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C_2-EQUIVARIANT AND $\mathbb{R}$-MOTIVIC STABLE STEMS II

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ABSTRACT. We show that the stable homotopy groups of the C_2-equivariant sphere spectrum and the $\mathbb{R}$-motivic sphere spectrum are isomorphic in a range. This result supersedes previous work of Dugger and the third author.

1. INTRODUCTION

This article is part of an ongoing project to make explicit computations of stable homotopy groups in the $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, C_2-equivariant, and classical stable homotopy theories, as depicted in the diagram

$$\begin{array}{ccc} \mathbb{R}\text{-motivic} & \xrightarrow{\text{extension of scalars}} & \mathbb{C}\text{-motivic} \\ {\scriptstyle\text{realization}}\downarrow & & \downarrow{\scriptstyle\text{realization}} \\ C_2\text{-equivariant} & \xrightarrow[\text{forgetful}]{} & \text{classical} \end{array} \tag{1.1}$$

The vertical arrows labelled "realization" refer to the Betti realization functors that take a variety over $\mathbb{C}$ (resp., over $\mathbb{R}$) to the space (resp., C_2-equivariant space) of $\mathbb{C}$-valued points. The horizontal arrow labelled "extension of scalars" refers to the functor that takes a variety over $\mathbb{R}$ and views it as a variety over $\mathbb{C}$. The horizontal arrow labelled "forgetful" refers to the functor that takes a C_2-equivariant object to its underlying nonequivariant object.

The goal of this paper is to study the top horizontal arrow in diagram (1.1). We show that there is an isomorphism

$$\pi^{\mathbb{R}}_{*,*} \to \pi^{C_2}_{*,*}$$

in a range of degrees. Here $\pi^{\mathbb{R}}_{*,*}$ are the stable homotopy groups of the $\mathbb{R}$-motivic sphere $S^{0,0}_{\mathbb{R}}$ completed at 2 and η, and $\pi^{C_2}_{*,*}$ are the stable homotopy groups of the C_2-equivariant sphere $S^0_{C_2}$. For the purposes of this paper, $\pi^{\mathbb{R}}_{*,*}$ and $\pi^{C_2}_{*,*}$ can be defined as the targets of the $\mathbb{R}$-motivic and C_2-equivariant Adams spectral sequences; in particular, these are 2-complete homotopy groups. The map is induced by equivariant Betti realization that takes a variety over $\mathbb{R}$ to the space of $\mathbb{C}$-valued points, equipped with the conjugation action [13, Section 3.3], [8, Section 4.4]. In practice, information typically flows from source to target along the isomorphism. Even though $\pi^{\mathbb{R}}_{*,*}$ is highly nontrivial [1], [3], it is somewhat easier to compute than $\pi^{C_2}_{*,*}$.

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See the introduction of [4] for a more thorough discussion of the objects and categories under consideration. We assume that the reader is familiar with the motivic and C_2-equivariant Adams spectral sequences. Relevant details appear in [1], [2], [6], [9].

Our work is a natural sequel to the paper [4], which establishes an isomorphism between $\mathbb{R}$-motivic and C_2-equivariant stable homotopy groups in a strictly smaller range. The method of [4] is to compare cobar complexes, which then yields a comparison of Adams E_2-pages. In turn, this leads to a comparison of stable homotopy groups. While the cobar complex has good formal properties, it is a wasteful construction in the sense that it is much larger than needed to compute Adams E_2-pages. The approach of this article, in effect, ignores large parts of the cobar complexes that do not contribute to Adams E_2-pages.

More specifically, we will compare the $\mathbb{R}$-motivic and C_2-equivariant ρ-Bockstein spectral sequences [1], [3], [6] that converge to the $\mathbb{R}$-motivic and C_2-equivariant Adams E_2-pages, respectively. We will show that these ρ-Bockstein spectral sequences are isomorphic in a range. As in [4], this implies that the Adams E_2-pages are isomorphic in a range, which further implies that the stable homotopy groups are isomorphic in a range as well.

Theorem 1.1. *The Betti realization map*

$$\pi^{\mathbb{R}}_{s,w} \to \pi^{C_2}_{s,w}$$

(1) *is an isomorphism if* $2w - s < 5$ *and* $(s, w) \neq (0, 2)$,
(2) *is an injection if* $2w - s = 5$.

Proof. Proposition 4.2 shows that Betti realization induces an isomorphism between the $\mathbb{R}$-motivic and C_2-equivariant Adams E_∞-pages when $2w - s < 5$ and $s \neq 0$. In other words, Betti realization maps the associated graded group of $\pi^{\mathbb{R}}_{s,w}$ isomorphically onto the associated graded group of $\pi^{C_2}_{s,w}$. It follows that Betti realization is an isomorphism before passing to associated graded groups. This establishes part (1).

The proof of part (2) is essentially the same. □

Theorem 1.1 is stated in terms of the stem s and the weight w. In some situations, it is more convenient to work with the stem s and the coweight $s - w$. In those terms, Theorem 1.1 says that Betti realization:

(1) is an isomorphism if $s < 2(s - w) + 5$ and $(s, s - w) \neq (0, -2)$,
(2) is an injection if $s = 2(s - w) + 5$.

In order to further illustrate Theorem 1.1, Figures 1–4 show C_2-equivariant Adams charts in coweights 0 through 3. These charts show the range of stems in which $\mathbb{R}$-motivic and $\mathbb{C}_2$-equivariant stable homotopy groups are isomorphic. The elements in green on the far right of each chart are the first C_2-equivariant classes that have no $\mathbb{R}$-motivic analogues. Explanations for the computations in these charts will appear elsewhere. Here is a key for reading the charts:

- Vertical lines indicate multiplications by h_0.
- Horizontal lines indicate multiplications by ρ.
- Lines of slope 1 indicate multiplications by h_1.
- Arrows indicate infinite sequences of elements that are related by multiplications.

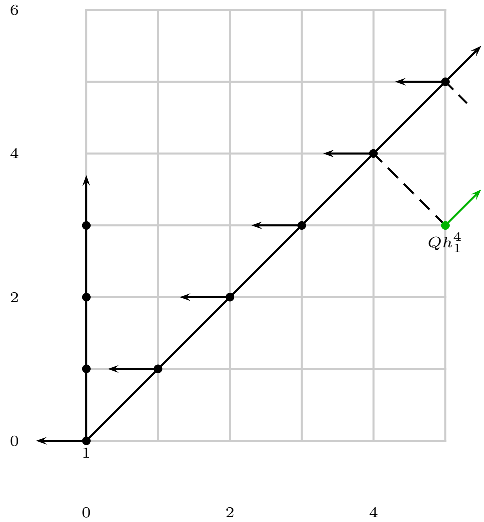

FIGURE 1. C_2-equivariant Adams E_∞-page in coweight 0

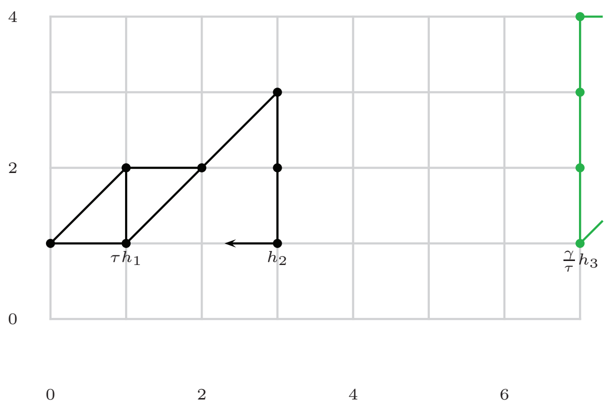

FIGURE 2. C_2-equivariant Adams E_∞-page in coweight 1

- Vertical dashed lines indicate hidden h_0 extensions.
- Dashed lines of negative slope indicate hidden ρ extensions.

Figure 5 describes some of the global structure of $\pi_{*,*}^{C_2}$ in graphical form; see [4, Section 1.2] for more discussion. The groups $\pi_{s,w}^{C_2}$ can be separated into different regions, with qualitatively different behavior in each region:

- zero: The groups in this region are all zero.
- $\mathbb{R}$-motivic: The groups in this region are isomorphic to $\pi_{*,*}^{\mathbb{R}}$, according to Theorem 1.1.

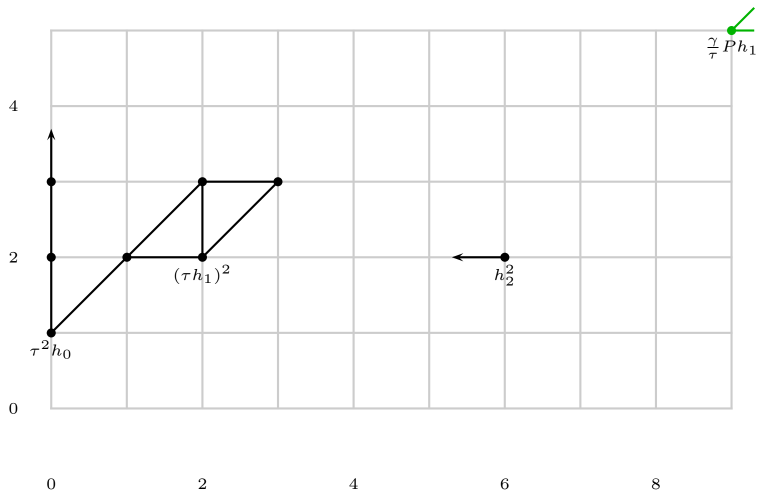

FIGURE 3. C_2-equivariant Adams E_∞-page in coweight 2

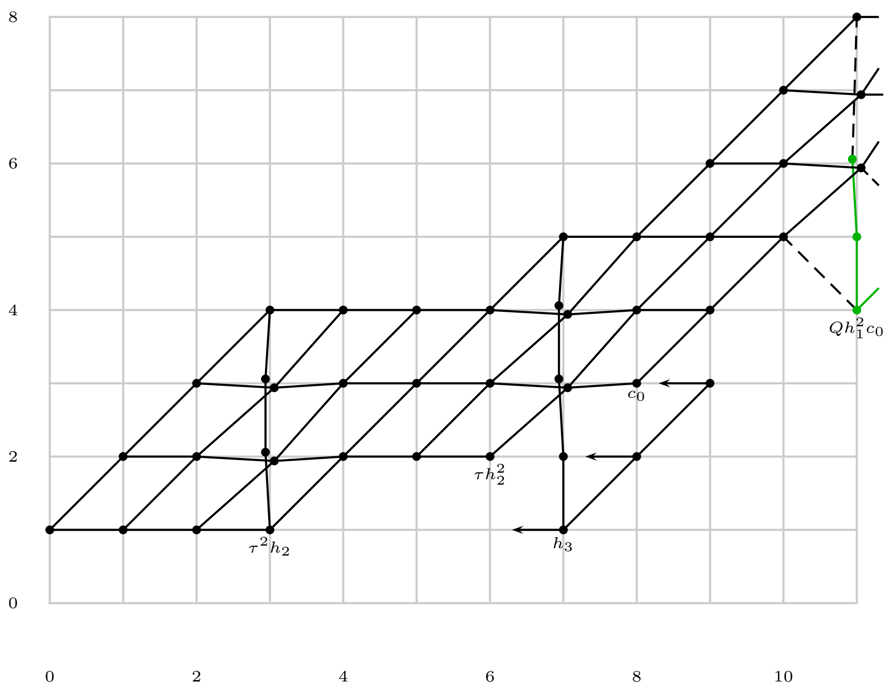

FIGURE 4. C_2-equivariant Adams E_∞-page in coweight 3

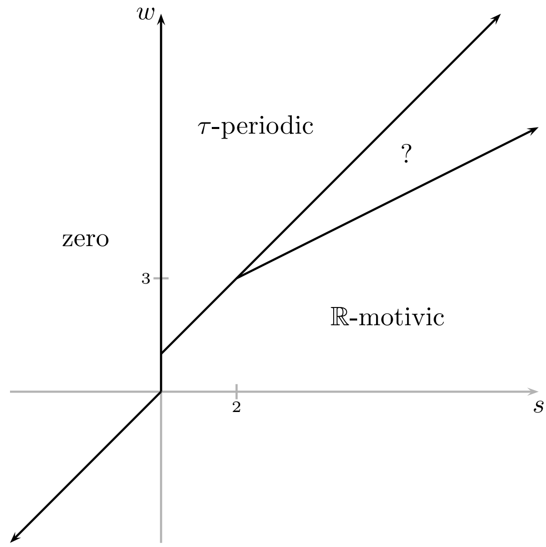

FIGURE 5. The structure of $\pi^{C_2}_{s,w}$

- τ-periodic: The groups in this region display a certain type of periodicity. In particular, they can be deduced from groups in the $\mathbb{R}$-motivic region.
- ?: The groups in this region are more complicated, with the occurrence of purely equivariant phenomena.

Our result is sharp in the following sense. In Example 4.3, we will describe an infinite family of elements in $\pi^{C_2}_{*,*}$ lying just outside the range under consideration that are not in the image of Betti realization.

We have at least two motivations for proving Theorem 1.1. First, this theorem is a self-evidently useful tool in the ongoing program to carry out explicit computations of motivic and equivariant stable homotopy groups. Second, the theorem is used in [1] to compute some classical Mahowald invariants from detailed information about $\mathbb{R}$-motivic stable homotopy groups.

1.2. **Notation.** We write $\mathrm{Ext}_{\mathbb{C}}$ (resp., $\mathrm{Ext}_{\mathbb{R}}$, Ext_{C_2}) for the $\mathbb{C}$-motivic (resp., $\mathbb{R}$-motivic, C_2-equivariant) Ext groups that serve as the E_2 page of the $\mathbb{C}$-motivic (resp., $\mathbb{R}$-motivic, C_2-equivariant) Adams spectral sequence. We grade these Ext groups in the form (s, f, w), where s is the stem (i.e., the total degree minus the homological degree), f is the Adams filtration (i.e., the homological degree), and w is the (motivic or equivariant) weight.

2. THE $\mathbb{C}$-MOTIVIC COFIBER OF τ

We recall some needed facts from $\mathbb{C}$-motivic stable homotopy theory [9], [10].

Let S/τ be the cofiber of τ in the 2-complete $\mathbb{C}$-motivic stable homotopy category, and let $\mathrm{Ext}_{\mathbb{C}}(S/\tau)$ be the E_2-page of the Adams spectral sequence that converges to the homotopy groups of S/τ. The cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} S/\tau \xrightarrow{p} S^{1,-1}$$

induces a long exact sequence
(2.1)
$$\cdots \longrightarrow \mathrm{Ext}_{\mathbb{C}}^{s,f,w+1} \xrightarrow{\tau} \mathrm{Ext}_{\mathbb{C}}^{s,f,w} \xrightarrow{i} \mathrm{Ext}_{\mathbb{C}}^{s,f,w}(S/\tau) \xrightarrow{p} \mathrm{Ext}_{\mathbb{C}}^{s-1,f+1,w+1} \longrightarrow \cdots.$$

Lemma 2.1. *Let $2w - s < 1$. The group $\mathrm{Ext}_{\mathbb{C}}^{s,f,w}(S/\tau)$ is:*

(1) *a copy of $\mathbb{F}_2$, generated by $i(h_0^f)$, if $s = 0$ and $w = 0$,*
(2) *zero otherwise.*

Proof. The algebraic Novikov spectral sequence converges to the Adams–Novikov E_2-page $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ [11], [12]. The group $\mathrm{Ext}_{\mathbb{C}}^{s,f,w}(S/\tau)$ is isomorphic to a part of the algebraic Novikov E_2-page that contributes to the Adams–Novikov E_2-page in stem s and filtration $2w-s$ [5, Theorem 1.14]. The result follows from an elementary analysis of the algebraic Novikov E_2-page in Adams–Novikov filtration zero. □

The vanishing result of Lemma 2.1 can be applied to the long exact sequence (2.1) to obtain information about multiplication by τ on $\mathrm{Ext}_{\mathbb{C}}$.

Proposition 2.2. *Let x be a nonzero element of $\mathrm{Ext}_{\mathbb{C}}$ of degree (s, f, w).*

(1) *If x is not divisible by τ, then $2w - s \geq 1$, or $x = h_0^f$.*
(2) *If x is annihilated by τ, then $2w - s \geq 4$.*

Proof. For part (1), the image $i(x)$ of x in $\mathrm{Ext}_{\mathbb{C}}(S/\tau)$ must be nonzero. By Lemma 2.1, if $2w - s$ is less than 1, then x must be h_0^f.

For part (2), x must lie in the image of p. The preimage of x has degree $(s + 1, f - 1, w - 1)$. Lemma 2.1 implies that
$$2(w-1) - (s+1) = 2w - s - 3 \geq 1.$$ □

3. The ρ-Bockstein spectral sequence

Recall from [6, Section 2] that Ext_{C_2} splits as $\mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{NC}$, where Ext_{NC} is associated to the "negative cone" in the C_2-equivariant cohomology of a point. Betti realization induces the natural inclusion

(3.1) $$\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{\mathbb{R}} \oplus \mathrm{Ext}_{NC}\,.$$

In order to obtain an isomorphism $\mathrm{Ext}_{\mathbb{R}} \to \mathrm{Ext}_{C_2}$ in a range of degrees, we must show that Ext_{NC} vanishes in that range.

The groups Ext_{NC} can be computed by a ρ-Bockstein spectral sequence, denoted E^- in [6]. The E_1^--page of this spectral sequence contains elements of two types.

First, there are elements of the form $\frac{\gamma}{\rho^a\tau^b}x$, where $0 \leq a$, $1 \leq b$, and x is an element of $\mathrm{Ext}_{\mathbb{C}}$ that is τ-free and not divisible by τ. If x has degree (s, f, w) in $\mathrm{Ext}_{\mathbb{C}}$, then $\frac{\gamma}{\rho^a\tau^b}x$ has degree $(s + a, f, w + a + b + 1)$.

The second type of element in E_1^- is of the form $\frac{Q}{\rho^a\tau^b}x$, where $0 \leq a$, $0 \leq b \leq k$, and x is an element of $\mathrm{Ext}_{\mathbb{C}}$ that is annihilated by τ and is divisible by τ^k but not by τ^{k+1}. If x has degree (s, f, w) in $\mathrm{Ext}_{\mathbb{C}}$, then $\frac{Q}{\rho^a\tau^b}x$ has degree $(s + a + 1, f - 1, w + a + b + 1)$.

Lemma 3.1.

(1) *If $\frac{\gamma}{\rho^a\tau^b}x$ is a nonzero element of E_1^- with degree (s, f, w), then $x = h_0^f$ or $2w - s \geq a + 2b + 3$.*

(2) *If* $\frac{Q}{\rho^a \tau^b} x$ *is a nonzero element of* E_1^- *with degree* (s, f, w)*, then*

$$2w - s \geq a + 2b + 5.$$

Proof. For part (1), the element x has degree $(s - a, f, w - a - b - 1)$. Since x is not divisible by τ, Proposition 2.2 implies that $x = h_0^f$ or

$$2(w - a - b - 1) - (s - a) = 2w - s - a - 2b - 2 \geq 1.$$

For part (2), the element x has degree $(s - a - 1, f + 1, w - a - b - 1)$. Since x is annihilated by τ, Proposition 2.2 implies that

$$2(w - a - b - 1) - (s - a - 1) = 2w - s - a - 2b - 1 \geq 4. \qquad \square$$

Lemma 3.2. *Amongst elements of the form* $\frac{\gamma}{\rho^a \tau^b} h_0^f$ *in* E_1^-*, the only nonzero permanent cycles are* $\frac{\gamma}{\tau^{2k+1}} h_0^f$ *in degree* $(0, f, 2k+2)$ *for all* $k \geq 0$.

Proof. As in [6, Proposition 7.7] or [7, Lemma 4.1], there are Bockstein differentials

$$d_1 \left(\frac{\gamma}{\rho \tau^{2k+1}} h_0^f \right) = \frac{\gamma}{\tau^{2k+2}} h_0^{f+1}. \qquad \square$$

Proposition 3.3. *Let* y *be a nonzero element of* Ext_{NC} *of degree* (s, f, w)*. Then* y *equals* $\frac{\gamma}{\tau} h_0^f$*, or* $2w - s \geq 5$.

Proof. The element y is represented by an element of the ρ-Bockstein E_1^--page. If y is of the form $\frac{\gamma}{\rho^a \tau^b} x$, then $0 \leq a$ and $1 \leq b$, so Lemma 3.2 and part (1) of Lemma 3.1 give the desired result.

On the other hand, if y is of the form $\frac{Q}{\rho^a \tau^b} x$, then $0 \leq a$ and $0 \leq b$, so part (2) of Lemma 3.1 gives the desired result. $\square$

Theorem 3.4. *Betti realization* $\mathrm{Ext}_{\mathbb{R}}^{s,f,w} \to \mathrm{Ext}_{C_2}^{s,f,w}$ *is:*

(1) *an injection in all degrees,*
(2) *an isomorphism if* $2w - s < 5$*, except when* $s = 0$ *and* $w = 2$.

Proof. Diagram (3.1) shows that Betti realization is an injection and induces an isomorphism if and only if Ext_{NC} vanishes. Proposition 3.3 provides the needed vanishing result for Ext_{NC}, since $\frac{\gamma}{\tau} h_0^f$ has degree $(0, f, 2)$. $\square$

4. The Adams spectral sequence

Theorem 3.4 shows that the $\mathbb{R}$-motivic and C_2-equivariant Adams E_2-pages are isomorphic in a range. Now we will extend this isomorphism to higher Adams pages and then to stable homotopy groups. We write $E_r^{\mathbb{R}}(s, f, w)$ and $E_r^{C_2}(s, f, w)$ for the $\mathbb{R}$-motivic and C_2-equivariant Adams E_r-pages in degree (s, f, w), respectively.

Lemma 4.1. *In the* C_2*-equivariant Adams spectral sequence, the element* $\frac{\gamma}{\tau} h_0^f$ *is a permanent cycle.*

Proof. Targets of possible differentials on $\frac{\gamma}{\tau} h_0^f$ lie in degrees $(-1, r + f, 2)$. The Adams E_2-page is zero in those degrees, as $\mathrm{Ext}_{\mathbb{R}}$ vanishes when the coweight $s - w$ is negative and Ext_{NC} vanishes when the stem s is negative. $\square$

Proposition 4.2. *Let $r \geq 2$, or let $r = \infty$. The Betti realization map*

$$E_r^{\mathbb{R}}(s,f,w) \to E_r^{C_2}(s,f,w)$$

(1) *is an isomorphism if $2w - s < 5$ and $(s,w) \neq (0,2)$,*
(2) *is an injection if $2w - s = 5$.*

Proof. The proof is by induction on r. The base case $r = 2$ is established in Theorem 3.4. For the sake of induction, assume that the result is known for r. Consider the diagram

$$\begin{array}{ccccc} E_r^{\mathbb{R}}(s+1,f-r,w) & \xrightarrow{d_r} & E_r^{\mathbb{R}}(s,f,w) & \xrightarrow{d_r} & E_r^{\mathbb{R}}(s-1,f+r,w) \\ \downarrow & & \downarrow & & \downarrow \\ E_r^{C_2}(s+1,f-r,w) & \xrightarrow[d_r]{} & E_r^{C_2}(s,f,w) & \xrightarrow[d_r]{} & E_r^{C_2}(s-1,f+r,w) \end{array}$$

For the induction step in part (1), suppose that $2w - s < 5$ and that $(s,w) \neq (0,2)$. Then the induction assumption implies that the left and middle vertical arrows are isomorphisms, while the right vertical arrow is an injection. A standard diagram chase implies that $E_{r+1}^{\mathbb{R}}(s,f,w) \to E_{r+1}^{C_2}(s,f,w)$ is an isomorphism.

The induction step for part (2) splits into two cases. Suppose that $2w - s < 6$ and that $(s,w) \neq (-1,2)$. The induction assumption implies that the left vertical arrow is an isomorphism and the middle vertical arrow is an injection (and nothing can be said about the right vertical arrow). Again, a diagram chase shows that $E_{r+1}^{\mathbb{R}}(s,f,w) \to E_{r+1}^{C_2}(s,f,w)$ is an injection.

Now suppose that $(s,w) = (-1,2)$. In this case, the left vertical arrow is known only to be an injection. However, Lemma 4.1 implies that this does not matter, and the same diagram chase gives the desired conclusion. This finishes the induction step for part (1).

Finally, the case $r = \infty$ follows from the previous cases, since $E_\infty^{\mathbb{R}}(s,f,w)$ and $E_\infty^{C_2}(s,f,w)$ are equal to $E_r^{\mathbb{R}}(s,f,w)$ and $E_r^{C_2}(s,f,w)$ for $r > N$, where N depends on (s,f,w). □

Example 4.3. Consider the elements $\frac{\gamma}{\tau}P^k h_1$ in degree $(8k+1, 4k+1, 4k+3)$. Note that

$$2(4k+3) - (8k+1) = 5,$$

so these elements lie just outside the range in part (1) of Theorem 1.1. These elements are permanent cycles in both the ρ-Bockstein and Adams spectral sequences, since they lie near the top of the Adams chart and there are no possible elements to serve as targets for differentials. Moreover, they are not hit by any ρ-Bockstein or Adams differentials since they are detected by the equivariant spectrum ko_{C_2} [6].

Therefore, for all $k \geq 1$,

$$\pi^{\mathbb{R}}_{8k+1,4k+3} \to \pi^{C_2}_{8k+1,4k+3}$$

is not an isomorphism. Thus, part (1) of Theorem 1.1 is sharp, in the sense that there is no larger range of degrees bounded by linear inequalities in which Betti realization is an isomorphism.

References

[1] Eva Belmont and Daniel C. Isaksen, *$\mathbb{R}$-motivic stable stems* (2020), preprint.

[2] Daniel Dugger and Daniel C. Isaksen, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010), no. 2, 967–1014, DOI 10.2140/gt.2010.14.967. MR2629898

[3] Daniel Dugger and Daniel C. Isaksen, *Low-dimensional Milnor-Witt stems over* $\mathbb{R}$, Ann. K-Theory **2** (2017), no. 2, 175–210, DOI 10.2140/akt.2017.2.175. MR3590344

[4] Daniel Dugger and Daniel C. Isaksen, *$\mathbb{Z}/2$-equivariant and $\mathbb{R}$-motivic stable stems*, Proc. Amer. Math. Soc. **145** (2017), no. 8, 3617–3627, DOI 10.1090/proc/13505. MR3652813

[5] Bogdan Gheorghe, Guozhen Wang, and Zhouli Xu, *The special fiber of the motivic deformation of the stable homotopy category is algebraic*, Acta. Math., to appear.

[6] Bertrand J. Guillou, Michael A. Hill, Daniel C. Isaksen, and Douglas Conner Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of* ko_{C_2}, Tunis. J. Math. **2** (2020), no. 3, 567–632, DOI 10.2140/tunis.2020.2.567. MR4041284

[7] Bertrand J. Guillou and Daniel C. Isaksen, *The Bredon-Landweber region in C_2-equivariant stable homotopy groups*, preprint, available at `arXiv:1907.01539`.

[8] J. Heller and K. Ormsby, *Galois equivariance and stable motivic homotopy theory*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 8047–8077, DOI 10.1090/tran6647. MR3546793

[9] Daniel C. Isaksen, *Stable stems*, Mem. Amer. Math. Soc. **262** (2019), no. 1269, viii+159, DOI 10.1090/memo/1269. MR4046815

[10] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, *More stable stems* (2020), preprint.

[11] Haynes Robert Miller, *Some algebraic aspects of the Adams-Novikov spectral sequence*, ProQuest LLC, Ann Arbor, MI, 1975. Thesis (Ph.D.)–Princeton University. MR2625232

[12] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 855–951. MR0221509

[13] Fabien Morel and Vladimir Voevodsky, $\mathbf{A}^1$*-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001). MR1813224

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