Eine Wanderung durch Rainer Vogt's mathematisches Schaffen

Clemens Berger

Bonn MPIM, February 11, 2016

1942: Rainer Max Vogt is born in Stuttgart
1960-1965: Graduate studies in Frankfurt am Main
1965-1968: PhD supervised by J. M. Boardman in Warwick
1968-1974: Visiting Professor in Aarhus, Heidelberg and Saarland
1974-2015: Professor for Topology in Osnabrück

PhD students: K. Below, O. Blömer, M. Brinkmeier, J. Hollender, T. Hüttemann, H. Wellen, X. Yang

50 publications with over *500 citations* in AMS MathSciNet basis

1. Convenient categories of topological spaces (1971)
2. Homotopy limits and colimits (1973)
3. Homotopy invariant algebraic structures (& Boardman 1973)
4. $THH(R) = R \otimes S^1$ (& McClure and Schwänzl 1997)
5. Iterated monoidal categories (BFSV 2003)
6. An additivity theorem for the interchange (& Fiedorowicz 2015)

Problem

Find a category $\mathcal{T}$ of topological spaces which is *cartesian closed*, i.e. such that $\mathcal{T}(X \times Y, Z) = \mathcal{T}(X, Z^Y)$ for a functional space Z^Y.

Proposition (Vogt 1971)

Let $\mathcal{C}$ be a class of topological spaces fulfilling

- $\mathcal{C}$ is closed under binary product in Top;
- for any X in $\mathcal{C}$ and Y in Top, evaluation (with respect to compact-open topology) $(Y^X)_{co} \times X \to Y$ is continuous.

Then the coreflective hull $\overline{\mathcal{C}}$ in Top is cartesian closed.

$\overline{\mathcal{C}}$ is the coreflective subcategory of Top consisting of the spaces with the final topology with respect to the class of maps out of $\mathcal{C}$.

Examples: $\mathcal{C}_1$=(compact Hausdorff spaces), and $\mathcal{C}_2$=(locally compact spaces), and $\mathcal{C}_3$=(exponentiable spaces, Day-Kelly 1970).

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Let $F : \mathcal{A} \to \mathrm{Top}$ be an $\mathcal{A}$-diagram of topological spaces.

Definition (Vogt 1973, cf. Segal 1968, Bousfield-Kan 1972)

$$\mathrm{hocolim}_{\mathcal{A}} F \overset{def}{=} \left(\coprod_{x_0 \in \mathcal{A}} \coprod_{n \geq 0} F(x_0) \times \mathcal{A}^{n+1}(x_0, x_{n+1}) \times [0,1]^n \right) / \sim$$

Theorem (Vogt 1973, cf. Segal 1968, Bousfield-Kan 1972)

$\mathrm{hocolim}_{\mathcal{A}} : \mathrm{Top}^{\mathcal{A}} \to \mathrm{Top}$ takes pointwise homotopy equivalences to homotopy equivalences.

Definition (W-construction of a category, Vogt 1973)

There is a *topologically enriched* category $W\mathcal{A}$ sth.

- $\mathrm{Ob}(W\mathcal{A}) = \mathrm{Ob}(\mathcal{A})$;
- $(W\mathcal{A})(x, y) = (\coprod_{n \geq 0} \mathcal{A}^{n+1}(x, y) \times [0,1]^n) / \sim$.

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A *homotopy coherent $\mathcal{A}$-diagram* is a top. functor $W\mathcal{A} \to \mathcal{T}$. Let $\mathcal{T}^{h\mathcal{A}}$ be the category of homotopy coherent $\mathcal{A}$-diagrams.

There is an enriched adjunction $\underline{\mathrm{colim}}_{W\mathcal{A}} : \mathcal{T}^{h\mathcal{A}} \leftrightarrows \mathcal{T} : \underline{c}_{\mathcal{A}}$. Define $\epsilon : W\mathcal{A} \to \mathcal{A}$ by $(W\mathcal{A})(x,y) \mapsto \mathcal{A}(x,y) = \pi_0(W\mathcal{A})(x,y)$.

Proposition (Vogt 1973)

$$\mathrm{hocolim}_{\mathcal{A}}(F) \cong \underline{\mathrm{colim}}_{W\mathcal{A}}\epsilon^*(F)$$

Remark (simplicial vs cubical, cf. Baues 1983)

There is a 2-category $\omega[n]$ with same objects as $[n]$ such that

$$(\omega[n])(k,l) = \mathrm{Fact}([n];k,l) \text{ if } k \leq l.$$

where $\mathrm{Fact}([n];k,l)$ is the factorization category of $k \to l$ in $[n]$.

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Lemma

- $\omega[n](k,l) \cong [1]^{l-k-1}$ if $k < l$
- $|\mathrm{nerve}(\omega[n])| \cong W[n]$
- $\Delta \to \mathrm{sCat} : [n] \mapsto \mathbb{C}[n] := \mathrm{nerve}(\omega[n])$

Definition (Homotopy coherent nerve)

$$\mathrm{Hom}_{\mathrm{sCat}}(\mathbb{C}[-],-) : \mathrm{sCat} \rightleftarrows \mathrm{sSets} : - \otimes_{\Delta} \mathbb{C}[-]$$

Theorem (Joyal 2007, Lurie 2009)

This is a Quillen equivalence between the Bergner model structure on simplicial cat's and the Joyal model structure on simplicial sets.

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Definition (Vogt 1968, Boardman-Vogt 1973)

An *operator category in normal form* is a strictly associative symmetric monoidal subcategory $(\mathcal{B}, \oplus, 0)$ of $(\mathcal{T}, \times, \star)$ such that

- The objects of $\mathcal{B}$ are the natural numbers sth. $m \oplus n = m + n$
- for all $n = n_1 + \cdots + n_k$ there is a canonical isomorphism $\mathcal{B}(n_1, 1) \times \cdots \times \mathcal{B}(n_k, 1) \times_{\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}} \mathfrak{S}_n \cong \mathcal{B}(n, k)$.

Definition (May 1972)

An operator category in normal form $(\mathcal{B}(n,k))_{n,k \in \mathbb{N}}$ determines, and is determined by, a *symmetric operad* $(\mathcal{O}(n) = \mathcal{B}(n,1))_{n \in \mathbb{N}}$.
The categorical structure of $\mathcal{B}$ amounts to a substitutional structure of $\mathcal{O}$, i.e. a unit $1 \in \mathcal{O}(1)$, and a multiplication

$$\mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

satisfying *associativity*, *unitarity* and *equivariance* constraints.

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Definition

Each topological space X has an endo-operad $\mathcal{E}_X(k) = \underline{\mathcal{T}}(X^k, X)$.
A *$\mathcal{O}$-algebra structure* on X is an operad map $\mathcal{O} \to \mathcal{E}_X$.

Remark

An $\mathcal{O}$-algebra structure on $X \iff \mathcal{O}(k) \times X^k \to X, \quad k \geq 0.$

Remark

Topological monoids are algebras over a symmetric operad; topological groups are not ! *Compare:* Monoids can be defined in any symmetric monoidal category, while groups cannot.

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Example (Iterated loop spaces and coendomorphism operads)

A k-ary operation on $\Omega^n X = \mathcal{T}_*(S^n, X)$ amounts to a map

$$\mathcal{T}_*(S^n \vee \cdots \vee S^n, X) = \mathcal{T}_*(S^n, X)^k \to \mathcal{T}_*(S^n, X).$$

Such k-ary operations are induced by points in

$$\mathrm{Coend}(S^n)(k) = \mathcal{T}_*(S^n, S^n \vee \cdots \vee S^n).$$

Any *suboperad* of $\mathrm{Coend}(S^n)$ acts on n-fold loop spaces.

Definition (operad of little n-cubes)

A little n-cube is an affine embedding $f : [0,1]^n \to [0,1]^n$ preserving the direction of the axes. $\mathcal{C}_n(k)$ is the space of k-tuples $(f_1, \dots, f_k)$ of little n-cubes with pairwise disjoint interiors. This defines a suboperad $\mathcal{C}_n$ of $\mathrm{Coend}(S^n)$ acting on n-fold loop spaces.

Example (Iterated loop spaces and coendomorphism operads)

A k-ary operation on $\Omega^n X = \mathcal{T}_*(S^n, X)$ amounts to a map

$$\mathcal{T}_*(S^n \vee \cdots \vee S^n, X) = \mathcal{T}_*(S^n, X)^k \to \mathcal{T}_*(S^n, X).$$

Such k-ary operations are induced by points in

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Theorem (Boardman-Vogt, May, Segal)

Any connected $\mathcal{C}_n$-algebra is weakly equivalent to an n-fold loop space.

Theorem (May, Segal)

The free $\mathcal{C}_n$-algebra generated by a pointed connected space X is weakly equivalent to $\Omega^n \Sigma^n X$. The operad inclusions $\mathcal{C}_n \subset \mathcal{C}_{n+1}$ correspond to the stabilization maps $\Omega^n \Sigma^n X \to \Omega^{n+1} \Sigma^{n+1} X$.

Definition (Boardman-Vogt, May)

An E_∞-operad is a top. operad $\mathcal{O}$ such that for each k, $\mathcal{O}(k)$ is a universal principal $\mathfrak{S}_k$-bundle.

Theorem (Boardman-Vogt, May)

The colimit $\mathcal{C}_\infty = \mathrm{colim}_n \mathcal{C}_n$ is an E_∞-operad. Every algebra over an E_∞-operad is (up to group completion) an infinite loop space.

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An $\mathcal{O}$-algebra structure on X is *homotopy-invariant* if it can be transported along any homotopy equivalence $X \simeq Y$.

B-V construct a functorial resolution $\epsilon : W(\mathcal{O}) \xrightarrow{\sim} \mathcal{O}$ such that $W(\mathcal{O})$-algebra structures are "homotopical" $\mathcal{O}$-algebra structures.

Theorem (Boardman-Vogt 1973)

For each top. operad $\mathcal{O}$ such that all $\mathcal{O}(k)$ are principal $\mathfrak{S}_k$-bundles, $W(\mathcal{O})$-algebra structures are homotopy-invariant.

Theorem (B.-Moerdijk 2003/2007)

In any monoidal model category with suitable interval H, operads have a functorial W-resolution $W_H(\mathcal{O}) \xrightarrow{\sim} \mathcal{O}$.
If $\mathcal{O}$ is a *well-pointed $\mathfrak{S}$-cofibrant* operad then $W_H(\mathcal{O})$ is a *cofibrant* operad and B-V's homotopy-invariance property holds.

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Remark (Brave new algebra, Waldhausen, EKMM 1997, HSS 2000)

- Infinite loop spaces are connective *Ω-spectra* $(X_n)_{n \geq 0}$. (i.e. $X_n \xrightarrow{\sim} \Omega X_{n+1}$ and $\pi_k(X_n) = *$ for $k < n$).
- abelian groups A induce EM-*spectra* $(K(A, n))_{n \geq 0}$.
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$\exists$ *symmetric monoidal structure* on spectra prolonging *smash product* of pointed spaces and *tensor product* of abelian groups.

Definition (THH, Böckstedt 1985)

$THH(E_\infty$-ring spectrum $R)$ defined like HH(comm. ring A) using the sphere spectrum $(S^n)_{n \geq 0}$ instead of the integers $\mathbb{Z}$.

Theorem (McClure-Schwänzl-Vogt 1997, cf. Loday 1980 for HH)

$$THH(R) = R \otimes S^1 \stackrel{def}{=} |R \wedge \Sigma^\infty(\Delta[1]/\partial\Delta[1])_+|.$$

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Remark

A double loop space is the same as a loop space in loop spaces:
$\Omega(\Omega X) = \mathcal{T}_*(S^1, \mathcal{T}_*(S^1, X)) = \mathcal{T}_*(S^1 \wedge S^1, X) = \mathcal{T}_*(S^2, X) = \Omega^2 X$

Definition (BFSV 2003)

A 2-monoidal category is a monoid in the category of strict monoidal categories and normal lax monoidal functors.

A 2-monoidal category is a category equipped with two strictly associative tensors $\otimes_1, \otimes_2$ sharing the same unit, and interrelated by an *interchange morphism*

$$(X \otimes_2 Y) \otimes_1 (Z \otimes_2 W) \to (X \otimes_1 Z) \otimes_2 (Y \otimes_1 W)$$

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The category of n-monoidal categories is the category of algebras for a categorical operad $\mathcal{M}_n$.
This operad is an E_n-operad in *posets*, in particular $|\mathcal{M}_n| \simeq \mathcal{C}_n$.

A typical morphism in $\mathcal{M}_2(4)$:

$$(((1 \otimes_1 2) \otimes_2 4) \otimes_1 3) \to (2 \otimes_2 ((1 \otimes_1 3) \otimes_2 4))$$

Theorem (BFSV 2003 and FSV 2014, cf. Thomason for $n = 1, \infty$)

The nerve of an n-monoidal category is up to group completion an n-fold loop space. Any n-fold loop space arises in this way.

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Remark (Special case $n = 2$)

$$|\mathcal{M}_2(k)| \simeq \mathcal{C}_2(k) \simeq F(\mathbb{R}^2, k) \simeq B(PBr(k))$$

Example (E_2-operads and braided monoidal categories)

$$\mathcal{M}_2 \xrightarrow{\sim} \mathrm{CoBr} \xleftarrow{\sim} \mathrm{PaBr}$$

2-monoidal	braided	braided monoidal
	strict monoidal	strict unit

Theorem (Drinfeld 1990, cf. Fresse 2015 and Horel 2015)

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \subseteq \widehat{GT} \stackrel{def}{=} \mathrm{Aut}^0_{\mathrm{Oper(Grpd)}}(\widehat{PaBr})$$

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Problem

The BV-tensor product $- \otimes_{BV} -$ does not preserve weak equivalences so that $E_m \otimes_{BV} E_n \not\simeq E_{m+n}$ although $(m+n)$-fold loop spaces are m-fold loop spaces in n-fold loop spaces.

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Let $\mathcal{A}, \mathcal{B}$ be top. operads. There exists a top. operad $\mathcal{A} \otimes_{BV} \mathcal{B}$ sth. $\mathcal{A} \otimes_{BV} \mathcal{B}$-algebras are the same as $\mathcal{A}$-algebras in $\mathcal{B}$-algebras.

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For the operad $\mathcal{D}_n$ of *decomposable little n-cubes*, one has an operad isomorphism $\mathcal{D}_m \otimes_{BV} \mathcal{D}_n \cong \mathcal{D}_{m+n}$.

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Theorem (Fiedorowicz-Vogt 2015)

If $\mathcal{A}$ is a cofibrant E_m-operad and $\mathcal{B}$ is a cofibrant E_n-operad then $\mathcal{A} \otimes_{BV} \mathcal{B}$ is an E_{m+n}-operad.

Proof.

Analysis of the cellular structure of $W_{red}|\mathcal{M}_m| \otimes_{BV} W_{red}|\mathcal{M}_n|$. □

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