COMPUTATION OF THE E_3-TERM OF THE ADAMS SPECTRAL SEQUENCE

HANS JOACHIM BAUES AND MAMUKA JIBLADZE

The algebra $\mathscr{B}$ of secondary cohomology operations is a pair algebra with Σ-structure which as a Hopf algebra was explicitly computed in [1]. In particular the multiplication map A of $\mathscr{B}$ was determined by an algorithm. In this paper we introduce algebraically the secondary Ext-groups $\mathrm{Ext}_{\mathscr{B}}$ and we prove that the E_3-term of the Adams spectral sequence (computing stable maps in $\{Y, X\}_p^*$) is given by

$$\mathrm{E}_3(Y, X) = \mathrm{Ext}_{\mathscr{B}}(\mathscr{H}X, \mathscr{H}Y).$$

Here $\mathscr{H}X$ is the secondary cohomology of the spectrum X which is the $\mathscr{B}$-module $\mathbb{G}^\Sigma$ if X is the sphere spectrum S^0. This leads to an algorithm for the computation of the group

$$\mathrm{E}_3(S^0, S^0) = \mathrm{Ext}_{\mathscr{B}}(\mathbb{G}^\Sigma, \mathbb{G}^\Sigma)$$

which is a new explicit approximation of stable homotopy groups of spheres improving the Adams approximation

$$\mathrm{E}_2(S^0, S^0) = \mathrm{Ext}_{\mathscr{A}}(\mathbb{F}, \mathbb{F}).$$

An implementation of our algorithm computed $\mathrm{E}_3(S^0, S^0)$ by now up to degree 40. In this range our results confirm the known results in the literature, see for example the book of Ravenel [6].

1. M

We here recall from [1] the notion of pair modules, pair algebras, and pair modules over a pair algebra B. The category B-**Mod** of pair modules over B is an additive track category in which we consider secondary resolutions as defined in [3]. Using such secondary resolutions we shall obtain the secondary derived functors Ext_B in section 3.

Let k be a commutative ring with unit and let **Mod** be the category of k-modules and k-linear maps. This is a symmetric monoidal category via the tensor product $A \otimes B$ over k of k-modules A, B. A *pair* of modules is a morphism

$$X = \left(X_1 \xrightarrow{\partial} X_0\right) \tag{1.1}$$

in **Mod**. We write $\pi_0(X) = \ker\partial$ and $\pi_1(X) = \operatorname{coker}\partial$. A *morphism* $f : X \to Y$ of pairs is a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ X_0 & \xrightarrow{f_0} & Y_0. \end{array}$$

Evidently pairs with these morphisms form a category $\mathscr{P}\!air(\mathbf{Mod})$ and one has functors

$$\pi_0, \pi_1 : \mathscr{P}\!air(\mathbf{Mod}) \to \mathbf{Mod}.$$

A pair morphism is called a *weak equivalence* if it induces isomorphisms on π_0 and π_1.

Clearly a pair in **Mod** coincides with a chain complex concentrated in degrees 0 and 1. For two pairs X and Y the tensor product of the complexes corresponding to them is concentrated in degrees in 0, 1 and 2 and is given by

$$X_1 \otimes Y_1 \xrightarrow{\partial_1} X_1 \otimes Y_0 \oplus X_0 \otimes Y_1 \xrightarrow{\partial_0} X_0 \otimes Y_0$$

with $\partial_0 = (\partial \otimes 1, 1 \otimes \partial)$ and $\partial_1 = (-1 \otimes \partial, \partial \otimes 1)$. Truncating $X \otimes Y$ we get the pair

$$X \bar{\otimes} Y = \left((X \bar{\otimes} Y)_1 = \operatorname{coker}(\partial_1) \xrightarrow{\partial} X_0 \otimes Y_0 = (X \bar{\otimes} Y)_0\right)$$

with ∂ induced by ∂_0.

(1.2) Remark. Note that the full embedding of the category of pairs into the category of chain complexes induced by the above identification has a left adjoint Tr given by truncation: for a chain complex

$$C = \left(... \to C_2 \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_0} C_0 \xrightarrow{\partial_{-1}} C_{-1} \to ...\right),$$

one has

$$\mathrm{Tr}(C) = \left(\mathrm{coker}(\partial_1) \xrightarrow{\bar\partial_0} C_0\right),$$

with $\bar\partial_0$ induced by ∂_0. Then clearly one has

$$X \bar\otimes Y = \mathrm{Tr}(X \otimes Y).$$

Using the fact that Tr is a reflection onto a full subcategory, one easily checks that the category $\mathscr{P}\!air(\mathbf{Mod})$ together with the tensor product $\bar\otimes$ and unit $k = (0 \to k)$ is a symmetric monoidal category, and Tr is a monoidal functor.

We define the tensor product $A \otimes B$ of two graded modules in the usual way, i. e. by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes B^j.$$

A *(graded) pair module* is a graded object of $\mathscr{P}\!air(\mathbf{Mod})$, i. e. a sequence $X^n = (\partial : X_1^n \to X_0^n)$ of pairs in **Mod**. We identify such a pair module X with the underlying morphism ∂ of degree 0 between graded modules

$$X = \left(X_1 \xrightarrow{\partial} X_0\right).$$

Now the tensor product $X \bar\otimes Y$ of graded pair modules X, Y is defined by

(1.3) $$(X \bar\otimes Y)^n = \bigoplus_{i+j=n} X^i \bar\otimes Y^j.$$

This defines a monoidal structure on the category of graded pair modules. Morphisms in this category are of degree 0.

For two morphisms $f, g : X \to Y$ between graded pair modules, a *homotopy* $H : f \Rightarrow g$ is a morphism $H : X_0 \to Y_1$ of degree 0 as in the diagram

(1.4) $$\begin{array}{ccc} X_1 & \overset{f_1}{\underset{g_1}{\rightrightarrows}} & Y_1 \\ {\scriptstyle\partial}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle\partial} \\ X_0 & \overset{f_0}{\underset{g_0}{\rightrightarrows}} & Y_0, \end{array}$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$.

A *pair algebra* B is a monoid in the monoidal category of graded pair modules, with multiplication

$$\mu : B \bar\otimes B \to B.$$

We assume that B is concentrated in nonnegative degrees, that is $B^n = 0$ for $n < 0$.

A *left B-module* is a graded pair module M together with a left action

$$\mu : B \bar\otimes M \to M$$

of the monoid B on M.

More explicitly pair algebras and modules over them can be described as follows.

(1.5) Definition. A *pair algebra* B is a graded pair

$$\partial : B_1 \to B_0$$

in **Mod** with $B_1^n = B_0^n = 0$ for $n < 0$ such that B_0 is a graded algebra in **Mod**, B_1 is a graded B_0-B_0-bimodule, and ∂ is a bimodule homomorphism. Moreover for $x, y \in B_1$ the equality

$$\partial(x)y = x\partial(y)$$

holds in B_1.

It is easy to see that there results an exact sequence of graded B_0-B_0-bimodules

$$0 \to \pi_1 B \to B_1 \xrightarrow{\partial} B_0 \to \pi_0 B \to 0$$

where in fact $\pi_0 B$ is a k-algebra, $\pi_1 B$ is a $\pi_0 B$-$\pi_0 B$-bimodule, and $B_0 \to \pi_0(B)$ is a homomorphism of algebras.

(1.6) Definition. A *(left) module* over a pair algebra B is a graded pair $M = (\partial : M_1 \to M_0)$ in **Mod** such that M_1 and M_0 are left B_0-modules and ∂ is B_0-linear. Moreover a B_0-linear map

$$\bar{\mu} : B_1 \otimes_{B_0} M_0 \to M_1$$

is given fitting in the commutative diagram

$$\begin{array}{ccc} B_1 \otimes_{B_0} M_1 & \xrightarrow{1\otimes\partial} & B_1 \otimes_{B_0} M_0 \\ {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\bar{\mu}} & \downarrow{\scriptstyle\mu} \\ M_1 & \xrightarrow[\partial]{} & M_0, \end{array}$$

where $\mu(b \otimes m) = \partial(b)m$ for $b \in B_1$ and $m \in M_1 \cup M_0$.

For an indeterminate element x of degree $n = |x|$ let $B[x]$ denote the B-module with $B[x]_i$ consisting of expressions bx with $b \in B_i$, $i = 0, 1$, with bx having degree $|b| + n$, and structure maps given by $\partial(bx) = \partial(b)x$, $\mu(b' \otimes bx) = (b'b)x$ and $\bar{\mu}(b' \otimes bx) = (b'b)x$.

A *free* B-module is a direct sum of several copies of modules of the form $B[x]$, with $x \in I$ for some set I of indeterminates of possibly different degrees. It will be denoted

$$B[I] = \bigoplus_{x\in I} B[x].$$

For a left B-module M one has the exact sequence of B_0-modules

$$0 \to \pi_1 M \to M_1 \to M_0 \to \pi_0 M \to 0$$

where $\pi_0 M$ and $\pi_1 M$ are actually $\pi_0 B$-modules.

Let B-**Mod** be the category of left modules over the pair algebra B. Morphisms $f = (f_0, f_1) : M \to N$ are pair morphisms which are B-equivariant, that is, f_0 and f_1 are B_0-equivariant and compatible with $\bar{\mu}$ above, i. e. the diagram

$$\begin{array}{ccc} B_1 \otimes_{B_0} M_0 & \xrightarrow{\bar{\mu}} & M_1 \\ {\scriptstyle 1\otimes f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_1 \otimes_{B_0} N_0 & \xrightarrow{\bar{\mu}} & N_1 \end{array}$$

commutes.

For two such maps $f, g : M \to N$ a track $H : f \Rightarrow g$ is a degree zero map

$$H : M_0 \to N_1 \tag{1.7}$$

satisfying $f_0 - g_0 = \partial H$ and $f_1 - g_1 = H\partial$ such that H is B_0-equivariant. For tracks $H : f \Rightarrow g$, $K : g \Rightarrow h$ their composition $K \square H : f \Rightarrow h$ is $K + H$.

(1.8) Proposition. *For a pair algebra B, the category B-**Mod** with the above track structure is a well-defined additive track category.*

Proof. For a morphism $f = (f_0, f_1) : M \to N$ between B-modules, one has

$$\mathrm{Aut}(f) = \{H \in \mathrm{Hom}_{B_0}(M_0, N_1) \mid \partial H = f_0 - f_0, H\partial = f_1 - f_1\} \cong \mathrm{Hom}_{\pi_0 B}(\pi_0 M, \pi_1 N).$$

Since this group is abelian, by [2] we know that B-**Mod** is a linear track extension of its homotopy category by the bifunctor D with $D(M, N) = \mathrm{Hom}_{\pi_0 B}(\pi_0 M, \pi_1 N)$. It thus remains to show that the homotopy category is additive and the bifunctor D is biadditive.

By definition the set of morphisms $[M, N]$ between objects M, N in the homotopy category is given by the exact sequence of abelian groups

$$\mathrm{Hom}_{B_0}(M_0, N_1) \to \mathrm{Hom}_B(M, N) \twoheadrightarrow [M, N].$$

This makes evident the abelian group structure on the hom-sets $[M,N]$. Bilinearity of composition follows from consideration of the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{B_0}(M_0,N_1)\otimes\mathrm{Hom}_B(N,P)\oplus\mathrm{Hom}_B(M,N)\otimes\mathrm{Hom}_{B_0}(N_0,P_1) & \xrightarrow{\mu} & \mathrm{Hom}_{B_0}(M_0,P_1)\\
\downarrow & & \downarrow\\
\mathrm{Hom}_B(M,N)\otimes\mathrm{Hom}_B(N,P) & \longrightarrow & \mathrm{Hom}_B(M,P)\\
\downarrow & & \downarrow\\
[M,N]\otimes[N,P] & \dashrightarrow & [M,P]
\end{array}$$

with exact columns, where $\mu(H\otimes g+f\otimes K)=g_1H+Kf_0$. It also shows that the functor B-$\mathbf{Mod}\to B$-$\mathbf{Mod}_{\simeq}$ is linear. Since this functor is the identity on objects, it follows that the homotopy category is additive.

Now note that both functors π_0, π_1 factor to define functors on B-$\mathbf{Mod}_{\simeq}$. Since these functors are evidently additive, it follows that $D=\mathrm{Hom}_{\pi_0B}(\pi_0,\pi_1)$ is a biadditive bifunctor. □

(1.9) Lemma. *If M is a free B-module, then the canonical map*

$$[M,N]\to\mathrm{Hom}_{\pi_0B}(\pi_0M,\pi_0N)$$

is an isomorphism for any B-module N.

Proof. Let $(g_i)_{i\in I}$ be a free generating set for M. Given a $\pi_0(B)$-equivariant homomorphism $f:\pi_0M\to\pi_0N$, define its lifting $\tilde f$ to M by specifying $\tilde f(g_i)=n_i$, with n_i chosen arbitrarily from the class $f([g_i])=[n_i]$.

To show monomorphicity, given $f:M\to N$ such that $\pi_0f=0$, this means that $\operatorname{im}f_0\subset\operatorname{im}\partial$, so we can choose $H(g_i)\in N_1$ in such a way that $\partial H(g_i)=f_0(g_i)$. This then extends uniquely to a B_0-module homomorphism $H:M_0\to N_1$ with $\partial H=f_0$; moreover any element of M_1 is a linear combination of elements of the form b_1g_i with $b_1\in B_1$, and for these one has $H\partial(b_1g_i)=H(\partial(b_1)g_i)=\partial(b_1)H(g_i)$. But $f_1(b_1g_i)=b_1f_0(g_i)=b_1\partial H(g_i)=\partial(b_1)H(g_i)$ too, so $H\partial=f_1$. This shows that f is nullhomotopic. □

2. Σ-

(2.1) Definition. The *suspension* ΣX of a graded object $X=(X^n)_{n\in\mathbb Z}$ is given by degree shift, $(\Sigma X)^n=X^{n-1}$.

Let $\Sigma:X\to\Sigma X$ be the map of degree 1 given by the identity. If X is a left A-module over the graded algebra A then ΣX is a left A-module via

$$a\cdot\Sigma x=(-1)^{|a|}\Sigma(a\cdot x) \tag{2.2}$$

for $a\in A$, $x\in X$. On the other hand if X is a right A-module then $(\Sigma x)\cdot a=\Sigma(x\cdot a)$ yields the right A-module structure on ΣX.

(2.3) Definition. A Σ-*module* is a graded pair module $X=(\partial:X_1\to X_0)$ together with an isomorphism

$$\sigma:\pi_1X\cong\Sigma\pi_0X$$

of graded k-modules. We then call σ a Σ-*structure* of X. A Σ-map between Σ-modules is a map f between pair modules such that $\sigma(\pi_1f)=\Sigma(\pi_0f)\sigma$. If X is a pair algebra then a Σ-structure is an isomorphism of π_0X-π_0X-bimodules. If X is a left module over a pair algebra B then a Σ-structure of X is an isomorphism σ of left π_0B-modules. Let

$$(B\text{-}\mathbf{Mod})^\Sigma\subset B\text{-}\mathbf{Mod}$$

be the track category of B-modules with Σ-structure and Σ-maps.

(2.4) Lemma. *Suspension of a B-module M has by* (2.2) *the structure of a B-module and ΣM has a Σ-structure if M has one.*

Proof. Given $\sigma:\pi_1M\cong\Sigma\pi_0M$ one defines a Σ-structure on ΣM via

$$\pi_1\Sigma M=\Sigma\pi_1M\xrightarrow{\Sigma\sigma}\Sigma\Sigma\pi_0M=\Sigma\pi_0\Sigma M.$$

□

Hence we get suspension functors between track categories

$$\begin{array}{ccc} B\text{-}\mathbf{Mod} & \xrightarrow{\Sigma} & B\text{-}\mathbf{Mod} \\ \uparrow & & \uparrow \\ (B\text{-}\mathbf{Mod})^{\Sigma} & \xrightarrow{\Sigma} & (B\text{-}\mathbf{Mod})^{\Sigma}. \end{array}$$

(2.5) Lemma. *The track category $(B\text{–}\mathbf{Mod})^{\Sigma}$ is $\mathbb{L}$-additive in the sense of [3], with $\mathbb{L} = \Sigma^{-1}$, or as well $\mathbb{R}$-additive, with $\mathbb{R} = \Sigma$.*

Proof. The statement of the lemma means that the bifunctor

$$D(M,N) = \mathrm{Aut}(0_{M,N})$$

is either left- or right-representable, i. e. there is an endofunctor $\mathbb{L}$, respectively $\mathbb{R}$ of $(B\text{-}\mathbf{Mod})^{\Sigma}$ and a binatural isomorphism $D(M,N) \cong [\mathbb{L}M, N]$, resp. $D(M,N) \cong [M, \mathbb{R}N]$.

Now by (1.7), a track in $\mathrm{Aut}(0_{M,N})$ is a B_0-module homomorphism $H : M_0 \to N_1$ with $\partial H = H\partial = 0$; hence

$$D(M,N) \cong \mathrm{Hom}_{\pi_0 B}(\pi_0 M, \pi_1 N) \cong \mathrm{Hom}_{\pi_0 B}(\pi_0 \Sigma^{-1} M, \pi_0 N) \cong \mathrm{Hom}_{\pi_0 B}(\pi_0 M, \pi_0 \Sigma N).$$

□

(2.6) Lemma. *If B is a pair algebra with Σ-structure then each free B-module has a Σ-structure.*

Proof. This is clear from the description of free modules in 1.6. □

3. T

For a pair algebra B with a Σ-structure, for a Σ-module M over B, and a module N over B we now define the *secondary differential*

$$d_{(2)} : \mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \pi_0 N) \to \mathrm{Ext}^{n+2}_{\pi_0 B}(\pi_0 M, \pi_1 N).$$

Here $d_{(2)} = d_{(2)}(M,N)$ depends on the B-modules M and N and is natural in M and N with respect to maps in $(B\text{–}\mathbf{Mod})^{\Sigma}$. For the definition of $d_{(2)}$ we consider secondary chain complexes and secondary resolutions. In [3] such a construction was performed in the generality of an arbitrary $\mathbb{L}$-additive track category. We will first present the construction of $d_{(2)}$ for the track category of pair modules and then will indicate how this construction is a particular case of the more general situation discussed in [3].

(3.1) Definition. For a pair algebra B, a *secondary chain complex* $M_\bullet$ in B-$\mathbf{Mod}$ is given by a diagram of the form

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & M_{n+2,1} & \xrightarrow{d_{n+1,1}} & M_{n+1,1} & \xrightarrow{d_{n,1}} & M_{n,1} & \xrightarrow{d_{n-1,1}} & M_{n-1,1} & \longrightarrow & \cdots \\ & {\scriptstyle H_{n+1}}\nearrow & \downarrow{\scriptstyle \partial_{n+2}} & {\scriptstyle H_n}\nearrow & \downarrow{\scriptstyle \partial_{n+1}} & {\scriptstyle H_{n-1}}\nearrow & \downarrow{\scriptstyle \partial_n} & \nearrow & \downarrow{\scriptstyle \partial_{n-1}} & \nearrow & \\ \cdots & \longrightarrow & M_{n+2,0} & \xrightarrow[d_{n+1,0}]{} & M_{n+1,0} & \xrightarrow[d_{n,0}]{} & M_{n,0} & \xrightarrow[d_{n-1,0}]{} & M_{n-1,0} & \longrightarrow & \cdots \end{array}$$

where each $M_n = (\partial_n : M_{n,1} \to M_{n,0})$ is a B-module, each $d_n = (d_{n,0}, d_{n,1})$ is a morphism in B-$\mathbf{Mod}$, each H_n is B_0-linear and moreover the identities

$$d_{n,0} d_{n+1,0} = \partial_n H_n$$
$$d_{n,1} d_{n+1,1} = H_n \partial_{n+2}$$

and

$$H_n d_{n+2,0} = d_{n,1} H_{n+1}$$

hold for all $n \in \mathbb{Z}$. We thus see that in this case a secondary complex is the same as a graded version of a *multicomplex* (see e. g. [5]) with only two nonzero rows.

One then defines the *total complex* $\mathrm{Tot}(M_\bullet)$ of the form

$$\dots \leftarrow M_{n-1,0}\oplus M_{n-2,1} \xleftarrow{\begin{pmatrix} d_{n-1,0} & -\partial_{n-1}\\ H_{n-2} & -d_{n-2,1}\end{pmatrix}} M_{n,0}\oplus M_{n-1,1} \xleftarrow{\begin{pmatrix} d_{n,0} & -\partial_{n}\\ H_{n-1} & -d_{n-1,1}\end{pmatrix}} M_{n+1,0}\oplus M_{n,1} \leftarrow \dots$$

Cycles and boundaries in this complex will be called secondary cycles, resp. secondary boundaries of $M_\bullet$. Thus a secondary n-cycle in $M_\bullet$ is a pair (c,γ) with $c\in M_{n,0}$, $\gamma\in M_{n-1,1}$ such that $d_{n-1,0}c=\partial_{n-1}\gamma$, $H_{n-2}c=d_{n-2,1}\gamma$ and such a cycle is a boundary iff there exist $b\in M_{n+1,0}$ and $\beta\in M_{n,1}$ with $c=d_{n,0}b+\partial_n\beta$ and $\gamma=H_{n-1}b+d_{n-1,1}\beta$. A secondary complex $M_\bullet$ is called *exact* if its total complex is, that is, if secondary cycles are secondary boundaries.

Let us now consider a secondary chain complex $M_\bullet$ in B-**Mod**. It is clear then that

$$\pi_0M_\bullet:\qquad \dots\to\pi_0M_{n+2}\xrightarrow{\pi_0d_{n+1}}\pi_0M_{n+1}\xrightarrow{\pi_0d_n}\pi_0M_n\xrightarrow{\pi_0d_{n-1}}\pi_0M_{n-1}\to\dots$$

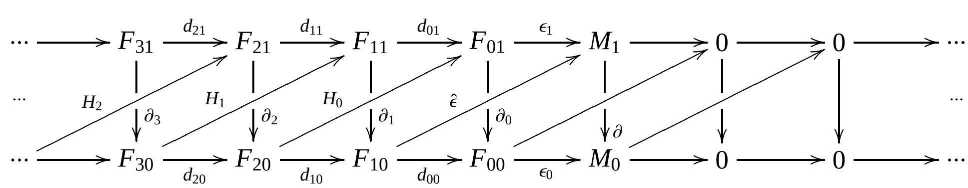
is a chain complex of π_0B-modules. The next result corresponds to [3, lemma 3.5].

(3.2) Proposition. *Let $M_\bullet$ be a secondary complex consisting of Σ-modules and Σ-maps between them. If $\pi_0(M_\bullet)$ is an exact complex then $M_\bullet$ is an exact secondary complex. Conversely, if $\pi_0M_\bullet$ is bounded below then secondary exactness of $M_\bullet$ implies exactness of $\pi_0M_\bullet$.*

Proof. The proof consists in translating the argument from the analogous general statement in [3] to our setting. Suppose first that $\pi_0M_\bullet$ is an exact complex, and consider a secondary cycle $(c,\gamma)\in M_{n,0}\oplus M_{n-1,1}$, i. e. one has $d_{n-1,0}c=\partial_{n-1}\gamma$ and $H_{n-2}c=d_{n-2,1}\gamma$. Then in particular $[c]\in\pi_0M_n$ is a cycle, so there exists $[b]\in\pi_0M_{n+1}$ with $[c]=\pi_0(d_n)[b]$. Take $b\in[b]$, then $c-d_{n,0}b=\partial_n\beta$ for some $\beta\in M_{n+1,1}$. Consider $\delta=\gamma-H_{n-1}b-d_{n-1,1}\beta$. One has $\partial_{n-1}\delta=\partial_{n-1}\gamma-\partial_{n-1}H_{n-1}b-\partial_{n-1}d_{n-1,1}\beta=d_{n-1,0}c-d_{n-1,0}d_{n,0}b-d_{n-1,0}\partial_n\beta=0$, so that δ is an element of π_1M_n. Moreover $d_{n-2,1}\delta=d_{n-2,1}\gamma-d_{n-2,1}H_{n-1}b-d_{n-2,1}d_{n-1,1}\beta=H_{n-2}c-H_{n-2}d_{n,0}b-H_{n-2}\partial_n\beta=0$, i. e. δ is a cycle in $\pi_1M_\bullet$. Since by assumption $\pi_0M_\bullet$ is exact, taking into account the Σ-structure $\pi_1M_\bullet$ is exact too, so that there exists $\psi\in\pi_1M_n$ with $\delta=d_{n-1,1}\psi$. Define $\tilde\beta=\beta+\psi$. Then $d_{n,0}b+\partial_n\tilde\beta=d_{n,0}b+\partial_n\beta=c$ since $\psi\in\ker\partial_n$. Moreover $d_{n-1,1}\tilde\beta=d_{n-1,1}\beta+d_{n-1,1}\psi=d_{n-1,1}\beta+\delta=\gamma-H_{n-1}b$, which means that (c,γ) is the boundary of $(b,\tilde\beta)$. Thus $M_\bullet$ is an exact secondary complex.

Conversely suppose $M_\bullet$ is exact, and $\pi_0M_\bullet$ bounded below. Given a cycle $[c]\in\pi_0(M_n)$, represent it by a $c\in M_{n,0}$. Then $\pi_0d_{n-1}[c]=0$ implies $d_{n-1,0}c\in\operatorname{im}\partial_{n-1}$, so there is a $\gamma\in M_{n-1,1}$ such that $d_{n-1,0}c=\partial_{n-1}\gamma$. Consider $\omega=d_{n-2,1}\gamma-H_{n-2}c$. One has $\partial_{n-2}\omega=\partial_{n-2}d_{n-2,1}\gamma-\partial_{n-2}H_{n-2}c=d_{n-2,0}\partial_{n-1}\gamma-d_{n-2,0}d_{n-1,0}c=0$, i. e. ω is an element of π_1M_{n-2}. Moreover $d_{n-3,1}\omega=d_{n-3,1}d_{n-2,1}\gamma-d_{n-3,1}H_{n-2}c=H_{n-3}\partial_{n-1}\gamma-H_{n-3}d_{n,0}c=0$, so ω is a $n-2$-dimensional cycle in $\pi_1M_\bullet$. Using the Σ-structure, this then gives a $n-3$-dimensional cycle in $\pi_0M_\bullet$. Now since $\pi_0M_\bullet$ is bounded below, we might assume by induction that it is exact in dimension $n-3$, so that ω is a boundary. That is, there exists $\alpha\in\pi_1M_{n-1}$ with $d_{n-2,1}\alpha=\omega$. Define $\tilde\gamma=\gamma-\alpha$; then one has $d_{n-2,1}\tilde\gamma=d_{n-2,1}\gamma-d_{n-2,1}\alpha=d_{n-2,1}\gamma-\omega=H_{n-2}c$. Moreover $\partial_{n-1}\tilde\gamma=\partial_{n-1}\gamma=d_{n-1,0}c$ since $\alpha\in\ker(\partial)n-1$. Thus $(c,\tilde\gamma)$ is a secondary cycle, and by secondary exactness of $M_\bullet$ there exists a pair (b,β) with $c=d_{n,0}b+\partial_n\beta$. Then $[c]=\pi_0(d_n)[b]$, i. e. c is a boundary. □

(3.3) Definition. Let B be a pair algebra with Σ-structure. A *secondary resolution* of a Σ-module $M=(\partial: M_1\to M_0)$ over B is an exact secondary complex $F_\bullet$ in $(B\text{–}\mathbf{Mod})^\Sigma$ of the form



where each $F_n=(\partial_n: F_{n1}\to F_{n0})$ is a free B-module.

It follows from 3.2 that for any secondary resolution $F_\bullet$ of a B-module M with Σ-structure, $\pi_0F_\bullet$ will be a free resolution of the π_0B-module π_0M, so that in particular one has

$$\mathrm{Ext}^n_{\pi_0B}(\pi_0M,U)\cong H^n\,\mathrm{Hom}(\pi_0F_\bullet,U)$$

for all n and any π_0B-module U.

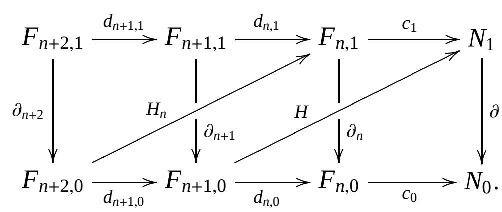
(3.4) Definition. Given a pair algebra B with Σ-structure, a Σ-module M over B, a module N over B and a secondary resolution $F_\bullet$ of M, we define the *secondary differential*

$$d_{(2)} : \mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \pi_0 N) \to \mathrm{Ext}^{n+2}_{\pi_0 B}(\pi_0 M, \pi_1 N)$$

in the following way. Suppose given a class $[c] \in \mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \pi_0 N)$. First represent it by some element in $\mathrm{Hom}_{\pi_0 B}(\pi_0 F_n, \pi_0 N)$ which is a cocycle, i. e. its composite with $\pi_0(d_n)$ is 0. By 1.9 we know that the natural maps

$$[F_n, N] \to \mathrm{Hom}_{\pi_0 B}(\pi_0 F, \pi_0 N)$$

are isomorphisms, hence to any such element corresponds a homotopy class in $[F_n, N]$ which is also a cocycle, i. e. value of $[d_n, N]$ on it is zero. Take a representative map $c : F_n \to N$ from this homotopy class. Then cd_n is nullhomotopic, so we can find a B_0-equivariant map $H : F_{n+1,0} \to N_1$ such that in the diagram



one has $c_0 d_{n,0} = \partial H$, $c_1 d_{n,1} = H\partial_{\underline{n}+1}$ and $\partial c_1 = c_0 \partial_n$. Then taking $\Gamma = c_1 H_n - H d_{n+1,0}$ one has $\partial\Gamma = 0$, $\Gamma\partial_{n+2} = 0$, so Γ determines a map $\overline{\Gamma} : \operatorname{coker}\partial_{n+2} \to \ker\partial$, i. e. from $\pi_0 F_{n+2}$ to $\pi_1 N$. Moreover $\overline{\Gamma}\pi_0(d_{n+2}) = 0$, so it is a cocycle in $\mathrm{Hom}(\pi_0(F_\bullet), \pi_1 N)$ and we define

$$d_{(2)}[c] = [\overline{\Gamma}] \in \mathrm{Ext}^{n+2}_{\pi_0 B}(\pi_0 M, \pi_1 N).$$

(3.5) Definition. Let M and N be B-modules with Σ-structure. Then also all the B-modules $\Sigma^k M$, $\Sigma^k N$ have Σ-structures and we get by 3.4 the secondary differential

$$\begin{array}{ccc}
\mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \pi_0 \Sigma^k N) & \xrightarrow{d_{(2)}(M,\Sigma^k N)} & \mathrm{Ext}^{n+2}_{\pi_0 B}(\pi_0 M, \pi_1 \Sigma^k N) \\
\| & & \| \\
\mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \Sigma^k \pi_0 N) & \xrightarrow{d} & \mathrm{Ext}^{n+2}_{\pi_0 B}(\pi_0 M, \Sigma^{k+1} \pi_0 N).
\end{array}$$

In case the composite

$$\mathrm{Ext}^{n-2}_{\pi_0 B}(\pi_0 M, \Sigma^{k-1}\pi_0 N) \xrightarrow{d} \mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \Sigma^k \pi_0 N) \xrightarrow{d} \mathrm{Ext}^{n+2}_{\pi_0 B}(\pi_0 M, \Sigma^{k+1}\pi_0 N)$$

vanishes we define the *secondary* Ext-*groups* to be the quotient groups

$$\mathrm{Ext}^n_B(M, N)^k := \ker d / \operatorname{im} d.$$

(3.6) Theorem. *For a Σ-algebra B, a B-module M with Σ-structure and any B-module N, the secondary differential $d_{(2)}$ in 3.4 coincides with the secondary differential*

$$d_{(2)} : \mathrm{Ext}^n_{\mathbf{a}}(M, N) \to \mathrm{Ext}^{n+2}_{\mathbf{a}}(M, N)$$

from [3, Section 4] as constructed for the $\mathbb{L}$-additive track category $(B\text{–}\mathbf{Mod})^\Sigma$ in 2.5, relative to the subcategory **b** *of free B-modules with $\mathbf{a} = \mathbf{b}_\simeq$.*

Proof. We begin by recalling the appropriate notions from [3]. There secondary chain complexes $A_\bullet = (A_n, d_n, \delta_n)_{n\in\mathbb{Z}}$ are defined in arbitrary additive track category $\mathbf{B}$. They consist of objects A_n, morphisms $d_n : A_{n+1} \to A_n$ and tracks $\delta_n : d_n d_{n+1} \Rightarrow 0_{A_{n+2},A_n}$, $n \in \mathbb{Z}$, such that the equality of tracks

$$\delta_n d_{n+2} = d_n \delta_{n+1}$$

holds for all n. For an object X, an X-valued n-cycle in a secondary chain complex $A_\bullet$ is defined to be a pair (c, γ) consisting of a morphism $c : X \to A_n$ and a track $\gamma : d_{n-1}c \Rightarrow 0_{X,A_{n-1}}$ such that the equality of tracks

$$\delta_{n-2} c = d_{n-2}\gamma$$

is satisfied. Such a cycle is called a *boundary* if there exists a map $b : X \to A_{n+1}$ and a track $\beta : c \Rightarrow d_n b$ such that the equality

$$\gamma = \delta_{n-1} b \square d_{n-1} \beta$$

holds. A secondary chain complex is called X-exact if every X-valued cycle in it is a boundary. Similarly it is called $\mathbf{b}$-*exact*, if it is X-exact for every object X in $\mathbf{b}$, where $\mathbf{b}$ is a track subcategory of $\mathbf{B}$. A secondary $\mathbf{b}$-resolution of an object A is a $\mathbf{b}$-exact secondary chain complex $A_\bullet$ with $A_n = 0$ for $n < -1$, $A_{-1} = A$, $A_n \in \mathbf{b}$ for $n \neq -1$; the last differentials will be then denoted $d_{-1} = \epsilon : A_0 \to A$, $\delta_{-1} = \hat{\epsilon} : \epsilon d_0 \to 0_{A_1,A}$ and the pair $(\epsilon, \hat{\epsilon})$ will be called *augmentation* of the resolution. It is clear that any secondary chain complex $(A_\bullet, d_\bullet, \delta_\bullet)$ in $\mathbf{B}$ gives rise to a chain complex $(A_\bullet, [d_\bullet])$, in the ordinary sense, in the homotopy category $\mathbf{B}_\simeq$ of $\mathbf{B}$. Moreover if $\mathbf{B}$ is Σ-additive, i. e. there exists a functor Σ and isomorphisms $\mathrm{Aut}(0_{X,Y}) \cong [\Sigma X, Y]$, natural in X, Y, then $\mathbf{b}$-exactness of $(A_\bullet, d_\bullet, \delta_\bullet)$ implies $\mathbf{b}_\simeq$-exactness of $(A_\bullet, [d_\bullet])$ in the sense that the chain complex of abelian groups $[X, (A_\bullet, [d_\bullet])]$ will be exact for each $X \in \mathbf{b}$. In [3], the notion of $\mathbf{b}_\simeq$-relative derived functors has been developed using such $\mathbf{b}_\simeq$-resolutions, which we also recall.

For an additive subcategory $\mathbf{a} = \mathbf{b}_\simeq$ of the homotopy category $\mathbf{B}_\simeq$, the $\mathbf{a}$-relative left derived functors $\mathrm{L}^{\mathbf{a}}_n F$, $n \geqslant 0$, of a functor $F : \mathbf{B}_\simeq \to \mathscr{A}$ from $\mathbf{B}_\simeq$ to an abelian category $\mathscr{A}$ are defined by

$$(\mathrm{L}^{\mathbf{a}}_n F)A = H_n(F(A_\bullet)),$$

where $A_\bullet$ is given by any $\mathbf{a}$-resolution of A. Similarly, $\mathbf{a}$-relative right derived functors of a contravariant functor $F : \mathbf{B}_\simeq^{\mathrm{op}} \to \mathscr{A}$ are given by

$$(\mathrm{R}^n_{\mathbf{a}} F)A = H^n(F(A_\bullet)).$$

In particular, for the contravariant functor $F = [\_, B]$ we get the $\mathbf{a}$-relative Ext-groups

$$\mathrm{Ext}^n_{\mathbf{a}}(A, B) := (\mathrm{R}^n_{\mathbf{a}}[\_, B])A = H^n([A_\bullet, B])$$

for any $\mathbf{a}$-exact resolution $A_\bullet$ of A. Similarly, for the contravariant functor $\mathrm{Aut}(0_{\_,B})$ which assigns to an object A the group $\mathrm{Aut}(0_{A,B})$ of all tracks $\alpha : 0_{A,B} \Rightarrow 0_{A,B}$ from the zero map $A \to * \to B$ to itself, one gets the groups of $\mathbf{a}$-derived automorphisms

$$\mathrm{Aut}^n_{\mathbf{a}}(A, B) := (\mathrm{R}^n_{\mathbf{a}}\,\mathrm{Aut}(0_{\_,B}))(A).$$

It is proved in [3] that under mild conditions (existence of a subset of $\mathbf{a}$ such that every object of $\mathbf{a}$ is a direct summand of a direct sum of objects from that subset) every object has an $\mathbf{a}$-resolution, and that the resulting groups do not depend on the choice of a resolution.

We next recall the construction of the secondary differential from [3]. This is the map of the form

$$d_{(2)} : \mathrm{Ext}^n_{\mathbf{a}}(A, B) \to \mathrm{Aut}^n_{\mathbf{a}}(0_{A,B});$$

it is constructed from any secondary $\mathbf{b}$-resolution $(A_\bullet, d_\bullet, \delta_\bullet, \epsilon, \hat{\epsilon})$ of the object A. Given an element $[c] \in \mathrm{Ext}^n_{\mathbf{a}}(A, B)$, one first represents it by an n-cocycle in $[(A_\bullet, [d_\bullet]), B]$, i. e. by a homotopy class $[c] \in [A_n, B]$ with $[cd_n] = 0$. One then chooses an actual representative $c : A_n \to B$ of it in $\mathbf{B}$ and a track $\gamma : 0 \Rightarrow cd_n$. It can be shown that the composite track $\Gamma = c\delta_n \square \gamma d_{n+1} \in \mathrm{Aut}(0_{A_{n+2},B})$ satisfies $\Gamma d_{n+1} = 0$, so it is an $(n+2)$-cocycle in the cochain complex $\mathrm{Aut}(0_{(A_\bullet,[d_\bullet]),B}) \cong [(\Sigma A_\bullet, [\Sigma d_\bullet]), B]$, so determines a cohomology class $d(2)([c]) = [\Gamma] \in \mathrm{Ext}^{n+2}_{\mathbf{a}}(\Sigma A, B)$. It is proved in [3, 4.2] that the above construction does not indeed depend on choices.

Now turning to our situation, it is straightforward to verify that a secondary chain complex in the sense of [3] in the track category B-**Mod** is the same as the 2-complex in the sense of 3.1, and that the two notions of exactness coincide. In particular then the notions of resolution are also equivalent.

The track subcategory $\mathbf{b}$ of free modules is generated by coproducts from a single object, so $\mathbf{b}_\simeq$-resolutions of any B-module exist. In fact it follows from [3, 2.13] that any B-module has a secondary $\mathbf{b}$-resolution too.

Moreover there are natural isomorphisms

$$\mathrm{Aut}(0_{M,N}) \cong \mathrm{Hom}_{\pi_0 B}(\pi_0 M, \pi_1 N).$$

Indeed a track from the zero map to itself is a B_0-module homomorphism $H : M_0 \to N_1$ with $\partial H = 0$, $H\partial = 0$, so H factors through $M_0 \twoheadrightarrow \pi_0 M$ and over $\pi_1 N \rightarrowtail N_1$.

Hence the proof is finished with the following lemma. □

(3.7) Lemma. *For any B-modules M, N there are isomorphisms*

$$\mathrm{Ext}^n_{\mathbf{a}}(M,N) \cong \mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \pi_0 N)$$

and

$$(\mathrm{R}^n_{\mathbf{a}}(\mathrm{Hom}_{\pi_0 B}(\pi_0(\_), \pi_1 N)))(M) \cong \mathrm{Ext}^n_{\pi_0 B}(\pi_0 M, \pi_1 N).$$

Proof. By definition the groups $\mathrm{Ext}^*_{\mathbf{a}}(M,N)$, respectively $(\mathrm{R}^n_{\mathbf{a}}(\mathrm{Hom}_{B_0}(\pi_0(\_), \pi_1 N)))(M)$, are cohomology groups of the complex $[F_\bullet, N]$, resp. $\mathrm{Hom}_{\pi_0 B}(\pi_0(F_\bullet), \pi_1 N)$, where $F_\bullet$ is some **a**-resolution of M. We can choose for $\mathbb{F}_\bullet$ some secondary **b**-resolution of M. Then $\pi_0 F_\bullet$ is a free $\pi_0 B$-resolution of $\pi_0 M$, which makes evident the second isomorphism. For the first, just note in addition that by 1.9 $[F_\bullet, N]$ is isomorphic to $\mathrm{Hom}_{B_0}(\pi_0(F_\bullet), \pi_0 N)$. □

4. T

In this section we introduce the notion of stable maps and stable tracks between spectra. This yields the track category of spectra. See also [1, section 2.5].

(4.1) Definition. A *spectrum X* is a sequence of maps

$$X_i \xrightarrow{r} \Omega X_{i+1}, \ i \in \mathbb{Z}$$

in the category $\mathbf{Top}^*$ of pointed spaces. This is an Ω-spectrum if r is a homotopy equivalence for all i.

A *stable homotopy class* $f : X \to Y$ between spectra is a sequence of homotopy classes $f_i \in [X_i, Y_i]$ such that the squares

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ \downarrow r & & \downarrow r \\ \Omega X_{i+1} & \xrightarrow{\Omega f_{i+1}} & \Omega Y_{i+1} \end{array}$$

commute in $\mathbf{Top}^*_{\simeq}$. The category **Spec** consists of spectra and stable homotopy classes as morphisms. Its full subcategory Ω-**Spec** consisting of Ω-spectra is equivalent to the usual homotopy category of spectra considered as a Quillen model category.

A *stable map* $f = (f_i, \tilde{f}_i)_i : X \to Y$ between spectra is a sequence of diagrams in the track category $[\![\mathbf{Top}^*]\!]$ $(i \in \mathbb{Z})$

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ r\downarrow & \tilde{f}_i \nearrow\!\!\!\nearrow & \downarrow r \\ \Omega X_{i+1} & \xrightarrow[\Omega f_{i+1}]{} & \Omega Y_{i+1}. \end{array}$$

Obvious composition of such maps yields the category

$$[\![\mathbf{Spec}]\!]_0 \, .$$

It is the underlying category of a track category $[\![\mathbf{Spec}]\!]$ with tracks $(H : f \Rightarrow g) \in [\![\mathbf{Spec}]\!]_1$ given by sequences

$$H_i : f_i \Rightarrow g_i$$

of tracks in $\mathbf{Top}^*$ such that the diagrams

$$\begin{array}{ccc} & \overset{g_i}{\frown} \; H_i \Uparrow & \\ X_i & \xrightarrow{f_i} & Y_i \\ r\downarrow & \tilde{f}_i \nearrow\!\!\!\nearrow & \downarrow r \\ \Omega X_{i+1} & \xrightarrow{\Omega f_{i+1}} & \Omega Y_{i+1} \\ & \Omega H_{i+1} \Downarrow \; \underset{\Omega g_{i+1}}{\smile} & \end{array}$$

paste to $\tilde{g}_i$. This yields a well-defined track category $[\![\mathbf{Spec}]\!]$. Moreover

$$[\![\mathbf{Spec}]\!]_{\simeq} \cong \mathbf{Spec}$$

is an isomorphism of categories. Let $[\![X, Y]\!]$ be the groupoid of morphisms $X \to Y$ in $[\![\mathbf{Spec}]\!]_0$ and let $[\![X, Y]\!]_1^0$ be the set of pairs (f, H) where $f : X \to Y$ is a map and $H : f \Rightarrow 0$ is a track in $[\![\mathbf{Spec}]\!]$, i. e. a stable homotopy class of nullhomotopies for f.

For a spectrum X let $\Sigma^k X$ be the *shifted spectrum* with $(\Sigma^k X)_n = X_{n+k}$ and the commutative diagram

$$\begin{array}{ccc} (\Sigma^k X)_n & \xrightarrow{r} & \Omega(\Sigma^k X)_{n+1} \\ \| & & \| \\ X_{n+k} & \xrightarrow{r} & \Omega(X_{n+k+1}) \end{array}$$

defining r for $\Sigma^k X$. A map $f : Y \to \Sigma^k X$ is also called a map f *of degree* k from Y to X.

5. T $\mathscr{B}$ $\mathscr{B}$-

The secondary cohomology of a space was introduced in [1, section 6.3]. We here consider the corresponding notion of secondary cohomology of a spectrum.

Let $\mathbb{F}$ be a prime field $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ and let Z denote the Eilenberg-Mac Lane spectrum with

$$Z^n = K(\mathbb{F}, n)$$

chosen as in [1]. Here Z^n is a topological $\mathbb{F}$-vector space and the homotopy equivalence $Z^n \to \Omega Z^{n+1}$ is $\mathbb{F}$-linear. This shows that for a spectrum X the sets $\left[\!\!\left[X, \Sigma^k Z\right]\!\!\right]_0$ and $\left[\!\!\left[X, \Sigma^k Z\right]\!\!\right]_1^0$, of stable maps and stable 0-tracks repectively, are $\mathbb{F}$-vector spaces.

We now recall the definition of the pair algebra $\mathscr{B} = (\partial : \mathscr{B}_1 \to \mathscr{B}_0)$ of secondary cohomology operations from [1]. Let $\mathbb{G} = \mathbb{Z}/p^2\mathbb{Z}$ and let

$$\mathscr{B}_0 = T_{\mathbb{G}}(E_{\mathscr{A}})$$

be the $\mathbb{G}$-tensor algebra generated by the subset

$$E_{\mathscr{A}} = \begin{cases} \left\{\mathrm{Sq}^1, \mathrm{Sq}^2, ...\right\} & \text{for } p = 2, \\ \left\{\mathrm{P}^1, \mathrm{P}^2, ...\right\} \cup \left\{\beta, \beta\mathrm{P}^1, \beta\mathrm{P}^2, ...\right\} & \text{for odd } p \end{cases}$$

of the mod p Steenrod algebra $\mathscr{A}$. We define $\mathscr{B}_1$ by the pullback diagram of graded abelian groups

$$\begin{array}{ccc} & & \Sigma\mathscr{A} \\ & & \downarrow \\ \mathscr{B}_1 & \longrightarrow & [\![Z, \Sigma^* Z]\!]_1^0 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \mathscr{B}_0 & \xrightarrow{s} & [\![Z, \Sigma^* Z]\!]_0 \\ & & \downarrow \\ & & \mathscr{A}. \end{array} \tag{5.1}$$

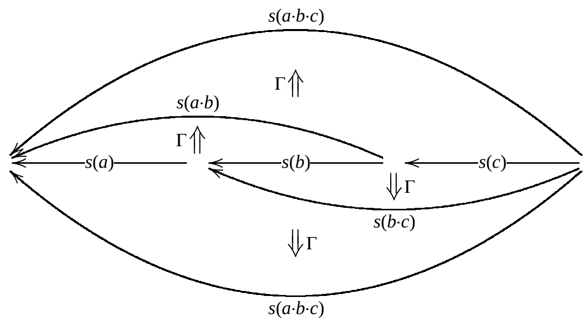
in which the right hand column is an exact sequence. Here we choose for $\alpha \in E_{\mathscr{A}}$ a stable map $s(\alpha) : Z \to \Sigma^{|\alpha|} Z$ representing α and we define s to be the $\mathbb{G}$-linear map given on monomials $a_1 \cdots a_n$ in the free monoid $\mathrm{Mon}(E_{\mathscr{A}})$ generated by $E_{\mathscr{A}}$ by the composites

$$s(a_1 \cdots a_n) = s(a_1) \cdots s(a_n).$$

It is proved in [1, 5.2.3] that s defines a pseudofunctor, that is, there is a well-defined track

$$\Gamma : s(a \cdot b) \Rightarrow s(a) \circ s(b)$$

for $a, b \in \mathscr{B}_0$ such that for any a, b, c pasting of tracks in the diagram



yields the identity track. Now $\mathscr{B}_1$ is a $\mathscr{B}_0$-$\mathscr{B}_0$-bimodule by defining

$$a(b,z) = (a \cdot b, a \bullet z)$$

with $a \bullet z$ given by pasting $s(a)z$ and Γ. Similarly

$$(b,z)a = (b \cdot a, z \bullet a)$$

where $z \bullet a$ is obtained by pasting $zs(a)$ and Γ. Then it is shown in [1] that $\mathscr{B} = (\partial : \mathscr{B}_1 \to \mathscr{B}_0)$ is a well-defined pair algebra with $\pi_0\mathscr{B} = \mathscr{A}$ and Σ-structure $\pi_1\mathscr{B} = \Sigma\mathscr{A}$.

For a spectrum X let

$$\mathscr{H}(X)_0 = \mathscr{B}_0 \, [\![X, \Sigma^* Z]\!]_0$$

be the free $\mathscr{B}_0$-module generated by the graded set $[\![X, \Sigma^* Z]\!]_0$. We define $\mathscr{H}(X)_1$ by the pullback diagram

$$\begin{array}{ccc} & & \Sigma H^* X \\ & & \downarrow \\ \mathscr{H}(X)_1 & \longrightarrow & [\![X, \Sigma^* Z]\!]_1^0 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \mathscr{H}(X)_0 & \xrightarrow{s} & [\![X, \Sigma^* Z]\!]_0 \\ & & \downarrow \\ & & H^* X \end{array}$$

where s is the $\mathbb{G}$-linear map which is the identity on generators and is defined on words $a_1 \cdots a_n \cdot u$ by the composite $s(a_1) \cdots s(a_n)s(u)$ for a_i as above and $u \in [\![X, \Sigma^* Z]\!]_0$. Again s is a pseudofunctor and with actions $\bullet$ defined as above we see that the graded pair module

$$\mathscr{H}(X) = \left(\mathscr{H}(X)_1 \xrightarrow{\partial} \mathscr{H}(X)_0 \right)$$

is a $\mathscr{B}$-module. We call $\mathscr{H}(X)$ the *secondary cohomology* of the spectrum X. Of course $\mathscr{H}(X)$ has a Σ-structure in the sense of 2.3 above.

(5.2) Example. Let $\mathbb{G}^\Sigma$ be the $\mathscr{B}$-module given by the augmentation $\mathscr{B} \to \mathbb{G}^\Sigma$ in [1]. Recall that $\mathbb{G}^\Sigma$ is the pair

$$\mathbb{G}^\Sigma = \left(\mathbb{F} \oplus \Sigma\mathbb{F} \xrightarrow{\partial} \mathbb{G} \right)$$

with $\partial|_{\mathbb{F}}$ the inclusion nad $\partial|_{\Sigma\mathbb{F}} = 0$. Then the sphere spectrum S^0 admits a weak equivalence of $\mathscr{B}$-modules

$$\mathscr{H}(S^0) \xrightarrow{\sim} \mathbb{G}^\Sigma.$$

Compare [1, 12.1.5].

6. T E_3- A

We now are ready to formulate our main result describing the algebraic equivalent of the E_3-term of the Adams spectral sequence. Let X be a spectrum of finite type and Y a finite dimensional spectrum. Then for each prime p there is a spectral sequence $\mathrm{E}_* = \mathrm{E}_*(Y, X)$ with

$$\mathrm{E}_* \Longrightarrow [Y, \Sigma^* X]_p$$
$$\mathrm{E}_2 = \mathrm{Ext}_{\mathscr{A}}(H^*X, H^*Y).$$

(6.1) Theorem. *The* E_3*-term* $\mathrm{E}_3 = \mathrm{E}_3(Y, X)$ *of the Adams spectral sequence is given by the secondary* Ext *group defined in 3.5*

$$\mathrm{E}_3 = \mathrm{Ext}_{\mathscr{B}}(\mathscr{H}^*X, \mathscr{H}^*Y).$$

(6.2) Corollary. *If X and Y are both the sphere spectrum we get*

$$\mathrm{E}_3(S^0, S^0) = \mathrm{Ext}_{\mathscr{B}}(\mathbb{G}^\Sigma, \mathbb{G}^\Sigma).$$

Since the pair algebra $\mathscr{B}$ is computed in [1] completely we see that $\mathrm{E}_3(S^0, S^0)$ is algebraically determined. This leads to the algorithm below computing $\mathrm{E}_3(S^0, S^0)$.

The proof of 6.1 is based on the following result in [1]. Consider the track categories

$$\mathbf{b} \subset [\![\mathbf{Spec}]\!]$$
$$\mathbf{b}' \subset (\mathscr{B}{-}\mathbf{Mod})^\Sigma$$

where $[\![\mathbf{Spec}]\!]$ is the track category of spectra in 4.1 and $(\mathscr{B}{-}\mathbf{Mod})^\Sigma$ is the track category of $\mathscr{B}$-modules with Σ-structure in 2.3 with the pair algebra $\mathscr{B}$ defined by (5.1). Let $\mathbf{b}$ be the full track subcategory of $[\![\mathbf{Spec}]\!]$ consisting of finite products of shifted Eilenberg-Mac Lane spectra $\Sigma^k Z^*$. Moreover let $\mathbf{b}'$ be the full track subcategory of $(\mathscr{B}{-}\mathbf{Mod})^\Sigma$ consisting of finitely generated free $\mathscr{B}$-modules. As in [3, 4.3] we obtain for spectra X, Y in 6.1 the track categories

$$\{Y, X\}\,\mathbf{b} \subset [\![\mathbf{Spec}]\!]$$
$$\mathbf{b}'\,\{\mathscr{H}X, \mathscr{H}Y\} \subset (\mathscr{B}{-}\mathbf{Mod})^\Sigma$$

with $\{Y, X\}\,\mathbf{b}$ obtained by adding to $\mathbf{b}$ the objects X, Y and all morphisms and tracks from $[\![X, Z]\!]$, $[\![Y, Z]\!]$ for all objects Z in $\mathbf{b}$. It is proved in [1, 5.5.6] that the following result holds which shows that we can apply [3, 5.1].

(6.3) Theorem [1]. *There is a strict track equivalence*

$$(\{Y, X\}\,\mathbf{b})^{\mathrm{op}} \xrightarrow{\sim} \mathbf{b}'\,\{\mathscr{H}X, \mathscr{H}Y\}\,.$$

□

Proof of 6.1. By the main result 7.3 in [3] we have a description of the differential $d_{(2)}$ in the Adams spectral sequence by the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_{\mathbf{a}^{\mathrm{op}}}(X, Y)^m & \xrightarrow{d_{(2)}} & \mathrm{Ext}^{n+2}_{\mathbf{a}^{\mathrm{op}}}(X, Y)^{m+1} \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{Ext}^n_{\mathscr{A}}(H^*X, H^*Y)^m & \xrightarrow{d_{(2)}} & \mathrm{Ext}^{n+2}_{\mathscr{A}}(H^*X, H^*Y)^{m+1}
\end{array}$$

where $\mathbf{a} = \mathbf{b}_{\simeq}$. On the other hand the differential $d_{(2)}$ defining the secondary Ext-group $\mathrm{Ext}_{\mathscr{B}}(\mathscr{H}X, \mathscr{H}Y)$ is by 3.6 given by the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^n_{\mathbf{a}'}(\mathscr{H}X, \mathscr{H}Y)^m & \longrightarrow & \mathrm{Ext}^{n+2}_{\mathbf{a}'}(\mathscr{H}X, \mathscr{H}Y)^{m+1} \\
\| & & \| \\
\mathrm{Ext}^n_{\mathscr{A}}(H^*X, H^*Y)^m & \longrightarrow & \mathrm{Ext}^{n+2}_{\mathscr{A}}(H^*X, H^*Y)^{m+1}
\end{array}$$

where $\mathbf{a}' = \mathbf{b}'_{\simeq}$. Now [3, 5.1] shows by 6.3 that the top rows of these diagrams coincide. □

7. T $\mathscr{B}$

We recall notation $\mathbb{G} = \mathbb{Z}/4\mathbb{Z}$, $\mathbb{F} = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ from [1]. The quotient homomorphism $\mathbb{G} \twoheadrightarrow \mathbb{F}$ will be denoted by π and the isomorphism $\mathbb{F} \cong 2\mathbb{G}$ by i. Moreover we will need the set-theoretic section $\chi : \mathbb{F} \hookrightarrow \mathbb{G}$ of π given by $\chi(0) = 0$, $\chi(1) = 1$. In the pair algebra $\mathscr{B} = (\partial : \mathscr{B}_1 \to \mathscr{B}_0)$, recall that $\mathscr{B}_0$ is the graded free associative $\mathbb{G}$-algebra on the generators Sq^n of degree n, for $n \geqslant 1$; there is thus a surjective homomorphism of graded algebras $\pi : \mathscr{B}_0 \twoheadrightarrow \mathscr{A}$ onto the mod 2 Steenrod algebra. Its kernel is denoted by R, so that we have the short exact sequence

$$0 \to R \to \mathscr{B}_0 \to \mathscr{A} \to 0.$$

It is well known that R is a graded two-sided ideal generated (as a two-sided ideal, i. e. as a $\mathscr{B}_0$-$\mathscr{B}_0$-bimodule) by $2\mathscr{B}_0$ and by the *Adem elements*

$$[a, b] := \mathrm{Sq}^a \mathrm{Sq}^b + \sum_{k=\max(0,a-b+1)}^{\min\left(b-1,\left[\frac{a}{2}\right]\right)} \chi\binom{b-k-1}{a-2k} \mathrm{Sq}^{a+b-k} \mathrm{Sq}^k,$$

for $0 < a < 2b$. As shown in [1], one can generate R as a right $\mathscr{B}_0$-module by $2 \in R^0$ and the *admissible relations* $\mathrm{Sq}^{a_k} \mathrm{Sq}^{a_{k-1}} \cdots \mathrm{Sq}^{a_1}[a_0, b] \in R^{a_k+\ldots+a_0+b}$, with $a_{j+1} \geqslant 2a_j$ for all $k > j \geqslant 0$ and $a_0 < 2b$.

As for the rest of the structure of $\mathscr{B}$, as an abelian group, $\mathscr{B}_1$ is $R \oplus \Sigma\mathscr{A}$, that is,

$$\mathscr{B}_1^n = R^n \oplus \mathscr{A}^{n-1},$$

and ∂ is the projection. Moreover the $\mathscr{B}_0$-bimodule structure of $\mathscr{B}_1$ is given by

$$(r, a)b = (rb, a\pi(b))$$

and

$$b(r, a) = (br, A(\pi(b), r) + \pi(b)a),$$

where

$$A : \mathscr{A} \otimes R \to \mathscr{A}$$

is the *multiplication map* of degree -1 described in [1]. Algebraic properties characterizing the multiplication map A are achieved in [1, theorem 16.3.3]. In [1, section 16.6] an algorithm is obtained which computes the multiplication map A.

Particular important elements of $\mathscr{B}_1$ get special notation; e. g. we have $[2] := (2, 0) \in R^0 \oplus 0 = \mathscr{B}_1^0$ and $\Sigma 1 := (0, 1) \in R^1 \oplus \mathscr{A}^0 = \mathscr{B}_1^1$. This pair algebra has an augmentation $\epsilon : \mathscr{B} \to \mathbb{G}^\Sigma$, where $\mathbb{G}^\Sigma = ((i, 0) : \mathbb{F} \oplus \Sigma\mathbb{F} \to \mathbb{G})$ is the graded $\mathscr{B}$-module equal to $i : \mathbb{F} \cong 2\mathbb{G} \subset \mathbb{G}$ in degree 0, to $\mathbb{F} \to 0$ in degree 1 and zero in all other degrees. Components of ϵ are the augmentation $\epsilon_0 : \mathscr{B}_0 \to \mathbb{G}$ and the homomorphism $\epsilon_1 : R \oplus \Sigma\mathscr{A} \to 2\mathbb{G} \oplus \Sigma\mathbb{F}$ given by $(r, a) \mapsto (\epsilon_0(r), \epsilon(a))$.

8. T $d_{(2)}$ $\mathrm{Ext}_{\mathscr{A}}(\mathbb{F}, \mathbb{F})$

Suppose now given some projective resolution of the left $\mathscr{A}$-module $\mathbb{F}$. For definiteness, we will work with the minimal resolution

$$\mathbb{F} \leftarrow \mathscr{A}\left\langle g_0^0 \right\rangle \leftarrow \mathscr{A}\left\langle g_1^{2^n} \mid n \geqslant 0 \right\rangle \leftarrow \mathscr{A}\left\langle g_2^{2^i+2^j} \mid |i-j| \neq 1 \right\rangle \leftarrow \ldots, \tag{8.1}$$

where g_m^d, $d \geqslant m$, is a generator of the m-th resolving module in degree d. Sometimes there are more than one generators with the same m and d, in which case the further ones will be denoted by ${}'g_m^d$, ${}''g_m^d$, $\cdots$.

These generators and values of the differential on them can be computed effectively; for example, $d(g_1^{2^n}) = \mathrm{Sq}^{2^n}\, g_0^0$ and $d(g_m^m) = \mathrm{Sq}^1\, g_{m-1}^{m-1}$; moreover e. g. an algorithm from [4] gives

$$
\begin{aligned}
d(g_2^4) &= \mathrm{Sq}^3\, g_1^1 + \mathrm{Sq}^2\, g_1^2\\
d(g_2^5) &= \mathrm{Sq}^4\, g_1^1 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, g_1^2 + \mathrm{Sq}^1\, g_1^4\\
d(g_2^8) &= \mathrm{Sq}^6\, g_1^2 + (\mathrm{Sq}^4 + \mathrm{Sq}^3\,\mathrm{Sq}^1) g_1^4\\
d(g_2^9) &= \mathrm{Sq}^8\, g_1^1 + (\mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1) g_1^4 + \mathrm{Sq}^1\, g_1^8\\
d(g_2^{10}) &= (\mathrm{Sq}^8 + \mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1) g_1^2 + (\mathrm{Sq}^5\,\mathrm{Sq}^1 + \mathrm{Sq}^4\,\mathrm{Sq}^2) g_1^4 + \mathrm{Sq}^2\, g_1^8\\
d(g_2^{16}) &= (\mathrm{Sq}^{12} + \mathrm{Sq}^9\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^8\,\mathrm{Sq}^3\,\mathrm{Sq}^1) g_1^4 + (\mathrm{Sq}^8 + \mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^6\,\mathrm{Sq}^2) g_1^8\\
&\cdots,\\
d(g_3^6) &= \mathrm{Sq}^4\, g_2^2 + \mathrm{Sq}^2\, g_2^4 + \mathrm{Sq}^1\, g_2^5\\
d(g_3^{10}) &= \mathrm{Sq}^8\, g_2^2 + (\mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1) g_2^5 + \mathrm{Sq}^1\, g_2^9\\
d(g_3^{11}) &= (\mathrm{Sq}^7 + \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) g_2^4 + \mathrm{Sq}^6\, g_2^5 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, g_2^8\\
d(g_3^{12}) &= \mathrm{Sq}^8\, g_2^4 + (\mathrm{Sq}^6\,\mathrm{Sq}^1 + \mathrm{Sq}^5\,\mathrm{Sq}^2) g_2^5 + (\mathrm{Sq}^4 + \mathrm{Sq}^3\,\mathrm{Sq}^1) g_2^8 + \mathrm{Sq}^3\, g_2^9 + \mathrm{Sq}^2\, g_2^{10}\\
&\cdots,\\
d(g_4^{11}) &= \mathrm{Sq}^8\, g_3^3 + (\mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1) g_3^6 + \mathrm{Sq}^1\, g_3^{10}\\
d(g_4^{13}) &= \mathrm{Sq}^8\,\mathrm{Sq}^2\, g_3^3 + (\mathrm{Sq}^7 + \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) g_3^6 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, g_3^{10} + \mathrm{Sq}^2\, g_3^{11}\\
&\cdots,\\
d(g_5^{14}) &= \mathrm{Sq}^{10}\, g_4^4 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, g_4^{11}\\
d(g_5^{16}) &= \mathrm{Sq}^{12}\, g_4^4 + \mathrm{Sq}^4\,\mathrm{Sq}^1\, g_4^{11} + \mathrm{Sq}^3\, g_4^{13}\\
&\cdots,\\
d(g_6^{16}) &= \mathrm{Sq}^{11}\, g_5^5 + \mathrm{Sq}^2\, g_5^{14}\\
&\cdots,
\end{aligned}
$$

etc.

By understanding the above formulæ *literally* (i. e. by applying χ degreewise to them), each such resolution gives rise to a sequence of $\mathscr{B}$-module homomorphisms

$$
\mathbb{G}^{\Sigma} \leftarrow \mathscr{B}\left\langle g_0^0\right\rangle \leftarrow \mathscr{B}\left\langle g_1^{2^n} \mid n \geqslant 0\right\rangle \leftarrow \mathscr{B}\left\langle g_2^{2^i+2^j} \mid |i-j| \neq 1\right\rangle \leftarrow \ldots, \tag{8.2}
$$

which is far from being exact — in fact even the composites of consecutive maps are not zero. In more detail, one has commutative diagrams

$$
\begin{array}{ccccccc}
2\mathbb{G} & \xleftarrow{\epsilon_0} & R^0 g_0^0 & \longleftarrow & 0 & \longleftarrow & \cdots\\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{G} & \xleftarrow{\epsilon_0} & \mathscr{B}_0^0 g_0^0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}
$$

in degree 0,

$$
\begin{array}{ccccccccc}
\mathbb{F} & \xleftarrow{(0,\epsilon)} & R^1 g_0^0 \oplus \mathscr{A}^0 g_0^0 & \xleftarrow{\binom{d}{0}} & R^0 g_1^1 & \longleftarrow & 0 & \longleftarrow & \cdots\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathscr{B}_0^1 g_0^0 & \xleftarrow{d} & \mathscr{B}_0^0 g_1^1 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}
$$

in degree 1,

$$\begin{array}{ccccccccc}
0 & \longleftarrow & R^2g_0^0\oplus\mathscr{A}^1g_0^0 & \xleftarrow{\left(\begin{smallmatrix}d&0\\0&d\end{smallmatrix}\right)} & \left(R^1g_1^1\oplus R^0g_1^2\right)\oplus\mathscr{A}^0g_1^1 & \xleftarrow{\left(\begin{smallmatrix}d\\0\end{smallmatrix}\right)} & R^0g_2^2 & \longleftarrow & 0 \longleftarrow \cdots\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longleftarrow & \mathscr{B}_0^2g_0^0 & \xleftarrow{\quad d\quad} & \mathscr{B}_0^1g_1^1\oplus\mathscr{B}_0^0g_1^2 & \xleftarrow{\quad d\quad} & \mathscr{B}_0^0g_2^2 & \longleftarrow & 0 \longleftarrow \cdots
\end{array}$$

in degree 2, ...

$$\begin{array}{ccccccc}
0 & \longleftarrow & R^ng_0^0\oplus\mathscr{A}^{n-1}g_0^0 & \xleftarrow{\left(\begin{smallmatrix}d&0\\0&d\end{smallmatrix}\right)} & \bigoplus_{2^i\leqslant n}R^{n-2^i}g_1^{2^i}\oplus\bigoplus_{2^i\leqslant n-1}\mathscr{A}^{n-1-2^i}g_1^{2^i} & \longleftarrow & \cdots\\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & \mathscr{B}_0^ng_0^0 & \xleftarrow{\quad d\quad} & \bigoplus_{2^i\leqslant n}\mathscr{B}_0^{n-2^i}g_1^{2^i} & \longleftarrow & \cdots
\end{array}$$

in degree n, etc.

Our task is then to complete these diagrams into an exact secondary complex via certain (degree preserving) maps

$$\delta_m=\begin{pmatrix}\delta_m^R\\ \delta_m^{\mathscr{A}}\end{pmatrix}:\mathscr{B}_0\left\langle g_{m+2}^n\mid n\right\rangle\to(R\oplus\Sigma\mathscr{A})\left\langle g_m^n\mid n\right\rangle.$$

Now for these maps to form a secondary complex, according to 3.1.1 one must have $\partial\delta=d_0d_0$, $\delta\partial=d_1d_1$, and $d_1\delta=\delta d_0$. One sees easily that these equations together with the requirement that δ be left $\mathscr{B}_0$-module homomorphism are equivalent to

(8.3) $$\delta^R=dd,$$

(8.4) $$\delta^{\mathscr{A}}(bg)=\pi(b)\delta^{\mathscr{A}}(g)+A(\pi(b),dd(g)),$$

(8.5) $$d\delta^{\mathscr{A}}=\delta^{\mathscr{A}}d,$$

for $b\in\mathscr{B}_0$, g one of the g_m^n, and $A(a,rg):=A(a,r)g$ for $a\in\mathscr{A}$, $r\in R$. Hence δ is completely determined by the elements

$$\delta_m^{\mathscr{A}}(g_{m+2}^n)\in\bigoplus_k\mathscr{A}^{n-k-1}\left\langle g_m^k\right\rangle$$

which, to form a secondary complex, are only required to satisfy

$$d\delta_m^{\mathscr{A}}(g_{m+2}^n)=\delta_{m-1}^{\mathscr{A}}d(g_{m+2}^n),$$

where on the right $\delta_{m-1}^{\mathscr{A}}$ is extended to $\mathscr{B}_0\left\langle g_{m+1}^*\right\rangle$ via 8.4. Then furthermore secondary exactness must hold, which by 3.1 means that the (ordinary) complex

$$\leftarrow\mathscr{B}_0\left\langle g_{m-1}^*\right\rangle\oplus(R\oplus\Sigma\mathscr{A})\left\langle g_{m-2}^*\right\rangle\leftarrow\mathscr{B}_0\left\langle g_m^*\right\rangle\oplus(R\oplus\Sigma\mathscr{A})\left\langle g_{m-1}^*\right\rangle\leftarrow\mathscr{B}_0\left\langle g_{m+1}^*\right\rangle\oplus(R\oplus\Sigma\mathscr{A})\left\langle g_m^*\right\rangle\leftarrow$$

with differentials

$$\left(\begin{smallmatrix}d_{m+1}&i_{m+1}&0\\ d_md_{m+1}&d_m&0\\ \delta_m^{\mathscr{A}}&0&d_m\end{smallmatrix}\right):\mathscr{B}_0\left\langle g_{m+2}^*\right\rangle\oplus R\left\langle g_{m+1}^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_{m+1}^*\right\rangle\to\mathscr{B}_0\left\langle g_{m+1}^*\right\rangle\oplus R\left\langle g_m^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_m^*\right\rangle$$

is exact. Then straightforward checking shows that one can eliminate R from this complex altogether, so that its exactness is equivalent to the exactness of a smaller complex

$$\leftarrow\mathscr{B}_0\left\langle g_{m-1}^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_{m-2}^*\right\rangle\leftarrow\mathscr{B}_0\left\langle g_m^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_{m-1}^*\right\rangle\leftarrow\mathscr{B}_0\left\langle g_{m+1}^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_m^*\right\rangle\leftarrow$$

with differentials

$$\left(\begin{smallmatrix}d_{m+1}&0\\ \delta_m^{\mathscr{A}}&d_m\end{smallmatrix}\right):\mathscr{B}_0\left\langle g_{m+2}^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_{m+1}^*\right\rangle\to\mathscr{B}_0\left\langle g_{m+1}^*\right\rangle\oplus\Sigma\mathscr{A}\left\langle g_m^*\right\rangle.$$

Note also that by 8.4 $\delta^{\mathscr{A}}$ factors through π to give

$$\bar{\delta}_m:\mathscr{A}\left\langle g_{m+2}^*\right\rangle\to\Sigma\mathscr{A}\left\langle g_m^*\right\rangle.$$

It follows that secondary exactness of the resulting complex is equivalent to exactness of the *mapping cone* of this $\bar{\delta}$, i. e. to the requirement that $\bar{\delta}$ is a quasiisomorphism. On the other hand, the complex $(\mathscr{A}\left\langle g_*^*\right\rangle,d_*)$ is acyclic by construction, so any of its self-maps is a quasiisomorphism. We thus obtain

(8.6) Theorem. *Completions of the diagram 8.2 to an exact secondary complex are in one-to-one correspondence with maps* $\delta_m : \mathscr{A}\left\langle g^*_{m+2}\right\rangle \to \Sigma\mathscr{A}\left\langle g^*_m\right\rangle$ *satisfying*

$$d\delta g = \delta d g, \tag{8.7}$$

with $\delta(ag)$ *for* $a \in \mathscr{A}$ *defined by*

$$\delta(ag) = a\delta(g) + A(a, ddg)$$

where $A(a, rg)$ *for* $r \in R$ *is interpreted as* $A(a, r)g$*.*

□

We can use this to construct the secondary resolution inductively. Just start by introducing values of δ on the generators as expressions with indeterminate coefficients; the equation (8.7) will impose linear conditions on these coefficients. These are then solved degree by degree. For example, in degree 2 one may have

$$\delta(g^2_2) = \eta^2_2(\mathrm{Sq}^1)\,\mathrm{Sq}^1\, g^0_0$$

for some $\eta^2_2(\mathrm{Sq}^1) \in \mathbb{F}$. Similarly in degree 3 one may have

$$\delta(g^3_3) = \eta^3_3(\mathrm{Sq}^1)\,\mathrm{Sq}^1\, g^1_1 + \eta^3_3(1)g^2_1.$$

Then one will get

$$d\delta(g^3_3) = \eta^3_3(\mathrm{Sq}^1)\,\mathrm{Sq}^1\, d(g^1_1) + \eta^3_3(1)d(g^2_1) = \eta^3_3(\mathrm{Sq}^1)\,\mathrm{Sq}^1\,\mathrm{Sq}^1\, g^0_0 + \eta^3_3(1)\,\mathrm{Sq}^2\, g^0_0 = \eta^3_3(1)\,\mathrm{Sq}^2\, g^0_0$$

and

$$\begin{aligned}\delta d(g^3_3) &= \delta(\mathrm{Sq}^1\, g^2_2)\\ &= \mathrm{Sq}^1\,\delta(g^2_2) + A(\mathrm{Sq}^1, dd(g^2_2)) = \eta^2_2(\mathrm{Sq}^1)\,\mathrm{Sq}^1\,\mathrm{Sq}^1\, g^0_0 + A(\mathrm{Sq}^1, d(\mathrm{Sq}^1\, g^1_1)) = A(\mathrm{Sq}^1, \mathrm{Sq}^1\,\mathrm{Sq}^1\, g^0_0) = 0;\end{aligned}$$

thus (8.7) forces $\eta^3_3(1) = 0$.

Similarly one puts $\delta(g^d_m) = \sum_{m-2\leqslant d'\leqslant d-1}\sum_a \eta^d_m(a)ag^{d'}_{m-2}$, with a running over a basis in $\mathscr{A}^{d-1-d'}$, and then substituting this in (8.7) gives linear equations on the numbers $\eta^d_m(a)$. Solving these equations and choosing the remaining η's arbitrarily then gives values of the differential δ in the secondary resolution.

Then finally to obtain the secondary differential

$$d_{(2)} : \mathrm{Ext}^n_{\mathscr{A}}(\mathbb{F}, \mathbb{F})^m \to \mathrm{Ext}^{n+2}_{\mathscr{A}}(\mathbb{F}, \mathbb{F})^{m+1}$$

from this δ, one just applies the functor $\mathrm{Hom}_{\mathscr{A}}(\_, \mathbb{F})$ to the initial minimal resolution and calculates the map induced by δ on cohomology of the resulting cochain complex, i. e. on $\mathrm{Ext}^*_{\mathscr{A}}(\mathbb{F}, \mathbb{F})$. In fact since (8.1) is a minimal resolution, the value of $\mathrm{Hom}_{\mathscr{A}}(\_, \mathbb{F})$ on it coincides with its own cohomology and is the $\mathbb{F}$-vector space of those linear maps $\mathscr{A}\left\langle g^*_*\right\rangle \to \mathbb{F}$ which vanish on all elements of the form ag^*_* with a of positive degree.

Let us then identify $\mathrm{Ext}^*_{\mathscr{A}}(\mathbb{F}, \mathbb{F})$ with this space and choose a basis in it consisting of elements $\hat{g}^d_m$ defined as the maps sending the generator g^d_m to 1 and all other generators to 0. One then has

$$(d_{(2)}(\hat{g}^d_m))(g^{d'}_{m'}) = \hat{g}^d_m\delta(g^{d'}_{m'}).$$

The right hand side is nonzero precisely when g^d_m appears in $\delta(g^{d'}_{m'})$ with coefficient 1, i. e. one has

$$d_{(2)}(\hat{g}^d_m) = \sum_{g^d_m \text{ appears in } \delta(g^{d+1}_{m+2})} \hat{g}^{d+1}_{m+2}.$$

For example, looking at the table of values of δ below we see that the first instance of a g^d_m appearing with coefficient 1 in a value of δ on a generator is

$$\delta(g^{17}_3) = g^{16}_1 + \mathrm{Sq}^{12}\, g^4_1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^4\, g^2_1 + (\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5 + \mathrm{Sq}^{11}\,\mathrm{Sq}^4)g^1_1.$$

This means

$$d_{(2)}(\hat{g}^{16}_1) = \hat{g}^{17}_3$$

and moreover $d_{(2)}(\hat{g}^d_m) = 0$ for all g^d_m with $d < 17$ (one can check all cases for each given d since the number of generators g^d_m for each given d is finite).

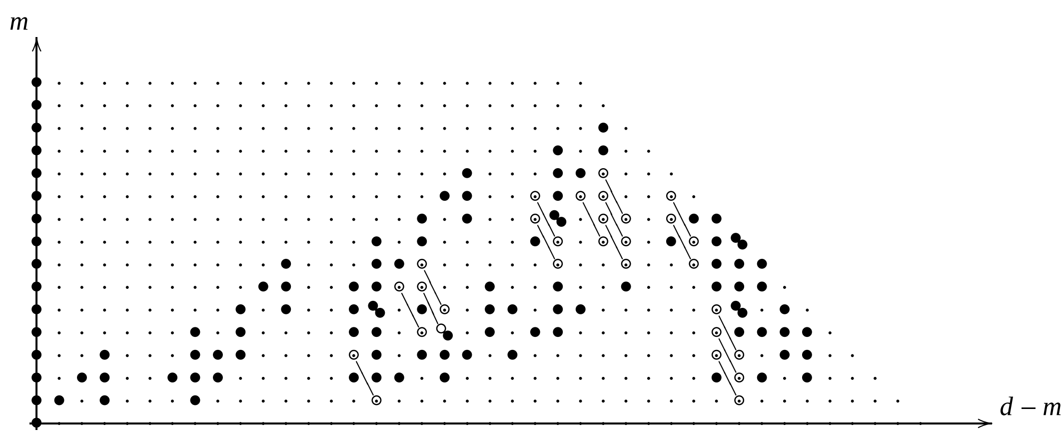
Treating similarly the rest of the table below we find that the only nonzero values of $d_{(2)}$ on generators of degree < 36 are as follows:

$$
\begin{aligned}
d_{(2)}(\hat{g}_1^{16}) &= \hat{g}_3^{17}\\
d_{(2)}(\hat{g}_4^{21}) &= \hat{g}_6^{22}\\
d_{(2)}(\hat{g}_4^{22}) &= \hat{g}_6^{23}\\
d_{(2)}(\hat{g}_5^{23}) &= \hat{g}_7^{24}\\
d_{(2)}(\hat{g}_7^{30}) &= \hat{g}_9^{31}\\
d_{(2)}(\hat{g}_8^{31}) &= \hat{g}_{10}^{32}\\
d_{(2)}(\hat{g}_1^{32}) &= \hat{g}_3^{33}\\
d_{(2)}(\hat{g}_2^{33}) &= \hat{g}_4^{34}\\
d_{(2)}(\hat{g}_7^{33}) &= \hat{g}_9^{34}\\
d_{(2)}(\hat{g}_8^{33}) &= \hat{g}_{10}^{34}\\
d_{(2)}({}'\hat{g}_3^{34}) &= \hat{g}_5^{35}\\
d_{(2)}(\hat{g}_8^{34}) &= \hat{g}_{10}^{35}.
\end{aligned}
$$

Presently a computer calculation continues reaching degree 39 and showing that up to that degree there are the following further nonzero differentials:

$$
\begin{aligned}
d_{(2)}(\hat{g}_7^{36}) &= \hat{g}_9^{37}\\
d_{(2)}(\hat{g}_8^{37}) &= \hat{g}_{10}^{38}.
\end{aligned}
$$

These data can be summarized in the following picture, thus confirming calculations presented in Ravenel's book [6].



9. T δ $\mathbb{G}^{\Sigma}$

The following table presents results of computer calculations of the differential δ. Note that it does not have invariant meaning since it depends on the choices involved in determination of the multiplication map A, of the resolution and of those indeterminate coefficients $\eta_m^d(a)$ which remain undetermined after the conditions (8.7) are satisfied. The resulting secondary differential $d_{(2)}$ however does not depend on these choices and is canonically determined.

$$\delta(g_2^2) = 0$$

$$\delta(g_3^3) = 0$$

$$
\begin{aligned}
\delta(g_2^4) &= 0\\
\delta(g_4^4) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_2^5) &= 0\\
\delta(g_5^5) &= 0
\end{aligned}
$$

$$\delta(g_3^6) = \mathrm{Sq}^4\, g_1^1$$

$$\delta(g_6^6) = 0$$

$$\delta(g_7^7) = 0$$

$$\begin{aligned}\delta(g_2^8) &= 0\\ \delta(g_8^8) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_2^9) &= 0\\ \delta(g_9^9) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_2^{10}) &= 0\\ \delta(g_3^{10}) &= (\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^7)g_1^2\\ &\quad + \mathrm{Sq}^8\,g_1^1\\ \delta(g_{10}^{10}) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_3^{11}) &= (\mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^8)g_1^2\\ &\quad + \mathrm{Sq}^6\,\mathrm{Sq}^3\,g_1^1\\ \delta(g_4^{11}) &= \mathrm{Sq}^5\,g_2^5\\ &\quad + \mathrm{Sq}^4\,\mathrm{Sq}^2\,g_2^4\\ \delta(g_{11}^{11}) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_3^{12}) &= \mathrm{Sq}^7\,\mathrm{Sq}^3\,g_1^1\\ \delta(g_{12}^{12}) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_4^{13}) &= \mathrm{Sq}^4\,g_2^8\\ &\quad +(\mathrm{Sq}^7 + \mathrm{Sq}^5\,\mathrm{Sq}^2)g_2^5\\ &\quad +(\mathrm{Sq}^8 + \mathrm{Sq}^6\,\mathrm{Sq}^2)g_2^4\\ &\quad +(\mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^{10})g_2^2\\ \delta(g_{13}^{13}) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_5^{14}) &= \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,g_3^6\\ &\quad +(\mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^2)g_3^3\\ \delta(g_{14}^{14}) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_2^{16}) &= 0\\ \delta(g_5^{16}) &= \mathrm{Sq}^3\,g_3^{12}\\ &\quad + \mathrm{Sq}^4\,g_3^{11}\\ &\quad + \mathrm{Sq}^5\,g_3^{10}\\ &\quad + \mathrm{Sq}^{10}\,\mathrm{Sq}^2\,g_3^3\\ \delta(g_6^{16}) &= 0\end{aligned}$$

$$\begin{aligned}\delta(g_2^{17}) &= 0\\ \delta(g_3^{17}) &= g_1^{16}\\ &\quad + \mathrm{Sq}^{12}\,g_1^4\\ &\quad + \mathrm{Sq}^{10}\,\mathrm{Sq}^4\,g_1^2\\ &\quad +(\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5 + \mathrm{Sq}^{11}\,\mathrm{Sq}^4)g_1^1\\ \delta(g_6^{17}) &= (\mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1)g_4^{11}\\ &\quad +(\mathrm{Sq}^{12} + \mathrm{Sq}^{10}\,\mathrm{Sq}^2)g_4^4\end{aligned}$$

$$\begin{aligned}\delta(g_2^{18}) &= 0\\ \delta(g_3^{18}) &= (\mathrm{Sq}^{11}\,\mathrm{Sq}^4 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_1^2\\ &\quad +(\mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{11}\,\mathrm{Sq}^5 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4 + \mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{16})g_1^1\end{aligned}$$

$$\begin{aligned}
\delta(g_4^{18}) &= (\mathrm{Sq}^6\,\mathrm{Sq}^1 + \mathrm{Sq}^7)g_2^{10}\\
&+(\mathrm{Sq}^6\,\mathrm{Sq}^3 + \mathrm{Sq}^7\,\mathrm{Sq}^2 + \mathrm{Sq}^9)g_2^8\\
&+\mathrm{Sq}^8\,\mathrm{Sq}^4\,g_2^5\\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{13} + \mathrm{Sq}^{11}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^1)g_2^4\\
&+(\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{15} + \mathrm{Sq}^{12}\,\mathrm{Sq}^3 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5)g_2^2\\
\delta(g_7^{18}) &= \mathrm{Sq}^2\,\mathrm{Sq}^1\,g_5^{14}
\end{aligned}$$

$$\begin{aligned}
\delta(g_4^{19}) &= \mathrm{Sq}^9\,g_2^9\\
&+(\mathrm{Sq}^{10} + \mathrm{Sq}^8\,\mathrm{Sq}^2)g_2^8\\
&+\mathrm{Sq}^{11}\,\mathrm{Sq}^2\,g_2^5\\
&+(\mathrm{Sq}^{11}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{13}\,\mathrm{Sq}^1 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^3\,\mathrm{Sq}^1)g_2^4\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4)g_2^2\\
\delta(g_5^{19}) &= \mathrm{Sq}^1\,g_3^{17}\\
&+\mathrm{Sq}^4\,\mathrm{Sq}^2\,g_3^{12}\\
&+\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,g_3^{11}\\
&+(\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^8)g_3^{10}\\
&+(\mathrm{Sq}^8\,\mathrm{Sq}^4 + \mathrm{Sq}^{11}\,\mathrm{Sq}^1)g_3^6\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5 + \mathrm{Sq}^{15} + \mathrm{Sq}^{11}\,\mathrm{Sq}^4)g_3^3
\end{aligned}$$

$$\begin{aligned}
\delta(g_2^{20}) &= 0\\
\delta(g_3^{20}) &= (\mathrm{Sq}^{15} + \mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_1^4\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^5 + \mathrm{Sq}^{13}\,\mathrm{Sq}^4 + \mathrm{Sq}^{16}\,\mathrm{Sq}^1)g_1^2\\
&+(\mathrm{Sq}^{11}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^3 + \mathrm{Sq}^{18} + \mathrm{Sq}^{12}\,\mathrm{Sq}^6)g_1^1\\
\delta(g_5^{20}) &= \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,g_3^{12}\\
&+(\mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^8)g_3^{11}\\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{13} + \mathrm{Sq}^{11}\,\mathrm{Sq}^2)g_3^6\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^3 + \mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{11}\,\mathrm{Sq}^5 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4)g_3^3\\
\delta('g_5^{20}) &= \mathrm{Sq}^5\,\mathrm{Sq}^2\,g_3^{12}\\
&+\mathrm{Sq}^7\,\mathrm{Sq}^2\,g_3^{10}\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{11}\,\mathrm{Sq}^2)g_3^6\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{13}\,\mathrm{Sq}^3 + \mathrm{Sq}^{11}\,\mathrm{Sq}^5 + \mathrm{Sq}^{16} + \mathrm{Sq}^{12}\,\mathrm{Sq}^4)g_3^3\\
\delta(g_6^{20}) &= (\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^8)g_4^{11}\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2 + \mathrm{Sq}^{15} + \mathrm{Sq}^{11}\,\mathrm{Sq}^4)g_4^4
\end{aligned}$$

$$\begin{aligned}
\delta(g_3^{21}) &= (\mathrm{Sq}^{15}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{17}\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6)g_1^2\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^4 + \mathrm{Sq}^{16}\,\mathrm{Sq}^3 + \mathrm{Sq}^{17}\,\mathrm{Sq}^2 + \mathrm{Sq}^{19})g_1^1\\
\delta(g_4^{21}) &= \mathrm{Sq}^3\,g_2^{17}\\
&+(\mathrm{Sq}^{10} + \mathrm{Sq}^9\,\mathrm{Sq}^1)g_2^{10}\\
&+(\mathrm{Sq}^9\,\mathrm{Sq}^3 + \mathrm{Sq}^{11}\,\mathrm{Sq}^1)g_2^8\\
&+(\mathrm{Sq}^{15} + \mathrm{Sq}^{13}\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5)g_2^5\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4 + \mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_2^4\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6 + \mathrm{Sq}^{15}\,\mathrm{Sq}^3)g_2^2\\
\delta(g_6^{21}) &= (\mathrm{Sq}^7 + \mathrm{Sq}^6\,\mathrm{Sq}^1)g_4^{13}\\
&+(\mathrm{Sq}^9 + \mathrm{Sq}^8\,\mathrm{Sq}^1)g_4^{11}\\
&+\mathrm{Sq}^{11}\,\mathrm{Sq}^5\,g_4^4
\end{aligned}$$

$$\begin{aligned}
\delta(g_3^{22}) &= \mathrm{Sq}^{17}\,g_1^4\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^1)g_1^2\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{17}\,\mathrm{Sq}^3 + \mathrm{Sq}^{18}\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_1^1
\end{aligned}$$

$$
\begin{aligned}
\delta(g_4^{22}) &= \mathrm{Sq}^4\, g_2^{17}\\
&+ \mathrm{Sq}^{11}\, g_2^{10}\\
&+(\mathrm{Sq}^{12} + \mathrm{Sq}^9\,\mathrm{Sq}^3)g_2^9\\
&+(\mathrm{Sq}^9\,\mathrm{Sq}^4 + \mathrm{Sq}^{13} + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^1)g_2^8\\
&+ \mathrm{Sq}^{12}\,\mathrm{Sq}^4\, g_2^5\\
&+ \mathrm{Sq}^{15}\,\mathrm{Sq}^2\, g_2^4\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{19} + \mathrm{Sq}^{13}\,\mathrm{Sq}^6 + \mathrm{Sq}^{14}\,\mathrm{Sq}^5)g_2^2\\
\delta('g_4^{22}) &= \mathrm{Sq}^2\,\mathrm{Sq}^1\, g_2^{18}\\
&+(\mathrm{Sq}^8\,\mathrm{Sq}^4 + \mathrm{Sq}^{12})g_2^9\\
&+(\mathrm{Sq}^9\,\mathrm{Sq}^4 + \mathrm{Sq}^{13} + \mathrm{Sq}^{12}\,\mathrm{Sq}^1)g_2^8\\
&+(\mathrm{Sq}^{16} + \mathrm{Sq}^{13}\,\mathrm{Sq}^3)g_2^5\\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^2 + \mathrm{Sq}^{16}\,\mathrm{Sq}^1 + \mathrm{Sq}^{13}\,\mathrm{Sq}^4 + \mathrm{Sq}^{11}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_2^4\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^5 + \mathrm{Sq}^{19} + \mathrm{Sq}^{17}\,\mathrm{Sq}^2)g_2^2\\
\delta(g_5^{22}) &= (\mathrm{Sq}^7\,\mathrm{Sq}^2 + \mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^6\,\mathrm{Sq}^3)g_3^{12}\\
&+ \mathrm{Sq}^{10}\, g_3^{11} + (\mathrm{Sq}^9\,\mathrm{Sq}^2 + \mathrm{Sq}^8\,\mathrm{Sq}^3 + \mathrm{Sq}^{11})g_3^{10}\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^1 + \mathrm{Sq}^{11}\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^3 + \mathrm{Sq}^{13}\,\mathrm{Sq}^2)g_3^6\\
&+ \mathrm{Sq}^{13}\,\mathrm{Sq}^5\, g_3^3\\
\delta(g_6^{22}) &= g_4^{21}\\
&+(\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^8 + \mathrm{Sq}^7\,\mathrm{Sq}^1)g_4^{13}\\
&+ \mathrm{Sq}^{10}\, g_4^{11}\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^4 + \mathrm{Sq}^{15}\,\mathrm{Sq}^2 + \mathrm{Sq}^{17})g_4^4\\
\delta(g_7^{22}) &= (\mathrm{Sq}^{13}\,\mathrm{Sq}^3 + \mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{16})g_5^5
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_5^{23}) &= \mathrm{Sq}^4\, g_3^{18}\\
&+ \mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1\, g_3^{12}\\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^1 + \mathrm{Sq}^{11})g_3^{11}\\
&+(\mathrm{Sq}^8\,\mathrm{Sq}^4 + \mathrm{Sq}^9\,\mathrm{Sq}^3)g_3^{10}\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^3 + \mathrm{Sq}^{15}\,\mathrm{Sq}^1 + \mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{11}\,\mathrm{Sq}^5 + \mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4)g_3^6\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^3 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6 + \mathrm{Sq}^{15}\,\mathrm{Sq}^4)g_3^3\\
\delta(g_6^{23}) &= g_4^{22}\\
&+ \mathrm{Sq}^9\, g_4^{13}\\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^1 + \mathrm{Sq}^{11} + \mathrm{Sq}^8\,\mathrm{Sq}^3)g_4^{11}\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4)g_4^4\\
\delta(g_7^{23}) &= (\mathrm{Sq}^6 + \mathrm{Sq}^4\,\mathrm{Sq}^2)g_5^{16}\\
&+ \mathrm{Sq}^7\,\mathrm{Sq}^1\, g_5^{14}\\
&+ \mathrm{Sq}^{15}\,\mathrm{Sq}^2\, g_5^5\\
\delta(g_8^{23}) &= \mathrm{Sq}^5\, g_6^{17}\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{13}\,\mathrm{Sq}^3)g_6^6
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_3^{24}) &= \mathrm{Sq}^{11}\,\mathrm{Sq}^4\, g_1^8\\
&+(\mathrm{Sq}^{19} + \mathrm{Sq}^{17}\,\mathrm{Sq}^2)g_1^4\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{17}\,\mathrm{Sq}^4 + \mathrm{Sq}^{21})g_1^2\\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^7 + \mathrm{Sq}^{14}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{18}\,\mathrm{Sq}^4 + \mathrm{Sq}^{22} + \mathrm{Sq}^{20}\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^5\,\mathrm{Sq}^2)g_1^1\\
\delta(g_4^{24}) &= \mathrm{Sq}^5\, g_2^{18}\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^9\,\mathrm{Sq}^4)g_2^{10}\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^2 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{11}\,\mathrm{Sq}^3)g_2^9\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^1)g_2^8\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{16}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4 + \mathrm{Sq}^{11}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^3 + \mathrm{Sq}^{13}\,\mathrm{Sq}^5)g_2^5\\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^4 + \mathrm{Sq}^{19} + \mathrm{Sq}^{14}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6 + \mathrm{Sq}^{18}\,\mathrm{Sq}^1 + \mathrm{Sq}^{13}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^3\,\mathrm{Sq}^1\\
&\quad + \mathrm{Sq}^{17}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^1)g_2^4\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^5 + \mathrm{Sq}^{14}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{18}\,\mathrm{Sq}^3 + \mathrm{Sq}^{15}\,\mathrm{Sq}^6 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{21})g_2^2
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_7^{24}) &= g_5^{23} \\
&+ \mathrm{Sq}^4\, g_5^{19} \\
&+(\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^7)g_5^{16} \\
&+(\mathrm{Sq}^9 + \mathrm{Sq}^8\,\mathrm{Sq}^1)g_5^{14} \\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^6 + \mathrm{Sq}^{16}\,\mathrm{Sq}^2 + \mathrm{Sq}^{13}\,\mathrm{Sq}^5 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4)g_5^5
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_5^{25}) &= \mathrm{Sq}^4\, g_3^{20} \\
&+ \mathrm{Sq}^6\, g_3^{18} \\
&+ \mathrm{Sq}^7\, g_3^{17} \\
&+(\mathrm{Sq}^8\,\mathrm{Sq}^4 + \mathrm{Sq}^9\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^9\,\mathrm{Sq}^3 + \mathrm{Sq}^{10}\,\mathrm{Sq}^2)g_3^{12} \\
&+(\mathrm{Sq}^9\,\mathrm{Sq}^4 + \mathrm{Sq}^{10}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^1)g_3^{11} \\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6 + \mathrm{Sq}^{13}\,\mathrm{Sq}^5 + \mathrm{Sq}^{18} + \mathrm{Sq}^{13}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{16}\,\mathrm{Sq}^2 + \mathrm{Sq}^{17}\,\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{14}\,\mathrm{Sq}^4 + \mathrm{Sq}^{15}\,\mathrm{Sq}^3)g_3^6 \\
&+(\mathrm{Sq}^{18}\,\mathrm{Sq}^3 + \mathrm{Sq}^{19}\,\mathrm{Sq}^2 + \mathrm{Sq}^{16}\,\mathrm{Sq}^5)g_3^3 \\
\delta(g_8^{25}) &= \mathrm{Sq}^7\, g_6^{17}
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_4^{26}) &= (\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^6\,\mathrm{Sq}^1)g_2^{18} \\
&+(\mathrm{Sq}^8 + \mathrm{Sq}^6\,\mathrm{Sq}^2)g_2^{17} \\
&+(\mathrm{Sq}^{15} + \mathrm{Sq}^{14}\,\mathrm{Sq}^1 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_2^{10} \\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^4 + \mathrm{Sq}^{14}\,\mathrm{Sq}^2)g_2^9 \\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^4 + \mathrm{Sq}^{14}\,\mathrm{Sq}^3 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^2)g_2^8 \\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^6 + \mathrm{Sq}^{18}\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^5 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^2)g_2^5 \\
&+(\mathrm{Sq}^{18}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{20}\,\mathrm{Sq}^1 + \mathrm{Sq}^{17}\,\mathrm{Sq}^4 + \mathrm{Sq}^{19}\,\mathrm{Sq}^2 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^7 + \mathrm{Sq}^{15}\,\mathrm{Sq}^6)g_2^4 \\
&+(\mathrm{Sq}^{17}\,\mathrm{Sq}^6 + \mathrm{Sq}^{14}\,\mathrm{Sq}^7\,\mathrm{Sq}^2 + \mathrm{Sq}^{19}\,\mathrm{Sq}^4 + \mathrm{Sq}^{16}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^6\,\mathrm{Sq}^3 + \mathrm{Sq}^{15}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{20}\,\mathrm{Sq}^3 \\
&\quad + \mathrm{Sq}^{17}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_2^2 \\
\delta(g_5^{26}) &= \mathrm{Sq}^5\, g_3^{20} \\
&+ \mathrm{Sq}^5\,\mathrm{Sq}^2\, g_3^{18} \\
&+ \mathrm{Sq}^6\,\mathrm{Sq}^2\, g_3^{17} \\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^1)g_3^{12} \\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2 + \mathrm{Sq}^{11}\,\mathrm{Sq}^4 + \mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^3 + \mathrm{Sq}^{10}\,\mathrm{Sq}^5)g_3^{10} \\
&+(\mathrm{Sq}^{17}\,\mathrm{Sq}^2 + \mathrm{Sq}^{11}\,\mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{16}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{19} + \mathrm{Sq}^{15}\,\mathrm{Sq}^4 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^1 \\
&\quad + \mathrm{Sq}^{12}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^5\,\mathrm{Sq}^2)g_3^6 \\
&+(\mathrm{Sq}^{18}\,\mathrm{Sq}^4 + \mathrm{Sq}^{14}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{16}\,\mathrm{Sq}^6 + \mathrm{Sq}^{19}\,\mathrm{Sq}^3 + \mathrm{Sq}^{17}\,\mathrm{Sq}^5 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6\,\mathrm{Sq}^3)g_3^3 \\
\delta(g_6^{26}) &= \mathrm{Sq}^{3\prime}\, g_4^{22} \\
&+ \mathrm{Sq}^3\, g_4^{22} \\
&+ \mathrm{Sq}^4\, g_4^{21} \\
&+ \mathrm{Sq}^6\, g_4^{19} \\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^2 + \mathrm{Sq}^9\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12})g_4^{13} \\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^1 + \mathrm{Sq}^{14} + \mathrm{Sq}^{11}\,\mathrm{Sq}^3 + \mathrm{Sq}^{12}\,\mathrm{Sq}^2)g_4^{11} \\
&+(\mathrm{Sq}^{17}\,\mathrm{Sq}^4 + \mathrm{Sq}^{15}\,\mathrm{Sq}^6 + \mathrm{Sq}^{18}\,\mathrm{Sq}^3 + \mathrm{Sq}^{21})g_4^4 \\
\delta(g_9^{26}) &= (\mathrm{Sq}^6\,\mathrm{Sq}^1 + \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_7^{18} \\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^3 + \mathrm{Sq}^{16}\,\mathrm{Sq}^2)g_7^7
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_4^{27}) &= \mathrm{Sq}^4\,\mathrm{Sq}^2\, g_2^{20} \\
&+(\mathrm{Sq}^7\,\mathrm{Sq}^2 + \mathrm{Sq}^9)g_2^{17} \\
&+ \mathrm{Sq}^{10}\, g_2^{16} \\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^4 + \mathrm{Sq}^{11}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{13}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{16})g_2^{10} \\
&+(\mathrm{Sq}^{17} + \mathrm{Sq}^{10}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{13}\,\mathrm{Sq}^4 + \mathrm{Sq}^{12}\,\mathrm{Sq}^5 + \mathrm{Sq}^{15}\,\mathrm{Sq}^2)g_2^9 \\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^4)g_2^8 \\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^6 + \mathrm{Sq}^{19}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3 + \mathrm{Sq}^{13}\,\mathrm{Sq}^6\,\mathrm{Sq}^2)g_2^5 \\
&+(\mathrm{Sq}^{17}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{20}\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^7 + \mathrm{Sq}^{14}\,\mathrm{Sq}^7\,\mathrm{Sq}^1)g_2^4 \\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^6\,\mathrm{Sq}^3 + \mathrm{Sq}^{18}\,\mathrm{Sq}^6 + \mathrm{Sq}^{16}\,\mathrm{Sq}^8 + \mathrm{Sq}^{20}\,\mathrm{Sq}^4 + \mathrm{Sq}^{18}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{15}\,\mathrm{Sq}^7\,\mathrm{Sq}^2)g_2^2
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_5^{28}) &= (\mathrm{Sq}^6\,\mathrm{Sq}^1+\mathrm{Sq}^7)g_3^{20}\\
&+\mathrm{Sq}^9\,g_3^{18}\\
&+\mathrm{Sq}^7\,\mathrm{Sq}^3\,g_3^{17}\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{11}\,\mathrm{Sq}^3\,\mathrm{Sq}^1+\mathrm{Sq}^{15}+\mathrm{Sq}^{14}\,\mathrm{Sq}^1+\mathrm{Sq}^{12}\,\mathrm{Sq}^3)g_3^{12}\\
&+\mathrm{Sq}^{12}\,\mathrm{Sq}^4\,g_3^{11}\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^3+\mathrm{Sq}^{11}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_3^{10}\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{15}\,\mathrm{Sq}^6+\mathrm{Sq}^{21}+\mathrm{Sq}^{13}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^{19}\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^3+\mathrm{Sq}^{15}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\\
&\quad+\mathrm{Sq}^{17}\,\mathrm{Sq}^4+\mathrm{Sq}^{14}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3+\mathrm{Sq}^{14}\,\mathrm{Sq}^7+\mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_3^6\\
&+(\mathrm{Sq}^{20}\,\mathrm{Sq}^4+\mathrm{Sq}^{24}+\mathrm{Sq}^{18}\,\mathrm{Sq}^6+\mathrm{Sq}^{18}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{16}\,\mathrm{Sq}^8)g_3^3\\
\delta(g_9^{28}) &= \mathrm{Sq}^4\,g_7^{23}\\
&+(\mathrm{Sq}^{20}+\mathrm{Sq}^{18}\,\mathrm{Sq}^2)g_7^7\\
\delta(g_{10}^{28}) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_5^{29}) &= \mathrm{Sq}^4\,\mathrm{Sq}^2\,g_3^{22}\\
&+(\mathrm{Sq}^7\,\mathrm{Sq}^1+\mathrm{Sq}^6\,\mathrm{Sq}^2)g_3^{20}\\
&+\mathrm{Sq}^{10}\,g_3^{18}\\
&+(\mathrm{Sq}^{11}+\mathrm{Sq}^9\,\mathrm{Sq}^2)g_3^{17}\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^4+\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{16}+\mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{12}\,\mathrm{Sq}^3\,\mathrm{Sq}^1+\mathrm{Sq}^{15}\,\mathrm{Sq}^1+\mathrm{Sq}^{14}\,\mathrm{Sq}^2)g_3^{12}\\
&+(\mathrm{Sq}^{17}+\mathrm{Sq}^{16}\,\mathrm{Sq}^1)g_3^{11}\\
&+(\mathrm{Sq}^{11}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{12}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{13}\,\mathrm{Sq}^5+\mathrm{Sq}^{18})g_3^{10}\\
&+(\mathrm{Sq}^{19}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{15}\,\mathrm{Sq}^6\,\mathrm{Sq}^1+\mathrm{Sq}^{15}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{14}\,\mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1\\
&\quad+\mathrm{Sq}^{15}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{17}\,\mathrm{Sq}^5+\mathrm{Sq}^{19}\,\mathrm{Sq}^3+\mathrm{Sq}^{22})g_3^6\\
&+(\mathrm{Sq}^{16}\,\mathrm{Sq}^6\,\mathrm{Sq}^3+\mathrm{Sq}^{20}\,\mathrm{Sq}^5+\mathrm{Sq}^{17}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^{15}\,\mathrm{Sq}^7\,\mathrm{Sq}^3+\mathrm{Sq}^{16}\,\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^5\,\mathrm{Sq}^2\\
&\quad+\mathrm{Sq}^{19}\,\mathrm{Sq}^6)g_3^3\\
\delta(g_6^{29}) &= (\mathrm{Sq}^{12}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{11}\,\mathrm{Sq}^4+\mathrm{Sq}^{13}\,\mathrm{Sq}^2+\mathrm{Sq}^{14}\,\mathrm{Sq}^1+\mathrm{Sq}^{15}+\mathrm{Sq}^{11}\,\mathrm{Sq}^3\,\mathrm{Sq}^1)g_4^{13}\\
&+(\mathrm{Sq}^{12}\,\mathrm{Sq}^5+\mathrm{Sq}^{15}\,\mathrm{Sq}^2+\mathrm{Sq}^{13}\,\mathrm{Sq}^4+\mathrm{Sq}^{16}\,\mathrm{Sq}^1+\mathrm{Sq}^{17}+\mathrm{Sq}^{11}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{14}\,\mathrm{Sq}^3)g_4^{11}\\
&+(\mathrm{Sq}^{17}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{22}\,\mathrm{Sq}^2+\mathrm{Sq}^{15}\,\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^4)g_4^4\\
\delta(g_{10}^{29}) &= (\mathrm{Sq}^5+\mathrm{Sq}^4\,\mathrm{Sq}^1)g_8^{23}\\
&+(\mathrm{Sq}^{18}\,\mathrm{Sq}^2+\mathrm{Sq}^{20})g_8^8
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_7^{30}) &= \mathrm{Sq}^3\,g_5^{26}\\
&+\mathrm{Sq}^4\,\mathrm{Sq}^2\,g_5^{23}\\
&+\mathrm{Sq}^7\,g_5^{22}\\
&+\mathrm{Sq}^9\,g_5^{20}\\
&+\mathrm{Sq}^9\,{}'g_5^{20}\\
&+\mathrm{Sq}^8\,\mathrm{Sq}^2\,g_5^{19}\\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^3+\mathrm{Sq}^{11}\,\mathrm{Sq}^2)g_5^{16}\\
&+(\mathrm{Sq}^{15}+\mathrm{Sq}^{12}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{10}\,\mathrm{Sq}^5+\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_5^{14}\\
&+(\mathrm{Sq}^{21}\,\mathrm{Sq}^3+\mathrm{Sq}^{19}\,\mathrm{Sq}^5+\mathrm{Sq}^{18}\,\mathrm{Sq}^6+\mathrm{Sq}^{17}\,\mathrm{Sq}^7+\mathrm{Sq}^{22}\,\mathrm{Sq}^2+\mathrm{Sq}^{24})g_5^5\\
\delta(g_8^{30}) &= \mathrm{Sq}^2\,\mathrm{Sq}^1\,g_6^{26}\\
&+\mathrm{Sq}^6\,g_6^{23}\\
&+(\mathrm{Sq}^6\,\mathrm{Sq}^1+\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_6^{22}\\
&+(\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^8\,\mathrm{Sq}^1+\mathrm{Sq}^6\,\mathrm{Sq}^3)g_6^{20}\\
&+(\mathrm{Sq}^{10}\,\mathrm{Sq}^2+\mathrm{Sq}^9\,\mathrm{Sq}^3+\mathrm{Sq}^8\,\mathrm{Sq}^4)g_6^{17}\\
&+(\mathrm{Sq}^{13}+\mathrm{Sq}^{12}\,\mathrm{Sq}^1+\mathrm{Sq}^{10}\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_6^{16}\\
&+(\mathrm{Sq}^{21}\,\mathrm{Sq}^2+\mathrm{Sq}^{17}\,\mathrm{Sq}^6+\mathrm{Sq}^{18}\,\mathrm{Sq}^5+\mathrm{Sq}^{16}\,\mathrm{Sq}^7+\mathrm{Sq}^{17}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_6^6\\
\delta(g_{11}^{30}) &= \mathrm{Sq}^2\,\mathrm{Sq}^1\,g_9^{26}
\end{aligned}
$$

$$\begin{aligned}
\delta(g_8^{31}) &= \mathrm{Sq}^7 g_6^{23}\\
&+(\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1+\mathrm{Sq}^7\mathrm{Sq}^1)g_6^{22}\\
&+(\mathrm{Sq}^7\mathrm{Sq}^3+\mathrm{Sq}^{10}+\mathrm{Sq}^8\mathrm{Sq}^2)g_6^{20}\\
&+(\mathrm{Sq}^{10}\mathrm{Sq}^3+\mathrm{Sq}^{13})g_6^{17}\\
&+(\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^2+\mathrm{Sq}^{13}\mathrm{Sq}^1+\mathrm{Sq}^{14}+\mathrm{Sq}^{11}\mathrm{Sq}^2\mathrm{Sq}^1)g_6^{16}\\
&+(\mathrm{Sq}^{17}\mathrm{Sq}^5\mathrm{Sq}^2+\mathrm{Sq}^{22}\mathrm{Sq}^2+\mathrm{Sq}^{21}\mathrm{Sq}^3+\mathrm{Sq}^{24}+\mathrm{Sq}^{18}\mathrm{Sq}^6+\mathrm{Sq}^{17}\mathrm{Sq}^7)g_6^{6}\\
\delta(g_9^{31}) &= g_7^{30}\\
&+\mathrm{Sq}^4\mathrm{Sq}^2 g_7^{24}\\
&+(\mathrm{Sq}^6\mathrm{Sq}^1+\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1+\mathrm{Sq}^7)g_7^{23}\\
&+(\mathrm{Sq}^{12}+\mathrm{Sq}^8\mathrm{Sq}^4+\mathrm{Sq}^9\mathrm{Sq}^2\mathrm{Sq}^1)g_7^{18}\\
&+(\mathrm{Sq}^{20}\mathrm{Sq}^3+\mathrm{Sq}^{18}\mathrm{Sq}^5)g_7^{7}
\end{aligned}$$

$$\begin{aligned}
\delta(g_2^{32}) &= 0\\
\delta(g_6^{32}) &= \mathrm{Sq}^4 g_4^{27}\\
&+\mathrm{Sq}^5 g_4^{26}\\
&+(\mathrm{Sq}^8\mathrm{Sq}^1+\mathrm{Sq}^9+\mathrm{Sq}^6\mathrm{Sq}^3)'g_4^{22}\\
&+\mathrm{Sq}^7\mathrm{Sq}^2 g_4^{22}\\
&+(\mathrm{Sq}^{10}+\mathrm{Sq}^8\mathrm{Sq}^2)g_4^{21}\\
&+(\mathrm{Sq}^{12}+\mathrm{Sq}^8\mathrm{Sq}^4)g_4^{19}\\
&+\mathrm{Sq}^{13} g_4^{18}\\
&+(\mathrm{Sq}^{16}\mathrm{Sq}^2+\mathrm{Sq}^{17}\mathrm{Sq}^1+\mathrm{Sq}^{13}\mathrm{Sq}^5+\mathrm{Sq}^{12}\mathrm{Sq}^4\mathrm{Sq}^2+\mathrm{Sq}^{15}\mathrm{Sq}^2\mathrm{Sq}^1+\mathrm{Sq}^{14}\mathrm{Sq}^3\mathrm{Sq}^1)g_4^{13}\\
&+(\mathrm{Sq}^{15}\mathrm{Sq}^4\mathrm{Sq}^1+\mathrm{Sq}^{14}\mathrm{Sq}^6+\mathrm{Sq}^{16}\mathrm{Sq}^4+\mathrm{Sq}^{14}\mathrm{Sq}^4\mathrm{Sq}^2+\mathrm{Sq}^{12}\mathrm{Sq}^6\mathrm{Sq}^2+\mathrm{Sq}^{13}\mathrm{Sq}^6\mathrm{Sq}^1)g_4^{11}\\
&+(\mathrm{Sq}^{24}\mathrm{Sq}^3+\mathrm{Sq}^{17}\mathrm{Sq}^8\mathrm{Sq}^2+\mathrm{Sq}^{20}\mathrm{Sq}^5\mathrm{Sq}^2+\mathrm{Sq}^{19}\mathrm{Sq}^6\mathrm{Sq}^2+\mathrm{Sq}^{18}\mathrm{Sq}^9+\mathrm{Sq}^{27})g_4^{4}\\
\delta(g_9^{32}) &= \mathrm{Sq}^7 g_7^{24}\\
&+\mathrm{Sq}^8 g_7^{23}\\
&+(\mathrm{Sq}^{11}\mathrm{Sq}^2+\mathrm{Sq}^{12}\mathrm{Sq}^1+\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^1+\mathrm{Sq}^9\mathrm{Sq}^4)g_7^{18}\\
&+(\mathrm{Sq}^{18}\mathrm{Sq}^4\mathrm{Sq}^2+\mathrm{Sq}^{21}\mathrm{Sq}^3+\mathrm{Sq}^{22}\mathrm{Sq}^2+\mathrm{Sq}^{24}+\mathrm{Sq}^{20}\mathrm{Sq}^4)g_7^{7}\\
\delta('g_9^{32}) &= (\mathrm{Sq}^7+\mathrm{Sq}^5\mathrm{Sq}^2)g_7^{24}\\
&+\mathrm{Sq}^8 g_7^{23}\\
&+(\mathrm{Sq}^8\mathrm{Sq}^4\mathrm{Sq}^1+\mathrm{Sq}^{10}\mathrm{Sq}^2\mathrm{Sq}^1)g_7^{18}\\
&+(\mathrm{Sq}^{18}\mathrm{Sq}^4\mathrm{Sq}^2+\mathrm{Sq}^{19}\mathrm{Sq}^5+\mathrm{Sq}^{22}\mathrm{Sq}^2+\mathrm{Sq}^{20}\mathrm{Sq}^4)g_7^{7}\\
\delta(g_{10}^{32}) &= g_8^{31}\\
&+\mathrm{Sq}^6 g_8^{25}\\
&+(\mathrm{Sq}^7\mathrm{Sq}^1+\mathrm{Sq}^6\mathrm{Sq}^2)g_8^{23}\\
&+(\mathrm{Sq}^{21}\mathrm{Sq}^2+\mathrm{Sq}^{19}\mathrm{Sq}^4+\mathrm{Sq}^{23})g_8^{8}
\end{aligned}$$

$$\begin{aligned}
\delta(g_2^{33}) &= 0\\
\delta(g_3^{33}) &= g_1^{32}\\
&+\mathrm{Sq}^{24} g_1^{8}\\
&+(\mathrm{Sq}^{28}+\mathrm{Sq}^{25}\mathrm{Sq}^3)g_1^{4}\\
&+(\mathrm{Sq}^{29}\mathrm{Sq}^1+\mathrm{Sq}^{30}+\mathrm{Sq}^{23}\mathrm{Sq}^7+\mathrm{Sq}^{22}\mathrm{Sq}^7\mathrm{Sq}^1+\mathrm{Sq}^{25}\mathrm{Sq}^5+\mathrm{Sq}^{23}\mathrm{Sq}^6\mathrm{Sq}^1+\mathrm{Sq}^{23}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)g_1^{2}\\
&+(\mathrm{Sq}^{29}\mathrm{Sq}^2+\mathrm{Sq}^{20}\mathrm{Sq}^8\mathrm{Sq}^3+\mathrm{Sq}^{22}\mathrm{Sq}^7\mathrm{Sq}^2+\mathrm{Sq}^{27}\mathrm{Sq}^4+\mathrm{Sq}^{21}\mathrm{Sq}^7\mathrm{Sq}^3+\mathrm{Sq}^{21}\mathrm{Sq}^8\mathrm{Sq}^2+\mathrm{Sq}^{26}\mathrm{Sq}^5\\
&\quad+\mathrm{Sq}^{28}\mathrm{Sq}^3+\mathrm{Sq}^{19}\mathrm{Sq}^9\mathrm{Sq}^3+\mathrm{Sq}^{25}\mathrm{Sq}^6+\mathrm{Sq}^{19}\mathrm{Sq}^8\mathrm{Sq}^4)g_1^{1}\\
\delta(g_7^{33}) &= \mathrm{Sq}^4\mathrm{Sq}^2 g_5^{26}\\
&+\mathrm{Sq}^7 g_5^{25}\\
&+(\mathrm{Sq}^7\mathrm{Sq}^2+\mathrm{Sq}^8\mathrm{Sq}^1+\mathrm{Sq}^6\mathrm{Sq}^3+\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)g_5^{23}\\
&+\mathrm{Sq}^8\mathrm{Sq}^2 g_5^{22}\\
&+(\mathrm{Sq}^8\mathrm{Sq}^4+\mathrm{Sq}^{11}\mathrm{Sq}^1)g_5^{20}\\
&+\mathrm{Sq}^{10}\mathrm{Sq}^2\,'g_5^{20}\\
&+(\mathrm{Sq}^{13}\mathrm{Sq}^5+\mathrm{Sq}^{15}\mathrm{Sq}^2\mathrm{Sq}^1)g_5^{14}\\
&+(\mathrm{Sq}^{18}\mathrm{Sq}^6\mathrm{Sq}^3+\mathrm{Sq}^{18}\mathrm{Sq}^7\mathrm{Sq}^2+\mathrm{Sq}^{17}\mathrm{Sq}^8\mathrm{Sq}^2+\mathrm{Sq}^{22}\mathrm{Sq}^5+\mathrm{Sq}^{23}\mathrm{Sq}^4+\mathrm{Sq}^{19}\mathrm{Sq}^6\mathrm{Sq}^2\\
&\quad+\mathrm{Sq}^{18}\mathrm{Sq}^9)g_5^{5}
\end{aligned}$$

$$
\begin{aligned}
\delta(g_8^{33}) &= \mathrm{Sq}^2\,\mathrm{Sq}^1\,g_6^{29}\\
&+ \mathrm{Sq}^6\,g_6^{26}\\
&+(\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^9+\mathrm{Sq}^6\,\mathrm{Sq}^3)g_6^{23}\\
&+(\mathrm{Sq}^{10}+\mathrm{Sq}^8\,\mathrm{Sq}^2)g_6^{22}\\
&+ \mathrm{Sq}^{11}\,g_6^{21}\\
&+(\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{10}\,\mathrm{Sq}^2)g_6^{20}\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2+\mathrm{Sq}^{11}\,\mathrm{Sq}^4+\mathrm{Sq}^{10}\,\mathrm{Sq}^5+\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{15})g_6^{17}\\
&+(\mathrm{Sq}^{15}\,\mathrm{Sq}^1+\mathrm{Sq}^{13}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_6^{16}\\
&+(\mathrm{Sq}^{20}\,\mathrm{Sq}^6+\mathrm{Sq}^{18}\,\mathrm{Sq}^8+\mathrm{Sq}^{19}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{21}\,\mathrm{Sq}^5+\mathrm{Sq}^{23}\,\mathrm{Sq}^3)g_6^6\\
\delta(g_{10}^{33}) &= (\mathrm{Sq}^7+\mathrm{Sq}^6\,\mathrm{Sq}^1)g_8^{25}\\
&+(\mathrm{Sq}^9+\mathrm{Sq}^8\,\mathrm{Sq}^1)g_8^{23}\\
&+ \mathrm{Sq}^{19}\,\mathrm{Sq}^5\,g_8^8
\end{aligned}
$$

$$
\begin{aligned}
\delta(g_2^{34}) &= 0\\
\delta(g_3^{34}) &= (\mathrm{Sq}^{21}\,\mathrm{Sq}^8+\mathrm{Sq}^{22}\,\mathrm{Sq}^5\,\mathrm{Sq}^2)g_1^4\\
&+(\mathrm{Sq}^{19}\,\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{23}\,\mathrm{Sq}^8+\mathrm{Sq}^{16}\,\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{21}\,\mathrm{Sq}^{10}+\mathrm{Sq}^{24}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{30}\,\mathrm{Sq}^1\\
&\quad+\mathrm{Sq}^{21}\,\mathrm{Sq}^7\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{20}\,\mathrm{Sq}^{10}\,\mathrm{Sq}^1+\mathrm{Sq}^{23}\,\mathrm{Sq}^7\,\mathrm{Sq}^1)g_1^2\\
&+(\mathrm{Sq}^{25}\,\mathrm{Sq}^7+\mathrm{Sq}^{22}\,\mathrm{Sq}^8\,\mathrm{Sq}^2+\mathrm{Sq}^{25}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{30}\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^9\,\mathrm{Sq}^3+\mathrm{Sq}^{29}\,\mathrm{Sq}^3+\mathrm{Sq}^{23}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\\
&\quad+\mathrm{Sq}^{23}\,\mathrm{Sq}^9+\mathrm{Sq}^{20}\,\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{21}\,\mathrm{Sq}^8\,\mathrm{Sq}^3+\mathrm{Sq}^{26}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^{10}\,\mathrm{Sq}^2\\
&\quad+\mathrm{Sq}^{24}\,\mathrm{Sq}^8+\mathrm{Sq}^{32})g_1^1\\
\delta('g_3^{34}) &= (\mathrm{Sq}^{26}\,\mathrm{Sq}^3+\mathrm{Sq}^{21}\,\mathrm{Sq}^8+\mathrm{Sq}^{27}\,\mathrm{Sq}^2+\mathrm{Sq}^{29})g_1^4\\
&+(\mathrm{Sq}^{23}\,\mathrm{Sq}^8+\mathrm{Sq}^{28}\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{21}\,\mathrm{Sq}^7\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{22}\,\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{27}\,\mathrm{Sq}^4+\mathrm{Sq}^{24}\,\mathrm{Sq}^7\\
&\quad+\mathrm{Sq}^{19}\,\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{16}\,\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_1^2\\
&+(\mathrm{Sq}^{24}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^{30}\,\mathrm{Sq}^2+\mathrm{Sq}^{27}\,\mathrm{Sq}^5+\mathrm{Sq}^{24}\,\mathrm{Sq}^8+\mathrm{Sq}^{21}\,\mathrm{Sq}^8\,\mathrm{Sq}^3+\mathrm{Sq}^{25}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{23}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\\
&\quad+\mathrm{Sq}^{18}\,\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{26}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{22}\,\mathrm{Sq}^8\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{32}+\mathrm{Sq}^{29}\,\mathrm{Sq}^3\\
&+\mathrm{Sq}^{23}\,\mathrm{Sq}^9)g_1^1\\
\delta(g_4^{34}) &= g_2^{33}\\
&+ \mathrm{Sq}^{13}\,g_2^{20}\\
&+ \mathrm{Sq}^{15}\,g_2^{18}\\
&+(\mathrm{Sq}^{16}+\mathrm{Sq}^{11}\,\mathrm{Sq}^5+\mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_2^{17}\\
&+ \mathrm{Sq}^{12}\,\mathrm{Sq}^5\,g_2^{16}\\
&+(\mathrm{Sq}^{23}+\mathrm{Sq}^{17}\,\mathrm{Sq}^6+\mathrm{Sq}^{16}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{15}\,\mathrm{Sq}^7\,\mathrm{Sq}^1+\mathrm{Sq}^{16}\,\mathrm{Sq}^6\,\mathrm{Sq}^1+\mathrm{Sq}^{18}\,\mathrm{Sq}^4\,\mathrm{Sq}^1)g_2^{10}\\
&+(\mathrm{Sq}^{21}\,\mathrm{Sq}^3+\mathrm{Sq}^{18}\,\mathrm{Sq}^6+\mathrm{Sq}^{16}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^{15}\,\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^{17}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{15}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\\
&+\mathrm{Sq}^{19}\,\mathrm{Sq}^5)g_2^9\\
&+(\mathrm{Sq}^{25}+\mathrm{Sq}^{23}\,\mathrm{Sq}^2+\mathrm{Sq}^{21}\,\mathrm{Sq}^4+\mathrm{Sq}^{19}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{22}\,\mathrm{Sq}^3)g_2^8\\
&+(\mathrm{Sq}^{26}\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^{22}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{21}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^8+\mathrm{Sq}^{19}\,\mathrm{Sq}^9+\mathrm{Sq}^{25}\,\mathrm{Sq}^3\\
&\quad+\mathrm{Sq}^{24}\,\mathrm{Sq}^4+\mathrm{Sq}^{17}\,\mathrm{Sq}^8\,\mathrm{Sq}^3+\mathrm{Sq}^{16}\,\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{18}\,\mathrm{Sq}^8\,\mathrm{Sq}^2)g_2^5\\
&+(\mathrm{Sq}^{19}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1+\mathrm{Sq}^{20}\,\mathrm{Sq}^9+\mathrm{Sq}^{27}\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^{23}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{28}\,\mathrm{Sq}^1+\mathrm{Sq}^{29}\\
&\quad+\mathrm{Sq}^{20}\,\mathrm{Sq}^8\,\mathrm{Sq}^1+\mathrm{Sq}^{22}\,\mathrm{Sq}^7+\mathrm{Sq}^{22}\,\mathrm{Sq}^6\,\mathrm{Sq}^1+\mathrm{Sq}^{24}\,\mathrm{Sq}^4\,\mathrm{Sq}^1+\mathrm{Sq}^{22}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\\
&\quad+\mathrm{Sq}^{18}\,\mathrm{Sq}^8\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{25}\,\mathrm{Sq}^4+\mathrm{Sq}^{19}\,\mathrm{Sq}^7\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{21}\,\mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{26}\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_2^4\\
&+(\mathrm{Sq}^{20}\,\mathrm{Sq}^9\,\mathrm{Sq}^2+\mathrm{Sq}^{19}\,\mathrm{Sq}^8\,\mathrm{Sq}^4+\mathrm{Sq}^{21}\,\mathrm{Sq}^7\,\mathrm{Sq}^3+\mathrm{Sq}^{26}\,\mathrm{Sq}^5+\mathrm{Sq}^{21}\,\mathrm{Sq}^{10}+\mathrm{Sq}^{31}+\mathrm{Sq}^{20}\,\mathrm{Sq}^8\,\mathrm{Sq}^3\\
&\quad+\mathrm{Sq}^{18}\,\mathrm{Sq}^9\,\mathrm{Sq}^4+\mathrm{Sq}^{19}\,\mathrm{Sq}^9\,\mathrm{Sq}^3+\mathrm{Sq}^{21}\,\mathrm{Sq}^8\,\mathrm{Sq}^2+\mathrm{Sq}^{22}\,\mathrm{Sq}^7\,\mathrm{Sq}^2)g_2^2\\
\delta(g_8^{34}) &= (\mathrm{Sq}^6\,\mathrm{Sq}^1+\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^7)g_6^{26}\\
&+(\mathrm{Sq}^9\,\mathrm{Sq}^2+\mathrm{Sq}^{10}\,\mathrm{Sq}^1+\mathrm{Sq}^8\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_6^{22}\\
&+ \mathrm{Sq}^{12}\,g_6^{21}\\
&+(\mathrm{Sq}^{11}\,\mathrm{Sq}^5+\mathrm{Sq}^{13}\,\mathrm{Sq}^3)g_6^{17}\\
&+(\mathrm{Sq}^{13}\,\mathrm{Sq}^4+\mathrm{Sq}^{12}\,\mathrm{Sq}^4\,\mathrm{Sq}^1+\mathrm{Sq}^{15}\,\mathrm{Sq}^2)g_6^{16}\\
&+(\mathrm{Sq}^{18}\,\mathrm{Sq}^9+\mathrm{Sq}^{21}\,\mathrm{Sq}^6+\mathrm{Sq}^{22}\,\mathrm{Sq}^5+\mathrm{Sq}^{27}+\mathrm{Sq}^{19}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^{24}\,\mathrm{Sq}^3)g_6^6\\
\delta(g_9^{34}) &= g_7^{33}\\
&+(\mathrm{Sq}^6\,\mathrm{Sq}^3+\mathrm{Sq}^9+\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_7^{24}\\
&+(\mathrm{Sq}^{14}\,\mathrm{Sq}^1+\mathrm{Sq}^{13}\,\mathrm{Sq}^2+\mathrm{Sq}^{15}+\mathrm{Sq}^{11}\,\mathrm{Sq}^4+\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1+\mathrm{Sq}^{11}\,\mathrm{Sq}^3\,\mathrm{Sq}^1)g_7^{18}\\
&+(\mathrm{Sq}^{19}\,\mathrm{Sq}^5\,\mathrm{Sq}^2+\mathrm{Sq}^{20}\,\mathrm{Sq}^4\,\mathrm{Sq}^2+\mathrm{Sq}^{24}\,\mathrm{Sq}^2+\mathrm{Sq}^{18}\,\mathrm{Sq}^8+\mathrm{Sq}^{21}\,\mathrm{Sq}^5+\mathrm{Sq}^{23}\,\mathrm{Sq}^3+\mathrm{Sq}^{26})g_7^7
\end{aligned}
$$

$\delta(g_{10}^{34}) = g_8^{33}$
$+(\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^8 + \mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^7\,\mathrm{Sq}^1)g_8^{25}$
$+\mathrm{Sq}^8\,\mathrm{Sq}^2\,g_8^{23}$
$+(\mathrm{Sq}^{19}\,\mathrm{Sq}^6 + \mathrm{Sq}^{21}\,\mathrm{Sq}^4 + \mathrm{Sq}^{25} + \mathrm{Sq}^{20}\,\mathrm{Sq}^5)g_8^8$
$\delta(g_{11}^{34}) = \mathrm{Sq}^1\,{}'g_9^{32}$
$+(\mathrm{Sq}^{21}\,\mathrm{Sq}^3 + \mathrm{Sq}^{22}\,\mathrm{Sq}^2 + \mathrm{Sq}^{24})g_9^9$

$\delta(g_4^{35}) = \mathrm{Sq}^{17}\,g_2^{17}$
$+(\mathrm{Sq}^{18} + \mathrm{Sq}^{16}\,\mathrm{Sq}^2)g_2^{16}$
$+(\mathrm{Sq}^{18}\,\mathrm{Sq}^6 + \mathrm{Sq}^{16}\,\mathrm{Sq}^7\,\mathrm{Sq}^1)g_2^{10}$
$+\mathrm{Sq}^{19}\,\mathrm{Sq}^6\,g_2^9$
$+(\mathrm{Sq}^{18}\,\mathrm{Sq}^8 + \mathrm{Sq}^{19}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{24}\,\mathrm{Sq}^2 + \mathrm{Sq}^{18}\,\mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^{16}\,\mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^{26})g_2^8$
$+(\mathrm{Sq}^{25}\,\mathrm{Sq}^4 + \mathrm{Sq}^{21}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{24}\,\mathrm{Sq}^5 + \mathrm{Sq}^{26}\,\mathrm{Sq}^3 + \mathrm{Sq}^{21}\,\mathrm{Sq}^8 + \mathrm{Sq}^{23}\,\mathrm{Sq}^6 + \mathrm{Sq}^{18}\,\mathrm{Sq}^9\,\mathrm{Sq}^2$
$+\mathrm{Sq}^{19}\,\mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^{20}\,\mathrm{Sq}^7\,\mathrm{Sq}^2)g_2^5$
$+(\mathrm{Sq}^{23}\,\mathrm{Sq}^7 + \mathrm{Sq}^{20}\,\mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^{20}\,\mathrm{Sq}^7\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{23}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{27}\,\mathrm{Sq}^2\,\mathrm{Sq}^1$
$+\mathrm{Sq}^{16}\,\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{22}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{23}\,\mathrm{Sq}^6\,\mathrm{Sq}^1 + \mathrm{Sq}^{19}\,\mathrm{Sq}^8\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{28}\,\mathrm{Sq}^2 + \mathrm{Sq}^{25}\,\mathrm{Sq}^5$
$+\mathrm{Sq}^{21}\,\mathrm{Sq}^7\,\mathrm{Sq}^2)g_2^4$
$+(\mathrm{Sq}^{30}\,\mathrm{Sq}^2 + \mathrm{Sq}^{20}\,\mathrm{Sq}^9\,\mathrm{Sq}^3 + \mathrm{Sq}^{22}\,\mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^{23}\,\mathrm{Sq}^6\,\mathrm{Sq}^3 + \mathrm{Sq}^{25}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{24}\,\mathrm{Sq}^8$
$+\mathrm{Sq}^{26}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{24}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{21}\,\mathrm{Sq}^9\,\mathrm{Sq}^2 + \mathrm{Sq}^{23}\,\mathrm{Sq}^9 + \mathrm{Sq}^{18}\,\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2)g_2^2$
$\delta(g_5^{35}) = {}'g_3^{34}$
$+\mathrm{Sq}^9\,\mathrm{Sq}^4\,g_3^{21}$
$+(\mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{10}\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{11}\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_3^{20}$
$+(\mathrm{Sq}^{14}\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4 + \mathrm{Sq}^{13}\,\mathrm{Sq}^3)g_3^{18}$
$+(\mathrm{Sq}^{12}\,\mathrm{Sq}^5 + \mathrm{Sq}^{17} + \mathrm{Sq}^{14}\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_3^{17}$
$+(\mathrm{Sq}^{16}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{14}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{17}\,\mathrm{Sq}^5 + \mathrm{Sq}^{14}\,\mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^{21}\,\mathrm{Sq}^1$
$+\mathrm{Sq}^{19}\,\mathrm{Sq}^3 + \mathrm{Sq}^{15}\,\mathrm{Sq}^7)g_3^{12}$
$+(\mathrm{Sq}^{20}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{17}\,\mathrm{Sq}^6 + \mathrm{Sq}^{15}\,\mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^{18}\,\mathrm{Sq}^5 + \mathrm{Sq}^{22}\,\mathrm{Sq}^1)g_3^{11}$
$+(\mathrm{Sq}^{24} + \mathrm{Sq}^{20}\,\mathrm{Sq}^4 + \mathrm{Sq}^{19}\,\mathrm{Sq}^5 + \mathrm{Sq}^{17}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{18}\,\mathrm{Sq}^6)g_3^{10}$
$+(\mathrm{Sq}^{20}\,\mathrm{Sq}^8 + \mathrm{Sq}^{23}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^{24}\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{24}\,\mathrm{Sq}^4 + \mathrm{Sq}^{18}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{21}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1$
$+\mathrm{Sq}^{19}\,\mathrm{Sq}^8\,\mathrm{Sq}^1 + \mathrm{Sq}^{21}\,\mathrm{Sq}^7 + \mathrm{Sq}^{25}\,\mathrm{Sq}^3 + \mathrm{Sq}^{18}\,\mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^{19}\,\mathrm{Sq}^9 + \mathrm{Sq}^{27}\,\mathrm{Sq}^1 + \mathrm{Sq}^{16}\,\mathrm{Sq}^8\,\mathrm{Sq}^4$
$+\mathrm{Sq}^{23}\,\mathrm{Sq}^5 + \mathrm{Sq}^{17}\,\mathrm{Sq}^8\,\mathrm{Sq}^3 + \mathrm{Sq}^{21}\,\mathrm{Sq}^6\,\mathrm{Sq}^1)g_3^6$
$+(\mathrm{Sq}^{31} + \mathrm{Sq}^{26}\,\mathrm{Sq}^5 + \mathrm{Sq}^{25}\,\mathrm{Sq}^6 + \mathrm{Sq}^{19}\,\mathrm{Sq}^8\,\mathrm{Sq}^4 + \mathrm{Sq}^{21}\,\mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^{19}\,\mathrm{Sq}^9\,\mathrm{Sq}^3 + \mathrm{Sq}^{29}\,\mathrm{Sq}^2$
$+\mathrm{Sq}^{24}\,\mathrm{Sq}^5\,\mathrm{Sq}^2 + \mathrm{Sq}^{20}\,\mathrm{Sq}^8\,\mathrm{Sq}^3 + \mathrm{Sq}^{25}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{22}\,\mathrm{Sq}^6\,\mathrm{Sq}^3 + \mathrm{Sq}^{20}\,\mathrm{Sq}^9\,\mathrm{Sq}^2 + \mathrm{Sq}^{23}\,\mathrm{Sq}^8$
$+\mathrm{Sq}^{27}\,\mathrm{Sq}^4 + \mathrm{Sq}^{24}\,\mathrm{Sq}^7 + \mathrm{Sq}^{28}\,\mathrm{Sq}^3)g_3^3$
$\delta(g_9^{35}) = \mathrm{Sq}^4\,g_7^{30}$
$+(\mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{10} + \mathrm{Sq}^8\,\mathrm{Sq}^2)g_7^{24}$
$+(\mathrm{Sq}^{18}\,\mathrm{Sq}^9 + \mathrm{Sq}^{19}\,\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^{23}\,\mathrm{Sq}^4 + \mathrm{Sq}^{24}\,\mathrm{Sq}^3 + \mathrm{Sq}^{21}\,\mathrm{Sq}^6 + \mathrm{Sq}^{19}\,\mathrm{Sq}^8)g_7^7$
$+(\mathrm{Sq}^{11} + \mathrm{Sq}^8\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^1)g_7^{23}$
$+\mathrm{Sq}^{12}\,g_7^{22}$
$+(\mathrm{Sq}^{13}\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^{12}\,\mathrm{Sq}^4 + \mathrm{Sq}^{12}\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^{11}\,\mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^9\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1)g_7^{18}$
$\delta(g_{10}^{35}) = g_8^{34}$
$+\mathrm{Sq}^3\,g_8^{31}$
$+(\mathrm{Sq}^{10}\,\mathrm{Sq}^1 + \mathrm{Sq}^{11} + \mathrm{Sq}^9\,\mathrm{Sq}^2)g_8^{23}$
$+(\mathrm{Sq}^{24}\,\mathrm{Sq}^2 + \mathrm{Sq}^{19}\,\mathrm{Sq}^7)g_8^8$
$\delta(g_{11}^{35}) = \mathrm{Sq}^2\,{}'g_9^{32}$
$+\mathrm{Sq}^3\,g_9^{31}$
$+\mathrm{Sq}^6\,g_9^{28}$
$+(\mathrm{Sq}^8 + \mathrm{Sq}^7\,\mathrm{Sq}^1)g_9^{26}$
$+\mathrm{Sq}^{25}\,g_9^9$
$\delta(g_{12}^{35}) = \mathrm{Sq}^5\,g_{10}^{29}$
$+(\mathrm{Sq}^{22}\,\mathrm{Sq}^2 + \mathrm{Sq}^{21}\,\mathrm{Sq}^3)g_{10}^{10}$

R

[1] Hans-Joachim Baues, *The algebra of secondary cohomology operations*, 2004.

[2] Hans-Joachim Baues and Mamuka Jibladze, *Classification of abelian track categories*, K-Theory **25** (2002), 299–311.

[3] ———, *Secondary derived functors and the Adams spectral sequence*, `arxiv:math.AT/0407031`.

[4] Robert R. Bruner, *Calculation of large Ext modules*, Computers in Geometry and Topology (Chicago, IL, 1986), Lecture Notes in Pure and Appl. Math., vol. 114, Dekker, New York, 1989, pp. 79–104.

[5] Jean-Pierre Meyer, *Acyclic models for multicomplexes*, Duke Math. J. **45** (1978), 67–85.

[6] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, American Mathematical Society, University of Rochester - AMS, 2004, ISBN 0-8218-2967-X.

M -P -I ¨ M , V 7, D-53111 B , G
E-mail address: `baues@mpim-bonn.mpg.de`

R M I , M. A . 1, T 0193, G
E-mail address: `jib@rmi.acnet.ge`