Journal of Topology 6 (2013) 285–310
doi:10.1112/jtopol/jts032

The multiplication on BP

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Abstract

BP is an E_4 ring spectrum. The E_4 structure is unique up to automorphism.

1. *Introduction*

For each prime p, Brown and Peterson [**5**] constructed a spectrum BP to capture the non-Bockstein part of the Steenrod algebra. They showed that BP is the unique finite type p-local spectrum with its cohomology. Soon afterward, Quillen [**23**] showed that BP is a ring spectrum, and a straightforward calculation shows that the ring spectrum multiplication is unique up to automorphism. In the decades since, BP has become fundamental in the study of stable homotopy theory, largely because it is the complex oriented ring spectrum that has the universal p-typical formal group law. The complex cobordism spectrum MU has the universal formal group law, and its localization $\mathrm{MU}_{(p)}$ splits as a sum of shifted copies of BP. Working p-locally, most statements involving MU then have equivalent but simpler formulations in terms of BP.

One fundamental difference between BP and $\mathrm{MU}_{(p)}$ involves the coherence of the multiplicative structure. The coherence of the multiplication on MU is well-understood: As a Thom spectrum, MU has a canonical (even prototypical) structure of an E_∞ ring spectrum [**21**]. This structure arises from a geometric model of MU completely independent of formal group law considerations, and BP does not admit a corresponding construction. In fact, recent work of Johnson and Noel [**14**] finds a fundamental incompatibility between the E_∞ ring structure on MU and p-typical orientations, proving that the canonical map from MU to BP cannot be E_∞ or even H_∞ (at the primes their work covers) for any possible H_∞ structure on BP. Earlier work of Hu, Kriz, and May [**13**] had revealed that no map $\mathrm{BP} \to \mathrm{MU}_{(p)}$ can be H_∞ for any possible H_∞ structure on BP, quashing the conjectured approach to constructing an H_∞ ring structure on BP. No E_∞ or H_∞ ring structure is currently known to exist on BP, and there is currently no plausible approach to constructing one.

In this paper, we study the coherence of the multiplication on BP from the perspective of the E_n hierarchy. Previous work by Goerss [**11**], Lazarev [**17**], and the authors (unpublished) shows that BP is an A_∞ (or E_1) ring spectrum. In this paper, we show that BP admits an E_4 ring structure, and that this structure is unique up to automorphism.

Theorem 1.1. *BP is an E_4 ring spectrum; the E_4 ring structure is unique up to automorphism in the homotopy category of E_4 ring spectra.*

As part of the argument, we construct an idempotent splitting of $\mathrm{MU}_{(p)}$ as an E_4 ring spectrum with the E_4 ring spectrum BP as the unit summand. We do not know if this

Received 15 September 2011; revised 4 October 2012; published online 18 December 2012.

2010 *Mathematics Subject Classification* 55P43 (primary), 55N22, 55S35 (secondary).

The second author was supported in part by NSF grant DMS-0804272.

idempotent can be chosen to be the Quillen idempotent; we hope to return to this question in a future paper.

One of the main reasons for interest in E_∞ structures is that the category of modules over an E_∞ ring spectrum has a symmetric monoidal smash product. The main result of the paper [**19**] is that for a symmetric monoidal structure on the homotopy category of modules, an E_4 structure suffices. Thus, applying the main theorem of [**19**] to the previous theorem, we obtain the following corollary.

Corollary 1.2. *The derived category of BP-modules has a symmetric monoidal smash product* $\wedge_{\mathrm{BP}}$.

The proof of Theorem 1.1 relies on an obstruction theory for E_n ring structures based on the obstruction theory pioneered by Kriz [**16**] and Basterra [**2**] for E_∞ ring structures. Lemma 5.1 gives the main computation: The E_4 ring spectrum obstruction groups for BP are concentrated in even degrees and are torsion free. This breaks down for the E_6 ring spectrum obstruction groups, making analysis of the E_6 and higher ring spectrum obstruction theory significantly more difficult.

Richter [**24**] presents a different approach to coherence of the multiplication of BP using the n-stage hierarchy of Robinson [**25**], which is a homotopy commutative generalization of the A_n hierarchy of Stasheff [**27**]. In this terminology, [**24**] shows that BP has a $(2p^2 + 2p - 2)$-stage structure (which in particular restricts to an A_{2p^2+2p-2} but not A_∞ structure). The relationship of this result to Theorem 1.1 and the relationship of the n-stage hierarchy to the E_n hierarchy are not understood at the present time.

1.1. *Outline*

For the proof of Theorem 1.1, we apply Postnikov tower obstruction theory for E_4 ring spectra; we review the details of this obstruction theory in Sections 2–4 before applying it to prove the main theorem in Section 5. Finally, Sections 6 and 7 prove some of the properties of the spectral sequences for computing Quillen homology and cohomology stated in Section 3.

1.2. *Conventions*

Although we state most results in homotopy theoretic terms that are independent of models, where the proofs require models, we will work in the following context. We use the category of EKMM S-modules [**10**] as the basic category of spectra, and we understand S-algebras and commutative S-algebras in this context. For E_n structures, we use the Boardman–Vogt little n-cubes operad $\mathcal{C}_n$. In this context, for a commutative S-algebra R, an E_n R-algebra consists of an R-module A and maps of R-modules

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} \underbrace{(A \wedge_R \cdots \wedge_R A)}_{m\,\text{factors}} \longrightarrow A$$

satisfying the usual associativity and unit properties.

2. *Quillen cohomology of E_n algebras*

For the obstruction theory for E_n algebra in Section 4, the obstruction groups take values in certain Quillen cohomology groups. In this section, we review the construction of these groups in the special case in which we apply them. For this, we fix a connective commutative S-algebra R and a connective commutative R-algebra H, and we consider the category of E_n R-algebras lying over H. In practice (in Section 4), H will be an Eilenberg–Mac Lane spectrum $H\pi$ for a commutative $\pi_0 R$-algebra π, but we do not need to assume that here. As we review in this section, the advantage of this setup is that Quillen cohomology with coefficients in an H-module becomes a cohomology theory and has a corresponding homology theory, each satisfying the appropriate analog of the Eilenberg–Steenrod axioms.

DEFINITION 2.1. Let $\mathfrak{A}\mathcal{C}_n^{R/H}$ denote the category of E_n R-algebras lying over H: An object of $\mathfrak{A}\mathcal{C}_n^{R/H}$ is an E_n R-algebra A together with a map of E_n R-algebras $\epsilon : A \to H$; a morphism of $\mathfrak{A}\mathcal{C}_n^{R/H}$ is a map of E_n R-algebras $A \to A'$ such that the composite map $A \to A' \to H$ is ϵ.

For an H-module M, we make $H \vee M$ into an object of $\mathfrak{A}\mathcal{C}_n^{R/H}$ with the 'square zero' multiplication: The structure maps

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} (H \vee M)^{(m)} \longrightarrow (H \vee M)^{(m)}/\Sigma_m \longrightarrow H \vee M$$

are induced by the multiplication on H and the H-module structure on M on the summands with one or fewer factors of M and the trivial map on the summands with two or more factors of M. The trivial map $M \to *$ induces the map $H \vee M \to H$.

DEFINITION 2.2. For A in $\mathfrak{A}\mathcal{C}_n^{R/H}$, let $H^*_{\mathcal{C}_n}(A; M) = \mathrm{Ho}\,\mathfrak{A}\mathcal{C}_n^{R/H}(A, H \vee \Sigma^* M)$, where $\mathrm{Ho}\,\mathfrak{A}\mathcal{C}_n^{R/H}$ denotes the homotopy category obtained from $\mathfrak{A}\mathcal{C}_n^{R/H}$ by formally inverting the weak equivalences.

We note that $H^*_{\mathcal{C}_n}(A; M)$ is actually a graded abelian group and in fact a $\pi_* H$-module. The fold map $M \vee M \to M$ (which is a map of H-modules) induces a map in $\mathfrak{A}\mathcal{C}_n^{R/H}$ from $H \vee (M \vee M)$ to $H \vee M$, which induces the natural addition since $H \vee (M \vee M)$ is isomorphic in $\mathrm{Ho}\,\mathfrak{A}\mathcal{C}_n^{R/H}$ to the product of two copies of $H \vee M$. Likewise an element of $\pi_q H$ induces a map in the derived category of H-modules $\Sigma^q M \to M$, which induces a map $H \vee \Sigma^p M \to H \vee \Sigma^{p-q} M$ in $\mathrm{Ho}\,\mathfrak{A}\mathcal{C}_n^{R/H}$, and hence a natural map $H^p_{\mathcal{C}_n}(A; M) \to H^{p-q}_{\mathcal{C}_n}(A; M)$.

We can also define relative groups as follows. For an object A in $\mathfrak{A}\mathcal{C}_n^{R/H}$, we consider $\mathfrak{A}\mathcal{C}_n^{R/H} \backslash A$, the under-category of A in $\mathfrak{A}\mathcal{C}_n^{R/H}$. We regard $H \vee M$ as an object of $\mathfrak{A}\mathcal{C}_n^{R/H} \backslash A$ using the map $A \to H$. For X in $\mathfrak{A}\mathcal{C}_n^{R/H} \backslash A$, we define the relative Quillen cohomology group as

$$H^*_{\mathcal{C}_n}(X, A; M) = \mathrm{Ho}(\mathfrak{A}\mathcal{C}_n^{R/H} \backslash A')(X, H \vee \Sigma^* M),$$

where A' is a cofibrant approximation of A. Using these relative groups and a connecting morphism $\delta : H^*_{\mathcal{C}_n}(A; M) \to H^{*+1}_{\mathcal{C}_n}(X, A; M)$ described below, we obtain a cohomology theory.

THEOREM 2.3. *The functors $H^*_{\mathcal{C}_n}(-; M)$ and connecting morphism δ define a cohomology theory on the category $\mathfrak{A}\mathcal{C}_n^{R/H}$ satisfying the following version of the Eilenberg–Steenrod axioms.*

(i) (*Homotopy*) *If $(X, A) \to (Y, B)$ is a weak equivalence of pairs, then the induced map $H^*_{\mathcal{C}_n}(Y, B; M) \to H^*_{\mathcal{C}_n}(X, A; M)$ is an isomorphism of graded abelian groups.*

(ii) (*Exactness*) *For any pair (X, A), the sequence*

$$\cdots \longrightarrow H^n_{\mathcal{C}_n}(X, A; M) \longrightarrow H^n_{\mathcal{C}_n}(X; M) \longrightarrow H^n_{\mathcal{C}_n}(A; M) \xrightarrow{\delta} H^{n+1}_{\mathcal{C}_n}(X, A; M) \longrightarrow \cdots$$

is exact.

(iii) (*Excision*) *If A is cofibrant, $A \to B$ and $A \to X$ are cofibrations, and Y is the pushout $X \amalg_A B$, then the map of pairs $(X, A) \to (Y, B)$ induces an isomorphism of graded abelian groups $H^*_{\mathcal{C}_n}(Y, B; M) \to H^*_{\mathcal{C}_n}(X, A; M)$.*

(iv) (*Product*) *If $\{X_\alpha\}$ is a set of cofibrant objects and X is the coproduct, then the natural map $H^*_{\mathcal{C}_n}(X; M) \to \prod H^*_{\mathcal{C}_n}(X_\alpha; M)$ is an isomorphism.*

Constructing the connecting homomorphism, proving the previous theorem, and constructing the corresponding homology theory requires the 'based' version of the construction obtained by working over and under the same commutative S-algebra. The functor $H \wedge_R (-)$ takes E_n R-algebras lying over H to E_n H-algebras lying over H and is left adjoint to the forgetful functor. Writing $H \wedge^{\mathbf{L}}_R (-)$ for the left-derived functor, we get a natural isomorphism

$$H^*_{\mathcal{C}_n}(A; M) = \operatorname{Ho} \mathfrak{A}\mathcal{C}_n^{R/H}(A, H \vee \Sigma^* M) \cong \operatorname{Ho} \mathfrak{A}\mathcal{C}_n^{H/H}(H \wedge^{\mathbf{L}}_R A, H \vee \Sigma^* M).$$

In addition, for $f : A \to X$, we can then identify $H^*_{\mathcal{C}_n}(X, A; M)$ as

$$\operatorname{Ho}(\mathfrak{A}\mathcal{C}_n^{H/H} \backslash (H \wedge_R A'))(H \wedge^{\mathbf{L}}_R X, H \vee \Sigma^* M) \cong \operatorname{Ho} \mathfrak{A}\mathcal{C}_n^{H/H}(Cf, H \vee \Sigma^* M),$$

where Cf denotes the 'cofiber'

$$Cf = (H \wedge_R X') \amalg_{(H \wedge_R A')} H$$

formed as the pushout of E_n H-algebras, where $A' \to X'$ is a cofibration modeling the map $A \to X$. The homotopy cofiber of the inclusion of X' in Cf is equivalent to the suspension of A' in the homotopy category of $\mathfrak{A}\mathcal{C}_n^{H/H}$; this produces the connecting homomorphism satisfying the Exactness axiom. The remaining axioms are clear from the construction.

To construct the corresponding homology theory, we switch from the category $\mathfrak{A}\mathcal{C}_n^{H/H}$ to the category $\mathfrak{N}\mathcal{C}_n^H$ of 'non-unital' E_n H-algebras. These are the algebras of H-modules over the operad $\widetilde{\mathcal{C}}_n$ where

$$\widetilde{\mathcal{C}}_n(m) = \begin{cases} \emptyset, & m = 0, \\ \mathcal{C}_n(m), & m > 0. \end{cases}$$

We obtain a functor $K(-) = H \vee (-)$ from $\mathfrak{N}\mathcal{C}_n^H$ to $\mathfrak{A}\mathcal{C}_n^{H/H}$ by attaching a new unit; this is left adjoint to the functor I from $\mathfrak{A}\mathcal{C}_n^{H/H}$ to $\mathfrak{N}\mathcal{C}_n^H$ which takes the (point-set) fiber of the augmentation $A \to H$. Since the functor I preserves fibrations and acyclic fibrations, we see that the adjoint pair K, I forms a Quillen adjunction. Since we can calculate the effect on homotopy groups of K on arbitrary non-unital E_n H-algebras and of I on fibrant objects of $\mathfrak{A}\mathcal{C}_n^{H/H}$, we see that when A is fibrant, a map of augmented $\mathcal{C}_n$-algebras $KN \to A$ is a weak equivalence if and only if the adjoint map $N \to IA$ is a weak equivalence; in other words, K, I is a Quillen equivalence.

Theorem 2.4. *The functors K and I form a Quillen equivalence between the category of non-unital E_n H-algebras and the category of E_n H-algebras lying over H (the category of augmented E_n H-algebras).*

The square zero E_n H-algebra $H \vee M$ is KZM for the non-unital E_n H-algebra ZM given by M with the trivial structure maps. Thus, we have the further description of $H^*_{\mathcal{C}_n}(A; M)$ as $\operatorname{Ho} \mathfrak{N}\mathcal{C}_n^H(I^{\mathbf{R}}(H \wedge^{\mathbf{L}}_R A), Z\Sigma^* M)$ where $I^{\mathbf{R}}$ denotes the right derived functor of I. The functor Z is a right adjoint, with left adjoint Q the indecomposables functor defined by the coequalizer

$$\bigvee_{m>0} \mathcal{C}_n(m)_+ \wedge_{\Sigma_m} N^{(m)} \rightrightarrows N \to QN.$$

Here, one map is the action map for N and the other map is the trivial map on the factors for $m > 1$ and the map

$$\mathcal{C}_n(1)_+ \wedge N \longrightarrow *_+ \wedge N \cong N$$

on the $m = 1$ factor. Since the functor Z preserves fibrations and weak equivalences, we see that Q, Z forms a Quillen adjunction.

THEOREM 2.5. *The functors Q and Z form a Quillen adjunction between the category of non-unital E_n H-algebras and the category of H-modules.*

Writing $Q^{\mathbf{L}}$ for the left-derived functor of Q, we get reduced cohomology and homology theories on non-unital E_n H-algebras as follows.

DEFINITION 2.6. For N a non-unital E_n H-algebra, let

$$D^*(N; M) = \mathrm{Ext}^*_H(Q^{\mathbf{L}}N, M),$$
$$D_*(N; M) = \mathrm{Tor}^H_*(Q^{\mathbf{L}}N, M).$$

In other words, $D^*(N; M)$ and $D_*(N; M)$ are the homotopy groups of the derived function H-module and derived smash product as in [**10**, IV§ 1]. We then have as a result that

$$H^*_{\mathcal{C}_n}(A; M) \cong D^*(I^{\mathbf{R}}(H \wedge^{\mathbf{L}}_R A); M)$$

and we make the definition

$$H^{\mathcal{C}_n}_*(A; M) = D_*(I^{\mathbf{R}}(H \wedge^{\mathbf{L}}_R A); M).$$

For $A \to X$, we let

$$H^{\mathcal{C}_n}_*(X, A; M) = D_*(I^{\mathbf{R}}(Cf); M),$$

for Cf the cofiber as above. Using the connecting morphism ∂ arising from cofibration sequences in $\mathfrak{N}\mathcal{C}^H_n$, we get a homology theory.

THEOREM 2.7. *The functors $H^{\mathcal{C}_n}_*(-; M)$ and connecting morphism ∂ define a homology theory on the category $\mathfrak{A}\mathcal{C}^{R/H}_n$ satisfying the following version of the Eilenberg–Steenrod axioms.*

(i) (*Homotopy*) *If $(X, A) \to (Y, B)$ is a weak equivalence of pairs, then the induced map $H^{\mathcal{C}_n}_*(X, A; M) \to H^{\mathcal{C}_n}_*(Y, B; M)$ is an isomorphism of graded abelian groups.*

(ii) (*Exactness*) *For any pair (X, A), the sequence*

$$\cdots \longrightarrow H^{\mathcal{C}_n}_{n+1}(X, A; M) \xrightarrow{\partial} H^{\mathcal{C}_n}_n(A; M) \longrightarrow H^{\mathcal{C}_n}_n(X; M) \longrightarrow H^{\mathcal{C}_n}_n(X, A; M) \longrightarrow \cdots$$

is exact.

(iii) (*Excision*) *If A is cofibrant, $A \to B$ and $A \to X$ are cofibrations, and Y is the pushout $X \amalg_A B$, then the map of pairs $(X, A) \to (Y, B)$ induces an isomorphism of graded abelian groups $H^{\mathcal{C}_n}_*(X, A; M) \to H^{\mathcal{C}_n}_*(Y, B; M)$.*

(iv) (*Sum*) *If $\{X_\alpha\}$ is a set of cofibrant objects and X is the coproduct, then the natural map $\bigoplus H^{\mathcal{C}_n}_*(X_\alpha; M) \to H^{\mathcal{C}_n}_*(X; M)$ is an isomorphism.*

For convenience in inductive statements, we let

$$H^*_{\mathcal{C}_0}(A;M) = \mathrm{Ext}^*_R(I^{\mathbf{R}}A, M),$$
$$H_*^{\mathcal{C}_0}(A;M) = \mathrm{Tor}_*^R(I^{\mathbf{R}}A, M),$$

the R-module cohomology and homology with coefficients in M of the augmentation ideal; this is a cohomology theory and a homology theory, respectively, on the category of R-modules over H.

As we mentioned in the introduction, in our main application, H will be an Eilenberg–Mac Lane spectrum $H\pi$ for a commutative π_0R-algebra π. For any π-module L, HL has a unique structure as an $H\pi$-module. In this context we will abbreviate notation for the coefficients by writing L in place of HL, for example,

$$H^*_{\mathcal{C}_n}(A;L) = H^*_{\mathcal{C}_n}(A;HL),$$
$$H_*^{\mathcal{C}_n}(A;L) = H_*^{\mathcal{C}_n}(A;HL).$$

3. *Properties of Quillen homology and cohomology*

In the previous section, we reviewed the construction of the Quillen cohomology groups that we use in the obstruction theory in the next section. In this section, we review some of their fundamental properties that we need to construct the obstruction theory and to calculate the obstruction groups.

Although our obstruction theory involves the Quillen cohomology groups, we use the Quillen homology groups to help work with them. Definition 2.6 provides the precise relationship between Quillen homology and Quillen cohomology in this context. Computationally, applying the universal coefficient spectral sequences of [**10**, IV§ 4], we obtain the following universal coefficient spectral sequences for Quillen homology and cohomology.

THEOREM 3.1 (Universal Coefficient Spectral Sequences). *Let A be an E_n R-algebra lying over H and let M be an H-module. There are spectral sequences with*

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*H}_{s,t}(H_*^{\mathcal{C}_n}(A;H), \pi_*M),$$
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*H}(H_*^{\mathcal{C}_n}(A;H), \pi_*M)$$

converging strongly to $H_*^{\mathcal{C}_n}(A;M)$ and conditionally to $H^*_{\mathcal{C}_n}(A;M)$, respectively. For $A \to X$ a map of E_n R-algebras lying over H, there are spectral sequences with

$$E^2_{s,t} = \mathrm{Tor}^{\pi_*H}_{s,t}(H_*^{\mathcal{C}_n}(X,A;H), \pi_*M),$$
$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*H}(H_*^{\mathcal{C}_n}(X,A;H), \pi_*M)$$

converging strongly to $H_*^{\mathcal{C}_n}(X,A;M)$ and conditionally to $H^*_{\mathcal{C}_n}(X,A;M)$, respectively.

For computing $H_*^{\mathcal{C}_n}(A;H)$, the main result [**3**, Theorem 1.3] provides an iterative method. For A a cofibrant E_n R-algebra lying over H, let N be a cofibrant non-unital E_n H-algebra with a weak equivalence $KN \to H \wedge_R A$ of E_n H-algebras lying over H. By definition then $H_*^{\mathcal{C}_n}(A;H)$ is π_*QN. Sections 5 and 6 of [**3**] study the bar construction BKN and show that it has the structure of an E_{n-1} H-algebra and the reduced construction $\tilde{B}N$ (constructed by $K\tilde{B}N \simeq BKN$) has the structure of a non-unital E_{n-1} H-algebra. Iterating the bar construction, we have a weak equivalence $\tilde{B}^nN \simeq \Sigma^n QN$. In terms of homotopy groups, we have

$$H_{*-n}^{\mathcal{C}_n}(A;H) = \pi_*\tilde{B}^nN,$$

or more generally with coefficients,

$$H^{\mathcal{C}_n}_{*-n}(A; M) = \mathrm{Tor}^H_*(\tilde{B}^n N, M),$$
$$H_{\mathcal{C}_n}^{*-n}(A; M) = \mathrm{Ext}^*_H(\tilde{B}^n N, M).$$

The filtration on the bar construction then gives us a spectral sequence for computing the Quillen homology and cohomology from the homology and cohomology of smash powers of $\tilde{B}^{n-1}N$. The next theorem states this result in a form that is useful for induction. The extra summand of π_*M in the statement derives from the fact that $B\tilde{B}^j N \simeq H \vee \tilde{B}^{j+1}N$, or equivalently, the convention that $(\tilde{B}^j N)^{(0)} = H$ rather than $*$.

THEOREM 3.2. *Let A be an E_n R-algebra lying over H and let N be a cofibrant non-unital E_n H-algebra with a weak equivalence $KN \to H \wedge_R A$ of E_n H-algebras lying over H. For an H-module M and $j < n$, there is a spectral sequence with*

$$E^1_{s,t} = \mathrm{Tor}^H_t(\underbrace{\tilde{B}^j N \wedge_H \ldots \wedge_H \tilde{B}^j N}_{s\,\text{factors}}, M)$$

converging strongly to $\pi_*M \oplus H^{\mathcal{C}_{j+1}}_{*-(j+1)}(A; M)$ and a spectral sequence with

$$E_1^{s,t} = \mathrm{Ext}^t_H(\underbrace{\tilde{B}^j N \wedge_H \ldots \wedge_H \tilde{B}^j N}_{s\,\text{factors}}, M)$$

converging conditionally to $\pi_*M \oplus H_{\mathcal{C}_{j+1}}^{*-(j+1)}(A; M)$.

Under flatness or projectivity hypotheses, we have a good description of the E^2 page of the spectral sequence.

THEOREM 3.3. *Let A be an E_n R-algebra lying over H and let $H \to F$ be a map of commutative S-algebras. For $0 \leqslant j < n$, let B^j_* be the associative π_*F-algebra $\pi_*F \oplus H^{\mathcal{C}_j}_{*-j}(A; F)$.*

(i) *If B^j_* is flat over π_*F, then the E^2 page of the homological spectral sequence of Theorem 3.2 with coefficients in F is*

$$E^2_{s,t} = \mathrm{Tor}^{B^j_*}_{s,t}(\pi_*F, \pi_*F).$$

(ii) *If B^j_* is projective over π_*F, then the E_2 page of the cohomological spectral sequence of Theorem 3.2 is*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{B^j_*}(\pi_*F, \pi_*F).$$

These spectral sequences also have multiplicative structures. For the homological spectral sequence, when $n > 1$ and $j < n-1$, the bar construction $B_\bullet(\tilde{B}^j N)$ is a simplicial (partial) non-unital E_{n-j-1}-algebra. The structure map

$$\mathcal{C}_{n-j-1}(2)_+ \wedge (B(\tilde{B}^j N) \wedge B(\tilde{B}^j N)) \longrightarrow B(\tilde{B}^j N)$$

preserves the filtration and so induces an action on the spectral sequence. Since $\mathcal{C}_{n-j-1}(2) \simeq S^{n-j-2}$, for $j = n-2$, we get an associative multiplication, and for $j < n-2$ we get an associative commutative multiplication and a $(n-j-2)$-shifted Lie bracket, which together give the structure of an $(n-j-2)$-Poisson algebra. However, since each $B_s(\tilde{B}^j N)$ is actually a (partial) E_{n-j}-algebra, the Lie bracket from the E_{n-j-1}-algebra structure is zero, and so the Lie bracket in the spectral sequence is zero. The Lie bracket therefore plays no role in spectral sequence computations here, though we do get the conclusion that it lowers total filtration in B^{j+1}_*. As we show in Section 7, the cohomology version of the spectral sequence always has an algebra structure.

THEOREM 3.4. *Let A be an E_n R-algebra lying over H and let $H \to F$ be a map of commutative S-algebras.*

(i) *If $n > 1$ and $j < n-1$, then the homological spectral sequence of Theorem* 3.2 *with coefficients in F has a natural multiplication for which the differential is a derivation. Furthermore, this multiplication converges to the multiplication on the target coming from the $E_{n-(j+1)}$-algebra structure on the iterated bar construction. Under the flatness hypothesis of Theorem* 3.3, *the multiplication on the E^2 page coincides with the usual multiplication on* $\mathrm{Tor}^{B^j_*}_{*,*}(\pi_*F, \pi_*F)$ *for the commutative algebra B^j_*.*

(ii) *For all $j < n$, the cohomological spectral sequence of Theorem* 3.2 *with coefficients in F has a natural multiplication for which the differential is a derivation. Furthermore, this multiplication converges to the multiplication on the target coming from the diagonal on the bar construction. Under the projectivity hypothesis of Theorem* 3.3, *the multiplication on the E_2 page coincides with the usual* (*Yoneda*) *multiplication on* $\mathrm{Ext}^{*,*}_{B^j_*}(\pi_*F, \pi_*F)$.

We need one more set of results from [**3**]. Section 8 of [**3**] studies the compatibility of power operations on the algebra and on the bar construction. We summarize what we need in the following theorem. In this, we write $\tilde{H}_*(A; \mathbb{F}_p)$ for the kernel of the map $H_*(A; \mathbb{F}_p) \to \mathbb{F}_p$ induced by $A \to H \to H\mathbb{F}_p$, where $H_*(A; \mathbb{F}_p)$ denotes the ordinary R-module homology of A with coefficients in $\mathbb{F}_p$ (defined as $\pi_*(A \wedge^{\mathbf{L}}_R H\mathbb{F}_p) = \mathrm{Tor}^R_*(A, H\mathbb{F}_p)$ [**10**, IV.3.2]).

THEOREM 3.5. *Let $F = H\mathbb{F}_p$, and suppose $2 \leqslant n \leqslant \infty$ and $1 < j < n$. With notation as in Theorem* 3.3, *the map $\tilde{H}_*(A; \mathbb{F}_p) \to B^j_{*+j}$ preserves the Dyer–Lashof operations defined for E_{n-j} $H\mathbb{F}_p$-algebras.*

We can generalize the above theorem by studying the interaction of the Dyer–Lashof operations with the spectral sequence in Theorem 3.2. For $n > 2$ and $0 \leqslant j < n-2$, each line in the bar construction $B(\tilde{B}^j N)$ is a (partial) E_{n-j}-algebra, and so its $\mathbb{F}_p$ homology admits additive operations Q^i and (for $p > 2$) βQ^i on elements in degree t for $2i < t + n - j - 1$ for $p > 2$ or $i < t + n - j - 1$ for $p = 2$. The operation $\beta^\epsilon Q^i$ is zero if $2i - \epsilon < t$ for $p > 2$ or $i < t$ for $p = 2$ [**6**, III.3.1]. These operations therefore act on the E^1 page of the homological spectral sequence of Theorem 3.2 (for $M = H\mathbb{F}_p$); they commute up to sign with the d_1 differential since this is an alternating sum of face maps, each of which is a map of (partial) E_{n-j-1}-algebras. (The top operations commute with the face maps because they are the restriction operation ξ_{n-j-1} and for $p > 2$ the operation ζ_{n-j-1} in the E_{n-j-1}-algebra homology operations [**6**, III.3.3.(2)].) We prove the following theorem in Section 6.

THEOREM 3.6. *Suppose $2 < n \leqslant \infty$ and $0 \leqslant j < n-2$. For $F = H\mathbb{F}_p$, the spectral sequence of Theorem* 3.2 *admits operations*

$$\beta^\epsilon Q^i : E^r_{s,t} \longrightarrow E^r_{s,t+2i(p-1)-\epsilon}, \quad \epsilon = 0,1, \ \ p > 2,$$
$$Q^i : E^r_{s,t} \longrightarrow E^r_{s,t+i} \quad p = 2$$

for $r \geqslant 1$ and

$$2i < t + n - j - 1, \quad p > 2,$$
$$i < t + n - j - 1 \quad p = 2,$$

satisfying the following properties:

(i) *for $r = 1$, $\beta^\epsilon Q^i$ is the Dyer–Lashof operation $\beta^\epsilon Q^i$ on $H_*(B_s(\tilde{B}^j N); \mathbb{F}_p)$;*

(ii) *$\beta^\epsilon Q^i$ (anti)commutes with the differential $d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ by the formula $\beta^\epsilon Q^i d_r = (-1)^\epsilon d_r \beta^\epsilon Q^i$;*

(iii) *if $x \in E^1_{s,t}$ is a permanent cycle and $\beta^\epsilon Q^i$ is defined on $E^1_{s,t}$, then the permanent cycle $\beta^\epsilon Q^i x$ represents $\beta^\epsilon Q^i x$ in B^{j+1}_*, and if $s + t > 2i - \epsilon$ for $p > 2$ or $s + t > i$ for $p = 2$, then $\beta^\epsilon Q^i x$ is hit by a differential.*

Finally, we have the following Hurewicz Theorem relating Quillen homology with R-module homology. For an H-module M, we write $H_*(A; M)$ for $\pi_*(A \wedge^{\mathbf{L}}_R M) = \mathrm{Tor}^R_*(A, M)$, the R-module homology with coefficients in M, and likewise, $H_*(X, A; M)$ for the relative homology. Thinking in terms of non-unital E_n H-algebras, the unit of the Q, Z adjunction $N \to ZQN$ induces a natural map from (reduced) homology to Quillen homology.

THEOREM 3.7 (Hurewicz Theorem). *Let M be a connective H-module and let $A \to X$ be a map in $\mathfrak{A}\mathcal{C}^{R/H}_n$ that is a q-equivalence for $q > 0$. Then the natural map from $H_*(X, A; M)$ to $H^{\mathcal{C}_n}_*(X, A; M)$ is an isomorphism for $* \leqslant q + 1$, that is, $H^{\mathcal{C}_n}_*(X, A; M) = 0$ for $* \leqslant q$ and $H_{q+1}(X, A; M) \cong H^{\mathcal{C}_n}_{q+1}(X, A; M)$.*

Proof. Using the homology version of the Universal Coefficient Theorem, it suffices to consider the case when $M = H$. We model the map $H \wedge^{\mathbf{L}}_R A \to H \wedge^{\mathbf{L}}_R X$ as a map of non-unital E_n H-algebras $N \to Y$. Looking at the filtration on the bar constructions $\tilde{B}N$ and $\tilde{B}Y$, we see that on the sth filtration level, the map $N^{(s)} \to Y^{(s)}$ is sq-connected, and for $s = 1$, the cofiber has $(q+1)$st homotopy group $H_{q+1}(X, A; H)$. (The zeroth filtration level is trivial.) Since $q > 0$, for $s > 1$, $sq \geqslant q + 1$, and so it follows that the map $\tilde{B}N$ to $\tilde{B}Y$ is a $(q+1)$-equivalence whose cofiber has π_{q+2} given by $H_{q+1}(X, A; H)$ (via the inclusion of the cofiber of $\Sigma N \to \Sigma Y$ from filtration level 1). Repeating this argument, we get that the map $\tilde{B}^n N$ to $\tilde{B}^n Y$ is a $(q+n)$-equivalence whose cofiber has π_{q+n+1} given by $H_{q+1}(X, A; H)$ (via the inclusion of the cofiber of $\Sigma \tilde{B}^{n-1} N \to \Sigma \tilde{B}^{n-1} Y$ from filtration level 1). □

4. *Obstruction theory for connective E_n algebras*

This section generalizes the work of Kriz [**16**] and Basterra [**2**] on Postnikov towers and obstruction theory on E_∞ ring spectra to the context of E_n ring spectra. We continue to work in the context of E_n R-algebras for a connective commutative S-algebra R.

We begin with the notion of a Postnikov tower. For a connective cofibrant E_n R-algebra A fix $H = H\pi_0 A$, and note that A has an essentially unique structure of an E_n R-algebra over H: An easy induction in terms of cells for a cell E_n R-algebra homotopy equivalent to A shows that A admits a map of E_n R-algebras to H which induces the identity map on π_0, and that this map is unique up to homotopy. Fixing a choice of augmentation, we can then consider towers in $\mathfrak{A}\mathcal{C}^{R/H}_n$ under A,

$$A \longrightarrow \cdots \longrightarrow A[q] \longrightarrow A[q-1] \longrightarrow \cdots \longrightarrow A[0],$$

where the map $A[0] \to H$ is a weak equivalence, $\pi_s A[q] = 0$ for $s > q$, and the map $A \to A[q]$ induces an isomorphism on homotopy groups π_s for $s \leqslant q$. We call any such tower a *Postnikov* *tower for A in $\mathfrak{AC}_n^{R/H}$*, and we say that two Postnikov towers for A in $\mathfrak{AC}_n^{R/H}$ are equivalent if they are weakly equivalent as towers in $\mathfrak{AC}_n^{R/H}\backslash A$, that is, in the category of E_n R-algebras lying over H and under A. Of course, the underlying tower of R-modules of a Postnikov tower in $\mathfrak{AC}_n^{R/H}$ is a Postnikov tower of R-modules. We have the following existence and uniqueness theorem for Postnikov towers in $\mathfrak{AC}_n^{R/H}$.

THEOREM 4.1. *Let A be a connective cofibrant E_n R-algebra over $H = H\pi_0 A$. Then A has a Postnikov tower in $\mathfrak{AC}_n^{R/H}$ and any two are weakly equivalent.*

Proof. We can construct $A[q]$ by attaching E_n R-algebra cells to kill off the higher homotopy groups; this constructs a Postnikov tower where each structure map $A \to A[q]$ is a relative cell complex. It is easy to see that any Postnikov tower is weakly equivalent to one where the tower maps $A[q] \to A[q-1]$ are fibrations, and then an easy cell-by-cell argument constructs a weak equivalence of Postnikov towers from any one where the structure maps are relative cell complexes to any one where the tower maps are fibrations. □

The R-module Postnikov tower is modeled by a tower of principal fibrations and this leads to an obstruction theory in the category of R-modules. Kriz's insight is that the same holds for E_∞ ring spectra, and we argue that the same holds for E_n R-algebras. Since H is the final object in $\mathfrak{AC}_n^{R/H}$ a principal fibration in $\mathfrak{AC}_n^{R/H}$ is one that is the pullback of a fibration whose source is weakly equivalent to H. To avoid using cofibrant and fibrant approximation in the statements below, for the purposes of this section, we will understand 'homotopy fiber' in terms of the notion of weak pullback in $\operatorname{Ho}\mathfrak{AC}_n^{R/H}$: We say that $A[q]$ (or more accurately, the map $A[q] \to A[q-1]$) is the *homotopy fiber* of a map $A[q-1] \to H \vee M$ in $\operatorname{Ho}\mathfrak{AC}_n^{R/H}$ when the square

$$\begin{array}{ccc} A[q] & \longrightarrow & H \\ \downarrow & & \downarrow \\ A[q-1] & \longrightarrow & H \vee M \end{array}$$

in $\operatorname{Ho}\mathfrak{AC}_n^{R/H}$ is a weak pullback square, that is, given any map from B to $A[q-1]$ in $\operatorname{Ho}\mathfrak{AC}_n^{R/H}$ such that the composite to $H \vee M$ factors through (the unique map to) H, there exists a compatible map from B to $A[q]$ in $\operatorname{Ho}\mathfrak{AC}_n^{R/H}$. The map from B to $A[q]$ will generally not be unique: When the homotopy fiber is constructed as the pullback along a fibration for a choice of point-set model of the map $A[q-1] \to H \vee M$ and a fibration model of the map $H \to H \vee M$, it is easy to see that the set of choices for the map $B \to A[q]$ (when one exists) has a free transitive action of $H^{-1}_{\mathcal{C}_n}(B; M) \cong H^0_{\mathcal{C}_n}(B; \Omega M)$. The following theorem in part asserts that a Postnikov tower in $\mathfrak{AC}_n^{R/H}$ can be constructed by iterated homotopy fibers of E_n R-algebra k-invariants.

THEOREM 4.2. *Let A be a connective cofibrant E_n R-algebra over $H = H\pi_0 A$. Then there exists a Postnikov tower for A in $\mathfrak{AC}_n^{R/H}$,*

$$A \longrightarrow \cdots \longrightarrow A[q] \longrightarrow A[q-1] \longrightarrow \cdots \longrightarrow A[0],$$

and a sequence of classes $k_q^n \in H^{q+1}_{\mathcal{C}_n}(A[q-1]; \pi_q A)$ such that $A[0] \to H$ is a weak equivalence and each $A[q]$ is the homotopy fiber of the map $k_q^n : A[q-1] \to H \vee \Sigma^{q+1} H\pi_q A$. Moreover, these data have the following consistency and uniqueness properties.

(i) *The natural map* $H^{q+1}_{\mathcal{C}_n}(A[q-1];\pi_q A) \to H^{q+1}(A[q-1];\pi_q A)$ *takes* k^n_q *to the* R*-module* k*-invariant* $k_q = k^0_q$.

(ii) *If* $\{A[q], k^n_q\}$ *and* $\{A'[q], k'^n_q\}$ *are two Postnikov towers for* A *in* $\mathfrak{AC}^{R/H}_n$ *constructed as iterated homotopy fibers as above, then any weak equivalence of Postnikov towers relating them sends* k^n_q *to* k'^n_q.

As a consequence of the uniqueness properties of the two theorems above, it makes sense to talk about the Postnikov tower $\{A[q]\}$ and E_n R-algebra k-invariants k^n_q for connective E_n R-algebras A that are not necessarily cofibrant, constructing them by using a cofibrant approximation.

Before proving Theorem 4.2, we state the following obstruction theoretic consequences. The identification of $A[q]$ as a homotopy fiber gives the following E_n R-algebra analog of the elementary obstruction lemma.

Corollary 4.3. *Let* A *and* B *be* E_n R*-algebras over* $H = H\pi_0 A$, *and let* $f_{q-1} : B \to A[q-1]$ *be a map in* $\mathrm{Ho}\,\mathfrak{AC}^{R/H}_n$. *Then* f_{q-1} *lifts in* $\mathrm{Ho}\,\mathfrak{AC}^{R/H}_n$ *to a map* $f_q : B \to A[q]$ *if and only if* $f^*_{q-1}k^n_q \in H^{q+1}_{\mathcal{C}_n}(B;\pi_q A)$ *is zero; if such a lift exists, then after choosing one, the set of such lifts is in one-to-one correspondence with elements of* $H^q_{\mathcal{C}_n}(B;\pi_q A)$.

Putting this together for all q, we obtain the following corollary.

Corollary 4.4. *Let* A *and* B *be* E_n R*-algebras over* $H = H\pi_0 A$, *and let* $f_q : B \to A[q]$ *be a map in* $\mathrm{Ho}\,\mathfrak{AC}^{R/H}_n$. *Then* f_q *lifts in* $\mathrm{Ho}\,\mathfrak{AC}^{R/H}_n$ *to a map* $f : B \to A$ *when an inductively defined sequence of obstructions* $o_s \in H^{s+1}_{\mathcal{C}_n}(B;\pi_s A)$ *are zero for all* $s \geqslant q+1$.

On the other hand, consider the case of an R-module A with a fixed commutative $\pi_0 R$-algebra structure on $\pi_0 A$ and $\pi_0 A$-module structure on $\pi_* A$. Suppose $A[q-1]$ has an E_n R-algebra structure over $H = H\pi_0 A$, compatible with the given $\pi_0 A$-module structure on $\pi_* A$. The R-module k-invariant k^0_q and the augmentation $A[q-1] \to H$ induce a map of R-modules

$$A[q-1] \longrightarrow H \vee \Sigma^{q+1}\pi_q A$$

such that the homotopy pullback of the inclusion of H in $H \vee \Sigma^{q+1}\pi_q A$ is weakly equivalent to $A[q]$. This map is represented by a map in $\mathrm{Ho}\,\mathfrak{AC}^{R/H}_n$ if and only if k^0_q is in the image of $H^{q+1}_{\mathcal{C}_n}(A[q-1];\pi_q A)$. Combining this observation with Theorem 4.2 above, we obtain the following corollary.

Corollary 4.5. *Let* A *be an* R*-module with a fixed commutative* $\pi_0 R$*-algebra structure on* $\pi_0 A$ *and* $\pi_0 A$*-module structure on* $\pi_* A$. *An* E_n R*-algebra structure on* $A[q-1]$ *over* $H = H\pi_0 A$ *compatible with the* $\pi_0 A$*-module structure on* $\pi_* A$ *extends to a compatible* E_n R*-algebra structure on* $A[q]$ *if and only if the* R*-module* k*-invariant* $k^0_q \in H^{q+1}(A[q-1];\pi_q A)$ *lifts to an element of* $H^{q+1}_{\mathcal{C}_n}(A[q-1];\pi_q A)$.

Since A is the homotopy inverse limit of its Postnikov tower, we obtain the following corollary.

Corollary 4.6. *Let* A *be an* R*-module with a fixed commutative* $\pi_0 R$*-algebra structure on* $\pi_0 A$ *and* $\pi_0 A$*-module structure on* $\pi_* A$. *There exists a compatible* E_n R*-algebra structure on* A *if each* R*-module* k*-invariant* $k^0_q \in H^{q+1}(A[q-1],\pi_q A)$ *lifts to an element of the inductively defined group* $H^{q+1}_{\mathcal{C}_n}(A[q-1],\pi_q A)$ *(which is defined only after lifting* k^0_{q-1}*) for all* $q \geqslant 1$.

We now proceed to prove Theorem 4.2. It is convenient to construct the $A[q]$ as cofibrant objects and the maps $A[q] \to A[q-1]$ as fibrations. At the bottom level, we construct $A[0]$ by factoring the map $A \to H$ as a cofibration $A \to A[0]$ followed by an acyclic fibration $A[0] \to H$.

Assume by induction we have constructed $A[q-1]$. Then the Hurewicz Theorem for H-modules [**10**, IV.3.6] and the Hurewicz Theorem 3.7 give us canonical isomorphisms

$$\begin{aligned} H^{\mathcal{C}_n}_{q+1}(A[q-1], A; H) &\cong H_{q+1}(A[q-1], A; H) \\ &\cong \pi_{q+1}(H \wedge_R A[q-1], H \wedge_R A) \end{aligned}$$

and tell us that $H^{\mathcal{C}_n}_s(A[q-1], A; H) = 0$ for $s \leqslant q$. The Universal Coefficient Theorem for R-modules [**10**, IV.4.5] then gives us a canonical isomorphism

$$\pi_{q+1}(H \wedge_R A[q-1], H \wedge_R A) \cong \pi_0 H \otimes_{\pi_0 R} \pi_{q+1}(A[q-1], A) \cong \pi_0 A \otimes_{\pi_0 R} \pi_q A$$

and the Universal Coefficient Theorem 3.1 gives us a canonical isomorphism

$$\begin{aligned} H^{q+1}_{\mathcal{C}_n}(A[q-1], A; \pi_q A) &\cong \mathrm{Hom}_{\pi_0 A}(\pi_{q+1}(H \wedge_R A[q-1], H \wedge_R A), \pi_q A) \\ &\cong \mathrm{Hom}_{\pi_0 A}(\pi_0 A \otimes_{\pi_0 R} \pi_q A, \pi_q A) \\ &\cong \mathrm{Hom}_{\pi_0 R}(\pi_q A, \pi_q A). \end{aligned}$$

We let ℓ^n_q be the element of $H^{q+1}_{\mathcal{C}_n}(A[q-1], A; \pi_q A)$ that corresponds to the identity element of $\mathrm{Hom}_{\pi_0 R}(\pi_q A, \pi_q A)$, and we let k^n_q be its image in $H^{q+1}_{\mathcal{C}_n}(A[q-1]; \pi_q A)$. Note that by construction, the natural map $H^{q+1}_{\mathcal{C}_n}(A[q-1]; \pi_q A) \to H^{q+1}(A[q-1]; \pi_q A)$ takes k^n_q to the R-module k-invariant k^0_q.

Let $M = \Sigma^{q+1} H\pi_q A$, and choose a fibrant approximation $(H \vee M)_f$ of $H \vee M$ in $\mathfrak{AC}^{H/H}_n$. Since we assumed A and $A[q-1]$ are cofibrant, a representative for k^n_q is then a map $\kappa\colon A[q-1] \to (H \vee M)_f$ in $\mathfrak{AC}^{R/H}_n$ and a representative for ℓ^n_q is a homotopy in $\mathfrak{AC}^{R/H}_n$ from the composite map $A \to (H \vee M)_f$ to the trivial map $A \to H \to (H \vee M)_f$. Factoring $H \to (H \vee M)_f$ as an acyclic cofibration $H \to H'$ and a fibration $H' \to (H \vee M)_f$, and applying the homotopy lifting property, we obtain a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & H' \\ \downarrow & & \downarrow \\ A[q-1] & \xrightarrow[\kappa]{} & (H \vee M)_f. \end{array}$$

We form the homotopy fiber F of k^n_q as the pullback of $H' \to (H \vee M)_f$ along κ, and we construct $A[q]$ by factoring the induced map $A \to F$ as a cofibration $A \to A[q]$ followed by an acyclic fibration $A[q] \to F$. By construction, $\pi_q A[q]$ is canonically isomorphic to $\pi_q A$ and the map $A \to A[q]$ induces this canonical isomorphism. This completes the inductive step and the construction of the Postnikov tower as a tower of iterated homotopy fibers.

The consistency statement (i) follows from the construction of the k^n_q above; thus, it remains to prove the uniqueness statement (ii). A zigzag of weak equivalences of Postnikov towers in particular restricts to a zigzag of weak equivalences of pairs

$$(A[q-1], A) \simeq (A'[q-1], A)$$

in $\mathfrak{AC}^{R/H}_n$, which induces an isomorphism on $H^*_{\mathcal{C}_n}$. Naturality of the Hurewicz homomorphism implies that the following diagram commutes:

$$\begin{array}{ccccc} H^{q+1}_{\mathcal{C}_n}(A'[q-1], A; \pi_q A) & \xrightarrow{\cong} & H^{q+1}(A'[q-1], A; \pi_q A) & \xrightarrow{\cong} & \mathrm{Hom}_{\pi_0 R}(\pi_q A, \pi_q A) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \mathrm{id} \\ H^{q+1}_{\mathcal{C}_n}(A[q-1], A; \pi_q A) & \xrightarrow[\cong]{} & H^{q+1}(A[q-1], A; \pi_q A) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\pi_0 R}(\pi_q A, \pi_q A) \end{array}$$

where the right vertical map is the identity. The hypothesis that $A[q]$ and $A'[q]$ are the homotopy fibers of k^n_q and k'^n_q implies that the images of k^n_q and k'^n_q in $H^{q+1}_{\mathcal{C}_n}(A;\pi_q A)$ are zero, which implies that k^n_q is in the image of $H^{q+1}_{\mathcal{C}_n}(A[q-1],A;\pi_q A)$ and k' is in the image of $H^{q+1}_{\mathcal{C}_n}(A'[q-1],A;\pi_q A)$. Choosing a lift for each, we can identify the corresponding element of $\mathrm{Hom}_{\pi_0 R}(\pi_q A,\pi_q A)$ as the map on π_q induced by the map $A\to A[q]$ for k^n_q and the map $A\to A'[q]$ for k'^n_q. Thus, both lifts must correspond to the identity map. It follows that the zigzag of weak equivalences sends k'^n_q to k^n_q. This completes the proof of Theorem 4.2.

5. *Proof of the Main Theorem*

We now apply the obstruction theory of the previous section (with $R=S_{(p)}$ and $H=H\mathbb{Z}_{(p)}$) to prove the main theorem. We fix a prime p and argue by induction up the Postnikov tower as in Corollary 4.6.

As a base case, we have

$$\mathrm{MU}_{(p)}[1]=\mathrm{MU}_{(p)}[0]\cong H\mathbb{Z}_{(p)}\quad\text{and}\quad \mathrm{BP}[1]=\mathrm{BP}[0]\cong H\mathbb{Z}_{(p)}$$

are E_∞ ring spectra and isomorphic. In particular, we have maps of E_4 ring spectra

$$\mathrm{MU}_{(p)}\longrightarrow \mathrm{BP}[1]\quad\text{and}\quad \mathrm{BP}[1]\longrightarrow \mathrm{MU}_{(p)}[1]$$

over $H\mathbb{Z}_{(p)}$. We assume by induction that $\mathrm{BP}[2n-1]$ is an E_4 ring spectrum and that we have constructed maps of E_4 ring spectra

$$\mathrm{MU}_{(p)}\longrightarrow \mathrm{BP}[2n-1]\quad\text{and}\quad \mathrm{BP}[2n-1]\longrightarrow \mathrm{MU}_{(p)}[2n-1]$$

over $H\mathbb{Z}_{(p)}$. We then argue that we can extend the E_4 ring structure on $\mathrm{BP}[2n-1]$ to an E_4 ring structure on $\mathrm{BP}[2n+1]$ and that we can obtain maps of E_4 ring spectra

$$\mathrm{MU}_{(p)}\longrightarrow \mathrm{BP}[2n+1]\quad\text{and}\quad \mathrm{BP}[2n+1]\longrightarrow \mathrm{MU}_{(p)}[2n+1],$$

over $H\mathbb{Z}_{(p)}$ which extend the previous pair of maps. This then will prove that BP has the structure of an E_4 ring spectrum.

As part of the inductive argument, we will have to prove the following lemma about the Quillen homology and cohomology of the Postnikov sections.

Lemma 5.1. *Let $A=\mathrm{MU}_{(p)}[2n+1]$ or $A=\mathrm{BP}[2n+1]$. In degrees at most $2n+1$, the E_4 Quillen homology $H^{\mathcal{C}_4}_*(A;\mathbb{Z}_{(p)})$ and E_4 Quillen cohomology $H^*_{\mathcal{C}_4}(A;\mathbb{Z}_{(p)})$ are concentrated in even degrees and torsion free.*

As a consequence of this lemma, we see that the E_4 Quillen cohomology of $\mathrm{MU}_{(p)}$ and of BP is concentrated in even degrees for the E_4 structure on BP constructed above. If BP' denotes BP with any E_4 structure, obstruction theory (Corollary 4.4) then says that there exists a map of E_4 ring spectra BP to BP' over $H\mathbb{Z}_{(p)}$. Since this map preserves the unit, as a self-map on the underlying spectrum BP, it is an isomorphism on $H^0(\mathrm{BP};\mathbb{F}_p)$, and so is an isomorphism on $H^*(\mathrm{BP};\mathbb{F}_p)$, and so is an isomorphism in the stable category. The map $\mathrm{BP}\to\mathrm{BP}'$ is therefore a weak equivalence of E_4 ring spectra. This shows that any two E_4 ring structures on BP are isomorphic in the homotopy category of E_4 ring spectra, finishing the proof of Theorem 1.1.

Before proceeding with the argument, we note that as a byproduct of the outline above, we get maps of E_4 ring spectra $\mathrm{MU}_{(p)}\to\mathrm{BP}$ and $\mathrm{BP}\to\mathrm{MU}_{(p)}$. (Such maps must also exist by Lemma 5.1 and obstruction theory Corollary 4.4.) The composite map $\mathrm{BP}\to\mathrm{BP}$ is a weak equivalence, and so pre-composing its inverse with the map $\mathrm{BP}\to\mathrm{MU}_{(p)}$, the composite self-map of $\mathrm{MU}_{(p)}$ is now an idempotent E_4 ring map, factoring through BP, and splitting off BP as the unit summand. The construction gives no hint or details about which idempotent map of $\mathrm{MU}_{(p)}$ this is. We hope to return to this question in a future paper.

We now begin the proof of the inductive step. We start with the inductively constructed map of E_4 ring spectra $\mathrm{BP}[2n-1] \to \mathrm{MU}_{(p)}[2n-1]$. Ignoring the E_4 structure, this map extends to a map in the stable category from BP to $\mathrm{MU}_{(p)}$; since this extension is an isomorphism on $H^0(-;\mathbb{F}_p)$, it is split by a map in the stable category $s : \mathrm{MU}_{(p)} \to \mathrm{BP}$. Choosing such a splitting, we have that the spectrum-level k-invariant

$$k^0_{2n}(\mathrm{BP}) \in H^{2n+1}(\mathrm{BP}[2n-1], \pi_{2n}\mathrm{BP})$$

for BP is the image of the spectrum-level k-invariant

$$k^0_{2n}(\mathrm{MU}_{(p)}) \in H^{2n+1}(\mathrm{MU}_{(p)}[2n-1], \pi_{2n}\mathrm{MU}_{(p)})$$

under the map induced by $\mathrm{BP}[2n-1] \to \mathrm{MU}_{(p)}[2n-1]$ and the map $s_* : \pi_*\mathrm{MU}_{(p)} \to \pi_*\mathrm{BP}$. Since $\mathrm{MU}_{(p)}$ is an E_4 ring spectrum, its spectrum-level k-invariant $k^0_{2n}(\mathrm{MU}_{(p)})$ lifts to its E_4 ring spectrum k-invariant

$$k^4_{2n}(\mathrm{MU}_{(p)}) \in H^{2n+1}_{\mathcal{C}_4}(\mathrm{MU}_{(p)}[2n-1], \pi_{2n}\mathrm{MU}_{(p)}).$$

The image of this under the E_4 ring map $\mathrm{BP}[2n-1] \to \mathrm{MU}_{(p)}[2n-1]$ and the map $s_* : \pi_*\mathrm{MU}_{(p)} \to \pi_*\mathrm{BP}$ provides a lift to $H^{2n+1}_{\mathcal{C}_4}(\mathrm{BP}[2n-1], \pi_{2n}\mathrm{BP})$ of $k^0_{2n}(\mathrm{BP})$. This constructs $\mathrm{BP}[2n+1] = \mathrm{BP}[2n]$ as an E_4 ring spectrum.

Looking at the construction in the previous paragraph, it does not follow immediately that the E_4 ring map $\mathrm{BP}[2n-1] \to \mathrm{MU}_{(p)}[2n-1]$ extends to an E_4 ring map $\mathrm{BP}[2n+1] \to \mathrm{MU}_{(p)}[2n+1]$. However, Lemma 5.1 stated above implies that the obstructions in

$$H^{2n+1}_{\mathcal{C}_4}(\mathrm{BP}[2n+1], \pi_{2n}\mathrm{MU}_{(p)}) \quad \text{and} \quad H^{2n+1}_{\mathcal{C}_4}(\mathrm{MU}_{(p)}[2n+1], \pi_{2n}\mathrm{BP})$$

to extending the E_4 ring maps

$$\mathrm{BP}[2n-1] \longrightarrow \mathrm{MU}_{(p)}[2n-1] \quad \text{and} \quad \mathrm{MU}_{(p)} \longrightarrow \mathrm{BP}[2n-1]$$

to E_4 ring maps

$$\mathrm{BP}[2n+1] \longrightarrow \mathrm{MU}_{(p)}[2n+1] \quad \text{and} \quad \mathrm{MU}_{(p)} \longrightarrow \mathrm{BP}[2n+1]$$

are both zero. Thus, the completion of the inductive step reduces to Lemma 5.1.

The rest of the section is devoted to the proof of Lemma 5.1. The role of the inductively hypothesized E_4 ring map $\mathrm{MU}_{(p)} \to \mathrm{BP}[2n-1]$ is that we need it to prove the part of the lemma concerned with BP. The part of the lemma concerned with $\mathrm{MU}_{(p)}$ is independent of any facts about BP, and so provides the extension $\mathrm{MU}_{(p)} \to \mathrm{BP}[2n+1]$. A rigorous organization would be to prove the result for $\mathrm{MU}_{(p)}$ first and then go back and prove the result for BP; however, to avoid needless repetition, we will do the argument all at once. In fact, since the Thom isomorphism gives a weak equivalence of augmented E_∞ (and hence E_4) $H\mathbb{Z}_{(p)}$-algebras

$$H\mathbb{Z}_{(p)} \wedge \mathrm{MU} \simeq H\mathbb{Z}_{(p)} \wedge \mathrm{BU}_+,$$

the lemma for MU follows from Singer's computation of the cohomology of $B^4\mathrm{BU}$ [**26**] (see [**1**, 4.7] for an easier computation of this special case). Nevertheless, we must go through the computation for MU to obtain the computation for BP.

For the proof of the lemma, we need a fact about the Dyer–Lashof operations on MU. Let a_s denote the polynomial generator in $H_{2s}(\mathrm{MU};\mathbb{F}_p)$, which under the Thom isomorphism corresponds to the standard generator b_s in $H_{2s}(\mathrm{BU};\mathbb{F}_p)$. According to [**18**, IX.7.4.(i)] and [**15**, Theorem 6], we have

$$Q^{s+1}a_s = a_{s+(s+1)(p-1)} + \text{decomposables} \quad p > 2,$$
$$Q^{2s+2}a_s = a_{s+(s+1)} + \text{decomposables} \quad p = 2.$$

For convenience, abbreviate this operation as $\mathcal{Q}$: On an element x in degree $2s$,

$$\mathcal{Q}x = Q^{s+1}x \quad (p > 2), \quad \mathcal{Q}x = Q^{2s+2}x \quad (p = 2);$$

this is a well-defined operation on the homology of E_4 ring spectra. For $\mathrm{BP}[2n+1]$, when $n \geqslant p-1$, we have an element ξ_1 (for $p > 2$) or ξ_1^2 (for $p = 2$) in $H_{2p-2}(\mathrm{BP}[2n+1]; \mathbb{F}_p)$ mapping to the correspondingly named element in $\mathfrak{A}_{2p-2} = H_{2p-2}(H\mathbb{F}_p; \mathbb{F}_p)$. Since the map from $\mathrm{BP}[2n+1]$ to $H\mathbb{F}_p$ is an E_4 ring map, we can compute the Dyer–Lashof operations on this class from the Dyer–Lashof operations in $H_*(H\mathbb{F}_p; \mathbb{F}_p)$, computed in [**6**, III.2.2–3]. In particular, we have that in the range of dimensions where $H_*(\mathrm{BP}[2n+1]; \mathbb{F}_p) = H_*(BP; \mathbb{F}_p)$ (that is, $* \leqslant 2n+1$), $H_*(\mathrm{BP}[2n+1]; \mathbb{F}_p)$ is the polynomial algebra on generators $\xi_1, \mathcal{Q}\xi_1, \mathcal{Q}^2\xi_1, \ldots$ for $p > 2$ or $\xi_1^2, \mathcal{Q}\xi_1^2, \mathcal{Q}^2\xi_1^2, \ldots$ for $p = 2$.

To take advantage of the Dyer–Lashof operation $\mathcal{Q}$, we reorganize the polynomial generators of $H_*(\mathrm{MU}; \mathbb{F}_p)$ and $H_*(\mathrm{BP}[2n+1]; \mathbb{F}_p)$ as follows. For $H_*(\mathrm{MU}; \mathbb{F}_p)$, let $x_{0,0}^{\mathrm{MU}} = a_{p-1}$, and choose elements $x_{i,j}^{\mathrm{MU}}$ for $i \geqslant 1$, $j \geqslant 0$ in $H_{2s_{i,j}}(\mathrm{MU}; \mathbb{F}_p)$ such that

(i) $x_{i,j+1}^{\mathrm{MU}} = \mathcal{Q}x_{i,j}^{\mathrm{MU}}$ (so $s_{i,j+1} = p(s_{i,j}+1) - 1$), and
(ii) $H_*(\mathrm{MU}; \mathbb{F}_p)$ is polynomial on $x_{i,j}^{\mathrm{MU}}$, $i \geqslant 0, j \geqslant 0$.

For example, having chosen $x_{0,0}^{\mathrm{MU}}, \ldots, x_{k,0}^{\mathrm{MU}}$, then $x_{k+1,0}^{\mathrm{MU}}$ can be chosen as a_ℓ for the smallest positive integer ℓ not among the numbers $s_{i,j}$ for $0 \leqslant i \leqslant k, j \geqslant 0$. By slight abuse, we will regard the elements $x_{i,j}^{\mathrm{MU}}$ with $2s_{i,j} \leqslant 2n+1$ as elements of $H_*(\mathrm{MU}_{(p)}[2n+1]; \mathbb{F}_p)$; then we have that in degrees at most $2n+1$, $H_*(\mathrm{MU}_{(p)}[2n+1]; \mathbb{F}_p)$ is polynomial on the elements $x_{i,j}^{\mathrm{MU}}$ with $i \geqslant 0, j \geqslant 0$, and $2s_{i,j} \leqslant 2n+1$.

For $\mathrm{BP}[2n+1]$, when $n > p-1$, we take $x_{0,0}^{\mathrm{BP}}$ to be ξ_1 for $p > 2$ or ξ_1^2 for $p = 2$, and $x_{0,j+1}^{\mathrm{BP}} = \mathcal{Q}x_{0,j}^{\mathrm{BP}}$. Then in degrees at most $2n+1$, $H_*(\mathrm{BP}[2n+1]; \mathbb{F}_p)$ is polynomial on the elements $x_{0,j}^{\mathrm{BP}}$ with $j \geqslant 0$ and $2s_{0,j} \leqslant 2n+1$ (whether or not $n > p-1$). Looking at cohomology, we see that the map $\mathrm{MU}_{(p)} \to H\mathbb{F}_p$ sends $a_{p-1} = x_{0,0}^{\mathrm{MU}}$ to ξ_1 or ξ_1^2, and so it follows that the given map $\mathrm{MU}_{(p)} \to \mathrm{BP}[2n+1]$ sends $x_{0,0}^{\mathrm{MU}}$ to $x_{0,0}^{\mathrm{BP}}$ when $n > p-1$.

We are now ready to start the proof of the lemma. Let A be one of $\mathrm{MU}_{(p)}[2n+1]$ or $\mathrm{BP}[2n+1]$ and write $x_{i,j}$ for the generator $x_{i,j}^{\mathrm{MU}}$ or $x_{i,j}^{\mathrm{BP}}$ above, where in the case of BP, we understand $i = 0$. The lemma concerns Quillen homology and cohomology with coefficients in $\mathbb{Z}_{(p)}$ and we will use the Universal Coefficient Theorem 3.1 to obtain this information from the Quillen homology with coefficients in $\mathbb{F}_p$. We compute

$$B_*^q = B_*^q(A) = \mathbb{F}_p \oplus H_{*-q}^{\mathcal{C}_q}(A; \mathbb{F}_p)$$

in degrees at most $2n+1$ for $q = 1, 2, 3, 4$ inductively as follows.

To compute B_*^1, we apply the bar construction spectral sequence (Theorem 3.2). Looking at the E^1 page and its differential, we see that the E^2 page for computing B_*^1 is the exterior algebra on generators $\sigma x_{i,j}$ (in bidegree $1, 2s_{i,j}$) in internal degrees $t \leqslant 2n+1$. Because the sequence is multiplicative (Theorem 3.4.(i)), there can be no non-zero differentials starting in these degrees, and therefore also no non-zero differentials hitting these degrees. Passing to E^∞, we see that the associated graded of B_*^1 is isomorphic to this exterior algebra in degrees at most $2n+2$. Because B_*^1 is a Hopf algebra with its generators (in this degree range) primitive and in odd degree (by Theorem 3.4.(ii)), it follows that B_*^1 is this exterior algebra in degrees at most $2n+2$.

Applying the bar construction spectral sequence again, we see that the E^2 page for computing B_*^2 is the divided power algebra on generators $\sigma^2 x_{i,j}$ (in bidegree $1, 2s_{i,j}+1$) for $t \leqslant 2n+2$. Equivalently, letting $\gamma_{p^k}(\sigma x_{i,j})$ denote the element of $E^2_{p^k, 2p^k(2s_{i,j}+1)}$ represented by

$$\underbrace{\langle \sigma x_{i,j} \mid \cdots \mid \sigma x_{i,j} \rangle}_{p^k \text{ factors}} \in B_{p^k}(B_*^1) = B_*^1 \otimes \cdots \otimes B_*^1,$$

then $\sigma^2 x_{i,j} = \gamma_1(\sigma x_{i,j})$ and $E^2_{*,*}$ is the tensor product of truncated polynomial algebras

$$\bigotimes_{i,j,k} \mathbb{F}_p[\gamma_{p^k}(\sigma x_{i,j})]/(\gamma_{p^k}(\sigma x_{i,j}))^p$$

in internal degrees $t \leqslant 2n+2$. Since this is concentrated in even degrees, nothing in internal degree $t \leqslant 2n+2$ can be hit by a non-zero differential, and nothing in total degree at most $2n+3$ can have a non-zero differential. Thus, in total degrees at most $2n+3$, $E^2 = E^\infty$, and we can identify the associated graded of B^2_* in these degrees as the tensor product of truncated polynomial algebras above. Applying Theorem 3.5 and using the fact that $\mathcal{Q}x_{i,j} = x_{i,j+1}$, we see that in B^2_*,

$$(\sigma^2 x_{i,j})^p = Q^s \sigma^2 x_{i,j} = \sigma^2 Q^s x_{i,j} = \sigma^2 x_{i,j+1}$$

(for $s = s_{i,j}+1$ if $p>2$ or $s = 2s_{i,j}+2$ if $p=2$). In the case when $A = \mathrm{MU}_{(p)}[2n+1]$, because A is an E_∞ ring spectrum, we have all Dyer–Lashof operations, and applying Theorem 3.6, we see more generally that

$$(\gamma_{p^k}(\sigma x_{i,j}))^p = \gamma_{p^k}(\sigma x_{i,j+1}).$$

It follows in this case that in degrees at most $2n+3$, B^2_* is the polynomial algebra on generators $z_{i,k} = \gamma_{p^k}(\sigma x_{i,0})$ for $i \geqslant 0, k \geqslant 0$. In the case when $A = \mathrm{BP}[2n+1]$, we use the map $B^2_*(\mathrm{MU}_{(p)}[2n+1]) \to B^2_*$. Since this map sends $x^{\mathrm{MU}}_{0,0}$ to $x_{0,0}$, it sends $x^{\mathrm{MU}}_{0,j}$ to $x_{0,j}$, and it sends $\gamma_{p^k}(\sigma x^{\mathrm{MU}}_{0,j})$ to $\gamma_{p^k}(\sigma x_{0,j})$. Thus, it again follows that

$$(\gamma_{p^k}(\sigma x_{0,j}))^p = \gamma_{p^k}(\sigma x_{0,j+1}).$$

and that in degrees at most $2n+3$, B^2_* is the polynomial algebra on generators $z_{0,k} = \gamma_{p^k}(\sigma x_{i,0})$ for $k \geqslant 0$.

We compute B^3_* just as B^1_* and see that in degrees at most $2n+4$, it is an exterior algebra on odd degree generators. The E^2 page for computing B^4_* is concentrated in even degrees for degrees at most $2n+5$, and therefore B^4_* is as well. Hence, $H^{\mathcal{C}_4}_*(A;\mathbb{F}_p)$ is concentrated in even degrees for degrees at most $2n+1$. It follows that $H^{\mathcal{C}_4}_*(A;\mathbb{Z}_{(p)})$ is concentrated in even degrees for degrees at most $2n+1$ and is torsion free in degrees $< 2n+1$ and hence in degrees at most $2n+1$. Finally, we conclude that $H^*_{\mathcal{C}_4}(A;\mathbb{Z}_{(p)})$ is concentrated in even degrees and torsion free for degrees at most $2n+1$. (In fact, we see it is also torsion free in degree $2n+2$.) This completes the proof of Lemma 5.1 and therefore also the inductive step.

6. *Proof of Theorem* 3.6

This section proves Theorem 3.6, which explains how the operations fit into the bar construction spectral sequence of Theorem 3.2. The $(j+1)$, times iterated bar construction on an E_n algebra is the geometric realization of a simplicial (partial) E_{n-j-1} algebra; the spectral sequence with coefficients in $H\mathbb{F}_p$ arises from the simplicial filtration on the geometric realization of a simplicial (partial) E_{n-j-1} $H\mathbb{F}_p$-algebra. Theorem 3.6 is a special case of the following theorem, which fits the Dyer–Lashof operations into general spectral sequences of this type.

Theorem 6.1. *Let $A_\bullet$ be a proper simplicial* (*partial*) *E_n $H\mathbb{F}_p$-algebra for $2 \leqslant n \leqslant \infty$, and let $A = |A_\bullet|$ be its geometric realization. The spectral sequence*

$$E^1_{s,t} = \pi_t A_s$$

converging strongly to $\pi_{s+t}A$ admits operations

$$\beta^\epsilon Q^i : E^r_{s,t} \longrightarrow E^r_{s,t+2i(p-1)-\epsilon}, \quad \epsilon = 0,1, \ \ p>2,$$
$$Q^i : E^r_{s,t} \longrightarrow E^r_{s,t+i}, \quad p=2$$

for $r \geqslant 1$ and

$$2i < t+n-1, \quad p>2,$$
$$i < t+n-1, \quad p=2.$$

If $n<\infty$ and the $(n-1)$-shifted Lie bracket is zero on each $\pi_* A_s$, then there are also 'top' operations $Q^{(n-1+t)/2}$ and $\beta Q^{(n-1+t)/2}$ for $p>2$ ($n-1+t$ even) and Q^{n-1+t} for $p=2$ (all t) defined on $E^r_{s,t}$.

These operations satisfy the following properties.

(i) *On $E^1_{s,*}$, the operation $\beta^\epsilon Q^i$ is the Dyer–Lashof operation $\beta^\epsilon Q^i$ on $\pi_* A_s$ or in the case of the top operations, ξ_{n-1} (for $Q^{(n-1+t)/2}$, $p>2$ or Q^{n-1+t}, $p=2$) or ζ_{n-1} (for $\beta Q^{(n-1+t)/2}$).*

(ii) *The operation $\beta^\epsilon Q^i$ (anti)commutes with the differential $d_r: E^r_{s,t}\to E^r_{s-r,t+r-1}$ by the formula $\beta^\epsilon Q^i d_r = (-1)^\epsilon d_r \beta^\epsilon Q^i$.*

(iii) *If $x\in E^1_{s,t}$ is a permanent cycle and $\beta^\epsilon Q^i$ is defined on $E^1_{s,t}$, then the permanent cycle $\beta^\epsilon Q^i x$ represents $\beta^\epsilon Q^i x$ in $\pi_* A$, or when $s=0$ and $\beta^\epsilon Q^i$ is a top operation, ξ_{n-1} or ζ_{n-1}. If $s+t>2i-\epsilon$ for $p>2$ or $s+t>i$ for $p=2$, then $\beta^\epsilon Q^i x$ is hit by a differential.*

In the statement 'proper' means that the union of the degeneracies is a cofibration of some sort and just ensures that the geometric realization has the correct homotopy type; if $A_\bullet$ is not proper, the spectral sequence actually converges to the homotopy groups of the thickened realization. We can always arrange for $A_\bullet$ to be proper, replacing it with a levelwise equivalent object if necessary.

For the proof of Theorem 6.1, it is convenient to have $A_\bullet$ be 'Reedy fibrant', meaning that at each level s, the map from A_s to the limit over the face maps of A_i, $i<s$ is a fibration (more on this in the proof of Lemma 6.2 below). We can arrange this, without loss of generality, by replacing $A_\bullet$ by a weakly equivalent simplicial E_n $H\mathbb{F}_p$-algebra if necessary. We now assume this and begin the proof of Theorem 6.1.

We are taking the sign convention that on $E^1_{s,t}$, the d_1 differential is

$$d_1 = (-1)^t \sum_{i=0}^{s} (-1)^i \partial_i,$$

where ∂_i denotes the ith face map in the simplicial structure. Since each face map preserves the E_n $H\mathbb{F}_p$-algebra Dyer–Lashof operations and these operations (except the top operations) are additive, it follows that the d_1 differential commutes up to sign with these operations (except the top operations). Under the hypothesis that the Lie bracket is zero, the top operations are also additive and the d_1 differential commutes up to sign with these as well. It follows that the operations are well defined on the E^2 page. Since part (i) of the statement holds by definition and inductively part (ii) of the statement for the d_r differential shows that the operations are well defined on the E^{r+1} page, it suffices to verify parts (ii) and (iii).

Each element in $E^r_{s,t}$ for $r\geqslant 2$ is represented by an element of $E^1_{s,t}$ with the property that the face maps $\partial_1,\ldots,\partial_s$ are zero, this subset is the *normalized* E^1 page. Now given such an element x that survives to $E^r_{s,t}$, we can represent x as a map of $H\mathbb{F}_p$-modules

$$\bar{x}_s : S^t_c \longrightarrow A_s,$$

where S^t_c denotes a cofibrant t-sphere in $H\mathbb{F}_p$-modules, $S^t_c \simeq \Sigma^t H\mathbb{F}_p$. Moreover, since we have assumed that $A_\bullet$ is Reedy fibrant, we can choose $\bar{x}_s$ such that for $1\leqslant i\leqslant t$, $\partial_i \bar{x}_s$ is the (point-set) trivial map $S^t_c\to A_{s-1}$; we prove the following lemma at the end of the section.

LEMMA 6.2. *Let $x\in\pi_t A_s$ such that $\partial_i x = 0$ in $\pi_t A_{s-1}$ for all $1\leqslant i\leqslant s$. Then x can be represented by a map $\bar{x}: S^t_c\to A_s$ such that $\partial_i\circ\bar{x} = *$ for all $1\leqslant i\leqslant s$.*

We also use a relative version of this result, which is proved below.

Lemma 6.3. *Let $\bar{x} : S^t_c \to A_s$ be a map such that $\partial_i \circ \bar{x} = *$ for all $1 \leqslant i \leqslant s$. If $\bar{x}$ is null-homotopic in A_s, then there exists a null-homotopy $\bar{y} : CS^t_c \to A_s$ such that $\partial_i \circ \bar{y} = *$ for all $1 \leqslant i \leqslant s$.*

The map $\partial_0 \circ \bar{x}_s : S^t_c \to A_{s-1}$ represents $\partial_0 x$, which is the d_1 differential of x up to sign. Since x is in $E^r_{s,t}$, $r \geqslant 2$, we must have that $\partial_0 \circ \bar{x}_s$ is null-homotopic. Applying the relative version of the lemma, we can choose a null-homotopy

$$\bar{x}_{s-1} : CS^t_c \longrightarrow A_{s-1}$$

such that $\partial_i \circ \bar{x}_{s-1} = *$ for $1 \leqslant i \leqslant s-1$. Since $\partial_0\partial_0 = \partial_0\partial_1$, the restriction of $\partial_0 \circ \bar{x}_{s-1}$ to $S^t_c \subset CS^t_c$ is the trivial map, and we can consider $\partial_0 \circ \bar{x}_{s-1}$ as a map from S^{t+1}_c to A_{s-2}. This map represents the d_2 differential of x up to sign. If $r > 2$, we have that this map is null-homotopic and we can choose

$$\bar{x}_{s-2} : CS^{t+1}_c \longrightarrow A_{s-2}.$$

We continue in this way to construct $\bar{x}_s, \ldots, \bar{x}_m$, where $m = \max(s-r+1, 0)$. In the case when $r \geqslant s+1$, x is a permanent cycle; in the other case, $r \leqslant s$, we write $\bar{y}$ for the map $\partial_0 \circ \bar{x}_m$ viewed as a map from S^{t+r-1}_c to A_{s-r}, representing $d_r x$ up to sign.

The previous paragraph implicitly constructs a simplicial $H\mathbb{F}_p$-module $X_\bullet$ and a map of simplicial $H\mathbb{F}_p$-modules $X_\bullet \to A_\bullet$, where the non-degenerate part of X_q is as follows.

(i) The trivial $H\mathbb{F}_p$-module $*$ if $q < s-r$ or $q > s$.
(ii) The $H\mathbb{F}_p$-module S^{t+r-1}_c if $q = s-r \geqslant 0$.
(iii) The $H\mathbb{F}_p$-module $CS^{t-q+s-1}_c$ if $q > s-r$ and $q < s$.
(iv) The $H\mathbb{F}_p$-module S^t_c if $q = s$.

On each of these, the maps ∂_i for $i > 0$ are trivial, and the map ∂_0 is the composite

$$CS^{t-q+s-1}_c \longrightarrow S^{t-q+s}_c \subset CS^{t-q+s}_c = CS^{t-(q-1)+s-1}_c$$

for $s-r+1 < q < s$, the inclusion $S^t_c \to CS^t_c$ for $q = s$ and the map $CS^{t+r-2}_c \to S^{t+r-1}_c$ for $q = s-r+1 > 0$. The maps $\bar{x}_q$ and (when it is defined) $\bar{y}$ define a map of simplicial $H\mathbb{F}_p$-modules $\bar{x} : X_\bullet \to A_\bullet$.

We now split up the case $r \leqslant s$ (to prove part (ii)) from the permanent cycle case $r \geqslant s+1$ (to prove part (iii)). In the case $r \leqslant s$, the spectral sequence for computing the homotopy groups of $X_\bullet$ becomes zero at E^{r+1}, and so the geometric realization of $X_\bullet$ is contractible. Since $A_\bullet$ is a simplicial E_n $H\mathbb{F}_p$-algebra, we obtain a map of simplicial $H\mathbb{F}_p$-modules

$$\mathcal{C}_n(p)_+ \wedge_{\Sigma_p} X^{(p)}_\bullet \longrightarrow A_\bullet$$

and we look at the associated spectral sequence. Let a denote the fundamental class of S^t_c in $\pi_t X_s$, and let b denote the fundamental class of S^{t+r-1}_c in $\pi_{t+r-1} X_{s-r}$. Since a maps to x in $\pi_s A_t$ and b maps to $d_r x$ up to sign, it suffices to show that in the spectral sequence for $\mathcal{C}_n(p)_+ \wedge_{\Sigma_p} X^{(p)}_\bullet$, the d_r differential takes an operation on a to the corresponding operation on b up to the appropriate sign as in part (ii) of the statement of the theorem. For this, we use the comparison maps

$$S^\infty_+ \wedge_{C_p} X^{(p)}_\bullet \longrightarrow \mathcal{C}_\infty(p)_+ \wedge_{C_p} X^{(p)}_\bullet \longrightarrow \mathcal{C}_\infty(p)_+ \wedge_{\Sigma_p} X^{(p)}_\bullet \longleftarrow \mathcal{C}_n(p)_+ \wedge_{\Sigma_p} X^{(p)}_\bullet,$$

where C_p denotes the cyclic group of order p. In the degree range, we are looking at for the operations, the map

$$\mathcal{C}_n(p)_+ \wedge_{\Sigma_p} (S^{t+r-1}_c)^{(p)} \longrightarrow \mathcal{C}_\infty(p)_+ \wedge_{\Sigma_p} (S^{t+r-1}_c)^{(p)}$$

is an isomorphism or nearly an isomorphism depending on n; see [7, III.5.2–3]. In the degree of the top operations ξ_{n-1} and ζ_{n-1}, it is an isomorphism, and otherwise, by working with $n-1$ in place of n if necessary, we can make it an isomorphism. This reduces the problem to the study of the spectral sequence for $\mathcal{C}_\infty(p)_+ \wedge_{\Sigma_p} X_\bullet^{(p)}$ and from there to the study of the spectral sequence for $S^\infty_+ \wedge_{C_p} X_\bullet^{(p)}$. The advantage of the latter spectral sequence is that $S^\infty_+ \wedge_{C_p} X_\bullet^{(p)}$ is canonically a simplicial object in the category of CW $H\mathbb{F}_p$-modules, and we can compute the spectral sequence by looking at the cellular chain complex. The remainder of the proof of part (ii) is an easy argument with the chain-level operations of [20], as follows.

We work with the normalized total complexes of $C_*(X_\bullet)$ and $C_*(S^\infty_+ \wedge_{C_p} X_\bullet^{(p)})$: An element of $C_*(X_\bullet)$ is an element of $C_t(X_s)$ for which the face maps ∂_i are zero for $i \geqslant 1$, with total differential $D = (-1)^t\partial + d$, where $\partial = \partial_0$ is the simplicial differential and d is the internal differential in C_*. We then have an isomorphism of spectral sequences between the spectral sequence of the double complex and the spectral sequence for computing the homotopy groups of the simplicial $H\mathbb{F}_p$-modules $X_\bullet$ and $S^\infty_+ \wedge_{C_p} X_\bullet^{(p)}$, consistent with one commonly used set of sign conventions (compatible with the convention specified above for d_1). Let a_s be the generator of $C_t(X_s)$ corresponding to the cell S^t_c and a_{s-q} for the generator of $C_{t+q}(X_{s-q})$ corresponding to the cell CS_c^{t+q-1}. Under the usual sign conventions for the cellular chains of $CS_c^{t+q-1} = S_c^{t+q-1} \wedge I$ (where I has basepoint 1), we have that da_{s-q} is $(-1)^{t+q}$ times the class representing the bottom cell S_c^{t+q-1}. This gives us the formula

$$\partial a_{s-q+1} = (-1)^{t+q} da_{s-q}.$$

Taking

$$\bar{a} = a_s + a_{s-1} + a_{s-2} + \cdots + a_{s-r+1},$$

we have that

$$D\bar{a} = (-1)^{t+r-1}\partial a_{s-r+1}$$

is $(-1)^{t+r-1}$ times the generator of $C_{s-r}(X_{s-r})$ corresponding to the cell S_c^{t+r-1}. Since $\bar{a}_s$ represents a and ∂a_{s-r+1} represents b in the E^1 page of the spectral sequence for $\pi_*|X_\bullet|$, this tells us that with these sign conventions, $d_r a = (-1)^{t+r-1}b$ in E^r. The chain-level operation $\beta^\epsilon Q^i$ commutes with ∂ and $(-1)^\epsilon$-commutes with d, adding ϵ to the parity of the internal degree; defining $\beta^\epsilon Q^i \bar{a}$ to be the class

$$\beta^\epsilon Q^i a_s + \beta^\epsilon Q^i a_{s-1} + \cdots + \beta^\epsilon Q^i a_{s-r+1},$$

in $C_*(S^\infty_+ \wedge_{C_p} X_\bullet^{(p)})$, we get the formula

$$D\beta^\epsilon Q^i \bar{a} = (-1)^{t+r-1+2i(p-1)-\epsilon}\beta^\epsilon Q^i \partial a_{s-r+1} = (-1)^{t+r-1+\epsilon}\beta^\epsilon Q^i \partial a_{s-r+1}.$$

Since $\beta^\epsilon Q^i a_s$ represents $\beta^\epsilon Q^i a$ and $\beta^\epsilon Q^i \partial a_{s-r+1}$ represents $\beta^\epsilon Q^i b$ in the E^1 page of the spectral sequence for $\pi_*|S^\infty_+ \wedge_{C_p} X_\bullet^{(p)}|$, we see that

$$d_r(\beta^\epsilon Q^i a) = (-1)^{t+r-1+\epsilon}\beta^\epsilon Q^i b = (-1)^\epsilon \beta^\epsilon Q^i d_r a.$$

This completes the proof of part (ii).

In the permanent cycle case $r \geqslant s+1$, the geometric realization of $X_\bullet$ is homotopy equivalent to an $(s+t)$-sphere $H\mathbb{F}_p$-module and the cycle

$$\bar{a} = a_s + a_{s-1} + a_{s-2} + \cdots + a_0$$

in the normalized total cellular chain complex (in the notation above) provides a fundamental class in $H_{s+t}|X_\bullet| \cong \pi_{s+t}|X_\bullet|$. We have a map from $X_\bullet$ to the constant simplicial object S_c^{s+t}

induced by the collapse map $CS^{t+s-1}_c \to S^{s+t}_c$ in simplicial degree 0; looking at normalized total chain complexes, we see that this map is a weak equivalence, sending $\bar{a}$ to the fundamental class c of the cell S^{s+t}_c in $C_{s+t}(S^{s+t}_c)$. Looking at the maps of simplicial $H\mathbb{F}_p$-modules,

$$\begin{array}{ccc} \mathcal{C}_n(p)_+ \wedge_{\Sigma_p} X_\bullet^{(p)} & \longrightarrow & A_\bullet \\ \simeq\downarrow & & \\ \mathcal{C}_n(p)_+ \wedge_{\Sigma_p} (S^{s+t}_c)^{(p)} & & \end{array}$$

to prove part (iii), it suffices to show that the element in

$$\pi_*(\mathcal{C}_n(p)_+ \wedge_{\Sigma_p} |X_\bullet|^{(p)}) \cong \pi_*|\mathcal{C}_n(p)_+ \wedge_{\Sigma_p} X_\bullet^{(p)}|$$

represented by $\beta^\epsilon Q^i a$ in the E^∞ page goes to the element $\beta^\epsilon Q^i c$ of

$$\pi_*(\mathcal{C}_n(p)_+ \wedge_{\Sigma_p} (S^{s+t}_c)^{(p)})$$

given by $\beta^\epsilon Q^i$ applied to the fundamental class of S^{s+t}_c. (For the second assertion of part (iii), note that when $s+t > 2i-\epsilon$ $(p>2)$ or $s+t>i$ $(p=2)$, this element is zero.) Again, we look at chain-level operations on the normalized total cellular chain complex of $S^\infty_+ \wedge_{C_p} X_\bullet^{(p)}$. Now we have that

$$\beta^\epsilon Q^i a_s + \beta^\epsilon Q^i a_{s-1} + \cdots + \beta^\epsilon Q^i a_0$$

is a cycle, representing $\beta^\epsilon Q^i a$. Since it maps to $\beta^\epsilon Q^i c$ in the cellular chain complex of $S^\infty_+ \wedge_{C_p} (S^{s+t}_c)^{(p)}$, this completes the proof of part (iii) and the proof of Theorem 6.1, assuming Lemmas 6.2 and 6.3.

We now turn to the proof of Lemmas 6.2 and 6.3. We begin by reviewing partial matching objects [**9**, 2.3], which correspond to the limit of the last several faces. Let $M^0_0 = M^1_0 = *$; for $s>0$, let $M^0_s = *$, and for $1 \leqslant i \leqslant s+1$, let M^i_s be the limit of the diagram $\mathfrak{D}^i_s$ which has objects.

(i) For each j with $s-i < j \leqslant s$, a copy of A_{s-1} labeled (∂_j, A_{s-1}).
(ii) For each (j,k) with $s-i<j<k\leqslant s$, a copy of A_{s-2} labeled $(\partial_j\partial_k, A_{s-2})$. (We understand $A_{-1} = *$.)

and maps

(i) For each (j,k) with $s-i<j<k\leqslant s$, a map $(\partial_k, A_{s-1}) \to (\partial_j\partial_k, A_{s-2})$ given by the map $\partial_j : A_{s-1} \to A_{s-2}$.
(ii) For each (j,k) with $s-i<j<k\leqslant s$, a map $(\partial_j, A_{s-1}) \to (\partial_j\partial_k, A_{s-2})$ given by the map $\partial_{k-1} : A_{s-1} \to A_{s-2}$.

Inductively, for $s>0$, we can identify M^{i+1}_s as the pullback

$$M^{i+1}_s = M^i_s \times_{M^i_{s-1}} A_{s-1}.$$

We have assumed that $A_\bullet$ is Reedy fibrant, and this means that for every s, the map $A_s \to M^{s+1}_s$ induced by $\partial_0, \ldots, \partial_s$ is a fibration. Using this, we prove the following proposition.

Proposition 6.4. *The maps $A_s \to M^i_s$ and $M^i_s \to M^{i-1}_s$ are fibrations. For $i < s+1$, they are surjections on homotopy groups.*

Proof. We work by induction on s; the base case $s=0$ is clear. Assuming the statements for $s-1$, and using the pullback description of M^{i+1}_s, we see that the maps $M^i_s \to M^{i-1}_s$ are fibrations and for $i<s+1$ surjections on homotopy groups. Starting from the Reedy fibrant hypothesis that the map $A_s \to M^{s+1}_s$ is a fibration, downward induction shows that the maps

$A_s \to M^i_s$ are fibrations. Using the fact that the maps $M^i_s \to M^{i-1}_s$ are surjections on homotopy groups for $i < s+1$, we can identify $\pi_q M^s_s$ as the partial matching object M^s_s for $\pi_q A_\bullet$. The fact that simplicial abelian groups satisfy the Kan condition implies that the map $\pi_q A_s \to \pi_q M^s_s$ is a surjection. Downward induction now implies that the maps $\pi_q A_s \to \pi_q M^i_s$ are surjections for $i < s+1$. □

We can now prove Lemmas 6.2 and 6.3. For Lemma 6.2, we show inductively that we can choose $\bar{x}_i : S^t_c \to A_s$ for $i = 0, \ldots, s$ representing x and having the property that the composite to M^i_s is the trivial map; then $\bar{x} = \bar{x}_s$ satisfies $\partial_i \bar{x} = *$ for $1 \leqslant i \leqslant s$. Choose any $\bar{x}_0$ representing x; since $M^0_s = *$, the composite to M^0_s is trivial. Assume by induction that we have constructed $\bar{x}_{i-1}$. Since $\partial_{s-i+1} x = 0$, we have that $\partial_{s-i+1} \circ \bar{x}_{i-1}$ is null-homotopic, and we choose a null-homotopy $a : CS^t_c \to A_{s-1}$. Since $\partial_{s-k+1} \circ \bar{x}_{i-1} = *$ for $k < i$, the composite of $\bar{x}_{i-1}$ to M^{i-1}_{s-1} is trivial, and so we can regard the composite of a to M^{i-1}_{s-1} as a map b from $\Sigma S^t_c \to M^{i-1}_{s-1}$. Since the map $A_{s-1} \to M^{i-1}_{s-1}$ is surjective on homotopy groups, after altering a by adding an appropriate map $\Sigma S^t_c \to A_{s-1}$ if necessary, we can arrange for b to be null-homotopic. Choosing a null-homotopy h of b, and regarding it as a homotopy of a rel S^t_c, we can lift h to a homotopy of a in A_{s-1} since the map $A_{s-1} \to M^{i-1}_{s-1}$ is a fibration. Looking at the other side of the homotopy, we obtain a map $c : CS^t_c \to A_{s-1}$ which restricts to the composite of $\bar{x}_{i-1}$ on S^t_c and whose composite map $CS^t_c \to M^{i-1}_{s-1}$ is the trivial map. Using the fact that $M^i_s \cong M^{i-1}_s \times_{M^{i-1}_{s-1}} A_{s-1}$, the map c together with the trivial map $CS^t_c \to M^{i-1}_s$ defines a map $CS^t_c \to M^i_s$ which restricts to the composite of $\bar{x}_{i-1}$ on S^t_c. Regarding this as a homotopy from the composite of $\bar{x}_{i-1}$ into M^i_s to the trivial map $S^t_c \to M^i_s$ and using the fact that the map $A_s \to M^i_s$ is a fibration, we can lift this to a homotopy $g : S^t_c \wedge I_+ \to A_s$ starting at $\bar{x}_{i-1}$. Let $\bar{x}_i : S^t_c \to A_s$ be the other side of this homotopy; by construction, it represents x and its composite to M^i_s is trivial. This completes the inductive argument and the proof of Lemma 6.2.

The proof of Lemma 6.3 is similar: Given $\bar{x}$, we argue inductively that we can choose maps $\bar{y}_i : CS^t_c \to A_s$ for $i = 0, \ldots, s$ restricting to $\bar{x}$ on S^t_c and having the properties that the composite to M^i_s is the trivial map and that the composite to M^s_s, viewed as a map from ΣS^t_c to M^s_s is null-homotopic. Choose any $\bar{y}_0$ null-homotopy of $\bar{x}$; since $A_s \to M^s_s$ is surjective on π_{t+1}, after altering $\bar{y}_0$ by adding a map $\Sigma S^t_c \to A_s$ if necessary, we can arrange for the composite of $\bar{y}_0$ to M^s_s to be null-homotopic as required. Inductively, having chosen $\bar{y}_{i-1}$, we can regard the composite map to A_{s-1} as a map $z : \Sigma S^t_c \to A_{s-1}$, which is null-homotopic. As above, we can choose a null-homotopy a whose composite to M^{i-1}_s is the trivial map. We get a homotopy $\Sigma S^t_c \wedge I_+ \to M^i_s$ starting at the composite map of $\bar{y}_{i-1}$ and ending at the trivial map; we can choose a lift $g : CS^t_c \wedge I_+ \to A_s$ rel S^t_c starting at $\bar{y}_{i-1}$. Letting $\bar{y}_i$ be the other side of the homotopy, by construction it restricts to $\bar{x}$ on S^t_c, its composite to M^i_s is trivial, and its composite to M^s_s is null-homotopic. This completes the induction and the proof of Lemma 6.3.

7. *Proof of Theorem* 3.4(ii)

In this section, we prove Theorem 3.4(ii), verifying that the cohomological spectral sequence of Theorem 3.2 is multiplicative. Following the notation from Section 3, let N be a cofibrant non-unital E_n H-algebra with a weak equivalence $KN \to H \wedge^{\mathbf{L}}_R A$, and let $C = \tilde{B}^{n-1} N$, a (partial) non-unital E_1 H-algebra. The spectral sequences of Theorem 3.2 are induced by the geometric realization filtration on the bar construction BC and the multiplication on cohomology is induced by the bar diagonal $BC \to BC \wedge_H BC$. We prove the following theorem.

Theorem 7.1. *For a* (*partial*) *non-unital E_1 H-algebra C, the bar diagonal $BC \to BC \wedge_H BC$ preserves filtration, where on the left we use the geometric realization filtration and on the* *right we use the smash product of the geometric realization filtrations. Moreover, the induced map on the homotopy groups of the filtration quotients*

$$\pi_*(\Sigma^n C^{(n)}) \longrightarrow \pi_*\left(\bigvee_{j=0}^{n} \Sigma^n (C^{(j)} \wedge_H C^{(n-j)})\right) \cong \bigoplus_{j=0}^{n} \pi_*(\Sigma^n C^{(n)})$$

is the diagonal map.

Applying $F_H(-,F)$ and using the induced filtration, we get a multiplicative spectral sequence. The formula for the map on filtration quotients above induces the standard multiplication

$$\pi_{-p-q} F_H(C^{(p)}, F) \otimes \pi_{-p'-q'} F_H(C^{(p')}, F) \longrightarrow \pi_{-(p+p')-(q+q')} F_H(C^{(p+p')}, F).$$

This then proves Theorem 3.4(ii).

To prove Theorem 7.1, we first recall the construction of the bar diagonal. We follow [**3**, § 8] and define the bar diagonal as the composite

$$BC \cong |\operatorname{sd}_2 B_\bullet C| \longrightarrow |B_\bullet C \wedge_H B_\bullet C| \cong BC \wedge_H BC,$$

where sd_2 denotes the edgewise subdivision construction of [**4**, 1.1] or [**12**, § 4]. Theorem 7.1 follows from the geometric properties of the natural isomorphism $|\cdot| \cong |\operatorname{sd}_2(\cdot)|$, as we now explain by reviewing the construction in detail.

Recall that for a simplicial object X, the edgewise subdivision $\operatorname{sd}_2 X$ is defined as the simplicial object

$$(\operatorname{sd}_2 X)_n = X_{2(n+1)-1}$$

with face map ∂_i given by $\partial_i \partial_{i+n+1}$ on $X_{2(n+1)-1}$ and degeneracy map s_i given by $s_i s_{i+n+1}$ on $X_{2(n+1)-1}$. We note that $\operatorname{sd}_2 X$ is the diagonal simplicial object of a bisimplicial object that we will denote by $\operatorname{Sd}_2 X$: This has p,q-simplices given by

$$(\operatorname{Sd}_2 X)_{p,q} = X_{(p+1)+(q+1)-1}$$

with face and degeneracy maps ∂_i and s_i given by ∂_i and s_i in the p-direction and by ∂_{i+p+1} and s_{i+p+1} in the q-direction.

These constructions have a formulation in terms of the category of standard simplices $\mathbf{\Delta}$. An object of $\mathbf{\Delta}$ is an ordered set $[n] = \{0, \ldots, n\}$ and a morphism is a weakly increasing map. A simplicial object is then a contravariant functor out of $\mathbf{\Delta}$ and a bisimplicial object is a contravariant functor out of $\mathbf{\Delta} \times \mathbf{\Delta}$. Concatenation (disjoint union) defines a functor $\mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta}$, sending $[p], [q]$ to $[(p+1)+(q+1)-1]$; for a simplicial object X, $\operatorname{Sd}_2 X$ is the bisimplicial object obtained by composing concatenation with X. The diagonal simplicial object $\operatorname{sd}_2 X$ can also be described as the composite with X of the doubling functor $\mathbf{\Delta} \to \mathbf{\Delta}$ (the diagonal followed by concatenation).

The standard n-simplex is the simplicial set $\Delta[n] = \mathbf{\Delta}(-, [n])$, where (in geometric and combinatorial terms) we view $[n] = \{0, \ldots, n\}$ as the set of vertices: a simplicial map from $\Delta[m]$ to $\Delta[n]$ corresponds to the map $[m]$ to $[n]$ in $\mathbf{\Delta}$ determined by where its vertices go. We can identify the bisimplicial set $\operatorname{Sd}_2 \Delta[n]$ as the union

$$(\Delta[0] \times \Delta[n]) \cup (\Delta[1] \times \Delta[n-1]) \cup \cdots \cup (\Delta[n] \times \Delta[0]),$$

glued along boundary pieces

$$\partial_j \Delta[j+1] \times \Delta[n-j-1] \sim \Delta[j] \times \partial_0 \Delta[n-j].$$

We certainly have a map from this union to $\operatorname{Sd}_2 \Delta[n]$ induced by the order preserving surjections $[j] \amalg [n-j] \to [n]$ (sending the highest element of $[j]$ and the lowest element of $[n-j]$ to the same element of $[n]$), and every order preserving map $[j]\amalg[k]\to[n]$ factors through one of these (uniquely through the union after making the gluing identifications). For convenience, we will often write $\Delta[p,q]$ for the bisimplicial set $\Delta[p]\times\Delta[q]$, the (p,q)-*bisimplex*.

The homeomorphism $|\Delta[n]|\cong||\operatorname{Sd}_2\Delta[n]||$ is the *prismatic decomposition* [**22**, § 2] of $\Delta[n]$. The (double) geometric realization of the bisimplex $\|\Delta[j,n-j]\|=|\Delta[j]|\times|\Delta[n-j]|$ maps to the geometric realization of $\Delta[n]$ by sending the point with barycentric coordinates $(s_0,\ldots,s_j),(t_0,\ldots,t_{n-j})$ to the point with barycentric coordinates

$$(s_0/2,\ldots,s_{j-1}/2,s_j/2+t_0/2,t_1/2,\ldots,t_{n-j}/2).$$

It follows from the formula that these maps are compatible with the gluing and define a continuous map from $\|\operatorname{Sd}_2\Delta[n]\|$ to $|\Delta[n]|$; to see that it is a bijection and therefore a homeomorphism, we note that the image of $\Delta[j,n-j]$ consists of precisely those points with barycentric coordinates $(s_0,\ldots,s_n)$ such that

$$s_0+\cdots+s_{j-1}\leqslant 1/2\quad\text{and}\quad s_0+\cdots+s_j\geqslant 1/2$$

(with equality holding for one or the other sum exactly when the point is on one of the glued boundary pieces). The inverse homeomorphism

$$S_n:|\Delta[n]|\longrightarrow\|\operatorname{Sd}_2\Delta[n]\|$$

is a cellular map for the natural cell structure on the geometric realization on the left and the double geometric realization on the right. This is easy to see from a reformulation in terms of *summation coordinates*: For barycentric coordinates $(s_0,\ldots,s_n)$, let

$$u_1=s_0,\quad u_2=s_0+s_1,\ldots,u_n=s_0+\cdots+s_{n-1}$$

so that $0\leqslant u_1\leqslant\cdots\leqslant u_n\leqslant 1$; the summation coordinates define a homeomorphism of $|\Delta[n]|$ with the space of weakly increasing sequences of length n in $[0,1]$. For $(s_0,\ldots,s_j),(t_0,\ldots,t_{n-j})$ barycentric coordinates of a point in $\Delta[j,n-j]=\Delta[j]\times\Delta[n-j]$, we get a weakly increasing sequence of length n in $[0,1]$ defined by

$$v_1=s_0/2,\quad v_2=(s_0+s_1)/2,\quad\ldots,\quad v_j=(s_0+\cdots+s_{j-1})/2,$$
$$v_{j+1}=1/2+t_0/2,\quad\ldots,\quad v_n=1/2+(t_0+\cdots+t_{n-j-1})/2.$$

Glued points in $|\partial_j\Delta[j+1]|\times|\Delta[n-j-1]|$ and $|\Delta[j]\times\partial_0\Delta[n-j]|$ give the same sequence, and this formula defines a homeomorphism from $\|\operatorname{Sd}_2\Delta[n]\|$ to the space of weakly increasing sequences of length n in $[0,1]$. For $\Delta[n]$, the cell structure on the space of sequences corresponds to sequence entries being zero, one, or equal to the next entry; for $\operatorname{Sd}_2\Delta[n]$, the cell structure corresponds to entries being zero, one, $1/2$, or equal to the next entry. It follows that the map S_n is cellular.

It is clear from the definition that subdivision takes the colimit of simplicial objects to the corresponding colimit of bisimplicial objects, and it is well known that geometric realization of simplicial H-modules and the double geometric realization of bisimplicial H-modules commutes with colimits. Since we can write any simplicial H-module X as the coequalizer

$$\bigvee_{f:[m]\to[n]}X_n\wedge\Delta[m]_+\rightrightarrows\bigvee_{[n]}X_n\wedge\Delta[n]_+\longrightarrow X,$$

we can write $|X|$ and $\|\operatorname{Sd}_2X\|$ as coequalizers

$$\bigvee_{f:[m]\to[n]}X_n\wedge|\Delta[m]|_+\rightrightarrows\bigvee_{[n]}X_n\wedge|\Delta[n]|_+\longrightarrow|X|,$$

$$\bigvee_{f:[m]\to[n]}X_n\wedge\|\operatorname{Sd}_2\Delta[m]\|_+\rightrightarrows\bigvee_{[n]}X_n\wedge\|\operatorname{Sd}_2\Delta[n]\|_+\longrightarrow\|\operatorname{Sd}_2X\|.$$

The concrete description of the homeomorphisms $S_n : |\Delta[n]| \to \| \operatorname{Sd}_2 \Delta[n]\|$ above makes it clear that they are compatible with the maps in the coequalizer and induce an isomorphism $S : |X| \to \| \operatorname{Sd}_2 X\|$. We see from the naturality of the coequalizer diagrams that S is natural.

Regarding the coequalizers above as taking place in the category of filtered H-modules (or even doubly filtered H-modules for $\operatorname{Sd}_2 X$), we see that S preserves filtrations since the maps S_n do. In order to understand the map on filtration quotients, it is useful to observe how the pieces $X_n \wedge \| \operatorname{Sd}_2 \Delta[n]\|$ fit into the usual description of the double geometric realization built in terms of the pieces

$$\operatorname{Sd}_2 X_{p,q} \wedge \|\Delta[p,q]\|_+ = X_{p+q+1} \wedge \|\Delta[p,q]\|_+.$$

As indicated above, this (p,q) piece arises from the surjection

$$[p+q+1] = [p] \amalg [q] \longrightarrow [p+q],$$

which in these terms corresponds to the degeneracy s_p. Naturality now implies that the $X_n \wedge \|\Delta[j, n-j]\|_+$ piece of $X_n \wedge \| \operatorname{Sd}_2 \Delta[n]\|_+$ in the coequalizer above maps to the $X_{n+1} \wedge \|\Delta[j, n-j]\|_+$ piece of the double geometric realization via the map $s_j : X_n \to X_{n+1}$. The induced map on filtration quotients therefore sends the nth filtration quotient

$$X_n/(s_0, \ldots, s_n) \wedge |\Delta[n]/\partial|$$

of $|X|$ to the nth filtration quotient

$$\begin{aligned} X_{n+1}/(s_1, \ldots, s_n) \wedge \|\Delta[0,n]/\partial\| \vee X_{n+1}/(s_0, s_2, \ldots, s_n) \wedge \|\Delta[1, n-1]/\partial\| \vee \cdots \\ \vee X_{n+1}/(s_0, s_1, \ldots, s_{n-1}) \wedge \|\Delta[n,0]/\partial\| \end{aligned}$$

of $\| \operatorname{Sd}_2 X\|$ by the map induced by S_n and the degeneracy s_j on the $\Delta[j, n-j]$ summand.

In the case when $X = BC$, we have

$$\begin{gathered} X_n/(s_0, \ldots, s_n) = C^{(n)}, \\ X_{n+1}/(s_0, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n) = C^{(j)} \wedge_H (H \vee C) \wedge_H C^{(n-j)} \end{gathered}$$

and the map is induced by the isomorphism $C^{(n)} \cong C^{(j)} \wedge_H H \wedge_H C^{(n-j)}$. The bisimplicial map from $\operatorname{Sd}_2 B_\bullet C$ to $B_\bullet C \wedge_H B_\bullet C$ in bidegree p, q is the map

$$(H \vee C)^{(p+q+1)} \longrightarrow (H \vee C)^{(p)} \wedge_H (H \vee C)^{(q)}$$

induced by the augmentation $H \vee C \to H$ (using the trivial map $C \to *$) on the $(p+1)$st factor. The composite map $BC \to BC \wedge_H BC$ therefore induces on nth filtration quotient the map

$$C^{(n)} \wedge |\Delta[n]/\partial| \longrightarrow C^{(0)} \wedge_H C^{(n)} \wedge \|\Delta[0,n]/\partial\| \vee \cdots \vee C^{(n)} \wedge_H C^{(0)} \wedge \|\Delta[n,0]/\partial\|$$

induced by S_n and the isomorphisms $C^{(n)} \cong C^{(j)} \wedge_H C^{(n-j)}$. On homotopy groups this induces the diagonal map as in the statement of Theorem 7.1. This completes the proof of Theorem 7.1.

Remark 7.2. In [**3**, § 8], we used the subdivision isomorphism $|\operatorname{sd}_2 X| \cong |X|$ defined in [**4**, 1.1] because of its multiplicative properties (see [**3**, 8.1]); a check of formulas shows that the isomorphism $|X| \cong \| \operatorname{Sd}_2 X\| \cong |\operatorname{sd}_2 X|$ constructed above is its inverse, q.v. [**12**, § 4].

Remark 7.3. The work above can also be formulated in terms of Drinfeld's description of the geometric realization [**8**], which writes

$$|X| = \operatorname*{colim}_{F \subset I} X(\pi_0(I \setminus F))$$

as the colimit over the finite subsets F of the unit interval $I = [0,1]$, where we regard $\pi_0(I \setminus F)$ as a finite ordered set, that is,

$$X(\pi_0(I \setminus F)) = X_{\#\pi_0(I \setminus F) - 1}.$$

The double geometric realization is then

$$\| \mathrm{Sd}_2 X \| = \operatorname*{colim}_{F_1, F_2 \subset I} X(\pi_0(I \setminus F_1) \amalg \pi_0(I \setminus F_2)).$$

The isomorphism $S : |X| \to \| \mathrm{Sd}_2 X \|$ defined above is induced by the map of diagrams sending $F = \{0 \leqslant u_1 < u_2 < \cdots < u_n \leqslant 1\}$ to

$$F_1 = \{0 \leqslant v_1 < \cdots < v_j < 1\} = \{0 \leqslant 2u_1 < \cdots < 2u_j < 1\},$$
$$F_2 = \{0 \leqslant w_1 < \cdots < w_{n-j} \leqslant 1\} = \{0 \leqslant (2u_{j+1} - 1) < \cdots < (2u_n - 1) \leqslant 1\},$$

together with the natural transformation

$$X(\pi_0(I \setminus F)) \longrightarrow X(\pi_0(I \setminus F_1) \amalg \pi_0(I \setminus F_2))$$

induced by the map of ordered sets

$$\pi_0([0,1] \setminus \{u_1, \ldots, u_n\}) \longleftarrow \pi_0([0,1] \setminus \{u_1, \ldots, u_j, 1/2, u_{j+1}, \ldots u_n\})$$
$$\cong \pi_0(([0,1] \setminus \{v_1, \ldots, v_j\}) \amalg ([0,1] \setminus \{w_1, \ldots, w_{n-j}\})).$$

After verifying that the induced map is continuous, the remaining verifications are essentially the same as the ones above written in terms of summation coordinates.

Acknowledgements. The second author would like to thank Andrew Blumberg, Tyler Lawson, and Jim McClure for helpful comments.

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