @>>> EG(V) \\ @VVV @VVV \\ Y(V) @>>> BG(V), \end{CD}$$

that is, the map $Z_V \to Y(V)$ has a section $i$, and $Tf(V) = Z_V / i(Y(V))$. Here $Tf$ is a prespectrum, and we define the Thom spectrum in $\mathcal{S}$ associated to $f$ as the spectrification $Mf = LTf$.

Other filtrations can also be used in this construction, but it can be shown that the choice of filtration does not matter up to isomorphism of spectra [**17**, 7.4.4].

Remark 4.6. Lewis treated only group-like commutative monoid $\mathscr{I}$-FCP $X$ augmented over $F$; this augmentation gives an action of $X$ on $S^V$, which allows the construction of $EX$. However, we can develop the theory of Thom spectra for other choices of fiber, as long as we specify a levelwise action of $X$ on the fiber. Such constructions will be useful for us when considering models of Eilenberg–Mac Lane spectra as Thom spectra in Section 9. We shall consider $p$-local and $p$-complete spherical fibrations, and employ 'localized' and 'completed' versions of $F$ formed from spaces of based self-equivalences of the $p$-local sphere $S^V_{(p)}$ and based self-equivalences of the $p$-complete sphere $(S^V)^\wedge_p$.

We have constructed the Thom spectrum as a continuous functor from $\mathcal{U}/BG$ to coordinate-free spectra $\mathcal{S}$. Working with $\mathcal{T}/BG$, we obtain a functor to $\mathcal{S}\backslash S$, unital spectra. Here the unit $S \to Mf$ is induced by the inclusion $* \to X$ over $BG$. In abuse of notation, we refer to both of these functors as $M$.

### 4.2. *Properties of M*

Lewis proves that the Thom spectrum functor $M$ preserves colimits in $\mathcal{U}/BG$ (see [**17**, 7.4.3]). It is straightforward to extend this to the functor $M$ from $\mathcal{T}/BG$ to $\mathcal{S}\backslash S$.

Lemma 4.7. *The Thom spectrum functor takes colimits in $\mathcal{T}/BG$ to colimits in the category $\mathcal{S}\backslash S$.*

*Proof.* A colimit over $\mathcal{D}$ in $\mathcal{T}/BG$ is given as the pushout in $\mathcal{U}/BG$

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{D}} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathrm{colim}_{\mathcal{D}} R_d & \longrightarrow & Z \end{array}$$

where the indicated colimits are also taken in the category $\mathcal{U}/BG$. Similarly, a colimit over $\mathcal{D}$ in $\mathcal{S}\backslash S$ is constructed as the pushout in $\mathcal{S}$

$$\begin{array}{ccc} \mathrm{colim}_{\mathcal{D}}\, S & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{colim}_{\mathcal{D}}\, R_d & \longrightarrow & Z \end{array}$$

where the indicated colimits are also taken in $\mathcal{S}$. The result follows from the fact that $M$ takes colimits in $\mathcal{U}/BG$ to colimits in spectra and $M(*) \cong S$. □

Lewis also shows that the functor $M$ also preserves tensors with unbased spaces in $\mathcal{T}/BG$ (see [**17**, 7.4.6]).

PROPOSITION 4.8. *The Thom spectrum associated to the composition $X \wedge A_+ \to X \to BG$ is naturally isomorphic to $Mf \wedge A_+$.*

When $A = I$, this implies that functor $M$ converts fiberwise homotopy equivalences into homotopy equivalences in $\mathcal{S}\backslash S$.

The question of invariance under weak equivalence is somewhat more delicate. Unfortunately, quasifibrations are not preserved under pullback along arbitrary maps. This can cause technical difficulty when working with $BF$, or any other monoid $\mathscr{I}$-FCP (which is not a group $\mathscr{I}$-FCP). Following Lewis *et al.* [**17**, 7.3.4], we make the following definition.

DEFINITION 4.9. Define a map $f : X \to BG$ to be good if the projections $Z_V \to X(V)$ (where $Z_V$ is the pullback of Definition 4.5) are quasifibrations and the sections $X(V) \to Z_V$ are Hurewicz cofibrations.

A map $f : X \to BG$ associated to a group $\mathscr{I}$-FCP $G$ is always good, and all Hurewicz fibrations are good [**17**, 7.3.4]. Therefore, it is sometimes useful to replace arbitrary maps by Hurewicz fibrations when working over $BF$ via the functor $\Gamma$ (see [**17**, 7.1.11]). This is compatible with the linear isometries operad — recall that given an $\mathscr{L}$-map $f : X \to BF$, the map $\Gamma f : \Gamma X \to BF$ is also an $\mathscr{L}$-map [**23**, 1.8]. Our discussion of $\Gamma$ is brief, as we do not use it extensively in this paper.

When the maps in question are good, the Thom spectrum functor preserves weak equivalences over $BG$ (see [**17**, 7.4.9]).

THEOREM 4.10. *If $f : X \to BG$ and $g : X' \to BG$ are good maps such that there is a weak equivalence $h : X \simeq X'$ over $BG$, then there is a stable equivalence $Mh : Mf \simeq Mg$.*

In this situation, $M$ also takes homotopic maps to stably equivalent spectra [**17**, 7.4.10]. However, note that the stable equivalence depends on the homotopy.

Theorem 4.11. *If $f : X \to BG$ and $g : X \to BG$ are good maps that are homotopic, then there is a stable equivalence $Mf \simeq Mg$.*

## 5. *The Thom spectrum functor is a left adjoint*

As discussed previously, spaces with actions by the linear isometries operad $\mathscr{L}$ can be regarded as the category $\mathcal{T}[\mathbb{K}]$ of algebras over the monad $\mathbb{K}$. Similarly, spectra in $\mathcal{S}\backslash S$ which are $E_\infty$-ring spectra structured by the linear isometries operad can be regarded as the category $(\mathcal{S}\backslash S)[\mathbb{C}]$ of algebras with respect to the monad $\tilde{\mathbb{C}}$.

One of the main results of Lewis' work is that the Thom spectrum functor $M$ 'commutes' with these monads. Specifically, Lewis [**17**, 7.7.1] prove the following.

Theorem 5.1. *Let $G$ be a group-like commutative monoid $\mathscr{I}$-FCP that is augmented over $F$.*

(i) *Given a map $f : X \to BG$, there is an isomorphism $\tilde{\mathbb{C}}Mf \cong M(\mathbb{K}_{BG}f)$, where the map*

$$\mathbb{K}_{BG}f : \mathbb{K}_{BG}X \longrightarrow BG$$

*is the natural map induced from $X \to BG$.*

(ii) *This isomorphism is coherently compatible with the unit and multiplication maps for these monads, in the sense of Lemma* 2.2.

As we have observed, a consequence of this result is that the Thom spectrum functor induces a functor $M_{E_\infty}$ from $(\mathcal{T}/BG)[\mathbb{K}_{BG}]$ to $E_\infty$-ring spectra structured by $\tilde{\mathbb{C}}$. Composing with the functor $S \wedge_{\mathscr{L}} -$, we obtain a Thom spectrum functor $M_{\mathcal{CA}_S}$ from $(\mathcal{T}/BG)[\mathbb{K}_{BG}]$ to commutative $S$-algebras. Now employing Theorem 1.2, we obtain the main result. Note that here and in the remainder of the paper, we assume, unless otherwise indicated, that $G$ is a group-like commutative monoid $\mathscr{I}$-FCP that is augmented over $F$.

Theorem 5.2. *The Thom spectrum functor*

$$M_{\mathcal{CA}_S} : (\mathcal{T}/BG)[\mathbb{L}_{BG}] \longrightarrow \mathcal{CA}_S$$

*commutes with indexed colimits.*

*Proof.* We have verified that the functor $M_{E_\infty}$ satisfies the hypotheses of Theorem 1.2, and so we can conclude that $M_{E_\infty}$ commutes with indexed colimits. Since $M_{\mathcal{CA}_S}$ is obtained from $M_{E_\infty}$ via composition with a continuous left adjoint, the result follows. □

Since the Thom spectrum functor $M_{\mathcal{CA}_S}$ preserves indexed colimits, one would expect that it should in fact be a continuous left adjoint. We will prove this by showing that the hypotheses of the second part of Theorem 1.2 are satisfied. However, our method of proof does not produce an explicit description of the right adjoint and so is somewhat unsatisfying.

Lemma 5.3. *The Thom spectrum functor from $\mathcal{T}/BG$ to $\mathcal{S}\backslash S$ is a left adjoint.*

*Proof.* We know that the Thom spectrum functor preserves colimits in $\mathcal{T}/BG$. Moreover, it is easy to verify that the category $\mathcal{T}/BG$ satisfies the hypotheses of the special adjoint functor theorem, since $\mathcal{T}$ does. Therefore $M$ is a left adjoint. □

Now we have the following diagram of categories:

$$\begin{array}{ccc} \mathcal{T}/BG[\mathbb{K}_{BG}] & \xrightarrow{M_{E_\infty}} & (\mathcal{S}\backslash S)[\tilde{\mathbb{C}}] \\ U \downarrow \uparrow F & & V \downarrow \uparrow G \\ \mathcal{T}/BG & \xrightarrow{M} & (\mathcal{S}\backslash S). \end{array}$$

Here $U$ and $V$ denote forgetful functors, and $F$ and $G$ denote the free algebra functors. Recall that $(\mathcal{S}\backslash S)[\tilde{\mathbb{C}}]$ is the category of $E_\infty$-ring spectra [**12**, 2.4.5]. The square commutes in the sense that $M \circ U = V \circ M_{E_\infty}$ and $M_{E_\infty} \circ F = G \circ M$.

Corollary 5.4. *The Thom spectrum functor $M_{\mathcal{CA}_S}$ from $\mathcal{T}/BG[\mathbb{K}_{BG}]$ to the category of commutative $S$-algebras is a continuous left adjoint.*

*Proof.* It follows from Theorem 1.2 that $M_{E_\infty}$ is a continuous left adjoint. Since $S \wedge_{\mathscr{L}} -$ is a continuous left adjoint, the composite functor to commutative $S$-algebras is a continuous left adjoint as well. □

When restricting attention to vector bundles (that is, when $G$ is a commutative group $\mathscr{I}$-FCP over $F$), we can improve this result as follows. Recall that the categories of $\mathscr{L}$-spaces, $E_\infty$-ring spectra, and commutative $S$-algebras are all categories of algebras over monads. In each case, a model structure is constructed by lifting a cofibrantly generated model structure from the base category. As a consequence, we have an explicit description of the cell objects.

In each case, the cell objects are colimits of pushouts of the form

$$\begin{array}{ccc} \mathbb{Z}A & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \mathbb{Z}CA & \longrightarrow & X_n \end{array}$$

where $\mathbb{Z}$ is the appropriate monad and where $A$ to $CA$ is a generating cofibration in the base category. For instance, in the case of $\mathscr{L}$-spaces, $A \to CA$ is a map of the form

$$\bigvee_i S_+^{n_i-1} \longrightarrow \bigvee_i D_+^{n_i}.$$

For the category of commutative $S$-algebras, $A \to CA$ is a map of the form

$$\bigvee_i \Sigma^\infty S_+^{n_i-1} \longrightarrow \bigvee_i \Sigma^\infty D_+^{n_i}$$

where the suspension spectrum functor takes values in $S$-modules. The description for $E_\infty$-ring spectra is analogous.

Corollary 5.5. *Let $G$ be a commutative group $\mathscr{I}$-FCP augmented over $F$. Then the functor $M_{\mathcal{CA}_S}$ is a Quillen left adjoint.*

*Proof.* In these cases all maps are good, and so $M$ preserves weak equivalences. Therefore, it will suffice to show that $M$ takes the generating cofibrations to cofibrations. The generating cofibrations in $\mathcal{T}[\mathbb{K}_{BG}]$ are maps of the form $h : \mathbb{K}_{BG}A \to \mathbb{K}_{BG}CA$, where $A$ is a wedge of $S_+^{n_i-1}$ and $CA$ the corresponding wedge of $D_+^n$. The maps $D_+^n \to BG$ are arbitrary, and these choices determine the maps $S^{n_i-1} \to BG$. Denote the map $\mathbb{K}_{BG}A \to BG$ by $h_1$ and the map $\mathbb{K}_{BG}CA \to BG$ by $h_2$. Recall that $M\mathbb{K}_{BG}f \cong \tilde{\mathbb{C}}Mf$. In addition, a map from a contractible space to $BG$ represents a bundle that is isomorphic to a trivial bundle. Therefore, there is a homeomorphism $Mh_1 \cong \tilde{\mathbb{C}}\Sigma^\infty A$ and $Mh_2 \cong \tilde{\mathbb{C}}\Sigma^\infty CA$. The induced map $Mh : Mh_1 \to Mh_2$ clearly yields a generating cofibration in the category of $E_\infty$-ring spectra structured by $\tilde{\mathbb{C}}$. □

## 6. *Computing THH*

The formula $M(f \otimes S^1) \cong Mf \otimes S^1$ is a point-set result; $Mf \otimes S^1$ is an object in the category of commutative $S$-algebras. In this section we discuss how to ensure that $Mf \otimes S^1$ has the correct homotopy type so that it represents $\mathrm{THH}(Mf)$.

For an $S$-algebra $R$, in analogy with the classical definition of Hochschild homology as Tor we define

$$\mathrm{THH}(R) = R \wedge^L_{R \wedge R^{\mathrm{op}}} R.$$

In the algebraic setting, this Tor can be computed via the Hochschild resolution. In spectra, this leads to a candidate point-set description of $\mathrm{THH}(R)$ as the cyclic bar construction $N^{\mathrm{cyc}}(R)$. The precise relationship between these is studied in [**12**, 9.2]; the main result is that when $R$ is cofibrant they are canonically isomorphic in the derived category of $R$-modules [**12**, 9.2.2].

First, observe that there is a derived version of the cyclic bar construction in $\mathscr{L}$-spaces. This is a consequence of the very useful fact that, for a simplicial set $A.$ and an $\mathscr{L}$-space $X$, there is a homeomorphism $X \otimes |A.| \cong |X \otimes A.|$ (see [**3**, 6.7]). When $A.$ has finitely many nondegenerate simplices in each simplicial degree, this provides a tractable description of the tensor with $|A.|$ in terms of tensors with finite sets, that is, finite coproducts.

Lemma 6.1. *Let $g : X \to X'$ be a weak equivalence of cofibrant $\mathscr{L}$-spaces. Then there is an induced weak equivalence $g \otimes S^1_+ : X \otimes S^1_+ \to X' \otimes S^1_+$.*

*Proof.* Since $X \otimes S^1_+$ is a proper simplicial space for any $\mathscr{L}$-space $X$, the result follows from the fact that the induced map $g \coprod g : X \coprod X \to X' \coprod X'$ is a weak equivalence when $X$ and $X'$ are cofibrant. □

One might hope that for cofibrant $X$, $Mf$ is necessarily cofibrant as a (commutative) $S$-algebra. Of course when $M$ is a left Quillen adjoint this holds, but in general it turns out that $Mf$ does belong to a class of commutative $S$-algebras for which the point-set smash product has the correct homotopy type.

Theorem 6.2. *Let $f : X \to BG$ be a good $\mathscr{L}$-map such that $X$ is a cell $\mathscr{L}$-space. Then $Mf \wedge Mf$ represents the derived smash product.*

Recall the notion of an extended cell module [**2**, 9.6]. An extended $S$-cell is a pair $(X \wedge B^n_+, X \wedge S^{n-1}_+)$, where $X = S \wedge_{\mathscr{L}} \mathscr{L}(i) \ltimes_G K$ for a $G$-spectrum $K$ indexed on $U^i$ which has the homotopy type of a $G$-CW-spectrum for some $G \subset \Sigma^i$. An extended cell $S$-module is an $S$-module $M = \operatorname{colim} M_i$, where $M_0 = 0$ and $M_n$ is obtained from $M_{n-1}$ by a pushout of

$S$-modules of the form

$$\begin{array}{ccc} \bigvee_j X_j \wedge S_+^{n_j-1} & \longrightarrow & M_{n-1} \\ \downarrow & & \downarrow \\ \bigvee_j X_j \wedge B_+^{n_j} & \longrightarrow & M_n. \end{array}$$

Extended cell $S$-modules have the correct homotopy type for the purposes of the smash product. Therefore, it will suffice to show the following result.

PROPOSITION 6.3. *Let $f : X \to BG$ be a good $E_\infty$-map over the linear isometries operad such that $X$ is a cell $\mathscr{L}$-space. Then the underlying $S$-module of the $S$-algebra $Mf$ has the homotopy type of an extended cell $S$-module.*

*Proof.* By hypothesis, $X = \operatorname{colim} X_i$, where $X_0 = *$ and $X_i$ is obtained from $X_{i-1}$ as the pushout

$$\begin{array}{ccc} \tilde{\mathbb{K}}A & \longrightarrow & X_{i-1} \\ \downarrow & & \downarrow \\ \tilde{\mathbb{K}}CA & \longrightarrow & X_i \end{array}$$

where $A$ is a wedge of spheres $S_+^{n_i-1}$ and $CA$ is the associated wedge of $D_+^{n_i}$. Since $M$ commutes with colimits and $M\mathbb{K}g \cong \tilde{\mathbb{C}}Mg$, we have that $Mf = \operatorname{colim} Mf_i$, where $Mf_0 = S$ and $Mf_i$ is obtained from $Mf_{i-1}$ as the pushout

$$\begin{array}{ccc} \tilde{\mathbb{C}}MA & \longrightarrow & Mf_{i-1} \\ \downarrow & & \downarrow \\ \tilde{\mathbb{C}}MCA & \longrightarrow & Mf_i. \end{array}$$

As $CA$ is a contractible space with a disjoint basepoint, $MCA$ is homotopy equivalent to a cell $S$-module. Here $MA$ is the wedge of a Thom spectrum over a suspension with $S$, and so we know that it is also a cell $S$-module [**17**, 7.3.8]. Temporarily assume that $\tilde{\mathbb{C}}MA$ and $\tilde{\mathbb{C}}MCA$ are extended cell $S$-modules. Then we proceed as in [**12**, 7.7.5]. We see that $Mf_i$ is isomorphic under $Mf_{i-1}$ to the two-sided bar construction $B(\tilde{\mathbb{C}}MCA, \tilde{\mathbb{C}}MA, MX_{i-1})$. This is a proper simplicial spectrum, and since each simplicial level is an extended cell module and the face and degeneracy maps are cellular, so is the bar construction. By passage to colimits, the result follows.

To see that $\tilde{\mathbb{C}}MA$ and $\tilde{\mathbb{C}}MCA$ are extended cell $S$-modules, we essentially argue as in [**12**, 7.7.5] but must account for the quotients since we are using the reduced monads. Recall that there is a standard filtration on the reduced monads [**17**, 7.3.6], which allows us to regard the free $\tilde{\mathbb{C}}$ algebra as the colimit of spectra formed by pushouts of layers of the form $Z^j/\Sigma^j$. These are extended cell $S$-modules, and then a similar induction as above allows us to conclude the result. □

There is an additional difficulty that arises when working over $BF$; it seems to be difficult to replace an arbitrary map of $\mathscr{L}$-spaces $X \to BF$ with a map $X' \to BF$ which is a Hurewicz fibration and such that $X'$ is cofibrant as an $\mathscr{L}$-space. However, we believe that it suffices to work with the following composite replacement: given an arbitrary map of $\mathscr{L}$-spaces $X \to BF$, we work with $\Gamma X' \to BF$, where $X'$ is a cofibrant replacement of $X$. This should be proved along the lines of the analogous statement for associative $S$-algebras in [4], but we do not discuss it further here as we are able to obtain our main applications without this replacement process.

## 7. *Splitting of* THH$(Mf)$

In the previous section, we have verified that by appropriate modification of the map $f : X \to BG$ we can ensure that we can identify THH$(Mf)$ as $M(f \otimes S^1)$. In this section, we study $M(f \otimes S^1)$. In particular, we discuss briefly a connection to the free loop space $LBX$ and then investigate in detail the splitting result THH$(Mf) \simeq Mf \wedge BX_+$.

The starting point for our analysis is the observation that the based cofiber sequence $S^0 \to S^1_+ \to S^1$ yields an associated sequence of $\mathscr{L}$-spaces

$$X \longrightarrow X \otimes S^1_+ \longrightarrow X \otimes S^1.$$

The map $X \to X \otimes S^1_+$ is split by the collapse map $S^1_+ \to S^0$, and this induces a map $\theta : X \otimes S^1_+ \to X \times (X \otimes S^1)$.

Remark 7.1. Recall that $X \otimes S^1_+$ is the realization of the simplicial object $X \otimes (S^1_+)_\bullet$ induced by the standard description of $S^1_+$ as a simplicial set. This is in fact a cyclic object, and therefore $X \otimes S^1_+$ has an action of $S^1$ induced by the cyclic structure. The adjoint of the action map composed with the projection $X \otimes S^1_+ \to X \otimes S^1$ yields a map $X \otimes S^1_+ \to L(X \otimes S^1)$, which is a weak equivalence for group-like $\mathscr{L}$-spaces. When working over a commutative group $\mathscr{I}$-FCP, this weak equivalence implies a weak equivalence of Thom spectra, and so we obtain a description of THH$(Mf)$ in terms of a map $L(BX) \to BG$. This relationship is studied in detail in the companion paper [4].

### 7.1. *The splitting arising from an $E_\infty$-map*

In this section, we assume that we have a fixed $\mathscr{L}$-map $f : X \to BG$ such that $X$ is a group-like $\mathscr{L}$-space and $G$ is a commutative group $\mathscr{I}$-FCP augmented over $F$. We require this latter hypothesis to ensure that all maps that arise are good.

Lemma 7.2. *Let $X$ be a group-like cofibrant $\mathscr{L}$-space. The map*

$$\theta : X \otimes S^1_+ \longrightarrow X \otimes S^1 \times X \otimes S^0$$

*is a weak equivalence.*

*Proof.* Since $\mathscr{L}$ is an $E_\infty$-operad, we can functorially associate an $\Omega$-prespectrum $Z$ to $X$ using an 'infinite loop space machine'. We will show that $X \otimes A$ is weakly equivalent to $\Omega^\infty(Z \wedge A)$. Assuming this fact, the lemma is now a consequence of the stable splitting of $S^1_+$. Specifically, there is a chain of equivalences $Z \wedge S^1_+ \simeq (Z \wedge S^0) \vee (Z \wedge S^1) \simeq (Z \wedge S^0) \times (Z \wedge S^1)$. Applying $\Omega^\infty$ to this composite yields an equivalence $\Omega^\infty(Z \wedge S^1_+) \to (\Omega^\infty Z) \times (\Omega^\infty(Z \wedge S^1))$, since $\Omega^\infty$ preserves products and weak equivalences of spectra. Under the equivalence between $X$ and $\Omega^\infty Z$, this map coincides with the map induced from the splitting and so the result follows.

To compare $X \otimes A$ and $Z \wedge A$, we use a technique from [3]. Let $\tilde{X}$ denote the functor that assigns to a finite set $\underline{n}$ the tensor $X \otimes \underline{n}$. Using the folding map, this specifies a $\Gamma$-object in $\mathscr{L}$-spaces. Recall that the construction of a prespectrum from a $\Gamma$-object proceeds by prolonging the $\Gamma$-object to a functor from the category of spaces of the homotopy type of finite

$CW$-complexes. Such a functor is called a $\mathcal{W}$-space, and is an example of a diagram spectrum [22]. In this situation, the associated $\mathcal{W}$-space can be specified simply as $A \mapsto X \otimes A$. For any $\mathcal{W}$-space $Y$ and based space $A$, there is a stable equivalence between the prespectrum $\{Y(S^n) \wedge A\}$ and the prespectrum $\{Y(A \wedge S^n)\}$ induced by the assembly map $Y(S^n) \wedge A \to Y(A \wedge S^n)$ (see [22, 17.6]). Since $X$ was a cofibrant group-like $\mathscr{L}$-space, it follows that $\tilde{X}$ is very special [3, 6.8]. Therefore the associated $\mathcal{W}$-space $\tilde{X}$ is fibrant, which means that the underlying prespectra $\{\tilde{X}(S^n \wedge A)\}$ are $\Omega$-prespectra for all $A$. Finally, this implies that there is an equivalence between $\Omega^\infty(Z \wedge A)$ and $Z(A)$. A similar result (with a different proof) appears in [33]. □

Proposition 7.3. *Let $f : X \to BG$ be an $\mathscr{L}$-map, where $G$ is a commutative group $\mathscr{I}$-FCP augmented over $F$ and $X$ is a group-like $\mathscr{L}$-space. Then there is a weak equivalence of commutative $S$-algebras*

$$Mf \otimes S^1 \simeq BX_+ \wedge Mf.$$

*Proof.* By inspection of the description of the map $f \otimes S^1_+ : X \otimes S^1_+ \to BG$, we see that it can be factored as

$$X \otimes S^1_+ \xrightarrow{\theta} (X \otimes S^0) \times (X \otimes S^1) \xrightarrow{\pi_1} X \otimes S^0 \cong X \xrightarrow{f} BG,$$

where $\pi_1$ is the projection onto the first factor. By the preceding lemma, the hypotheses imply that the map $\theta : X \otimes S^1_+ \to (X \otimes S^1) \times (X \otimes S^0)$ is a weak equivalence. Therefore, there is an equivalence of Thom spectra $M\theta : M(f \otimes S^1_+) \to M(f \circ \pi_1)$. By the standard description of the Thom spectrum of a projection (Proposition 4.8), we know that $M(f \circ \pi_1) \cong Mf \wedge (X \otimes S^1)_+$. Moreover, Theorem 1.1 implies that $M(f \otimes S^1_+) \cong Mf \otimes S^1$. Finally, $X \otimes S^1$ is a model of $BX$; this follows by considering the $\Gamma$-space associated to $X$ as in the previous lemma [3, 6.5]. □

## 7.2. *Splitting arising from an $E_2$-map $f : X \to BG$*

It is sometimes the case that even though $f : X \to BG$ is not an $E_\infty$-map, $Mf$ is equivalent to a commutative $S$-algebra. We consider the situation in which $f : X \to BG$ is an $E_2$-map such that there is an equivalence of $E_2$-ring spectra from $Mf$ to an $E_\infty$-ring spectrum. Although this may seem at first like an artificial hypothesis, in fact this situation arises when considering the Thom spectra that yield Eilenberg-Mac Lane spectra. We will show that the splitting result holds here as well.

We drop the requirement here that $G$ is a commutative group $\mathscr{I}$-FCP. Fix an $E_2$-operad $\mathcal{C}_2$ which is augmented over the linear isometries operad. Then $BG$ is a $\mathcal{C}_2$-space and Lewis' theorem (see [17, 7.7.1]) shows that the Thom spectrum associated to a $\mathcal{C}_2$-map $f : X \to BG$ is a $\mathcal{C}_2$-ring spectrum.

Recall that there is a two-sided bar construction for spectra [12, 4.7.2]. Let $R$ be a commutative $S$-algebra. If $A$ is a left $R$-module and $N$ a right $R$-module, then the bar construction $B(A, R, N)$ is the realization of a simplicial spectrum in which the $k$-simplices are given by $A \wedge R^k \wedge N$ and the faces are given by the multiplication. When $R$ is a cofibrant commutative $S$-algebra and $A$ is a cofibrant $R$-module, the bar construction is naturally weakly equivalent to $A \wedge_R N$ and weak equivalences in each variable induce weak equivalences of bar constructions.

Remark 7.4. A simplicial spectrum $K$ is proper if the 'inclusion' $sK_q \to K_q$ is a cofibration, where $sK_q$ is the 'union' of the subspectra $s_j K_{q-1}$, with $0 \leqslant j < q$ (see [12, 10.2.2]). Of course, the 'union' denotes an appropriate pushout, and the 'inclusion' associated maps, but the terms are useful to emphasize the analogy with the situation in spaces. Maps between proper simplicial spectra, which induce levelwise equivalences, produce weak equivalences upon realization [**12**, 10.2.4]. When $R$ is a cofibrant commutative $S$-algebra and $A$ is a cofibrant $R$-module, the bar construction is a proper simplicial spectrum.

THEOREM 7.5. *Let $f : X \to BG$ be a good $\mathcal{C}_2$-map, where $G$ is a group-like commutative monoid $\mathscr{I}$-FCP augmented over $F$. Assume that $Mf$ is equivalent as a homotopy commutative $S$-algebra to some (strictly) commutative $S$-algebra $M'$. Then there is an isomorphism in the derived category as follows:*

$$\mathrm{THH}(Mf) \simeq BX_+ \wedge Mf.$$

*Proof.* One can describe $\mathrm{THH}(A)$ as the derived smash product $A \wedge^L_{A\wedge A^{\mathrm{op}}} A$ (see [**12**, 9.1.1]). Of course if $A$ is commutative, then $A \wedge A^{\mathrm{op}} \cong A \wedge A$. In our situation, this specializes to the derived smash product

$$\mathrm{THH}(Mf) = Mf \wedge^L_{Mf\wedge Mf^{\mathrm{op}}} Mf.$$

If $Mf$ were a commutative $S$-algebra, then we could use the Thom isomorphism to replace $Mf \wedge Mf^{\mathrm{op}} \cong Mf \wedge Mf$. We will show that in fact it suffices for $Mf$ to be weakly equivalent to a commutative $S$-algebra. We can assume without loss of generality that $Mf$ is cofibrant. Moreover, the hypotheses provide an equivalence of $S$-algebras $Mf \to M'$, where $M'$ can be taken to be a cofibrant commutative $S$-algebra.

The composite

$$Mf^{\mathrm{op}} \longrightarrow Mf^{\mathrm{op}} \wedge S^0 \longrightarrow Mf^{\mathrm{op}} \wedge X_+^{\mathrm{op}} \longrightarrow (M')^{\mathrm{op}} \wedge X_+^{\mathrm{op}} \simeq M' \wedge X_+^{\mathrm{op}}$$

is a map of $S$-algebras, and the map $M' \to M' \wedge S^0 \to M' \wedge X_+^{\mathrm{op}}$ is central [**12**, 7.1.2]. Therefore extension of scalars yields an induced map of $M'$-algebras $M' \wedge Mf^{\mathrm{op}} \to M' \wedge X_+^{\mathrm{op}}$, and the Thom isomorphism theorem implies that this map is a weak equivalence.

We will model the derived smash product using the two-sided bar construction. The preceding discussion implies that the composite

$$B(Mf, Mf \wedge Mf^{\mathrm{op}}, Mf) \longrightarrow B(M', Mf \wedge Mf^{\mathrm{op}}, M') \longrightarrow B(M', M' \wedge X_+^{\mathrm{op}}, M')$$

is a weak equivalence. Therefore we have an isomorphism

$$Mf \wedge^L_{Mf\wedge Mf^{\mathrm{op}}} Mf \longrightarrow M' \wedge^L_{M'\wedge X_+^{\mathrm{op}}} M'.$$

The $k$th simplicial level of $B(M', M' \wedge X_+^{\mathrm{op}}, M')$ is the product

$$M' \wedge (M' \wedge X_+^{\mathrm{op}})^k \wedge M',$$

where the actions of $M' \wedge X_+^{\mathrm{op}}$ on $M'$ are given by projecting $M' \wedge X_+^{\mathrm{op}} \to M'$ and then using the multiplication on $M'$. Clearly, there is an isomorphism

$$M' \wedge (M' \wedge X_+^{\mathrm{op}})^k \wedge M' \longrightarrow (M' \wedge (M')^k \wedge M' \wedge (X_+^{\mathrm{op}})^k$$

given by permuting the $X_+^{\mathrm{op}}$ factors to the right, and this map commutes with the simplicial identities. Thus, there is an equivalence

$$B(M', M' \wedge X_+^{\mathrm{op}}, M') \simeq B(M', M', M') \wedge B(S, \Sigma^\infty X_+^{\mathrm{op}}, S),$$

using the fact that the smash product commutes with realization. However, since $\Sigma^\infty$ commutes with the bar construction for monoids [**12**], we have weak equivalences

$$B(S, \Sigma^\infty X_+^{\mathrm{op}}, S) \simeq \Sigma^\infty BX_+^{\mathrm{op}} \simeq \Sigma^\infty BX_+.$$

We also know that $B(M', M', M')$ is homotopic to $M'$. □

## 8. *Calculation of* THH($\mathbb{Z}$), THH($\mathbb{Z}/p$), *and* THH($MU$)

In this section, we use the splitting results of the previous section to provide easy calculations of THH for various interesting Thom spectra. First, we recover the results of Bokstedt for $H\mathbb{Z}/p$ and $H\mathbb{Z}$ (see [**6**]). Next, we compute THH($MU$), recovering a calculation of McClure and Staffeldt [**30**]. Further calculations of bordism spectra are discussed in the companion paper [**4**].

### 8.1. THH($\mathbb{Z}$) *and* THH($\mathbb{Z}/p$)

There is an identification by Mahowald of $H\mathbb{Z}/2$ as the Thom spectrum associated to a certain map $\Omega^2 S^3 \to BO$ (see [**10**, **19**]). A modification of this approach by Hopkins allows the construction of $H\mathbb{Z}/p$ as the Thom spectrum associated to a certain $p$-local bundle over $\Omega^2 S^3$. Finally, $H\mathbb{Z}$ can be obtained as the Thom spectrum of a map $\Omega^2 S^3\langle 3\rangle \to BSF$. We discuss these constructions in the following section, in particular verifying that all of these Thom spectra are $E_2$-ring spectra associated to $E_2$ maps structured by the little 2-cubes operad. Using standard 'change of operad' techniques discussed in Appendix A, we can functorially convert these to classifying maps structured by an $E_2$ operad augmented over the linear isometries operad.

We have the following proposition, which will allow us to apply Theorem 7.5.

PROPOSITION 8.1. *For any connective $E_2$-ring spectrum $R$, there is a map of $E_2$-ring spectra from $R$ to $H\pi_0(R)$, unique up to homotopy, which induces an isomorphism on $\pi_0$. Here $H\pi_0(R)$ is regarded as an $E_2$-ring spectrum via the commutative $S$-algebra structure.*

Recall that THH($HR$) for $R$ a commutative ring is a product of Eilenberg–Mac Lane spectra [**6**, **12**, 9.1.3]. This implies that we can read off the homotopy type from the homotopy groups. Thus to compute THH($H\mathbb{Z}/2$), we must compute $\pi_*(B(\Omega^2 S^3) \wedge H\mathbb{Z}/2)$. This is just the homology of $\Omega S^3$ with $\mathbb{Z}/2$ coefficients, which can be easily calculated via inspection of the James construction. One immediately recovers the result

$$\mathrm{THH}(H\mathbb{Z}/2) = \prod_{i=0}^{\infty} K(\mathbb{Z}/2, 2i).$$

A similar argument applies to THH($H\mathbb{Z}/p$).

Finally, to compute THH($H\mathbb{Z}$), we must compute $\pi_*(B(\Omega^2 S^3\langle 3\rangle) \wedge H\mathbb{Z})$. Once more, this is just the ordinary homology with integral coefficients of $\Omega S^3\langle 3\rangle$. Computing again, we find

$$\mathrm{THH}(H\mathbb{Z}) = K(\mathbb{Z}, 0) \times \prod_{i=1}^{\infty} K(\mathbb{Z}/i, 2i-1).$$

### 8.2. THH($MU$)

The splitting formula implies that

$$\mathrm{THH}(MU) \simeq MU \wedge BBU_+ \simeq MU \wedge SU_+.$$

We can compute $MU_*(SU)$ via a standard Atiyah–Hirzebruch spectral sequence calculation, and it turns out to be $MU_*(pt) \otimes \Lambda(x_1, x_2, \ldots)$, with the generators in odd degrees. This agrees with the answer obtained by McClure and Staffeldt [**30**] and, as they observe, implies that THH($MU$) is a product of suspensions of $MU$. Other bordism spectra are analogous; see the companion paper [**4**] for further discussion.

## 9. *Realizing Eilenberg–Mac Lane spectra as Thom spectra*

In this section, we review and extend the classical realizations of Eilenberg–Mac Lane spectra as Thom spectra associated to certain bundles over $\Omega^2 S^3$ and $\Omega^2 S^3\langle 3\rangle$. Our main purpose is to ensure that we can obtain these Thom spectra as ring spectra that are sufficiently structured so as to permit the construction of THH and the application of our splitting theorem. In particular, improving on [**10**], we give a new description of $H\mathbb{Z}$ as the Thom spectrum associated to a double loop map $\Omega^2 S^3\langle 3\rangle \to BSF$.

### 9.1. *$H\mathbb{Z}/2$ as the Thom spectrum of a double loop map*

The construction of $H\mathbb{Z}/2$ as a Thom spectrum was the first to be extensively studied [**10**, **19**, **31**]. We briefly review the construction. Consider the map $\psi : S^1 \to BO$ representing the non-trivial element of $\pi_1(BO)$. The Thom spectrum associated to this map is the Moore spectrum $M\mathbb{Z}/2$. There is an induced map $\gamma : \Omega^2 S^3 \to BO$, as $BO$ is an infinite loop space (and in particular a double loop space). The Thom spectrum of $\gamma$ is $H\mathbb{Z}/2$.

A sketch of the proof for this is as follows. There is a map $\mathcal{A} \to H^*(M\gamma;\mathbb{Z}/2)$ given by evaluation on the Thom class, which is a map of modules over the Steenrod algebra. As $M\gamma$ is 2-local, it suffices to show that this map is an isomorphism. Dualizing, we can consider the corresponding map $H_*(M\gamma;\mathbb{Z}/2) \to \mathcal{A}^*$ of comodules over the dual Steenrod algebra $\mathcal{A}^*$. Next, by the Thom isomorphism we know that $H_*(M\gamma;\mathbb{Z}/2) \cong H_*(\Omega^2 S^3;\mathbb{Z}/2)$. The homology of $\Omega^2 S^3$ is $P\{x_n \mid n \geqslant 0\}$, where $x_0$ comes from the inclusion of $H_*(S^1;\mathbb{Z}/2)$ and the action of the Dyer–Lashof operations is known [**10**]; specifically, $x_0$ generates the homology as a module over the Dyer–Lashof algebra. Now, note that since the dimensions of $\mathcal{A}$ and $H^*(\Omega^2 S^3;\mathbb{Z}/2)$ are the same, it is enough to show that the evaluation map is either an injection or a surjection.

There are a variety of arguments to establish this fact; we shall review the technique used by [**31**]. First, we observe that both the Thom isomorphism and the map $\gamma_* : H_*(\Omega^2 S^3;\mathbb{Z}/2) \to H_*(BO;\mathbb{Z}/2)$ commute with the Dyer–Lashof operations. Recall that $H_*(BO;\mathbb{Z}/2)$ is generated by the images of the class in degree 1 under the first Dyer–Lashof operation. Therefore the behavior of $\gamma_*$ is completely determined by the fact that $\gamma_*(x_0)$ is that generating class in degree 1. Finally, we note that under the evaluation map $H_*(MO;\mathbb{Z}/2) \to \mathcal{A}$ the images of the iterates of $\gamma_*(x_0)$ under the Dyer–Lashof operation hit all of the generators of $\mathcal{A}$.

### 9.2. *$H\mathbb{Z}/p$ as the Thom spectrum of a double loop map*

Unfortunately, no stable spherical fibration can have $H\mathbb{Z}/p$ as its associated Thom spectrum: $\pi_0(Mf)$ is either $\mathbb{Z}$ or $\mathbb{Z}/2$, depending on whether $f$ represents an orientable bundle or not. Nonetheless, in [**20**] there is a brief discussion of an argument by Hopkins for realizing $H\mathbb{Z}/p$ as the Thom spectrum associated to a $p$-local stable spherical fibration.

In the bulk of this paper, we studied Thom spectra associated to the monoid $\mathscr{I}$-FCP that were augmented over $F$. The map to $X \to F$ was used to give an action of $X(V)$ on the sphere $S^V$, the fiber of the universal quasifibration $B(*, X(V), S^V) \to B(*, X(V), *)$. However, as we noted previously, this theory can be carried out with other choices of fiber, in particular the collection of $p$-local spheres $S^V_{(p)}$ or $p$-complete spheres $(S^V)^\wedge_p$. Rather than an augmentation over $F$, we will in this setting require augmentation over the appropriate '$p$-local' or '$p$-complete' analog. We rely on the careful treatment of fiberwise localization and completion given by May [**25**].

Definition 9.1. (i) Let $F_{(p)}$ denote the commutative monoid $\mathscr{I}$-FCP specified by taking $V$ to the based homotopy self-equivalences of $S^V_{(p)}$. Denote by $BF_{(p)}$ the commutative monoid $\mathscr{I}$-FCP obtained by passing to classifying spaces levelwise.

(ii) Let $(F)^{\wedge}_{p}$ denote the commutative monoid $\mathscr{I}$-FCP specified by taking $V$ to the based homotopy self-equivalences of $(S^V)^{\wedge}_{p}$. Denote by $B(F)^{\wedge}_{p}$ the commutative monoid $\mathscr{I}$-FCP obtained by passing to classifying spaces levelwise.

Here $BF_{(p)}(V)$ classifies spherical fibrations with fiber $S^V_{(p)}$ and $B(F)^{\wedge}_{p}(V)$ classifies spherical fibrations with fiber $(S^V)^{\wedge}_{p}$ (see [**25**]). Note that we must use continuous versions of localization and completion in order to ensure that we have continuous functors [**15**].

Remark 9.2. The notation we are using is potentially confusing, as the spaces $BF_{(p)}(V)$ are not the p-localizations of $BF(V)$ and the spaces $B(F)^{\wedge}_{p}$ are not the p-completions of $BF(V)$. Such equivalences are only true after passage to universal covers, as there is an evident difference at $\pi_1$.

In this setting, we can set up the theory of Thom spectra as discussed in previous sections of the paper with minimal modifications. For oriented bundles there is a Thom isomorphism with $\mathbb{Z}_{(p)}$ or $Z^{\wedge}_{p}$ and for unoriented bundles there is a $\mathbb{Z}/p$ Thom isomorphism [**25**].

Now, $\pi_1(BF_{(p)})$ is the group of $p$-local units $\mathbb{Z}_p^{\times}$. Consider a map $\phi : S^1 \to BF_p$ associated to a choice of unit $u$. The Thom spectrum associated to $\phi$ is the Moore spectrum obtained as the cofiber of the map $S_p \to S_p$ given by multiplication by $u - 1$. This identification follows immediately from the general description of the Thom spectrum of a bundle over a suspension [**17**, 9.3.8]. Taking $u = p + 1$, which is a $p$-local unit, we obtain the Moore spectrum $M(\mathbb{Z}/p)$. As before, there is an induced map $\gamma : \Omega^2 S^3 \to BF_{(p)}$ since $BF_{(p)}$ in an infinite loop space.

We will show that the Thom spectrum associated to this map is $H\mathbb{Z}/p$. Once again, the Thom class specifies a map $\mathcal{A}_p \to H^*(M\gamma)$ of modules over the Steenrod algebra. For odd $p$, we have $H_*(\Omega^2 S^3; \mathbb{Z}/p) = E\{x_n \mid n \geqslant 0\} \otimes P\{\beta x_n \mid n \geqslant 1\}$, where $x_0$ comes from the inclusion of $H_*(S^1; \mathbb{Z}/p)$, and is generated as a module over the Dyer–Lashof algebra by $x_0$ (see [**10**]). Again, note that since the dimensions of $\mathcal{A}$ and $H^*(\Omega^2 S^3; \mathbb{Z}/p)$ are the same, it is enough to show that the evaluation map is either an injection or a surjection. This can be shown by an argument analogous to the one described for $p = 2$.

### 9.3. *$H\mathbb{Z}$ as the Thom spectrum of a double loop map*

Finally, we consider the case of $H\mathbb{Z}$. It has long been known that $H\mathbb{Z}$ arises as the Thom spectrum associated to a certain map $\gamma : \Omega^2(S^3\langle 3\rangle) \to BSF$ (see [**10**, **19**]). However, the best published results obtain a description of this map as an $H$-map [**10**], which is inadequate for construction of THH. Moreover, it is not clear how to adapt the existing construction to improve this; the map $\gamma$ is constructed a prime at a time, and the localized maps $\gamma_p$ are seen to be $H$-maps because certain obstructions vanish.

Therefore, we give a new construction, based on a suggestion of Mike Mandell, which enables us to see that there is a suitable map that is a double loop map. Both $\Omega^2 S^3\langle 3\rangle$ and $BSF$ are rationally trivial, and so split as the product of their completions. Therefore a map $\Omega^2 S^3\langle 3\rangle \to BSF$ can be specified by the construction of a collection of maps $\Omega^2 S^3\langle 3\rangle \to (BSF)^{\wedge}_{p}$. Note that the $p$-completion of $BSF$ is weakly equivalent to $\operatorname{colim}_V B((SF)^{\wedge}_{p})$, where $(SF)^{\wedge}_{p}$ is the monoid $\mathscr{I}$-FCP constructed analogously to $(F)^{\wedge}_{p}$. The following lemma is standard.

Lemma 9.3. *Let $f : \Omega^2 S^3\langle 3\rangle \to BSF$ be a map specified by a collection of maps*

$$f_p : \Omega^2 S^3\langle 3\rangle \longrightarrow (BSF)^{\wedge}_{p}.$$

*If each $f_p$ is an $n$-fold loop map, then $f$ is an $n$-fold loop map.*

Next, we observe that it will suffice to show that at each prime, the map given by evaluation on the Thom class induces an equivalence between the Thom spectrum associated to $\Omega^2 S^3 \langle 3 \rangle \to B(SF)^\wedge_p$ and $H\mathbb{Z}^\wedge_p$. For this it suffices to show that the evaluation map induces an equivalence in $\mathbb{Z}/p$ cohomology for each $p$.

For $p = 2$ we can use the map induced by the composite

$$\Omega^2 S^3 \langle 3 \rangle \longrightarrow \Omega^2 S^3 \longrightarrow BO \longrightarrow B(SF)^\wedge_p.$$

This is a double loop map, and the associated Thom spectrum is $H\mathbb{Z}^\wedge_2$ (see [**10**]). For odd primes we proceed as follows. We know that $\pi_1(B(F)^\wedge_p)$ is the group of $p$-adic units $(\mathbb{Z}^\wedge_p)^\times$. Explicitly, for odd primes this is $(\mathbb{Z}^\wedge_p)^\times \cong \mathbb{Z}/(p-1) \oplus \mathbb{Z}^\wedge_p$. Take a map $\phi$ representing an element of $\pi_1(B(F)^\wedge_p)$ which is 0 on the $\mathbb{Z}/(p-1)$ factor and induces an isomorphism on the other component. We can equivalently regard $\phi$ as a map $\phi : S^3 \to B^3(F)^\wedge_p$. Now we can lift to a map $S^3 \langle 3 \rangle \to B^3(SF)^\wedge_p$. Since $\phi$ is trivial on the $\mathbb{Z}/(p-1)$ component of $\pi_3(B^3(F)^\wedge_p)$, we can lift the map to the fiber over the map $B^3(F)^\wedge_p \to K(\mathbb{Z}/(p-1), 3)$. The induced map is an isomorphism on $\pi_3$ by construction, and so now we can pass to fibers over $K((Z)^\wedge_p, 3)$ to obtain the desired map. Looping twice, denote by $\gamma$ the resulting map $\Omega^2 S^3 \to B(F)^\wedge_p$ and by $\gamma'$ the resulting map $\Omega^2 S^3 \langle 3 \rangle \to BS(F)^\wedge_p$.

We begin by identifying the Thom spectrum $M\gamma$; we will then use this to determine $M\gamma'$. The analysis of $M\gamma$ proceeds essentially as in the previous examples. Specifically, the Thom spectrum associated to the map $\phi$ is the Moore spectrum obtained as the cofiber of the map which is multiplication by $u - 1$, where $u$ is the chosen $p$-adic unit. This Moore spectrum is determined by the $p$-adic valuation of $u - 1$. To compute this, let us recall the identification of the $p$-adic units. A unit in $(Z)^\wedge_p$ is a $p$-adic integer with an expansion such that the first digit is nonzero. The projection onto the units of $\mathbb{Z}/p$ induces the first component of the identification. In our case, we are requiring a choice where the first component is 1. Subtracting 1 from this, we find that the first component must be 0 and the later components are arbitrary. Combining with the constraint that the projection of $u$ generates the $(Z)^\wedge_p$, we find that we have the Moore spectra $M(\mathbb{Z}/p)$. A similar argument to the one employed above implies that $M\gamma$ is $H\mathbb{Z}/p$.

Finally, we use this identification to determine the Thom spectrum $M\gamma'$. Let us first consider the case of $p$ an odd prime. Essentially by construction, there is a commutative diagram of Thom spectra

$$\begin{array}{ccc} Mf & \longrightarrow & M((SF)^\wedge_p) \\ \downarrow & & \downarrow \\ H\mathbb{Z}/p & \longrightarrow & M((F)^\wedge_p) \end{array}$$

associated to the commutative diagram of spaces

$$\begin{array}{ccc} \Omega^2 S^3 \langle 3 \rangle & \longrightarrow & B((SF)^\wedge_p) \\ \downarrow & & \downarrow \\ \Omega^2 S^3 & \longrightarrow & B((F)^\wedge_p). \end{array}$$

By the naturality of the Thom isomorphism, this implies that we have a commutative diagram of modules over the Steenrod algebra as follows:

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & H^*(M\gamma) \\ \downarrow & & \downarrow \\ \mathcal{A}/\beta\mathcal{A} & \longrightarrow & H^*(M\gamma') \end{array}$$

The map $\mathcal{A} \to \mathcal{A}/\beta\mathcal{A}$ is a surjection; we have seen that the map $\mathcal{A} \to H^*(M\gamma)$ is an isomorphism, and $\Omega^2 S^3 \langle 3 \rangle \to \Omega^2 S^3$ induces a surjection on cohomology (and on homology a map of comodules over the dual Steenrod algebra). This implies that the bottom horizontal map must be a surjection. Since the dimensions of $\mathcal{A}/\beta\mathcal{A}$ and $H^*(\Omega^2 S^3 \langle 3 \rangle ; \mathbb{Z}/p)$ are the same, this map must in fact be an isomorphism.

REMARK 9.4. If we work at the prime 2, we have that $\pi_1$ is $(\mathbb{Z}_2^\wedge)^\times = \mathbb{Z}/2 \oplus \mathbb{Z}_2^\wedge$. Following the outline above, we would like to identify the Thom spectrum associated to $\phi$. The projection onto the units of $\mathbb{Z}/4$ induces the first component of the identification of $(\mathbb{Z}_2^\wedge)^\times$. The two choices are expansions that begin $1, 1, \ldots$ and $1, 0, \ldots$. Since we want something that projects to 0, we must have the latter. Subtracting 1 from this, we find we end up with a $p$-adic number that begins $0, 0, \ldots$ and therefore has $p$-adic valuation 2 or higher. Thus the associated Thom spectrum is the Moore spectrum $M(\mathbb{Z}/4)$.

However, consideration of the Dyer–Lashof operations tells us that the Thom spectrum of $\gamma$ is not $H(\mathbb{Z}/4)$. In general, we cannot obtain $H(\mathbb{Z}/p^n)$ as a Thom spectrum over $\Omega^2 S^3$. This can be seen by considering the element $x_0$ in $H_1(\Omega^2 S^3)$. The last Dyer–Lashof operation takes this to $Q_2 x$, but since the classifying map takes $x$ to 0, it must take $Q_2 x$ to zero and thus must be 0 on $H^3$ as well, which implies that the Thom spectrum cannot be the Eilenberg–Mac Lane spectrum. It is also possible to deduce the impossibility of realizing $H(\mathbb{Z}/p^n)$ as such a Thom spectrum by observing that the computations of [**9**] are incompatible with our splitting results.

## *Appendix A. Change of operads*

The linear isometries operad arises naturally when considering the infinite loop space structure on $BG$. Moreover, since we are interested in a Thom spectrum functor that takes values in the EKMM category of spectra, the presence of the linear isometries operad is to be expected. However, it is useful to be able to accept a somewhat broader range of input data.

In the examples above, the initial input was the maps $X \to B^n(BF)$, which were looped down to produce the $n$-fold loop maps $\Omega^n X \to \Omega^n B^n(BF)$. To specify the multiplicative structure carefully, we need to choose a precise model of the delooping $B$. Let us assume that we are working with a specified choice of $BF$, where the $E_\infty$ structure is described by an action of the linear isometries operad $\mathscr{L}$. By pullback, we regard this as a space structured by the product operad $\mathcal{C}_n \times \mathscr{L}$, where $\mathcal{C}_n$ is the little $n$-cubes operad. Denote by $\mathbb{D}$ the monad associated to this operad. Following [**23**, 13.1], for any $\mathbb{D}$-space $Z$ we have the diagram

$$Z \xleftarrow{\simeq} B(\mathbb{D}, \mathbb{D}, Z) \xrightarrow{\simeq} \Omega^n B(\Sigma^n, \mathbb{D}, Z)$$

in which the maps are maps of $\mathbb{D}$-spaces, and the action of $\mathbb{D}$ on $\Omega^n \Sigma^n$ comes from the augmentation of $\mathbb{D}$ over the monad associated to the little $n$-cubes operad. The $\mathbb{D}$-space action on $\Omega^n B(\Sigma^n, \mathbb{D}, Z)$ is produced by pullback from the $\mathcal{C}_n$ action on $B(\Omega^n \Sigma^n, \mathbb{D}, Z)$. Thus, we use $B(\Sigma^n, \mathbb{D}, BF)$ as our model of $B^n BF$.

Given a map $X \to B(\Sigma^n, \mathbb{D}, BF)$, the associated map $\Omega^n X \to \Omega^n B(\Sigma^n, \mathbb{D}, BF)$ is a map of $\mathbb{D}$-spaces with regard to the geometric action of the little $n$-cubes operad; and on $\Omega^n B(\Sigma^n, \mathbb{D}, BF)$, this is precisely the action that arises in the diagram above. Replacing the map by a fibration and pulling back, we get a map of $\mathbb{D}$-spaces $X' \to B(\mathbb{D}, \mathbb{D}, BF)$, and pushing forward we get a map of $\mathbb{D}$-spaces $X' \to BF$, where the $\mathbb{D}$ action on $BF$ comes from the augmentation over the linear isometries operad.

*Acknowledgements.* These results appeared as part of the author's 2005 University of Chicago thesis. I would like to thank Peter May for his support and suggestions throughout the conduct of this research. I would also like to express my gratitude to Michael Mandell — this paper could not have been written without his generous assistance. In addition, I would like thank Christian Schlichtkrull and Ralph Cohen for agreeing to join forces in the preparation of [4]. The paper was improved by comments from Christopher Douglas, Halvard Fausk, and an anonymous referee.

## *References*


**1.** C. Ausoni and J. Rognes, 'Algebraic $K$-theory of topological $K$-theory', *Acta Math.* 188 (2002) 1–39.
**2.** M. Basterra, 'Andre-Quillen cohomology of commutative S-algebras', *J. Pure Appl. Algebra* 144 (1999) 111–143.
**3.** M. Basterra and M. A. Mandell, 'Homology and cohomology of $E_\infty$ ring spectra', *Math. Z.* 249 (2005) 903–944.
**4.** A. J. Blumberg, R. Cohen and C. Schlichtkrull, 'Topological Hochschild homology and the free loop space', *Geom. Topol.* 14 (2010) 1165–1242.
**5.** J. M. Boardman and R. M. Vogt, 'Homotopy everything $h$-spaces', *Bull. Amer. Math. Soc.* 74 (1968) 1117–1121.
**6.** M. Bokstedt, 'The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$', Preprint, 1991.
**7.** M. Bokstedt, W. C. Hsiang and I. Madsen, 'The cyclotomic trace and algebraic K-theory of spaces', *Invent. Math.* 111 (1993) 465–539.
**8.** F. Borceux, *Handbook of categorical algebra 2: categories and structures*, Encyclopedia of Mathematics and its Applications 51 (Cambridge University Press, Cambridge, 1994).
**9.** M. Brun, 'Topological Hochschild homology of $\mathbf{Z}/p^n$', *J. Pure Appl. Algebra* 148 (2000) 29–76.
**10.** F. R. Cohen, J. P. May and L .R. Taylor, '$K(\mathbb{Z}, 0)$ and $K(\mathbb{Z}_2, 0)$ as Thom spectra', *Illinois J. Math.* 25 (1981) 99–106.
**11.** B. I. Dundas, 'Relative $K$-theory and topological cyclic homology', *Acta Math.* 179 (1997) 223–242.
**12.** A. D. Elmendorf, I. Kriz, M. A. Mandell and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47 (American Mathematical Society, Providence, RI, 1997).
**13.** M. J. Hopkins, 'Notes on $E_\infty$ ring spectra', Typed Notes, 1993.
**14.** M. Intermont and M. W. Johnson, 'Model structures on the category of ex-spaces', *Topol. Appl.* 119 (2002) 325–353.
**15.** N. Iwase, 'A continuous localization and completion', *Trans. Amer. Math. Soc.* 320 (1990) 77–90.
**16.** L. G. Lewis, 'Open maps, colimits, and a convenient category of fibre spaces', *Topol. Appl.* 19 (1985) 75–89.
**17.** L. G. Lewis, J. P. May and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics 1213 (Springer, New York, 1986).
**18.** J. Lurie, 'Derived algebraic geometry', Thesis, MIT, 2004.
**19.** M. Mahowald, 'Ring spectra which are Thom complexes', *Duke Math. J.* 46 (1979) 549–559.
**20.** M. Mahowald, D. C. Ravenel and P. Shick, *The Thomified Eilenberg–Moore spectral sequence*, Progress in Mathematics 196 (Birkhäuser, Basel, 2001) 249–262.
**21.** M. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Memoirs of the American Mathematical Society 755 (American Mathematical Society, Providence, RI, 2002).
**22.** M. Mandell, J. P May, S. Schwede and B. Shipley, 'Model categories of diagram spectra', *Proc. London Math. Soc.* 3 (2001) 441–512.
**23.** J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics 271 (Springer, New York, 1972).
**24.** J. P. May, *Classifying spaces and fibrations*, Memoirs of the American Mathematical Society 155 (American Mathematical Society, Providence, RI, 1975).
**25.** J. P. May, 'Fiberwise localization and completion', *Trans. Amer. Math. Soc.* 258 (1980) 127–146.
**26.** J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs 132 (American Mathematical Society, Providence, RI, 2006).
**27.** J. P. May, F. Quinn and N. Ray, *$E_\infty$ ring spaces and $E_\infty$ ring spectra*, Lectures Notes in Mathematics 577 (Springer, New York, 1977).
**28.** R. McCarthy, 'Relative algebraic $K$-theory and topological cyclic homology', *Acta Math.* 179 (1997) 197–222.
**29.** J. E. McClure, R. Schwanzl and R. Vogt, 'THH$(R) \cong R \otimes S^1$ for $E_\infty$ ring spectra', *J. Pure Appl. Algebra* 121 (1997) 137–159.
**30.** J. E. McClure and R. E. Staffeldt, 'On the topological Hochschild homology of *bu*. I', *Amer. J. Math.* 115 (1993) 1–45.
**31.** S. Priddy, 'K$(Z/2)$ as a Thom spectrum', *Proc. Amer. Math. Soc.* 70 (1978) 207–208.
**32.** J. Rognes, 'Galois extensions of structured ring spectra', *Mem. Amer. Math. Soc.* 192 (2008) 898.
**33.** C. Schlichtkrull, 'Units of ring spectra and their traces in algebraic $K$-theory', *Geom. Topol.* 8 (2004) 645–673.

**34.** C. Schlichtkrull, 'Higher topological Hochschild homology of Thom spectra', Preprint, 2008, http://arxiv.org/abs/0811.0597.
**35.** F. Waldhausen, 'Algebraic $K$-theory of spaces, localization, and the chromatic filtration of stable homotopy', *Algebraic topology*, Lecture Notes in Mathematics 1051 (Springer, Berlin, 1984) 173–195.
**36.** F. Waldhausen, 'Algebraic $K$-theory of spaces', *Algebraic and geometric topology*, Lecture Notes in Mathematics 1126 (Springer, Berlin, 1985) 318–419.

*Andrew J. Blumberg*
*Department of Mathematics*
*University of Texas*
*Austin, TX 78712*
*USA*

blumberg@math.utexas.edu