THH and traces of enriched categories

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Abstract

We prove that topological Hochschild homology (THH) arises from a presheaf of circles on a certain combinatorial category, which gives a universal construction of THH for any enriched ∞-category.

Our results rely crucially on an elementary, model-independent framework for enriched higher category theory, which may be of independent interest.

1 Introduction

Those interested only in enriched category theory, read Sections 1.3 and 2.

1.1 Background

The topological Hochschild homology (THH) of a ring or ring spectrum R is an object much studied in recent years, because it can be used in favorable cases to compute the algebraic K-theory of R. The strategy takes advantage of a rich *cyclotomic* structure on $\mathrm{THH}(R)$, which is a refinement of a natural circle action. This so-called *trace methods* approach to K-theory originated with the Dennis trace in the 70s and Bokstedt's work [6] in the 80s, and has taken off since then. A modern account is [1].

THH has a variety of other structure in addition to the circle action. First, it is Morita invariant, and therefore lifts to an invariant of ∞-categories enriched in spectra. (Even more, THH is an invariant of noncommutative motives [4] 10.2.)

If $\mathcal{C}$ is a spectral ∞-category, then $\mathrm{THH}(\mathcal{C})$ is the geometric realization of the simplicial spectrum

$$\mathrm{THH}(\mathcal{C})_n = \coprod_{X_0,\dots,X_n\in\mathcal{C}} \mathrm{Hom}(X_0,X_1)\otimes\cdots\otimes\mathrm{Hom}(X_n,X_0),$$

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which is called the cyclic bar construction. If $\mathcal{C}$ has a single object, we may identify it with an $\mathbb{E}_1$-ring spectrum R, and we recover the original notion of THH as the geometric realization of

$$\mathrm{THH}(R)_n = R^{\otimes n+1}.$$

Second, THH is a trace functor [12] [7]. That is, we may further generalize, defining $\mathrm{THH}(F)$ for any endomorphism $F : \mathcal{C} \to \mathcal{C}$ of a spectral ∞-category. Usual THH is recovered via $\mathrm{THH}(\mathcal{C}) = \mathrm{THH}(\mathrm{id}_{\mathcal{C}})$, and THH satisfies the trace identity $\mathrm{THH}(FG) \cong \mathrm{THH}(GF)$.

Working at this level of generality is useful because the trace identity on THH implies many other useful properties, including Morita invariance [11]. It is therefore natural to ask:

Question 1.1. *To what extent does the trace identity uniquely determine THH? That is, to what extent is THH the universal trace functor on the ∞-category Cat^{Sp} of spectral ∞-categories?*

The trace identity asserts that the functor $\mathrm{THH} : \coprod_{\mathcal{C}} \mathrm{End}(\mathcal{C}) \to \mathrm{Sp}$ coequalizes (up to homotopy) the diagram

$$\coprod_{\mathcal{C}_0,\mathcal{C}_1} \mathrm{Fun}^{\mathrm{Sp}}(\mathcal{C}_0,\mathcal{C}_1) \otimes \mathrm{Fun}^{\mathrm{Sp}}(\mathcal{C}_1,\mathcal{C}_0) \rightrightarrows \coprod_{\mathcal{C}} \mathrm{Fun}^{\mathrm{Sp}}(\mathcal{C},\mathcal{C}).$$

This diagram itself is the 1-skeleton of a cyclic bar construction, taking place in $\mathrm{Cat}^{\mathrm{Sp}}$ (rather than Sp). We might hope that THH is compatible with the entire cyclic bar construction, not just its 1-skeleton. To formalize this situation, we are forced to study THH of more general enriched ∞-categories.

Definition 1.2. *If $\mathcal{V}$ is a symmetric monoidal ∞-category*[1] *and $\mathcal{C}$ is a $\mathcal{V}$-enriched category, THH$(\mathcal{C}) \in \mathcal{V}$ is the geometric realization of the cyclic bar construction (in $\mathcal{V}$):*

$$THH(\mathcal{C})_n = \coprod_{X_0,\ldots,X_n \in \mathcal{C}} Hom(X_0, X_1) \otimes \cdots \otimes Hom(X_n, X_0).$$

Definition 1.3. *If $\mathcal{C}$ is a $\mathcal{V}$-enriched category and $X \in \mathcal{V}$, a* homotopy-coherent trace functor *from $\mathcal{C}$ to X is a morphism $THH(\mathcal{C}) \to X$.*

Then we might ask, en route to answering Question 1.1:

[1] Since $\mathrm{THH}(\mathcal{C})$ is a colimit, we should either assume $\mathcal{V}$ is presentable, or define $\mathrm{THH}(\mathcal{C})$ as a presheaf on $\mathcal{V}$. We will ignore this unimportant subtlety in the introduction.

Question 1.4. *Can THH be promoted to a homotopy-coherent trace functor* $THH(Cat^{Sp}) \xrightarrow{THH} Sp$*?*

We will not answer these questions. However, if we hope to study questions of a formal nature like these, Definition 1.2 is not ideal. It is fundamentally a calculation of THH, when we would prefer a universal property. For example, it obscures the circle action (and ensuing cyclotomic structure) on THH, and it makes explicit reference to the set of objects of $\mathcal{C}$.

It also relies implicitly on enriched ∞-categories, for which the state of the art (largely due to Gepner and Haugseng [10]) is rather technical and dependent on the particular model of quasicategories.

Motivated by these objections, we will:

1. present a combinatorial, model-independent framework for enriched higher category theory; a $\mathcal{V}$-enriched category is a symmetric monoidal functor $\mathrm{Bypass}_S \to \mathcal{V}$ from a category of combinatorial graphs;

2. present a universal construction of THH, using this framework; THH is the pushforward along $\mathrm{Bypass}_S \to \mathcal{V}$ of a certain presheaf $\mathcal{O}_{\mathrm{thh}}$ on Bypass_S obtained as a push-pull construction applied to the circle;

3. explicitly calculate the presheaf $\mathcal{O}_{\mathrm{thh}}$.

1.2 First results

As a warm-up, a special case of (2) can be stated without any enriched category theory.

Suppose that A is an associative algebra in $\mathcal{V}$ (equivalently, a $\mathcal{V}$-enriched category with one object). We regard A as a symmetric monoidal functor

$$\mathrm{Ass} \xrightarrow{A} \mathcal{V},$$

where Ass is the *associative PROP* ([14] 4.1.1): An object of Ass is a finite set, and a morphism a function $f : X \to Y$ with a total ordering of each $f^{-1}(y)$.

Let Λ be Connes' cyclic category [8]; roughly, the category of cyclically ordered finite sets. There is a functor $\Lambda \xrightarrow{k} \mathrm{Ass}$ which forgets the cyclic ordering (see Proposition 5.1). Moreover, the classifying space of Λ is BS^1 [8], so there is a functor of ∞-categories $\Lambda \xrightarrow{r} BS^1$ which formally inverts all the morphisms of Λ. In summary, we have the diagram:

$$\begin{array}{ccccc} \Lambda & \xrightarrow{k} & \mathrm{Ass} & \xrightarrow{A} & \mathcal{V} \\ \downarrow{\scriptstyle r} & & & & \\ BS^1. & & & & \end{array}$$

Theorem (5.3)**.** *If $\mathcal{V}$ is presentable and A is an associative algebra in $\mathcal{V}$,*

$$THH(A) \cong A_* k_* r^*(S^1).$$

The notation requires some explanation:

- S^1 is the circle with the free S^1-action, regarded as an S^1-space and thus a presheaf $(BS^1)^{\mathrm{op}} \to \mathrm{Top}$;
- $r^* : \mathcal{P}(BS^1) \to \mathcal{P}(\Lambda)$ denotes precomposition of a presheaf by r;
- $k_* : \mathcal{P}(\Lambda) \to \mathcal{P}(\mathrm{Ass})$ denotes left Kan extension along k;
- $A_* : \mathcal{P}(\mathrm{Ass}) \to \mathcal{V}$ is the unique functor extending A to $\mathcal{P}(\mathrm{Ass}) \supseteq \mathrm{Ass}$ which preserves small colimits.

The upshot is that we have a universal construction of THH which makes explicit the S^1-action. In particular, there is a presheaf $\mathcal{O}_{\mathrm{thh}} = k_* r^* S^1$ on the associative PROP Ass, described by a push-pull procedure applied to the circle, and

$$\mathrm{THH}(\mathcal{C}) \cong \mathcal{C}_* \mathcal{O}_{\mathrm{thh}}.$$

Our main results will be:

- (Theorem 6.1) a generalization of Theorem 5.3, replacing the associative algebra by any $\mathcal{V}$-enriched category;
- (Theorem 8.6) an explicit calculation of the presheaf $\mathcal{O}_{\mathrm{thh}}$.

Remark 1.5. *We will extend Theorem 5.3 by categorification, replacing associative algebras by enriched categories. We conjecture that it may also be generalized in other directions. For instance, we believe there is an analogue calculating factorization homology $\int_M A$ when M is an n-manifold and A is an $\mathbb{E}_n$-algebra.*

If so, Theorem 5.3 would recover the well-known identification between THH and factorization homology over S^1. Ayala-Mazel-Gee-Rozenblyum [2] have results related to ours in the factorization homology setting.

1.3 Enriched categories

We will use the following straightforward definition of enriched ∞-categories. For each set S, there is a symmetric monoidal category Bypass_S, and:

Definition (2.4)**.** *If $\mathcal{V}$ is a symmetric monoidal ∞-category, a $\mathcal{V}$-enriched category with set S of objects is a symmetric monoidal functor*

$$\mathcal{C} : \mathit{Bypass}_S \to \mathcal{V}.$$

We will now describe Bypass_S. An object is a directed graph on the fixed set S of vertices. These are really *multigraphs*, in that they may include multiple edges between the same two vertices, as well as loops from a vertex to itself.

A morphism in Bypass_S is a combination of the following combinatorial moves that we call *bypass operations*:

- choose an ordered set of edges forming a path $X_0 \to X_1 \to \cdots \to X_n$, and replace them by a single edge $X_0 \to X_n$;
- choose a vertex X, and add a new edge (loop) $X \to X$.

If $\Gamma, \Gamma' \in \mathrm{Bypass}_S$, we write $\Gamma \otimes \Gamma'$ for the graph whose set of edges is the disjoint union of edges in Γ and edges in Γ'. In this way, Bypass_S is symmetric monoidal, and the unit is the empty graph $\varnothing$ (no edges).

We also write $(X, Y) \in \mathrm{Bypass}_S$ for the graph with a single edge from X to Y. Essentially by construction, Bypass_S admits a presentation as a symmetric monoidal category by:

- objects (X, Y) for $X, Y \in S$;
- morphisms $(X, Y) \otimes (Y, Z) \to (X, Z)$ for $X, Y, Z \in S$;
- morphisms $\varnothing \to (X, X)$ for $X \in S$;
- associative and unital relations.

This presentation encodes the classical axioms of an enriched category. The notation has been set up conveniently so that, if $\mathcal{C} : \mathrm{Bypass}_S \to \mathcal{V}$ is an enriched category, $\mathcal{C}(X, Y)$ is the object of morphisms from X to Y.

In Section 2, we will prove:

Proposition (2.7)**.** *Definition 2.4 agrees with that of Gepner-Haugseng [10].*

Now suppose we have a graph $\Gamma \in \mathrm{Bypass}_S$. An *Eulerian tour* on Γ is a cyclic ordering on the edges of Γ so that they form a single cycle. We let $\mathrm{Bypass}_S^{\mathrm{Eul}}$ denote the category of nonempty graphs in Bypass_S with specified Eulerian tour, and $\mathrm{Bypass}_S^{\mathrm{Eul}} \xrightarrow{k} \mathrm{Bypass}_S$ the forgetful functor.

Our first main result generalizes Theorem 5.3 to a universal construction of THH of enriched categories:

Theorem (6.1)**.** *If $\mathcal{C} : Bypass_S \to \mathcal{V}$ is a $\mathcal{V}$-enriched category with set S of objects,*

$$THH(\mathcal{C}) \cong \mathcal{C}_* k_* r^*(S^1),$$

with maps as in the diagram

$$\begin{array}{ccccc} Bypass_S^{Eul} & \xrightarrow{k} & Bypass_S & \xrightarrow{\mathcal{C}} & \mathcal{V} \\ \downarrow{\scriptstyle r} & & & & \\ BS^1. & & & & \end{array}$$

The functor r is that which exhibits BS^1 as the classifying space of $\mathrm{Bypass}_S^{\mathrm{Eul}}$ (Corollary 7.4).

In the case $S = *$, then $\mathrm{Bypass}_* = \mathrm{Ass}$, because a graph on one vertex can be identified with a finite set (of loops at that vertex). On the other hand, an Eulerian tour is a cyclic ordering, so $\mathrm{Bypass}_*^{\mathrm{Eul}} = \Lambda$, and we recover Theorem 5.3.

Remark 1.6. *The description $THH(\mathcal{C}) \cong \mathcal{C}_* k_* r^*(S^1)$ has a few benefits: Most obviously, it makes explicit the S^1-action. It also isolates the cyclic bar construction as $k_* r^*(S^1)$, divorcing it from the enriched category theory (which is encoded in $\mathcal{C}_*$). Formal arguments involving the cyclic bar construction can now be encoded as properties of the presheaf $k_* r^*(S^1)$.*

1.4 Calculations

If $\mathcal{C} : \mathrm{Bypass}_S \to \mathcal{V}$ is an enriched category with set S of objects, $\mathrm{THH}(\mathcal{C})$ is a colimit of terms which can be defined in Bypass_S. Therefore, there is a formal colimit in Bypass_S, namely the geometric realization of

$$(\mathcal{O}_{\mathrm{thh}})_n = \coprod_{X_0, \ldots, X_n} (X_0, X_1) \otimes \cdots \otimes (X_n, X_0),$$

for which $\mathcal{C}(\mathcal{O}_{\mathrm{thh}}) \cong \mathrm{THH}(\mathcal{C})$. By an extension of the Yoneda lemma, formal colimits can be identified with presheaves of spaces $\mathrm{Bypass}_S^{\mathrm{op}} \to \mathrm{Top}$, so we think of $\mathcal{O}_{\mathrm{thh}}$ as a presheaf.

Then Theorem 6.1 can be restated $\mathcal{O}_{\mathrm{thh}} \cong k_* r^*(S^1)$. This is a useful universal property of THH. However, we can also do more: We can explicitly calculate the presheaf $\mathcal{O}_{\mathrm{thh}} : \mathrm{Bypass}_S^{\mathrm{op}} \to \mathrm{Top}$:

Proposition (Corollary 8.3)**.** *We have*

$$\mathcal{O}_{thh}(\Gamma) \cong \begin{cases} (S^1)^{\amalg Eul(\Gamma)}, & \text{if } \Gamma \neq \varnothing \\ S, & \text{if } \Gamma = \varnothing \end{cases},$$

where the disjoint union of circles is taken over the set $Eul(\Gamma)$ *of Eulerian tours on* Γ*, and* S *is the ambient set of objects.*

Remark 1.7. *Because of the different behavior at* $\varnothing$*, we will clean up our exposition by restricting away from the empty graph.*

Let $\mathcal{O}^+_{\mathrm{thh}}$ denote $\mathcal{O}_{\mathrm{thh}}$ restricted away from the empty graph. This is a presheaf on the full subcategory $\mathrm{Bypass}_S^+ \subseteq \mathrm{Bypass}_S$ of nonempty graphs.

The last proposition describes $\mathcal{O}^+_{\mathrm{thh}}(\Gamma)$ for each Γ; however, the restriction maps $\mathcal{O}^+_{\mathrm{thh}}(f)$ for $f : \Gamma \to \Gamma'$ may be nontrivially twisted.

The following theorem will give a complete description of $\mathcal{O}^+_{\mathrm{thh}}$. If $\mathrm{Eul}(\Gamma)$ denotes the set of Eulerian tours on Γ, then Eul is a presheaf of sets defined on nonempty graphs Bypass_S^+.

Theorem (8.6)**.** *If* $\mathcal{O}^+_{thh}$ *is as above, then:*

1. $\mathcal{O}^+_{thh}$ *has a canonical* S^1*-action;*
2. $(\mathcal{O}^+_{thh})_{hS^1} \cong Eul$*;*
3. *The moduli space of presheaves satisfying (1)-(2) is* $\mathbb{Z} \times BS^1$*; that is, such presheaves are determined up to equivalence by an integer invariant we call* degree*;*
4. *The degree of* $\mathcal{O}^+_{thh}$ *is* 1*.*

Notice that the last proposition (Corollary 8.3) follows from the theorem. Indeed, any $\mathcal{O}$ satisfying (1)-(2) evaluates on $\Gamma \neq \varnothing$ by

$$\mathcal{O}(\Gamma) \cong (S^1)^{\amalg \mathrm{Eul}(\Gamma)}.$$

Property (4) describes the twisting of the sheaf $\mathcal{O}_{\mathrm{thh}}$.

In general, write $\mathrm{Eul}(n)$ for the S^1-bundle over Eul of degree n.

Corollary 1.8. *If $\mathcal{C}$ is $\mathcal{V}$-enriched, then*

$$\mathcal{C}_*(Eul(\pm 1)) \cong THH(\mathcal{C}).$$

We can also identify $\mathcal{C}_*(Eul(n))$ for all n:

$$\mathcal{C}_*(Eul(\pm n)) \cong THH(\mathcal{C})_{hC_n},$$

$$\mathcal{C}_*(Eul(0)) \cong S^1 \otimes THH(\mathcal{C})_{hS^1},$$

$$\mathcal{C}_*(Eul) \cong THH(\mathcal{C})_{hS^1}.$$

Theorem 8.6 is the most technical part of the paper. It relies on a key combinatorial lemma that we sketch now.

Namely, the following data is equivalent:

- a graph $\Gamma \in \mathrm{Bypass}_S$ with a chosen Eulerian tour;
- a cyclically ordered set of edges $E \in \Lambda$ with a labeling of its vertices in S.

Categorically, this combinatorial argument implies that $\mathrm{Bypass}_S^{\mathrm{Eul}}$ is both the (right fibrational) Grothendieck construction applied to the presheaf Eul on Bypass_S, as well as the (left fibrational) Grothendieck construction applied to a certain presheaf on Λ.

In the case $S = *$, this recovers the observation $\mathrm{Bypass}_*^{\mathrm{Eul}} \cong \Lambda$.

We will use this combinatorial argument to prove that $\mathrm{Bypass}_S^{\mathrm{Eul}}$ has classifying space BS^1 (Corollary 7.4), and to pass back and forth between presheaves on Bypass_S and presheaves on $\mathrm{Bypass}_S^{\mathrm{Eul}}$ (Corollary 7.6). This is the technical material necessary to prove Theorem 8.6(3), and the rest follows by Theorem 6.1.

1.5 Organization

Section 2 concerns enriched higher categories. We want to emphasize that our construction is relatively elementary and model-independent. For that reason, this section is written for a wide audience, including those who may not be interested in THH.

In Section 3, we review the cyclic category Λ. The results are not new, but there is one crucial idea (Lemma 3.4): There are canonical equivalences $\Lambda \cong \mathrm{Bypass}_*^{\mathrm{Eul}} \cong \Lambda^{\mathrm{op}}$. Throughout the paper, we will frequently make the second identification $\mathrm{Bypass}_*^{\mathrm{Eul}} \cong \Lambda^{\mathrm{op}}$.

In the very short Section 4, we define THH of an enriched category via the cyclic bar construction. The cyclic bar construction itself is formally encoded by a presheaf $\mathcal{O}_{\text{thh}}$ on Bypass_S.

Section 5 contains the proof of the first main result (Theorem 5.3), which provides a universal construction for THH of associative algebras.

Although the cyclic bar construction of $\text{THH}(\mathcal{C})$ makes reference to the set of objects of $\mathcal{C}$, THH does not actually depend in any meaningful way on the object-set. In Section 6, we prove a weak version of this statement, and use it to generalize our construction of THH to the case of enriched ∞-categories.

The final two sections are the calculation, Theorem 8.6. For this, we need the combinatorial counting-in-two-ways argument described above, which is in Section 7, followed by the proof of Theorem 8.6 and its corollaries in Section 8.

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1.7 Not included in this paper

Everything in this paper is done for enriched categories with fixed sets of objects. That is, we define THH as a functor on $\text{Cat}_S^{\mathcal{V}}$, the ∞-category of $\mathcal{V}$-enriched categories with set S of objects (and functors between them which act as the identity on objects).

In fact, THH is functorial on $\text{Cat}^{\mathcal{V}}$ (the ∞-category of small $\mathcal{V}$-enriched categories). The author has chosen not to include a discussion of the functoriality of THH because it would lengthen the paper and distract from the main results.

We hope to include these details in a sequel on cyclotomic structures. Until then, the interested reader can derive the functoriality of THH from the two properties:

- If $f : S \to T$ is a function inducing $F : \text{Bypass}_S \to \text{Bypass}_T$, then $F_*\mathcal{O}_{\text{thh}} \cong \mathcal{O}_{\text{thh}}$. Hence, if $\mathcal{C} \to \mathcal{D}$ is fully faithful, there is a functor $\text{THH}(\mathcal{C}) \to \text{THH}(\mathcal{D})$.
- If f is surjective, then also $F^*\mathcal{O}_{\text{thh}} \cong \mathcal{O}_{\text{thh}}$. Hence, if $\mathcal{C} \to \mathcal{D}$ is also essentially surjective, then $\text{THH}(\mathcal{C}) \to \text{THH}(\mathcal{D})$ is an equivalence.

1.8 Notation

We use ∞-categorical language throughout, writing Top for the ∞-category of spaces, or homotopy types, and $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Top})$ for presheaves on a small ∞-category $\mathcal{C}$.

Via the Yoneda embedding $\mathcal{C} \subseteq \mathcal{P}(\mathcal{C})$, we will identify $\mathcal{C}$ with the ∞-category of representable presheaves. That is, we will use the same symbol to refer to an object $X \in \mathcal{C}$ or the associated representable presheaf $X \in \mathcal{P}(\mathcal{C})$.

By [13] 5.1.5.6, $\mathcal{P}(\mathcal{C})$ can also be identified with the ∞-category of *formal colimits* in $\mathcal{C}$. Hence, if $F : \mathcal{C} \to \mathcal{D}$ is a functor, $\mathcal{C}$ is a small ∞-category, and $\mathcal{D}$ is a presentable ∞-category, then there is an essentially unique extension

$$F_* : \mathcal{P}(\mathcal{C}) \to \mathcal{D}$$

which preserves colimits and restricts to F on representables. Also, F_* has a right adjoint F^* by the adjoint functor theorem [13] 5.5.2.9.

Here is a special case: If $F : \mathcal{C} \to \mathcal{D}$ is a functor between small ∞-categories, we embed $\mathcal{D} \subseteq \mathcal{P}(\mathcal{D})$, which is presentable, so we can apply the above construction to produce an adjoint pair

$$F_* : \mathcal{P}(\mathcal{C}) \leftrightarrows \mathcal{P}(\mathcal{D}) : F^*.$$

As before, the left adjoint F_* is the unique functor preserving colimits which restricts to F on representables.

The right adjoint F^* is precomposition of a presheaf $\mathcal{D}^{\mathrm{op}} \to \mathrm{Top}$ by $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Example 1.9. *For any ∞-category $\mathcal{C}$, let $c : \mathcal{C} \to *$ denote the trivial functor. Then $c^* : Top \to \mathcal{P}(\mathcal{C})$ sends a space X to the constant presheaf at X. Hence its left adjoint is*

$$c_*(\mathcal{O}) \cong colim_{\mathcal{C}^{op}}(\mathcal{O}).$$

Example 1.10. *If $\mathcal{C}$ is an ∞-category, it has a classifying space $|\mathcal{C}|$, which has the following universal property: If X is an ∞-groupoid, any functor $\mathcal{C} \to X$ factors uniquely through $|\mathcal{C}|$.*

If $\mathcal{C}$ is an ordinary category, $|\mathcal{C}|$ is classically the geometric realization of the nerve of $\mathcal{C}$.

By [13] 3.3.4.6, $|\mathcal{C}| \cong c_*(*)$, where $c : \mathcal{C} \to *$ as before, and $*$ is the trivial constant presheaf on $\mathcal{C}$.

2 Enriched categories

In this section, we give a combinatorial description of enriched categories as symmetric monoidal functors $\mathrm{Bypass}_S \to \mathcal{V}$, where Bypass_S is a category of graphs and bypass operations. Specifically:

Definition 2.1. *A* directed multigraph *(or just* graph*)* Γ *consists of a set* E *of edges, a set* S *of vertices, and two functions* $s, t : E \to S$ *called the source and target. We say* Γ *is* finite *if* E *is finite (even if* S *is not).*

Given two directed multigraphs Γ, Γ' *with identical sets of vertices, a* bypass operation $f : \Gamma \to \Gamma'$ *is a function* $f : E \to E'$ *along with a total ordering of the set* $f^{-1}(e)$ *for each* $e \in E'$*, satisfying the properties:*

- *If* $f^{-1}(e)$ *is empty, then* e *is a* loop*; that is,* $s(e) = t(e)$*;*
- *If* $f^{-1}(e) = \{e_1 < \cdots < e_k\}$*, then* $e_1 \to \cdots \to e_k$ *is a* path *from* $s(e)$ *to* $t(e)$*; that is,* $t(e_i) = s(e_{i+1})$ *for* $1 \leqslant i < k$*,* $s(e_1) = s(e)$*, and* $t(e_k) = t(e)$*.*

We think of a graph Γ as a set S of vertices and a set E of edges, such that each $e \in E$ is a directed edge from $s(e)$ to $t(e)$. A bypass operation is a series of operations of the following forms (corresponding respectively to the two properties above) which transform Γ into Γ':

- Add a loop (an edge from a vertex to itself);
- Given edges which form a path $e_1 \to \cdots \to e_n$, replace all of them by a single edge from $s(e_1)$ to $t(e_n)$.

The second operation is the origin of our term *bypass*.

Definition 2.2. *Given a set* S *(not necessarily finite),* Bypass_S *is the category of* finite *directed multigraphs with fixed object set* S*, and bypass operations for morphisms.*

Given two directed multigraphs Γ, Γ' on a fixed set S of vertices, denote by $\Gamma \otimes \Gamma'$ the graph with edge set $E \amalg E'$ and the induced source and target maps. Then $\otimes$ is a symmetric monoidal operation on Bypass_S. The unit of the symmetric monoidal structure is the empty graph $\varnothing$ with no edges.

We will introduce notation for a few special graphs that will be important in all that follows:

Definition 2.3. *If $X_i \in S$, then $(X_1, X_2) \in Bypass_S$ denotes the graph with a single edge from X_1 to X_2. More generally,*

$$(X_1, X_2, \ldots, X_n) = (X_1, X_2) \otimes (X_2, X_3) \otimes \cdots \otimes (X_{n-1}, X_n)$$

denotes the graph with a single path $X_1 \to \cdots \to X_n$.

As a symmetric monoidal category, $\mathrm{Bypass}_S^{\otimes}$ admits a presentation by:

- objects (X, Y),
- morphisms $(X, Y, Z) = (X, Y) \otimes (Y, Z) \to (X, Z)$;
- morphisms $\varnothing \to (X, X)$;
- relations (commuting diagrams)

$$\begin{array}{ccc} (a,b,c,d) & \longrightarrow & (a,b,d) \\ \downarrow & & \downarrow \\ (a,c,d) & \longrightarrow & (a,d) \end{array} \qquad \begin{array}{ccccc} (a,b,b) & \longrightarrow & (a,b) & \longleftarrow & (a,a,b) \\ \uparrow & \nearrow & & \nwarrow & \uparrow \\ (a,b) & & & & (a,b) \end{array}$$

Notice that this presentation corresponds exactly to the axioms of an enriched category. Hence, the following definition agrees with the usual one when $\mathcal{V}$ is an ordinary category:

Definition 2.4. *If $\mathcal{V}$ is a symmetric monoidal ∞-category, a $\mathcal{V}$-enriched category with set S of objects is a symmetric monoidal functor*

$$\mathcal{C} : Bypass_S \to \mathcal{V}.$$

Example 2.5. *If $S = *$ is a singleton, then $Bypass_*$ is the category of finite sets, whose morphisms come with prescribed total orderings of the fibers. This is the symmetric monoidal envelope of the associative operad ([14] 4.1.1.1), so symmetric monoidal functors $Bypass_* \to \mathcal{V}$ may be identified with associative algebras in $\mathcal{V}$.*

This is consistent with the principle that associative algebras are enriched categories with one object.

Remark 2.6. *The notation has been set up conveniently so that when the functor $\mathcal{C}$ is evaluated at $(X, Y) \in Bypass_S$, we get $\mathcal{C}(X, Y) \cong Map(X, Y)$. If 1 denotes the unit in $\mathcal{V}$, we also have*

$$\mathcal{C}(\varnothing) \cong 1,$$

$$\mathcal{C}(X_1, \ldots, X_n) \cong \mathcal{C}(X_1, X_2) \otimes \cdots \otimes \mathcal{C}(X_{n-1}, X_n).$$

We end this section with a comparison to the definition of Gepner-Haugseng:

Proposition 2.7. *Definition 2.4 agrees with that of Gepner-Haugseng [10].*

Proof. Gepner and Haugseng ([10] 2.2.17) define $\mathrm{Cat}_S^{\mathcal{V}} = \mathrm{Alg}_{\mathcal{O}_S^{\otimes}}(\mathcal{V})$, where $\mathcal{O}_S^{\otimes}$ is a nonsymmetric ∞-operad defined in [10] 2.1. For them, $\mathcal{V}$ is just monoidal, but for us it is even *symmetric* monoidal. Hence $\mathrm{Cat}_S^{\mathcal{V}} \cong \mathrm{Alg}_{\bar{\mathcal{O}}_S^{\otimes}}(\mathcal{V})$, where $\bar{\mathcal{O}}_S^{\otimes}$ is the *symmetrization* of $\mathcal{O}_S^{\otimes}$ ([10] Definition 3.7.6).

Unpacking definitions, the colors of $\bar{\mathcal{O}}_S^{\otimes}$ are symbols (X, Y) for $X, Y \in S$, and active morphisms (multilinear morphisms) from the unordered tuple $((X_1, Y_1), \ldots, (X_n, Y_n))$ to (A, B) are canonically in bijection with Bypass_S-morphisms of the form

$$(X_1, Y_1) \otimes \cdots \otimes (X_n, Y_n) \to (A, B).$$

The symmetric monoidal envelope $\mathrm{Env}(\bar{\mathcal{O}}_S^{\otimes})$ is the subcategory of $\bar{\mathcal{O}}_S^{\otimes}$ spanned by just the active morphisms ([14] 2.2.4). Because objects of Bypass_S can be written in a unique way as tensor products of the elementary objects (X, Y), unpacking definitions shows that $\mathrm{Env}(\bar{\mathcal{O}}_S^{\otimes}) \cong \mathrm{Bypass}_S$, as symmetric monoidal categories. Verification is elementary because both of these are 1-categories.

The theorem is then the universal property of symmetric monoidal envelopes ([14] 2.2.4.9). □

The reader may object: We have defined a $\mathcal{V}$-enriched category, but a bit more work is needed to define enriched functors (that is, to define an ∞-category $\mathrm{Cat}^{\mathcal{V}}$). This won't be important for this paper, but the construction is a standard one, which we sketch below. See [10] for details.

We build $\mathrm{Cat}^{\mathcal{V}}$ out of $\mathrm{Cat}_S^{\mathcal{V}}$ as follows. The construction $S \mapsto \mathrm{Bypass}_S$ is functorial $\mathrm{Bypass}_{(-)} : \mathrm{Set} \to \mathrm{Cat}$, and the associated Grothendieck construction (or cocartesian fibration) is $p : \mathrm{preCat}^{\mathcal{V}} \to \mathrm{Set}$.

In $\mathrm{preCat}^{\mathcal{V}}$, the objects are enriched categories, and morphisms are composites $F : \mathcal{C} \to \mathrm{im}(F) \subseteq \mathcal{D}$, where the first functor acts as the identity on objects, and the second is fully faithful (that is, a p-cocartesian morphism in $\mathrm{preCat}^{\mathcal{V}}$).

Now we need to insist that fully faithful essentially surjective functors are equivalences. We do this by inverting those p-cocartesian morphisms f in $\mathrm{preCat}^{\mathcal{V}}$ for which $p(f)$ is a surjection. The result is $\mathrm{Cat}^{\mathcal{V}}$.

Remark 2.8. *Actually, Gepner-Haugseng's enriched categories have underlying* spaces *of objects, while ours have underlying* sets *of objects. By [10] 5.3.17, these two theories are equivalent.*

See Section 1.7 for a comment on how THH interacts with the construction of $\mathrm{Cat}^{\mathcal{V}}$.

3 The cyclic category

In this section, we will review the cyclic category with an eye towards our application to enriched category theory.

Definition 3.1. *If Γ is a* nonempty *directed multigraph, an* Eulerian tour *on Γ is a total ordering of the edges in such a way that they form a single cycle. (That is, it is a path which begins and ends at the same vertex, visiting each edge exactly once.)*

We will regard two Eulerian tours on Γ as the same if they differ only by cyclic permutation.

Suppose that $\Gamma \xrightarrow{f} \Gamma'$ is a bypass operation between nonempty graphs. Given an Eulerian tour on Γ', then the total orderings of fibers of f (part of the data of the morphism f) induces an Eulerian tour on Γ. A cyclic shift of the tour on Γ' induces a cyclic shift of the induced tour on Γ.

Definition 3.2. *Let $\mathit{Bypass}_S^{\mathit{Eul}}$ denote the category of (finite, nonempty) directed multigraphs with a chosen Eulerian tour, along with those morphisms $\Gamma \to \Gamma'$ in Bypass_S which pull back the chosen Eulerian tour on Γ' to the chosen Eulerian tour on Γ.*

To understand the structure of $\mathrm{Bypass}_S^{\mathrm{Eul}}$, we first consider the case $S = *$. In this case, $\mathrm{Bypass}_*^{\mathrm{Eul}}$ may be identified with Connes' cyclic category.

Definition 3.3 (Connes' cyclic category [8])**.** *Let $\mathbb{T}_n$ denote the category on objects v_i generated by* irreducible morphisms *e_i:*

$$v_0 \xrightarrow{e_0} \cdots \xrightarrow{e_{n-1}} v_n \xrightarrow{e_n} v_0.$$

We call this category a cyclically ordered set*. Each object has a* degree 1 *endomorphism given by passing around this cycle once.*

The cyclic category Λ is the category of finite cyclically ordered sets and functors between them which send degree 1 endomorphisms to degree 1 endomorphisms. We call these degree 1 functors.

There are two functors

$$i_v : \Lambda \to \mathrm{Bypass}_*^{\mathrm{Eul}},$$
$$i_e : \Lambda^{\mathrm{op}} \to \mathrm{Bypass}_*^{\mathrm{Eul}},$$

where $i_v(\mathbb{T}_n)$ is the graph on one vertex whose edges are given by the objects v_i of $\mathbb{T}_n$, and $i_e(\mathbb{T}_n)$ is the graph whose edges are given by the irreducible morphisms e_i of $\mathbb{T}_n$.

These two functors treat morphisms as follows: If $f : \mathbb{T}_m \to \mathbb{T}_n$, then $i_v(f)$ acts as the function f. On the other hand, $i_e(f)$ sends an edge $e \in \mathbb{T}_n$ to the unique edge $e' \in \mathbb{T}_m$ for which $f(e')$, written as a composite of elementary morphisms, includes the morphism e.

Lemma 3.4. *Both i_v and i_e are equivalences of categories.*

In particular, we have a canonical equivalence $\Lambda \cong \Lambda^{\mathrm{op}}$, known from [8].

Proof. Let $\Gamma \in \mathrm{Bypass}_S^{\mathrm{Eul}}$ (notice S is not necessarily $*$). The Eulerian tour endows the edges of Γ with a cyclic ordering, so there is a forgetful functor $\mathrm{Bypass}_S^{\mathrm{Eul}} \to \Lambda$ which remembers just the set of edges. When $S = *$, this is i_v^{-1}, so i_v is an equivalence.

On the other hand, suppose that $\Gamma, \Gamma' \in \mathrm{Bypass}_S^{\mathrm{Eul}}$ have cyclically ordered sets of edges $\mathbb{T}_m, \mathbb{T}_n$, respectively. Given a map $f : \Gamma \to \Gamma'$, there is an induced map $f^* : \mathbb{T}_n \to \mathbb{T}_m$ which sends an edge $e \in \Gamma'$ to the unique edge $e' \in \Gamma$ such that

$$f(e') \geqslant e > f(e'_-).$$

Here, e'_- is the edge immediately preceding e' in the cyclic ordering.

The significance is that e' and e have the same source vertex, so we are 'remembering an Eulerian tour by its vertices'. Although there may be other edges in Γ which share the same source with e, e' is the only one which (roughly speaking) has the same source for purely formal reasons.

Define $i_e^{-1}(f) = f^*$, so that i_e^{-1} is functorial $\mathrm{Bypass}_S^{\mathrm{Eul}} \to \Lambda^{\mathrm{op}}$. We claim that i_e^{-1} is inverse to i_e when $S = *$.

In this case, $i_e^{-1} i_e(\mathbb{T}_m)$ is the cyclic set of elementary edges of $\mathbb{T}_m$, and if $f : \mathbb{T}_m \to \mathbb{T}_n$, then $i_e^{-1} i_e(f)$ sends an elementary edge $v_i \xrightarrow{e_i} v_{i+1}$ of $\mathbb{T}_m$ to the first elementary edge whose source is the same as $f(v_i)$.

In other words, there is a natural equivalence $i_e^{-1} i_e(\mathbb{T}_m) \to \mathbb{T}_m$ given by relabeling the vertices (labeled by elementary edges e_i) by vertices v_i. Similarly, $i_e i_e^{-1}(\mathbb{T}_m) \cong \mathbb{T}_m$, so i_e, i_e^{-1} are inverse, completing the proof. □

The cyclic category also a few other useful properties, which we review now. We will say that:

Definition 3.5. *A* cyclic space *X is a presheaf on Λ. Call $cTop = \mathcal{P}(\Lambda)$.*

First, notice that there is a functor $i : \Delta \to \Lambda$ from the simplex category (of finite, nonempty, totally ordered sets), which sends the totally ordered set $\{0 < \cdots < n\}$ to the cyclically ordered set $\{0 < \cdots < n < 0\}$. Hence we may regard any cyclic space X as having an underlying simplicial space i^*X.

We write $|X|$ for the geometric realization of the simplicial space. If c denotes the functor $\Delta \to *$, then by definition

$$|X| = c_* i^* X.$$

(For our pushforward and pullback notation, see Section 1.8.)

Lemma 3.6 ([9] Proposition 2.7)**.** *If $\mathbb{T}_n \in \Lambda \subseteq cTop$ denotes the representable cyclic space, then $|\mathbb{T}_n| \cong S^1$, and the functor $c_* i^* : \mathbb{T}_n \to Top$ factors through the subcategory $BS^1 \subseteq Top$ of S^1-torsors.*

Hence we have a functor $r : \Lambda \to BS^1$.

Proposition 3.7. *In the commutative square*

$$\begin{array}{ccc} \Delta & \xrightarrow{\ i\ } & \Lambda \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle r} \\ * & \xrightarrow[\ i\ ]{} & BS^1, \end{array}$$

there is a natural equivalence $i^* r_* \cong c_* i^*$ of functors $cTop \to Top$.

Proof. Lemma 3.6 asserts that $i^* r_* \cong c_* i^*$ when restricted to $\Lambda \subseteq \mathcal{P}(\Lambda)$. Moreover, r_*, c_* each have right adjoints (r^*, c^*), and each i^* has right adjoint (given by right Kan extension). Therefore, $i^* r_*$ and $c_* i^*$ each preserve colimits by the adjoint functor theorem. Since all presheaves are colimits of representables, the proposition follows. □

Corollary 3.8. *If $X \in cTop$, then $|X|$ has a canonical S^1-action, and*

$$colim_{\Lambda^{op}}(X) \cong |X|_{hS^1}.$$

Proof. By Proposition 3.7, $|X| \cong i^* r_* X$. As a presheaf over BS^1, $r_* X$ can be identified with an S^1-space, and $i^* r_* X$ with the underlying space (forgetting the S^1-action). Therefore, $|X|$ has a canonical S^1-action.

Moreover, if j denotes the map $BS^1 \to *$, then $j_*(-) \cong (-)_{hS^1}$ by Example 1.9. Therefore,

$$|X|_{hS^1} \cong j_* r_* X \cong (jr)_* X \cong \mathrm{colim}(X).$$

□

Proposition 3.9. *The functor $r : \Lambda \to BS^1$ exhibits BS^1 as the classifying space of Λ.*

Proof. We have $|\Lambda| \cong |\Lambda^{\text{op}}| \cong \text{colim}_{\Lambda^{\text{op}}}(*)$, the first equivalence by Lemma 3.4 ($\Lambda \cong \Lambda^{\text{op}}$), and the second by [13] 3.3.4.6. Once again, consider the functors

$$\Lambda \xrightarrow{r} BS^1 \xrightarrow{j} *.$$

Then $|\Lambda| \cong (jr)_*(*) \cong j_* r_*(*)$. By Proposition 3.7, $r_*(*)$ has underlying space the geometric realization of $*$, so

$$|\Lambda| \cong j_* r_*(*) \cong j_*(*) \cong (*)_{hS^1} \cong BS^1.$$

□

4 The presheaf $\mathcal{O}_{\text{thh}}$

In the next two sections, we will prove Theorem 5.3, describing the presheaf $\mathcal{O}_{\text{thh}}$. We begin in this section by defining $\mathcal{O}_{\text{thh}}$.

Suppose that $\mathcal{V}$ is a closed symmetric monoidal, presentable ∞-category and $\mathcal{C} : \text{Bypass}_S \to \mathcal{V}$ is a $\mathcal{V}$-enriched category with set S of objects. As in Section 1.8, we can extend $\mathcal{C}$ continuously to a functor

$$\mathcal{C}_* : \mathcal{P}(\text{Bypass}_S) \to \mathcal{V}$$

which preserves colimits.

Let $(\mathcal{O}_{\text{thh}})_\bullet : \Lambda^{\text{op}} \to \mathcal{P}(\text{Bypass}_S)$ be the cyclic presheaf defined by

$$(\mathcal{O}_{\text{thh}})_n = \coprod_{X_0,\ldots,X_n \in S} (X_0, X_1, \ldots, X_n, X_0),$$

and let $\mathcal{O}_{\text{thh}} \in \mathcal{P}(\text{Bypass}_S)$ be its geometric realization.

To be explicit, the cyclic structure is as follows: If $f : \mathbb{T}_m \to \mathbb{T}_n$ is a map in Λ, write e_i for an elementary morphism of $\mathbb{T}_m$, and $f(e_i) = e_{i1} \cdots e_{ik}$ is a composite of elementary morphisms in $\mathbb{T}_n$. Then there is a bypass operation from $(X_0, \ldots, X_n, X_0)$ to some $(Y_0, \ldots, Y_m, Y_0)$ given by replacing each path $e_{i1} \cdots e_{ik}$ by a single edge, or by introducing a loop if $k = 0$.

Definition 4.1 (THH of an enriched category)**.** *If $\mathcal{C} : Bypass_S \to \mathcal{V}$ is a $\mathcal{V}$-enriched category with set S of objects, then $THH(\mathcal{C}) = \mathcal{C}_*(\mathcal{O}_{thh})$.*

In particular, because $\mathcal{C}_*$ preserves colimits, $\mathrm{THH}(\mathcal{C})$ is the geometric realization

$$\mathrm{THH}(\mathcal{C}) = \left| [n] \mapsto \coprod_{X_0,\ldots,X_n} \mathcal{C}(X_0,X_1) \otimes \cdots \otimes \mathcal{C}(X_n,X_0) \right|.$$

Remark 4.2. *When $\mathcal{V} = Sp$, a $\mathcal{V}$-enriched category is a spectral category, and Definition 4.1 is the usual cyclic bar construction computing THH ([5] Section 3).*

When $\mathcal{V} = Top$, a $\mathcal{V}$-enriched category is an ∞-category, and Definition 4.1 is the usual cyclic bar construction computing unstable THH *([1] pg. 857).*

Remark 4.3. *For any $X_0,\ldots,X_n \in S$, there is a canonical Eulerian tour on $(X_0,\ldots,X_n,X_0)$ given by the cycle $X_0 \to \cdots \to X_n \to X_0$. Write $[X_0,\ldots,X_n,X_0] \in Bypass_S^{Eul}$ when we wish to remember the canonical Eulerian tour.*

Then the cyclic presheaf $(\mathcal{O}_{thh})_\bullet$ lifts to a cyclic presheaf

$$(\overline{\mathcal{O}}_{thh})_\bullet : \Lambda^{op} \to \mathcal{P}(Bypass_S^{Eul})$$

given by $(\overline{\mathcal{O}}_{thh})_n = \coprod_{X_0,\ldots,X_n \in S}[X_0,X_1,\ldots,X_n,X_0]$, such that

$$\mathcal{O}_{thh} \cong k_* \overline{\mathcal{O}}_{thh}.$$

Here $k : Bypass_S^{Eul} \to Bypass_S$ is the forgetful functor.

5 THH of associative algebras

In this section, we will identify $\mathcal{O}_{\mathrm{thh}}$ when $S = *$. For the rest of this section, set $S = *$, and identify $\mathrm{Bypass}_*^{\mathrm{Eul}} \cong \Lambda^{\mathrm{op}}$ via Lemma 3.4.

Let $\mathcal{Y} : \Lambda^{\mathrm{op}} \times \Lambda \to \mathrm{Top}$ denote the Yoneda map $\mathcal{Y}(X,Y) = \mathrm{Map}(X,Y)$, which we may regard as a presheaf $\mathcal{Y} \in \mathcal{P}(\Lambda \times \Lambda^{\mathrm{op}})$.

Proposition 5.1. *Consider the diagram*

$$\begin{array}{ccccccc}
\Delta \times \Lambda^{op} & \xrightarrow{c} & \Lambda^{op} & = & Bypass_*^{Eul} & \xrightarrow{k} & Bypass_* \\
\downarrow i & & \downarrow i & & & & \\
\Lambda \times \Lambda^{op} & \xrightarrow{r} & BS^1 \times \Lambda^{op}, & & & &
\end{array}$$

where k is the forgetful functor, and the square is Λ^{op} times that of Proposition 3.7. Then

$$\mathcal{O}_{thh} \cong k_* c_* i^* \mathcal{Y} \cong k_* i^* r_* \mathcal{Y}.$$

Proof. If $S = *$, then as in Remark 4.3, $(\overline{\mathcal{O}}_{\mathrm{thh}})_n = [*, \cdots, *]$, which is $\mathbb{T}_n$ via the identification $\mathrm{Bypass}_*^{\mathrm{Eul}} \cong \Lambda^{\mathrm{op}}$. Indeed, $(\overline{\mathcal{O}}_{\mathrm{thh}})_\bullet$ is identified with the Yoneda embedding

$$\Lambda^{\mathrm{op}} \to \mathcal{P}(\Lambda^{\mathrm{op}}) \cong \mathcal{P}(\mathrm{Bypass}_*^{\mathrm{Eul}}).$$

Since $\overline{\mathcal{O}}_{\mathrm{thh}}$ is the geometric realization, then

$$\overline{\mathcal{O}}_{\mathrm{thh}} \cong c_* i^* \mathcal{Y}.$$

By Lemma 3.4, we also have

$$\overline{\mathcal{O}}_{\mathrm{thh}} \cong i^* r_* \mathcal{Y}.$$

Since $\mathcal{O}_{\mathrm{thh}} \cong k_* \overline{\mathcal{O}}_{\mathrm{thh}}$ (Remark 4.3), the proposition follows. □

Corollary 5.2. *There is a canonical S^1-action on $\overline{\mathcal{O}}_{thh}$, and $(\overline{\mathcal{O}}_{thh})_{hS^1} \cong *$, the constant presheaf on $Bypass_*^{Eul}$.*

Proof. This follows from Proposition 5.1 and Corollary 3.8. □

We will now turn to the first of our main results from the introduction. We can identify presheaves on BS^1 with S^1-equivariant spaces:

$$\mathcal{P}(BS^1) \cong \mathcal{P}((BS^1)^{\mathrm{op}}) \cong \mathrm{Top}^{S^1}.$$

Write $S^1 \in \mathcal{P}(S^1)$ for the torsor (that is, S^1 acting freely on itself).

Theorem 5.3. *Let $k : \Lambda^{op} \cong Bypass_*^{Eul} \to Bypass_*$ denote the forgetful functor and $r : \Lambda^{op} \to |\Lambda^{op}| \cong BS^1$. Then*

$$\mathcal{O}_{thh} \cong k_* r^*(S^1).$$

Lemma 5.4. *Suppose $\mathcal{O} \in \mathcal{P}(\Lambda^{op})$ has an S^1-action for which $\mathcal{O}_{hS^1} \cong *$ is the constant presheaf.*

If $colim_\Lambda(\mathcal{O}) \cong *$, then $\mathcal{O} \cong r^*(S^1)$.

Proof of lemma. Since $\mathcal{O}$ has an S^1-action, it may be regarded as a functor $\mathcal{O} : \Lambda \to \mathrm{Top}^{S^1}$ to S^1-spaces. Since $\mathcal{O}_{hS^1} \cong *$, it lands in the full subcategory spanned by the torsor S^1, which is equivalent to the ∞-groupoid BS^1. Hence we have

$$\mathcal{O} : \Lambda \to BS^1 \subseteq \mathrm{Top}^{S^1}.$$

Since BS^1 is an ∞-groupoid, $\mathcal{O}$ factors through $|\Lambda| \cong BS^1$, as

$$\Lambda \xrightarrow{r} BS^1 \xrightarrow{f} BS^1 \subseteq \mathrm{Top}^{S^1}$$

for some $f : BS^1 \to BS^1$. If f is the degree n map (multiplication by n on $\pi_2 \cong \mathbb{Z}$), then $\mathcal{O} \cong r^*(S^1_{(n)})$, where $S^1_{(n)}$ denotes the circle with S^1-action

$$\theta \cdot z = \theta^n z.$$

Note that $S^1_{(n)} \cong S^1_{(-n)}$.

Let c denote the functor $\Lambda \to *$ and $\mathcal{O}_n = r^* S^1_{(n)}$. We are to prove the following: If $c_* \mathcal{O}_n = \operatorname{colim}_\Lambda(\mathcal{O}_n) \cong *$, then $n = \pm 1$.

First we show $n \neq 0$. Indeed, since r^* has a right adjoint (right Kan extension), it preserves colimits, so

$$\mathcal{O}_0 \cong S^1 \otimes r^*(*),$$

which is the constant presheaf S^1. Thus

$$c_*\mathcal{O}_0 \cong S^1 \times c_*(*) \cong S^1 \times BS^1 \neq *.$$

Thus $n \neq 0$. Moreover, since $\mathcal{O}_n \cong \mathcal{O}_{-n}$, assume $n > 0$. Since r^* preserves colimits, $\mathcal{O}_n \cong (\mathcal{O}_1)_{hC_n}$, with C_n the cyclic subgroup of S^1 of order n. Hence

$$* \cong c_*\mathcal{O}_n \cong c_*(\mathcal{O}_1)_{hC_n}.$$

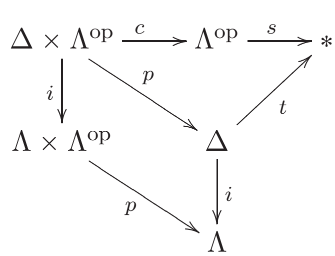
As a left adjoint functor, c_* also preserves colimits, so $c_*(\mathcal{O}_1) \cong C_n$.

But we also know that $(\mathcal{O}_1)_{hS^1} \cong r^*(*) \cong *$, the constant presheaf, so

$$c_*(\mathcal{O}_1)_{hS^1} \cong BS^1.$$

Given that $c_*(\mathcal{O}_1) \cong C_n$, we know $(C_n)_{hS^1} \cong BS^1$, which implies $n = 1$. $\square$

Proof of Theorem 5.3. By Corollary 5.2, $(\overline{\mathcal{O}}_{\text{thh}})_{hS^1} \cong *$. By the lemma, we need only show that $\operatorname{colim}(\overline{\mathcal{O}}_{\text{thh}}) \cong *$. Consider the diagram



Then

$$\operatorname{colim}(\overline{\mathcal{O}}_{\text{thh}}) \cong s_*\overline{\mathcal{O}}_{\text{thh}} \cong s_*c_*i^*\mathcal{Y} \cong t_*p_*i^*\mathcal{Y} \cong t_*i^*p_*\mathcal{Y}.$$

But $p_*\mathcal{Y}(\mathbb{T}_n) \cong |(\mathrm{Map}(\mathbb{T}_n, -))|_{hS^1}$ by Corollary 3.8, which is $(S^1)_{hS^1} \cong *$ by Lemma 3.6. Hence $p_*\mathcal{Y} \cong *$, the constant presheaf, so $i^*p_*\mathcal{Y} \cong *$ is the constant simplicial space, which has contractible geometric realization.

The hypothesis of the lemma follows, so by the lemma $\overline{\mathcal{O}}_{\mathrm{thh}} \cong r^*(S^1)$, and therefore

$$\mathcal{O}_{\mathrm{thh}} = k_*\overline{\mathcal{O}}_{\mathrm{thh}} \cong k_*r^*(S^1).$$

□

6 THH of enriched categories

Now we will identify $\mathcal{O}_{\mathrm{thh}} \in \mathcal{P}(\mathrm{Bypass}_S)$ for an arbitrary set S, generalizing Theorem 5.3 to Theorem 6.1.

Let r denote the composite

$$\mathrm{Bypass}_S^{\mathrm{Eul}} \xrightarrow{F} \mathrm{Bypass}_*^{\mathrm{Eul}} \to |\mathrm{Bypass}_*^{\mathrm{Eul}}| \cong BS^1,$$

where F forgets the labeling of the vertices. (We will later prove that r exhibits BS^1 as the classifying space of $\mathrm{Bypass}_S^{\mathrm{Eul}}$; see Remark 7.5.)

Theorem 6.1. *Let the functor $r : \mathrm{Bypass}_S^{Eul} \to BS^1$ be as above, and write $k : \mathrm{Bypass}_S^{Eul} \to \mathrm{Bypass}_S$ for the forgetful functor. Then*

$$\mathcal{O}_{thh} \cong k_*r^*(S^1).$$

Notice that this precisely recovers Theorem 5.3 when $S = *$. In general, we will derive this theorem from the $S = *$ case by means of a lemma asserting (roughly) that THH does not depend on the ambient set S.

Recall the presheaf $\overline{\mathcal{O}}_{\mathrm{thh}} \in \mathcal{P}(\mathrm{Bypass}_S^{\mathrm{Eul}})$ of Remark 4.3. In this section, we will use a superscript (as in $\overline{\mathcal{O}}_{\mathrm{thh}}^S$) to emphasize we are working over Bypass_S for a particular S.

Lemma 6.2. *If $F : \mathrm{Bypass}_S^{Eul} \to \mathrm{Bypass}_*^{Eul}$ is the functor which forgets the labeling of vertices, then $F^*\overline{\mathcal{O}}_{thh}^* \cong \overline{\mathcal{O}}_{thh}^S$.*

Proof. Recall that $\overline{\mathcal{O}}_{\mathrm{thh}}^S$ is the geometric realization of the cyclic presheaf

$$(\overline{\mathcal{O}}_{\mathrm{thh}}^S)_n = \coprod_{X_0,\ldots,X_n} [X_0, \ldots, X_n, X_0].$$

Given $\Gamma \in \mathrm{Bypass}_S^{\mathrm{Eul}}$, we may evaluate

$$(\overline{\mathcal{O}}_{\mathrm{thh}}^S)_n(\Gamma) \cong \coprod_{X_0,\ldots,X_n} \mathrm{Map}(\Gamma, [X_0, \ldots, X_n, X_0]).$$

A map $\Gamma \to [X_0, \dots, X_n, X_0]$ is an *itinerary* of the Eulerian tour on Γ; that is, it is a way to regard the Eulerian tour as a tour with stops at $X_0, X_1, \dots, X_n$ in that order.

Hence, $(\overline{\mathcal{O}}^S_{\text{thh}})_n(\Gamma)$ is the set of all n-stop itineraries for the specified Eulerian tour.

On the other hand, $F^*(\overline{\mathcal{O}}^*_{\text{thh}})_n(\Gamma)$ is the set of all n-stop itineraries for the Eulerian tour where we have forgotten the exact location of each stop.

In order to remember an itinerary, we need only remember the *order* of our stops, not the exact location. Therefore, $(\overline{\mathcal{O}}^S_{\text{thh}})_n \cong F^*(\overline{\mathcal{O}}^*_{\text{thh}})_n$. Moreover, F^* preserves all colimits (in particular geometric realizations), so $\overline{\mathcal{O}}^S_{\text{thh}} \cong F^*\overline{\mathcal{O}}^*_{\text{thh}}$, as desired. □

Proof of Theorem 6.1. Using Remark 4.3, $\mathcal{O}^S_{\text{thh}} \cong k_*\overline{\mathcal{O}}^S_{\text{thh}}$, which by the lemma and Theorem 5.3 is

$$k_*F^*\overline{\mathcal{O}}^*_{\text{thh}} \cong k_*F^*r^*(S^1) \cong k_*r^*(S^1),$$

as desired. (There is a potential for confusion in the notation: We are using r to refer first to $\text{Bypass}^{\text{Eul}}_* \to BS^1$, and second to $\text{Bypass}^{\text{Eul}}_S \to BS^1$, which is really the composite rF.) □

7 Counting Eulerian tours two ways

In section 3, we saw that $\text{Bypass}^{\text{Eul}}_* \cong \Lambda$. For sets $S \neq *$, there is still a close relationship between $\text{Bypass}^{\text{Eul}}_S$ and Λ, which we explore in this section.

Recall we are regarding the objects of Λ as categories $\mathbb{T}_n$.

Definition 7.1. *Given a category $\mathcal{C}$, its cyclic nerve is the cyclic set*

$$N_{cyc}\mathcal{C} = Fun(-, \mathcal{C}) : \Lambda^{op} \to Set.$$

Write S_{triv} for the category with set S of objects and exactly one morphism between each object (which is equivalent to the trivial category $*$). In this case, $(N_{\text{cyc}}S_{\text{triv}})_n \cong S^{n+1}$ is the set of labelings of $\{0, \dots, n\}$ by S.

We now turn to the main result in this section. Notice that an object of $\text{Bypass}^{\text{Eul}}_S$ can be described in two ways:

1. as a nonempty graph $\Gamma \in \text{Bypass}_S$ along with an Eulerian tour; or
2. as a cyclically ordered set of edges $\mathbb{T}_n \in \Lambda$ along with a labeling of its vertices in S.

Write $\text{Bypass}_S^+ \subseteq \text{Bypass}_S$ for the full subcategory of nonempty graphs.

Categorically, (1) and (2) amount to the following two lemmas:

Lemma 7.2. *The forgetful functor $Bypass_S^{Eul} \to Bypass_S^+$ is a right fibration, and the associated straightening $Eul\colon (Bypass_S^+)^{op} \to Set$ sends a graph Γ to its set $Eul(\Gamma)$ of Eulerian tours.*

Proof. The (right fibrational) Grothendieck construction of Eul is the functor $\mathcal{G} \to \text{Bypass}_S^+$, where $\mathcal{G}$ is the category of nonempty graphs along with a chosen Eulerian tour. By construction, this right fibration is equivalent to $\text{Bypass}_S^{\text{Eul}} \to \text{Bypass}_S^+$. □

Lemma 7.3. *The forgetful functor*

$$Bypass_S^{Eul} \to Bypass_*^{Eul} \cong \Lambda^{op}$$

is a left fibration, and the associated straightening is $N_{cyc}S_{triv} : \Lambda^{op} \to Set$, which sends $\mathbb{T}_n$ to the set of ways to label its vertices in S.

Proof. Recall the explicit description of the equivalence $i_e^{-1} : \text{Bypass}_*^{\text{Eul}} \to \Lambda^{\text{op}}$ from Lemma 3.4: $i_e^{-1}(\Gamma)$ is the set of edges of Γ, cyclically ordered by the Eulerian tour. If $f : \Gamma \to \Gamma'$ is a bypass operation, then $i_e^{-1}(f) : e \mapsto e'$ if and only if

$$f(e'_-) < e \leqslant f(e'),$$

where e'_- is the edge immediately preceding e' in the Eulerian tour on Γ.

The composite $\text{Bypass}_*^{\text{Eul}} \cong \Lambda^{\text{op}} \to \text{Set}$ sends the graph Γ_n with n edges to the set of ways to label vertices in S.

The associated (left fibrational) Grothendieck construction is the functor $\mathcal{G} \to \text{Bypass}_*^{\text{Eul}}$, where $\mathcal{G}$ is defined as follows: An object is a graph $\Gamma_n \in \text{Bypass}_*^{\text{Eul}}$ with n edges and for each edge e, a labeling $s(e) \in S$ which we call its *source vertex*. A morphism is $f : \Gamma_m \to \Gamma_n$ such that:

- if the preimage of an edge $e \in \Gamma_n$ is the path $v_0 \xrightarrow{e_0} \cdots \xrightarrow{e_{n-1}} v_n$, then $s(e) = s(e_0)$;

- if the preimage of $e \in \Gamma_n$ is empty, and $f^{-1}(e)_+$ is the first edge in Γ_m whose image under f comes after e, then $s(e) = s(f^{-1}(e)_+)$.

This is also a description of $\text{Bypass}_S^{\text{Eul}}$: the objects are all of the form $[X_0, X_1, \ldots, X_n, X_0]$, and a map is an order-presering function on the edge set which satisfies the properties above. Hence $\mathcal{G} \cong \text{Bypass}_S^{\text{Eul}}$, and this completes the proof. □

We end this section with two corollaries that will be important later.

Corollary 7.4. *The classifying space of* $Bypass_S^{Eul}$ *is* $|Bypass_S^{Eul}| \cong BS^1$.

Proof. By [13] 3.3.4.6 and Lemma 7.3, $|\mathrm{Bypass}_S^{\mathrm{Eul}}| \cong \mathrm{colim}(N_{\mathrm{cyc}} S_{\mathrm{triv}})$. By Corollary 3.8, this is equivalent to $|N_{\mathrm{cyc}} S_{\mathrm{triv}}|_{hS^1}$. In the particular case of S_{triv}, the cyclic nerve is identical to the ordinary nerve (as a simplicial set), and $|NS_{\mathrm{triv}}| \cong *$ since $S_{\mathrm{triv}} \cong *$. Therefore, $|\mathrm{Bypass}_S^{\mathrm{Eul}}| \cong (*)_{hS^1} \cong BS^1$. □

Remark 7.5. *Notice that we have actually proven the stronger statement that the functor* $Bypass_S^{Eul} \to Bypass_*^{Eul} \cong \Lambda$ *is an equivalence on geometric realizations of nerves. Therefore, the functor* r *of Theorem 6.1 exhibits* BS^1 *as the classifying space of* $Bypass_S^{Eul}$.

Corollary 7.6. *The forgetful functor* $k : Bypass_S^{Eul} \to Bypass_S^{+}$ *induces an equivalence of* ∞*-categories*

$$k_* : \mathcal{P}(Bypass_S^{Eul}) \to \mathcal{P}(Bypass_S^{+})_{/Eul}.$$

In particular, $k_*(*) \cong \mathrm{Eul}$.

Proof. If $\mathcal{D}$ is a quasicategory, then $\mathcal{P}(\mathcal{D}) \cong \mathrm{RFib}(\mathcal{D})$ by [3] 1.4. Here $\mathrm{RFib}(\mathcal{D})$ is the full subcategory of $(\mathrm{Cat}_\infty)_{/\mathcal{D}}$ spanned by the right fibrations, and a presheaf F is sent to its Grothendieck construction $\int F \to \mathcal{D}$.

Therefore, if F is a presheaf of spaces on $\mathcal{D}$, the Grothendieck construction also induces an equivalence

$$\mathcal{P}(\mathcal{D})_{/F} \cong \mathrm{RFib}(\mathcal{D})_{/\int F}.$$

Now suppose that $\mathcal{D}$ is a 1-category and F is discrete (a presheaf of sets). Then $\int F$ is a 1-category with an explicit model: an object is a pair (X, x) with $X \in \mathcal{D}$ and $x \in F(X)$. A morphism $(X, x) \to (Y, y)$ is a morphism $X \xrightarrow{f} Y$ such that $F(f)(y) = x$.

Identifying categories with their nerves, the functor $\int F \to \mathcal{D}$ is a *strict* right fibration; that is, given $n \geqslant 1$, $0 < k \leqslant n$, and a diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \int F \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{D}, \end{array}$$

there is a *unique* lift $\Delta^n \to \int F$. It follows that, if $G : \mathcal{A} \to \int F$ is a functor for which the composite $\mathcal{A} \to \mathcal{D}$ is a right fibration, then G is a right fibration.

(A simplex σ in $\int F$ can be projected down to $\mathcal{D}$ and then lifted to some σ' in $\mathcal{A}$. Because the simplex in $\int F$ is suitably unique, σ' is a lift of σ.)

Therefore,

$$\mathcal{P}(\mathcal{D})_{/F} \cong \mathrm{RFib}(\mathcal{D})_{/\int F} \cong \mathrm{RFib}(\textstyle\int F) \cong \mathcal{P}(\textstyle\int F).$$

When $\mathcal{D} = \mathrm{Bypass}_S^+$ and $F = \mathrm{Eul}$, we have the desired equivalence by Lemma 7.2:

$$\mathcal{P}(\mathrm{Bypass}_S^+)_{/\mathrm{Eul}} \cong \mathcal{P}(\mathrm{Bypass}_S^{\mathrm{Eul}}).$$

□

8 THH as a circle bundle

In this section, we will prove Theorem 8.6, which identifies the presheaf $\mathcal{O}_{\mathrm{thh}} \in \mathcal{P}(\mathrm{Bypass}_S)$ explicitly, restricted away from the empty graph. It will turn out to be essentially an immediate corollary of Theorem 6.1 and Corollaries 7.4 and 7.6.

First, it is not hard to calculate $\mathcal{O}_{\mathrm{thh}}$ at the empty graph:

Proposition 8.1. *There is an equivalence $\mathcal{O}_{thh}(\varnothing) \cong S$.*

Proof. We know

$$(\mathcal{O}_{\mathrm{thh}})_n(\varnothing) \cong \coprod_{X_0,\ldots,X_n} \mathrm{Map}(\varnothing, (X_0, \ldots, X_n, X_0)),$$

and $\mathrm{Map}(\varnothing, (X_0, \ldots, X_n, X_0))$ is contractible if $X_0 = \cdots = X_n$, and otherwise empty. Hence $(\mathcal{O}_{\mathrm{thh}})_\bullet(\varnothing)$ is the constant cyclic set with value S, and therefore the geometric realization is $\mathcal{O}_{\mathrm{thh}}(\varnothing) \cong S$. □

Henceforth, we will restrict to $\mathcal{O}_{\mathrm{thh}}^+ \in \mathcal{P}(\mathrm{Bypass}_S^+)$. By Theorem 6.1, we know $\mathcal{O}_{\mathrm{thh}}^+ \cong k_* r^*(S^1)$, with functors as in

$$\begin{array}{ccc} \mathrm{Bypass}_S^{\mathrm{Eul}} & \xrightarrow{k} & \mathrm{Bypass}_S^+ \\ {\scriptstyle r}\downarrow & & \\ BS^1. & & \end{array}$$

We will now prove our main result calculating $\mathcal{O}_{\mathrm{thh}}^+$, divided into three propositions.

Proposition 8.2. *$\mathcal{O}_{thh}^+$ has a canonical S^1-action and $(\mathcal{O}_{thh}^+)_{hS^1} \cong Eul$.*

Proof. The circle action comes from the description $\mathcal{O}^+_{\mathrm{thh}} \cong k_* r^*(S^1)$ of Theorem 6.1. (As an S^1-bimodule, S^1 has an S^1-action even as an object of Top^{S^1}.) Since k_* and r^* each have right adjoints (given by k^* and right Kan extension along r), they preserve colimits. Therefore,

$$(\mathcal{O}^+_{\mathrm{thh}})_{hS^1} \cong k_* r^*((S^1)_{hS^1}) \cong k_* r^*(*) \cong k_*(*) \cong \mathrm{Eul}.$$

The last step is by Corollary 7.6. ∎

Corollary 8.3. *If* $\Gamma \in \mathit{Bypass}_S$*, then*

$$\mathcal{O}_{thh}(\Gamma) \cong \begin{cases} (S^1)^{\amalg Eul(\Gamma)}, & \text{if } \Gamma \neq \varnothing \\ S, & \text{if } \Gamma = \varnothing \end{cases},$$

Proof. The $\Gamma = \varnothing$ case is Proposition 8.1, so assume $\Gamma \neq \varnothing$.

By Proposition 8.2,

$$\mathcal{O}_{\mathrm{thh}}(\Gamma)_{hS^1} \cong (\mathcal{O}_{\mathrm{thh}})_{hS^1}(\Gamma) \cong \mathrm{Eul}(\Gamma).$$

Therefore, $\mathcal{O}_{\mathrm{thh}}(\Gamma)$ is an S^1-space with homotopy orbits equivalent to a *set* $\mathrm{Eul}(\Gamma)$. It must be that $\mathcal{O}_{\mathrm{thh}}(\Gamma)$ is a disjoint union of circles indexed by $\mathrm{Eul}(\Gamma)$. ∎

We call an S^1-equivariant presheaf $\mathcal{O} \in \mathrm{Bypass}^+_S$ satisfying (as in Proposition 8.2) $\mathcal{O}_{hS^1} \cong \mathrm{Eul}$ an *S^1-bundle over Eul*.

In other words, an S^1-bundle over Eul is an S^1-equivariant object of $\mathcal{P}(\mathrm{Bypass}^{\mathrm{Eul}}_S) \cong \mathcal{P}(\mathrm{Bypass}^+_S)_{/\mathrm{Eul}}$ for which $\mathcal{O}_{hS^1}$ is the terminal object. Equivalently, it is a functor

$$\mathcal{O} : (\mathrm{Bypass}^{\mathrm{Eul}}_S)^{\mathrm{op}} \to \mathrm{Top}^{S^1}$$

which lands in the full subcategory $BS^1 \subseteq \mathrm{Top}^{S^1}$ spanned by the torsor S^1.

Proposition 8.4. *The moduli space of S^1-bundles over Eul is $BS^1 \times \mathbb{Z}$.*

Proof. By the discussion above, the moduli space of S^1-bundles over Eul is the full subcategory

$$\mathrm{Fun}((\mathrm{Bypass}^{\mathrm{Eul}}_S)^{\mathrm{op}}, BS^1) \subseteq \mathrm{Fun}((\mathrm{Bypass}^{\mathrm{Eul}}_S)^{\mathrm{op}}, \mathrm{Top}^{S^1}).$$

Since BS^1 is an ∞-groupoid, any functor $(\mathrm{Bypass}^{\mathrm{Eul}}_S)^{\mathrm{op}} \to BS^1$ factors through the classifying space, which by Corollary 7.4 is $|\mathrm{Bypass}^{\mathrm{Eul}}_S| \cong BS^1$.

Hence, the moduli space of S^1-bundles over Eul is

$$\mathrm{Map}(BS^1, BS^1) \cong BS^1 \times \mathbb{Z}.$$

∎

As in the proof of Lemma 5.4, let $S^1_{(n)} \in \mathrm{Top}^{S^1}$ denote the circle with its S^1-action

$$\theta \cdot z = \theta^n z.$$

Writing $BS^1 \xrightarrow{n} BS^1$ for the degree n map, then $S^1_{(n)} = n^*(S^1)$.

The proof of Proposition 8.4 asserts that every S^1-bundle $\mathcal{O}$ over Eul is of the form

$$(\mathrm{Bypass}_S^{\mathrm{Eul}})^{\mathrm{op}} \to |\mathrm{Bypass}_S^{\mathrm{Eul}}| \cong BS^1 \xrightarrow{n} BS^1 \subseteq \mathrm{Top}^{S^1}$$

for some $n \in \mathbb{Z}$, or equivalently

$$\mathcal{O} \cong r^*(S^1_{(n)})$$

for some $n \in \mathbb{Z}$, where $r : \mathrm{Bypass}_S^{\mathrm{Eul}} \to BS^1$ as usual. We call n the *degree* of $\mathcal{O}$.

Proposition 8.5. *As an S^1-bundle over Eul, $\mathcal{O}^+_{thh}$ has degree 1.*

Proof. This is just the assertion that $\mathcal{O}^+_{\mathrm{thh}} \cong k_* r^*(S^1)$, which is Theorem 6.1, remembering that k_* specializes to an equivalence $\mathcal{P}(\mathrm{Bypass}_S^{\mathrm{Eul}}) \to \mathcal{P}(\mathrm{Bypass}_S^+)_{/\mathrm{Eul}}$. ☐

Our main theorem is a summary of Propositions 8.2, 8.4, and 8.5:

Theorem 8.6. *If $\mathcal{O}^+_{thh}$ is as above, then:*

1. *$\mathcal{O}^+_{thh}$ has a canonical S^1-action;*
2. *$(\mathcal{O}^+_{thh})_{hS^1} \cong Eul$;*
3. *The moduli space of presheaves satisfying (1)-(2) is equivalent to $\mathbb{Z} \times BS^1$; that is, such presheaves are determined up to equivalence by an integer invariant we call* degree*;*
4. *The degree of $\mathcal{O}^+_{thh}$ is* 1*.*

Let $\mathrm{Eul}(n) \in \mathcal{P}(\mathrm{Bypass}_S^+)$ denote the S^1-bundle of degree n over Eul.

Corollary 8.7. *If $\mathcal{V}$ is presentable and symmetric monoidal, and $\mathcal{C} : Bypass_S \to \mathcal{V}$ is a $\mathcal{V}$-enriched category, then*

$$\mathcal{C}_* Eul(\pm 1) \cong THH(\mathcal{C}),$$

$$\mathcal{C}_* Eul \cong THH(\mathcal{C})_{hS^1},$$

$$\mathcal{C}_* Eul(\pm n) \cong THH(\mathcal{C})_{C_n}, \text{ if } n \geqslant 1,$$

$$\mathcal{C}_* Eul(0) \cong S^1 \otimes THH(\mathcal{C})_{hS^1}.$$

Proof. The first statement summarizes the theorem. The later statements are because $\mathrm{Eul}(n) \cong k_* r^*(S^1_{(n)})$, $\mathrm{Eul} \cong k_* r^*(*)$, each of r^*, k_*, and $\mathcal{C}_*$ preserves colimits, and we have the identities in Top^{S^1}:

$$* \cong (S^1_{(1)})_{hS^1}$$

$$S^1_{(n)} \cong (S^1_{(1)})_{hC_n}$$

$$S^1_{(0)} \cong S^1 \otimes *.$$

□

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