ON CYCLIC FIXED POINTS OF SPECTRA

Marcel Bökstedt, Bob Bruner, Sverre Lunøe–Nielsen and John Rognes

Abstract. For a finite p-group G and a bounded below G-spectrum X of finite type mod p, the G Segal conjecture for X asserts that the canonical map $X^G \to X^{hG}$ is a p-adic equivalence. We show that if the C_p Segal conjecture holds for a C_{p^n} spectrum X, as well as for each of its C_{p^k} geometric fixed points for $0 < k < n$, then then C_{p^n} Segal conjecture holds for X. Similar results hold for weaker forms of the Segal conjecture, asking only that the canonical map induces an equivalence in sufficiently high degrees, on homotopy groups with suitable finite coefficients.

The Segal conjecture for a finite p-group G asserts that when $X = S_G$ is the genuinely G-equivariant sphere spectrum, the canonical map $X^G \to X^{hG} = F(EG_+, X)^G$ is a p-adic equivalence. For cyclic groups $C = C_p$ of prime order the conjecture was proved by Lin [LDMA] and Gunawardena [AGM]. Thereafter Ravenel [Ra] gave an inductive proof of the Segal conjecture for finite cyclic p-groups $G = C_{p^n}$, with $n \geq 2$, starting from Lin and Gunawardena's theorems.

Ravenel's result was superseded by Carlsson's proof [Ca] of the Segal conjecture for all finite groups, but as we shall show here and elsewhere, Ravenel's methods are also of interest in a more general context where X is a quite general G-spectrum. As was elucidated by Miller and Wilkerson [MW], Ravenel's methods give two proofs, one Ext-computational using the modified Adams spectral sequence, and one non-computational, using explicit geometric constructions.

We shall generalize Ravenel and Miller–Wilkerson's non-computational inductive proof of the Segal conjecture to a study of when $\Gamma_n \colon X^G \to X^{hG}$ is "close to" a p-adic equivalence for $G = C_{p^n}$, assuming that $\Gamma_1 \colon X^C \to X^{hC}$ is "close to" such an equivalence for $C = C_p$. In the special case when $X = THH(B)$ is the topological Hochschild homology of a suitable connective ring spectrum, our results generalize the main theorem of Tsalidis [Ts].

The main technical result is Theorem 1.13, from which Corollary 1.14 follows by an easy induction. The two special cases of principal interest, with $G = C_{p^n}$ and $X = B^{\wedge p^n}$ or $X = THH(B)$, are made explicit in Theorems 1.15 and 1.16. See also Examples 1.17 and 1.18 for further elaboration.

Let p be a prime, $n \geq 1$ and $G = C_{p^n} \subset S^1$. Let $C = C_p \subset G$ and let $\bar{G} = G/C \cong C_{p^{n-1}}$. Let $\lambda = \mathbb{C}(1)$ be the basic faithful G-representation of complex rank one, and S^λ its one-point compactification. Let $\infty\lambda$ be the direct sum of a countably infinite set of copies of λ. The unit sphere $S(\infty\lambda) = EG$ is a free contractible G-CW space, and $S^{\infty\lambda} = \widetilde{EG}$ sits in a G-homotopy cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$, where the first map collapses EG to the non-basepoint.

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Let X be any genuine G-spectrum [LMS], and consider the vertical map

$$\begin{array}{ccccc} EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \wedge X \\ \downarrow{\scriptstyle \simeq_G} & & \downarrow & & \downarrow \\ EG_+ \wedge F(EG_+, X) & \longrightarrow & F(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge F(EG_+, X) \end{array} \tag{1.1}$$

of horizontal G-homotopy cofiber sequences. The left hand vertical map is a G-equivalence, so the right hand square is G-homotopy Cartesian. Passing to G-fixed point spectra we obtain a vertical map

$$\begin{array}{ccccc} X_{hG} & \xrightarrow{N} & X^G & \xrightarrow{R} & \Phi^C(X)^{\bar{G}} \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat{\Gamma}_n} \\ X_{hG} & \xrightarrow{N^h} & X^{hG} & \xrightarrow{R^h} & X^{tG} \end{array} \tag{1.2}$$

of horizontal homotopy cofiber sequences, called the norm–restriction sequences. Here $X_{hG} = EG_+ \wedge_G X$ (homotopy orbits), $X^{hG} = F(EG_+, X)^G$ (homotopy fixed points) and $X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$ (Tate construction), and there is a $\bar{G}$-equivariant equivalence $[\widetilde{EG} \wedge X]^C \simeq \Phi^C(X)$ (geometric fixed points).

Let $\lambda^p = \mathbb{C}(p)$ be the p-th tensor power of λ. Its unit circle is $S(\lambda^p) \cong S^1/C$ as a G-CW space. Let $z\colon S^0 \to S^{\lambda^p}$ be the one-point compactification of the inclusion $\{0\} \subset \lambda^p$. Then there is a G-homotopy cofiber sequence

$$(S^1/C)_+ \to S^0 \xrightarrow{z} S^{\lambda^p} . \tag{1.3}$$

Desuspending X by multiples of λ^p we get a tower of G-spectra

$$\dots \xrightarrow{z} \Sigma^{-j\lambda^p} X \xrightarrow{z} \dots \xrightarrow{z} \Sigma^{-\lambda^p} X \xrightarrow{z} X ,$$

where $j \geq 0$. Substituting $\Sigma^{-j\lambda^p} X$ for X in the right-hand square of (1.2) we get a tower of homotopy Cartesian squares, with j-th term

$$\begin{array}{ccc} (\Sigma^{-j\lambda^p} X)^G & \longrightarrow & \Phi^C(\Sigma^{-j\lambda^p} X)^{\bar{G}} \\ \downarrow & & \downarrow \\ (\Sigma^{-j\lambda^p} X)^{hG} & \longrightarrow & (\Sigma^{-j\lambda^p} X)^{tG} . \end{array} \tag{1.4}$$

Lemma 1.5. *There are homotopy cofiber sequences*

$$(\Sigma^{-\lambda^p} X)^G \xrightarrow{z} X^G \to F((S^1/C)_+, X)^G$$

and

$$\Sigma^{-1} X^C \to F((S^1/C)_+, X)^G \to X^C .$$

The connecting map of the latter is $(1 - \bar{T})\colon X^C \to X^C$, *where* $\bar{T}$ *generates* $\bar{G}$.

Proof. This is clear from the G-homotopy cofiber sequence (1.3) above, the G-cofiber sequence

$$(G/C)_+ \to (S^1/C)_+ \to \Sigma(G/C)_+$$

with connecting map $(1 - T)\colon \Sigma(G/C)_+ \to \Sigma(G/C)_+$, where T generates G, and the $\bar{G}$-equivalence $F((G/C)_+, X)^G \simeq X^C$. $\square$

Lemma 1.6. *(a) The homotopy cofiber of the map z from the $(j+1)$-th to the j-th instance of the square (1.4) is the homotopy Cartesian square*

$$\begin{array}{ccc} F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^G & \longrightarrow & F((S^1/C)_+, \Phi^C(\Sigma^{-j\lambda^p} X))^{\bar{G}} \\ \downarrow & & \downarrow \\ F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^{hG} & \longrightarrow & F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^{tG} . \end{array}$$

(b) There is a homotopy cofiber sequence of homotopy Cartesian squares, mapping from the $(2j+1)$-th desuspension of

$$\begin{array}{ccc} X^C & \xrightarrow{R} & \Phi^C(X) \\ \downarrow{\scriptstyle \Gamma_1} & & \downarrow{\scriptstyle \hat{\Gamma}_1} \\ X^{hC} & \xrightarrow{R^h} & X^{tC} \end{array}$$

to the square in (a), with homotopy cofiber the $(2j)$-th desuspension of the square just displayed. The connecting map of the latter homotopy cofiber sequence is induced by $(1-\bar{T})$ in each corner.

Proof. In the lower right-hand corner of (1.1) we have a G-equivalence

$$\Sigma^{-\lambda^p} \widetilde{EG} \wedge F(EG_+, \Sigma^{-j\lambda^p} X) \simeq \widetilde{EG} \wedge F(EG_+, \Sigma^{-(j+1)\lambda^p} X)$$

since S^{λ^p} is G-dualizable, and similarly for the other corners of the square. Hence the first part of Lemma 1.5 applies to the map z of squares. In view of the G-equivalence

$$F((S^1/C)_+, \widetilde{EG} \wedge F(EG_+, \Sigma^{-j\lambda^p} X)) \simeq \widetilde{EG} \wedge F(EG_+, F((S^1/C)_+, \Sigma^{-j\lambda^p} X)) ,$$

which uses that $(S^1/C)_+$ is G-dualizable, we can rewrite the homotopy cofiber in the lower right-hand corner as $F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^{tG}$, and similarly for the other corners.

Next, we can apply the second part of Lemma 1.5 to get a homotopy cofiber sequence

$$\Sigma^{-(2j+1)} X^{tC} \to F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^{tG} \to \Sigma^{-2j} X^{tC} ,$$

where we have used that C acts trivially on λ^p, and that $[\widetilde{EG} \wedge F(EG_+, X)]^C = X^{tC}$ (since $EG = EC$ and $\widetilde{EG} = \widetilde{EC}$ as C-spaces). The same argument applies in the three other corners, and produces the asserted homotopy cofiber sequence of homotopy Cartesian squares. □

Definition 1.7. Let W be any spectrum, and k an integer or $-\infty$. We shall say that X is (W, k)**-coconnected** if $\pi_* F(W, X) = 0$ for all $* \geq k$. If W is a finite CW spectrum then $F(W, S) \wedge X \simeq F(W, X)$, and the condition is equivalent to asking that $V_*(X) = \pi_*(V \wedge X) = 0$ for $* \geq k$, where $V = F(W, S)$ is the Spanier–Whitehead dual of W. If $W = S^{-1}/p^\infty = \operatorname{hocolim}_e F(S/p^e, S)$ is the Moore spectrum with integral homology $\mathbb{Z}/p^\infty$ concentrated in degree -1, then

$$F(W, X) = \operatorname*{holim}_e X/p^e = X_p^\wedge$$

is the p-adic completion of X, and the condition is equivalent to asking that $\pi_*(X_p^\wedge) = 0$ for $* \geq k$.

Returning to the general case, if $X' \to X \to X''$ is a homotopy cofiber sequence, and X' and X'' are (W,k)-coconnected, then so is X. Any homotopy limit of (W,k)-coconnected spectra is again (W,k)-coconnected. We say that a map $f\colon X \to Y$ is (W,k)-coconnected if its homotopy fiber is (W,k)-coconnected, in the above sense. This is equivalent to asking that

$$f_*\colon \pi_* F(W,X) \to \pi_* F(W,Y)$$

is injective for $* = k$ and an isomorphism for $* > k$. For example, a map is a p-adic equivalence if and only if it is $(S^{-1}/p^\infty, -\infty)$-coconnected. Note that $\Gamma_n\colon X^G \to X^{hG}$ is (W,k)-coconnected if and only if $\hat{\Gamma}_n\colon \Phi^C(X)^{\bar{G}} \to X^{tG}$ is (W,k)-coconnected, since these maps have equivalent homotopy fibers.

Proposition 1.8. *Suppose that*

$$\Gamma_1\colon X^C \to X^{hC}$$

is (W,k)-coconnected. Then the map of horizontal homotopy cofibers induced by the square

$$\begin{array}{ccc} \operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^G & \longrightarrow & X^G \\ \downarrow & & \downarrow \Gamma_n \\ \operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^{hG} & \longrightarrow & X^{hG} \end{array}$$

is (W,k)-coconnected, and the same conclusion holds for the map of horizontal homotopy cofibers induced by the square

$$\begin{array}{ccc} \operatorname{holim}_j \Phi^C(\Sigma^{-j\lambda^p} X)^{\bar{G}} & \longrightarrow & \Phi^C(X)^{\bar{G}} \\ \downarrow & & \downarrow \hat{\Gamma}_n \\ \operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^{tG} & \longrightarrow & X^{tG} . \end{array}$$

Proof. Each desuspension of Γ_1 is (W,k)-coconnected or better, so by Lemma 1.6(b) each map

$$F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^G \to F((S^1/C)_+, \Sigma^{-j\lambda^p} X)^{hG}$$

is (W,k)-coconnected, for $j \geq 0$. By Lemma 1.6(a), this is the vertical map of horizontal homotopy cofibers induced by the square

$$\begin{array}{ccc} (\Sigma^{-(j+1)\lambda^p} X)^G & \xrightarrow{z} & (\Sigma^{-j\lambda^p} X)^G \\ \downarrow & & \downarrow \\ (\Sigma^{-(j+1)\lambda^p} X)^{hG} & \xrightarrow{z} & (\Sigma^{-j\lambda^p} X)^{hG} . \end{array}$$

By induction on j, using homotopy cofiber sequences of the form

$$\operatorname{hocofib}(z) \to \operatorname{hocofib}(z^{j+1}) \to \operatorname{hocofib}(z^j) ,$$

the vertical map of horizontal homotopy cofibers induced by the square

$$\begin{array}{ccc} (\Sigma^{-j\lambda^p} X)^G & \xrightarrow{z^j} & X^G \\ \downarrow & & \downarrow \Gamma_n \\ (\Sigma^{-j\lambda^p} X)^{hG} & \xrightarrow{z^j} & X^{hG} \end{array}$$

is (W, k)-coconnected, for each $j \geq 0$. The first part of the proposition then follows by passage to homotopy limits.

The second part of the result follows by the same proof, or by noting that the two displayed squares assemble to a cube with two homotopy Cartesian faces. □

Proposition 1.9. *There are homotopy cofiber sequences*

$$\operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^G \to (X^C)^{\bar{G}} \xrightarrow{\Gamma_{n-1}} (X^C)^{h\bar{G}}$$

and

$$\operatorname{holim}_j \Phi^C(\Sigma^{-j\lambda^p} X)^{\bar{G}} \to \Phi^C(X)^{\bar{G}} \xrightarrow{\Gamma_{n-1}} \Phi^C(X)^{h\bar{G}} ,$$

where the right hand maps are Γ_{n-1} for the $\bar{G}$-spectrum X^C and the $\bar{G}$-spectrum $\Phi^C(X)$, respectively.

Proof. We have equivalences

$$(\Sigma^{-j\lambda^p} X)^G = (\Sigma^{-j\bar{\lambda}} X^C)^{\bar{G}} \simeq F(S^{j\bar{\lambda}}, X^C)^{\bar{G}} ,$$

where $\bar{\lambda}$ is the faithful $\bar{G}$-representation of rank one that pulls back to λ^p along $G \to \bar{G}$. So $S^{\infty\bar{\lambda}} = \widetilde{E\bar{G}}$, and in the homotopy limit

$$\operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^G \simeq F(\widetilde{E\bar{G}}, X^C)^{\bar{G}} .$$

In view of the $\bar{G}$-homotopy cofiber sequence $E\bar{G}_+ \to S^0 \to \widetilde{E\bar{G}}$, this is the homotopy fiber of the map

$$\Gamma_{n-1} \colon (X^C)^{\bar{G}} \to (X^C)^{h\bar{G}} ,$$

for the $\bar{G}$-spectrum X^C. Replacing X by $\widetilde{EG} \wedge X$ gives the second case. □

We write $H_*(X) = X_*(X; \mathbb{F}_p)$ for mod p homology. The Greenlees filtration of $\widetilde{EG}$ is an integer-indexed G-cellular filtration, whose $2i$-th term is $S^{i\lambda}$ for each integer i. It induces an increasing filtration of $X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$, and a tower of homotopy cofibers with $2i$-th term

$$X^{tG}[i] = [\widetilde{EG}/S^{i\lambda} \wedge F(EG_+, X)]^G . \tag{1.10}$$

We omit the description of the odd-indexed terms. The associated spectral sequence is the homological G-Tate spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(G; H_t(X))$$

converging to the **continuous homology groups**

$$H_*^c(X^{tG}) = \lim_i H_*(X^{tG}[i])$$

of X^{tG}, at least when X is a bounded below spectrum with $H_*(X)$ of finite type. Note that i tends to $-\infty$ in this limit. We shall also refer to the continuous cohomology groups

$$H_c^*(X^{tG}) = \operatorname{colim}_i H^*(X^{tG}[i])\,,$$

and note that $H_*^c(X^{tG}) \cong H_c^*(X^{tG})^*$ (the Hom dual) when $H_*(X)$ is of finite type, because then each $H_*(X^{tG}[i])$ is also of finite type.

Let the G-map $\xi\colon S^\lambda \to S^{\lambda^p}$ be the suspension of the degree p covering map $\pi\colon S^1 = S(\lambda) \to S(\lambda^p) = S^1/C$ of unit spheres, as in the following vertical map of G-homotopy cofiber sequences:

$$\begin{array}{ccccc} S(\lambda)_+ & \longrightarrow & S^0 & \longrightarrow & S^\lambda \\ \big\downarrow{\scriptstyle \pi_+} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \xi} \\ S(\lambda^p)_+ & \longrightarrow & S^0 & \xrightarrow{z} & S^{\lambda^p} \end{array}$$

Then ξ has degree p on the top cell, so $\xi_*\colon H_*(S^\lambda) \to H_*(S^{\lambda^p})$ is the zero homomorphism.

Proposition 1.11. *Let X be a G-spectrum with $H_*(X)$ bounded below. Then*

$$\lim_j H_*^c((\Sigma^{-j\lambda^p} X)^{tG}) = 0$$

and

$$\operatorname{colim}_j H_c^*((\Sigma^{-j\lambda^p} X)^{tG}) = 0\,.$$

Proof. In the notation of (1.10) we have a natural equivalence

$$(\Sigma^{-j\lambda^p} X)^{tG}[i] \xrightarrow{\simeq} (\Sigma^{j(\lambda-\lambda^p)} X)^{tG}[i-j]$$

for each i and j, since $S^{j\lambda}$ is G-dualizable. Under this identification, the tower map

$$z\colon (\Sigma^{-(j+1)\lambda^p} X)^{tG}[i] \to (\Sigma^{-j\lambda^p} X)^{tG}[i]$$

induced by smashing with $z\colon S^0 \to S^{\lambda^p}$ corresponds to the composite of the map

$$(\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG}[i-j-1] \to (\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG}[i-j]$$

induced by the inclusion $S^{(i-j-1)\lambda} \to S^{(i-j)\lambda}$ and the map

$$\xi\colon (\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG}[i-j] \to (\Sigma^{j(\lambda-\lambda^p)} X)^{tG}[i-j]$$

induced by smashing with $\xi\colon S^\lambda \to S^{\lambda^p}$. Passing to the limit over i, the homomorphism

$$z_*\colon H_*^c((\Sigma^{-(j+1)\lambda^p} X)^{tG}) \to H_*^c((\Sigma^{-j\lambda^p} X)^{tG})$$

is identified with the homomorphism

$$\xi_* \colon H^c_*((\Sigma^{(j+1)(\lambda-\lambda^p)} X)^{tG}) \to H^c_*((\Sigma^{j(\lambda-\lambda^p)} X)^{tG}) \,, \tag{1.12}$$

so it suffices to show that the limit over j of the latter homomorphisms is zero. Let

$$\hat{E}^2_{s,t}(j) = \hat{H}^{-s}(G; H_t(\Sigma^{j(\lambda-\lambda^p)} X)) \Longrightarrow H^c_{s+t}((\Sigma^{j(\lambda-\lambda^p)} X)^{tG})$$

be the Tate spectral sequence for the j-th term in the ξ-tower.

The map ξ_* above is compatible with the spectral sequence map $\hat{E}^2_{**}(j+1) \to \hat{E}^2_{**}(j)$ that is induced on Tate cohomology by the G-module homomorphism

$$\xi_* \colon H_*(\Sigma^{(j+1)(\lambda-\lambda^p)} X) \to H_*(\Sigma^{j(\lambda-\lambda^p)} X) \,.$$

But this homomorphism is zero, since $\xi_* \colon H_*(S^\lambda) \to H_*(S^{\lambda^p})$ is zero. Hence the map of spectral sequences is also zero. It follows that the homomorphism ξ_* in (1.12) strictly reduces the Tate filtration ($= s$) of each nonzero continuous homology class. Equivalently, ξ_* strictly increases the vertical degree ($= t$) of the spectral sequence representative of each nonzero class.

By assumption, there is an integer ℓ such that $H_t(X) = 0$ for all $t < \ell$. Then $\hat{E}^2_{s,t}(j) = 0$ for $t < \ell$ and any j, so each nonzero class x_j in $H^c_*((\Sigma^{j(\lambda-\lambda^p)} X)^{tG})$ is represented in vertical degree $\ge \ell$ in $\hat{E}^\infty_{**}(j)$. If $x_j = \xi^k_*(x_{j+k})$, then x_{j+k} must be represented in vertical degree $\ge \ell$ in $\hat{E}^\infty_{**}(j+k)$, so by k-fold application of what we have just proved, x_j must be represented in vertical degree $\ge \ell + k$ in $\hat{E}^\infty_{**}(j)$. If $(x_j)_j$ is a ξ_*-compatible string representing an arbitrary element of $\lim_j H^c_*((\Sigma^{j(\lambda-\lambda^p)} X)^{tG})$, then each x_j is in the image of ξ^k_* for arbitrarily large k, hence is represented in an arbitrarily large vertical degree, or equivalently, in an arbitrarily small ($=$ negative) Tate filtration. Hence each x_j must be zero in the continuous homology, and $(x_j)_j$ is the zero string. Thus $\lim_j H^c_*((\Sigma^{j(\lambda-\lambda^p)} X)^{tG})$ must be the trivial group.

Let $M = \operatorname{colim}_j H_c^*((\Sigma^{-j\lambda^p} X)^{tG})$. Then the Hom dual M^* is the limit group we just showed is zero, and M injects into the double Hom dual M^{**}, so $M = 0$ too. □

For X of suitably finite type, Proposition 1.11 will tell us that the p-adic completion of

$$Y = \operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^{tG}$$

is contractible. To relate this to Propositions 1.8 and 1.9, we will assume that W is such that $F(W, Y)$ is contractible whenever $Y^\wedge_p$ is contractible. To ensure this, we assume that W is in the localizing subcategory of spectra generated by S^{-1}/p^∞, i.e., the smallest thick subcategory of spectra that contains S^{-1}/p^∞ and is closed under arbitrary wedge sums. This obviously includes the case $W = S^{-1}/p^\infty$, but also the cases $W = F(V, S)$ where V is a finite CW spectrum whose integral homology is p-torsion.

The following statements presume our standing conventions that p is a prime, $H_*(X) = H_*(X; \mathbb{F}_p)$, $n \ge 1$ (but only $n \ge 2$ will be interesting), $C = C_p \subset G = C_{p^n}$, $\bar{G} = G/C = C_{p^{n-1}}$, and X is a genuine G-spectrum. A map f is (W, k)-coconnected if $\pi_* F(W, \operatorname{hofib}(f)) = 0$ for $* \ge k$.

Theorem 1.13. *Assume that $\pi_*(X)$ is bounded below and $H_*(X)$ is of finite type. Let W be in the localizing subcategory of spectra generated by S^{-1}/p^∞, and let k be any integer or $-\infty$. Suppose that*

$$\Gamma_1 \colon X^C \to X^{hC}$$

and

$$\Gamma_{n-1} \colon \Phi^C(X)^{\bar{G}} \to \Phi^C(X)^{h\bar{G}}$$

are (W, k)-coconnected maps. Then

$$\Gamma_n \colon X^G \to X^{hG}$$

and

$$\hat{\Gamma}_n \colon \Phi^C(X)^{\bar{G}} \to X^{tG}$$

are (W, k)-coconnected maps.

The hypothesis on Γ_1 may be replaced by the equivalent assumption that $\hat{\Gamma}_1 : \Phi^C(X) \to X^{tC}$ is (W, k)-coconnected.

Less formally, the theorem asserts that if $X^C \to X^{hC}$ is close to an equivalence, and we can inductively prove that $(X')^{\bar{G}} \to (X')^{h\bar{G}}$ is close to an equivalence for $X' = \Phi^C(X)$, then $X^G \to X^{hG}$ is close to an equivalence.

Proof. We abbreviate the second square diagram in Proposition 1.8, together with its vertical map of horizontal homotopy cofibers, as follows:

$$\begin{array}{ccccc} Y' & \longrightarrow & \Phi^C(X)^{\bar{G}} & \longrightarrow & Z' \\ \downarrow & & \downarrow{\scriptstyle \hat{\Gamma}_n} & & \downarrow \\ Y & \longrightarrow & X^{tG} & \longrightarrow & Z \end{array}$$

Here

$$Y = \operatorname*{holim}_j (\Sigma^{-j\lambda^p} X)^{tG}$$

is the homotopy limit over i and j of the spectra

$$\begin{aligned} (\Sigma^{-j\lambda^p} X)^{tG}[i] &= [\widetilde{EG}/S^{i\lambda} \wedge F(EG_+, \Sigma^{-j\lambda^p} X)]^G \\ &\simeq (\widetilde{EG}/S^{i\lambda} \wedge \Sigma^{-j\lambda^p} X)_{hG}\,, \end{aligned}$$

each of which is bounded below and has mod p homology of finite type. (To see this, note that $\widetilde{EG}/S^{i\lambda}$ is a free G-CW spectrum, so taking G-fixed points is equivalent to forming G-homotopy orbits.) Hence there is an inverse limit Adams spectral sequence

$$E_2^{**} = \operatorname{Ext}_A^{**}(M, \mathbb{F}_p) \Longrightarrow \pi_*(Y_p^\wedge)$$

converging to the p-adic homotopy of that homotopy limit (see [CMP] and [LN1]), where

$$M = \operatorname*{colim}_j H_c^*((\Sigma^{-j\lambda^p} X)^{tG})$$

is the A-module that was shown to be zero in Proposition 1.11. Hence the E_2-term is zero and $Y_p^\wedge$ is contractible. By the assumption on W, it follows that Y is $(W, -\infty)$-coconnected.

By the hypothesis on Γ_{n-1} for $\Phi^C(X)$, and the second case of Proposition 1.9, we find that

$$Y' = \operatorname{holim}_j \Phi^C(\Sigma^{-j\lambda^p} X)^{\bar{G}}$$

is (W, k)-coconnected. Hence the vertical map $Y' \to Y$ is (W, k)-coconnected.

By the hypothesis on Γ_1 for X, and Proposition 1.8, the vertical map $Z' \to Z$ of horizontal homotopy cofibers is also (W, k)-coconnected. It follows by the extension property that $\hat{\Gamma}_n$ for X is (W, k)-coconnected, and this is equivalent to $\Gamma_n \colon X^G \to X^{hG}$ being (W, k)-coconnected. □

Corollary 1.14. *Let X be a C_{p^n}-spectrum, let W be in the localizing subcategory of spectra generated by S^{-1}/p^∞, and let $k \geq -\infty$. Suppose for each of the geometric fixed point spectra*

$$Y = X\,,\ \Phi^{C_p}(X)\,,\ \dots\,,\ \Phi^{C_{p^{n-1}}}(X)$$

that Y is bounded below with $H_*(Y)$ of finite type and that

$$\Gamma_1 \colon Y^{C_p} \to Y^{hC_p}$$

is (W, k)-coconnected. Then

$$\Gamma_n \colon X^{C_{p^n}} \to X^{hC_{p^n}}$$

and

$$\hat{\Gamma}_n \colon \Phi^{C_p}(X)^{C_{p^{n-1}}} \to X^{tC_{p^n}}$$

are (W, k)-coconnected maps.

Proof. By induction on n, using Theorem 1.13 and the observation that

$$\Phi^{C_p}(\Phi^{C_{p^e}}(X)) \cong \Phi^{C_{p^{e+1}}}(X)$$

for all $0 \leq e < n$. □

As usual, the hypothesis on Γ_1 may be replaced by the equivalent condition that $\hat{\Gamma}_1 \colon \Phi^{C_p}(Y) \to Y^{tC_p}$ is (W, k)-coconnected. Inspection of the proof shows that we do not really need to assume that the last spectrum in the list, $\Phi^{C_{p^{n-1}}}(X)$, is bounded below and has mod p homology of finite type.

Theorem 1.15. *Let B be a spectrum with $\pi_*(B)$ bounded below and $H_*(B)$ of finite type. Let $B^{\wedge p^n}$ be the C_{p^n}-spectrum $sd_{p^n} THH(B)_0$, with the group action that cyclically permutes the smash product factors. Then*

$$\Gamma_n \colon (B^{\wedge p^n})^{C_{p^n}} \to (B^{\wedge p^n})^{hC_{p^n}}$$

and

$$\hat{\Gamma}_n \colon (B^{\wedge p^{n-1}})^{C_{p^{n-1}}} \to (B^{\wedge p^n})^{tC_{p^n}}$$

are p-adic equivalences.

Proof. This follows from Corollary 1.14 in the case $X = B^{\wedge p^n}$, $W = S^{-1}/p^\infty$ and $k = -\infty$, since for each $0 \le e < n$ there is a $C_{p^{n-e}}$-equivalence

$$Y = \Phi^{C_{p^e}}(B^{\wedge p^n}) \simeq B^{\wedge p^{n-e}} ,$$

the right hand side is clearly bounded below with mod p homology of finite type, and $\hat{\Gamma}_1 \colon \Phi^{C_p}(Y) \to Y^{tC_p}$ is a p-adic equivalence. Writing $Y \simeq Z^{\wedge p}$, where $Z = B^{\wedge p^{n-e-1}}$ is bounded below with $H_*(Z)$ of finite type, the first claim follows since $\Phi^{C_p}(Z^{\wedge p}) \simeq Z$ by Hesselholt and Madsen [HM, Prop. 2.5], and the second claim follows since $\hat{\Gamma}_1 \colon \Phi^{C_p}(Z^{\wedge p}) \to (Z^{\wedge p})^{tC_p}$ is a p-adic equivalence by Lunøe–Nielsen [LN2], generalizing the results of [BMMS, II.5]. □

Theorem 1.16. *Let B be a connective S-algebra with $H_*(B)$ of finite type, and suppose that*

$$\Gamma_1 \colon THH(B)^{C_p} \to THH(B)^{hC_p}$$

is (W, k)-coconnected, for some W in the localizing subcategory of spectra generated by S^{-1}/p^∞ and $k \ge -\infty$. Then

$$\Gamma_n \colon THH(B)^{C_{p^n}} \to THH(B)^{hC_{p^n}}$$

and

$$\hat{\Gamma}_n \colon THH(B)^{C_{p^{n-1}}} \to THH(B)^{tC_{p^n}}$$

are (W, k)-coconnected maps.

Proof. There is a $C_{p^{n-1}}$-equivalence

$$r \colon \Phi^{C_p} THH(B) \xrightarrow{\simeq} THH(B)$$

(the cyclotomic structure map of $THH(B)$, see [HM, §2.5]), whose e-th iterate is a $C_{p^{n-e}}$-equivalence $\Phi^{C_{p^e}}(THH(B)) \simeq THH(B)$. It is clear from the simplicial definition that $THH(B)$ is connective and has mod p homology of finite type, so the assertion follows from Corollary 1.14. □

Example 1.17. Taking $W = S^{-1}/p^\infty$, the assumption in Theorem 1.16 is that the p-completed map $\Gamma_1 \colon (THH(B)^{C_p})^\wedge_p \to (THH(B)^{hC_p})^\wedge_p$ is k-coconnected, i.e., that it induces an injection on π_k and an isomorphism on π_* for $* > k$, and the conclusion is that the p-completed map

$$\Gamma_n \colon (THH(B)^{C_{p^n}})^\wedge_p \to (THH(B)^{hC_{p^n}})^\wedge_p$$

is also k-coconnected, for all $n \ge 2$. This recovers Tsalidis' theorem [Ts, Thm. 2.4].

Example 1.18. Taking $W = F(V, S)$ and $V = V(1)$ (the Smith–Toda complex), the assumption is that $V(1)_*\Gamma_1 \colon V(1)_*THH(B)^{C_p} \to V(1)_*THH(B)^{hC_p}$ is k-coconnected, and the conclusion is that

$$V(1)_*\Gamma_n \colon V(1)_*THH(B)^{C_{p^n}} \to V(1)_*THH(B)^{hC_{p^n}}$$

is also k-coconnected, for all $n \ge 2$. This recovers the extension of Tsalidis' theorem silently used by Ausoni and Rognes [AR, Thm. 5.7], in the special case when $B = \ell$, the connective Adams summand of p-local complex K-theory, and $k = (2p - 2)$.

References

[AGM] Adams, J. F.; Gunawardena, J. H.; Miller, H., *The Segal conjecture for elementary abelian p-groups*, Topology **24** (1985), 435–460.

[AR] Ausoni, Christian; Rognes, John, *Algebraic K-theory of topological K-theory*, Acta Math. **188** (2002), 1–39.

[BMMS] Bruner, R. R.; May, J. P.; McClure, J. E.; Steinberger, M., *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer–Verlag, Berlin, 1986.

[Ca] Carlsson, Gunnar, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. (2) **120** (1984), 189–224.

[CMP] Caruso, J.; May, J. P.; Priddy, S. B., *The Segal conjecture for elementary abelian p-groups. II. p-adic completion in equivariant cohomology*, Topology **26** (1987), 413–433.

[HM] Hesselholt, Lars; Madsen, Ib, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), 29–101.

[LMS] Lewis, L. G., Jr.; May, J. P.; Steinberger, M.; McClure, J. E., *Equivariant stable homotopy theory. With contributions by J. E. McClure*, Lecture Notes in Mathematics, vol. 1213, Springer–Verlag, Berlin, 1986.

[LDMA] Lin, W. H.; Davis, D. M.; Mahowald, M. E.; Adams, J. F., *Calculation of Lin's Ext groups*, Math. Proc. Cambridge Philos. Soc. **87** (1980), 459–469.

[LN1] Lunøe–Nielsen, Sverre, *Continuous cohomology of towers of bounded below spectra of finite type*, Preprint at http://folk.uio.no/sverreln/.

[LN2] Lunøe–Nielsen, Sverre, *On the Tate construction of topological Hochschild homology and its relation to the construction of Singer*, Preprint at http://folk.uio.no/sverreln/.

[MW] Miller, Haynes; Wilkerson, Clarence, *On the Segal conjecture for periodic groups*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982), Contemp. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1983, pp. 233–246.

[Ra] Ravenel, Douglas C., *The Segal conjecture for cyclic groups and its consequences. With an appendix by Haynes R. Miller*, Amer. J. Math. **106** (1984), 415–446.

[Ts] Tsalidis, Stavros, *Topological Hochschild homology and the homotopy descent problem*, Topology **37** (1998), 913–934.

Department of Mathematics, University of Oslo, Box 1053, Blindern, NO–0316 Oslo, Norway

E-mail address: `rognes@math.uio.no`