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André–Quillen cohomology of commutative S-algebras

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Abstract

We define topological André–Quillen cohomology of commutative S-algebras and construct a spectral sequence that calculates the André–Quillen cohomology of commutative S-algebras over $H\mathbb{F}_p$. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

The notion of commutative S-algebra, which is equivalent to the traditional notion of E_∞-ring spectrum, is a generalization to stable homotopy theory of the algebraic notion of commutative ring. The purpose of this paper is to construct and begin the analysis of the topological analogue of André–Quillen cohomology [11]. Our initial motivation was to develop and expand the mathematical theory suggested by the work of Igor Kriz in "Towers of E_∞-ring spectra with an application to BP" [6]. We develop the theory of André–Quillen cohomology of commutative S-algebras in the framework provided by Elmendorf et al. [5]. McClure and Hunter studied the André–Quillen cohomology of the Eilemberg–Mac Lane E_∞-ring spectrum corresponding to the field of two elements before the necessary foundations on spectra were in place. We hope to apply some of their ideas in future calculations. Alan Robinson and Sarah Whitehouse have also worked on a cohomology theory for E_∞-ring spectra. At the moment we are not sure how it would relate to our theory.

The following is an outline of the structure of the paper.

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In the first section we introduce the model categories in which our work takes place. These are

- $\mathscr{M}_A$, the category of A-modules for a commutative S-algebra A.
- $\mathscr{C}_{A/B}$, the category of commutative A-algebras with A-algebra maps to B.
- $\mathscr{N}_A$, the category of non-unital commutative A-algebras.

All of the above are complete and cocomplete topological model categories. In general, the classical notions of cofibration and fibration will not coincide with the respective concepts that describe our model categories. Hence, we retain the terminology of [5] and refer to the model theoretic concepts as q-cofibrations and q-fibrations. Given a model category $\mathscr{C}$, we let $\bar{h}\mathscr{C}$ denote its homotopy category, that is, the category obtained by inverting the weak equivalences of the model category structure of $\mathscr{C}$.

In Sections 2 and 3, we will exhibit a series of adjunctions relating the categories listed above. These adjunctions pass to the homotopy categories and, in particular, we find that $\bar{h}\mathscr{C}_{A/A}$ is equivalent to $\bar{h}\mathscr{N}_A$. If B is a q-cofibrant commutative A-algebra, the adjunctions and equivalence just mentioned will allow us to describe a left adjoint to the trivial B-algebra extension functor

$$B \vee - : \bar{h}\mathscr{M}_B \to \bar{h}\mathscr{C}_{A/B}.$$

We will refer to this left adjoint as the "abelianization" functor for commutative A-algebras over B.

In Section 4, we define André–Quillen cohomology of commutative S-algebras in terms of the "abelianization" functor. If B is a commutative A-algebra, let $\Omega_{B/A}$ denote the B-module obtained by applying the "abelianization" functor to B so that we have

$$\bar{h}\mathscr{C}_{A/B}(B, B \vee M) \cong \bar{h}\mathscr{M}_B(\Omega_{B/A}, M).$$

Given a B-module M, we define the André–Quillen cohomology of B relative to A with coefficients in M by

$$AQ^*(B/A; M) = \mathrm{Ext}^*_B(\Omega_{B/A}, M) \equiv \pi_{-*}F_B(\Omega_{B/A}, M).$$

We observe that there is a forgetful map from AQ-cohomology relative to A to ordinary A-module cohomology.

We finish the section with topological analogues of Quillen's transitivity exact sequence and flat base change [11].

In Section 5 we construct a spectral sequence suitable for computing the AQ-cohomology relative to S of commutative S-algebras B with S-algebra maps to $H\mathbb{F}_p$. In Section 6 we identify its E_2-term as the derived functors of a certain composite functor and in Section 7, we exhibit a Grothendieck-type spectral sequence to calculate those.

In Section 8 we present an application of the theory due to Igor Kriz. We construct the Postnikov towers of connective algebras in the category of commutative S-algebras. The k-invariants lie in their André–Quillen cohomology relative to S and map to their ordinary k-invariants via the forgetful map exhibited in Section 4.

In Section 9 we prove a lemma needed in Section 5. Basically, we overcome the fact that the underlying module of a q-cofibrant non-unital commutative algebra is not q-cofibrant.

The numbering of the theorems, definitions and such describe the section and order in which they appear within that section. That is, Definition 2.3 refers to a definition in Section 2 which is the numbered item within that section.

1. Categories of A-algebras

Let A be a commutative S-algebra, and let $\mathscr{M}_A$ denote the category of A-modules. Recall from [5] that $\mathscr{C}_A$, the category of commutative A-algebras, coincides with $\mathscr{M}_A[\mathbb{P}]$, the category of $\mathbb{P}$-algebras, where $\mathbb{P}:\mathscr{M}_A \to \mathscr{M}_A$ is the monad given by

$$\mathbb{P}M = \bigvee_{j\geq 0} M^j/\Sigma_j.$$

Here M^j denotes the j-fold smash power over A and $M^0 = A$.

By [5] we know that $\mathscr{C}_A$ is enriched over topological spaces and has a topological model category structure where the weak equivalences are those morphisms which are weak equivalences when viewed as maps of spectra. The q-fibrations are those morphisms which, when viewed as maps of A-modules, satisfy the RLP with respect to the inclusions of A-modules $CS^n_A \wedge \{0_+\} \to CS^n_A \wedge I_+$, and the q-cofibrations are those morphisms which are retracts of relative cell $\mathbb{P}$-algebras.

The category of commutative A-algebras over B. For a commutative A-algebra B, let $\mathscr{C}_{A/B}$ denote the category of commutative A-algebras over B, that is, the subcategory of $\mathscr{C}_A$ whose objects C come equipped with a map of A-algebras $\varepsilon: C \to B$ and whose morphisms are maps of A-algebras over B.

$\mathscr{C}_{A/B}$ inherites a topological model category structure from $\mathscr{C}_A$. The case $A = B$ is important to us and we note that $\mathscr{C}_{A/A}$ is a pointed category and hence it is enriched over based topological spaces.

The category of non-unital commutative A-algebras. Define $\mathbb{A}:\mathscr{M}_A \to \mathscr{M}_A$ by $\mathbb{A}M = \bigvee_{j>0} M^j/\Sigma_j$; $\mathbb{A}$ is a monad with unit $\eta: M \to \mathbb{A}M$ the inclusion on the first summand, and product $\mu: \mathbb{A}\mathbb{A}M \to AM$ induced by the maps

$$M^{j_1} \wedge_A \cdots \wedge_A M^{j_k} \to M^{j_1+\cdots+j_k}$$

given by the evident identifications.

Let $\mathscr{N}_A = \mathscr{M}_A[\mathbb{A}]$ be the category of $\mathbb{A}$-algebras and note that this is the category of non-unital commutative A-algebras. We will refer to an object of $\mathscr{N}_A$ as an A-NUCA or an $\mathbb{A}$-algebra if the monadic description of the category is relevant.

Let $K:\mathscr{N}_A \to \mathscr{C}_A$ denote the functor which assigns to an $\mathbb{A}$-algebra M the commutative A-algebra $A \vee M$ with multiplication

$$(A \vee M) \wedge_A (A \vee M) \cong A \vee M \vee M \vee M \wedge_A M \to A \vee M$$

given by the obvious maps on the first summands and the multiplication of M on the last one. Note that for any A-module X, $K(\mathbb{A}X)=\mathbb{P}X$.

Proposition 1.1. *$\mathscr{N}_A$ has a topological model structure where the weak equivalences are those maps which are weak equivalences when viewed as maps of spectra. The q-fibrations are those maps which, when viewed as maps of A-modules, satisfy the RLP with respect to the inclusions of A-modules $CS_A^n \wedge \{0_+\} \to CS_A^n \wedge I_+$. A map is a q-cofibration if and only if it is a retract of a relative cell object. A q-cofibration is a cofibration (but not a q-cofibration) of underlying A-modules.*

Proof. In [5], Theorem VII.4.9 asserts that if A is a commutative S-algebra and $\mathbb{T}: \mathscr{M}_A \to \mathscr{M}_A$ is a continuous monad that preserves reflexive coequalizers and which satisfies the "Cofibration Hypothesis", then $\mathscr{M}_A$ creates a topological model category structure in $\mathscr{M}_A[\mathbb{T}]$.

The functor $\mathbb{A}: \mathscr{M}_A \to \mathscr{M}_A$ is a continuous monad which, by the argument in [5, II.7.2], preserves reflexive coequalizers in $\mathscr{M}_A$. The relevant formulation of the "Cofibration Hypothesis" in our case says that, given a pushout in $\mathscr{M}_A[\mathbb{A}]$ of the general form

$$\begin{array}{ccc} \mathbb{A}E & \longrightarrow & D \\ \downarrow & & \downarrow{\scriptstyle i} \\ \mathbb{A}CE & \longrightarrow & B \end{array}$$

with E a wedge of A-module spheres S_A^n, the map i is a cofibration of A-modules. The underlying spectrum of the $\mathbb{A}$-algebra colimit of a sequence of cofibrations of $\mathbb{A}$-algebras is their colimit as a sequence of maps of spectra.

Given a pushout diagram in $\mathscr{M}_A[\mathbb{A}]$ as above, one checks that

$$\begin{array}{ccc} K(\mathbb{A}E)=\mathbb{P}E & \longrightarrow & K(D) \\ \downarrow & & \downarrow{\scriptstyle A\vee i} \\ K(\mathbb{A}CE)=\mathbb{P}CE & \longrightarrow & K(B) \end{array}$$

is a pushout in the category of commutative A-algebras and, by [5, VII.3.9], $A\vee i$ is a cofibration of A-modules. Since i is a retract of this map it is also a cofibration of A-modules.

Now, let $\{M_i\}$ be a sequence of maps of $\mathbb{A}$-algebras that are cofibrations of spectra. The colimit in the category of spectra computes the colimit in the category of A-modules.

Note that we have the following map:

$$(\operatorname{colim} M_i) \wedge_A (\operatorname{colim} M_j) \cong \operatorname{colim}(M_i \wedge_A M_i) \to \operatorname{colim} M_i.$$

Thus, the colimit in the category of spectra comes with a multiplication that gives it an $\mathbb{A}$-algebra structure.

Hence, $\mathscr{M}_A$ creates the topological model category structure on $\mathscr{M}_A[\mathbb{A}] = \mathscr{N}_A$ claimed in the proposition. The statement about the q-cofibrations follows from [5, VII.4.14]. $\square$

2. The augmentation ideal functor

We will exhibit an equivalence between the homotopy categories $\bar{h}\mathscr{N}_A$ and $\bar{h}\mathscr{C}_{A/A}$.

Let $I : \mathscr{C}_{A/A} \to \mathscr{N}_A$ assign to (B, η, ε) its "augmentation ideal": $I(B)$ is given by the pullback diagram in $\mathscr{M}_A$,

$$\begin{tikzcd} I(B) \arrow[r, "i"] \arrow[d] & B \arrow[d, "\varepsilon"] \\ * \arrow[r] & A. \end{tikzcd}$$

Note that since the following is a commutative diagram

$$\begin{tikzcd} I(B) \wedge_A I(B) \arrow[r, "i \wedge i"] \arrow[d] & B \wedge_A B \arrow[r, "\phi_B"] \arrow[d, "\varepsilon \wedge \varepsilon"] & B \arrow[d, "\varepsilon"] \\ * = * \wedge_A * \arrow[r] & A \wedge_A A \arrow[r, "\cong"'] & A \end{tikzcd}$$

$I(B)$ comes with a multiplication ϕ given by the universal property of pullbacks. On morphisms f, $I(f)$ is also given by the universal property of pullbacks.

Recall the functor $K : \mathscr{N}_A \to \mathscr{C}_A$ from the previous section. The projection map π : $K(M) = A \vee M \to A$ is a map of A-algebras, and the composite

$$A \xrightarrow{\eta} A \vee M \xrightarrow{\pi} A$$

is the identity. Hence, we may view K as a functor $\mathscr{N}_A \to \mathscr{C}_{A/A}$. We have the following adjunction. We thank the referee who suggested the following line of argument for the proofs of this fact and Proposition 3.1.

Proposition 2.1. $\mathscr{C}_{A/A}(K(M), B) \cong \mathscr{N}_A(M, I(B))$.

Proof. If $M = \mathbb{A}X$ for an A-module X, we have

$$\mathscr{C}_{A/A}(K(\mathbb{A}X), B) \cong \mathscr{C}_{A/A}(\mathbb{P}X, B) \cong \mathscr{M}_A(X, I(B)) \cong \mathscr{N}_A(\mathbb{A}X, B).$$

Since for any A-NUCA M,

$$\mathbb{A}\mathbb{A}M \underset{\mathbb{A}\xi}{\overset{\mu}{\rightrightarrows}} \mathbb{A}M \xrightarrow{\xi} M$$

is a split coequalizer, the result follows for general M. $\square$

Proposition 2.2. *The adjunction described above gives an equivalence of homotopy categories.*

Proof. We will use Dwyer and Spalinski's refinement of Quillen's total derived functor theorem [10, I.4]. To show that the adjunction passes to the homotopy categories, it is enough to show that K preserves q-cofibrations and acyclic q-cofibrations.

By Proposition 1.1, a map in $\mathscr{N}_A$ is a q-cofibration if and only if it is a retract of a relative cell $\mathbb{A}$-algebra. A map in $\mathscr{C}_{A/A}$ is a q-cofibration if and only if it is a q-cofibration in $\mathscr{C}_A$, that is, if and only if it is a retract of a relative cell $\mathbb{P}$-algebra in $\mathscr{C}_{A/A}$.

Let Y be a relative cell $\mathbb{A}$-algebra under the $\mathbb{A}$-algebra X. Thus, $Y = \operatorname{colim} Y_n$, where $Y_0 = X$ and Y_{n+1} is obtained from Y_n by a pushout of the form

$$\begin{tikzcd}
\mathbb{A}E \arrow[r] \arrow[d] & Y_n \arrow[d, "i"] \\
\mathbb{A}CE \arrow[r] & Y_{n+1}
\end{tikzcd}$$

with E a wedge of A-module spheres S_A^m.

Since K is a left adjoint, it preserves all colimits. Further, $\mathbb{P}X = K\mathbb{A}X$ so, if $(\operatorname{colim} Y_n, X)$ is a relative cell $\mathbb{A}$-algebra $(K\operatorname{colim} Y_n, KX) = (\operatorname{colim} KY_n, KX)$ is a relative cell $\mathbb{P}$-algebra. Hence, K takes retracts of relative cell $\mathbb{A}$-algebras to retracts of relative cell $\mathbb{P}$-algebras in $\mathscr{C}_{A/A}$. Since K clearly preserves weak equivalences, we conclude that the adjunction passes to the homotopy categories.

To prove the proposed equivalence, it is left to show that if M is a q-cofibrant object of $\mathscr{N}_A$ and B is a q-fibrant object of $\mathscr{C}_{A/A}$, then $g : M \to I(B)$ is a weak equivalence if and only if its adjoint $\tilde{g} : A \vee M \to B$ is a weak equivalence. But this is clear. Given g as above, its adjoint is the composite

$$\tilde{g} : A \vee M \xrightarrow{A \vee g} A \vee I(B) \xrightarrow{\eta \vee i} B.$$

If g is a weak equivalence, $A \vee g$ is also a weak equivalence. If B is q-fibrant, that is, if the augmentation map $B \to A$ is a fibration, then its fiber is $I(B) \xrightarrow{i} B$ and $\eta \vee i$ is a weak equivalence. Hence, the composite $\tilde{g}$ is a weak equivalence as well.

Now, suppose that $\tilde{g}: A \vee M \to B$ is a weak equivalence in $\mathscr{C}_{A/A}$. We have the following commutative diagram of A-modules:

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d] & I(B) \arrow[d] \\ A \vee M \arrow[r, "A \vee g"] \arrow[dr, "\tilde{g}"'] & A \vee I(B) \arrow[d, "\cong"] \\ & B. \end{tikzcd}$$

Since $\tilde{g}$ is a weak equivalence, $A \vee g$ is a weak equivalence and the diagram

$$\begin{tikzcd} \pi_* M \arrow[r, "g_*"] \arrow[d] & \pi_* I(B) \arrow[d] \\ \pi_* A \oplus \pi_* M \arrow[r, "\cong"'] & \pi_* A \oplus \pi_* I(B) \end{tikzcd}$$

shows that g is a weak equivalence.

Hence, by [4, 9.7–9.8], the total derived functors $\boldsymbol{L}K$ and $\boldsymbol{R}I$ exist and are inverse equivalences between the homotopy categories $\bar{h}\mathscr{N}_A$ and $\bar{h}\mathscr{C}_{A/A}$. $\square$

3. The indecomposables and abelianization functors

Let $Q: \mathscr{N}_A \to \mathscr{M}_A$ denote the "indecomposables" functor that assigns to an object N of $\mathscr{N}_A$ the A-module $Q(N)$ given by the pushout of A-modules

$$\begin{tikzcd} N \wedge_A N \arrow[r] \arrow[d] & * \arrow[d] \\ N \arrow[r, "r"'] & Q(N). \end{tikzcd}$$

Let $Z: \mathscr{M}_A \to \mathscr{N}_A$ be the functor given by considering A-modules as objects in $\mathscr{N}_A$ with zero multiplication and which is the identity on morphisms.

Proposition 3.1. $\mathscr{M}_A(Q(N), M) \cong \mathscr{N}_A(N, Z(M))$.

Proof. If $N = \mathbb{A}X$ for some A-module X, then $Q(N) = Q(\mathbb{A}X) = X$ and we have that

$$\mathscr{N}_A(\mathbb{A}X, Z(M)) \cong \mathscr{M}_A(X, M) \cong \mathscr{M}_A(Q(\mathbb{A}X), M).$$

For a general A-NUCA the result follows as in Proposition 2.1. $\square$

Since Z is the identity on morphisms and $\mathcal{M}_A$ creates the model category structure on $\mathcal{N}_A$, Z preserves q-fibrations and acyclic q-fibrations, so the total derived functors $\boldsymbol{R}Z$ and $\boldsymbol{L}Q$ exist and are adjoint. We have

$$\bar{h}\mathcal{M}_A(\boldsymbol{L}Q(N), M) \cong \bar{h}\mathcal{N}_A(N, \boldsymbol{R}Z(M)).$$

If B is a commutative A-algebra, the usual adjunction

$$\mathcal{C}_A(C, D) \cong \mathcal{C}_B(C \wedge_A B, D)$$

restricts to the categories of algebras over B and passes to their homotopy categories. We have

$$\bar{h}\mathcal{C}_{A/B}(C, D) \cong \bar{h}\mathcal{C}_{B/B}(C \wedge_A^{\boldsymbol{L}} B, D)$$

where $C \wedge_A^{\boldsymbol{L}} B$ denotes the total derived functor of $- \wedge_A B$ applied to C. That is, $C \wedge_A^{\boldsymbol{L}} B \cong \Gamma C \wedge_A B$ for ΓC a q-cofibrant commutative A-algebra over B weakly equivalent to C.

Let B be a q-cofibrant commutative A-algebra. We now have all the ingredients to describe the *abelianization* functor

$$Ab_A^B : \bar{h}\mathcal{C}_{A/B} \to \bar{h}\mathcal{M}_B$$

adjoint to the trivial B-algebra extension functor

$$B \vee - : \bar{h}\mathcal{M}_B \to \bar{h}\mathcal{C}_{A/B}.$$

Proposition 3.2. *Let C be a commutative A-algebra over B and M a B-module. Then*:

$$\bar{h}\mathcal{C}_{A/B}(C, B \vee M) \cong \bar{h}\mathcal{M}_B(\boldsymbol{L}Q\boldsymbol{R}I(C \wedge_A^{\boldsymbol{L}} B), M).$$

Proof. We have the following diagram, where the parallel arrows are adjunctions:

$$\begin{tikzcd}
\bar{h}\mathcal{C}_{A/B} \arrow[r, shift left, "- \wedge_A^{\boldsymbol{L}} B"] & \bar{h}\mathcal{C}_{B/B} \arrow[l, shift left] \arrow[r, shift right, "\boldsymbol{R}I"'] & \bar{h}\mathcal{N}_B \arrow[l, shift right, "\boldsymbol{L}K"'] \arrow[d, shift left, "\boldsymbol{L}Q"] \\
 & & \bar{h}\mathcal{M}_B. \arrow[u, shift left, "\boldsymbol{R}Z"] \arrow[ull, "B\vee-"]
\end{tikzcd}$$

The unlabeled arrow is given by the forgetful functor.

For C a commutative A-algebra over B we have

$$\begin{aligned}
\bar{h}\mathcal{M}_B(\boldsymbol{L}Q\boldsymbol{R}I(C \wedge_A^{\boldsymbol{L}} B), M) &\cong \bar{h}\mathcal{N}_B(\boldsymbol{R}I(C \wedge_A^{\boldsymbol{L}} B), \boldsymbol{R}Z(M)) \\
&\cong \bar{h}\mathcal{C}_{B/B}(\boldsymbol{L}K\boldsymbol{R}I(C \wedge_A^{\boldsymbol{L}} B), \boldsymbol{L}K\boldsymbol{R}Z(M)) \\
&\cong \bar{h}\mathcal{C}_{B/B}(C \wedge_A^{\boldsymbol{L}} B, B \vee M) \\
&\cong \bar{h}\mathcal{C}_{A/B}(C, B \vee M),
\end{aligned}$$

where the second and third isomorphisms come from the fact that the pair $(\boldsymbol{L}K, \boldsymbol{R}I)$ gives an adjoint equivalence of homotopy categories. $\square$

The above proposition tells us that the right adjoint to the trivial B-algebra extension functor $B \vee - : \bar{h}\mathcal{M}_B \to \bar{h}\mathcal{C}_{A/B}$ is given by

$$Ab_A^B(\ ) = \boldsymbol{L}Q\boldsymbol{R}I(- \wedge_A^{\boldsymbol{L}} B). \tag{1}$$

The reader should be aware that the functor $B \vee -$ and the composite functor $QI(- \wedge_A B)$ are not adjoint before passing to the homotopy categories.

4. André–Quillen cohomology of commutative A-algebras

Let B be a commutative A-algebra and M a B-module.

Definition 4.1. Let $\Omega_{B/A}$ denote the B-module $Ab_A^B(B) = \boldsymbol{L}Q\boldsymbol{R}I(B \wedge_A^{\boldsymbol{L}} B)$, so that

$$\bar{h}\mathcal{C}_{A/B}(B, B \vee M) \cong \bar{h}\mathcal{M}_B(\Omega_{B/A}, M).$$

Define the André–Quillen cohomology of B relative to A with coefficients in M by

$$AQ^*(B/A; M) = Ext_B^*(\Omega_{B/A}, M) \equiv \pi_{-*}F_B(\Omega_{B/A}, M).$$

We can relate AQ-cohomology relative to A to ordinary A-module cohomology as follows. For a q-cofibrant A-algebra B, consider the composite map of A-modules

$$\psi : B \to B \vee \Omega_{B/A} \to \Omega_{B/A},$$

where the first map is the A-algebra map, viewed as a map of A-modules, adjoint to the identity and the second map is the projection. The map ψ induces a forgetful map from AQ-cohomology to ordinary A-module cohomology:

$$\psi^* : Ext_B^*(\Omega_{B/A}, M) \to Ext_A^*(\Omega_{B/A}, M) \to Ext_A^*(B, M). \tag{2}$$

We have a topological analogue of Quillen's transitivity exact sequence [11, Theorem 5.1].

Proposition 4.2. *Let $A \to B \to C$ be maps of q-cofibrant commutative S-algebras. Then*

$$\Omega_{B/A} \wedge_B C \to \Omega_{C/A} \to \Omega_{C/B}$$

is a homotopy cofiber sequence of C-modules.

Corollary 4.3. *If $A \to B \to C$ are maps of q-cofibrant S-algebras and M is a C-module, we have a long exact sequence*

$$\cdots \to AQ^n(C/B; M) \to AQ^n(C/A; M) \to AQ^n(B/A; M) \to AQ^{n+1}(C/B; M) \cdots$$

Before we can prove the proposition we need some facts about the functors $\boldsymbol{R}I$ and $\boldsymbol{L}Q$. We fix the following notation: $\rightarrowtail$ denotes maps which are q-cofibrations and $\twoheadrightarrow$ denotes maps which are q-fibrations. We also append subscripts to the functors I, Q and K and their derived functors to avoid confusion as to what categories these functors refer to.

Lemma 4.4. *Let A be a q-cofibrant commutative S-algebra, B a q-cofibrant commutative A-algebra, and C a q-fibrant and q-cofibrant A-algebra over A. In the homotopy category $\bar{h}\mathcal{N}_B$,*

$$\boldsymbol{R}I_A(C)\wedge^{\boldsymbol{L}}_A B \cong \boldsymbol{R}I_B(C\wedge^{\boldsymbol{L}}_A B).$$

That is, $\boldsymbol{R}I$ commutes with $(-)\wedge^{\boldsymbol{L}}_A B$.

Proof. For an object J in $\mathcal{N}_A$ let γJ denote a q-cofibrant approximation and for an object D in $\mathscr{C}_{B/B}$ let fD denote a q-fibrant approximation.

We will show that, in $\mathcal{N}_B$,

$$\boldsymbol{R}I_A(C)\wedge^{\boldsymbol{L}}_A B = \gamma I_A(C)\wedge_A B \xrightarrow{\sim} I_B(f(C\wedge_A B)) = \boldsymbol{R}I_B(C\wedge^{\boldsymbol{L}}_A B).$$

We have the following diagram in $\mathscr{C}_{A/A}$:

$$A \rightarrowtail A\vee \gamma I_A(C) \xrightarrow{\sim} C \twoheadrightarrow A.$$

The functor $-\wedge_A B$ takes q-cofibrations in $\mathscr{C}_{A/A}$ to q-cofibrations in $\mathscr{C}_{B/B}$ and preserves weak equivalences between q-cofibrant objects. Hence, applying $-\wedge_A B$, we obtain the following diagram in $\mathscr{C}_{B/B}$:

$$B \rightarrowtail B\vee (\gamma I_A(C)\wedge_A B) \xrightarrow{\sim} C\wedge_A B \to B.$$

We factor the augmentation $C\wedge_A B\to B$ by an acyclic q-cofibration and a q-fibration to obtain the following diagram:

$$B \rightarrowtail B\vee (\gamma I_A(C)\wedge_A B) \xrightarrow{\sim} C\wedge_A B \overset{\sim}{\rightarrowtail} f(C\wedge_A B) \twoheadrightarrow B.$$

Hence, we have that $K_B(\gamma I_A(C)\wedge_A B) = B\vee(\gamma I_A(C)\wedge_A B) \xrightarrow{\sim} f(C\wedge_A B)$. Note that $\gamma I_A(C)\wedge_A B$ is q-cofibrant and $f(C\wedge_A B)$ is q-fibrant. Hence, since the total derived functors $\boldsymbol{L}K_B$ and $\boldsymbol{R}I_B$ give an adjoint equivalence of homotopy categories, the adjoint

$$\gamma I_A(C)\wedge_A B \to I_B f(C\wedge_A B)$$

to the weak equivalence above is also a weak equivalence. □

Lemma 4.5. *Let A be a q-cofibrant commutative S-algebra, B a q-cofibrant commutative A-algebra, and N a q-cofibrant A-NUCA. In the homotopy category $\bar{h}\mathscr{M}_B$,*

$$\boldsymbol{L}Q_A(N)\wedge^{\boldsymbol{L}}_A B \cong \boldsymbol{L}Q_B(N\wedge^{\boldsymbol{L}}_A B).$$

That is, $\boldsymbol{L}Q$ commutes with $(-)\wedge^{\boldsymbol{L}}_A B$.

Proof. The functor $-\wedge_A B$, takes q-cofibrant objects to q-cofibrant objects and, being a left adjoint, it commutes with colimits. The functor Q also takes q-cofibrant objects to q-cofibrant objects and, since it is defined by a pushout diagram, it commutes with $-\wedge_A B$. □

We now give the proof of Proposition 4.2.

Proof of Proposition 4.2. We will use the following notation:
- ΓD denotes a q-cofibrant A-algebra approximation of D.
- γM denotes a q-cofibrant C-NUCA approximation of M.
- fR denotes a q-fibrant C-algebra over C approximation of R.

Consider the following factorization by q-cofibrant A-algebras:

$$\begin{tikzcd}
 & \Gamma B \arrow[rr, tail] \arrow[dr, two heads, "\cong"] & & \Gamma C \arrow[dr, two heads, "\cong"] & \\
A \arrow[ur, tail] \arrow[rr] & & B \arrow[rr] & & C
\end{tikzcd}$$

Applying $-\wedge_A C$ we obtain a diagram of q-cofibrant C-algebras over C

$$C \rightarrowtail \Gamma B \wedge_A C \overset{g}{\rightarrowtail} \Gamma C \wedge_A C.$$

The cofiber of the map g is given by the homotopy pushout in the category of C-algebras over C of

$$\Gamma C \wedge_A C \leftarrowtail \Gamma B \wedge_A C \to C.$$

By [5, VII.7.3], since all the algebras are q-cofibrant and the map g is a q-cofibration, this is homotopy equivalent to the actual pushout

$$\Gamma C \wedge_A C \wedge_{\Gamma B \wedge_A C} C \cong \Gamma C \wedge_{\Gamma B} C,$$

which is a q-cofibrant C-algebra over C. We obtain a cofiber sequence of q-cofibrant C-algebras over C:

$$\Gamma B \wedge_A C \to \Gamma C \wedge_A C \to \Gamma C \wedge_{\Gamma B} C.$$

This gives us a homotopy cofiber sequence of C-modules

$$Q_C \gamma I_C f(\Gamma B \wedge_A C) \to Q_C \gamma I_C f(\Gamma C \wedge_A C) \to Q_C \gamma I_C f(\Gamma C \wedge_{\Gamma B} C)$$

since the derived functors $(\boldsymbol{L}K_C, \boldsymbol{R}I_C)$ give an equivalence between the homotopy categories $\bar{h}\mathscr{C}_{C/C}$ and $\bar{h}\mathscr{N}_C$, and since Q_C preserves cofiber sequences.

By the two previous lemmas, $Q_C \gamma I_C f(\Gamma B \wedge_A C) \cong \Omega_{B/A} \wedge_B C$.

By definition, $Q_C \gamma I_C f(\Gamma C \wedge_A C) = \Omega_{C/A}$.

Hence, it is left to show that $Q_C \gamma I_C f(\Gamma C \wedge_{\Gamma B} C) = \Omega_{C/B}$.

By definition $\Omega_{C/B} = Q_C \gamma I_C f(\Gamma_B C \wedge_B C)$ where $\Gamma_B C$ denotes a q-cofibrant B-algebra approximation of C. Thus, it is enough to show that $\Gamma C \wedge_{\Gamma B} C$ is weakly equivalent to $\Gamma_B C \wedge_B C$.

Factoring the arrow $B \to C$ in the original factorization of $A \to B \to C$, we obtain the following commutaive diagram:

$$\begin{tikzcd}
\Gamma B \arrow[r, two heads] \arrow[d, tail] & B \arrow[r, tail] & \Gamma_B C \arrow[d, two heads, "\cong"] \\
\Gamma C \arrow[rru] \arrow[rr, two heads, "\cong"'] & & C.
\end{tikzcd}$$

The existence of the diagonal arrow is guaranteed by the properties of q-fibrations and q-cofibrations in a model category, and it is a weak equivalence.

The diagram

$$\begin{array}{ccccc}
C & \longleftarrow & \Gamma B & \rightarrowtail & \Gamma C \\
\Vert & & \downarrow \cong & & \downarrow \cong \\
C & \longleftarrow & B & \rightarrowtail & \Gamma_B C
\end{array}$$

gives that the homotopy pushouts of the top and bottom rows are weakly equivalent. But, all the algebras involved are q-cofibrant commutative S-algebras hence the homotopy pushouts are weakly equivalent to the actual pushouts. Thus, we get $\Gamma C \wedge_{\Gamma B} C \simeq \Gamma_B C \wedge_B C$ as desired. $\square$

We also have the analogue of Quillen's Theorem 5.3 [11] which he calls "flat base change".

Proposition 4.6. *Let A be a q-cofibrant commutative S-algebra and let B and C be q-cofibrant commutative A-algebras. There are isomorphisms in the homotopy category of $B \wedge_A C$-modules*

$$\Omega_{B \wedge_A C/C} \cong \Omega_{B/A} \wedge_A C,$$

$$\Omega_{B \wedge_A C/A} \cong \Omega_{B/A} \wedge_A C \vee \Omega_{C/A} \wedge_A B.$$

Hence, if M is a $B \wedge_A C$-module there are isomorphisms

$$AQ^*(B \wedge_A C/C; M) \cong AQ^*(B/A; M),$$

$$AQ^*(B \wedge_A C/A; M) \cong AQ^*(B/A; M) \oplus AQ^*(C/A; M).$$

Proof. The first isomorphism follows from Lemmas 4.4 and 4.5. The second isomorphism follows from comparing the two homotopy cofiber sequences that we obtain by applying the previous theorem to the sequences

$$A \to B \to B \wedge_A C; \;\; A \to C \to B \wedge_A C.$$

The isomorphisms on cohomology follow from the properties of *Ext*. $\square$

5. Construction of the spectral sequence

Let B be a q-cofibrant connective commutative S-algebra with an S-algebra map to $H\mathbb{F}_p$ which we take to be a q-cofibration.

Let A denote the q-cofibrant B algebra $H\mathbb{F}_p$. We will also consider $H\mathbb{F}_p$ as a B-module and in this case we will write $H\mathbb{F}_p$ instead of A.

We will construct a spectral sequence to calculate

$$AQ^*(B/S, H\mathbb{F}_p) = Ext_B^*(\Omega_{B/S}, H\mathbb{F}_p) = Ext_B^*(\boldsymbol{L}Q_B\boldsymbol{R}I_B(B \wedge^{\boldsymbol{L}} B), H\mathbb{F}_p).$$

Theorem 5.1. *There exists a spectral sequence that satisfies*

$$E_2^{s,t} = \mathrm{Hom}_{\mathbb{F}_p}(\mathscr{L}_s^{\mathbb{F}}(\mathbb{F}_p \otimes_{\mathscr{R}} Q^{\mathrm{alg}})(H\mathbb{F}_{p*}B)_t, \mathbb{F}_p)$$

and converges strongly to $AQ^*(B/S, H\mathbb{F}_p)$.

Here $\mathscr{L}_s^{\mathbb{F}}(-)$ denotes the sth comonad $\mathbb{F}$-left derived functor which we will define shortly. We shall first prove the following proposition.

Proposition 5.2. $AQ^*(B/S, H\mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(\pi_*(\boldsymbol{L}Q_A\boldsymbol{R}I_A(B \wedge^{\boldsymbol{L}} A)), \mathbb{F}_p)$.

Proof. This follows from Lemmas 4.3 and 4.4:

$$\begin{aligned} AQ^*(B/S; H\mathbb{F}_p) &= \mathrm{Ext}_B^*(\boldsymbol{L}Q_B\boldsymbol{R}I_B(B \wedge^{\boldsymbol{L}} B), H\mathbb{F}_p) \\ &= \mathrm{Ext}_A^*(\boldsymbol{L}Q_B\boldsymbol{R}I_B(B \wedge^{\boldsymbol{L}} B) \wedge_B^{\boldsymbol{L}} A, H\mathbb{F}_p) \\ &= Ext_A^*(\boldsymbol{L}Q_A(\boldsymbol{R}I_B(B \wedge^{\boldsymbol{L}} B) \wedge_B^{\boldsymbol{L}} A), H\mathbb{F}_p) \\ &= \mathrm{Ext}_A^*(\boldsymbol{L}Q_A\boldsymbol{R}I_A(B \wedge^{\boldsymbol{L}} A), H\mathbb{F}_p) \\ &= \mathrm{Hom}_{\mathbb{F}_p}(\pi_*(\boldsymbol{L}Q_A\boldsymbol{R}I_A(B \wedge^{\boldsymbol{L}} A)), \mathbb{F}_p). \qquad \square \end{aligned}$$

We now proceed to describe a simplicial resolution of $\boldsymbol{L}Q_A\boldsymbol{R}I_A(B \wedge A)$. We will use the bar filtration to construct the spectral sequence and identify its E_2-term.

Suppose that N is a q-cofibrant A-NUCA and recall that $(\mathbb{A}, \mu, \eta)$ is the free non-unital commutative A-algebra monad in the category of A-modules.

Let $B_*N = B_*(\mathbb{A}, \mathbb{A}, N)$ denote the simplicial object with nth term $B_nN = \mathbb{A}^{n+1}N$ and face and degeneracy operators given by

$$d_i = \mathbb{A}^i \mu_{\mathbb{A}^{n-i-1}}, \quad 0 \le i < n, \;\; d_n = \mathbb{A}^n \xi,$$

$$s_i = \mathbb{A}^{i+1} \eta_{\mathbb{A}^{n-i}}, \quad 0 \le i \le n.$$

There are several ways we can interpret this construction.

If we consider $\mathbb{A} : \mathscr{M}_A \to \mathscr{M}_A[\mathbb{A}]$ as the functor left adjoint to the forgetful functor $U : \mathscr{M}_A[\mathbb{A}] \to \mathscr{M}_A$, then

$$B_*N = B_*(\mathbb{A}, U\mathbb{A}, UN)$$

is the construction of [8] for the monad $U\mathbb{A}$ in $\mathscr{M}_A$ and the $U\mathbb{A}$-functor $\mathbb{A}$.

On the other hand, $(\mathbb{A}U, \xi, \mathbb{A}\eta U)$ is a comonad on $\mathscr{M}_A[\mathbb{A}]$ and

$$B_*(\mathbb{A}, U\mathbb{A}, UN) = B_*(\mathbb{A}U, \mathbb{A}U, N)$$

is the usual bar construction applied to the comonad $\mathbb{A}U$.

In any case, we continue to omit the forgetful functor and note that B_*N is a simplicial A-NUCA. The degeneracies s_i are of the form $\mathbb{A}^{i+1}\eta$ and hence B_*N is proper. Further, the action map of N provides an augmentation for B_*N. We can consider $\xi : B_*N \to N$ as an augmented simplicial A-module for which it is routine to define a contraction using the unit map $\eta : N \to \mathbb{A}N$.

An argument similar to that of [5, VII.3.3] shows that the geometric realization in the category of $\mathbb{A}$-algebras is isomorphic to the geometric realization in the category of A-modules. Hence, $|B_*N|$ and N are homotopy equivalent as A-modules and weakly equivalent as $\mathbb{A}$-algebras.

Note that, even though N is a q-cofibrant A-NUCA, $|B_*N|$ might not be q-cofibrant. The functor $\mathbb{A}$ takes q-cofibrant A-modules to q-cofibrant A-NUCAs, but a q-cofibrant A-NUCA is not necessarily q-cofibrant as an A-module. Hence, the simplices $\mathbb{A}^nN$ might not be q-cofibrant.

We construct a simplicial A-NUCA ΓB_*N with q-cofibrant simplices weakly equivalent to those of B_*N and whose geometric realization is a q-cofibrant A-NUCA weakly equivalent to $|B_*N|$ and hence weakly equivalent to the q-cofibrant A-NUCA N.

Explicitly, let ΓB_* be the simplicial A-NUCA with $\Gamma B_n = \Gamma\mathbb{A}^{n+1}N$ where Γ denotes the functorial cell $\mathbb{A}$-algebra approximation described in [5, VII 5.8.] By construction, ΓB_*N satisfies the hypothesis of [5, X.2.7(i)] in the NUCA setting. That is, each ΓB_qN is a cell object, each degeneracy operator is the inclusion of a subcomplex, and each face operator is sequentially cellular. By the proof of the above mentioned result, we can conclude that $|\Gamma B_*N|$ is a cell, hence q-cofibrant, A-NUCA.

Since both ΓB_*N and B_*N are proper simplicial A-NUCA's with degreewise weakly equivalent simplices, [5, X.2.4(ii)] gives that their geometric realizations are weakly equivalent as desired.

We have the following composite weak equivalence:

$$|\Gamma B_*N| \to |B_*N| \to N.$$

Therefore, since $|\Gamma B_*N|$ and N are q-cofibrant, the composite

$$\boldsymbol{L}Q_A|\Gamma B_*N| \cong Q_A|\Gamma B_*N| \to Q_A|B_*N| \to Q_AN \cong \boldsymbol{L}Q_AN \tag{3}$$

is a weak equivalence.

Since Q_A is a left adjoint, it commutes with categorical colimits and hence it commutes with geometric realization. Thus, we could use the bar filtration of $|Q_A\Gamma B_*N|$ to construct a spectral sequence converging to $\pi_*(\boldsymbol{L}Q_AN) \cong \pi_*Q_AN$. But we would have very little knowledge of its E_2 term:

$$E^2_{s,t} = H^s(\pi_t(Q_A\Gamma B_*N)),$$

the sth homology group of the chain complex associated to the simplicial group obtained by applying the functor π_t degree-wise to $Q_A\Gamma B_*N$.

On the other hand, if we knew that $|Q_AB_*N| \to Q_AN$ were a weak equivalence, we could take advantage of the fact that for any $\mathbb{A}$-algebra T, we have a natural isomorphism $Q_A\mathbb{A}T \cong T$ and, as we will see in the next section, $\pi_*\mathbb{A}X$ can be described algebraically as a functor of π_*X. So, we would have a much better handle on the E^2-term of the spectral sequence obtained from the bar filtration of $|Q_AB_*N|$.

We will show that the left arrow in (3) is a weak equivalence, and hence $Q_A|B_*N|$ and Q_AN are weakly equivalent as desired.

We will need the following key lemma which we shall prove in Section 9.

Lemma 5.3. *Let N be a q-cofibrant A-NUCA and let $\gamma : X \to \mathbb{A}^nN$ be a cell A-module approximation. Then $\mathbb{A}\gamma : \mathbb{A}X \to \mathbb{A}\mathbb{A}^nN$ is a weak equivalence.*

Proposition 5.4. *$Q_A|B_*N| \to Q_AN$ is a weak equivalence of A-modules.*

Proof. Let $\gamma : X \to \mathbb{A}^nN$ be a cell A-module approximation. By the previous lemma, $\mathbb{A}\gamma : \mathbb{A}X \to \mathbb{A}\mathbb{A}^nN$ is a weak equivalence. Thus, $\Gamma\mathbb{A}X \to \Gamma\mathbb{A}\mathbb{A}^nN$ is a homotopy equivalence of $\mathbb{A}$-algebras since it is a weak equivalence of q-cofibrant $\mathbb{A}$-algebras.

We have the following commutative diagram:

$$\begin{array}{ccc} Q_A\Gamma\mathbb{A}X & \longrightarrow & Q_A\Gamma\mathbb{A}\mathbb{A}^nN \\ \Big\downarrow & & \Big\downarrow \\ X \cong Q_A\mathbb{A}X & \longrightarrow & \mathbb{A}^nN \cong Q_A\mathbb{A}\mathbb{A}^nN. \end{array}$$

The top horizontal and the left vertical arrows are homotopy equivalences of A-modules. The bottom horizontal arrow is a weak equivalence and hence, the right arrow is also a weak equivalence.

Thus, $Q_A\Gamma B_nN$ is weakly equivalent to Q_AB_nN for each n. Since both $Q_A\Gamma B_*N$ and Q_AB_*N are proper, their geometric realizations are weakly equivalent. Since Q_A commutes with geometric realization, we have that $Q_A|\Gamma B_*N|$ is weak equivalent to $Q_A|B_*N|$ which gives the desired result. $\square$

To construct our spectral sequence we take $B \wedge A$ to be q-fibrant and q-cofibrant, and we let N above be a q-cofibrant approximation $\gamma I_A(B \wedge A)$ of $I_A(B \wedge A)$.

As in [5, X.2.9], the bar filtration yields a natural homological spectral sequence with E^2-term given by

$$E^2_{s,t} = H_s(\pi_t(Q_AB_*N)),$$

the sth homology group of $\pi_t(Q_AB_*N)$, the chain complex associated to the simplicial group obtained by applying the functor π_t degree-wise to Q_AB_*N.

This spectral sequence converges strongly to

$$\pi_*(|Q_A B_* N|) \cong \pi_*(Q_A N) \cong \pi_*(\boldsymbol{L}Q_A \boldsymbol{R}I_A(B \wedge^{\boldsymbol{L}} A)).$$

We apply the exact functor $\mathrm{Hom}_{\mathbb{F}_p}(-, \mathbb{F}_p)$ to the spectral sequence above and we obtain a first quadrant cohomological spectral sequence with E_2-term

$$E_2^{s,t} = H_s(\mathrm{Hom}_{\mathbb{F}_p}(\pi_t(Q_A B_* N), \mathbb{F}_p))$$

and differentials $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$; it converges strongly to

$$\mathrm{Hom}_{\mathbb{F}_p}(\pi_*(Q_A N), \mathbb{F}_p) \cong AQ^*(B/S; A).$$

6. Algebraic identification of the E_2-term

Let A, as before, denote the q-cofibrant commutative S-algebra $H\mathbb{F}_p$. We can give a more algebraic description of the E_2-term of our spectral sequence once we note that for any object X in $\mathscr{N}_A$ there is an algebraic functor of $\pi_* X$ describing $\pi_*(Q_A \mathbb{A} X)$.

We will use the following notation:

- $\mathscr{V}$ will denote the category of graded $\mathbb{F}_p$-vector spaces.
- $\mathscr{R}$ will denote the ($\mathbb{Z}$-graded) Dyer–Lashof algebra at the prime p.
- $\mathscr{D}$, $\mathscr{N}\mathscr{D}$ and $\mathscr{U}$ will denote, respectively, the categories of algebras, non-unital algebras, and modules with an allowable action by $\mathscr{R}$.

Let $\mathbb{P}$ denote the free commutative A-algebra monad in the category of A-modules. Mandell's description of the homotopy operations on the category of commutative A-algebras [7] together with McClure's work in [1] shows that π_* is a functor from $\mathscr{C}_A$ to $\mathscr{D}$. Further, for any A-module Y, $\pi_* \mathbb{P} Y$ is the free allowable algebra over $\mathscr{R}$ generated by the graded $\mathbb{F}_p$-vector space $\pi_* Y$.

Let X be a q-cofibrant object in $\mathscr{N}_A$. Then, $\pi_* X = I^{\mathrm{alg}} \pi_*(K_A X)$ where I^{alg} denotes the algebraic "augmentation ideal" functor. Hence, π_* is a functor from $\mathscr{N}_A$ to $\mathscr{N}\mathscr{D}$.

Let $F : \mathscr{V} \to \mathscr{D}$ denote the free allowable algebra over $\mathscr{R}$ functor constructed by May in [1] and let $\bar{F} : \mathscr{V} \to \mathscr{N}\mathscr{D}$ denote the left adjoint to the forgetful functor. One can see that $\bar{F} = I^{\mathrm{alg}} F$. Thus, we have a natural isomorphism

$$\pi_* \mathbb{A} X \cong I^{\mathrm{alg}} \pi_* \mathbb{P} X = \bar{F}(\pi_* X).$$

Let $\bar{\mathbb{F}} = \bar{F}\bar{U}$ where $\bar{U} : \mathscr{N}\mathscr{D} \to \mathscr{V}$ denotes the forgetful functor. Note that $\bar{\mathbb{F}}$ is a comonad on $\mathscr{N}\mathscr{D}$.

If X is an $\mathbb{A}$-algebra, $\pi_* \mathbb{A} X \cong \bar{\mathbb{F}}(\pi_* X)$ and the isomorphism is natural not only with respect to X but also with respect to $\mathbb{A} X$. If X and Y are $\mathbb{A}$-algebras and $f : \mathbb{A} X \to \mathbb{A} Y$

is a map of $\mathbb{A}$-algebras, the following diagram commutes:

$$\begin{tikzcd}
\bar{\mathbb{F}}(\pi_*X) \arrow[r, "\bar{\mathbb{F}}(\eta_*)"] \arrow[d, "\cong"'] & \bar{\mathbb{F}}(\pi_*\mathbb{A}X) \arrow[r, "\bar{\mathbb{F}}(f_*)"] & \bar{\mathbb{F}}(\pi_*\mathbb{A}Y) \arrow[r, "\bar{\mathbb{F}}(\xi_*)"] & \bar{\mathbb{F}}(\pi_*Y) \arrow[d, "\cong"] \\
\pi_*(\mathbb{A}X) \arrow[rrr, "f_*"'] & & & \pi_*(\mathbb{A}Y)
\end{tikzcd}$$

Let Q^{alg} and $\bar{Q}^{\mathrm{alg}}$ denote the "algebra indecomposables" functor in $\mathscr{D}$ and $\mathscr{N}\mathscr{D}$, respectively. Note that their target is $\mathscr{U}$. By the comments above, for an A-module Y we have

$$\pi_*Y = \mathbb{F}_p \otimes_{\mathscr{R}} (Q^{\mathrm{alg}}\pi_*\mathbb{P}Y) = \mathbb{F}_p \otimes_{\mathscr{R}} (\bar{Q}^{\mathrm{alg}}\pi_*\mathbb{A}Y).$$

Let $\bar{Q}$ denote the composite functor $\mathbb{F}_p \otimes_{\mathscr{R}} \bar{Q}^{\mathrm{alg}}(-)$. Since $Q_A\mathbb{A}X$ and X are naturally isomorphic we see that there is a composite natural isomorphism of functors:

$$\pi_*(Q_A\mathbb{A}X) \cong \pi_*X \cong \bar{Q}(\pi_*\mathbb{A}X) \cong \bar{Q}\bar{\mathbb{F}}(\pi_*X).$$

Further, when X and Y are $\mathbb{A}$-algebras and $f:\mathbb{A}X \to \mathbb{A}Y$ is a map of $\mathbb{A}$-algebras, the following diagram commutes:

$$\begin{tikzcd}
\bar{Q}\bar{\mathbb{F}}(\pi_*X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \bar{Q}(\pi_*\mathbb{A}X) \arrow[r, "\bar{Q}(f_*)"] & \bar{Q}(\pi_*\mathbb{A}Y) \arrow[r, "\cong"] & \bar{Q}\bar{\mathbb{F}}(\pi_*Y) \arrow[d, "\cong"] \\
\pi_*Q_A\mathbb{A}X \arrow[rrr, "(Q_Af)_*"'] & & & \pi_*Q_A\mathbb{A}Y
\end{tikzcd}$$

We now remind the reader of the definition of comonad derived functors [3, 9].

Definition 6.1. Let $F:\mathscr{C} \to \mathscr{A}$ be a functor to some abelian category and let $(\mathbb{T}, \varepsilon, \delta)$ denote a comonad on $\mathscr{C}$. For an object C of $\mathscr{C}$, let $\mathbb{T}_*C$ denote the augmented simplicial object with

$$\mathbb{T}_nC = \mathbb{T}^{n+1}C,$$

$$d_i = \mathbb{T}^i\varepsilon_{\mathbb{T}^{n-i}}, \quad 0 \le i \le n$$

$$s_i = \mathbb{T}^i\delta_{\mathbb{T}^{n-i}}, \quad 0 \le i \le n$$

and augmentation given by the counit $\varepsilon:\mathbb{T}C \to C$.

Let $F\mathbb{T}_*C$ denote the augmented simplicial object obtained by applying F to $\mathbb{T}_*C$ degreewise. The comonad $\mathbb{T}$-left derived functors of F applied to C, $\mathscr{L}_*^{\mathbb{T}}F(C)$, are given by the homology of the chain complex associated to $F\mathbb{T}_*C$.

For every object C of $\mathscr{C}$, there is a natural map

$$\varepsilon : \mathscr{L}_0^{\mathbb{T}} F(C) \to FC$$

and we say that C is $\mathscr{L}_*^{\mathbb{T}} F$-acyclic if ε is an isomorphism and $\mathscr{L}_i^{\mathbb{T}} F(C) = 0$ for $i > 0$.

The functor $\bar{\mathbb{F}} = \bar{F}\bar{U}$ defines a comonad on $\mathscr{N}\mathscr{D}$; we will denote its counit and comultiplication by $\bar{\varepsilon}$ and $\bar{\delta}$, respectively.

The natural isomorphisms

$$\pi_* \mathbb{A} X \cong \bar{\mathbb{F}} \pi_* X \quad \text{and} \quad \pi_*(Q_A \mathbb{A} X) \cong \bar{Q} \bar{\mathbb{F}} \pi_* X$$

give an isomorphism of simplicial objects

$$\pi_* Q_A B_* N \cong \bar{Q} \bar{\mathbb{F}}_* (\pi_* N),$$

where the object on the left is obtained by applying $\pi_* Q_A$ degreewise to the simplicial A-NUCA $B_*(\mathbb{A}, \mathbb{A}, N)$. Hence, we have an equivalence of chain complexes and an isomorphism of their homologies. The homology of the chain complex obtained from $\bar{Q}\bar{\mathbb{F}}_*(\pi_* N)$ calculates $\mathscr{L}_*^{\bar{\mathbb{F}}} \bar{Q}(\pi_* N)$, the comonad $\bar{\mathbb{F}}$-left derived functors of $\bar{Q}$, applied to $\pi_* N$.

Recall that $N = \gamma I_A(B \wedge A)$ is a q-cofibrant approximation to $I_A(B \wedge A)$. Thus, we have

$$\pi_*(N) = \pi_*(\gamma I_A(B \wedge A)) = I^{\mathrm{alg}} \pi_*(A \vee \gamma I_A(B \wedge A)) = I^{\mathrm{alg}} \pi_*(B \wedge A) = I^{\mathrm{alg}} H_*(B; \mathbb{F}_p).$$

Hence, we can rewrite

$$E_2^{s,t} = H_s(\mathrm{Hom}_{\mathbb{F}_p}(\pi_t(Q_A B_* N), \mathbb{F}_p)) = \mathrm{Hom}_{\mathbb{F}_p}(\mathscr{L}_s^{\bar{\mathbb{F}}} \bar{Q}(I^{\mathrm{alg}} H_*(B; \mathbb{F}_p)_t, \mathbb{F}_p).$$

To obtain Theorem 5.1, we let $(\mathbb{F} = UF, \varepsilon, \delta)$ be the comonad associated to the free functor $F : \mathscr{V} \to \mathscr{D}$. The following proposition shows that the $\mathbb{F}$-left derived functors of the composite $\bar{Q} I^{\mathrm{alg}}$ applied to an object C of $\mathscr{D}$ coincide with the $\bar{\mathbb{F}}$-left derived functors of $\bar{Q}$ applied to the object $I^{\mathrm{alg}} C$ of $\mathscr{N}\mathscr{D}$.

Proposition 6.2. *For each t and each object C of $\mathscr{D}$, there is a natural isomorphism of $\mathbb{F}_p$-vector spaces*

$$\mathscr{L}_t^{\mathbb{F}} (\bar{Q} I^{\mathrm{alg}})(C) \cong \mathscr{L}_t^{\bar{\mathbb{F}}} \bar{Q}(I^{\mathrm{alg}} C).$$

Proof. Consider the maps of double-chain complexes associated to the augmentations:

$$\bar{Q} I^{\mathrm{alg}} \mathbb{F}_* C \xleftarrow{\bar{Q}\bar{\varepsilon}_*} \bar{Q}\bar{\mathbb{F}}_*(I^{\mathrm{alg}} \mathbb{F}_* C) \xrightarrow{\bar{Q}\bar{\mathbb{F}}_* I^{\mathrm{alg}} \varepsilon} \bar{Q}\bar{\mathbb{F}}_* I^{\mathrm{alg}} C.$$

We will show that both maps induce homology isomorphisms.

For each object C of $\mathscr{D}$ we have

$$\tilde{\eta} : I^{\mathrm{alg}} \mathbb{F} C = I^{\mathrm{alg}} FUC \to I^{\mathrm{alg}} F \bar{U} I^{\mathrm{alg}} FUC = \bar{\mathbb{F}} I^{\mathrm{alg}} \mathbb{F} C$$

given by $\tilde{\eta} = I^{\mathrm{alg}} F \bar{\eta}_{UC}$ where $\bar{\eta}$ is the unit for the monad $\bar{U} I^{\mathrm{alg}} F$. Since $\bar{\varepsilon} \circ \tilde{\eta} = \bar{\varepsilon}_{I^{\mathrm{alg}} FUC} \circ I^{\mathrm{alg}} F \bar{\eta}_{UC} = Id_{I^{\mathrm{alg}} FUC}$, there is a contraction for

$$\bar{Q}\bar{\varepsilon} : \bar{Q}\bar{\mathbb{F}}_*(I^{\mathrm{alg}}\mathbb{F}^n C) \to \bar{Q} I^{\mathrm{alg}} \mathbb{F}^n C.$$

Hence, filtering the double-chain complex associated to

$$\bar{Q}\bar{\varepsilon}_* : \bar{Q}\bar{\mathbb{F}}_*(I^{\mathrm{alg}}\mathbb{F}_* C) \to \bar{Q} I^{\mathrm{alg}} \mathbb{F}_* C$$

by degree in $\mathbb{F}$ we obtain an isomorphism at E_1.

We also have $s : \bar{U} I^{\mathrm{alg}} C \to \bar{U} I^{\mathrm{alg}} FUC$ with $s = \bar{\eta}_{UC} \circ i$ where $i : \bar{U} I^{\mathrm{alg}} C \to UC$ is the obvious inclusion. Note that $\bar{U} I^{\mathrm{alg}} \varepsilon \circ s = Id_{\bar{U} I^{\mathrm{alg}} C}$ and hence there exists a contraction for

$$\bar{U} I^{\mathrm{alg}} \varepsilon : \bar{U} I^{\mathrm{alg}} \mathbb{F}_* C \to \bar{U} I^{\mathrm{alg}} C.$$

Since, for each $n \geq 0$, $\bar{Q}\bar{\mathbb{F}}^{n+1} I^{\mathrm{alg}} = \bar{Q}\bar{\mathbb{F}}^{n} I^{\mathrm{alg}} F \bar{U} I^{\mathrm{alg}}$ there exists a contraction for

$$\bar{Q}\bar{\mathbb{F}}^{n+1} I^{\mathrm{alg}} \varepsilon : \bar{Q}\bar{\mathbb{F}}^{n+1} I^{\mathrm{alg}} \mathbb{F}_* C \to \bar{Q}\bar{\mathbb{F}}^{n+1} I^{\mathrm{alg}} C.$$

Hence, filtering the double-chain complex associated to

$$\bar{Q}\bar{\mathbb{F}}_* I^{\mathrm{alg}} \varepsilon : \bar{Q}\bar{\mathbb{F}}_* I^{\mathrm{alg}} \mathbb{F}_* C \to \bar{Q}\bar{\mathbb{F}}_* I^{\mathrm{alg}} C$$

by degree in $\bar{\mathbb{F}}$, we obtain an isomorphism at E_1. Therefore both maps of total complexes above are homology isomorphisms. $\square$

Note that

$$\bar{Q} I^{\mathrm{alg}}(-) = \mathbb{F}_p \otimes_{\mathscr{R}} \bar{Q}^{\mathrm{alg}} I^{\mathrm{alg}}(-) = \mathbb{F}_p \otimes_{\mathscr{R}} Q^{\mathrm{alg}}(-).$$

Hence we can rewrite the E_2 term in the form proposed by Theorem 5.1:

$$E_2^{s,t} = \mathrm{Hom}_{\mathbb{F}_p}(\mathscr{L}_s^{\mathbb{F}}(\mathbb{F}_p \otimes_{\mathscr{R}} Q^{\mathrm{alg}}(-))(H_*(B;\mathbb{F}_p))_t, \mathbb{F}_p).$$

7. A Grothendieck spectral sequence

We construct a spectral sequence that converges to the comonad left derived functors of the composite functor

$$Q = \mathbb{F}_p \otimes_{\mathscr{R}} Q^{\mathrm{alg}}(-).$$

The existence of such a spectral sequence is an application to left derived functors of proposition 2.13 in Miller's paper "The Sullivan Conjecture on Maps from Classifying Spaces" [9].

We consider the following general situation:

$$\mathscr{C} \xrightarrow{F} \mathscr{B} \xrightarrow{E} \mathscr{A}$$

where F and E are covariant functors and $\mathscr{B}$ and $\mathscr{A}$ are abelian categories. We let $(\mathbb{T}, \varepsilon, \delta)$ denote a comonad on $\mathscr{C}$ and $(\mathbb{S}, \varepsilon', \delta')$ a comonad on $\mathscr{B}$.

In this setting Miller's Proposition 2.13 reads:

Proposition 7.1. *Assume that*:

(a) *For every object C of $\mathscr{C}$, $F\mathbb{T}C$ is $\mathscr{L}_*^{\mathbb{S}}E$-acyclic.*

(b) *$E\mathbb{S}^{n+1}$ is exact for all $n \geq 0$.*

Then there is a convergent homological spectral sequence

$$E^2_{s,t} = \mathscr{L}_s^{\mathbb{S}}E(\mathscr{L}_t^{\mathbb{T}}F(C)) \Rightarrow \mathscr{L}_{s+t}^{\mathbb{T}}(EF)(C).$$

Before we apply this proposition to our composite functor $\mathbb{F}_p \otimes_{\mathscr{R}} Q^{\mathrm{alg}}(-)$, we need the following observation about the target of Q^{alg}.

Recall that if B is an algebra with an allowable action of the Dyer–Lashof algebra $\mathscr{R}$ at the prime p, then $Q^n x = x^p$ if $|x| = n$ and $p = 2$ or if $2|x| = n$ and $p > 2$. Hence, $Q^{\mathrm{alg}}B$ is a module with an allowable action of $\mathscr{R}$ that satisfies $Q^{|x|}x = 0$ if $p = 2$ and $Q^{2|x|}x = 0$ if $p > 2$. Let $a\mathscr{U}$ denote the full subcategory of $\mathscr{U}$ consisting of modules satisfying the above condition. Then we have the following setting:

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "Q^{\mathrm{alg}}"] \arrow[d, shift left, "U'"] & a\mathscr{U} \arrow[r, "\mathbb{F}_p \otimes_{\mathscr{R}}"] \arrow[d, shift left, "i"] & \mathscr{V} \\
\mathscr{U} \arrow[u, shift left, "V"] \arrow[d, shift left, "U"] & \mathscr{U} \arrow[u, shift left, "a"] \arrow[d, shift left, "U"] & \\
\mathscr{V} \arrow[u, shift left, "D"] & \mathscr{V}. \arrow[u, shift left, "D"] &
\end{tikzcd}$$

Here U' and U are forgetful functors and V and D are their respective left adjoints constructed by May in [1]; i denotes the inclusion and a is its left adjoint given by $aM = M/\{Q^s x\}$ with $s = |x|$ if $p = 2$ and $s = 2|x|$ if $p > 2$. Observe that $ai = Id$ and, $Q^{\mathrm{alg}}VM = aM$ if M is an object of $\mathscr{U}$.

Let $(\mathbb{F}, \varepsilon, \delta)$ denote the comonad on $\mathscr{D}$ appearing in Theorem 3.1. Note that it is given by $\mathbb{F} = VD\bar{U}$, where $\bar{U}$ is the forgetful functor to $\mathscr{V}$.

Let $(\mathbb{D}, \varepsilon', \delta')$ denote the comonad on $a\mathscr{U}$ given by $\mathbb{D} = aDUi$.

Let $Tor_*^{\mathbb{D}}(\mathbb{F}_p -)$ denote the comonad $\mathbb{D}$-left derived functors of $\mathbb{F}_p \otimes_{\mathscr{R}} (-)$.

Proposition 7.2. *There exists a convergent homological spectral sequence*:

$$E^2_{s,t} = Tor_s^{\mathbb{D}}(\mathbb{F}_p, \mathscr{L}_t^{\mathbb{F}} Q^{\mathrm{alg}}(H_*(B; \mathbb{F}_p)) \Rightarrow \mathscr{L}_{s+t}^{\mathbb{F}}(\mathbb{F}_p \otimes_{\mathscr{R}} Q^{\mathrm{alg}}(-))(H\mathbb{F}_{p*}B).$$

We must show that our functors and comonads satisfy the conditions on Miller's proposition, and we break the argument into the following two lemmas. Let $E = \mathbb{F}_p \otimes_{\mathscr{R}} -$.

Lemma 7.3. *$Q^{\mathrm{alg}}\mathbb{F}A$ is $\mathscr{L}_*^S E$-acyclic for all A in $\mathscr{D}$.*

Proof. Let

$$f : Q^{\mathrm{alg}}\mathbb{F}A = Q^{\mathrm{alg}}VD\bar{U}A = aD\bar{U}A \to aDUiaD\bar{U}A = \mathbb{D}Q^{\mathrm{alg}}\mathbb{F}A$$

be given by $f = aD\eta_{\bar{U}A}$ where η is the unit of the monad $UiaD$. Then, $\varepsilon' \circ f = \varepsilon' \circ aD\eta_{\bar{U}A} = Id_{aD\bar{U}A}$. Hence, there is a contraction for $\varepsilon_* : \mathbb{D}_* Q^{\mathrm{alg}}\mathbb{F}A \to Q^{\mathrm{alg}}\mathbb{F}A$ which gives the desired result. $\square$

Lemma 7.4. *$E\mathbb{D}^{n+1}$ is exact for all $n \geq 0$.*

Proof. We show that $\mathbb{D} : a\mathscr{U} \to a\mathscr{U}$ and $E\mathbb{D} : a\mathscr{U} \to \mathscr{V}$ are exact.

Let $M' \to M \to M''$ be exact in $a\mathscr{U}$. Since $\mathbb{D} = aDUi$, we first must forget down to the category of vector spaces. Then the exact sequence above becomes $UiM' \to UiM' \oplus UiM'' \to UiM''$. Hence $\mathbb{D}M' \to \mathbb{D}M \to \mathbb{D}M''$ is the same as $\mathbb{D}M' \to \mathbb{D}M' \oplus \mathbb{D}M'' \to \mathbb{D}M''$ which is exact. Applying $\mathbb{F}_p \otimes_{\mathscr{R}} (-)$ to this sequence it remains exact. Hence, the result follows. $\square$

8. Postnikov towers of connective commutative S-algebras

In this section we give the proof of Theorem 1.6 of Kriz's unpublished paper [6]. That is, given a connective commutative S-algebra we construct its Postnikov tower in the category of commutative S-algebras.

In the paper mentioned above, Kriz gives a different description of $\Omega_{B/A}$ from ours. He claims that, as in our framework,

$$\bar{h}\mathscr{C}_{A/B}(B, B \vee M) \cong \bar{h}\mathscr{M}_B(\Omega_{B/A}, M)$$

but we have been unable to verify this bijection using his description of $\Omega_{B/A}$. This is what prompted us to develop our theory. Kriz defines E_∞-cohomology in terms of $\Omega_{B/A}$ and our definition of André–Quillen cohomology mimics his. Thus, his observations and in particular his construction of the Postnikov towers of E_∞-ring spectra translate directly to our framework with some differences in the proofs.

We will use the following definitions and observations:

Let B be a connective, q-cofibrant commutative S-algebra with $\pi_0 B = K$. Then, by [5, IV.3.1], there is a map of commutative S-algebras $\pi : B \to HK$ which we can take to be a q-cofibration and that realizes the identity homomorphism $\pi_0 B = K$. By construction, its underlying map of spectra is the bottom k-invariant.

Let A be a q-cofibrant commutative S-algebra. Suppose that B as above is also a q-cofibrant commutative A-algebra. Let J be a K-module. Then HJ is a B-module and for $n \geq 0$

$$AQ^n(B/A; HJ) = \pi_{-n} F_B(\Omega_{B/A}, HJ) \cong \bar{h}\mathscr{M}_B(\Omega_{B/A}, \Sigma^n HJ) \cong \bar{h}\mathscr{C}_{A/B}(B, B \vee \Sigma^n HJ).$$

Let $[k]$ be a class in $AQ^n(B/A; HJ)$ represented by a map of A-algebras over B

$$k : B \to B \vee \Sigma^n HJ$$

and let $1 : B \to B \vee \Sigma^n HJ$ be the inclusion in the first wedge summand. The homotopy pullback in the category of commutative A-algebras

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d, "1"] \\ B \arrow[r, "k"'] & B \vee \Sigma^n HJ \end{tikzcd}$$

produces B', a commutative A-algebra over B. Following Kriz's terminology we say that B' is an extension of B by k.

Let k' denote the composite of k with the projection $B \vee \Sigma^n HJ \to \Sigma^n HJ$. The class $[k']$ coincides with the class $\psi^*([k])$ where

$$\psi^* : AQ^n(B/A, HJ) \to H^n(B, A; J)$$

is the forgetful map of (2). Moreover, B' is the homotopy fiber of k' in the category of A-modules.

We have the following theorem.

Theorem 8.1 (Kriz [6]). *For any connective commutative S-algebra A, there is a sequence of commutative S-algebras A_i, $\pi_0 A$-modules Q_i, and elements*

$$k_i \in AQ^*(A_i/S, HQ_i)$$

such that

(a) *$A_0 = H\pi_0 A$ and A_{i+1} is the extension of A_i by the k-invariant k_i,*

(b) *$\pi_j A_i = 0$ for $j > i$,*

(c) *there are maps of commutative S-algebras $\lambda_i : A \to A_i$ which are $(i+1)$-equivalences and such that*

$$\begin{tikzcd} A \arrow[r, "\lambda_{i+1}"] \arrow[dr, "\lambda_i"'] & A_{i+1} \arrow[d] \\ & A_i \end{tikzcd}$$

commutes in the homotopy category of commutative S-algebras.

Furthermore, this data is unique up to (non-canonical) isomorphism.

In order to prove this theorem we need the following lemma.

Lemma 8.2. *Let B be a connective commutative S-algebra and let $\pi : A \to B$ be a map of S-algebras which is an n-equivalence, where $n \geq 1$. Then, $\Omega_{B/A}$ is n-connected and $\pi_{n+1}(\Omega_{B/A}) \cong \pi_n A$.*

Proof. Consider the composite map of A-modules

$$u : B \to B \vee \Omega_{B/A} \to \Omega_{B/A},$$

where the first map is a map of A-algebras adjoint to the identity and the second is the projection. Then the composite map of A-modules

$$A \xrightarrow{\pi} B \xrightarrow{u} \Omega_{B/A}$$

is zero since, by functoriality, it factors through $\Omega_{A/A} = *$. Thus, we have a commutative diagram of A-modules

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] \arrow[dr] & C\pi \arrow[d, "\tau"] \\ & & \Omega_{B/A}, \end{tikzcd}$$

where $C\pi$ is the cofiber of π. Hence $C\pi$ is n-connected. We will show that $\Omega_{B/A}$ is also n-connected and that τ induces an isomorphism on π_{n+1}.

Consider the following commutative diagram of A-modules:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C\pi \arrow[d] \\ B \arrow[r] & B \wedge_A B \arrow[r] & B \wedge_A C\pi. \end{tikzcd}$$

The top and bottom rows are cofiber sequences and the vertical maps are the obvious inclusions.

The spectral sequence in [5, IV.4.1] shows that $\pi_i(B, \wedge_A C\pi) = 0$ for $i \leq n$ and that the map $C\pi \to B \wedge_A C\pi$ gives an isomorphism on π_{n+1}. Since

$$B \vee \mathbf{R}I_B(B \wedge_A^{\boldsymbol{L}} B) \to B \wedge_A B$$

is a weak equivalence, we see that $B \wedge_A C\pi$, the cofiber of $B \to B \wedge_A B$, is weakly equivalent to $\boldsymbol{R}I_B(B \wedge_A^{\boldsymbol{L}} B)$.

The following diagram commutes in the homotopy category of A-modules:

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d] & B \arrow[r] \arrow[d] & C\pi \arrow[r, "\tau"] \arrow[d] & \Omega_{B/A} \\ B \arrow[r] & B \wedge_A B \arrow[r] & B \wedge_A C\pi \arrow[r, "\cong"'] & \boldsymbol{R}I_B(B \wedge_A^{L} B). \arrow[u, "i"'] \end{tikzcd}$$

Hence, the effect of τ on π_{n+1} coincides with the effect of i.

Recall from Section 5 that there is a spectral sequence with

$$E^2_{s,t} = H_s(\pi_t(Q_B B_*(\mathbb{A}, \mathbb{A}, \gamma \boldsymbol{R} I_B(B \wedge^{\boldsymbol{L}}_A B)))$$

that calculates $\pi_*(\Omega_{B/A})$. Since $\boldsymbol{R} I_B(B \wedge^{\boldsymbol{L}}_A B)$ is n-connected,

$$E^1_{s,t} = \pi_t(Q_B(\mathbb{A}^{s+1}(\gamma \boldsymbol{R} I_B(B \wedge^{\boldsymbol{L}}_A B)))) = \pi_t(\mathbb{A}^s(\gamma \boldsymbol{R} I_B(B \wedge^{\boldsymbol{L}}_A B))) = 0$$

if $t \leq n$ and

$$E^2_{0,n+1} = E^\infty_{0,n+1} = \pi_{n+1} \boldsymbol{R} I_B(B \wedge^{\boldsymbol{L}}_A B) \cong \pi_{n+1}(\Omega_{B/A}),$$

where the isomorphism is induced by the map i, as desired. □

Proof of Theorem 8.1. As mentioned above, there is a map of commutative S-algebras $\pi : A \to A_0 = H\pi_0 A$ which is a 1-equivalence and realizes the bottom k-invariant in the category of S-modules. To construct A_1, the next stage of the tower in the category of S-modules, we let $B = A_0$ in the lemma above. Since $\Omega_{B/A}$ is 1 connected and $\pi_2(\Omega_{B/A}) \cong \pi_1 A$, we can construct a map

$$c_1 : \Omega_{B/A} \to \Sigma^2 H\pi_1 A$$

by attaching cells to $\Omega_{B/A}$ to kill its higher homotopy groups. We define A_1 to be the homotopy fiber of the composite

$$B \to \Omega_{B/A} \xrightarrow{c_1} \Sigma^2 H\pi_1 A.$$

Then A_1 is equivalent to the extension of B by the map of A-algebras

$$k_1 : B \to B \vee \Sigma^2 H\pi_1 A$$

adjoint to c_1. Hence, A_1 is a commutative A-algebra, thus a commutative S-algebra. The required map $\lambda_1 : A \to A_1$ is given by the unit and k_1 gives a class in $AQ^2(B/S; H\pi_1 A)$.

If the ith stage of the tower has been constructed, letting $B = A_i$, we construct A_{i+1} in an analogous way. The uniqueness of these k-invariants follows from the uniqueness of the classical k-invariants. □

9. Extended cell R-modules

The object of this section is to prove the following theorem from which Lemma 5.3 follows.

Theorem 9.1. *Let R be a q-cofibrant commutative S-algebra. Let N be a q-cofibrant R-NUCA and $n > 0$. If $\gamma : Y \to \mathbb{A}^n N$ is a cell R-module approximation then*

$$\gamma^i/\Sigma_i : Y^i/\Sigma_i \to (\mathbb{A}^n N)^i/\Sigma_i$$

is a weak equivalence for all $i > 0$.

We restate Lemma 5.3 as a corollary:

Corollary 9.2. *If N is a q-cofibrant R-NUCA and $Y \to \mathbb{A}^n N$ is a cell R-module approximation, then $\mathbb{A}Y \to \mathbb{A}\mathbb{A}^n N$ is a weak equivalence.*

To prove the theorem using induction on n does not seem to be possible. We are able to prove the case $n=1$ relying on a convenient filtration of the q-cofibrant R-NUCA N. However, for $n \geq 1$ $\mathbb{A}^n N$ is not q-cofibrant and we lack the above mentioned filtration to show the inductive step. Our proof of the theorem relies on the elementary observation that $\mathbb{A}^n N$ is a wedge of R-modules of the form N^j/H where $j \geq 1$ and $H \subset \Sigma_j$.

We will show that when N is a q-cofibrant R-NUCA, arbitrary wedges of R-modules N^j/H with j and H as above have cell R-module approximations $Y_\alpha \to N^{j_\alpha}/H_{j_\alpha}$ such that $(\bigvee Y_\alpha)^i/\Sigma_i \to (\bigvee N^{j_\alpha}/H_{j_\alpha})^i/\Sigma_i$ is a weak equivalence.

We begin by generalizing the class of R-modules $\bar{\bar{\mathscr{E}}}_R$ described in [5, VII.6.4] to a class $\bar{\bar{\mathscr{F}}}_R$ whose objects retain the desirable property specified in [5, VII.6.7]. That is, the derived smash product of objects in $\bar{\bar{\mathscr{F}}}_R$ is represented by their point set-level smash product. Then we describe a subclass of $\bar{\bar{\mathscr{F}}}_R$, the class of extended cell R-modules, to which q-cofibrant R-NUCA's belong. We will show that if M is an extended cell R-module, M has a cell R-module approximation $Y \to M$ such that for all $i \geq 1$ and all $H \subset \Sigma_i$, $Y^i/H \to M^i/H$ is a weak equivalence. Furthermore, we will see that arbitrary wedges of these M^i/H have cell R-module approximations which behave as desired with respect to the symmetric power operation $(-)^j/\Sigma_j$.

We will use the following notation and definitions:

$X^{(i)}$ will denote the i-fold external smash product of spectra.

R will denote a q-cofibrant commutative S-algebra.

Definition 9.3. Let $\mathscr{F}_R$ be the class of all R-modules of the form

$$R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(i) \ltimes_G K$$

where K is any G-spectrum indexed on the universe U^i that has the homotopy type of a G-CW spectrum for some $G \subset \Sigma_i$.

Define $\bar{\bar{\mathscr{F}}}_R$ to be the closure of $\mathscr{F}_R$ under the following operations in the category of R-modules: finite $\wedge_R$, wedges, pushouts along cofibrations, colimits of countable sequences of cofibrations, homotopy equivalence and the following operation which we call stabilization: if ΣM is in $\bar{\bar{\mathscr{F}}}_R$ then M is in $\bar{\bar{\mathscr{F}}}_R$.

We have the following observations:

(1) $\bar{\bar{\mathscr{E}}}_R \subset \bar{\bar{\mathscr{F}}}_R$.

(2) R is in $\bar{\bar{\mathscr{F}}}_S$.

(3) Stabilization implies that if $M \to N \to C$ is a cofiber sequence of R-modules with M and C in $\bar{\bar{\mathscr{F}}}_R$, then N is in $\bar{\bar{\mathscr{F}}}_R$.

We have the following generalization of [5, VII.6.3]:

Theorem 9.4. *Let X_i, $1 \le i \le n$, be an H_i-spectrum indexed on U^{j_i} which has the homotopy type of an H_i-CW spectrum for some $H_i \subset \Sigma_{j_i}$. Then*

$$\bigwedge_{\mathscr{L}} S \wedge_{\mathscr{L}} \mathbb{L}(\mathscr{L}(j_i) \ltimes_{H_i} X\, i)$$

has the homotopy type of CW S-module and

$$\wedge_{\mathscr{L}}(Id \wedge_{\mathscr{L}} \xi) : \bigwedge_{\mathscr{L}} S \wedge_{\mathscr{L}} \mathbb{L}(\mathscr{L}(j_i) \ltimes_{H_i} X_i) \to \bigwedge_{\mathscr{L}} S \wedge_{\mathscr{L}} \mathscr{L}(j_i) \ltimes_{H_i} X_i$$

is a homotopy equivalence of spectra and a weak equivalence of S-modules.

Proof. By [5, XI.2.5], $\bigwedge_{\mathscr{L}} \mathscr{L}(j_i) \ltimes_{H_i} X_i$ has the homotopy type of a CW-spectrum indexed in U. The assertion about the homotopy type of $\bigwedge_{\mathscr{L}} S \wedge_{\mathscr{L}} \mathbb{L}(\mathscr{L}(j_i) \ltimes_{H_i} X_i)$ follows from the fact that $\mathbb{L}$ and $S \wedge_{\mathscr{L}}(-)$ preserve CW- homotopy types and the smash product of CW-S-modules is a CW-S-module.

As in [5, VII.6.3] we have the following commutative diagram of $\mathbb{L}$-spectra:

$$\begin{CD}
\bigwedge_{\mathscr{L}}(S \wedge_{\mathscr{L}} \mathbb{L}(\mathscr{L}(j_i) \ltimes_{H_i}, X_i)) @>{\wedge_{\mathscr{L}}\lambda}>> \bigwedge_{\mathscr{L}} \mathbb{L}(\mathscr{L}(j_i) \ltimes_{H_i} X_i)\\
@V{\wedge_{\mathscr{L}}(Id \wedge_{\mathscr{L}} \xi)}VV @VV{\wedge_{\mathscr{L}}\xi}V\\
\bigwedge_{\mathscr{L}}(S \wedge_{\mathscr{L}} \mathscr{L}(j_i) \ltimes_{H_i}, X_i) @>>{\wedge_{\mathscr{L}}\lambda}> \mathscr{L}(j_i) \ltimes_{H_i} X_i.
\end{CD}$$

The top horizontal arrow is a homotopy equivalence of $\mathbb{L}$-spectra and the bottom horizontal arrow is a homotopy equivalence of spectra. By [5, I.5.4 and I.5.6] the right vertical arrow is isomorphic to the map

$$\begin{CD}
(\mathscr{L}(n) \times \mathscr{L}(j_i) \times \cdots \times \mathscr{L}(j_n)) \ltimes_{H_1 \times \cdots \times H_n} X_1 \wedge \cdots \wedge X_n\\
@V{\gamma \ltimes Id}VV\\
\mathscr{L}(j_1 + \cdots + j_n) \ltimes_{H_1 \times \cdots \times H_n} X_1 \wedge \cdots \wedge X_n.
\end{CD}$$

Since γ is an $(H_1 \times \cdots \times H_n)$-equivariant homotopy equivalence, [2, XXII.1.8] gives that $\gamma \ltimes Id$ is an equivariant homotopy equivalence before passing to orbits. Thus, after passing to orbits, $\gamma \ltimes Id$ is a homotopy equivalence and so is the left vertical arrow. $\square$

We also have the following generalization of [5, VII.6.7].

Theorem 9.5. *Choose cell R-modules ΓM and weak equivalences of R-modules $\gamma : \Gamma M \to M$ for each $M \in \bar{\bar{\mathscr{F}}}_R$. Then, for any finite subset $\{M_1, \ldots, M_n\}$ of $\bar{\bar{\mathscr{F}}}_R$*

$$\gamma \wedge_R \cdots \wedge_R \gamma : \Gamma M_1 \wedge_R \cdots \wedge_R \Gamma M_n \to M_1 \wedge_R \cdots \wedge_R M_n$$

is a weak equivalence of R-modules.

Proof. We adapt the proof of [5, VII.6.7] to our situation.

Let $R = S$. By standard commutation formulas relating smash products with the chosen operations and the fact that this operations preserve weak equivalences, it is enough to show that the claim is true when each $\mathscr{M}_i$ is in $\mathscr{F}_S$. For these the conclusion follows by the previous theorem.

For general R and $M_i = R \wedge_S N_i$, where $N_i \in \mathscr{F}_S$ has cell S-module approximation ΓN_i, $R \wedge_S \Gamma N_i$ is a cell R-module approximation of M_i. Using the commuting properties of smash products we have that

$$(R \wedge_S N_1) \wedge_R \cdots \wedge_R (R \wedge_S N_n) \cong R \wedge_S N_1 \wedge_S \cdots \wedge_S N_n$$

and similarly for the ΓN_i. Hence we must show that

$$R \wedge_S \Gamma N_1 \wedge_S \cdots \wedge_S \Gamma N_n \to R \wedge_S N_1 \wedge_S \cdots \wedge_S N_n$$

is a weak equivalence. The conclusion follows from the result for S since, as mentioned above, $R \in \bar{\mathscr{F}}_S$. For general M_i the result follows as in the case $R = S$. $\square$

We now introduce the class of extended cell R-modules.

Definition 9.6. An extended cell is a pair of the form $(X \wedge B^n_+, X \wedge S^{n-1}_+)$, where $n \geq 0$ and $X = R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(i) \ltimes_G K$ for a G-spectrum K indexed on U^i and which has the homotopy type of a G-CW spectrum for some $G \subset \Sigma_i$. Here $S^{-1} = \emptyset$.

An extended cell R-module is an R-module $M = \operatorname{colim} M_i$ with $M_0 = *$ and M_n obtained from M_{n-1} by a pushout of R-modules of the form

$$\begin{array}{ccc} \bigvee_\alpha X_\alpha \wedge S^{n_\alpha - 1}_+ & \longrightarrow & M_{n-1} \\ \downarrow & & \downarrow \\ \bigvee_\alpha X_\alpha \wedge B^{n_\alpha}_+ & \longrightarrow & M_n. \end{array}$$

The sequence M_i is called the sequential filtration of M. Note that the maps $M_i \to M_{i+1}$ are cofibrations of R-modules.

The following lemma explains our interest in extended cell R-modules.

Lemma 9.7. *If N is a cell R-NUCA then it is an extended cell R-module.*

Proof. Supposed that $N = \operatorname{colim} M_i$ where $M_0 = *$ and M_i is obtained from M_{i-1} as a pushout of the form

$$\begin{array}{ccc} \mathbb{A}E & \longrightarrow & M_{i-1} \\ \downarrow & & \downarrow \\ \mathbb{A}CE & \longrightarrow & M_i, \end{array}$$

where E is a wedge of sphere R-modules.

Then $M_i \cong |\beta_*^R(\mathbb{A}CE, \mathbb{A}E, M_{i-1})|$, where the bar construction is the one described in [5, VII.3.5] with respect to the coproduct $A \amalg B = A \vee B \vee A \wedge_R B$. A similar argument to that of [5, VII.7.5] shows that the qth filtration of the bar construction is an extended cell R-module and by passage to colimits so is any cell R-NUCA. $\square$

Clearly, extended cell R-modules are in $\bar{\bar{\mathscr{F}}}_R$. They also have the following key property.

Theorem 9.8. *If M is an extended cell R-module, $i > 0$ and $H \subset \Sigma_i$, then M^i/H is in $\bar{\bar{\mathscr{F}}}_R$ and the projection*

$$\pi : (EH)_+ \wedge_H M^i \to M^i/H$$

is a weak equivalence of spectra.

We will see below that this theorem is an immediate consequence of the case $n = 1$ of the following proposition:

Proposition 9.9. *Let X_i, $1 \le i \le n$, be extended cell R-modules with finite sequential filtrations of respective lengths d_i and let $m > 0$. Then for each partition $m = m_1 + \cdots + m_n$ and each $H \subset \Sigma_m$, $((\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X_n^{m_n})/H$ is in $\bar{\bar{\mathscr{F}}}_R$ and*

$$(EH)_+ \wedge_H ((\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X_n^{m_n}) \to ((\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X_n^{m_n})/H$$

is a weak equivalence of spectra. Here $\times \Sigma_{m_i} = \Sigma_{m_1} \times \cdots \times \Sigma_{m_n}$.

Proof. We will describe a filtration of $(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X_n^{m_n}$,

$$G_0 \subset G_1 \subset \cdots \subset G_t = (\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X_n^{m_n},$$

where each of the inclusions is a Σ_m-cofibration of R-modules. If we know the result for G_0 and each of the subquotients we can conclude the result for

$$(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X^{m_n}.$$

We use induction on $d =$ maximum length of filtration of the X_i's. The base case $d = 1$ will be the hardest to prove and we postpone it until the end.

The inductive step is proved by induction on $c =$ number of X_i's with sequential filtration of length d. Let $d > 1$ and assume inductively that the results are true for $f < d$. The case $c = 0$ occurs when all the X_i's have filtration at most $d - 1$, in which case the statement is true by the inductive hypothesis. Now, assume the results for $l < c$. Without loss of generality assume that X_1 has sequential filtration of length d and let Z be the $(d-1)$-st filtration level of X_1. Then X_1/Z is an extended cell R-module with filtration length 1.

Filter $X_1^{m_1}$ by

$$Z^{m_1} = F_0 \subset F_1 \subset \cdots \subset F_{m_1 - 1} \subset F_{m_1} = X_1^{m_1},$$

where F_k is the union of the subcomplexes $M_1 \wedge_R \cdots \wedge_R M_{m_1}$ such that each M_j is either X_1 or Z and k of the M_j are X_1. As in [5, III.5.1], the inclusions $F_k \subset F_{k+1}$ are Σ_{m_1}-cofibrations and the subquotients can be identified equivariantly as

$$(F_k/F_{k-1}) \cong (\Sigma_{m_1})_+ \wedge_{\Sigma_{m_1-k} \times \Sigma_k} (Z)^{m_1-k} \wedge_R (X_1/Z)^k.$$

This filtration induces a filtration on $(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} X_1^{m_1} \wedge_R \cdots \wedge_R X_n^{m_n}$. The initial R-submodule is $(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} Z^{m_1} \wedge_R X^{m_2} \wedge_R \cdots \wedge_R X_n^{m_n}$, the inclusions are Σ_m-cofibrations and the subquotients are identified equivariantly as

$$\begin{aligned}
&(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} (F_k/F_{k-1}) \wedge_R X^{m_2} \wedge_R \cdots \wedge_R X_n^{m_n} \\
&\quad \cong (\Sigma_m)_+ \wedge_{\Sigma_{m_1-k} \times \Sigma_k \times \Sigma_{m_2} \times \cdots \times \Sigma_{m_n}} (Z)^{m_1-k} \wedge_R (X_1/Z)^k \wedge_R X^{m_2} \wedge_R \cdots \wedge_R X_n^{m_n}.
\end{aligned}$$

Since Z and X_1/Z have filtration length less than d we have reduced c by one hence by induction the statement is true for these modules. Inducting up the filtration we prove the inductive step for d.

We are left with showing the result for $d=1$. That is, each X_i is an extended cell R-module with filtration length 1. Then each X_i is a wedge of things of the form

$$(R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(p) \ltimes_G K) \wedge (B^n/S^{n-1})_+$$

and this reduces to the case when each X_i consists of a single wedge summand. Let

$$X_i = (R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(p_i) \ltimes_{G_i} K_i) \wedge (B^{n_i}/S^{n_i-1})_+$$

where K_i is a G_i-spectrum of the homotopy type of a G_i-CW-spectrum indexed on U^{p_i} and $G_i \subset \Sigma_{p_i}$. To simplify notation we suppress $(B^{n_i}/S^{n_i-1})_+$ in what follows.

We want to show that

$$\left[(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} \left(\bigwedge_R R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(m_i p_i) \ltimes_{G_i^{m_i}} K_i^{(m_i)} \right) \right] \Big/ H$$

is in $\bar{\bar{\mathscr{F}}}_R$ and that it is weak equivalent to

$$(EH)_+ \wedge_H \left[(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} \left(\bigwedge_R R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(m_i p_i) \ltimes_{G_i^{m_i}} K_i^{(m_i)} \right) \right].$$

Note that

$$\begin{aligned}
&(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} \left(\bigwedge_R R \wedge_S S \wedge_{\mathscr{L}} \mathscr{L}(m_i p_i) \ltimes_{G_i^{m_i}} K_i^{(m_i)} \right) \\
&\quad = R \wedge_S S \wedge_{\mathscr{L}} \left[(\Sigma_m)_+ \wedge_{\times \Sigma_{m_i}} \mathscr{L}(\Sigma m_i p_i) \ltimes_{\times G_i^{m_i}} \bigwedge_i K_i^{(m_i)} \right] \\
&\quad = R \wedge_S S \wedge_{\mathscr{L}} \left[(\Sigma_m \times \mathscr{L}(\Sigma m_i p_i)) \ltimes_{\times (\Sigma_{m_i} \int G_i)} \bigwedge_i K_i^{(m_i)} \right].
\end{aligned}$$

We simplify the notation further and consider the problem for an R-module of the form

$$R\wedge_{\mathscr{L}}[(\Sigma_m)_+\wedge_G(\mathscr{L}(q)\ltimes_Q K)]$$

where $G=\Sigma_{m_1}\times\cdots\times\Sigma_{m_k}$, $Q=G_1^{m_1}\times\cdots\times G_k^{m_k}$, $m=\Sigma m_i$, $q=\Sigma p_i m_i$ and K is a spectrum indexed on U^q that has the homotopy type of a $G\int Q$-CW spectrum. Here we have let $G\int Q$ denote $(\Sigma_{m_1}\int G_{m_1})\times\cdots\times(\Sigma_{m_k}\int G_{m_k})\subset\Sigma_q$.

For $H\subset\Sigma_m$ we can write Σ_m as the disjoint union of double cosets $Hx_\alpha G$ and we have that H-equivariantly

$$(\Sigma_m)_+\wedge_G(\mathscr{L}(q)\ltimes_Q K)=\bigvee_\alpha(Hx_\alpha G)_+\wedge_G(\mathscr{L}(q)\ltimes_Q K).$$

We will show that for each α

$$(EH)_+\wedge_H(Hx_\alpha G)_+\wedge_G(\mathscr{L}(q)\ltimes_Q K)\to[(Hx_\alpha G)_+\wedge_G(\mathscr{L}(q)\ltimes_Q K)]/H$$

is a homotopy equivalence of spectra.

Let $J_\alpha=\{g\in G\mid x_\alpha g=hx_\alpha,\ h\in H\}$. Note that the map $g\mapsto x_\alpha g(x_\alpha)^{-1}$ embeds J_α in H. We recall from elementary group theory that

$$EH\times_H Hx_\alpha G\cong EH\times_{J_\alpha}G$$

as G-spaces. Letting $X=\mathscr{L}(q)\ltimes_Q K$, this implies that

$$(EH)_+\wedge_H(Hx_\alpha G)_+\wedge_G X=(EH)_+\wedge_{J_\alpha}X.$$

We also have that

$$\begin{aligned}[(Hx_\alpha G)_+\wedge_G X]/H&=S^0\wedge_H(Hx_\alpha G)_+\wedge_G X\\&=(*\times_H HxG)\wedge_G X\cong G/J_\alpha\wedge_G X=X/J_\alpha.\end{aligned}$$

Thus,

$$[(\Sigma_m)_+\wedge_G\mathscr{L}(q)\ltimes_Q K]/H\cong\bigvee_\alpha(\mathscr{L}(q)\ltimes_Q K)/J_\alpha=\bigvee_\alpha\mathscr{L}(q)\ltimes_{Q\rtimes J_\alpha}K,$$

which shows that

$$\left[(\Sigma_m)_+\wedge_{\times\Sigma_{m_i}}\left(\bigwedge_R R\wedge_S S\wedge_{\mathscr{L}}\mathscr{L}(m_ip_i)\ltimes_{G_i^{m_i}}K_i^{(m_i)}\right)\right]\Big/H$$

is a wedge of R-modules in $\bar{\bar{\mathscr{F}}}_R$ and hence is in $\bar{\bar{\mathscr{F}}}_R$ as desired. We need to show that

$$\pi:(EH)_+\wedge_{J_\alpha}X\cong(EH)_+\wedge_{J_\alpha}(\mathscr{L}(q)\ltimes_Q K)\to(\mathscr{L}(q)\ltimes_Q K)/J_\alpha\cong X/J_\alpha$$

is a homotopy equivalence of spectra. But this is the case because

$$(EH)_+\wedge(\mathscr{L}(q)\ltimes_Q K)=(EH\times\mathscr{L}(q))\ltimes_Q K$$

and π is induced from the homotopy equivalence of $G\int Q$-spaces $EH\times\mathscr{L}(q)\to\mathscr{L}(q)$ by passage to orbits.

Further, by [5, I.8.5] and the equivariant version of [5, I.2.5],

$$S \wedge_{\mathscr{L}} ((EH)_+ \wedge_{J_\alpha} (\mathscr{L}(q) \ltimes_Q K)) \to S \wedge_{\mathscr{L}} ((\mathscr{L}(q) \ltimes_Q K)/J_\alpha)$$

is also a homotopy equivalence of spectra. Since R is q-cofibrant, $R \wedge_S (-)$ takes this homotopy equivalence to a weak equivalence giving the desired result. □

As mentioned above, Theorem 9.8 follows from the case $n=1$ in the previous proposition. The restriction to extended cell R-modules with finite sequential filtration does not present a problem. If M is a general extended cell R-module, then $M = \operatorname{colim} M_j$, where each of the M_j is an extended cell R-module with finite sequential filtration. The previous proposition proves the theorem for each of the M_i's. Since we have that $M^i = \operatorname{colim} M^i_j$ and the maps $M^i_{j-1} \to M^i_j$ are Σ_i-cofibrations, Theorem 9.8 follows for M.

We now see that extended cell R-modules have the following good properties.

Theorem 9.10. *Let M be an extended cell R-module and let $Y \to M$ be a cell R-module approximation. Then, for all $i \geq 1$ and all $H \subset \Sigma_i$, $Y^i/H \to M^i/H$ is a weak equivalence. Further, there is a cell R-module approximation $Z \to M^i/H$ such that $Z^j/G \to (M^i/H)^j/G$ is a weak equivalence for all $j \geq 1$ and all $G \subset \Sigma_j$.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} (EH)_+ \wedge_H Y^i & \longrightarrow & (EH)_+ \wedge_H M^i \\ \downarrow & & \downarrow \\ Y^i/H & \longrightarrow & M^i/H, \end{array}$$

where Y^i is a cell R-module approximation of M^i and the vertical arrows are weak equivalences by Theorem 9.8. We will show that

$$(EH)_+ \wedge_H Y^i \to (EH)_+ \wedge_H M^i$$

is a weak equivalence.

We use the skeletal filtration of EH to set up a natural spectral sequence

$$H_*(H, \pi_*(Y^i)) \Rightarrow \pi_*((EH)_+ \wedge_H Y^i),$$

which proves the first part of the theorem.

To prove the second part of the theorem, let $Z \to (EH)_+ \wedge_H M^i$ be a cell R-module approximation. Then, the composite

$$Z \to (EH)_+ \wedge_H Y^i \to Y^i/H$$

is a cell R-module approximation of Y^i/H. Hence, since M^i/H is in $\bar{\bar{\mathscr{F}}}_R$, the following composite is also a weak equivalence:

$$Z^j \to [(EH)_+ \wedge_H M^i]^j \to (M^i/H)^j \cong M^{ij}/H^j.$$

Note that

$$[(EH)_+ \wedge_H M^i]^j \cong (EH)_+^j \wedge_{H^j} M^{ij} \cong (EH^j)_+ \wedge_{H^j} M^{ij}.$$

Hence, the second arrow of the above composite is a weak equivalence by Theorem 9.8. Therefore, the first arrow is also a weak equivalence.

We also have that

$$\begin{aligned}(EG)_+ \wedge_G [(EH)_+^j \wedge_{H^j} M^{ij}] &\cong [EG \times (EH)^i]_+ \wedge_{G\int H} M^{ij} \\ &\cong \left[E\left(G\int H\right)\right]_+ \wedge_{G\int H} M^{ij}\end{aligned}$$

and hence Theorem 9.8 gives that the vertical arrows in the following commutative diagram are weak equivalences:

$$\begin{tikzcd}
(EG)_+ \wedge_G Z^j \arrow[r] \arrow[d] & (EG)_+ \wedge_G [(EH)_+^j \wedge_{H^j} M^{ij}] \arrow[r, "\cong"] & E(G\int H)]_+ \wedge_{G\int H} M^{ij} \arrow[d] \\
Z^j/G \arrow[r] & (M^i/H)^j/G \arrow[r, "\cong"'] & M^{ij}/(G\int H).
\end{tikzcd}$$

The top horizontal arrow is also a weak equivalence and hence the bottom horizontal arrow is a weak equivalence as desired. □

We need one last proposition.

Proposition 9.11. *Let $M = K^k/H$ and $N = L^l/J$ be R-modules for some extended cell R-modules K and L, $k, l \geq 1$, $H \subset \Sigma_k$ and $J \subset \Sigma_l$. Then there exists a cell R-module approximation $Z \to M \vee N$ such that, for all $i \geq 1$,*

$$Z^i/\Sigma_i \to (M \vee N)^i/\Sigma_i$$

is a weak equivalence.

Proof. By the previous theorem we have cell R-module approximations $X \to M$ and $Y \to N$ such that for all $j \geq 1$, $X^j/\Sigma_j \to M^j/\Sigma_j$ and $Y^j/\Sigma_j \to N^j/\Sigma_j$ are weak equivalences. For $i > 0$

$$(M \vee N)^i = \bigvee_j [(\Sigma_i)_+ \times_{\Sigma_{i-j} \times \Sigma_j} M^{i-j} \wedge_R N^j]$$

thus, passing to orbits, we have

$$(M \vee N)^i/\Sigma_i = \bigvee_j M^{i-j}/\Sigma_{i-j} \wedge_R N^j/\Sigma_j,$$

and simmilarly with M and N replaced by X and Y. By Theorem 9.8 we know that for each j, $M^{i-j}/\Sigma_{i-j} = K^{k(i-j)}/(\Sigma_{i-j} \int H)$ and $N^j/\Sigma_j = L^{lj}/(\Sigma_j \int J)$ are in $\bar{\mathscr{F}}_R$.

Since, X^{i-j}/Σ_{i-j}, Y^j/Σ_j are also in $\bar{\mathscr{F}}_R$ the weak equivalences above smash to a weak equivalence

$$X^{i-j}/\Sigma_{i-j} \wedge_R Y^j/\Sigma_j \to M^{i-j}/\Sigma_{i-j} \wedge_R N^j/\Sigma_j.$$

Therefore,

$$Z = X \vee Y \to M \vee N$$

is a cell R-module approximation with the desired property. □

Theorem 9.1 follows from the previous work since if N is a q-cofibrant R-NUCA, then N is an extended cell R-module and $\mathbb{A}^n N$ is a wedge of R-modules of the form N^i/H with $i \geq 1$ and $H \subset \Sigma_i$.

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