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THE HOMOTOPY SPECTRAL SEQUENCE OF A SPACE WITH COEFFICIENTS IN A RING†

A. K. BOUSFIELD and D. M. KAN

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§1. INTRODUCTION

IN THIS paper we start the investigation of a spectral sequence $\{E_r(X; R)\}$ defined for every space X (with base point) and ring R, which, very roughly speaking, *goes from R-homology to R-homotopy*. Before trying to explain its construction we list its

1.1. Main properties

(i) *Convergence*. Under suitable hypotheses (a.o. $\pi_1 X = 0$)

$$\{E_r(X; Z)\} \Rightarrow \pi_* X$$

$$\{E_r(X; Q)\} \Rightarrow \pi_* X \otimes Q$$

$$\{E_r(X; Z_p)\} \Rightarrow \pi_* X/(\text{torsion prime to } p).$$

(ii) *The E_2-term*. If R is a field, then the E_1-term depends functorially on $\tilde{H}_*(X; R)$ ($\tilde{H}_*$ denotes reduced homology) and the differential d_1 (and hence the E_2-term) depends on only primary operations. In particular, for $R = Z_p$, the E_2-term is an "unstable Ext" depending only on the structure of $H_*(X; Z_p)$ as a coalgebra over the Steenrod algebra.

(iii) $\{E_r(X; Z_p)\}$ *is an unstable Adams spectral sequence*; i.e. for $R = Z_p$ our spectra sequence coincides, in the stable range, with the Adams spectral sequence.

(iv) *Comparison with other unstable Adams spectral sequences*. These are:

(a) *The Massey–Peterson spectral sequence* (see [15] for $p = 2$, §13 for p odd), which has the right E_2-term (i.e. the unstable Ext mentioned above), but is only defined for "very nice" (see 13.1) spaces.

(b) *The accelerated* 2-*lower central series spectral sequence* [6, 20], whose E_2-term depends in general on higher order operations, but is "right" for "nice" spaces (these include loop spaces, but not wedges of spheres).

(c) *The p-derived spectral sequence* (1.3), whose E_2-term also depends, in general, on higher order operations, but is "right" for "nice" spaces.

(d) *The Hopf tower spectral sequence* [21] which is defined for various pseudo Hopf spaces and has the "right" E_2-term in the case where it is known.

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As mentioned above our spectral sequence is defined for all spaces and always has the "right" E_2-term. In a future note we expect to show for "nice" spaces that it coincides from E_2 on with (b) and (c). Presumably for "very nice" spaces it coincides from E_2 on with all the above spectral sequences.

(v) *Generalization to function complexes.* As usual there is such a generalization.

(vi) *Pairings and products.* The spectral sequence admits smash and composition pairings as well as Whitehead products.

We now try to explain the

1.2. Construction of the spectral sequence.

This is based on the following homotopy version of the Hurewicz homomorphism:

For any ring R and space X with base point $*$ one can (if one works in a suitable category of topological spaces [23]) *define a topological left R-module RX as the left R-module with a generator for every point of X and one relation $1 \, . \, * = 0$, topologized by the requirement that the inclusion $\phi\colon X \to RX$ be continuous and open.* The usefulness of this construction lies in the fact that:

(i) *There is a natural isomorphism*

$$\pi_* RX \approx \tilde{H}_*(X; R).$$

(ii) *As RX is abelian it has trivial k-invariants and hence its homotopy type depends only on its homotopy groups and therefore on $H_*(X; R)$; if R is a field, this dependence is functorial.*

(iii) *The homomorphism*

$$\pi_* X \xrightarrow{\phi_*} \pi_* RX \approx \tilde{H}_*(X; R)$$

is the Hurewicz homomorphism.

Our spectral sequence then is the homotopy spectral sequence of the natural tower of fibre maps

$$\cdots \to D_{s+1}X \xrightarrow{\delta} D_s X \to \cdots \to D_1 X \xrightarrow{\delta} D_0 X = X$$

obtained by defining $\delta\colon D_{s+1}X \to D_s X$ as the fibre map induced by the map $D_s \phi\colon D_s X \to D_s RX$ (from the path fibration over $D_s RX$). Observe that:

(a) *The $\pi_* D_s RX$, and hence the E_1-term, depends only on $\tilde{H}_*(X; R)$, although not functorially (unless R is a field), and*

(b) *as always the images of the $\pi_* D_s X$ filter $\pi_* X$ and the associated graded group is naturally isomorphic with the E_∞-term or a subgroup thereof.*

This is what we meant by our statement in the beginning that the spectral sequence goes from R-homology to R-homotopy.

Remark 1.3. The homotopy spectral sequence of the more obvious tower

$$\cdots \to D'_{s+1} X \xrightarrow{\delta'} D'_s X \to \cdots \to D'_1 X \xrightarrow{\delta'} D'_0 X = X$$

obtained by defining δ': $D'_{s+1}X \to D_s'X$ as the fibre map induced by the map ϕ: $D'_sX \to RD'_sX$ is nothing but the *derived* spectral sequence (which, for $R = Z_p$, coincides with the p-derived series spectral sequence). It has, of course, the just mentioned property (b), but not property (a), as in general $\pi_* RD_s'X$ does not depend only on $H_*(X; R)$.

The paper is written semi-simplicially and freely uses the notation and results of [11], [14], [16]. This is, of course, not essential and a more topologically oriented reader should have no problems translating the results into for him more understandable language. There are two chapters and an appendix.

In §2 and §3 we lay the foundations for the definition of the spectral sequence (§4). In §5 we show that our spectral sequence, for $R = Z_p$, is an unstable Adams spectral sequence (by comparing it with the derived spectral sequence), while §6 contains a more precise formulation of the above convergence statements as well as their proofs; the latter rely heavily on the Curtis–Rector and Curtis convergence theorems [10], [20] for the (p)-lower central series spectral sequences. And in §7 we show how all this can be generalized to function complexes. We end the first chapter with the observation (§8) that the spectral sequence for arbitrary commutative R is completely determined by the spectral sequences for $R = Z$ and $R = Z_{p^j}$ (p prime).

Chapter II deals with the E_2-term, mainly for $R = Z_p$. In §9 and §10 we give a cosimplicial description of $E_2(X; R)$ which we use in §11 to show that $E_2(X; Z_p)$ depends only on $H_*(X; Z_p)$ as an unstable coalgebra over the Steenrod algebra. (We state our results in terms of *homology* as this seems the natural thing to do. However a reader who prefers a cohomological approach should have no problem translating our results into cohomological terms, *if he is willing to impose suitable finiteness conditions on* X.) In §12 $E_2(X; Z_p)$ is described as an unstable Ext which (§13) for "very nice" spaces coincides with the Massey–Peterson Ext. And in §14 we give, for "very nice" spaces and p odd, a convenient E_1-term (the case $p = 2$ was done in [6]).

An appendix contains the calculation of $E_2(K(G, n); R)$ for abelian G and commutative R.

The construction of the various pairings and products requires completely different techniques and will be published separately [8].

CHAPTER 1. THE SPECTRAL SEQUENCE

§2. PRELIMINARIES

We start with a quick review of some well known constructions in the category $\mathscr{S}_*$ of simplicial sets with base point $*$ and its full subcategory $\mathscr{S}_{*K}$ of Kan complexes with base point.

2.1. Simplicial modules generated by simplicial sets

Let $X \in \mathscr{S}_*$ and let R be a ring (with unit). Then $RX \in \mathscr{S}_{*K}$ will denote *the simplicial R-module* (with 0 as base point) *generated by the simplices of* X, *with the base point of* X (and its degeneracies) *put equal to* 0, and we will write

$$X \xrightarrow{\phi} RX \quad \text{and} \quad RRX \xrightarrow{\psi} RX$$

for the map given by $\phi x = 1 \cdot x$ for all $x \in X$ and the (left) R-module homomorphism given by $\psi(1 \cdot y) = y$ for all $y \in RX$.

The usefulness of this construction is due to the following two properties:

(i) (R, ϕ, ψ) *is a triple in the sense of* [12].

(ii) *There is a natural isomorphism*

$$\pi_* RX \approx \tilde{H}_*(X; R)$$

(*where* $\tilde{H}_*$ *denotes reduced homology*) *such that*

$$\pi_* X \xrightarrow{\phi_*} \pi_* RX \approx \tilde{H}_*(X; R)$$

is the Hurewicz map.

We also need:

2.2. The (standard) path fibration

For $X \in \mathscr{S}_{*K}$ we mean by *the* (*standard*) *path fibration over* X the map

$$\Lambda X \xrightarrow{\lambda} X \in \mathscr{S}_{*K}$$

where ΛX is *the* (*standard*) *path complex*, i.e. the simplicial set of which an n-simplex is any $x \in X_{n+1}$ such that $d_1 \cdots d_{n+1} x = *$ and of which the face and degeneracy maps

$$\Lambda X_n \xrightarrow{d_i} \Lambda X_{n-1} \qquad \Lambda X_n \xrightarrow{s_i} \Lambda X_{n+1} \qquad 0 \le i \le n$$

are the functions induced by the maps

$$X_{n+1} \xrightarrow{d_{i+1}} X_n \qquad X_{n+1} \xrightarrow{s_{i+1}} X_{n+2} \qquad 0 \le i \le n$$

and where λ is the map induced by the 0-face operator

$$\Lambda X_n \subset X_{n+1} \xrightarrow{d_0} X_n.$$

Clearly λ *is a fibre map with contractible total complex.*

§3. DERIVATION OF A FUNCTOR WITH RESPECT TO A RING

For an efficient definition of our spectral sequence and proof of its convergence (under suitable hypotheses, of course) we need the notion of

3.1. Derivation of a functor with respect to a ring

Let R be a ring (with unit) and let $T: \mathscr{S}_* \to \mathscr{S}_*$ be a covariant functor which *respects* $\mathscr{S}_{*K}$ (i.e. $X \in \mathscr{S}_{*K}$ implies $TX \in \mathscr{S}_{*K}$). Then we define a functor

$$D_1 T: \mathscr{S}_* \to \mathscr{S}_*$$

(*the derivation of* T *with respect to* R) and a natural transformation

$$\delta: D_1 T \to T$$

by requiring that *for each* $X \in \mathscr{S}_*$ *the map* $\delta X\colon (D_1T)X \to TX$ *is the fibre map induced by the map* $T\phi\colon TX \to TRX$ *from the path fibration* (2.2) *over* TRX, i.e., D_1T and δ are determined by the pull back diagram

$$\begin{array}{ccc} D_1T & \longrightarrow & \Lambda TR \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\lambda} \\ T & \xrightarrow{T\varphi} & TR\,. \end{array}$$

This definition is *natural* with respect to T, i.e. a natural transformation $\gamma\colon T \to T'$ induces a natural transformation $D_1\gamma\colon D_1T \to D_1T'$.

One readily verifies

PROPOSITION 3.2. *If* T *respects* $\mathscr{S}_{*K}$, *then so does* D_1T.

Therefore we can make the following:

3.3. Notational convention

(i) If $X \in \mathscr{S}_*$ and Id: $\mathscr{S}_* \to \mathscr{S}_*$ denotes the identity, then we write

$$D_sX \quad \text{for} \quad (D_1 \cdots (D_1Id)\cdots)X.$$

(ii) If $X \in \mathscr{S}_*$ and $T_1, \ldots, T_n\colon \mathscr{S}_* \to \mathscr{S}_*$ are functors which respect $\mathscr{S}_{*K}$, then we write

$$D_sT_1 \cdots T_nX \quad \text{for} \quad (D_1 \cdots (D_1(T_1 \cdots T_n))\cdots)X.$$

Note that this implies that in general

$$D_s(TX) \neq D_sTX.$$

Other obvious properties of the derivation are:

3.4. Preservation of weak homotopy equivalences

If $\gamma\colon T \to T'$ *is a natural weak homotopy equivalence, then so is* $D_1\gamma\colon D_1T \to D_1T'$.

3.5. Preservation of fibrations

If

$$T' \xrightarrow{i} T \xrightarrow{p} T''$$

is a natural fibration (i.e. for every $X \in \mathscr{S}_*$, $pX\colon TX \to T''X$ is a fibre map with $iX\colon T'X \to TX$ as fibre), *then so is*

$$DT' \xrightarrow{Di} DT \xrightarrow{Dp} DT''.$$

For later reference we also state (a).

TWISTING LEMMA 3.6. *Let* $s > 0$. *Then the natural transformations*

$$D_i\delta\colon D_{s+1}T \to D_sT \qquad 0 \le i \le s$$

are weakly homotopic.

Proof. For $X \in \mathcal{S}_*$ let $j\colon X \to \operatorname{Sin}|X|$ denote the natural map of X into the singular complex of its realization [16]. Then one readily verifies that the compositions

$$D_2 T \xrightarrow{D_i \delta} D_1 T \xrightarrow{D_1 j} D_1 \operatorname{Sin}|T| \qquad i = 0, 1$$

are naturally homotopic, which proves the lemma for $s = 1$. The general case now follows from 3.4.

§4. THE SPECTRAL SEQUENCE

Now we can define the homotopy spectral sequence of a space $X \in \mathcal{S}_*$ with coefficients in a ring R and discuss some of the immediate consequences of this definition.

4.1. The spectral sequence

Let $X \in \mathcal{S}_*$ and let R be a ring. *The homotopy spectral sequence* $\{E_r(X; R)\}$ (or short $\{E_r X\}$) *of* X *with coefficients in* R is the homotopy spectral sequence of the sequence of fibre maps

$$\cdots \xrightarrow{\delta} D_s X \xrightarrow{\delta} D_{s-1} X \to \cdots \to D_1 X \xrightarrow{\delta} D_0 X = X$$

fringed in dimension 1. By this we mean that

$$\begin{aligned} E_1^{s,t} X &= \pi_{t-s} D_s(RX) \qquad & t - 1 \geq s \geq 0 \\ &= 0 & \text{otherwise} \end{aligned}$$

and that

$$E_r^{s,t} X = \ker d_{r-1} / \operatorname{im} d_{r-1} \qquad t - 1 > s \geq 0$$

but in dimension 1

$$E_r^{s,s+1} X \subset E_{r-1}^{s,s+1} X / \operatorname{im} d_{r-1} \qquad s \geq 0$$

as we define $E_r^{s,s+1} X$ by

$$E_r^{s,s+1} X = Z_{r-1}^{s,s+1} X / \operatorname{im} d_{r-1} \qquad s \geq 0$$

where $Z_{r-1}^{s,s+1} X \subset E_{r-1}^{s,s+1} X$ consists of what would have been the cycles, i.e. the elements for which the image under the boundary map $\partial\colon \pi_1 D_s(RX) \to \pi_0 D_{s+1} X$ lifts to $\pi_0 D_{s+r} X$.

One has of course, to verify that $Z_{r-1}^{s,s+1} X$ *is indeed a group*; but this readily follows, by induction on r, from the observation that:

(i) A spherical 1-simplex $y \in D_s(RX)$ can be considered as an $(s+1)$-simplex $v \in R^{s+1} X$ (R^{s+1} denotes the $(s+1)$-fold iteration of R) such that $d_i v$ lies in the image of the map $R^i \phi\colon R^s X \to R^{s+1} X$ for $0 \leq i < s$ and $d_s v = d_{s+1} v = *$, and

(ii) the simplex y represents an element of $Z_{r-1}^{s,s+1} X$ if and only if there is an $(s+r)$-simplex $w \in R^{s+r} X$ such that $d_i w$ lies in the image of the map $R^i \phi\colon R^{s+r-1} X \to R^{s+r} X$ for $0 \leq i < s + r$, $d_{s+r} w = *$ and

$$d_{s+1} \cdots d_{s+r-1} w = (R^{s+r-1}\phi) \cdots (R^{s+1}\phi) v.$$

4.2. Why the fringe and not an edge

We *fringed* the above spectral sequence (i.e. defined E_r in the bottom dimension in the most natural way without worrying whether in this dimension also $E_r = H(E_{r-1}, d_{r-1})$) instead of *edging* it (i.e. defining E_r in the bottom dimension in such a manner that in this dimension also $E_r = H(E_{r-1}, d_{r-1})$) because:

(i) The edging can be done in several different ways, each of which has some advantages as well as disadvantages which seems to suggest that edging may not be the "right" thing to do.

(ii) There will no longer be any need for special statements about the bottom dimension.

An immediate consequence of the definition of the spectral sequence is

4.3. Dependence of $E_1 X$ on $\tilde{H}_*(X; R)$

The E_1-term $E_1 X$ depends only on the homotopy type of RX, i.e. on $\tilde{H}_*(X; R)$ as a graded abelian group. Moreover, if R is a field, then this dependence is functorial.

For simplicial R-modules we have the following

4.4. Collapsing lemma

Let $X \in \mathcal{S}_*$ be a simplicial (left) R-module. Then

$$E_2^{s,t}X = E_\infty^{s,t}X = 0 \qquad \text{for } s > 0$$

$$E_2^{0,t}X = E_\infty^{0,t}X \approx \pi_t X \qquad \text{for } t > 0$$

Proof. Let $\psi' : RX \to X$ be the (unique) R-module homomorphism such that

$$\psi'\phi = id: X \to X.$$

Then for all $s > 0$

$$(D_s\psi')(D_s\phi) = id: D_s X \to D_s X$$

and the lemma readily follows.

We end with:

4.5. Some trivialities about $E_r X$ and $E_\infty X$

(i) $d_r : E_r^{s,t}X \to E_r^{s+r,\,t+r-1}X.$

(ii) $E_{r+1}^{s,t}X \subset E_r^{s,t}X \qquad$ *for $r > s$.*

(iii) $E_\infty^{s,t}X = \bigcap_{r>s} E_r^{s,t}X.$

(iv) *for $t - 1 \geq s \geq 0$ there is a natural short exact sequence*

$$0 \to (F^s/F^{s+1})\pi_{t-s}X \xrightarrow{\ e\ } E_\infty^{s,t}X \to F^\infty\pi_{t-s-1}D_{s+1}X \cap \ker \delta_* \to 0$$

where $F^u\pi_g D_s X = im(\pi_g D_{s+u} X \to \pi_g D_s X)$

and $F^\infty\pi_g D_s X = \bigcap_u F^u\pi_g D_s X$.

Remark 4.6. The only property of the functor $R : \mathscr{S}_* \to \mathscr{S}_*$ that was used (except in 4.2) was that (R, ϕ, ψ) is a triple in the sense of [12]. Similar results thus hold for other triples such as, for instance, Milnor's free group functor F [11].

§5. COMPARISON WITH THE ADAMS SPECTRAL SEQUENCE

We will show in this section that our spectral sequence coincides in the stable range with the well known derived spectral sequence and hence, for $R = Z_p$, with the Adams spectral sequence.

First we recall the definition of:

5.1. The derived spectral sequence

For $X \in \mathscr{S}_*$ and R a ring let

$$\cdots \to D'_{s+1}X \xrightarrow{\delta'} D_s'X \to \cdots \to D_1'X \xrightarrow{\delta'} D_0'X = X$$

be the tower of fibrations where *each δ' is the fibre map induced by the map $\phi\colon D'_s X \to RD'_s X$ from the path fibration* over RD'_sX, i.e. δ' is given by the pull back diagram

$$\begin{array}{ccc} D'_{s+1}X & \longrightarrow & \Lambda RD_s'X \\ \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \lambda} \\ D_s'X & \xrightarrow{\varphi} & RD_s'X . \end{array}$$

The derived spectral sequence of X with coefficients in R then is the homotopy spectral sequence of this tower, fringed and indexed as in §4.

For $R = Z_p$ *and in the stable range this* clearly *is the Adams spectral sequence*

In order to compare the derived spectral sequence with ours we need the fact that *$D_s(RX)$ can be turned into a simplicial R-module.* To be precise

LEMMA 5.2. *For $s \geq 0$ there is a natural isomorphism*

$$h_s : (D_s R)X \approx D_s(RX).$$

Proof. Let

$$RRX \xrightarrow{t} RRX$$

denote the natural twisting map given by

$$tu = \phi\psi u - u + (R\phi)\psi u$$

for all $u \in RRX$. Then one readily verifies that $tt = id$ and that the diagram

$$\begin{array}{ccccc} & & RX & & \\ & {\scriptstyle \varphi}\swarrow & & \searrow{\scriptstyle R\varphi} & \\ RRX & & \overset{t}{\approx} & & RRX \end{array}$$

commutes. Now let $h_0 = id$ and inductively define h_{s+1} as induced by the diagram

$$\begin{array}{ccc}
(D_s R)X & \xrightarrow{\quad h_s \quad} & D_s(RX) \\
\downarrow{\scriptstyle (D_s R)\varphi} & & \downarrow{\scriptstyle D\varphi} \\
(D_s R)RX \xrightarrow{h_s} D_s(RRX) & \xrightarrow{D_s t} & D_s(RRX).
\end{array}$$

5.3. Comparison of the spectral sequences

In order to compare the derived spectral sequence with ours we construct a commutative ladder

$$\begin{array}{ccccccc}
\cdots \to & D'_{s+1}X & \xrightarrow{\delta'} & D_s'X & \to \cdots \to & X \\
 & \downarrow{\scriptstyle f_{s+1}} & & \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f_0} \\
\cdots \to & D_{s+1}X & \xrightarrow{\delta} & D_s X & \to \cdots \to & X
\end{array}$$

by putting $f_0 = id$ and inductively defining f_{s+1} as the map induced by the diagram

$$\begin{array}{ccc}
D_s'X & \xrightarrow{\qquad f_s \qquad} & D_s X \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle D_s\varphi} \\
RD_s'X \xrightarrow{Rf_s} RD_s X & \xrightarrow{g_s} (D_s R)X \xrightarrow{h_s} & D_s(RX)
\end{array}$$

where g_s is the extension of the (obvious) map $D_s X \to (D_s R)X$ to a homomorphism of simplicial R-modules. This ladder induces a map from the derived spectral sequence to ours which, according to the following lemma, is an isomorphism in the "stable range". We leave to the reader the task of interpreting the term "stable range" precisely.

LEMMA 5.4. *The maps*

$$D_s' X \xrightarrow{f_s} D_s X$$

$$RD_s'X \xrightarrow{g_s} (D_s R)X$$

induce isomorphisms of the homotopy groups in the "stable range".

The proof is straightforward using induction on s first for g_s and then for f_s.

COROLLARY 5.5 *The homotopy spectral sequence with coefficients in Z_p coincides, in the "stable range", with the Adams spectral sequence.*

§6. CONVERGENCE STATEMENTS

It is clear from 4.4 (iv) that, in order that our spectral sequence has some use, one needs more information about $F^\infty \pi_g D_s X$. In general there is, of course, not much one can say, but for *simply connected* $X \in \mathscr{S}_*$ we can make the following convergence statements.

6.1. The integral case.

Let $R = Z$, *the ring of the integers. Then for* $t - 1 \geq s \geq 0$

$$F^{\infty}\pi_{t-s} D_s X = 0.$$

COROLLARY 6.2. *For* $t - s > 1$

$$F^{\infty}\pi_{t-s} X = 0$$

$$(F^s/F^{s+1})\pi_{t-s} X \overset{e}{\approx} E_{\infty}{}^{s,t} X.$$

It is not hard to show that this implies

6.3. The case of subrings of the rationals

Let J be a set of primes and let $R = Z[J^{-1}]$ *the ring of those rationals whose denominators involve only primes in J. Then the above isomorphisms hold when tensored with* $Z[J^{-1}]$.

Furthermore we can state (still assuming that $X \in \mathcal{S}_*$ *is simply connected*)

6.4. The mod-*h* case

Let h be an integer > 1 *and let* $R = Z_h$, *the ring of the integers modulo h. Then for* $t - 1 \geq s \geq 0$

$$F^{\infty}\pi_{t-s} D_s X = \bigcap_u h^u \pi_{t-s} D_s X.$$

COROLLARY 6.5. *If* $\pi_g X$ *is finitely generated for all g (and X is simply connected, of course), then for* $t - s > 1$

$$F^{\infty}\pi_{t-s} X = \bigcap_u h^u \pi_{t-s} X$$

$$(F^s/F^{s+1})\pi_{t-s} X \overset{e}{\approx} E_{\infty}{}^{s,t} X.$$

Remark 6.6. These convergence statements are *not* best possible. For instance the condition imposed on X in 6.5 can be somewhat relaxed. And it seems likely (see [19]) that the simple connectivity of X can be replaced by some weaker condition.

We first give a proof of 6.4 and then indicate what changes should be made therein to obtain a proof of 6.1.

Proof of 6.4. Obviously for $t - s = g > 0$

$$F^{\infty}\pi_g D_s X \supset \bigcap_u h^u \pi_g D_s X$$

The proof of the inclusion in the other direction essentially consists in constructing (for every prime p that divides h) a map from our spectral sequence to the p-lower central series one and then using the following slight generalization of the Curtis–Rector convergence theorem:

Let $X \in \mathcal{S}_*$ *be simply connected, let* $s \geq 0$, *let p be a prime, let*

$$\cdots \to \Gamma_{i+1} \to \Gamma_i \to \cdots \to \Gamma_1 = Id$$

be the p-lower central series functors [20] *and let G be the loop group functor* [16]. *Then, for* $g \geq 0$, *an element of* $\pi_g D_s GX$ *is in the image of* $\pi_g D_s \Gamma_i GX$ *for all i if and only if it is divisible by* p^i *for all i.*

Proof. The "if" part is obvious. For the "only if" part we need the following property of the pth power map $\xi\colon \Gamma_r GX \to \Gamma_{pr} GX$ [20, §4]: There exists an integer N (depending on $g+s$) such that $\xi_*\colon \pi_j \Gamma_r GX \to \pi_j \Gamma_{pr} GX$ is an isomorphism whenever $r \geq N$ and $j \leq g+s$ This implies that

$$(D_s \xi)_* : \pi_g D_s \Gamma_r GX \to \pi_g D_s \Gamma_{pr} GX$$

is an isomorphism for $r \geq N$ and the "only if" part follows as in [20].

We now return to the proof of 6.4. In view of 3.5 it suffices to show that for $g \geq 0$

$$F^\infty \pi_g D_s GX \subset \bigcap_u h^u \pi_g D_s GX.$$

Thus, by Curtis–Rector, all one has to do is, construct for every prime p that divides h, every $s \geq 0$ and $g \geq 0$ and every reduced $X \in \mathscr{S}_*$ (i.e. X has only one vertex) a commutative ladder

$$\begin{array}{ccccccc} \cdots \to \pi_g D_{s+i} GX & \to & \pi_g D_{s+i-1} GX & \to \cdots \to & \pi_g D_s GX \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\ \cdots \to \pi_g D_s \Gamma_{i+1} GX & \to & \pi_g D_s \Gamma_i GX & \to \cdots \to & \pi_g D_s GX. \end{array} \tag{6.8}$$

And for this it suffices to construct natural commutative ladders in $\mathscr{S}_{*K}$

$$\begin{array}{ccccccc} \cdots \to D_i GX & \to & D_{i-1} GX & \to \cdots \to & D_0 GX = GX \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\ \cdots \to C_{i+1} X & \longrightarrow & C_i X & \to \cdots \to & C_1 X = GX \end{array} \tag{6.9}$$

and

$$\begin{array}{ccccccc} \cdots \to \Gamma_{i+1} GX & \to & \Gamma_i GX & \to \cdots \to & GX \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\ \cdots \to C_{i+1} X & \longrightarrow & C_i X & \to \cdots \to & GX \end{array} \tag{6.10}$$

such that in (6.10) the vertical maps are homotopy equivalences, because then (6.8) is readily obtained by applying $\pi_g D_s$ to (6.9) and (6.10) using the twisting lemma (3.6).

It thus remains to construct the ladders (6.9) and (6.10). To construct (6.9) we assume that there exist functors N_i from simplicial $\mathbf{Z}_p$-modules to $\mathscr{S}_{*K}$ and natural transformations $C_i \to N_i Z_p$ such that

$$\begin{array}{ccc} C_{i+1} & \longrightarrow & \Lambda N_i Z_p \\ \downarrow & & \downarrow{\scriptstyle \lambda} \\ C_i & \longrightarrow & N_i Z_p \end{array}$$

is a pull back diagram. The map $D_i GX \to C_{i+1}X$ then can be defined as the one induced by the composition

$$D_{i-1}GZ_h X \to D_{i-1}GZ_p X \to C_i Z_p X \to N_i Z_p Z_p X \xrightarrow{N_i\psi} N_i Z_p X.$$

And finally to construct a ladder (6.10) satisfying the above extra assumption we observe [11] that there exist functors M_i from simplicial Z_p-modules to $\mathcal{S}_{*K}$ such that

$$\Gamma_i GX/\Gamma_{i+1}GX = M_i Z_p X.$$

Supposing inductively that $\Gamma_i GX \to C_i X$ is a trivial cofibration (i.e. a 1–1 weak homotopy equivalence), it will suffice to construct N_i together with a natural diagram in $\mathcal{S}_{*K}$

$$\begin{array}{ccc} \Gamma_i GX & \longrightarrow & C_i X \\ \downarrow & & \downarrow \\ M_i Z_p X & \longrightarrow & N_i Z_p X \end{array}$$

such that the bottom map is also a trivial cofibration, which is not hard to do since the map on the left factors through $\Gamma_i GZ_p X$ and since trivial cofibrations are preserved under co-base extensions, i.e. push-outs, in $\mathcal{S}_*$.

Proof of 6.1. This is essentially the same as the above proof of 6.4 except that one uses.

(i) Z instead of Z_h and Z_p

(ii) the integral lower central series functors [10], and

(iii) the following slight variation on the Curtis convergence theorem:

Let $X \in \mathcal{S}_*$ be simply connected and let $s \geq 0$. Then for $g \geq 0$, no non-zero element of $\pi_g D_s GX$ is in the image of $\pi_g D_s \Gamma_i GX$ for all i.

For $s = 0$ this is the main result of [10]. The rest is an easy induction on s.

Remark 6.7. Using similar arguments it is not hard to show that the homotopy spectral sequence that resulted from Milnor's free group functor F (4.5) has the same convergence properties (6.1 and 6.2) as the integral homotopy spectral sequence.

§7. GENERALIZATION TO FUNCTION COMPLEXES

The results of the preceeding sections will now be generalized to function complexes. We start with recalling the notion of

7.1. Function complexes with base point

For W, $X \in \mathcal{S}_*$ the *function complex with base point*

$$hom(W, X) \in \mathcal{S}_*$$

has as n-simplices the maps

$$\Delta[n] \wedge W \to X \in \mathcal{S}_*$$

where $\Delta[n] \wedge W$ is obtained from $\Delta[n] \times W$ by collapsing $\Delta[n] \times \{*\}$, and has face and degeneracy operators induced by the standard maps [16]

$$\Delta[n-1] \xrightarrow{\delta i} \Delta[n] \qquad \Delta[n+1] \xrightarrow{\sigma_i} \Delta[n].$$

Its main property is:

If $X \in \mathscr{S}_{*K}$, then $hom(W, X) \in \mathscr{S}_{*K}$ (i.e. $hom(W, \ )$ respects $\mathscr{S}_{*K}$ and the elements of $\pi_g\, hom(W, X)$ are in 1–1 correspondence with the homotopy classes (rel $*$) of maps $S^g W \to X$ (where $S^g W$ denotes the g-fold suspension of W).

Now we can define

7.2. The spectral sequence for function complexes

Let $W, X \in \mathscr{S}_*$ and let R be a ring. *The homotopy spectral sequence* $\{E_r(W, X; R)\}$ (or short $\{E_r(W, X)\}$) *of $hom(W, X)$ with coefficients in R* is the homotopy spectral sequence of the sequence of fibre maps

$$\cdots \to D_s hom(W, X) \xrightarrow{\delta} D_{s-1} hom(W, X) \to \cdots \to hom(W, X)$$

again indexed and *fringed* as in §4. Thus

$$\begin{aligned} E_1^{s,t}(W,X) &= \pi_{t-s} D_s hom(W, RX) \qquad & t-1 \geq s \geq 0 \\ &= 0 & \text{otherwise.} \end{aligned}$$

The results of §4 and §5 readily generalize to function complexes. The same holds for §6 except that

(i) for the generalizations of 6.1, 6.2, 6.3 and 6.4 one has to assume that $X \in \mathscr{S}_{*K}$ (otherwise $hom(W, X)$ may have the "wrong" homotopy type), that X is simply connected and that W has the weak homotopy type of a finite dimensional complex.

(ii) for the generalization of 6.5 one has to assume that $X \in \mathscr{S}_{*K}$, that X is simply connected, that $\pi_g X$ is finitely generated for all g, and that W has the weak homotopy type of a finite complex.

The proofs use the same arguments.

§8. THE COEFFICIENT RING

We end this chapter with some results which imply that the spectral sequences $\{E_r(X; Z)\}$ and $\{E_r(X; Z_{p^j})\}$(p prime) determine all the spectral sequences $\{E_r(X; R)\}$ with *commutative* R.

Throughout this section all rings will be commutative and $\otimes$ will denote $\otimes_z$. The proofs (of 8.2 and 8.6) will be given in [9].

8.1. The core of a ring

The *core* of a ring R is the subring

$$cR = \{x \mid 1 \otimes x = x \otimes 1 \in R \otimes R\}.$$

The usefulness of this notion is due to

REDUCTION THEOREM 8.2. *For $X \in \mathcal{S}_*$ and R a ring, the inclusion $cR \subset R$ induces isomorphisms*

$$E_r(X; cR) \approx E_r(X\ ; R) \qquad for\ r \geq 2.$$

Moreover

$$E_r^{0,n}(K(Z, n); R) \approx cR \qquad for\ r \geq 2$$

i.e. this reduction is best possible.

COROLLARY 8.3. $ccR = cR$.

To find out which rings can serve as cores we therefore define

8.4. Solid rings

A ring R is called *solid* if $cR = R$.

8.5. Examples of solid rings

(i) The cyclic rings Z_h for $h \geq 2$.
(ii) The subrings of the rationals, i.e. the rings $Z[J^{-1}]$ where J is any set of primes (6.3).
(iii) The product rings $Z[J^{-1}] \times Z_h$ where each prime factor of h is in J.

Moreover we have

8.6. Description of all solid rings

(i) *A ring R is solid if and only if the multiplication map $R \otimes R \to R$ is an isomorphism.*
(ii) *Every solid ring is isomorphic to a direct limit (over a directed system) of the rings of* 8.5.

The statement at the beginning of this section now follows immediately from the following proposition (of which the verification is straightforward).

PROPOSITION 8.7. *Let $X \in \mathcal{S}_*$ and $r \geq 2$.*

(i) *if $R = \varinjlim R_i$ is a direct limit over a directed system, then*

$$E_r(X; R) \approx \varinjlim E_r(X; R_i).$$

(ii) *If $R = Z[J^{-1}]$, then*

$$E_r(X; R) \approx E_r(X; Z) \otimes R.$$

(iii) *If either $R \times R' = Z_m \times Z_n$ with $(m, n) = 1$ or $R \times R' = Z[J^{-1}] \times Z_h$ with h and J as in* 8.5 (iii), *then*

$$E_r(X; R \times R') \approx E_r(X; R) \oplus E_r(X; R').$$

CHAPTER II. THE E_2-TERM FOR $R = Z_p$

§9. COSIMPLICIAL OBJECTS

We will investigate $E_2(X; R)$ using cosimplicial methods and therefore start with recalling the notion of an *(augmented) cosimplicial object* and (following Godement [13] and Eilenberg–Moore [12]) giving our *prime example*: the resolution of a space with respect to a ring. For the moment we will *not* yet assume that $R = Z_p$.

9.1. Cosimplicial objects

A *cosimplicial object* $\mathbf{X}$ (over a category $\mathscr{C}$) consists of

(i) for every integer $n \geq 0$ an object $\mathbf{X}^n \in \mathscr{C}$.
(ii) for every pair of integers (i, n) with $0 \leq i \leq n$ *coface* and *codegeneracy* maps

$$d^i\colon \mathbf{X}^{n-1} \to \mathbf{X}^n \qquad s^i\colon \mathbf{X}^{n+1} \to \mathbf{X}^n$$

in $\mathscr{C}$ satisfying the identities

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{for } i < j \\ s^j d^i &= d^i s^{j-1} && \text{for } i < j \\ &= id && \text{for } i = j, j+1 \\ &= d^{i-1} s^j && \text{for } i > j+1 \\ s^j s^i &= s^{i-1} s^j && \text{for } i > j. \end{aligned}$$

A *cosimplicial map* $f\colon \mathbf{X} \to \mathbf{Y}$ consists of maps

$$f\colon \mathbf{X}^n \to \mathbf{Y}^n \in \mathscr{C}$$

which commute with all the cofaces and codegeneracies. *A cosimplicial object (map) over $\mathscr{C}$ thus corresponds to a simplicial object (map) over the dual category $\mathscr{C}^*$.*

9.2. Augmentations

An *augmentation* of a cosimplicial object $\mathbf{X}$ (over $\mathscr{C}$) consists of a map

$$d^0\colon \mathbf{X}^{-1} \to \mathbf{X}^0 \in \mathscr{C}$$

such that

$$d^1 d^0 = d^0 d^0\colon \mathbf{X}^{-1} \to \mathbf{X}^1$$

We now turn to our prime example.

9.3. The resolution of a space with respect to a ring

Let $X \in \mathscr{S}_*$ and let R be a ring (with unit). Then *the resolution of X with respect to R* is the augmented cosimplicial object $\mathbf{R}X$ over $\mathscr{S}_*$ given by

$$\mathbf{R}X^n = R^{n+1}X \qquad n \geq -1$$

$$\mathbf{R}X^{n-1} \xrightarrow{d^i} \mathbf{R}X^n = R^n X \xrightarrow{R^i \varphi R^{n-i}} R^{n+1}X$$

$$\mathbf{R}X^{n+1} \xrightarrow{s^i} \mathbf{R}X^n = R^{n+2}X \xrightarrow{R^i \psi R^{n-i}} R^{n+1}X.$$

Clearly $\mathbf{R}X$ *is natural in* X *as well as in* R.

Remark 9.4. In verifying that $\mathbf{R}X$ is indeed an augmented cosimplicial object, one only has to use the fact that (R, ϕ, ψ) is a *triple* in the sense of [12]. The same construction thus can be made using other triples.

A way of constructing more cosimplicial objects is by

9.5. Applying a functor

Let $\mathbf{X}$ be an (augmented) cosimplicial object over a category $\mathscr{C}$ and let $T: \mathscr{C} \to \mathscr{C}'$ be a covariant functor. *Application of* T *to* $\mathbf{X}$ then yields an (augmented) cosimplicial object $T\mathbf{X}$ over $\mathscr{C}'$ with

$$(T\mathbf{X})^n = T(\mathbf{X}^n) \qquad \text{for all } n.$$

In particular for $\mathbf{R}X$ as above, $\pi_i \mathbf{R}X (i \geq 1)$ is an (augmented) cosimplicial abelian group.

§10. A COSIMPLICIAL DESCRIPTION OF $E_2(X; R)$

We now give a very useful cosimplicial description of $E_2(X; R)$ valid for all R. For this we need

10.1. The cohomotopy groups of a cosimplicial abelian group

These are dual to "the homotopy groups of a simplicial abelian group": for an (augmented or not) cosimplicial abelian group $\mathbf{A}$ we denote by ch $\mathbf{A}$ its *cochain complex* given by

$$\begin{aligned} (\text{ch } \mathbf{A})^n &= \mathbf{A}^n \qquad n \geq 0 \\ &= 0 \qquad n < 0 \end{aligned}$$

$$\delta = \sum_{i=0}^{n} (-1)^i d^i : \mathbf{A}^{n-1} \to \mathbf{A}^n$$

and define its *cohomotopy groups* $\pi^s \mathbf{A}$ by

$$\pi^s \mathbf{A} = H^s(\text{ch } \mathbf{A}).$$

Then we have

10.2. Cosimplicial description of $E_2(X; R)$

Let $X \in \mathscr{S}_*$ *and let* R *be a ring. Then there are natural isomorphisms*

$$\begin{aligned} E_2{}^{s,t}(X; R) &\approx \pi^s \pi_t \mathbf{R}X \qquad \textit{for } t > s \geq 0 \\ &= 0 \qquad \textit{otherwise} \end{aligned}$$

and similarly:

10.3. The function complex case

Let $W, X \in \mathscr{S}_*$ and let R be a ring. Then there are natural isomorphisms

$$\begin{aligned} E_2^{s,t}(W, X; R) &\approx \pi^s\pi_t hom(W, \mathbf{R}X) && \text{for } t > s \geq 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Remark 10.4. These statements, as well as their proof, only use the fact that (R, ϕ, ψ) is a *triple* (4.6). *The above description of the E_2-term* thus *remains valid for arbitrary triples.*

Remark 10.5. For arbitrary triples, and even for arbitrary rings, there is not much one can do to improve on the above description of E_2. However considerable simplifications are possible *if R is a field* and the remainder of this chapter will be devoted to the case $R = Z_p$.

For $R = Q$, the rationals, our spectral sequence is closely related to *the rational cobar spectral sequence* [1]; our E_2-term consists of the primitive elements in the E_2-term of the latter. As this involves "the Whitehead product in E_2", we postpone a full account of this case till [8].

To prove 10.2 and 10.3 we need a

COLLAPSE LEMMA 10.6. *Let $X \in \mathscr{S}_*$, let R be a ring and let*

$$B: \mathscr{S}_* \to (\textit{abelian groups})$$

be a functor such that the natural transformation $B\phi: B \to BR$ has a left inverse. Then

$$\begin{aligned} \pi^s B\mathbf{R}X &\approx BX && \text{for } s = 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

This is proved by constructing a contracting homotopy for ch($B\mathbf{R}X$)

Proof of 10.2 (the proof of 10.3 is similar). Consider, for each $k \geq 1$, the double (cochain) complex C with

$$\begin{aligned} C^{m,n} &= \pi_{k-m} D_m R(\mathbf{R}X)^n && k \geq m \geq 0, n \geq 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

and the obvious coboundary maps. By 4.4 and 10.6 both spectral sequences for computing the total cohomology of C collapse and hence, in the required range this total cohomology is isomorphic to $E_2(X; R)$ as well as to $\pi^*\pi_* \mathbf{R}X$.

Remark 10.7. The above determination of $E_2(X; R)$ uses only two general properties of our spectral sequence, namely

(i) *The E_2-term collapses to $\pi_* X$ whenever X is a simplicial R-module,*
(ii) *The E_1-term depends functorially on RX.*

§11. UNSTABLE COALGEBRAS OVER THE STEENROD ALGEBRA

In this section we consider the category $\mathscr{C}\mathscr{A}$ of *unstable coalgebras over the Steenrod algebra* $\mathscr{A}$ and observe that the Z_p-homology functor is actually a functor

$$H_*(\ ; Z_p): \mathscr{S}_{*c} \to \mathscr{C}\mathscr{A}$$

where $\mathscr{S}_{*c} \subset \mathscr{S}_*$ is the full subcategory of *connected* complexes. Moreover this functor has some nice properties which imply that *$E_2(X; Z_p)$ depends only on $H_*(X; Z_p)$ as an unstable coalgebra over the Steenrod algebra $\mathscr{A}$.* First we consider

11.1. Unstable $\mathscr{A}$-modules

Let $\mathscr{A}$ denote the mod-p Steenrod algebra graded with *upper* indices; so $\mathscr{A}^i = 0$ for $i < 0$. An *unstable right-$\mathscr{A}$-module* then consists of

(i) a graded Z_p-module M (with $M_n = 0$ for $n < 0$).

(ii) a multiplication map $M \otimes \mathscr{A} \to M$ (with $M_n \otimes \mathscr{A}^i \to M_{n-i}$) which, in addition to the usual module properties, has the unstable property

$$xSq^n = 0 \qquad p = 2,\ \deg x < 2n$$

$$xP^n = 0 \qquad p \text{ odd},\ \deg x < 2pn$$

$$x\beta P^n = 0 \qquad p \text{ odd},\ \deg x = 2pn + 1.$$

Note that, *for $X \in \mathscr{S}_*$, $H_*(X; Z_p)$ is an unstable right $\mathscr{A}$-module*, if the right $\mathscr{A}$ action on $H_*(X; Z_p)$ is defined as in [6].

11.2. A tensor product

For unstable right $\mathscr{A}$-modules M and N one can turn $M \otimes N$ into an unstable right $\mathscr{A}$-module by defining the right $\mathscr{A}$-action with the Cartan formula

$$(x \otimes y)Sq^n = \sum_{i=0}^{n} xSq^i \otimes ySq^{n-i} \qquad = 2$$

$$(x \otimes y)P^n = \sum_{i=0}^{n} xP^i \otimes yP^{n-i} \qquad p \text{ odd}$$

$$(x \otimes y)\beta = x\beta \otimes y + (-1)^{\deg x} x \otimes y\beta \qquad p \text{ odd}$$

For $X,\ Y \in \mathscr{S}_*$ *there then* clearly *is a natural isomorphism of unstable right $\mathscr{A}$-modules*

$$H_*(X; Z_p) \otimes H_*(Y; Z_p) \approx H_*(X \times Y; Z_p).$$

11.3. The category $\mathscr{C}\mathscr{A}$ of unstable $\mathscr{A}$-coalgebras

An object C in this category is both an *unstable right $\mathscr{A}$-module* and a *connected co-commutative Z_p-coalgebra* (see [17]) where these two structures are compatible in the sense that

(i) the comultiplication map $C \to C \otimes C$ is a right $\mathscr{A}$-module map

(ii) the p-th root map $(\ \ )\xi\colon C_{pk} \to C_k$ (dual to the p-th power map for commutative Z_p-algebras) satisfies

$$x\xi = xSq^n \qquad p = 2,\ \deg x = 2n$$

$$x\xi = xP^n \qquad p \text{ odd},\ \deg x = 2pn.$$

Note that, *for* $X \in \mathcal{S}_{*c}$ *(i.e. X connected) we have*

$$H_*(X; Z_p) \in \mathcal{C}\mathcal{A}$$

where the comultiplication map is induced by the diagonal $X \to X \times X$.

11.4 A triple on $\mathcal{C}\mathcal{A}$

Let $\mathcal{M}\mathcal{Z}_p$ denote the category of *connected* graded Z_p modules (i.e. trivial in degrees ≤ 0). Then the *forgetful functor*

$$J: \mathcal{C}\mathcal{A} \to \mathcal{M}\mathcal{Z}_p$$

(with $(JC)_n = C_n$ for $n \geq 1$) has a *right adjoint*

$$V: \mathcal{M}\mathcal{Z}_p \to \mathcal{C}\mathcal{A}$$

given by

$$VM = H_*\left(\prod_{n=1}^{\infty} K(M_n, n); Z_p\right)$$

and as usual [12] such a pair of adjoint functors gives rise to a *triple* (T, ϕ, ψ) *on* $\mathcal{C}\mathcal{A}$ with $T = VJ$.

In view of 9.4 we can, for an object $C \in \mathcal{C}\mathcal{A}$ form a *cosimplicial object* **T**C (over $\mathcal{C}\mathcal{A}$). Using this functor we now state the result mentioned at the beginning of this section.

THEOREM 11.5. *Let* $X \in \mathcal{S}_{*c}$ *and* $W \in \mathcal{S}_*$. *Then, for* $t > s \geq 0$, *there are natural isomorphisms*

$$E_2^{s,t}(X; Z_p) = \pi^s Hom_{\mathcal{C}\mathcal{A}}(H_*(S^t; Z_p), \mathbf{T}H_*(X; Z_p))$$
$$E_2^{s,t}(W, X; Z_p) \approx \pi^s Hom_{\mathcal{C}\mathcal{A}}(H_*(S^t W; Z_p), \mathbf{T}H_*(X; Z_p)).$$

To prove this we first observe that the triple (T, ϕ, ψ) on $\mathcal{C}\mathcal{A}$ is closely related to the triple (R, ϕ, ψ) on $\mathcal{S}_{*c}$. In fact

$$\begin{array}{ccc} \mathcal{S}_{*c} & \xrightarrow{H_*(\ ;Z_p)} & \mathcal{C}\mathcal{A} \\ {\scriptstyle Z_p}\downarrow\uparrow & & {\scriptstyle J}\downarrow\uparrow{\scriptstyle V} \\ \mathcal{S}_{p,c} & \xrightarrow{\pi^*} & \mathcal{M}\mathcal{Z}_p. \end{array}$$

LEMMA 11.6. *The diagram* (where $\mathcal{S}_{p,c}$ is the category of connected simplicial Z_p-modules and the unnamed functor is the forgetful one) *commutes in the obvious sense.*

This immediately implies

COROLLARY 11.7. *Let* $X \in \mathcal{S}_{*c}$. *Then there is a natural isomorphism of cosimplicial objects over* $\mathcal{C}\mathcal{A}$

$$H_*(\mathbf{Z}_p X; Z_p) \approx \mathbf{T}H_*(X; Z_p).$$

In view of 10.2 and 10.3 it thus remains to prove the following lemma which readily follows from our knowledge of $H_*(K(Z_p, n); Z_p)$ and the fact that each $Y \in \mathcal{S}_{p,c}$ (11.6) is a product of $K(Z_p, n)$'s.

LEMMA 11.8. *Let $Y \in \mathcal{S}_{p,c}$ (11.6) and $W \in \mathcal{S}_*$. Then for $t \geq 1$, the functor $H_*(\ ;Z_p)$ induces isomorphisms*

$$\pi_t Y = [S^t, Y] \approx Hom_{\mathcal{CA}}(H_*(S^t; Z_p), H_*(Y; Z_p))$$
$$\pi_t\, hom(W, O) = [S^tW, Y] \approx Hom_{\mathcal{CA}}(H_*(S^tW; Z_p), H_*(Y; Z_p))$$

where [,] denotes homotopy classes of maps (rel.*).

§12. $E_2(X; Z_p)$ AS AN UNSTABLE *Ext*

In this section we define the "unstable Ext" functors $Ext_{\mathcal{CA}}^s$ $(s \geq 0)$ which are, roughly speaking, *the right derived functors of the functor* $Hom_{\mathcal{CA}}$. Their definition (12.3), together with 11.5, immediately implies

THEOREM 12.1. *Let $X \in \mathcal{S}_{*c}$ and $W \in \mathcal{S}_*$. Then, for $t > s \geq 0$, there are natural isomorphisms*

$$E_2^{s,t}(X; Z_p) \approx Ext_{\mathcal{CA}}^s(H_*(S^t; Z_p), H_*(X; Z_p))$$
$$E_2^{s,t}(W, X; Z_p) \approx Ext_{\mathcal{CA}}^s(H_*(S^tW; Z_p), H_*(X; Z_p)).$$

First we consider:

12.2. Right derived function on $\mathcal{CA}$

The theory of derived functors of non-additive functors is presented at length in [2, 3, 18], but for a brief account the reader may consult [4]. In view of these sources one can for a covariant functor

$$F\colon \mathcal{CA} \to (abelian\ groups)$$

define its *right derived functors* $\mathcal{R}^sF$ as the functors

$$\mathcal{R}^sF = \pi^s F\mathbf{T}\colon \mathcal{CA} \to (abelian\ groups) \qquad s \geq 0$$

where $\mathbf{T}$ is as in 11.4.

One can also use a more flexible approach by putting, for $C \in \mathcal{CA}$

$$(\mathcal{R}^sF)C = \pi^s\mathbf{C} \qquad s \geq 0$$

where $\mathbf{C}$ is any so-called *cosimplicial resolution* of C, i.e. augmented cosimplicial object over $\mathcal{CA}$ such that (in the notation of 11.4)

(i) $\mathbf{C}^{-1} = C$
$\mathbf{C}^s \approx VG^s$ for some $G^s \in \mathcal{MX}_p$ $(s \geq 0)$

(ii) $\pi^0 J\mathbf{C} = JC$
$\pi^s J\mathbf{C} = 0$ $(s > 0)$.

Note that $\mathbf{T}C$ *is such a resolution.*

Now we define:

12.3. The functors $Ext_{\mathscr{C}\mathscr{A}}{}^s$

If $B \in \mathscr{C}\mathscr{A}$ has trivial comultiplication (for instance if $B = H_*(S^tW; Z_p)$ for some $W \in \mathscr{S}_*$ and $t > 0$), then the functor $Hom_{\mathscr{C}\mathscr{A}}(B, \ \ )$ is actually a functor

$$Hom_{\mathscr{C}\mathscr{A}}(B, \ \ ) : \mathscr{C}\mathscr{A} \to (Z_p\text{-}modules).$$

For such B we thus can (and will) apply the above and define the "unstable Ext" functors $Ext_{\mathscr{C}\mathscr{A}}{}^s(B, \ \ )$ by

$$Ext_{\mathscr{C}\mathscr{A}}{}^s(B, \ \ ) = \mathscr{R}^s Hom_{\mathscr{C}\mathscr{A}}(B, \ \ ).$$

§13. THE E_2-TERM IN THE MASSEY-PETERSON CASE

In [15] Massey and Peterson constructed for "very nice" spaces an unstable Adams spectral sequence and succeeded in describing their E_2-term as an ordinary *Ext* in a category of unstable modules over the Steenrod algebra. We now apply Theorem 12.1 to show that for "very nice" spaces our E_2-term (which is an $Ext_{\mathscr{C}\mathscr{A}}$) reduces to the Massey–Peterson Ext.

We start by recalling the notion of:

13.1. "Very nice" spaces

Let $\mathscr{M}\mathscr{A}$ denote the category of *connected* (i.e. trivial in degrees ≤ 0) unstable right $\mathscr{A}$-modules, let (see 11.1)

$$J' : \mathscr{C}\mathscr{A} \to \mathscr{M}\mathscr{A}$$

be the *forgetful functor* (with $(J'C)_n = C_n$ for $n \geq 1$) and let

$$U : \mathscr{M}\mathscr{A} \to \mathscr{C}\mathscr{A}$$

be its *right adjoint* (if $M \in \mathscr{M}\mathscr{A}$ is of finite type, then UM is just dual to the free unstable $\mathscr{A}$-algebra generated by M^* ([24], p. 29)). A complex $X \in \mathscr{S}_*$ then is called *very nice* (mod p) if

$$H_*(X; Z_p) \approx UM \in \mathscr{C}\mathscr{A} \qquad \text{for some } M \in \mathscr{M}\mathscr{A}.$$

For example *the sphere S^n, $n \geq 1$, is very nice if $p = 2$ or if p odd, n odd.*

13.2. The Massey–Peterson *Ext*

The category $\mathscr{M}\mathscr{A}$ of connected unstable right $\mathscr{A}$-modules is an abelian category with enough injectives. Thus for $N \in \mathscr{M}\mathscr{A}$ we may define $Ext_{\mathscr{M}\mathscr{A}}{}^s(N, \ \ )$ $(s \geq 0)$ as *the usual s-th right derived functor of $Hom_{\mathscr{M}\mathscr{A}}(N, \ \ )$.*

The Massey–Peterson result [15] then becomes in our framework

THEOREM 13.3. *Let $X, W \in \mathscr{S}_*$ and let $M \in \mathscr{M}\mathscr{A}$ be such that $H_*(X; Z_p) \approx UM \in \mathscr{C}\mathscr{A}$. Then, for $t > s \geq 0$, there are natural isomorphisms*

$$E_2{}^{s,t}(X; Z_p) \approx Ext_{\mathscr{M}\mathscr{A}}{}^s(\tilde{H}_*(S^t; Z_p), M)$$

$$E_2{}^{s,t}(W, X; Z_p) \approx Ext_{\mathscr{M}\mathscr{A}}{}^s(\tilde{H}_*(S^tW; Z_p), M).$$

This follows immediately from 12.1 and

AN ALGEBRAIC LEMMA 13.4. *Let $B \in \mathscr{C}\mathscr{A}$ have trivial comultiplication and let $M \in \mathscr{M}\mathscr{A}$. Then, for $s \geq 0$, there is a natural isomorphism*

$$Ext_{\mathscr{C}\mathscr{A}}{}^s(B, UM) \approx Ext_{\mathscr{M}\mathscr{A}}{}^s(J'B, M).$$

To prove this observe that (as in 11.4) the *forgetful functor*

$$J'': \mathscr{M}\mathscr{A} \to \mathscr{M}\mathscr{Z}_p$$

and its *right adjoint*

$$V'': \mathscr{M}\mathscr{Z}_p \to \mathscr{M}\mathscr{A}$$

give rise to a *triple* (T'', ϕ, ψ) *on* $\mathscr{M}\mathscr{A}$ with $T'' = V''J''$ and hence to a *cosimplicial object* $\mathbf{T}''M$ over $\mathscr{M}\mathscr{A}$. Consequently [2, 3, 4, 18] there exists, for $s \geq 0$, a natural isomorphism

$$Ext_{\mathscr{M}\mathscr{A}}{}^s(J'B, M) \approx \pi^s Hom_{\mathscr{M}\mathscr{A}}(J'B, \mathbf{T}''M)$$

and in view of the adjunction isomorphism

$$\pi^s Hom_{\mathscr{M}\mathscr{A}}(J'B, \mathbf{T}''M) \approx \pi^s Hom_{\mathscr{C}\mathscr{A}}(B, U\mathbf{T}''M)$$

it thus remains to show that $U\mathbf{T}''M$ is a cosimplicial resolution in the sense of 12.2. Part (i) of this is easy, and part (ii) also not hard to prove using the fact that the map $J''d^0: J''M \to J''T''M$ has a left inverse together with the following.

LEMMA 13.5. *Let $M \in \mathscr{M}\mathscr{A}$. Then there is a natural filtration*

$$UM = F^0UM \supset F^1UM \supset F^2UM \supset \cdots$$

such that, for every $k \geq 0$, there is a natural isomorphism

$$Sym^k J''M \approx J''(F^kUM/F^{k+1}UM)$$

where

$$SymX = \sum_{k \geq 0} Sym^k X \qquad X \in \mathscr{M}\mathscr{Z}_p$$

denotes the quotient of the tensor algebra on X by the ideal generated by all x^p abd $xx' - (-1)^{mn}x'x$ for $x \in X_m$, $x' \in X_n$, and $Sym^k x \subset SymX$ is generated by the k-fold products of elements of X

Proof. For $M, N \in \mathscr{M}\mathscr{A}$ there is a natural isomorphism

$$U(M \oplus N) \approx UM \otimes UN \in \mathscr{C}\mathscr{A}$$

since U, as a right adjoint, preserves (categorical) direct products. We therefore can *turn the coalgebra UM into a Hopf algebra* by defining a multiplication by

$$UM \otimes UM \approx U(M \oplus M) \xrightarrow{U(+)} UM$$

and take as filtration *the augmentation filtration* [17, p. 252]

$$UM = F^0UM \supset F^1UM \supset F^2UM \supset \cdots$$

where F^1UM is the augmentation ideal and

$$F^kUM = (F^1UM)^k \qquad k \geq 1.$$

It is not hard to show for the Hopf algebra UM that $PUM \to QUM$ is a monomorphism and that $QUM \approx M$. Thus the associated bigraded Hopf algebra

$$E^0UM = \sum_{k \geq 0} F^k UM / F^{k+1} UM$$

has the property

$$PE^0UM \approx QE^0UM \approx M.$$

The lemma now follows since [17, 6.11] E^0UM is the universal enveloping algebra of the restricted Lie algebra $PE^0UM \approx M$, and the Lie operations in PE^0UM are trivial.

Remark 13.6. The natural isomorphism

$$Ext_{\mathscr{C}A}{}^s(B, UM) \approx Ext_{\mathscr{M}\mathscr{A}}{}^s(J'B, M)$$

of 13.4 can be constructed explicitly as the composite

$$\pi^s Hom_{\mathscr{C}\mathscr{A}}(B, \mathbf{T}UM) \approx \pi^s Hom_{\mathscr{C}\mathscr{A}}(B, U\mathbf{T}''M) \approx \pi^s Hom_{\mathscr{M}\mathscr{A}}(J'B, \mathbf{T}''M)$$

where the first isomorphism is induced by the cosimplicial map

$$\mathbf{T}UM \to U\mathbf{T}''M$$

which sends $(UV''J''J')^n UM$ into $U(V''J'')^n M$ by the iterated adjunction map $J'U \to id$, and where the second isomorphism is as in 13.4.

§14. A CONVENIENT E_1-TERM FOR VERY NICE SPACES

In [7] an E_1-term (much smaller than that from the bar resolution) was constructed for the mod-p Adams spectral sequence; and subsequently in [6] a similar unstable E_1-term was obtained for very nice spaces in case $p = 2$. Using theorem 13.3 this result of [6] will now be extended to *odd* p, thereby making possible mod-p computations of the sort done mod-2 in [11], e.g. computing $E_2(S^{2n+1}; Z_p)$ in a range. Throughout this section p *will* thus *be odd.*

We first recall from [7], with a slight change in sign,

14.1. The algebra Λ

This will be the *differential graded associative algebra* with unit (over Z_p) having:

(i) A *generator* λ_i of degree $2i(p-1)-1$ for each $i > 0$.

(ii) A *generator* μ_i of degree $2i(p-1)$ for each $i \geq 0$.

(iii) For every $i > 0$ and $k \geq 0$ the *relations*

$$\lambda_i \lambda_{pi+k} = \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \lambda_{i+k-j} \lambda_{pi+j}$$

$$\lambda_i \mu_{pi+k} = \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \lambda_{i+k-j} \mu_{pi+j} + \sum_{j \geq 0} (-1)^{j} \binom{(p-1)(k-j)}{j} \mu_{i+k-j} \lambda_{pi+j}$$

and for every $i \geq 0$ and $k \geq 0$ the *relations*

$$\mu_i \lambda_{pi+k+1} = \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \mu_{i+k-j} \lambda_{pi+j+1}$$

$$\mu_i \mu_{pi+k+1} = \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \mu_{i+k-j} \mu_{pi+j+1}.$$

(iv) A *differential* ∂ given by

$$\partial \lambda_k = \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \lambda_{k-j} \lambda_j \qquad k \geq 1$$

$$\partial \mu_k = \sum_{j \geq 0} (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \lambda_{k-j} \mu_j + \sum_{j \geq 1} (-1)^j \binom{(p-1)(k-j)}{j} \mu_{k-j} \lambda_j \qquad k \geq 0$$

$$\partial(xy) = (\partial x)y + (-1)^{\deg x} x(\partial y) \qquad x, y \in \Lambda.$$

A monomial $v_I = v_{i_1} \cdots v_{i_s}$ of generators (with each $v = \lambda$ or μ) is called *allowable* if $i_{k+1} \leq p i_k$ whenever $v_{i_k} = \mu_{i_k}$ $(1 \leq k \leq s-1)$ and if $i_{k+1} \leq p i_k - 1$ whenever $v_{i_k} = \lambda_{i_k}$ $(1 \leq k \leq s-1)$. Then Λ *has a Z_p-basis given by all allowable monomials (including the empty monomial* 1). Note that

$$\Lambda = \bigoplus_{s \geq 0} \Lambda^s$$

where Λ^s is generated by the monomials of length s.

Remark 14.2. Actually there is a slight difference between Λ and the E^1S of [7, p. 340] since the latter has

$$d^1(xy) = (-1)^{\deg y}(d^1x)y + x(d^1y).$$

To be precise, there is an isomorphism $(E^1S)^\# \approx \Lambda$ of differential graded algebras, where $(E^1S)^\#$ equals E^1S as a differential graded Z_p-module but has a new multiplication $\#$ defined by

$$x \# y = (-1)^{(\deg x)(\deg y)} xy.$$

We have also expressed our formulae in allowable form and used v_i instead of v_{i-1}. Finally, we confess that the right side of the formula for $d^1\mu_{n-1}$ in [7, p. 340] should have been multiplied by -1.

14.3. The cochain complex $M \hat{\otimes} \Lambda$

For $M \in \mathcal{M}\mathcal{A}$ let $M \hat{\otimes} \Lambda^s$ denote the subspace of $M \otimes \Lambda^s$ generated by all $x \otimes v_I$ with $v_I = v_{i_1} \cdots v_{i_s}$ allowable, $\deg x \geq 2i_1$ if $v_{i_1} = \lambda_{i_1}$ and $\deg x \geq 2i_1 + 1$ if $v_{i_1} = \mu_{i_1}$. Then $M \hat{\otimes} \Lambda$ will be the cochain complex with

$$(M \hat{\otimes} \Lambda)^s = M \hat{\otimes} \Lambda^s \qquad s \geq 0$$

$$\delta(x \otimes v_I) = (-1)^{\deg x} \sum_{(i>0)} xP^i \otimes \lambda_i v_I + \sum_{i \geq 0} x\beta P^i \otimes \mu_i v_I + (-1)^{\deg x} x \otimes \partial v_I$$

bigraded by giving $x \otimes v_I \in M \hat{\otimes} \Lambda^s$ bidegree (s, t) with $t = s + \deg x + \deg v_I$.

The main result of this section then is

THEOREM 14.4. *For $M \in \mathcal{M}\mathcal{A}$ and p odd, there is a natural isomorphism*

$$Ext^s_{\mathcal{M}\mathcal{A}}(\tilde{H}(S^t; Z_p), M) \approx H^{s,t}(M \hat{\otimes} \Lambda).$$

And this together with 13.3 yields

COROLLARY 14.5. *Let $X \in \mathcal{S}_*$ and $M \in \mathcal{M}\mathcal{A}$ be such that $H_*(X; Z_p) \approx UM \in \mathcal{C}\mathcal{A}$ Then, for $t > s \geq 0$, there is a natural isomorphism*

$$E_2^{s,t}(X; Z_p) \approx H^{s,t}(M \hat{\otimes} \Lambda).$$

The mod-2 version of 14.4 was proved in [6, 3.3] using functors Ω and Ω^1 of Massey–Peterson. A similar proof works in our case using mod-p functors Ω and Ω^1 defined as follows:

For p odd and $M \in \mathcal{M}\mathcal{A}$ let $SM \in \mathcal{M}\mathcal{A}$ be given by $(SM)_i = M_{i-1}$ with the same $\mathcal{A}$-action as M. This suspension functor has a right adjoint, the loop functor Ω, and this functor Ω has a first derived functor Ω^1. A more explicit description of ΩM and $\Omega^1 M$ is by means of the exact sequence

$$0 \to S\Omega M \to M \xrightarrow{\zeta} DM \to S\Omega^1 M \to 0$$

where $DM \in \mathcal{M}\mathcal{A}$ is given by

$$\begin{aligned} (DM)_q &= M_{2n} && \text{for } q = 2pn \\ &= M_{2n+1} && \text{for } q = 2pn + 2 \\ &= 0 && \textit{otherwise} \end{aligned}$$

with right operators P^{pk} and P^{pk+1} corresponding to

$$P^k: M_{m+2k(p-1)} \to M_n \qquad m, k \geq 0$$

$$\beta P^k: M_{2n+2k(p-1)+2} \to M_{2n+1} \qquad n, k \geq 0$$

and β and all other P^t vanishing, and $\zeta: M \to DM \in \mathcal{M}\mathcal{A}$ is the map corresponding to

$$P^n: M_{2pn} \to M_n \qquad n \geq 0$$

$$\beta P^n: M_{2pn+2} \to M_{2n+1} \qquad n \geq 0.$$

As in the mod-2 case one then obtains an algebraic EHP sequence [6, 3.5] involving $H(\Omega M \hat{\otimes} \Lambda)$, $H(M \hat{\otimes} \Lambda)$ and $H(\Omega^1 M \hat{\otimes} \Lambda)$ and this readily leads to a proof of 14.4.

Remark 14.6. One often writes

$$Ext^{s,t}_{\mathcal{M}\mathcal{A}}(Z_p, M) \quad \text{for} \quad Ext_{\mathcal{M}\mathcal{A}}{}^s(\tilde{H}(S^t; Z_p), M).$$

Remark 14.7. Actual computation of $H(M \hat{\otimes} \Lambda)$ is greatly facilitated by "separating off the towers" [5, 2.3]. For $M \in \mathcal{M}\mathcal{A}$ and p odd, let $OM \subset M \hat{\otimes} \Lambda$ be the subcomplex generated by all $x \otimes v_I \in M \hat{\otimes} \Lambda^s$ with $v_I = v_{i_1} \cdots v_{i_s}$ allowable and $v_{i_s} = \lambda_{i_s}$ and let TM be the quotient complex of $M \hat{\otimes} \Lambda$ such that

$$(TM)^0 = M$$
$$(TM)^1 = M \otimes \mu_0$$
$$(TM)^s = M \otimes (\mu_0)^s \oplus \sum_{t>0} M_{2t} \otimes \lambda_t(\mu_0)^{s-1} \qquad s \geq 2.$$

Then one has a long exact sequence

$$\cdots \to H^{s-1}TM \to H^sOM \to H^s(M \hat{\otimes} \Lambda) \to H^sTM \to \cdots$$

§15. APPENDIX: E_2-TERM FOR A $K(G, n)$

For the computation of $E_2(K(G, n); R)$ for G abelian and R any commutative ring, it suffices, in view of §8, to calculate $E_2(K(G, n); Z)$ (which was done in 4.4) and $E_2(K(G, n); Z_h)$ for $h \geq 2$. This will be done below.

Let $ker(G, Z_h)$ and $coker(G, Z_h)$ denote the kernel and cokernel of the obvious composition

$$Tor(G, Z_h) \to G \to G \otimes Z_h.$$

Then we have

Theorem. 15.1 *For G an abelian group, $n \geq 1$ and $h \geq 2$ there are natural isomorphisms*

$$\begin{aligned} E_2^{s,t}(K(G, n); Z_h) &\approx G \otimes Z_h && \textit{for } s = 0, \quad t = n \\ &\approx coker(G, Z_h) && \textit{for } s > 0, \quad t - s = n \\ &\approx ker(G, Z_h) && \textit{for } s \geq 0, \quad t - s = n + 1 \\ &\approx 0 && \textit{otherwise.} \end{aligned}$$

Our proof of this theorem will depend on the following result which is of interest in its own right. Let M denote the Moore space given by the mapping cone of $S^1 \xrightarrow{h} S^1$ and for $X \in \mathcal{S}_*$ let

$$\pi_n(X; Z_h) = \pi_{n-2}\, hom(M, X) \qquad n \geq 2.$$

Then we have

Lemma 15.2. *For $X \in \mathcal{S}_*$ connected, $h \geq 2$, $t \geq 3$ and $s \geq 0$, the ring homomorphism $Z \to Z_h$ induces on isomorphism*

$$\pi^s\pi_t(\mathbf{Z}X; Z_h) \approx \pi^s\pi_t(\mathbf{Z}_h X; Z_h).$$

Proof of 15.1. Let $Y = K(G, n)$ and $t \geq 3$. Then by 15.2 and the function complex version of 4.4

$$\begin{aligned} \pi^s\pi_t(\mathbf{Z}_h Y; Z_h) &\approx \pi_t(Y; Z_h) && \text{for } s = 0 \\ &\approx 0 && \textit{otherwise.} \end{aligned}$$

Moreover

$$\begin{aligned} \pi_t(Y; Z_h) &\approx G \otimes Z_h && \text{for } t = n \\ &\approx Tor(G, Z_h) && \text{for } t = n + 1 \\ &\approx 0 && \textit{otherwise} \end{aligned}$$

and the theorem now follows readily from 10.2 and the cohomotopy long exact sequence of the short exact sequence of cosimplicial abelian groups

$$0 \to \pi_t \mathbf{Z}_h Y \to \pi_t(\mathbf{Z}_h Y;\, Z_h) \to \pi_{t-1} \mathbf{Z}_h Y \to 0.$$

To prove 15.2 we need in turn

LEMMA 15.3. *For $X \in \mathscr{S}_*$ connected, $m \geq 1$ and $t \geq 3$ the map*

$$\pi_t(Z^m X;\, Z_h) \to \pi_t(Z^m Z_h X;\, Z_h)$$

induced by the map $\phi\colon X \to Z_h X$ has a natural left inverse r_m.

Proof of 15.2. The function complex version of 4.4 and the collapse lemma 10.6 together with 15.3 show that for $t \geq 3$ and $j \geq 1$

$$\begin{aligned} \pi^s \pi_t(\mathbf{Z}(Z_h)^j X;\, Z_h) &\approx \pi_t((Z_h)^j X;\, Z_h) & s &= 0 \\ &\approx 0 & s &> 0 \\ \pi^s \pi_t(Z^j \mathbf{Z}_h X;\, Z_h) &\approx \pi_t(Z^j X;\, Z_h) & s &= 0 \\ &\approx 0 & s &> 0. \end{aligned}$$

Hence both spectral sequences of the double cosimplicial abelian groups $\pi_t(\mathbf{Z}\mathbf{Z}_h X;\, Z_h)$ collapse. This readily implies the desired result.

Finally, to prove 15.3 we need

15.4. The functor Z^+

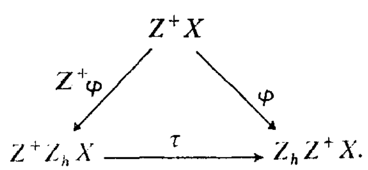
For $X \in \mathscr{S}_*$ let $Z^+ X$ be the free abelian monoid generated by X with the base point (and its degeneracies) put equal to 0 and let $\phi\colon X \to Z^+ X$ be the usual inclusion. Then [22] *for $X \in \mathscr{S}_*$ connected, the natural map*

$$\pi_* Z^+ X \to \pi_* Z X$$

is an isomorphism.

Another useful property is:

PROPOSITION 15.5. *For $X \in \mathscr{S}_*$ there is a natural map $\tau\colon Z^+ Z_h X \to Z_h Z^+ X$ such that the following diagram commutes*



Proof. Replace the usual functor Z_h by the equivalent functor

$$Z_h X = \left\{ \sum_i n_i x_i \,\middle|\, n_i \in Z_h,\ x_i \in X,\ \sum_i n_i = 1 \right\}.$$

Now $Z_h Z^+ X$ is an abelian monoid with monoid multiplication induced by that of the mod-h group ring on $Z^+ X$. The map

$$Z_h \phi\colon Z_h X \to Z_h Z^+ X$$

of pointed sets extends to a unique map τ of abelian monoids and this τ has the desired properties.

Proof of 15.3. For any $X \in \mathcal{S}_*$ there is a natural ismorphism $\pi_t(ZX; Z_h) \approx \pi_t Z_h X$ given (for example) by the composition

$$\pi_t(ZX; Z_h) \to H_t(ZX; Z_h) = \pi_t Z_h Z Y \to \pi_t Z_h Y$$

where the first map is the Hurewicz map. This easily implies the case $m = 1$ and we proceed inductively. Given r_{m-1} for some $m \geq 2$, it will suffice to construct a natural left inverse to the map

$$\pi_t(Z^{m-1}Z^+X; Z_h) \to \pi_t(Z^{m-1}Z^+Z_hX; Z_h)$$

induced by ϕ: $X \to Z_h X$. Such an inverse is given by the composition

$$\pi_t(Z^{m-1}Z^+Z_hX; Z_h) \to \pi_t(Z^{m-1}Z_hZ^+X; Z_h) \xrightarrow{r_{m-1}} \pi_t(Z^{m-1}Z^+X; Z_h)$$

where the first map is induced by τ: $Z^+Z_hX \to Z_hZ^+X$.

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Brandeis University
Waltham, Mass.
Massachusetts Institute of Technology
Cambridge, Mass.