A CARTAN-EILENBERG SPECTRAL SEQUENCE FOR A NON-NORMAL EXTENSION

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Abstract. Let $\Phi \to \Gamma \to \Sigma$ be a conormal extension of Hopf algebras over a commutative ring k, and let M be a Γ-comodule. The Cartan-Eilenberg spectral sequence

$$E_2 = \mathrm{Ext}_\Phi(k, \mathrm{Ext}_\Sigma(k, M)) \implies \mathrm{Ext}_\Gamma(k, M)$$

is a standard tool for computing the Hopf algebra cohomology of Γ with coefficients in M in terms of the cohomology of the pieces Φ and Σ. Bruner and Rognes, generalizing a construction of Davis and Mahowald, have introduced a generalization of the Cartan-Eilenberg spectral sequence converging to $\mathrm{Ext}_\Gamma(k, M)$ that can be defined when $\Phi = \Gamma \,\square_\Sigma k$ is compatibly an algebra and a Γ-comodule. We offer a concrete cobar-like construction that fits into their framework, and show how this work fits into a larger story. In particular, we show that this spectral sequence is isomorphic, starting at the E_1 page, to both the Adams spectral sequence in the stable category of Γ-comodules as studied by Margolis and Palmieri, and to a filtration spectral sequence on the cobar complex for Γ originally due to Adams. We obtain a description of the E_2 term under an additional flatness assumption. We discuss applications to computing localizations of the Adams spectral sequence E_2 page.

1. Introduction

Suppose Γ is a Hopf algebra over a commutative ring k and we wish to calculate its Hopf algebra cohomology $\mathrm{Ext}_\Gamma(k, k)$. If $k = \mathbb{F}_p$ and Γ is a group ring $\mathbb{F}_p[G]$, then this cohomology is by definition the mod-p group cohomology $H^*(G, \mathbb{F}_p)$, and a short exact sequence of groups $1 \to N \to G \to G/N \to 1$ gives rise to a Lyndon-Hochschild-Serre spectral sequence

$$E_2^{**} = H^*(G/N, H^*(N, \mathbb{F}_p)) \implies H^*(G, \mathbb{F}_p)$$

(equivalently $\mathrm{Ext}^*_{\mathbb{F}_p[G/N]}(\mathbb{F}_p, \mathrm{Ext}^*_{\mathbb{F}_p[N]}(\mathbb{F}_p, \mathbb{F}_p)) \implies \mathrm{Ext}^*_{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p)$). The analogue for more general Γ is called the Cartan-Eilenberg spectral sequence (alternatively the extension spectral sequence or the change-of-rings spectral sequence); there are a number of variants that are defined in various settings.

In the setting described in Cartan and Eilenberg's classic book (see [CE99, XVI.5$(2)_4$]), one begins with an algebra map $\Gamma \to \Sigma$ and a Γ-module M. Then the composite functor spectral sequence associated to the functors $\mathrm{Hom}_\Sigma(k, -) \circ \mathrm{Hom}_\Gamma(\Sigma, -)$ has the form

$$E_2 = \mathrm{Ext}_\Sigma(k, \mathrm{Ext}_\Gamma(\Sigma, M)) \implies \mathrm{Ext}_\Gamma(k, M).$$

Given a normal algebra map $i : \Phi \to \Gamma$ (i.e., the left ideal $\Gamma \cdot i(\Phi)$ is also a right ideal), one can define a quotient $\Gamma \otimes_\Phi k$ that is an algebra. If Σ can be expressed as such a quotient where Γ is projective as a Φ-module, then one may apply the change of rings theorem to obtain the more familiar form [CE99, Theorem XVI.6.1]

$$E_2 = \mathrm{Ext}_\Sigma(k, \mathrm{Ext}_\Phi(k, M)) \implies \mathrm{Ext}_\Gamma(k, M). \tag{1.1}$$

This reduces to the group cohomology example above in the case when the sequence $\Phi \to \Gamma \to \Sigma$ is $\mathbb{F}_p[N] \to \mathbb{F}_p[G] \to \mathbb{F}_p[G/N]$. Normality of i, which guarantees the quotient Σ is an algebra, is the analogue of normality of the subgroup N.

The case where Γ is a Hopf algebra (or Hopf algebroid) is of particular interest to stable homotopy theory. A detailed account of the Cartan-Eilenberg spectral sequence in the setting of Hopf algebroids can be found in [Rav86, Appendix A1.3]. The basic setup involves a normal extension of Hopf algebras, i.e., a sequence of Hopf algebra maps $\Phi \overset{i}{\hookrightarrow} \Gamma \overset{\pi}{\twoheadrightarrow} \Sigma := \Gamma \otimes_\Phi k$ such that i is a normal map of k-algebras, and i and π split over k. The E_2 page has the same form as (1.1). One motivating example is the extension generated by the quotient $A \to A/\beta$ of the Steenrod algebra obtained by modding out by the action of the Bockstein. Working in the Hopf algebra setting with more general coefficient comodules, Singer [Sin06, Chapter 4] identifies the Cartan-Eilenberg spectral sequence as the spectral sequence of a bi-cosimplicial commutative algebra and describes a general theory of power operations acting on it.

In this paper we assume weaker hypotheses: in particular, Σ need not be an algebra, in which case the classical E_2 page and the first of the composite functors is not even defined. Davis and Mahowald [DM82] were the first to develop a Cartan-Eilenberg type spectral sequence when they studied $\mathrm{Ext}_{A(2)}(M, \mathbb{F}_2)$ for $A(2)$-modules M using the non-normal map $A(1) \to A(2)$ of Hopf algebras. In this paper we study a generalization of their construction due to Bruner and Rognes [BR].

As our main application is to the localized cohomology of the dual Steenrod algebra, where it is more convenient to work with comodules than modules, the rest of this discussion will pertain to the dual setting to that described above. Henceforth, Ext_Γ will denote comodule Ext—that is, derived functors of Hom in the category of comodules over a Hopf algebra Γ. The classical Cartan-Eilenberg spectral sequence in this setting has the form

$$E_2 = \mathrm{Ext}_\Phi(k, \mathrm{Ext}_\Sigma(k, M)) \implies \mathrm{Ext}_\Gamma(k, M)$$

where M is a Φ-comodule and $\Phi \to \Gamma \to \Sigma$ is a conormal extension of Hopf algebras (i.e., an extension such that $\Gamma \,\square_\Sigma k = k \,\square_\Sigma \Gamma$ as sub-vector spaces).

Suppose $\Gamma \to \Sigma$ is a surjection of Hopf algebras, and the map $\Phi := \Gamma \,\square_\Sigma k \to \Gamma$ is a map of Γ-comodule algebras. For a Γ-comodule M, we consider three spectral sequences converging to $\mathrm{Ext}^*_\Gamma(k, M)$.

(1) Given a Γ-comodule resolution of k of the form $\Phi \otimes R^0 \to \Phi \otimes R^1 \to \dots$ for Γ-comodules R^n, Bruner and Rognes describe a Cartan-Eilenberg type spectral sequence converging to $\mathrm{Ext}^*_\Gamma(k, M)$.[1] While Bruner and Rognes are primarily interested in the case where Σ is an exterior algebra and the complex $\Phi \otimes R^*$ is a minimal resolution, we construct a resolution $R^n = \Phi^{\otimes n}$ which, in the case Φ is a coalgebra, reduces to the Φ-cobar resolution of M. Our construction just depends on a Γ-comodule map $\Phi \to \Gamma$ where Φ is a Γ-comodule algebra; the presence of Σ such that $\Phi = \Gamma \,\square_\Sigma k$ simply gives a more convenient form for the E_1 page.

(2) Margolis [Mar83] and Palmieri [Pal01] have studied the generalized Adams spectral sequence constructed in the category of stable Γ-comodules, a close cousin of the derived

[1] The spectral sequence we actually discuss computes $\mathrm{Cotor}_\Gamma(M, N)$, not $\mathrm{Ext}_\Gamma(M, N)$. We gloss over this difference because these are naturally isomorphic when $M = k$, which is our main case of interest. An analogous construction would work for comodule Ext in general, however.

category of Γ-comodules. If Φ is a Γ-comodule algebra and M is a Γ-comodule, then the Φ-based Adams spectral sequence in $\mathrm{Stable}(\Gamma)$ for M converges to $\mathrm{Ext}_\Gamma(k, M)$.

(3) The third construction is a filtration spectral sequence on the cobar complex on Γ due to Adams [Ada60]. Though originally studied in the case where $\Phi \to \Gamma \to \Sigma$ is an extension of Hopf algebras, the filtration spectral sequence itself may be defined in the setting here (where Φ is a Γ-comodule algebra) without modification.

The main results of this paper can be summarized as follows.

Theorem 1.1. *The spectral sequences (1), (2), and (3) coincide at the E_1 page, which has the form*

$$E_1^{s,t} = \mathrm{Ext}^t_\Sigma(k, \overline{\Phi}^{\otimes s} \otimes M)$$

where $\overline{\Phi}$ denotes the coaugmentation ideal.

These three constructions have different advantages: since (1) is the spectral sequence associated to a bi-cosimplicial commutative algebra, it has power operations due to the general theory of Sawka [Saw82]. The Adams spectral sequence presentation (2) naturally comes with an E_2 term under an analogue of the classical Adams flatness condition (see Corollary 1.2 below). The filtration spectral sequence presentation (3) is convenient for explicit computations in low degrees. Our comparison theorems enable one to use all of these desirable properties without regard to a choice of underlying model.

Corollary 1.2. *If $\mathrm{Ext}^*_\Sigma(k, \Phi)$ is flat as a module over $\mathrm{Ext}^*_\Sigma(k, k)$, then the spectral sequences (1), (2), and (3) have E_2 page*

$$E_2^{**} \cong \mathrm{Ext}^*_{\mathrm{Ext}_\Sigma(k,\Phi)}(\mathrm{Ext}_\Sigma(k, k), \mathrm{Ext}_\Sigma(k, M)).$$

In Section 2 we give an example of a setting in which the new E_2 page is defined but the classical Cartan-Eilenberg spectral sequence is not; in general, we expect such settings to involve computing localized Ext groups. Given a non-nilpotent element $x \in \mathrm{Ext}_\Gamma(k, k)$ whose image in $\mathrm{Ext}_\Sigma(k, k)$ is non-nilpotent, one may localize the entire spectral sequence construction, obtaining a spectral sequence that, in good cases, converges to $x^{-1}\,\mathrm{Ext}_\Gamma(k, M)$ (though we note that convergence must be checked separately). In many cases of interest, the E_2 condition holds only after inverting x, in which case the E_2 page of the localized spectral sequence has the form

$$E_2^{**} \cong \mathrm{Ext}^*_{x^{-1}\,\mathrm{Ext}_\Sigma(k,\Phi)}(x^{-1}\,\mathrm{Ext}_\Sigma(k, k),\ x^{-1}\,\mathrm{Ext}_\Sigma(k, M)).$$

(The idea is that the flatness condition may hold for the x-local part of $\mathrm{Ext}_\Sigma(k, \Phi)$ but not the torsion part.)

Localized Ext groups have been studied in many contexts, often because the localization represents the more tractable part of an otherwise complicated Ext group of interest. For example, Davis and Mahowald's work on v_1-local Ext groups over the Steenrod algebra [DM88] is an important part of understanding the E_2 page of the bo-based Adams spectral sequence (also see [BBB$^+$18] for a detailed study of the relationship between the v_1-periodic and v_1-torsion parts). More generally, for a type n spectrum X, the v_n-localized Adams E_2 page for X is the algebraic analogue of the chromatic localization $\pi_*(v_n^{-1}X)$. In Section 2 we use the techniques of this paper to give a new proof of May and Milgram's calculation of the p-towers in the Adams spectral sequence E_2 term for a finite spectrum; this calculates the E_2 page above a line of slope $\frac{1}{2p-2}$. This is a much easier analogue of the author's study [Bel19] of a different localization of the Adams E_2 page for the sphere at $p = 3$.

Outline. We begin in Section 2 by discussing the motivating application for this work, a localization of the Adams E_2 page for the sphere. In Section 3, we review the classical construction of the Cartan-Eilenberg spectral sequence for an extension of Hopf algebras $\Phi \to \Gamma \to \Sigma$, and define a variation (the spectral sequence mentioned in (1)) that makes sense when Φ is only a Γ-comodule algebra. The main step is to replace the Φ-cobar resolution of a Φ-comodule M (which does not make sense when Φ does not have a coalgebra structure) with a Γ-comodule resolution. This amounts to describing a specific resolution $\Phi \otimes R^*$ for use in Bruner and Rognes' construction.

In Section 4, we review Margolis and Palmieri's Adams spectral sequence, and prove that the spectral sequence (1) coincides with this one at E_1. This extends a remark of Palmieri [Pal01, §1.4], who notes that the spectral sequence he studies coincides with the Cartan-Eilenberg spectral sequence in the case that the extension is conormal (the coalgebra analogue of the normality condition mentioned above).

Section 5 is devoted to comparing the spectral sequences (1) and (3). Adams [Ada60, §2.3] mentions (without proof) that (3) coincides with the classical Cartan-Eilenberg spectral sequence in the case when the latter is defined. A proof of this fact is given in [Rav86, A1.3.16], attributed to Ossa. Our comparison proof is based on Ossa's. This involves the use of explicit formulas for the iterated shear isomorphism and its inverse, which are established in the appendix.

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2. Application: Localized Adams E_2 page

The motivation for the work in this paper was an attempt to calculate a localization of $\mathrm{Ext}_P(\mathbb{F}_3, \mathbb{F}_3)$, where $P = \mathbb{F}_3[\xi_1, \xi_2, \dots] \subset A$ is the dual algebra of reduced powers, using a Cartan-Eilenberg spectral sequence based on the extension

$$B := \mathbb{F}_3[\xi_1^3, \xi_2, \xi_3, \dots] \to P \to \mathbb{F}_3[\xi_1]/\xi_1^3.$$

Since B is not a sub-coalgebra of P (e.g. $\Delta(\xi_n) = \xi_{n-1}^3 \otimes \xi_1 + \dots$ is not contained in $P \otimes P$), the standard Cartan-Eilenberg spectral sequence is not defined; however, B is a P-comodule algebra, and so the generalized Cartan-Eilenberg spectral sequence described in this paper can be used. Furthermore, while the Adams flatness condition in Corollary 1.2 does not hold for this extension, it does hold after inverting the polynomial class $b_{10} \in \mathrm{Ext}^2_{\mathbb{F}_3[\xi_1]/(\xi_1^3)}(\mathbb{F}_3, \mathbb{F}_3)$. The computation of the resulting localized spectral sequence is the subject of [Bel19] which, at various points, crucially uses each of the three forms of the spectral sequence discussed in this paper.

In the rest of this section we give a complete calculation of a simpler version of this problem, which recovers a classical result on the rationalization of the sphere as a straightforward consequence of the techniques in this paper. Recall that the dual Steenrod algebra at $p > 2$ has the form $A = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$, and let $a_0 = [\tau_0] \in \mathrm{Ext}^1_A(\mathbb{F}_p, \mathbb{F}_p)$. This class survives the Adams spectral sequence $\mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) \implies \pi_* \widehat{S_p}$ and is detected in homotopy by the multiplication-by-p map $(S \xrightarrow{p} S) \in \pi_0 S$. The following result, which should be seen as the algebraic version of Serre's calculation [Ser53] of the homotopy of the rationalized sphere, is originally due to May and Milgram [MM81].

Proposition 2.1. *Let M be an A-comodule. For $p > 2$,*

$$a_0^{-1} \mathrm{Ext}_A(\mathbb{F}_p, M) = a_0^{-1} \mathrm{Ext}_{E[\tau_0]}(\mathbb{F}_p, M).$$

In particular, the a_0-localized Adams E_2 page for the sphere is $a_0^{-1} \mathrm{Ext}_A(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[a_0^{\pm 1}]$.

Proof. Let $E[x]$ denote the exterior algebra over $\mathbb{F}_p$ on a class x. The sequence of algebras

$$C := \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_1, \dots] \to A \to E[\tau_0]$$

is not an extension of Hopf algebras, as the diagonal on $\tau_n \in C$ does not lie in $C \otimes C$. However, $C = A \,\square_{E[\tau_0]} \mathbb{F}_p$ is an A-comodule algebra, and so the spectral sequences of Theorem 1.1 are defined. In particular, we have an E_1 page

$$E_1^{s,t} = \mathrm{Ext}^t_{E[\tau_0]}(\mathbb{F}_p, \overline{C}^{\otimes s} \otimes M).$$

Lemma 2.2. *$\overline{C}$ is free over $E[\tau_0]$.*

Proof. The Milnor diagonal $\Delta : A \to A \otimes A$ induces a right $E[\tau_0]$-coaction given by

$$\psi(\tau_n) = \tau_n \otimes 1 + \xi_n \otimes \tau_0$$
$$\psi(\xi_n) = \xi_n \otimes 1$$

and in particular, ξ_n^i is primitive for all i and $\psi(\xi_n^i \tau_n) = \xi_n^i \tau_n \otimes 1 + \xi_n^{i+1} \otimes \tau_0$. We have a $E[\tau_0]$-comodule decomposition of C:

$$\begin{aligned} C = \bigotimes_{n=1}^{\infty} \mathbb{F}_p[\xi_n, \tau_n]/\tau_n^2 &= \bigotimes_{n=1}^{\infty} \left(\mathbb{F}_p[\xi_n, \tau_n]/\tau_n^2 \right) \\ &= \bigotimes_{n=1}^{\infty} \left(\mathbb{F}_p\{1\} \oplus \mathbb{F}_p\{\xi_n^i,\ \tau_n \xi_n^i \ : \ i \geq 0\} \right) \\ &= \bigotimes_{n=1}^{\infty} \left(\mathbb{F}_p\{1\} \oplus \bigoplus_{i=1}^{\infty} \mathbb{F}_p\{\xi_n^i, \xi_n^{i-1}\tau_n\} \right) \end{aligned}$$

where all of the summands $\mathbb{F}_p\{\xi_n^i, \xi_n^{i-1}\tau_n\}$ are free. Thus, $C = \bigotimes_{n=1}^{\infty} (\mathbb{F}_p\{1\} \oplus F_n) = \mathbb{F}_p\{1\} \oplus F$ for free comodules F and F_n, and hence $\overline{C}$ is free. ∎

If F is free over $E[\tau_0]$, then $\mathrm{Ext}^*_{E[\tau_0]}(\mathbb{F}_p, F)$ is concentrated in homological degree zero. Thus $E_1^{s,t} = 0$ unless $s = 0$ or $t = 0$. For degree reasons, the a_0-localized Cartan-Eilenberg spectral sequence

$$a_0^{-1} E_1^{s,t} = a_0^{-1} \mathrm{Ext}^*_{E[\tau_0]}(\mathbb{F}_p, \overline{C}^{\otimes s} \otimes M) \implies a_0^{-1} \mathrm{Ext}_A(\mathbb{F}_p, M)$$

converges. Furthermore, $a_0^{-1}E_1^{s,t} = 0$ for $s > 0$, and the a_0-localized spectral sequence collapses at E_1 for degree reasons, giving the isomorphism

$$a_0^{-1}E_1^{0,*} = a_0^{-1}\operatorname{Ext}^*_{E[\tau_0]}(\mathbb{F}_p, M) \cong a_0^{-1}\operatorname{Ext}_A(\mathbb{F}_p, M). \qquad \square$$

3. The Cartan-Eilenberg spectral sequence

3.1. **Notation and preliminaries.** Throughout the paper, Γ will be a Hopf algebra over a commutative ring k, and M and N will be Γ-comodules. Coproducts will be denoted by Δ, and comodule coactions will be denoted by ψ.

Notation 3.1. We write $\sum m' \otimes m'' := \psi(m)$ and $\sum \gamma' \otimes \gamma'' := \Delta(\gamma)$ for $m \in M$ and $\gamma \in \Gamma$ when there is no ambiguity which coaction is in play.

An essential technical point in this paper is the comparison between two different Γ-comodule structures on a tensor product of Γ-comodules; we use the following nonstandard notation to clarify which structure is in play at a given time.

Definition 3.2. Let M and N be left Γ-comodules, with coaction denoted by $\psi(m) = \sum m' \otimes m''$ and $\psi(n) = \sum n' \otimes n''$. There are two natural ways to put a Γ-comodule structure on their tensor product $M \otimes N$: the *left coaction* $M \otimes N \to \Gamma \otimes (M \otimes N)$ is given by $m \otimes n \mapsto \sum m' \otimes m'' \otimes n$, and the *diagonal coaction* is given by $m \otimes n \mapsto \sum m'n' \otimes m'' \otimes n''$. To distinguish these, we write $M \overset{L}{\otimes} N$ for the tensor product $M \otimes N$ endowed with the left Γ-coaction, and $M \overset{\Delta}{\otimes} N$ for the diagonal coaction.

For a pair of right Γ-comodules one can analogously define the right and diagonal coactions, denoted $\overset{R}{\otimes}$ and $\overset{\Delta}{\otimes}$, respectively.

These constructions are isomorphic in the following special case:

Lemma 3.3. *If M is a left Γ-comodule, there is an isomorphism $S : \Gamma \overset{\Delta}{\otimes} M \to \Gamma \overset{L}{\otimes} M$ (called the* shear isomorphism*) given by:*

$$S : a \otimes m \mapsto \sum am' \otimes m''$$
$$S^{-1} : a \otimes m \mapsto \sum ac(m') \otimes m''$$

where c is the antipode on Γ. Analogously, if M is a right Γ-comodule, there is an isomorphism $S_c : M \overset{\Delta}{\otimes} \Gamma \to M \overset{R}{\otimes} \Gamma$ given by:

$$S_c : m \otimes a \mapsto \sum m' \otimes m''a$$
$$S_c^{-1} : m \otimes a \mapsto \sum m' \otimes c(m'')a.$$

The proof is straightforward. Now suppose $\Phi = \Gamma \,\square_\Sigma k$ for a Hopf algebra Σ such that Γ is injective as a Σ-comodule.

Lemma 3.4. *Let M be a Γ-comodule. Then $\Gamma \square_\Sigma M \subset \Gamma \overset{L}{\otimes} M$ inherits a left Γ-comodule structure, and the shear isomorphism $S : \Gamma \overset{\Delta}{\otimes} M \to \Gamma \overset{L}{\otimes} M$ restricts to an isomorphism*

$$\Phi \overset{\Delta}{\otimes} M \xrightarrow{\cong} \Gamma \square_\Sigma M.$$

The shear isomorphism $S_c : M \overset{\Delta}{\otimes} \Gamma \to M \overset{R}{\otimes} \Gamma$ restricts to an isomorphism $M \overset{\Delta}{\otimes} \Phi \xrightarrow{\cong} M \square_\Sigma \Gamma$.

Using this, we produce a useful variant of the usual change of rings theorem

$$\mathrm{Ext}_\Gamma(M, \Gamma \square_\Sigma N) \cong \mathrm{Ext}^*_\Sigma(M, N)$$

(see, e.g., [CE99, §VI.4]).

Corollary 3.5 (Change of rings theorem)**.** *Let M be a right Γ-comodule and N a left Γ-comodule, and let $\Phi = \Gamma \square_\Sigma N$. Then there is an isomorphism*

$$\mathrm{Ext}^*_\Gamma(M, \Phi \overset{\Delta}{\otimes} N) \cong \mathrm{Ext}^*_\Sigma(M, N).$$

Both change of rings statements hold for Cotor in addition to comodule Ext.

3.2. **Background: Classical Cartan-Eilenberg spectral sequence.** We begin by reviewing the classical construction of the Cartan-Eilenberg spectral sequence, expressed in the language of coalgebras. Following the treatment in [Rav86, A1.3.14], we will describe a spectral sequence that converges to $\mathrm{Cotor}_\Gamma(M, N)$. Many other treatments define the Cartan-Eilenberg spectral sequence in the coalgebra context using comodule Ext, but this does not matter for our cases of interest due to the isomorphism

$$\mathrm{Hom}_\Gamma(k, M) \cong k \square_\Gamma M$$

which implies $\mathrm{Ext}_\Gamma(k, M) \cong \mathrm{Cotor}_\Gamma(k, M)$. We choose Cotor because it is easier to work with: both slots are covariant, and it can be computed using an injective resolution of either side (just as projective resolutions are more convenient for modules, injective resolutions are more convenient for comodules).

Given an extension of Hopf algebras

$$\Phi \to \Gamma \to \Sigma$$

over a commutative ring k (so in particular $\Phi = \Gamma \square_\Sigma k$), a right Γ-comodule M, and a left Φ-comodule N, the Cartan-Eilenberg spectral sequence for computing $\mathrm{Cotor}_\Gamma(M, N)$ arises from the double complex (Γ-resolution of M) $\square_\Gamma$ (Φ-resolution of N). Our choice of injective resolution is the cobar resolution, which we will describe in some detail because an essential technical point in our spectral sequence construction comes down to the difference between two versions of the cobar resolution ($\overset{\Delta}{D}{}^*_\Gamma(N)$ and $\overset{L}{D}{}^*_\Gamma(N)$) which are isomorphic via the shear isomorphism.

Definition 3.6. Define the cobar resolution $\overset{L}{D}{}^*_\Gamma(N)$ of N to be the chain complex associated to the augmented cosimplicial object

(3.1)
$$\begin{tikzcd} N \arrow[d, "\simeq"] \\ \overset{L}{D}{}^\bullet_\Gamma(N) = \big(\Gamma \overset{L}{\otimes} N \arrow[r, shift left=2, "\Delta"] \arrow[r, shift right=2, "\psi"'] & \Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} N \arrow[l, "\varepsilon" description] \arrow[r, shift left=3, "\Delta_1"] \arrow[r, shift right=3, "\psi"'] & \Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} \Gamma \overset{L}{\otimes} N \arrow[l, shift right=1, "\varepsilon_2" description] \arrow[l, shift left=1, "\varepsilon_1" description] & \cdots & \big). \end{tikzcd}$$

Here the codegeneracies ε_i come from applying the coaugmentation ε to the i^{th} spot, and the coface maps $\Delta_i : \Gamma^{\otimes n} \otimes N \to \Gamma^{\otimes n+1} \otimes N$ for $1 \leq i \leq n$ come from applying Δ to the i^{th} slot; the last coface map comes from the coaction $\psi : N \to \Gamma \otimes N$. For a right Γ-comodule M, let $\overset{R}{D}{}^*_\Gamma(M)$ denote the analogous resolution $M \otimes \Gamma^{\otimes *} \overset{R}{\otimes} \Gamma$, and similarly for $\overset{R}{C}{}^*_\Gamma(M)$.

Let the *(non-normalized) cobar complex* $\overset{L}{C}{}^*_\Gamma(M,N)$ be the complex $M \,\square_\Gamma \overset{L}{D}{}^*_\Gamma(N)$.

There are also normalized versions $\mathcal{N}\overset{L}{D}{}^*_\Gamma(N)$ (with terms $\Gamma \overset{L}{\otimes} \overline{\Gamma}^{\otimes *} \otimes N$) and $\mathcal{N}\overset{L}{C}{}^*_\Gamma(M,N) = M \otimes \overline{\Gamma}^{\otimes *} \otimes N$ (with terms $M \,\square_\Gamma (\Gamma \overset{L}{\otimes} \overline{\Gamma}^{\otimes *} \otimes N)$), obtained by applying the normalization functor $\mathcal{N} : \mathrm{Ch} \to \mathrm{Ch}$ defined on terms by

$$\mathcal{N}A^n = \bigcap_{i=0}^{n-1} \ker(d_i : A^{n+1} \to A^n) \subset A^n.$$

(Here $\overline{\Gamma}$ denotes $\ker(\varepsilon : \Gamma \to k)$ but we will later also use that symbol to denote the quotient $\mathrm{coker}(\eta_L : k \to \Gamma)$.)

For our purposes, the classical Cartan-Eilenberg spectral sequence is the spectral sequence associated to the double complex $\mathcal{N}\overset{R}{D}{}^*_\Gamma(M) \,\square_\Gamma \mathcal{N}\overset{L}{D}{}^*_\Phi(N)$.

(3.2)
$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & (M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s} \otimes N) \arrow[r, "(-1)^t \mathbb{1} \otimes d_\Phi"] \arrow[d, "d_\Gamma \otimes \mathbb{1}"'] & (M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s+1} \otimes N) \arrow[r] \arrow[d, "d_\Gamma \otimes \mathbb{1}"] & \cdots \\ \cdots \arrow[r] & (M \otimes \overline{\Gamma}^{\otimes t+1} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s} \otimes N) \arrow[r, "(-1)^{t+1} \mathbb{1} \otimes d_\Phi"] \arrow[d] & (M \otimes \overline{\Gamma}^{\otimes t+1} \otimes \Gamma) \,\square_\Gamma (\Phi \otimes \overline{\Phi}^{\otimes s+1} \otimes N) \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd}$$

Taking homology in the vertical direction first, one has

$$\begin{aligned} E_1^{s,t} &= \mathrm{Cotor}^t_\Gamma(M,\, \Phi \otimes \overline{\Phi}^{\otimes s} \otimes N) \\ &\cong \mathrm{Cotor}^t_\Gamma(M,\, (\Gamma \,\square_\Sigma k) \otimes \overline{\Phi}^{\otimes s} \otimes N) \\ &\cong \mathrm{Cotor}^t_\Sigma(M,\, \overline{\Phi}^{\otimes s} \otimes N) \end{aligned}$$

where the last isomorphism is by the change of rings theorem. For the spectral sequence that starts by taking homology in the horizontal direction first, exactness of the functor

$(M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_{\Gamma} -$ gives

$$E_1^{*,t} \cong H^*((M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_{\Gamma} (\Phi \otimes \overline{\Phi}^{\otimes *} \otimes N)) \cong (M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_{\Gamma} H^*(\Phi \otimes \overline{\Phi}^{\otimes *} \otimes N)$$

and by the exactness of the resolution $\Phi \otimes \overline{\Phi}^{\otimes *} \otimes N$ of N, this is concentrated in degree zero as $(M \otimes \overline{\Gamma}^{\otimes t} \otimes \Gamma) \square_{\Gamma} N$. The E_2 page then takes cohomology in the t direction, obtaining $E_2 \cong E_\infty \cong \mathrm{Cotor}_{\Gamma}(M, N)$. The Cartan-Eilenberg spectral sequence is the vertical-first spectral sequence, and we have just shown that it has

$$E_1^{s,t} = \mathrm{Cotor}^t_{\Sigma}(M, \overline{\Phi}^{\otimes s} \otimes N) \implies \mathrm{Cotor}^{s+t}_{\Gamma}(M, N).$$

If Φ has trivial Σ-coaction, then we have $E_1^{s,t} \cong \mathrm{Cotor}^t_{\Sigma}(M, N) \otimes \overline{\Phi}^{\otimes s}$, whose cohomology is:

$$E_2 = \mathrm{Cotor}^s_{\Phi}(k, \mathrm{Cotor}^t_{\Sigma}(M, N)).$$

The spectral sequence converges because it is a first-quadrant double complex spectral sequence.

Remark 3.7. The E_2 page is independent of the Φ-resolution of N and the Γ-resolution of M, but the E_1 page does depend on the Φ-resolution of N.

3.3. **Weakening the hypotheses.** We define a related construction that makes sense when Φ is not a coalgebra. More precisely, let Γ be a Hopf algebra and let Φ be a Γ-comodule-algebra. The main issue with defining an analogue of (3.2) is that the cosimplicial object $\overset{L}{D}{}^{\bullet}_{\Phi}(N)$ is not defined, because the coface maps would be defined in terms of the coproduct on Φ.

To remedy this, we turn to a different construction of the cobar complex which is defined for unital algebras, and which is isomorphic to Definition 3.6 when we are working with a Hopf algebra. For a Γ-comodule N, define the resolution $\overset{\triangle}{D}{}^*_{\Gamma}(N)$ of N to be the chain complex associated to the augmented cosimplicial object

(3.3)
$$\begin{tikzcd}[column sep=large]
 & N \arrow[d, "\simeq"] & & & \\
\overset{\triangle}{D}{}^{\bullet}_{\Gamma}(N) = \big(& \Gamma \overset{\triangle}{\otimes} N \arrow[r, shift left=2, "\eta_1"] \arrow[r, shift right=2, "\eta_2"'] & \Gamma \overset{\triangle}{\otimes} \Gamma \overset{\triangle}{\otimes} N \arrow[l, "\mu"'] \arrow[r, shift left=3, "\eta_1"] \arrow[r] \arrow[r, shift right=3, "\eta_3"'] & \Gamma \overset{\triangle}{\otimes} \Gamma \overset{\triangle}{\otimes} \Gamma \overset{\triangle}{\otimes} N \arrow[l, shift left=1.5, "\mu_1"'] \arrow[l, shift right=1.5, "\mu_2"'] & \cdots \quad \big)
\end{tikzcd}$$

where the codegeneracies μ_i are multiplication of the i^{th} and $(i+1)^{st}$ copies of Γ, and the coface maps η_i are given by insertion of 1 into the i^{th} spot. Substituting Φ for Γ above, if N is a Γ-comodule we may still define $\overset{\triangle}{D}{}^*_{\Phi}(N)$, a complex of Γ-comodules free over Φ and quasi-isomorphic to N.

We can also describe $\overset{\triangle}{D}{}^{\bullet}_{\Phi}(N)$ in a more natural way. Since Φ is a monoid object in Comod_{Γ}, we can define the category Mod_{Φ} of Φ-modules in Comod_{Γ}. There is a free-forgetful adjunction

$$F_{\Phi} : \mathrm{Comod}_{\Gamma} \rightleftarrows \mathrm{Mod}_{\Phi} : U$$

where $F_{\Phi}(N) = \Phi \overset{\triangle}{\otimes} N$. Then $\overset{\triangle}{D}{}^{\bullet}_{\Phi}(N)$ is the cosimplicial object associated to the monad UF_{Φ}.

Definition 3.8. The *(non-normalized) cobar complex* $\overset{\triangle}{C}{}^*_\Gamma(M,N)$ is the complex $M \,\square_\Gamma \overset{\triangle}{D}{}^*_\Gamma(N)$. Similarly, define $\overset{L}{C}{}^*_\Gamma(M,N) = M \,\square_\Gamma \overset{L}{D}{}^*_\Gamma(N)$.

Proposition 3.9. *The shear isomorphism (Lemma 3.3) gives rise to an isomorphism of cosimplicial objects $\overset{\triangle}{D}{}^\bullet_\Gamma(N) \to \overset{L}{D}{}^\bullet_\Gamma(N)$, and hence isomorphisms of chain complexes $\overset{\triangle}{D}{}^*_\Gamma(N) \to \overset{L}{D}{}^*_\Gamma(N)$ and $\overset{\triangle}{C}{}^*_\Gamma(N) \to \overset{L}{C}{}^*_\Gamma(N)$.*

In the appendix, we write out explicit formulas for these isomorphisms.

Definition 3.10. If Γ is a Hopf algebra, Φ is a Γ-comodule-algebra, and M and N are Γ-comodules, define the Cartan-Eilenberg spectral sequence to be the spectral sequence associated to the double complex

$$(\mathcal{N}\overset{\triangle}{D}{}^*_\Gamma(M)) \,\square_\Gamma\, (\mathcal{N}\overset{\triangle}{D}{}^*_\Phi(N)).$$

The spectral sequence is unchanged starting at E_1 if we replace the complex on the right by a chain-homotopic one, and in Section 5 we will find it more convenient to use the complex

$$\overset{\triangle}{D}{}^*_\Gamma(M) \,\square_\Gamma\, (\mathcal{N}\overset{\triangle}{D}{}^*_\Phi(N)). \tag{3.4}$$

By definition, we have the E_1 term

$$E_1^{s,t} = \mathrm{Cotor}^t_\Gamma(M, \mathcal{N}\overset{\triangle}{D}{}^*_\Phi(N))$$

and it converges to $\mathrm{Cotor}_\Gamma(M,N)$ as with the usual construction of the Cartan-Eilenberg spectral sequence. As in the classical case, if $\Phi = \Gamma \,\square_\Sigma k$ for some coalgebra Σ, then by the version of the change of rings theorem in Corollary 3.5 we may write

$$E_1^{s,t} \cong \mathrm{Cotor}^t_\Gamma(M, \Phi \overset{\triangle}{\otimes} (\overline{\Phi}^{\otimes s} \overset{\triangle}{\otimes} N)) \cong \mathrm{Cotor}^t_\Sigma(M, \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N). \tag{3.5}$$

In particular, if $M = k$, then $E_1^{s,t} \cong \mathrm{Ext}^t_\Sigma(k, \overline{\Phi}^{\otimes s} \otimes N)$ and the spectral sequence converges to $\mathrm{Ext}_\Gamma(k,N)$.

Remark 3.11. If Φ did have a coalgebra structure, we can also define the spectral sequence in Section 3.2, and by Proposition 3.9 the two spectral sequences are isomorphic via the shear isomorphism.

Remark 3.12. Davis and Mahowald [DM82] studied a Cartan-Eilenberg type spectral sequence in the setting $\Gamma = A(2)_*$, $\Sigma = A(1)_*$ (so $\Phi = \Gamma \,\square_\Sigma k$ is not a sub-Hopf algebra of Γ). Instead of the cobar-inspired resolution described here, they work with a minimal resolution, which is more convenient for computational purposes. Bruner and Rognes [BR] construct a spectral sequence converging to $\mathrm{Ext}_\Gamma(k,N)$ given the data of a Γ-comodule algebra resolution of k of the form $\Phi \overset{\triangle}{\otimes} R^*$ where $\Phi = \Gamma \,\square_\Sigma k$ for a Hopf algebra Σ and R^* is a sequence of Σ-comodules. They show that their spectral sequence is multiplicative, given suitable multiplicative properties of R^*. The construction in this section can be seen as a special case of theirs, where $R^n = \overline{\Phi}^{\overset{\triangle}{\otimes} n}$. Our construction does not use the presentation of Φ as $\Gamma \,\square_\Sigma k$ except to obtain the nicer form of the E_1 page in (3.5).

4. Margolis-Palmieri Adams spectral sequence

4.1. **Background: Adams spectral sequence in** Stable(Γ)**.** Given a finite spectrum X and a ring spectrum E, the classical Adams spectral sequence is the spectral sequence obtained by applying $\pi_*(-)$ to the tower of fibrations

$$\begin{tikzcd} X \arrow[d] & \overline{E}\wedge X \arrow[l, dotted] \arrow[d] & \overline{E}\wedge\overline{E}\wedge X \arrow[l, dotted] & \dots \arrow[l, dotted] \\ E\wedge X \arrow[ur] & E\wedge\overline{E}\wedge X \arrow[ur] & & \end{tikzcd} \tag{4.1}$$

where $\overline{E}$ is the cofiber of the unit map $S \to E$. If E_*E is flat as an E_*-algebra, then the E_2 page is given by $\mathrm{Ext}_{E_*E}(E_*, E_*X)$.

This construction makes sense in the context of an arbitrary tensor triangulated category $(\mathcal{C}, \otimes, \mathbb{1})$. Given a ring object E and another object X of $\mathcal{C}$, let $\overline{E}$ be the cofiber of the unit map $\mathbb{1} \to E$. Then one can construct the same tower of fibrations (4.1) and apply the functor $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, -)$, giving rise to a spectral sequence which, under favorable conditions, converges to (a completion of) $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, X)$.

Following Palmieri [Pal01], we study this generalized Adams spectral sequence in the case $\mathcal{C} = \mathrm{Stable}(\Gamma)$, the category whose objects are unbounded cochain complexes of injective Γ-comodules and whose morphisms are chain complex morphisms modulo chain homotopy. The reason to work in this setting is the fact that

$$\mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(M, N) = \mathrm{Ext}_{\Gamma}(M, N)$$

for Γ-comodules M and N (we abuse notation by identifying M with its image under the functor $\mathrm{Comod}_{\Gamma} \to \mathrm{Stable}(\Gamma)$ given by taking injective resolutions). Thus one can use techniques from homotopy theory to study Ext groups.

Remark 4.1. The reader may wonder why we have chosen Stable(Γ) instead of the more familiar derived category $D(\Gamma)$, as there is also an identification $\mathrm{Hom}_{D(\Gamma)}(M, N) = \mathrm{Ext}_{\Gamma}(M, N)$. The reason is that Stable(Γ) is a better setting for studying localized Ext groups: if $x \in \mathrm{Ext}_{\Gamma}(k, k)$ is a non-nilpotent element, M and N are Γ-comodules, and $x^{-1}N$ is the colimit of multiplication by x in Stable(Γ), then

$$\mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(M, x^{-1}N) = x^{-1}\,\mathrm{Ext}_{\Gamma}(M, N)$$

but the analogous statement in $D(\Gamma)$ is not guaranteed to hold. As we care most about the constructions defined in this paper after localizing at a non-nilpotent element, and so this property is essential to ensuring that this localization is well-behaved.

The Adams spectral sequence in this setting was first studied by Margolis [Mar83] in the case where Γ is the dual Steenrod algebra, work which was extended and generalized by Palmieri. If $E \in \mathrm{Stable}(\Gamma)$ is a ring object (for example, a Γ-comodule algebra) and $X \in \mathrm{Stable}(\Gamma)$ we refer to the E-based Adams spectral sequence in Stable(Γ) computing $\mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(k, X)$ as the *E-based Margolis-Palmieri Adams spectral sequence* (MPASS). Analogously to the classical Adams flatness condition, if $\mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(k, E \overset{\vartriangle}{\otimes} E)$ is flat over $\mathrm{Hom}_{\mathrm{Stable}(\Gamma)}(k, E)$, then the E_2 page of the MPASS is

$$E_2 \cong \operatorname{Ext}_{\operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(k, E \overset{\vartriangle}{\otimes} E)}(\operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(k,E),\ \operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(k, E \overset{\vartriangle}{\otimes} X)). \tag{4.2}$$

Palmieri [Pal01, Proposition 1.4.3] identifies finiteness conditions on E and X under which the MPASS converges to $\operatorname{Hom}_{\mathrm{Stable}(\Gamma)}(k,X)$. The motivating application is the case where Γ is the dual Steenrod algebra, $X = H_*(Y)$ for a finite spectrum Y, and E is a subalgebra of A. Then the MPASS

$$E_2 \cong \operatorname{Ext}_{\operatorname{Ext}_A(k, E\otimes E)}(\operatorname{Ext}_A(k,E), \operatorname{Ext}_A(k, E\otimes H_*Y)) \implies \operatorname{Ext}_A(k, H_*Y)$$

converges to the E_2 page of the Adams spectral sequence for π_*Y.

4.2. **Comparison: MPASS vs. Cartan-Eilenberg spectral sequence.**

Theorem 4.2. *Given a left Γ-comodule-algebra Φ and a left Γ-comodule N, the Cartan-Eilenberg spectral sequence*

$$\overset{\vartriangle}{E}{}_1^{s,*} = H^*\big(\overset{\vartriangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N))\big) \implies \operatorname{Cotor}^*_\Gamma(k,N) \cong \operatorname{Ext}^*_\Gamma(k,N)$$

coincides starting at E_1 with the Φ-based MPASS

$$E_1^{s,*} = \operatorname{Ext}^*_\Gamma(k, \Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N) \implies \operatorname{Ext}^*_\Gamma(k,N).$$

Proof. Given a chain complex A^*, let $\mathcal{Q}A$ denote the quotient chain complex whose terms are given by $\mathcal{Q}A^n = A^n \big/ \sum_{i=1}^n \operatorname{im}(d^i : A^{n-1} \to A^n)$. In particular, $\mathcal{Q}\overset{\vartriangle}{D}{}^*_\Phi(N) \cong \Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} *} \overset{\vartriangle}{\otimes} N$. It is a general fact (see [GJ09, Theorem III.2.1 and Theorem III.2.4] for the dual version) that there is an isomorphism of chain complexes $\mathcal{N}^*A \xrightarrow{\cong} \mathcal{Q}^*A$, so instead of the double complex $\overset{\vartriangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N))$ we may use

$$\overset{\vartriangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\mathcal{Q}\overset{\vartriangle}{D}{}^*_\Phi(N)) = \Gamma^{\overset{\vartriangle}{\otimes} t+1} \,\square_\Gamma (\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N).$$

We will express the exact couples for both spectral sequences as coming from applying $\operatorname{Ext}_\Gamma(k,-)$ to fiber sequences in $\mathrm{Stable}(\Gamma)$, and show that there is a quasi-isomorphism connecting those fiber sequences. We begin by describing the exact couple for the Cartan-Eilenberg spectral sequence more explicitly. Let T^* be the total complex defined by $T^n = \bigoplus_{s+t=n} \Gamma^{\overset{\vartriangle}{\otimes} t+1} \,\square_\Gamma (\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N)$. The Cartan-Eilenberg spectral sequence arises from the filtration F^s on this total complex defined by:

$$F^{s_0}T^n = \bigoplus_{\substack{s+t=n \\ s\geq s_0}} \Gamma^{\overset{\vartriangle}{\otimes} t+1} \,\square_\Gamma (\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N).$$

For the associated graded we have:

$$F^{s_0}/F^{s_0+1}T^n = \Gamma^{\overset{\vartriangle}{\otimes} n-s_0+1} \,\square_\Gamma (\Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s_0} \overset{\vartriangle}{\otimes} N)$$

$$H^*(F^{s_0}/F^{s_0+1}T^*) = \operatorname{Cotor}^*_\Gamma(k, \Phi \overset{\vartriangle}{\otimes} \overline{\Phi}^{\overset{\vartriangle}{\otimes} s} \overset{\vartriangle}{\otimes} N).$$

By definition, the Cartan-Eilenberg spectral sequence arises from the exact couple

$$\begin{tikzcd} H^*(F^sT^*) \arrow[dr] & & H^*(F^{s+1}T^*) \arrow[ll] \\ & H^*(F^s/F^{s+1}T^*). \arrow[ur, dotted] & \end{tikzcd} \tag{4.3}$$

On the other hand, the MPASS comes from the exact couple obtained by applying the functor $\operatorname{Ext}_\Gamma(k,-)$ to the cofiber sequence

$$\overline{\Phi}^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N \to (\overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N)[1] \to (\Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\otimes s} \overset{\triangle}{\otimes} N)[1]. \tag{4.4}$$

in $\mathrm{Stable}(\Gamma)$. Since $\operatorname{Ext}_\Gamma(k,-) \cong \operatorname{Cotor}_\Gamma(k,-)$, this is isomorphic to the exact couple

$$\begin{tikzcd} H^*(\overset{\triangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N)) \arrow[dr] & & H^*(\overset{\triangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\overline{\Phi}^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N)) \arrow[ll] \\ & H^*(\overset{\triangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N)) \arrow[ur, dotted] & \end{tikzcd} \tag{4.5}$$

We claim that (4.5) and (4.3) are isomorphic exact couples. For the same reason that $H^*(T^*) \cong \operatorname{Ext}_\Gamma(k, M)$, we have $H^*(F^sT^*) \cong \operatorname{Ext}_\Gamma(k, \overline{\Phi}^{\otimes s} \otimes M)$. Moreover, there is a map $\overset{\triangle}{D}{}^*_\Gamma(k) \,\square_\Gamma (\overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} M) \to F^sT^*$ induced by the unit map $\overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N \to \Phi \overset{\triangle}{\otimes} \overline{\Phi}^{\overset{\triangle}{\otimes} s} \overset{\triangle}{\otimes} N$ that induces this isomorphism in cohomology compatibly with the rest of the exact couple. $\square$

The comparison statement shows that the E_2 page of the Cartan-Eilenberg spectral sequence coincides with the MPASS E_2 page (4.2).

Corollary 4.3. *If $\operatorname{Ext}_\Gamma(k, \Phi \otimes \Phi)$ is flat as a module over $\operatorname{Ext}_\Gamma(k, \Phi)$, then the Cartan-Eilenberg spectral sequence of Definition 3.10 has E_2 term given by*

$$E_2^{**} \cong \operatorname{Ext}^*_{\operatorname{Ext}_\Gamma(k,\Phi\otimes\Phi)}(\operatorname{Ext}_\Gamma(k,\Phi), \operatorname{Ext}^*_\Gamma(k, \Phi \overset{\triangle}{\otimes} N)).$$

If $\Phi = \Gamma \,\square_\Sigma k$ for some coalgebra Σ, then by the change of rings theorem (Corollary 3.5) this has the form

$$E_2^{**} = \operatorname{Ext}_{\operatorname{Ext}_\Sigma(k,\Phi)}(\operatorname{Ext}_\Sigma(k,k),\ \operatorname{Ext}_\Sigma(k,N)).$$

For $x \in \operatorname{Ext}_\Gamma(k,k)$, the x-localized Cartan-Eilenberg spectral sequence has E_2 term

$$\operatorname{Ext}_{x^{-1}\operatorname{Ext}_\Sigma(k,\Phi)}(x^{-1}\operatorname{Ext}_\Sigma(k,k),\ x^{-1}\operatorname{Ext}_\Sigma(k,N)).$$

Note that, for the localized spectral sequence, one must additionally check convergence.

5. Cartan-Eilenberg vs. filtration spectral sequence

It is a classical fact [Ada60, §2.3] that the Cartan-Eilenberg spectral sequence associated to the Hopf extension $\Phi \to \Gamma \to \Sigma$ computing $\operatorname{Cotor}_\Gamma(M,N)$ coincides with a filtration spectral sequence on the cobar complex $C_\Gamma(M,N)$ defined by

$$F^sC^n_\Gamma(M,N) = \{m[a_1|\dots|a_n]\nu \in C^n_\Gamma(M,N) \,:\, \#(\{a_1,\dots,a_n\}\cap G) \geq s\}$$

where

$$G := \ker(\Gamma \to \Sigma).$$

As G is an ideal in Γ and the cobar complex $C^*_\Gamma(k,k)$ is a ring under the concatenation product, one can say this filtration of $C^*_\Gamma(M,N) = M \otimes C^*_\Gamma(k,k) \otimes N$ comes from the G-adic filtration of $C^*_\Gamma(k,k)$. In this section, we adopt the notation of the previous sections, but also impose the additional condition that $\Phi = \Gamma \,\square_\Sigma k$ where $\Gamma \to \Sigma$ is a map of Hopf algebras.

Let E_r^{**} denote this filtration spectral sequence, and let $\overset{\vartriangle}{E}{}_r^{**}$ denote the generalized Cartan-Eilenberg spectral sequence described in Definition 3.10. Adapting an argument for the classical Cartan-Eilenberg spectral sequence, we will show that these agree starting at $r = 1$. As a double complex spectral sequence can be viewed as a filtration spectral sequence on the total complex, it suffices to show the following:

Theorem 5.1. *There is a filtration-preserving chain map*

$$\theta : \bigoplus_{s+t=n} (M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N)) \longrightarrow C^n_\Gamma(M,N)$$

whose induced map of spectral sequences $\overset{\vartriangle}{E}{}_r^{**} \to E_r^{**}$ is an isomorphism on E_1.

We begin by defining the comparison map θ.

Definition 5.2. Let $\widetilde{\theta}$ denote the composition

$$\begin{aligned}\widetilde{\theta} : (M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\Phi^{\overset{\vartriangle}{\otimes} s+1} \overset{\vartriangle}{\otimes} N) &\xrightarrow{S^n_c \otimes S^n} (M \otimes \Gamma^{\otimes t} \overset{R}{\otimes} \Gamma) \,\square_\Gamma (\Gamma \overset{L}{\otimes} \Gamma^{\otimes s} \otimes N) \\ &\xrightarrow{e} M \otimes \Gamma^{\otimes s+t} \otimes N\end{aligned}$$

where S^n_c is n-fold composition of the shear isomorphism $S_c : M \overset{\vartriangle}{\otimes} \Gamma \to M \overset{R}{\otimes} \Gamma$, S^n is the n-fold composition of the iterated shear isomorphism $S : \Gamma \overset{\vartriangle}{\otimes} N \to \Gamma \overset{L}{\otimes} N$, and e is given by

$$(m|a_1|\dots|a_t|a) \otimes (b|b_1|\dots|b_s|n) \mapsto \varepsilon(ab)m|a_1|\dots|a_t|b_1|\dots|b_s|n.$$

Define θ to be the restriction of $\widetilde{\theta}$ to $(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N))$.

In Lemma 5.4, we will show that this restriction lands in $(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\Gamma \,\square_\Sigma G(s) \,\square_\Sigma N)$, where

$$G(s) := \underbrace{G \,\square_\Sigma \dots \square_\Sigma G}_{s}.$$

We will see that $E_0^{0,*}(M,N)$ is easy to describe (and in particular it is easy to show that θ induces an isomorphism $\overset{\vartriangle}{E}{}_0^{0,*}(M,N) \cong E_0^{0,*}(M,N)$), and most of the work involves identifying $E_0^{s,*}(M,N)$ (for $s > 0$) with $E_0^{0,*}(M,N')$ for a different comodule N', in a way that is

compatible with a similar identification for $\overset{\triangle}{E}{}_0^{s,*}$. More precisely, we will show that there is a map β of chain complexes making the following diagram commute.
(5.1)

$$\begin{tikzcd}
(M \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes}*}) \square_\Gamma \mathcal{N}\overset{\triangle}{D}{}^0_\Phi(G(s) \square_\Sigma N) \arrow[r, equal] \arrow[d, "\cong", "\mathbb{1}\otimes S^{-1}"'] & \overset{\triangle}{E}{}_0^{0,*}(M, G(s) \square_\Sigma N) \arrow[r, "\theta", "\simeq"'] & E_0^{0,*}(M, G(s) \square_\Sigma N) \arrow[d, "\beta", "\simeq"'] \\
(M \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes}*}) \square_\Gamma \mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N) \arrow[r, equal] & \overset{\triangle}{E}{}_0^{s,*}(M, N) \arrow[r, "\theta"] & E_0^{s,*}(M, N)
\end{tikzcd}$$

It suffices to show the following:

(1) θ is a filtration-preserving chain map;

(2) S^{-1} gives rise to an isomorphism $\mathcal{N}\overset{\triangle}{D}{}^0_\Phi(G(s) \square_\Sigma N) \to \mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N)$;

(3) there exists a chain equivalence β making the diagram commute;

(4) θ is a chain equivalence for $s = 0$.

(1) says we have written down a filtration-preserving map between total complexes, and (2)–(4) allow us to use the diagram to show that θ is a chain equivalence for all $s \geq 0$. We prove (1) in Lemma 5.3 and Corollary 5.5, (2) in Corollary 5.6, (3) in Corollary/ Definition 5.9, and (4) in Proposition 5.11.

Both the structure of the proof and the argument for (2) are adapted from an argument attributed to Ossa appearing as [Rav86, A1.3.16], showing that the classical Cartan-Eilenberg spectral sequence coincides with the filtration spectral sequence under discussion. Our proof is more complicated than Ossa's original, as the spectral sequence of Definition 3.10 generalizes the classical Cartan-Eilenberg spectral sequence only after the iterated shear isomorphism has been applied. It is not natural to describe the cobar filtration spectral sequence after applying the isomorphism, so we must translate between the two contexts using explicit formulas for the iterated shear isomorphism.

Lemma 5.3. *$\widetilde{\theta}$ is a chain map $\bigoplus_{s+t=n}(M \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes}t+1}) \square_\Gamma \overset{\triangle}{D}{}^s_\Phi(N) \to C^n_\Gamma(M, N)$.*

Proof. Since S^n and S^n_c are maps of chain complexes of Γ-comodules, there is an induced map on the tensor product of chain complexes

$$(M \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes}*+1}) \otimes (\Phi^{\overset{\triangle}{\otimes}*+1} \overset{\triangle}{\otimes} N) \to (M \otimes \Gamma^{*+1}) \otimes (\Gamma^{\otimes *+1} \otimes N)$$

and since these are maps of chain complexes of Γ-comodules, this passes to a map on the cotensor product

$$(M \overset{\triangle}{\otimes} \Gamma^{\overset{\triangle}{\otimes}*+1}) \square_\Gamma (\Phi^{\overset{\triangle}{\otimes}*+1} \overset{\triangle}{\otimes} N) \to (M \otimes \Gamma^{*+1}) \square_\Gamma (\Gamma^{\otimes *+1} \otimes N).$$

Then $\widetilde{\theta}$ is formed by post-composing with the map

$$e : (M \otimes \Gamma^{\otimes t+1}) \square_\Gamma (\Gamma^{\otimes s+1} \otimes N) \to M \otimes \Gamma^{t+s} \otimes N$$

which takes $m[a_1|\dots|a_t]a_{t+1}\otimes b_0[b_1|\dots|b_s]n \mapsto \varepsilon(a_{t+1}b_0)m[a_1|\dots|a_t|b_1|\dots|b_s]n$. To see this is a chain map, it suffices to check the following diagram commutes.

$$\begin{tikzcd}
(M\otimes\Gamma^{\otimes t+1})\,\square_\Gamma(\Gamma^{\otimes s+1}\otimes N) \arrow[r, "\mathbb{1}\otimes\varepsilon\otimes\mathbb{1}"] \arrow[d, "d_{\substack{\text{double}\\\text{complex}}}"'] & M\otimes\Gamma^{\otimes t}\otimes\Gamma^{\otimes s}\otimes N \arrow[d, "d_{\text{cobar}}"] \\
\begin{array}{c}(M\otimes\Gamma^{\otimes t+1})\,\square_\Gamma(\Gamma^{\otimes s+2}\otimes N)\\ \oplus\ (M\otimes\Gamma^{\otimes t+2})\,\square_\Gamma(\Gamma^{\otimes s+1}\otimes N)\end{array} \arrow[r, "\mathbb{1}\otimes\varepsilon\otimes\mathbb{1}"] & M\otimes\Gamma^{\otimes t+s+1}\otimes N
\end{tikzcd}$$

This requires keeping track of signs: the double complex differential is $d_\Gamma\otimes\mathbb{1}+(-1)^t\mathbb{1}\otimes d_\Phi$, or more explicitly:

$$\begin{aligned}a_0[a_1|\dots|a_t]a_{t+1}\otimes b_0[b_1|\dots|b_s]b_{s+1} \mapsto &\sum_i(-1)^i a_0[\dots|a_i'|a_i''|\dots]a_{t+1}\otimes b_0[b_1|\dots|b_s]b_{s+1}\\ &+\sum_i(-1)^{i+t}a_0[a_1|\dots|a_t]a_{t+1}\otimes b_0[\dots|b_i'|b_i''|\dots]b_{s+1}\end{aligned}$$

and the cobar differential is

$$\begin{aligned}a_0[a_1|\dots|a_t|b_1|\dots|b_s]b_{s+1} \mapsto &\sum_i(-1)^i a_0[a_1|\dots|a_i'|a_i''|\dots|b_1|\dots|b_s]b_{s+1}\\ &+\sum_i(-1)^{t+i}a_0[a_1|\dots|a_t|b_1|\dots|b_i'|b_i''|\dots|b_s]b_{s+1}.\end{aligned}$$

In particular, notice that, on the bottom left composition, the terms corresponding to $a_0[\dots|a_{t+1}']a_{t+1}''\otimes b_0[\dots]b_{s+1}$ cancel in $M\otimes\Gamma^{\otimes t+s+1}\otimes N$ with the terms corresponding to $a_0[\dots]a_{t+1}\otimes b_0'[b_0''|\dots]b_{s+1}$. □

While $\widetilde{\theta}$ is not filtration-preserving, we will show that its restriction to $(M\overset{\vartriangle}{\otimes}\Gamma^{\overset{\vartriangle}{\otimes}t+1})\,\square_\Gamma\,\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi$ is.

Lemma 5.4. *The iterated shear map $S:\Gamma^{\overset{\vartriangle}{\otimes}s+1}\overset{\vartriangle}{\otimes}N\to\Gamma^{\overset{L}{\otimes}s+1}\otimes N$ restricts to an isomorphism $\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N)\to\Gamma\,\square_\Sigma G(s)\,\square_\Sigma N$.*

The proof is postponed to the appendix.

Corollary 5.5. *θ is filtration-preserving.*

Proof. This is a direct consequence of Lemma 5.4. □

Corollary 5.6. *There are isomorphisms*

$$\mathcal{N}\overset{\vartriangle}{D}{}^0_\Phi(G(s)\,\square_\Sigma N)=\Phi\overset{\vartriangle}{\otimes}(G(s)\,\square_\Sigma N)\xrightarrow{S\otimes\mathbb{1}}\Gamma\,\square_\Sigma G(s)\,\square_\Sigma N\xrightarrow{S^{-1}}\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N).$$

This gives the left vertical isomorphism in (5.1).

Our next task is to define the map β in (5.1) and show it is a chain equivalence. Most of the work for that is done in Lemma 5.8; the next lemma is helpful for that, and the result is summarized in Corollary/ Definition 5.9.

Lemma 5.7. *For fixed s, there is an isomorphism of complexes $F^s/F^{s+1}C_\Gamma(M,N) = E_0^{s,*}(M,N) \cong M \,\square_\Sigma E_0^{s,*}(M,\Sigma)\,\square_\Sigma N$.*

In particular, $E_0^{s,*}(M,N)$ only depends on the Σ-coaction on N, not the full Γ-coaction. We will abuse notation by writing $E_0^{s,*}(M,N)$ where N has a Σ-coaction and not a Γ-coaction (specifically, we do this for $N = G$).

Proof. We begin by showing that $F^s/F^{s+1}C_\Gamma(M,N)$ only depends on the Σ-coaction on N: given $x = m[\gamma_1|\dots|\gamma_n]\nu$ in $F^sC_\Gamma(M,N)$, the term $m[\gamma_1|\dots|\gamma_n|\nu']\nu''$ in $d(x)$ is in F^{s+1} if $\nu' \in G$. So, if we write $\psi(\nu) = \sum \underline{\nu'}|\underline{\nu''}$ for the coaction $\psi : N \to \Sigma \otimes N$, we can say that $d(x) \equiv \sum m[\gamma_1|\dots|\gamma_n|\underline{\nu'}]\underline{\nu''}$ in $F^s/F^{s+1}C_\Gamma^{n+1}(M,N)$.

We have an isomorphism $\psi : N \xrightarrow{\cong} \Sigma\,\square_\Sigma N$ of Σ-comodules, where the coaction on the right hand side is $\sigma \otimes \nu \mapsto \sigma' \otimes \sigma'' \otimes \nu$. This shows that the following diagram commutes

$$\begin{tikzcd} E_0^{s,t}(M,N) \arrow[r, "\psi"] \arrow[d, "d"'] & E_0^{s,t}(M,\Sigma)\,\square_\Sigma N \arrow[d, "d"] \\ E_0^{s,t+1}(M,N) \arrow[r, "\psi"] & E_0^{s,t+1}(M,\Sigma)\,\square_\Sigma N \end{tikzcd}$$

and so there is chain complex isomorphism $E_0^{s,*}(M,N) \cong E_0^{s,*}(M,\Sigma)\,\square_\Sigma N$ for every s. ☐

Lemma 5.8 ([Rav86, A1.3.16])**.** *The map*

$$\begin{aligned} \delta : E_0^{s-1,*}(M,G) &\longrightarrow E_0^{s,*}(M,\Sigma) \\ m[a_1|\dots|a_{s-1}]g &\longmapsto m[a_1|\dots|a_{s-1}|g']g''. \end{aligned}$$

is a chain equivalence, where $\sum g' \otimes g''$ is the image of $g \in G$ along the map $\Gamma \xrightarrow{\Delta} \Gamma\otimes\Gamma \to \Gamma\otimes\Sigma$.

Proof. We introduce a second filtration $\widetilde{F}^s$ which is defined only on $C_\Gamma(M,\Gamma)$:

$$\widetilde{F}^sC_\Gamma^n(M,\Gamma) = \{m[\gamma_1|\dots|\gamma_n]\gamma : \text{ at least } s \text{ of } \{\gamma,\gamma_1,\dots,\gamma_n\} \text{ are in } G\}^{2}.$$

There is a short exact sequence of complexes

$$0 \to F^s/\widetilde{F}^{s+1}C_\Gamma^*(M,\Gamma) \to \widetilde{F}^s/\widetilde{F}^{s+1}C_\Gamma^*(M,\Gamma) \to \widetilde{F}^s/F^sC_\Gamma^*(M,\Gamma) \to 0. \tag{5.2}$$

Unlike F, the new filtration $\widetilde{F}$ preserves the contracting homotopy on $C_\Gamma^*(M,\Gamma)$ given by $m[\gamma_1|\dots|\gamma_n]\gamma \mapsto \varepsilon(\gamma)m[\gamma_1|\dots|\gamma_{n-1}]\gamma_n$. So $\widetilde{F}^*C_\Gamma(M,\Gamma)$ is contractible, and so is the quotient complex $\widetilde{F}^*/\widetilde{F}^{*+1}C_\Gamma(M,\Gamma)$. The short exact sequence (5.2) gives rise to a long exact sequence in cohomology, and contractibility of the middle complex means that the boundary map

$$\delta : H^*(\widetilde{F}^s/F^sC_\Gamma^*(M,\Gamma)) \to H^*(F^s/\widetilde{F}^{s+1}C_\Gamma^{*+1}(M,\Gamma)) \tag{5.3}$$

is an isomorphism. We will identify $\widetilde{F}^s/F^sC_\Gamma^*(M,\Gamma)$ and $F^s/\widetilde{F}^{s+1}C_\Gamma^{*+1}(M,\Gamma)$ with the source and target of the desired map in the lemma statement, and show that δ can be lifted to a map on chains.

[2] This is off by one from the grading convention used in [Rav86, A1.3.16].

Levelwise, we can write

$$\widetilde{F}^{s+1}C^n_\Gamma(M,\Gamma) = F^{s+1}C^n_\Gamma(M,\Gamma) + F^sC^n_\Gamma(M,G) \tag{5.4}$$

but this is an abuse of notation—as G is not a Γ-comodule, $C^*_\Gamma(M,G)$ is not a complex (but we can still talk about $C^n_\Gamma(M,G) \subset C^n_\Gamma(M,\Gamma)$ as a sub-module). We will see that this will cease to be a problem upon passing to the associated graded E_0.

For each n, we have

$$\begin{aligned} \widetilde{F}^s/F^sC^*_\Gamma(M,\Gamma) &\cong \big(F^sC^n_\Gamma(M,\Gamma) + F^{s-1}C^n_\Gamma(M,G)\big)/F^sC^n_\Gamma(M,\Gamma) \\ &\cong F^{s-1}/F^sC^n_\Gamma(M,G) \end{aligned} \tag{5.5}$$

$$\begin{aligned} F^s/\widetilde{F}^{s+1}C^{*+1}_\Gamma(M,\Gamma) &\cong F^sC^n_\Gamma(M,\Gamma)/\big(F^{s+1}C^n_\Gamma(M,\Gamma) + F^sC^n_\Gamma(M,G)\big) \\ &= \big(F^sC^n_\Gamma(M,\Gamma)/F^{s+1}C^n_\Gamma(M,\Gamma)\big)/F^sC^n_\Gamma(M,G) \\ &\cong F^s/F^{s+1}C^n_\Gamma(M,\Sigma). \end{aligned} \tag{5.6}$$

While $F^sC^*_\Gamma(M,G)$ is not a complex, Lemma 5.8 shows that $F^{s-1}/F^sC^*_\Gamma(M,G)$ is a complex, and the isomorphisms $\widetilde{F}^s/F^sC^n_\Gamma(M,\Gamma) \cong F^{s-1}/F^sC^n_\Gamma(M,G)$ and $F^s/\widetilde{F}^{s+1}C^{*+1}_\Gamma(M,\Gamma) \cong F^s/F^{s+1}C^n_\Gamma(M,\Sigma)$ extend to isomorphisms of complexes. I claim the boundary map (5.3) can be identified as the map

$$\begin{aligned} H^*(F^{s-1}/F^sC^*_\Gamma(M,G)) &\xrightarrow{\delta} H^*(F^s/F^{s+1}C^*_\Gamma(\Sigma,N)) \\ m[a_1|\dots|a_n]g &\longmapsto \sum m[a_1|\dots|a_n|\underline{\underline{g}}']\underline{\underline{g}}'' \end{aligned}$$

where $\sum \underline{\underline{g}}'|\underline{\underline{g}}''$ is the image of g under the right Σ-coaction. As the boundary map, this is just given by the cobar differential, but in order for $m[a_1|\dots|a_n]g$ to be a cycle, the sum of all the terms except the one in the formula for δ is in $F^sC^{n+1}_\Gamma(M,\Gamma)$. Furthermore, I claim this can be extended to a map on chains:

$$\begin{aligned} \delta : F^{s-1}/F^sC^*_\Gamma(M,G) &\longrightarrow F^s/F^{s+1}C^*_\Gamma(\Sigma,N) \\ m[a_1|\dots|a_n]g &\longmapsto \sum m[a_1|\dots|a_n|\underline{\underline{g}}']\underline{\underline{g}}''. \end{aligned}$$

It suffices to show that the image of $m[a_1|\dots|a_n]g \in F^sC^*_\Gamma(M,G)$ lies in $F^{s+1}C^*_\Gamma(M,\Sigma)$, and this holds because $\underline{\underline{g}}''$ is the $(s+1)^{st}$ term in G. $\square$

Using Lemma 5.7, we can write this as a map

$$\begin{aligned} &E_0^{s-1,*}(M,G(s)\,\Box_\Sigma N) \\ &= E_0^{s-1,*}(M,G)\,\Box_\Sigma N \xrightarrow{\delta} E_0^{s,*}(M,\Sigma)\,\Box_\Sigma N = E_0^{s,*}(M,\Sigma\,\Box_\Sigma N) \xrightarrow{\cong} E_0^{s,*}(M,N) \\ &m[a_1|\dots|a_n]g|\nu \longmapsto \sum m[a_1|\dots|a_n\underline{\underline{g}}']\underline{\underline{g}}''\nu \longmapsto \sum m[a_1|\dots|a_n|g]\nu. \end{aligned}$$

Corollary/ Definition 5.9. Iterating δ gives rise to a chain equivalence

$$E_0^{0,*}(M,G(s)\,\Box_\Sigma N) \xrightarrow{\delta} E_0^{1,*}(M,G(s-1)\,\Box_\Sigma N) \xrightarrow{\delta} \dots \xrightarrow{\delta} E_0^{s,*}(M,N)$$

sending

$$m[a_1|\dots|a_n]g_1|\dots|g_s|\nu \longmapsto m[a_1|\dots|a_n|g_1|\dots|g_s]\nu.$$

Let β denote this composition.

It is now easy to see that (5.1) commutes. Our final task is to show (4) after (5.1); first we need an easy lemma.

Lemma 5.10. *Let Γ be a Hopf algebra and M be an Γ-comodule. Then the coaction $\psi : M \to \Gamma \,\square_\Gamma M$ is an isomorphism with inverse $T : \Gamma \,\square_\Gamma M \to M$ sending $a \otimes m \mapsto \varepsilon(a)m$.*

Proof. First we check that the coaction ψ lands in the cotensor product $\Gamma \,\square_\Gamma M$: we need to check that $\psi(m) = \sum m' \otimes m''$ lands in the kernel of $\Delta \otimes \mathbb{1} - \mathbb{1} \otimes \psi : \Gamma \otimes M \to \Gamma \otimes \Gamma \otimes M$. But $\sum (m')' \otimes (m')'' \otimes m'' - \sum m' \otimes (m'')' \otimes (m'')'' = 0$ by coassociativity.

Next, we check that T is an inverse. We have $T\psi(m) = T(\sum m' \otimes m'') = \sum \varepsilon(m')m''$. This is equal to m by general Hopf algebra properties. For the other composition, we have $\psi T(a \otimes m) = \sum \varepsilon(a) m' \otimes m''$. Since $a \otimes m$ is in $\Gamma \,\square_\Gamma M$, we have $\sum a \otimes m' \otimes m'' = \sum a' \otimes a'' \otimes m$. Applying $\varepsilon \cdot \mathbb{1} \otimes \mathbb{1}$ to this, we have $\sum \varepsilon(a) m' \otimes m'' = \sum \varepsilon(a') a'' \otimes m = \sum a \otimes m$. So $\psi \circ T = \mathbb{1}$. $\square$

Proposition 5.11. *θ induces an isomorphism $\overset{\vartriangle}{E}{}_1^{0,*} \to E_1^{0,*}$.*

Proof. First notice that we have an isomorphism

$$F^0/F^1(M \otimes \Gamma^{\otimes t} \otimes N) \cong M \otimes \Sigma^{\otimes t} \otimes N$$

since $m[\gamma_1|\dots|\gamma_s]\nu$ is in F^1 if any of the γ_i's are in G. On the other hand, we have

$$H^*(\overset{\vartriangle}{E}{}_1^{0,*}) = H^*((M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\Phi \overset{\vartriangle}{\otimes} N)) = \mathrm{Cotor}^*_\Gamma(M, \Phi \overset{\vartriangle}{\otimes} N) \cong \mathrm{Cotor}^*_\Sigma(M, N)$$

by the change of rings isomorphism. In the rest of this proof we make this isomorphism more explicit, enough to see that the isomorphism $\overset{\vartriangle}{E}{}_1^{0,*} \to E_1^{0,1}$ is induced by θ.

Since the shear map $\Gamma \overset{\vartriangle}{\otimes} \Gamma \to \Gamma \overset{L}{\otimes} \Gamma$ commutes with the map $\Gamma \otimes \Gamma \xrightarrow{q \otimes q} \Sigma \otimes \Sigma$, we have a commutative diagram

$$\begin{tikzcd}
(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\Phi \overset{\vartriangle}{\otimes} N) \arrow[d, "\mathbb{1}^{t+2} \otimes S"'] & & \\
(M \overset{\vartriangle}{\otimes} \Gamma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\Gamma \,\square_\Sigma N) \arrow[r, "\mathbb{1} \otimes q^{t+1} \otimes \mathbb{1}^2"] \arrow[d, "S_c^t \otimes \varepsilon \cdot \varepsilon \cdot \mathbb{1}"] & (M \overset{\vartriangle}{\otimes} \Sigma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Gamma (\Gamma \,\square_\Sigma N) \arrow[r, "\cong"] & (M \overset{\vartriangle}{\otimes} \Sigma^{\overset{\vartriangle}{\otimes} t+1}) \,\square_\Sigma N \arrow[d, "S_c^{t+1} \otimes \varepsilon \cdot \mathbb{1}"] \\
F^0/F^1(M \otimes \Gamma^t \otimes N) \arrow[rr, "\cong"] & & M \otimes \Sigma^t \otimes N
\end{tikzcd}$$

Note that the left vertical composition is θ, by definition. The middle horizontal composition is the chain equivalence inducing the change of rings isomorphism $\mathrm{Cotor}^*_\Gamma(M, \Gamma \,\square_\Sigma N) \cong \mathrm{Cotor}_\Sigma(M, N)$. By Lemma 5.10, the right vertical map is $S_c^{t+1} \otimes T$, an isomorphism. So the bottom left vertical map is a chain equivalence. The top left vertical map is an isomorphism, so θ is a chain equivalence. $\square$

Appendix: The cobar complex and the shear isomorphism

In this appendix we record some technical facts about the iterated shear isomorphism that are needed for the comparison proof in section 5.

First we need notation for the iterated coproduct.

Definition A.1. For a Hopf algebra Γ and Γ-comodule M, let Δ^n denote the iterated coproduct

$$\Delta^n : \Gamma \xrightarrow{\Delta} \Gamma^{\otimes 2} \xrightarrow{\Delta} \dots \xrightarrow{\Delta} \Gamma^{\otimes n+1}$$

and let ψ^n denote the iterated coaction $M \xrightarrow{\psi^n} \Gamma^{\otimes n} \otimes M$. Write $\sum m_{(1)}|\dots|m_{(n+1)} := \psi^n(m)$ and $\sum \gamma_{(1)}|\dots|\gamma_{(n+1)} := \Delta^n(\gamma)$. (Note that this notation is well-defined because of coassociativity.)

For example, $\Delta(\gamma) = \sum \gamma'|\gamma'' = \sum \gamma_{(1)}|\gamma_{(2)}$, and $\sum \Delta(\gamma_{(1)})|\gamma_{(2)} = \sum \gamma_{(1)}|\gamma_{(2)}|\gamma_{(3)}$.

Lemma A.2. *The iterated shear isomorphism* $S^n : \Gamma^{\overset{\Delta}{\otimes} n} \overset{\Delta}{\otimes} M \to \Gamma \overset{L}{\otimes} \Gamma^{\otimes n-1} \otimes M$ *is given by*

$$S^n : x_1|\dots|x_n|m \mapsto \sum x_{1(1)}x_{2(1)}\dots x_{n(1)}m_{(1)}|x_{2(2)}\dots x_{n(2)}m_{(2)}|x_{3(3)}\dots x_{n(3)}m_{(3)}|\dots|m_{(n+1)}.$$

The iterated shear isomorphism $S_c^n : M \overset{\Delta}{\otimes} \Gamma^{\overset{\Delta}{\otimes} n} \to M \otimes \Gamma^{\otimes n-1} \overset{R}{\otimes} \Gamma$ *is given by*

$$S_c^n : m|x_n|\dots|x_1 \mapsto \sum m_{(1)}|m_{(2)}x_{n(1)}|m_{(3)}x_{n(2)}x_{n-1(1)}|\dots|m_{(n+1)}x_{n(n)}x_{n-1(n-1)}\dots x_{2(2)}x_1.$$

Proof. We prove just the first statement, as the second is analogous. Use induction on n. If $n = 2$ this is true by definition of S. Now suppose S^{n-1} is given by the formula above. We can write S^n as the composition

$$\Gamma^{\overset{\Delta}{\otimes} n} \overset{\Delta}{\otimes} M \xrightarrow{S^{n-1}} \Gamma \overset{\Delta}{\otimes} (\Gamma^{\overset{L}{\otimes} n-1} \otimes M) \xrightarrow{S} \Gamma \overset{L}{\otimes} (\Gamma^{\overset{L}{\otimes} n-1} \otimes M)$$

and by the inductive hypothesis the first map sends

$$x_1|x_2|\dots|x_n|m \mapsto \sum x_1|x_{2(1)}x_{3(1)}\dots x_{n(1)}m_{(1)}|x_{3(2)}\dots x_{n(2)}m_{(2)}|\dots|m_{(n)}.$$

If we write this as $x_1|y$, then the second map sends this to $\sum x_1 y_{(1)}|y_{(2)}$; remembering that the coaction on y just comes from the first component, this is:

$$\sum x_1 x_{2(1)}x_{3(1)}\dots x_{n(1)}m_{(1)}|x_{2(2)}x_{3(2)}\dots x_{n(2)}m_{(2)}|x_{3(3)}\dots x_{n(3)}m_{(3)}|\dots|m_{(n+1)}. \qquad \square$$

Lemma A.3. *The iterated inverse shear isomorphism* $S^{-n} : \Gamma \overset{L}{\otimes} \Gamma^{\otimes n-1} \otimes M \to \Gamma^{\overset{\Delta}{\otimes} n} \overset{\Delta}{\otimes} M$ *is given by*

$$S^{-n} : x_1|\dots|x_n|m \mapsto \sum x_1 c(x_2')|x_2''c(x_3')|x_3''c(x_4')|\dots|x_n''c(m')|m''.$$

The iterated inverse shear isomorphism $S_c^{-n} : M \otimes \Gamma^{\otimes n-1} \overset{R}{\otimes} \Gamma \to M \overset{\Delta}{\otimes} \Gamma^{\overset{\Delta}{\otimes} n}$ *is given by*

$$S_c^{-n} : m|x_n|\dots|x_1 \mapsto \sum m'|c(m'')x_n'|c(x_n')x_{n-1}'|\dots|c(x_2')x_1.$$

Proof. Again we only prove the first statement, and again this is by induction on n. If $n = 1$, this is the definition of S^{-1} in Lemma 3.3.

Assume the formula holds for $n-1$. Write S^{-n} as the composition

$$\Gamma \overset{L}{\otimes} (\Gamma^{\overset{L}{\otimes} n-1} \otimes M) \xrightarrow{S^{-(n-1)}} \Gamma \overset{L}{\otimes} (\Gamma^{\overset{\triangle}{\otimes} n} \overset{\triangle}{\otimes} M) \xrightarrow{S^{-1}} \Gamma \overset{\triangle}{\otimes} (\Gamma^{\overset{\triangle}{\otimes} n} \overset{\triangle}{\otimes} M)$$

and by the inductive hypothesis the first map sends

$$x_1|x_2|\dots|x_n|m \mapsto \sum x_1|x_2c(x_3')|x_3''c(x_4')|\dots|x_n''c(m')|m''.$$

If we write this as $x_1|y$, then the second map sends this to $\sum x_1 c(y_{(1)})|y_{(2)}$, which is

$$\begin{aligned}
&\sum x_1 c\big((x_2c(x_3')x_3''c(x_4')\dots x_n''c(m')m'')'\big)|x_2''c(x_3')''|(x_3'')''c(x_4')''|\dots|(x_n'')''c(m')''|(m'')'' \\
&\quad= \sum x_1 c\big(x_{2(1)}c(x_{3(2)})x_{3(3)}c(x_{4(2)})x_{4(3)}\dots c(m_{(2)})m_{(3)}\big)|x_{2(2)}c(x_{3(1)})|x_{3(4)}c(x_{4(1)})| \\
&\qquad\qquad \dots|x_{n(4)}c(m_{(1)})|m_{(4)} \\
&\quad= \sum x_1 c\big(x_{2(1)}\varepsilon(x_{3(2)}\dots x_{n(2)}m_{(2)})\big)|x_{2(2)}c(x_{3(1)})|x_{3(3)}c(x_{4(1)})| \\
&\qquad\qquad \dots|x_{n(3)}c(m_{(1)})|m_{(3)} \\
&\quad= \sum x_1 c(x_{2(1)})|x_{2(2)}c(x_{3(1)})|x_{3(2)}c(x_{4(1)})|\dots|x_{n(2)}c(m_{(1)})|m_{(2)}.
\end{aligned}$$

Here the first equality uses the fact that $\sum c(x')|c(x'') = \sum c(x)''|c(x)'$, the second uses the fact that $c(x')x'' = \varepsilon(x)$, and the third uses the fact that $\sum \varepsilon(x')|x'' = \sum 1|x$. $\square$

Lemma A.4. *The iterated shear isomorphism $S : \Gamma^{\overset{\triangle}{\otimes} *+1} \overset{\triangle}{\otimes} N \to \Gamma^{\overset{L}{\otimes} *+1} \otimes N$ restricts to an isomorphism of chain complexes*

$$S : \Phi^{\overset{\triangle}{\otimes} *+1} \overset{\triangle}{\otimes} N \to \underbrace{\Gamma \,\square_\Sigma \dots \square_\Sigma \Gamma}_{*+1} \,\square_\Sigma N. \tag{A.7}$$

Proof. For any Γ-comodule M, by Lemma 3.4 the shear isomorphism gives an isomorphism $\Phi \overset{\triangle}{\otimes} N \xrightarrow{\cong} \Gamma \,\square_\Sigma N$, and iterating the shear map gives an isomorphism $\Phi^{\overset{\triangle}{\otimes} s+1} \overset{\triangle}{\otimes} N \xrightarrow{\cong} \underbrace{\Gamma \,\square_\Sigma \dots \square_\Sigma \Gamma}_{s+1} \,\square_\Sigma N$. $\square$

Proof of Lemma 5.4. It suffices to check the inclusions $S^{-1}(\Gamma \,\square_\Sigma G(s) \,\square_\Sigma N) \subset \mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N)$ and $S(\mathcal{N}\overset{\triangle}{D}{}^s_\Phi(M)) \subset \Gamma \,\square_\Sigma G(s) \,\square_\Sigma N$. For the first inclusion, use Lemma A.3 to observe that

$$S^{-1}(a|g_1|\dots|g_s|n) = \sum ac(g_1')|g_1''c(g_2')|g_2''c(g_3')|\dots|g_s''c(n')|n'' \tag{A.8}$$

and for $1 \le i \le s$ we have

$$\begin{aligned}
\mu_i(\textstyle\sum ac(g_1')|g_1''c(g_2')|g_2''c(g_3')|\dots|g_s''c(n')|n'') &= \textstyle\sum ac(g_1')|g_1''c(g_2')|\dots|g_{i-1}''c(g_i')g_i''c(g_{i+1}')|\dots|n'' \\
&= \textstyle\sum ac(g_1')|g_1''c(g_2')|\dots|g_{i-1}''\varepsilon(g_i)c(g_{i+1}')|\dots|n''
\end{aligned}$$

which is zero since $g_i \in G$ (and so $g_i \notin k$). This shows (A.8) is in $\mathcal{N}\overset{\triangle}{D}{}^s_\Phi(N)$.

For the other direction, let $x_0|\dots|x_s|n \in \mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(N) \subset \Phi^{\overset{\vartriangle}{\otimes}s+1} \otimes N$. By Lemma A.2, we have

$$S(x_0|\dots|x_s|n) = \sum x_{0(1)}x_{1(1)}\dots x_{s(1)}n_{(1)}|x_{1(2)}\dots x_{s(2)}n_{(2)}|x_{2(3)}\dots n_{(3)}|\dots|n_{(s+2)}. \tag{A.9}$$

The goal is to show that each component $x_{k(k+1)}x_{k+1(k+1)}\dots x_{s(k+1)}n_{(k+1)}$ is in G for $1 \leq k \leq s$. Since Φ is a left Γ-comodule, if $x \in \Phi$ then $\Delta^j(x) = x_{(1)}|\dots|x_{(j)}$ and so $x_{(j)} \in \Phi$. By assumption, all of the x_i's are in Φ, and since (A.9) involves the iterated coproduct $\Delta^{i+1}(x_i) = x_{i(1)}|\dots|x_{i(i+1)}$ for every i, we have $x_{i(i+1)} \in \Phi$. If we could guarantee $x_{k(k+1)}$ were in $\overline{\Phi}$, then we would be done (since $G = \overline{\Phi}\Gamma$). Instead, we show that the terms where $x_{k(k+1)} = 1$ sum to zero.

The terms where $x_{k(k+1)} = 1$ are:

$$\begin{aligned}\sum x_{0(1)}x_{1(1)}\dots x_{k-1(1)}x_{k(1)}\dots x_{s(1)}n_{(1)}|\dots|x_{k-2(k-1)}x_{k-1(k-1)}x_{k(k-1)}\cdots \\ |x_{k-1(k)}x_{k(k)}x_{k+1(k)}\cdots|x_{k+1(k+1)}x_{k+2(k+1)}\cdots|\dots|n_{(s+2)}.\end{aligned} \tag{A.10}$$

The assumption that $x_0|\dots|x_s$ is in $\mathcal{N}\overset{\vartriangle}{D}{}^s_\Phi(M)$ implies that $x_{k-1}x_k = 0$ (this is where we use the fact that $k \geq 1$), and hence

$$0 = \Delta^k(x_{k-1}x_k) = \sum x_{k-1(1)}x_{k(1)}|\dots|x_{k-1(k-1)}x_{k(k-1)}|x_{k-1(k)}x_{k(k)}.$$

Observing how $\Delta(x_{k-1}x_k)$ is embedded in (A.10), we have (A.10) $= 0$. $\square$

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