Simply Connected Five-Manifolds
Author(s): D. Barden
Source: *Annals of Mathematics*, Vol. 82, No. 3 (Nov., 1965), pp. 365-385
Published by: Mathematics Department, Princeton University
Stable URL: https://www.jstor.org/stable/1970702
Accessed: 11-01-2025 18:24 UTC

Simply connected five-manifolds

By D. Barden

S. Smale, using his theory of handlebodies, has classified, under diffeomorphism closed, simply connected, smooth 5-manifolds with vanishing second Stiefel-Whitney class. C.T.C. Wall has given a classification of $(n-1)$-connected $(2n+1)$-manifolds which does not however cover the case $n=2$. In this paper we complete the classification of simply connected 5-manifolds. A.A.Markov has proved that a general classification of 5-manifolds is impossible, but it seems reasonable to hope for results in the case of 5-manifolds with a given fundamental group.

The second Stiefel-Whitney class of a simply connected manifold may be regarded as a homomorphism $w: H_2(M; Z) \rightarrow Z_2$ and we may arrange w to be non-zero on at most one element of a 'basis' (0.5), this element having order 2^i for some i (Lemma C). Then i is a diffeomorphism invariant $i(M)$ of M.

If $H_2(X) \cong H(M)$, and $i(X) = i(M)$, where X and M are simply connected 5-manifolds, then there are (0.8) isomorphisms $\theta: H_2(X) \rightarrow H_2(M)$ which preserve the linking form b on the torsion subgroups (0.7), and which satisfy $w(M)\circ\theta = w(X)$. The basic theorem (2.2) states that any such isomorphism may be realized by a diffeomorphism of X with M. Thus $H_2(M)$ and $i(M)$ form a complete set of invariants for the diffeomorphism classification.

On the other hand b imposes restrictions on the second homology group (Lemma E), and hence on the decomposability of the manifolds. Using results of C.T.C.Wall on diffeomorphisms of 4-manifolds, it is possible to construct an example of an indecomposable manifold for each possible homology group (§1) and, using these, to give a canonical manifold in each diffeomorphism class (Theorem 2.3).

In addition to the main theorems, §2 contains some corollaries and applications of them. The manner of construction of the indecomposable manifolds and manifolds similar to them produces minimal handle decompositions and allows the computation of embedding and immersion dimensions. The nature of the invariants also allows an extension of the results.

The proof of Theorem 2.2 is omitted from §2 and occupies the remainder of the paper. X and M as above are necessarily cobordant (Lemma F), and the first step is to find a cobordism with minimal homotopy groups (§3), i.e., one which is simply connected and with second homology group zero or, if $w(X) \neq 0$, Z_2. In §4 modifications are described of which one enlarges the second homology group in a controlled manner, and the other removes certain elements of the third. These are used in § 5 to obtain an h-cobordism V between X and M such that, for all elements x in $H_2(X)$, x and $\theta(x)$ map to the same element of $H_2(V)$. Since V is 6-dimensional, Smale's Theorem (A) applies to give a diffeomorphism of V with $X \times I$, and this induces the required diffeomorphism of X with M.

This paper constitutes a revision and enlargement of part of my thesis (Cambridge, 1963). I am grateful to C.T.C.Wall for drawing my attention to this problem, and for his helpful suggestions throughout. In particular, I am indebted to him for making [23] and the results of [25] available to me before their publication.

0. Preliminary results and definitions

We shall be concerned throughout with compact C^∞-manifolds M^n of dimension n, though non-compact manifolds may occur as submanifolds of compact ones. M, whose boundary ∂M need not be vacuous, will be assumed orientable with chosen orientations μ, $\partial\mu$, generators respectively of $H_n(M, \partial M; Z)$ and $H_{n-1}(\partial M; Z)$. Most of the fundamental definitions and results required may be found in [24]. For the rounding of corners etc., see also [3].

0.1. There are several methods of combining two manifolds to form a third. We let $A \cup B$ denote the disjoint union of the two manifolds A^n, B^n. If A and B have non-empty boundaries ∂A, ∂B, then $A + B$ is formed from $A \cup B$ by embedding $(n-1)$-discs in ∂A and ∂B, identifying them under an orientation reversing diffeomorphism (the orientations of the embedded discs being induced from those of ∂A and ∂B), and smoothing the corners. More generally we form $A + fB$, where f is an orientation reversing diffeomorphism of any $(n-1)$-dimensional submanifold of ∂B with one of ∂A. $A \# B$ (see [9], [11]) is formed from $A \cup B$ by embedding an n-disc in each, avoiding the boundaries, removing the interiors of these discs, identifying the bounding $(n-1)$-spheres of the resulting holes under an orientation reversing diffeomorphism and rounding the corners. Note that $\partial(A + B) = \partial A \# \partial B$, the $\#$ taking place between the boundaries of the discs across which the $+$ was effected.

0.2. The manifold $A^n + h^r$, described as "A with an r-handle attached", is formed from $A \cup (D^r \times D^{n-r})$ by identifying $S^{r-1} \times D^{n-r}$ with its image under some embedding in ∂A, and rounding the corners. A *decomposition of A on B* is the presentation of A^n as

$$B^n + h_1^0 + \cdots + h_{i_0}^n + \cdots + h_{i_n}^n .$$

Usually B would be D^n or $Q^{n-1} \times I$ for some submanifold Q of ∂A. The following theorem is due to Smale ([18], see also [28]).

THEOREM A. *If $n > 5$ and M^n is compact and simply connected, all its boundary components are simply connected, $\partial M = Q_1 \cup Q_2$ where each Q_i is the union of connected components, and $H_k(M, Q_1; Z)$ has a direct sum decomposition with β_k infinite cyclic summands and σ_k finite cyclic summands, then M has a decomposition on $Q \times I$ with $\beta_k + \sigma_k + \sigma_{k-1}$ k-handles for each k.*

(If $n = 5$ under similar hypotheses, a decomposition may be found with the stated number of k-handles for $k = 0, 1, 4$ and 5. This follows from the methods used by Smale and is also proved explicity by A.H.Wallace in [28].)

There is a close connection between decompositions of a manifold M and 'nice functions' on it, that is non-degenerate differentiable real functions which are transverse to ∂M on a neighborhood of it and which take values $-1/2$ on Q_1, $n + (1/2)$ on Q_2, and k at each critical point of index k. To each such function correspond decompositions of M on $Q_1 \times I$ with exactly one k-handle for each critical point of index k. Conversely for each decomposition there are nice functions with the corresponding number of critical points.

If A^n has a decomposition on $\partial A \times I$ with α_k k-handles and B^n has one on $\partial B \times I$ with β_k k-handles, then for any orientation reversing diffeomorphism h between ∂A and ∂B, $A + h\, B$ has a decomposition with $\alpha_k + \beta_{n-k}$ k-handles. For, if $f : A \to R$ and $f' : B \to R$ correspond to the given decompositions with constants adjusted so that $f(\partial A) = -f'(\partial B)$, then g defined by $g(A) = f(A)$ and $g(B) = -f'(B)$ has precisely $\alpha_k + \beta_{n-k}$ critical points of index k, since none are introduced along $\partial A = \partial B$. Then g may be modified to have the correct critical and boundary values or, equivalently, the handles themselves may be rearranged (see [18], [28].)

0.3. THEOREM B (Haefliger [5]). *Let V^v be a closed manifold and M^m an arbitrary (C^∞-) manifold.*

(a) *If $f : V \to M$ is a continuous map with $\pi_i(f) = 0$ for $i < 2v - m + 2$, then f is homotopic to an embedding provided $2m > 3v + 2$.*

(b) *If $f,\ g : V \to M$ are homotopic embeddings with $\pi_i(f) = \pi_i(g) = 0$ for $i < 2v - m + 3$, then g is isotopic to f provided $2m > 3v + 3$.*

Here $\pi_i(f)$ denotes $\pi_i(C_f,\ V)$ where C_f, the mapping cylinder of f, is formed from $(V \times [0,\ 1]) \cup M$ by identifying, for each v in V, $v \times \{1\}$ with $f(v)$. $V \times \{0\}$ in C_f is referred to as V.

From this theorem it follows that any continuous map of a 3-sphere into a simply connected 6-manifold is homotopic to an embedding, and that two

2-spheres embedded in a simply connected 5-manifold are isotopic if and only if they are homotopic (if and only if they are homologous).

0.4. In an orientable n-manifold, the second Stiefel-Whitney class w^2 is the obstruction to parallelisability over the 2-skeleton, since it is the obstruction to the existence of an $(n-1)$-field over the 2-skeleton [21] and the complementary 1-field must be continuous.

In a simply connected manifold M, $H^2(M; Z_2)$ is isomorphic with $\mathrm{Hom}\big(H_2(M; Z), Z_2\big)$ and so $w^2(M)$ may be regarded as a homomorphism $w^2(M): H_2(M; Z) \to Z_2$.

In a 6-manifold M the obstruction to s-parallelisability over an embedded 2-sphere is the obstruction to triviality of its normal bundle. For $\tau(M) \mid S^2 = \tau(S^2) + \nu(S^2 \subset M)$ and so $\tau(M) | S^2 + \varepsilon^1 = \varepsilon^7 \Leftrightarrow \nu(S \subset M) + \varepsilon^3 = \varepsilon^7$; i.e., $\Leftrightarrow$ $\nu(S \subset M)$ is stably trivial; but it is already stable. Thus, in a simply connected 6-manifold, the value of w^2 on the homology class carried by an embedded 2-sphere is the obstruction to the triviality of the normal bundle of this sphere.

For orientable 5-manifolds $\langle \mu, w^2 \smile w^3 \rangle$ is the only possible non-zero Stiefel-Whitney number, and thus the oriented cobordism group Ω_5 is Z_2 (see [19]). w^3 is the mod 2 reduction of the integer class $W^3 = \delta^* w^2$ where δ^* is the Bockstein associated with the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{\;2\;} Z \longrightarrow Z_2 \longrightarrow 0 .$$

The fifth spinor cobordism group is zero (see [14]).

0.5. In an abelian group G, a set of r non-zero elements will be termed *independent* if the subgroup which they generate together is the direct sum of the r cyclic subgroups which they generate separately. By a *basis* of a finitely generated abelian group G will be meant an independent set which generates G. The set of non-zero elements $x_1, \cdots, x_r$ is independent if and only if $n_1x_1 + \cdots + n_rx_r = 0$ implies $n_ix_i = 0$ for each i. Any maximal pure independent set forms a basis (see [8]).

Clearly the number of elements in a basis is not in general an invariant of G. However the number of elements of infinite order is invariant, and for the torsion subgroup the orders and the number of elements in a basis with the most or fewest possible elements are invariants. In the first case the number of elements of order p^i is the i^{th} Ulm invariant of the p-primary component of G, and in the second case we have the classical decomposition of a finite abelian group as a direct sum $Z_{k_1} + \cdots + Z_{k_r}$[1] where k_i divides k_{i+1}.

[1] Throughout this paper if A and B are abelian groups, we shall denote by $A + B$ the direct sum of A and B.

DEFINITION. In a finitely generated abelian group A, a basis with the most possible elements will be called a *U-basis*; a basis with the fewest possible, a *minimal basis*. If w is a homomorphism of A into a group B, a *w-basis* will be a basis on all of whose elements except possibly one w is zero.

LEMMA C. *If A is a finitely generated abelian group, and w a homomorphism into Z_p, the cyclic group of order p, then A has a U-basis which is also a w-basis. If the order of the exceptional element in such a basis is p^i, $0 \leqq i \leqq \infty$, then i depends only on w, and $i(w) = i(w \circ \alpha)$ for any α in* $\mathrm{Aut}(A)$.

PROOF. Let e_1, e_2 be elements of a U-basis $\{e_1, \cdots, e_r\}$, of orders p^r, p^s with $r \leqq s$ and such that $w(e_1) = u$, $w(e_2) = ku$ for some u in Z_p. Then the set $\{e_2 - ke_1, e_1, e_3, \cdots, e_r\}$ is a U-basis and $w(e_2 - ke_1) = 0$. (The set clearly spans A and $gp\{e_1, e_2\} = gp\{e_1, e_2 - ke_1\} = gp\{e_1\} + gp\{e_2 - ke_1\}$, since if $n_1e_1 + n_2(e_2 - ke_1) = 0$ then p^s divides n_2 and so also p^r divides n_2 and thence n_1.) Since the basis is finite, such changes will eventually produce a U-basis having the required property. The possibilities $r, s = \infty$ are not excluded.

If w is not the zero homomorphism, then in every basis there is at least one element on which w is non-zero. If the distinguished element of a w-basis is of infinite order, then w is zero on the torsion subgroup of A. Since w is necessarily zero on q-primary components for $q \neq p$, to show that $i(w)$ is well defined it therefore suffices to consider A_p, the p-primary component of A.

Let $\{f_1, f_2, \cdots, f_r\}$ be a w-basis (and necessarily a U-basis) of A_p such that $w(f_1) \neq 0$. If e is any element of A_p with order less than that of f_1, then $w(e) = 0$, since for $e = n_1f_1 + \cdots + n_rf_r$ to have order less than f_1, p must divide n_1. Thus, if $w(e) \neq 0$, the order of e is at least as great as that of f_1; and if e is the exceptional element of a w-basis, then its order must be precisely that of f_1.

The image of a U-basis under an automorphism α of A is another U-basis. For independent elements map to independent elements since α is monomorphic and the images of the basis elements generate A since α is epimorphic, and clearly the new basis is also a U-basis. Thus the above argument also shows that $i(w) = i(w \circ \alpha)$.

COROLLARY. *If M is a simply connected 5-manifold then $i(M) = i(w^2(M))$ is a diffeomorphism invariant of M.*

0.6. There are two 3-disc bundles over the 2-sphere since $\pi_1(\mathrm{SO}_3) = Z_2$; denote these A, B where A is the trivial bundle $D^3 \times S^2$. Thus $\partial A = S^2 \times S^2$, $\partial B = P \# Q$ where P denotes the complex projective plane, and Q is the same space with the opposite orientation (see [21]). $H_2(A) = Z = H_2(B)$ and $H_2(\partial A) = Z + Z = H_2(\partial B)$ and we choose generators once for all as follows; choose generators u of $H_2(A)$, v of $H_2(B)$ and then choose generators a, b of $H_2(\partial A)$ corresponding to the factor S^2's, and generators p, q of $H_2(\partial B)$ corresponding to the summands P, Q, such that $i(a) = u$, $i(b) = 0$, $i(p) = v = i(q)$, where i is the homomorphism induced by the relevant inclusion. If $\cdot$ denote the intersection number of homology classes then $a \cdot b = 1$, $p \cdot p = 1$, $q \cdot q = -1$, and $a \cdot a = b \cdot b = p \cdot q = 0$.

Since $S^2 = D_N + \mathrm{id}\, D_S$, the union of its northern and southen hemispheres, and since any bundle over a disc is trivial, both A and B are of the form $D + fD$ with identification f along certain $D^3 \times S^1$'s in the bounding S^4 of each disc; i.e., they are of the form $D^5 + h^2$. Conversely any manifold of this form is one or another of these disc bundles (cf. [17]).

$w^2(A) = 0$ and $w^2(B) \neq 0$, i.e. $w^2(u) = 0$ and $w^2(v) \neq 0$ and so, if M is a simply connected manifold with $A \subset M$ or $B \subset M$, then $w^2(i(u)) = 0$ and $w^2(i(v)) \neq 0$ on account of the interpretation of w^2 as the obstruction to parallelisability over the 2-skeleton.

0.7. The following description of linking numbers is based on those of [16], [9] and [23], and we refer to [2] and [6] for the properties of the $\smile$- and $\frown$-products that we require.

Let M be a manifold with boundary ∂M having orientations μ, $\partial\mu$, so that $\mu\frown$ and $(\partial\mu)\frown$ are the duality isomorphisms in $(M, \partial M)$ and ∂M respectively, let β be the Bockstein in cohomology associated with the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{k} Q \xrightarrow{j} Q/Z \longrightarrow 0,$$

and let ε, ε' denote the homomorphisms $H_0(M; G) \to G$ and $H_0(\partial M; G) \to G$ respectively, induced by the augmentations of the chain complexes.

Let ξ be an element of $H_p(M, \partial M; Z)$ and x' of $H^{q+1}(M; Z)$ such that $\mu \frown x' = \xi$, where $p + q + 1 = n$. If ξ is a torsion element so is x', and in the exact sequence

$$H^q(M; Q) \xrightarrow{j^*} H^q(M; Q/Z) \xrightarrow{\beta} H^{q+1}(M; Z) \xrightarrow{k^*} H^{q+1}(M; Q),$$

associated with the above coefficient sequence, $k^*(x') = 0$ so that $x' = \beta(x)$ for some x in $H^q(M; Q/Z)$. For any η in $H_q(M; Z)$, $\varepsilon(\eta \frown x)$ is an element of Q/Z. If $\beta(x_1) = x'$, then $x_1 - x = j^*y$ for some y in $H^q(M; Q)$ and

$$\varepsilon(\eta \frown x_1) - \varepsilon(\eta \frown x) = \varepsilon(\eta \frown j^*y) = j^*(\eta \frown y) = j\varepsilon(\eta \frown y).$$

If η has finite order, this difference is zero since $\varepsilon\circ(\frown y)$ is a homomorphism from $H_q(M; Z)$ to Q, and so in this case $\varepsilon(\eta \frown x)$ is determined by η and ξ; it is called the *linking number* $b(\eta, \xi)$ of η with ξ. The definition of $b(\xi, \eta)$ is similar and there is an obvious simplification when $\partial M = \varnothing$.

LEMMA D. (i) *b is a non-singular bilinear form on the torsion subgroups of $H_q(M; Z)$ and $H_p(M, \partial M; Z)$.*

(ii) *$b(\xi, \eta) + (-1)^{pq} b(\eta, \xi) = 0$ where ξ, η are torsion elements of $H_p(M, \partial M; Z)$ and $H_q(M; Z)$ respectively.*

(iii) *$b(\partial\xi, \eta) = b(\xi, i_*\eta)$ where ξ, η are torsion elements of $H_p(M, \partial M; Z)$ and $H_q(\partial M; Z)$ respectively and ∂, i_* the corresponding homomorphisms of the homology sequence of $(M, \partial M)$.*

Proofs are straightforward and we give here only that of (iii).

Let $\eta = \partial\mu \frown \beta y$, then $i_*\eta = \mu \frown \delta(\beta y) = \mu \frown \beta(\delta y)$. Thus

$$b(\xi, i_*\eta) = \varepsilon(\xi \frown \delta y) = \varepsilon i_*(\partial\xi \frown y) = \varepsilon'(\partial\xi \frown y) = b(\partial\xi, \eta).$$

0.8. In particular when M is a $(4k + 1)$-dimensional closed manifold, linking numbers give a skew-symmetric non-singular bilinear form on $\operatorname{tors}(H_{2k}(M))$. In [23] Wall shows that this form determines completely the possibility of killing the middle homotopy groups of such manifolds. It also imposes restrictions on decompositions of and diffeomorphisms between the manifolds. We shall show (Theorem 2.2 and Corollary 2.2.1) that these and similar ones imposed by w^2 are, for 5-manifolds, the only restrictions. We need the following notation and results.

Let $b: G \otimes G \to Q/Z$ be a skew-symmetric non-singular bilinear form on the finite abelian group G.

DEFINITION. A *b-basis* of G is a basis $\{z_1, z_2, x_1, y_1, \cdots, x_k, y_k\}$ where z_1 has odd order φ, z_2 has order 2φ and $b(z_1, z_2) = 1/\varphi$, x_i and y_i have the same order θ_i and $b(x_i, y_i) = 1/\theta_i$, while all other linking numbers between these elements are zero except possibly $b(y_i, y_i)$ and $b(z_2, z_2)$. If G is a finitely generated abelian group and b is defined on its torsion subgroup, then a basis of G will be called a *b-basis* if it contains a b-basis of $\operatorname{tors}(G)$.

Note. In general, G will not have a basis as above, either z_1, or both z_1 and z_2 may be absent, but it will be convenient to insists on their presence and allow z_1 or both to be zero, adopting a similar convention for the terms 'basis', 'U-basis' and 'w-basis' when that is necessary. Note that the properties of b imply $b(z_1, z_1) = 0$ and $b(z_2, z_2) = 1/2$ if $z_2 \neq 0$.

For b as above the map $x \to b(x, x)$ determines a homomorphism of $\operatorname{tors}(G)$ into the subgroup $\{0, 1/2\}$ of Q/Z, which is isomorphic to Z_2. We denote this homomorphism of $\operatorname{tors}(G)$ to Z_2 by $w(b)$. A basis of G containing a $w(b)$-basis of $\operatorname{tors}(G)$ will be called a *$w(b)$-basis of G*.

LEMMA E. *Let* $b\colon G \otimes G \to Q/Z$ *be a skew-symmetric non-singular bilinear form on the torsion subgroup of the finitely generated abelian group* G. *Then* G *has a* U*-basis which is also a* b*-basis and* $w(b)$*-basis.*

PROOF. The elements of infinite order in any U-basis may be left as they are. It is shown in [23, Lemma 4(ii)] how to replace those of finite order by a b-basis of the torsion subgroup. The proof there is carried out separately on each p-primary component so the result is still a U-basis. If it is not a $w(b)$-basis, then we have, say, $b(y, y_1) = 1/2 = b(y_2, y_2)$. Thus 2 divides θ_1 and θ_2 and, being the order of an element of a U-basis, each is a power of 2 and if, say, $\theta_1 \leqq \theta_2$ we may define an automorphism of G by

$$x_1,\ x_2 \to x_1 - (\theta_2/\theta_1)x_2,\ x_2$$
$$y_1,\ y_2 \to y_1,\ y_1 + y_2 + (\theta_1/2)x_1$$

other basis elements being left fixed. The result is another U-, b-basis such that $w(b)$ is non-zero on one fewer element and, repeating this process as often as possible and then replacing y_1 by $y_1 + z_2$ ($z_1 = 0$ in a U-basis), we obtain a U-, b-, $w(b)$-basis.

Complement. G *has a* b-, $w(b)$-, *minimal basis.*

PROOF. Start from the b-, $w(b)$-, U-basis of the lemma. If $(\theta_1, \theta_2) = 1$, then there are k, q such that $k(\theta_1 + \theta_2) + q\theta_1\theta_2 = 1$. Then $k(x_1 + x_2)$ and $(y_1 + y_2)$ form a basis of $gp\{x_1, x_2, y_1, y_2\}$ and they have order $\theta_1\theta_2$ and linking number $k/\theta_1 + k/\theta_2 = 1/\theta_1\theta_2$ (in Q/Z). Similarly if θ_1 is odd x_1, y_1, z_2 may be replaced by z_1', $z_2' = x_1$, $y_1 + z_2$ so that $b(z_1, z_2) = 1/\theta_1$ and z_2 has order $2\theta_1$. These changes will not affect the property of being a $w(b)$-basis so we may use them to obtain a b-, $w(b)$-, minimal basis.

COROLLARY. $\text{Tors}(H_{2k}(M^{4k+1})) \cong B + B$ or $B + B + Z_2$ *for some finite abelian group* B.

In [23] Wall proves (Propositions 1 and 2):

LEMMA F. *Let* M *be a simply connected closed* 5*-manifold and* x *a torsion element of* $H_2(M; Z)$, *then* $b(x, x) \neq 0$ *if and only if* $w^2(x) \neq 0$. *The extra* Z_2 *appears in the above corollary if and only if* M *is in the non-zero cobordism class.*

Finally we note

LEMMA G. *If* M *is a simply connected closed* 5*-manifold, then* $H_2(M; Z)$ *has a* w^2-, b-, U*-basis and a* w^2-, b-, *minimal basis.*

PROOF. By Lemma F we may start from the $w(b)$-, b-, U-basis of Lemma E, respectively $w(b)$-, b-, minimal basis of its complement, and then make any alternative choices of the elements of infinite order that are necessary to obtain w^2-bases. These will clearly still be b- and U-, respectively minimal, bases.

1. The manifolds

For the notation used below see, 0.1, 0.6. Generators of the second homology groups of various copies of the disc bundles A, B will carry the same suffixes as the bundles.

Let M be an oriented 5-manifold and f an orientation preserving diffeomorphism of ∂M onto itself. Then, if M^* is a second copy of M but with the opposite orientation, f may be regarded as an orientation reversing diffeomorphism of ∂M^* onto ∂M and we may form the oriented closed manifold $M + fM^*$. We shall require f to realize given automorphisms f_* of $H_2(\partial M)$, obtaining f from the results of Wall in [25].

We note first the matrices

$$\mathbf{A}(k) = \begin{bmatrix} 1 & 0 & 0 & -k \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{B}(n) = \begin{bmatrix} 1 & n & -n & 0 \\ n & 1 & 0 & n \\ n & 0 & 1 & n \\ 0 & -n & n & 1 \end{bmatrix} \quad \mathbf{C}(n) = \begin{bmatrix} 1 & n & -n & 0 \\ 0 & n & -n & -1 \\ n & 0 & 1 & n \\ n & 1 & 0 & n \end{bmatrix}.$$

Construction. $M_1 = S^5$, $M_\infty = S^2 \times S^3$ and for $1 < k < \infty$, $M_k = (A_1 + A_2) + f_k(A_1 + A_2)^*$ where f_k realizes the automorphism $(f_k)_*(a_1, b_1, a_2, b_2) = (a_1, b_1, a_2, b_2)\mathbf{A}(k)$.

$X_{-1} = B + g_{-1}B^*$, $X_0 = S^5$, $X_\infty = B + g_\infty B^*$ and for $0 < j < \infty$, $X_j = (B_1 + B_2) + g_j(B_1 + B_2)^*$ where g_{-1} realizes the automorphism $(g_{-1})_*(p, q) = (p, -q)$, g_∞ realizes $(g_\infty)_*(p, q) = (p, q)$ and for $0 < j < \infty$, g_j realizes $(g_j)_*(p_1, q_1, p_2, q_2) = (p_1, q_1, p_2, q_2)\mathbf{B}(2^{j-1})$.

More generally construct $X[\mathbf{B}(n)]$ and $X[\mathbf{C}(n)]$ as were X_j, $0 < j < \infty$, but using the matrices $\mathbf{B}(n)$ and $\mathbf{C}(n)$ respectively instead of $\mathbf{B}(2^{j-1})$.

LEMMA 1.1. *All the manifolds are simply connected and*

(i) $H_2(M_k) = Z_k + Z_k$ *for* $k \neq 1, \infty$.

(ii) $H_2(X_{-1}) = Z_2$, $H_2(X_\infty) = Z$, $H_2(M_\infty) = Z$.

(iii) $H_2(X[\mathbf{B}(n)]) = Z_{2n} + Z_{2n}$, $H_2(X[\mathbf{C}(n)]) = Z_{2n-1} + Z_{4n-2}$.

(iv) $w^2(M_k) = 0$ *for all* k.

(v) $w^2(X) \neq 0$ *for* $X = X_{-1}, X_\infty, X[\mathbf{B}(n)]$ *or* $X[\mathbf{C}(n)]$.

In particular for $0 < j < \infty$, $H_2(X_j) = Z_{2^j} + Z_{2^j}$ *and* $w^2(X_j) \neq 0$.

PROOF. (i) Generators for $H_2(M_k)$ are carried by the images under inclusion of u_1 and u_2. This can be seen for example from the decomposition of M_k obtained from those of $A_1 + A_2$ and $(A_1 + A_2)^*$ as in 0.2. In this the only 2-

handles are those corresponding to A_1 and A_2. If i denote the homomorphism induced by the relevant inclusion map into M_k, then by the choice of f_k, $i(b_1{}^*) = i(ka_2 + b_1)$ and since, by the choice of these generators, $i(b_1) = 0 = i(b_1{}^*)$ and $i(a_2) = i(u_2)$ we get $k \cdot i(u_2) = 0$. Similarly $k \cdot i(u_1) = 0$, and as there are no other relations, (i) is proved; (ii) and (iii) follow by a similar argument. Thus for $1 < j < \infty$, the relations $i(p_1{}^*) = i(p_1 + 2^{j-1}q_1 + 2^{j-1}p_2)$ and $i(q_1{}^*) = i(2^{j-1}p_1 + q_1 - 2^{j-1}q_2)$ lead, since $i(p_1{}^*) = i(q_1{}^*)$, $i(p_1) = i(q_1)$ and $i(p_2) = i(q_2) = i(v_2)$, to the relation $2^i \cdot i(v_2) = 0$. The other relations give $2^j \cdot i(v_1) = 0$; (iv) follows from 0.4 since, by the handle decompositions mentioned above, the M_k are simply connected and generators of their second homology groups are carried by copies of the trivial 3-disc bundle A; (v) is similar. In particular $w^2(i(v_1)) \neq 0$.

Remark. (1) The manifolds M_k are those also called M_k by Smale in [19]. This follows from Lemma 1.1 and Smale's classification theorem.

(2) X_∞ is the non-trivial 3-sphere bundle over the 2-sphere. This can be seen by pulling the latter apart as in [21]. X_{-1} is the Wu manifold (cf. [4]).

(3) $X_1 = X_{-1} \# X_{-1}$, which will follow from Lemma 1.1 by Theorem 2.3, but otherwise the manifolds X_j and M_k are not decomposable. Except for M_2, this follows from Lemma E. However if M_2 were decomposable, one factor M' would have $H_2(M') = Z_2$, $w^2(M') = 0$ which, by Lemma F, is impossible.

LEMMA 1.2. (i) *$W^3(X_j)$ is non-zero except when $j = 0, \infty$.*

(ii) *$w^3(X_j)$ is non-zero only when $j = -1, 1$.*

PROOF. $W^3 = \delta^* w^2$ and as w^2 is known it is sufficient to calculate δ^*. Corresponding to the obvious handle decompositions the X_j, for $0 < j < \infty$, have cell decompositions with cells e^0, e_1^2, e_2^2, e_1^3, e_2^3, e^5. Denote (e_i) the integer chain carried by e_i, $[e_i]$ the dual cochain, $[e_i]_2$ its mod 2 reduction and let $\{\ \}$ denote the cohomology class. Thus (e_i^2) represent the generators of $H_2(X_j; Z)$ and $\partial(e_i^3) = 2^j(e_i^2)$; (cf. the proof of 1.1 (iii). For convenience we have swapped the labels of the 3-cells.)

The Bockstein is calculated from

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^2(X_j\,;Z) & \xrightarrow{\times 2} & C^2(X_j\,;Z) & \longrightarrow & C^2(X_j\,;Z_2) & \longrightarrow & 0 \\ & & \downarrow \delta & & \downarrow \delta & & \uparrow \delta & & \\ 0 & \longrightarrow & C^3(X_j\,;Z) & \xrightarrow{\times 2} & C^3(X_j\,;Z) & \longrightarrow & C^3(X_j\,;Z_2) & \longrightarrow & 0 \end{array}$$

$[e_i^2]_2$ lifts to $[e_i^2]$ in $C^2(X_j\,;Z)$ and, as $\delta[e_i^2] = 2^j[e_i^3] = (2^{j-1}[e_i^3]) \times 2$, $\delta^*\{[e_i^2]_2\} = 2^{j-1}\{[e_i^3]\}$. However the 2-cells e_i^2 correspond to the non-trivial bundles B_i, so $w^2(X_j) = \{[e_1^2]_2\} + \{[e_2^2]_2\}$ and $W^3(X_j) = \delta^* w^2(X_j) = 2^{j-1}\{[e_1^3] + [e_2^3]\}$. $w^3(X_j)$ is zero unless $j = 1$.

The results for $j = -1, \infty$ are proved similarly, there being only one 2-cell and one 3-cell and, in X_∞, $\partial[e^3] = 0$.

It remains to show how the diffeomorphisms required for building the manifolds are realized. Using a diffeomorphism of CP(2) which reverses the orientation of CP(1), it is clear how to realize g_{-1}, g_∞ is of course realized by the identity, and the automorphism determined by $\mathbf{C}(n)$ may be realized by a diffeomorphism which realizes the automorphism given by $\mathbf{B}(n)$ followed by a diffeomorphism which takes Q_2 onto Q_1 reversing the orientation of the projective line, takes Q_1 naturally onto Q_2 and leaves P_1 and P_2 alone.

For the rest, we need the results of Wall. We describe briefly the general diffeomorphism of [25], referring to that paper for the details. Let T denote $S^2 \times S^2$ or $P \# Q$ with generators x, y of $H_2(T)$ taken respectively as b, a, and p, $p - q$, i.e., such that $x \cdot y = 1$ and $y \cdot y = 0$. Then if N is a simply connected 4-manifold and β in $H_2(N)$ has $\beta \cdot \beta = 0$ there is diffeomorphism of $N \# T$ onto itself inducing the following automorphism of $H_2(N \# T)$:

$$
\begin{aligned}
y &\longrightarrow y \\
x &\longrightarrow x + \beta \\
\gamma \in H_2(N) &\longrightarrow \gamma - (\gamma \cdot \beta) y \, .
\end{aligned}
$$

To produce this diffeomorphism note that $\# T$ is equivalent to a spherical modification in N of some circle $f(S^1 \times 0)$ isotopic to zero. It is shown in [25] that β, being spherical, is also carried by an isotopy of $f(S^1 \times 0)$ finishing in its original position. The extended isotopy of N gives a diffeomorphism $h: N \to N$ taking, after some adjustment, a tubular neighborhood of $f(S^1 \times 0)$ to itself. The required diffeomorphism d, of $N \# T = (N - f(S^1 \times D^3)) + \mathrm{id}\, D^2 \times S^2$ is given by h on $N - f(S^1 \times D^3)$ and the identity on $D^2 \times S^2$. Clearly y, carried by $0 \times S^2$, is mapped to itself while x, carried in $N \# T$ by $D^2 \times 0$ together with an isotopy of $f(S^1 \times 1)$, $1 \in \partial D^3$, to zero in $N - f(S^1 \times D^3)$, maps to $x + \beta$. Now $\gamma \to \gamma + ay$ since the identity is induced on $H_2(N)$ and $0 = x \cdot \gamma = d(x) \cdot d(\gamma)$ shows that $a = -\beta \cdot \gamma$.

For $N = T = S^2 \times S^2$ take $\beta = -k a_1$. Then the resulting diffeomorphism realizes the automorphism determined by $\mathbf{A}(k)$.

Similarly taking $T = P_1 \# Q_2$, $N = Q_1 \# P_2$ and $\beta = n(q_1 + p_2)$ we may realize the automorphism given by $\mathbf{B}(n)$.

2. The theorems

THEOREM 2.0 (Markov [10]). *The class $\mathcal{C}$ of closed orientable C^∞-, n-manifolds ($n > 4$) is not classifiable under diffeomorphism, combinatorial equivalence, homeomorphism or homotopy type.*

Here each element of $\mathcal{C}$ is given by a triangulation, and by a classification would be understood the finding of a class of pairwise inequivalent manifolds and a finite algorithm to determine, from its triangulation, to which canonical manifold an arbitrary manifold of $\mathcal{C}$ is equivalent. The theorem follows trivially from

THEOREM 2.1. *There is no algorithm for $\mathcal{C}$ to determine whether an arbitrary member is simply connected.*

OUTLINE PROOF (for details see [10]). Given any group $G(r, k)$ with r generators and k relations between them, construct the manifold $M(r, k) = D^n + h_1^1 + \cdots + h_r^1 + h_1^2 + \cdots + h_k^2$ where the attaching maps for the 2-handles wind around the 1-handles according to the k relations. Then such an algorithm applied to $M(r, k) + \mathrm{id}\, M(r, k)$ would lead to an algorithm to determine whether $G(r, k) = 1$. Adyan [1] has shown that this is not possible.

THEOREM 2.2. *Let X and M be simply connected closed 5-manifolds and $\theta : H_2(X) \to H_2(M)$ an isomorphism preserving linking numbers and such that $w^2(M) \circ \theta = w^2(X)$. Then there is an orientation preserving diffeomorphism $f : X \to M$ such that $f_* = \theta$.*

The proof will occupy paragraphs 3, 4 and 5.

COROLLARY 2.2.1. *For each b-basis $\{z_1, z_2 ; x_1, y_1, \cdots, x_r, y_r ; e_1, \cdots, e_s\}$ of $H_2(M ; Z)$, there is a diffeomorphism f of M onto $N = M_{z_1,z_2} \# M_{x_1,y_1} \# \cdots \# M_{e_s}$, where, if $(u, v) = (z_1, z_2)$ or (x_i, y_i), f induces a b-preserving isomorphism of $gp\{u, v\}$ onto $H_2(M_{u,v})$ and $H_2(M_{e_i}) = Z$ with $w^2(M_{e_i}) = 0 \Leftrightarrow w^2(e_i) = 0$.*

Construction. Take $M_{z_1,z_2} = X[\mathrm{C}((\varphi - 1)/2)]$ if the order φ of z_1 is not 1, and $M_{z_1,z_2} = X_{-1}$ if $\varphi = 1$. Take $M_{x_i,y_i} = M_{\theta_i}$, θ_i the order of x_i and y_i, when $b(y_i, y_i) = 0$ and $M_{x_i,y_i} = X[\mathbf{B}(\theta_i/2)]$ if $b(y_i, y_i) \neq 0$. Take $M_{e_i} = M_\infty$ if $w^2(e_i) = 0$ and $M_{e_i} = X_\infty$ if $w^2(e_i) \neq 0$.

PROOF. These manifolds have the correct second homology group and second Stiefel-Whitney class. Thus, taking a b-, minimal basis u', v' of $H_2(M_{u,v})$, the isomorphism $u, v \to u', v'$ is b-preserving by Lemma F. Taking these together with isomorphisms mapping each e_i to either generator of $H_2(M_{e_i})$ we have an isomorphism of $H_2(M)$ onto $H_2(N)$ satisfying the hypotheses of the theorem which we may therefore apply to obtain f.

COROLLARY 2.2.2. *Corresponding to any b-basis of $H_2(M ; Z)$, M has a handle decomposition with one 0-handle, one 5-handle, and one 2-handle, and one 3-handle for each element of the basis. In particular, M has a decomposition with the minimum number of handles consistent with its homology.*

PROOF. This follows from the preceding corollary since each factor manifold has such a decomposition, as is clear from their method of construction; and, since $\#$ is equivalent to removing a 0-handle and a 5-handle from the disjoint sum, the same is therefore true of M. To obtain a minimal decomposition, use a b-, minimal basis (Lemma G).

THEOREM 2.3. *The class of simply connected, closed, smooth, oriented* 5-*manifolds is classifiable under diffeomorphism. A canonical set is* $X_j \# M_{k_1} \# \cdots \# M_{k_s}$ *where* $-1 \leqq j \leqq \infty$, $s \geqq 0$, $1 < k_1$ *and* k_i *divides* k_{i+1} *or* $k_{i+1} = \infty$. *A complete set of invariants is provided by* $H_2(M)$ *and* $i(M)$.

Remark. When $i(M) = 0$, this reduces to the classification of Smale [19].

PROOF. For $i(M)$ see 0.5. That this, with $H_2(M)$, distinguishes between the canonical manifolds, follows from Lemma 1.1 and the restrictions on the k_i. For two of these manifolds can only have the same second homology group if X_j in one is replaced by M_{2^j} in the other ($j > 0$), in which case $i(M)$ is j for the first, and zero for the second. That an arbitrary manifold of the class is diffeomorphic to one of the canonical manifolds may be seen as follows. Take a w^2-, b-, U-basis of $H_2(M)$, remove from it z_2 and x_1, y_1 if $b(y_1, y_1) \neq 0$ or e_1 if $w^2(e_1) \neq 0$ and let G denote the group generated by the remaining elements. Then $w^2(G) = 0$ and b restricts to a non-singular form on G and a b-, minimal basis of this (Lemma G) together with the excluded elements gives a basis of $H_2(M)$ which determines an obvious b-preserving, w^2-preserving isomorphism onto the second homology group of one of the canonical manifolds and so, by Theorem 2.2, a diffeomorphism of M onto this manifold. Note that in the canonical decomposition X_j is determined by $j = -1$ if $z_2 \neq 0$ and $j = i(M)$ otherwise, the remaining factors being determined by $H_2(M)$.

There is no difficulty in describing an algorithm to determine $H_2(M)$ and $i(M)$ from the triangulation. For $i(M)$, $w^2(M)$ must be calculated on a set of generators of $H_2(M)$; this could be done using the Wu formulas or, from Lemma F, by calculating the linking numbers.

COROLLARY 2.3.1. *The same classification is valid for homotopy type, for combinatorial equivalence or for homeomorphism.*

PROOF. The invariants, being obtainable from homology, cohomology and cap-products, are homotopy type invariants. Conversely, diffeomorphism implies each of the above relations.

COROLLARY 2.3.2. *The same classification applies to closed, simply connected combinatorial* 5-*manifolds.*

PROOF. J. Cerf has shown that $\Gamma^4 = 0$, and so every combinatorial 5-

manifold has a compatible differential structure.

LEMMA 2.4. *Let M be a simply connected 5-manifold, then*
(i) $M \propto R^8$ (*M is immersible in* R^8).
(ii) $M \propto R^7 \Leftrightarrow W^3(M) = 0$.
(iii) $M \propto R^6 \Leftrightarrow w^2(M) = 0$.

PROOF (see Hirsch [7]). The only possible non-zero Stiefel-Whitney classes are W^3 and w^2. Hence if ν is the normal bundle of an embedding of M in R^{11}, the total Whitney class (mod 2) is $w(\nu) = 1 + w^2 + w^3$. Moreover $W^3(\nu) = W^3(M)$ since each is $\delta^*(w^2)$.

If $w^2 = 0$, ν has a 5-frame cross-section over the 2-skeleton. The obstructions to extending this over M which correspond to non-zero $H^r(M)$ have coefficients in $\pi_2(V_{6,5})$ and $\pi_4(V_{6,5})$ which are both zero [15], so there is a 5-frame field over all M. Thus by [7] it is possible to immerse M in space of five fewer dimensions, that is in R^6. Similarly if $W^3 = 0$, ν has a 4-frame field over M ($\pi_4(V_{6,4}) = 0$), and there is always a 3-frame section of ν. Thus if $W^3(M) = 0$, M is immersible in R^7, and in any case it is immersible in R^8.

Conversely if $M \propto R^7$ then, multiplying by R^4, $W \propto R^{11}$ with a 4-frame field in the normal bundle and so $W^3(\nu)$ is zero. Similarly $M \propto R^6$ implies that $w^2(\nu)$ is zero.

COROLLARY 2.4.1. (i) $i(M) = 0$ *or* $\infty \Leftrightarrow M \propto R^7$,
(ii) $i(M) = 0 \Leftrightarrow M \propto R^6$.

PROOF (By Lemmas 1.2 and 2.4 and Theorem 2.3). Since by the last $M = X_j \# M'$ with $i(M') = 0$ and so $w^2(X_j) = 0 \Leftrightarrow w^2(M) = 0$, $W^3(X_j) = 0 \Leftrightarrow W^3(M) = 0$.

THEOREM 2.5. *Let M be a simply connected 5-manifold then*
(i) $M \subset R^9$ (*M is embeddable in* R^9).
(ii) $M \subset R^8 \Leftrightarrow i(M) = 0$ *or* ∞.
(iii) $M \subset R^6 \Leftrightarrow i(M) = 0 \Leftrightarrow M \subset R^7$.

PROOF. (i) Follows from Haefliger's Theorem B.

(ii) By Theorem 2.3, if $i(M) = \infty$, then $M = X_\infty \# M'$ where $i(M) = 0$, so assuming the embedding of (iii), it suffices to embed X_∞ in R^8. Look at the non-trivial disc bundle B. This has $W^3(B) = 0$ and so by Lemma 2.4 $B \propto R^7$. After a suitable isotopy we may assume the zero cross-section of B, and with it a sufficiently small tubular neighborhood, is embedded. But this neighborhood is again B, so $B \subset R^7$. Now if $h : B \to R^7$ is such an embedding then $(h(B) \times \{-1\}) + \mathrm{id}\,(h(\partial B) \times [-1, 1]) + \mathrm{id}\,(h(B) \times \{1\})$ is an embedding of $X_\infty = B + \mathrm{id}\, B$ in R^8 once it has been smoothed. Conversely by [13, Theorem 14] if

$M^5 \subset R^{5+k}$ with normal bundle ν, then $\chi(\nu) = 0$, where χ denotes the Euler class. However if $k = 3$, then $\chi(\nu) = W^3(M)$ (cf. [13, Theorem 22]) and so by Lemma 1.2, $i(M) = 0$ or ∞ (cf. the proof of 2.4.1.).

(iii) By a similar argument $M \subset R^7$ implies that $w^2(M) = 0$ and so $i(M) = 0$, and it remains to construct the embeddings of these manifolds in R^6. Let $V \subset R^6$ be $D^4 \times S^2$. For any k, k times the generator of $H_2(V)$ is carried by a 2-sphere S_k embedded in the boundary. The closure X of the complement of V in R^6 is 2-connected and so S_k is isotopic to zero in X by Theorem B. The 3-disc formed by the isotopy may intersect itself, however using Haefliger's theorem again, it may be replaced by an embedded 3-disc. This disc together with an embedded normal bundle form a 3-handle h^3 on V and, if $W = V + h^3$, then $H_0(W) = Z$, $H_2(W) = Z_k$, and $H_r(W) = 0$ otherwise. ∂W is clearly simply connected, and computation of the homology sequence of $(W, \partial W)$ shows that $H_2(\partial W)$ is an extension of Z_k by Z_k and so must be $Z_k + Z_k$ since it has an element of order k and is of the form $B + B$. Thus, since $w^2(W) = w^2(V) = 0$, by Theorem 2.3, $\partial W \cong M_k$, and by the same theorem the connected sum of these for suitable k gives the general M with the $i(M) = 0$.

3. Cobordisms between the manifolds

We shall prove Theorem 2.2 by obtaining an h-cobordism between X and M and then using Theorem A. The existence of a b-preserving w^2-preserving isomorphism between their second homology groups ensures that X and M are cobordant by Lemma F, and the first step is to find a cobordism which is simply connected and has the smallest possible second homology group. This will then be modified into an h-cobordism.

LEMMA 3.1. *If X and M are cobordant simply connected 5-manifolds with second Stiefel-Whitney classes either* (i) *both zero, or* (ii) *both non-zero, then a simply connected cobordism V between them may be found with $H_1(V) = 0$ in case* (i) *and $H_2(V) = Z_2$ in case* (ii).

PROOF. (i) When $w^2(X) = 0$, then since the 5-dimensional spinor cobordism group is zero, there is a cobordism V' between X and M with $w^2(V') = 0$. That is V' is 2-parallelisable and, by [13, Theorem 3], may be replaced by a 2-connected cobordism V. Alternatively we may use Lemma 1.3 of [19] to construct V directly; that lemma says that each of X and M bounds a manifold of the form $D^6 + 3$-handles, and the connected sum of these with suitable orientations will provide a V.

(ii) When $w^2(X) \neq 0$, then neither is $w^2(V)$. For

$$i^*w(V) = w(\tau(V) \mid X) = w(\tau(X) \oplus \varepsilon^1) = w(X) \cdot w(\varepsilon^1) = w(X) ,$$

where w is the total Stiefel-Whitney class, and the trivial factor ε^1 is the inward normal to V along X. In particular $w^2(X) = i^*w^2(V)$. Choosing any cobordism V' then on account of dimensions, e.g., by Theorem B, any map of a 1-sphere or 2-sphere into V' may be replaced by an embedding, and since V' is orientable any embedded 1-sphere has trivial normal bundle. Surgery [12], [9], [27], can therefore be applied to obtain a simply connected cobordism V''. Now if S^2 embedded in V'' carries x in $H_2(V'')$, then $w^2(x)$ is the obstruction to triviality of the normal bundle (see 0.4). Thus all the generators of $H_2(V'')$ may be killed except those on which w^2 is non-zero. By 0.5 this need only be one, say x. Since $w^2(2x) = 2\cdot w^2(x) = 0$, $2x$ may also be removed leaving a cobordism V with $H_2(V) = Z_2$.

4. Modification of the cobordisms

The cobordism of Lemma 3.1 is not necessarily an h-cobordism, having in general too small a second homology group and too large a third. In this section we describe the modification which will be used to rebuild the second homology group, and that which is required to trim the third. For Theorem 2.2 we require not just an h-cobordism but one V, such that the induced diffeomorphism between X and M will realize a given isomorphism of second homology groups. Thus a corresponding pair of elements from these groups must map to the same element of $H_2(V)$, or in other words their difference must lie in the kernel of the homomorphism induced by inclusion. Lemma 4.3 is therefore stated in a way which shows just how much control we have of this kernel when each modification is made. Two preparatory lemmas are necessary.

LEMMA 4.1. *If V is a simply connected 6-manifold, then any element of $H_3(V)$ is carried by an embedded sphere.*

PROOF. Since $\pi_1(V) = 1$, it follows by G.W.Whitehead's extension of the Hurewicz theorem [29] that the Hurewicz homomorphism $\pi_3(V) \to H_3(V)$ is an epimorphism. That is, the elements of $H_3(V)$ are carried by maps of spheres which, by Theorem A, are homotopic and so homologous to embedded spheres.

LEMMA 4.2. *If $\delta : G \to X + H$ is an epimorphism of finitely generated abelian groups with X cyclic and $\delta(\mathrm{tors}(G)) \subset H$, and if there is an indivisible element in the kernel of δ, then G may be written as the direct sum $Z + G'$ where Z is infinite cyclic and $\delta Z = X$, $\delta G' = H$.*

PROOF. Let x be a generator of X, z' an indivisible element of the kernel of δ, and write $G = gp\{z'\} + A$. Then $x = \delta a$ for some a in A and so $x = \delta z$ where $z = z' + a$ is an indivisible element of infinite order which therefore generates an infinite cyclic direct summand Z of G. Let $\{z, y'_1, \cdots, y'_r, t_1, \cdots, t_s\}$ be a basis of G with the y'_i of infinite and the t_j of finite order. If $\delta y'_i =$

$\lambda_i x + h_i$ with h_i in H, and if $y_i = y_i' - \lambda_i z$ then $\{z, y_1, \cdots, y_r, t_1, \cdots, t_s\}$ also is a basis with δy_i in H by construction and δt_j in H by hypothesis. Thus if $G' = gp\{y_1, \cdots, y_r, t_1, \cdots, t_s\}$ the decomposition $G = Z + G'$ satisfies the requirements of the lemma.

We are now in a position to prove the main lemmas.

LEMMA 4.3. *Let V be a simply connected compact 6-manifold with $H_2(\partial V) = X + H$ where X is cyclic, the image of the torsion subgroup of $H_3(V, \partial V)$ is contained in H and the kernel of $i: H_2(\partial V) \to H_2(V)$ is the sum of subgroups of X and H. Then there is a manifold U with the same boundary such that* $\ker(j) = \ker(i) \cap H$, *and there is a natural epimorphism $k: H_2(U) \to H_2(V)$ with $k \circ j = i$ and* $\ker(k) \subset j(X)$, *where $j: H_2(\partial V) \to H_2(U)$ is the homomorphism induced by the inclusion.*

PROOF. Let x generate X and λ be the least positive integer such that $i(\lambda x) = 0$, then by hypothesis $\ker(i) = \lambda X + K$ with $\partial(\mathrm{tors}(H_3(V, \partial V))) \subset K \subset H$, where ∂ is the boundary homomorphism. Then, taking the connected sum of V with $S^3 \times S^3$ if necessary to provide an indivisible element in the kernel of ∂, we may apply Lemma 4.2 to find a decomposition $H_3(V, \partial V) = Y + G$ such that Y is infinite cyclic and $\partial Y = \lambda X$, $\partial G = K$. Choose a basis $\{y, g_1, \cdots, g_k\}$ of $H_3(V, \partial V)$ with y in Y and g_i in G, and then find v in $H_3(V)$ with intersection number 1 with y and zero with each g_i. By Lemma 4.1, v is carried by an embedded 3-sphere S_v and as this necessarily has a trivial normal bundle we may carry out a spherical modification over it.

Let $S_v \times D^3$ be a tubular neighborhood of S_v avoiding ∂V, and form $W = V - (S_v \times D^3)$ and $U = W + \mathrm{id}(D^4 \times S^2)$. Note that W and U are simply connected and that, since U may be obtained from W by adding a 4-handle and a 6-handle so that $H_2(U, W) = 0$, $H_3(U, W) = 0$, the inclusion induces an isomorphism of $H_2(W)$ onto $H_2(U)$. We shall use this isomorphism to identify these groups and in particular replace j by the homomorphism $H_2(\partial V) \to H_2(W)$ induced by inclusion. In the sequence

$$H_3(V) \longrightarrow H_3(V, W) \longrightarrow H_2(W) \xrightarrow{k} H_3(V) \longrightarrow H_2(V, W),$$

$H_2(V, W) = 0$ and $H_3(V, W) = Z$ with its generator carried by a disc which is a fibre of the tubular neighborhood of S_v, the homomorphism from $H_3(V)$ is given by the inclusion in $H_3(V, \partial V)$ followed by taking the intersection number with v, and the image of $H_3(V, W)$ in $H_2(W)$ is generated by the element u carried by the boundary of the above disc fibre. Thus k is an epimorphism with kernel generated by u, and $k \circ j = i$ since all three are induced by inclusions.

The element y in $H_3(V, \partial V)$ has a representative chain which meets $S_v \times D^3$ just once, along a disc fibre $* \times D^3$; any representative is homologous to one 'pushed off' the disc $(S_v - *) \times D^3$ and then copies of $* \times D^3$ with opposite sign may be deleted from the result leaving eventually, since the intersection number of y with v is one, the required chain (cf. [23]). Thus u, carried by $* \times \partial D^3$, is homologous in W to $\partial y = \lambda x$, that is, $j(\lambda x) = u$ and so the kernel of k is in $j(X)$.

$\mathrm{Ker}(i) \cap H = K$, which by the choice of G is ∂G, so it remains to be shown that $\partial G = \ker(j)$. But any element of G has, as above, a representative chain avoiding $S_v \times D^3$ so that the elements of ∂G are homologous to zero in W. Conversely any element of $\ker(j)$ is the image of an element of $H_3(W, \partial V)$ any representative chain of which, *a fortiori*, misses $S_v \times D^3$. This chain represents in $H_3(V, \partial V)$ an element $\mu z + \mu_i g_i$ having intersection number μ with v. Thus $\mu = 0$ and the chosen element of $\ker(j)$ is in ∂G.

Remark. The conclusions of the lemma concerning 'k' ensure that j will be an epimorphism if i was.

LEMMA 4.4. *Let V be a simply connected cobordism between simply connected 5-manifolds X and M such that the inclusions induce isomorphisms $H_2(X) \xrightarrow{i} H_2(V)$ and $H_2(M) \xrightarrow{j} H_2(V)$ where $j^{-1} \circ i = \theta$. Then V may be replaced by an h-cobordism with the inclusions inducing the same isomorphism θ between second homology groups.*

PROOF. $H_3(V, X)$ and $H_3(V, M)$ are free, and intersection numbers provide a non-singular pairing between them. Choose x in $H_3(V, X)$ and m in $H_3(V, M)$ such that $x \cdot m = 1$. These elements lift into $H_3(V)$ and so, by Lemma 4.1, may be carried by embedded spheres S_x, S_m which we may assume meet transversely and, using Whitney's method for the removal of intersections [30], in just one point P. Moreover we may choose tubular neighborhoods N_x, N_m of the spheres which meet only over small discs about P in S_x and S_m respectively, the corresponding subbundles over these discs coinciding, with fibres of one being cross-sections of the other. Let T' be $N_x + N_m$ with the natural identification across their intersection. $\partial T'$ is a homotopy sphere and so [17], [20], [31] a sphere and, since $\Theta_5 = 0$ [11], by Theorem A diffeomorphic to the standard sphere. Thus we may form $T = T' + \mathrm{id}\, D^6$, $U = (V - \mathring{T}') + \mathrm{id}\, D^6$. Then $V = U \# T$ and the inclusions of X and M in V factor through that of $(U - \mathring{D}^6)$ in V. Thus i and j factor through the isomorphism $k: H_2(U) \to H_2(V)$ giving isomorphisms $i' = k^{-1} \circ i$, $j' = k^{-1} \circ j$ onto $H_2(U)$, and clearly $(j')^{-1} \circ i' = \theta$. But $H_3(V, X) = H_3(U, X) + H_3(T)$ so $H_3(U, X)$ has smaller rank than $H_3(V, X)$ and we may repeat this procedure until $H_3(U, X) = 0$ when we shall have the required h-cobordism.

5. Proof of Theorem 2.2

Recall that X and M are simply connected closed 5-manifolds, $\theta : H_2(X) \to H_2(M)$ is a w^2-preserving, b-preserving isomorphism and we require an orientation preserving diffeomorphism $f: X \to M$ such that $f_* = \theta$. For any cobordism V between X and M, $H_2(\partial V) = H_2(X) + H_2(M)$, and the homomorphisms of $H_2(X)$ and $H_2(M)$ into $H_2(V)$ induced by inclusions are just the restrictions of $i : H_2(\partial V) \to H_2(V)$. Note that the inclusion of M in ∂V is orientation reversing and so the sign of all linking numbers is changed. Let $\{z, x_i, y_i, e_k\}$ be a w^2-, b-, U-basis of $H_2(X)$ (see Lemma G; we omit z_1 which is necessarily zero), and $\{\theta(z), \theta(x_i), \theta(y_i), \theta(e_k)\}$ the corresponding basis of $H_2(M)$. Then starting from the 'minimal' cobordism of Lemma 3.1, we construct by repeated application of 4.3 a cobordism W such that $H_2(W)$ has a basis $\{i(z), i(x_i), i(y_i), i(e_k)\}$ where, for each x in $H_2(X)$, $i(x)$ has the same order as x and $i(x) = i(\theta(x))$. We use Lemma 4.4 to obtain an h-cobordism with the same property and then Theorem A to obtain the diffeomorphism f.

Take the basis

$$\{z,\ z - \theta(z),\ x_i,\ x_i - \theta(x_i),\ y_i,\ y_i - \theta(y_i),\ e_k,\ e_k - \theta(e_k)\}$$

for $H_2(\partial V)$ and adopt the convention that when Y is the cyclic subgroup generated by one of these elements, then H shall be the subgroup generated by the rest. We shall describe first the general step in the construction of W and then indicate how the initial step may differ from this.

Assume V is a cobordism with $H_2(V)$ having a basis

$$\{i(z),\ i(x_1),\ i(y_1),\ \cdots,\ i(x_n),\ i(y_n),\ i(e_1),\ \cdots,\ i(e_m)\}\ ,$$

where all these elements have the same order as in $H_2(X)$, and i is zero on the other basis elements of $H_2(\partial V)$ chosen above. Thus in particular $i(x) = i(\theta(x))$ for all x in $H_2(X)$. Write $H_2(\partial V) = Y + H$, with the above convention, where $Y = gp\{x_{n+1}\}$. Then the kernel of i is the sum of Y and a subgroup of H. For the torsion elements $i(z)$, $i(x_1)$, $\cdots$, $i(y_n)$ of the basis of $H_2(V)$, let $t(z)$, $t(x_1)$, $\cdots$, $t(y_n)$ be the corresponding elements of the 'dual' basis of $\operatorname{tors}(H_3(V, \partial V))$. Then if ∂ denotes the boundary homomorphism, using Lemma D(iii) we get $b(\partial t(x_1), x_1) = b(t(x_1), i(x_1)) = 1/\theta_1$, $b(\partial t(x_1), \theta(x_1)) = 1/\theta_1$, $b(\partial t(x_1), y_1) = 0$, etc., where θ_1 is the order of x_1 and y_1. But, since we have b-bases of $H_2(X)$, and $H_2(M)$, this identifies $\partial t(x_1)$ as $y_1 - \theta(y_1)$ if $b(y_1, y_1) = 0$, or $y_1 - \theta(y_1) - (\theta_1/2)(x_1 - \theta(x_1))$ if $b(y_1, y_1) \neq 0$. Similarly we may identify $\partial t(y_1) = x_1 - \theta(x_1)$, $\partial t(z) = z - \theta(z)$, etc., all of which are in the group H. Thus we may apply Lemma 4.3 to this decomposition of $H_2(\partial V)$. In the resulting cobordism U, $j : H_2(\partial V) \to H_2(U)$ is monomorphic on all those generators on which i was and also, since $\ker(j) \cap Y = 0$, on x_{n+1} and, since i and j have the same kernel on H, j is zero on the remaining generators. Moreover j is an epimorphism since i was (cf. the remark after Lemma 4.3) and so $H_2(U)$ has a basis similar to that which we started from in $H_2(V)$ but having one extra generator, that corresponding to x_{n+1}. The generators $i(y_{n+1})$ and $i(e_{m+1})$ are realized in a similar manner and we may repeat the procedure until we have a cobordism W for which i restricts to an isomorphism of $H_2(X)$ onto $H_2(W)$, and the kernel of i is generated by those elements, $z - \theta(z)$, $x_1 - \theta(x_1)$, etc., which ensure that $i(x) = i(\theta(x))$ for all x in $H_2(x)$.

If $w^2(X) = 0$, the initial step in the construction of W does not differ from the general one since the cobordism V of Lemma 3.1 then has zero second homology group. When $w^2(X) \neq 0$, and so $H_2(V) = Z_2$, three cases can arise. Note first that, writing second Stiefel-Whitney classes as homomorphisms, $w^2(\partial V) = w^2(V) \circ i$ (see the proof of 3.1(ii)), and so $i(x) = 0$ if and only if $w^2(x) = 0$ for any x in $H_2(X)$ and, by Lemma F, if and only if $b(x, x) = 0$ for torsion elements. Thus if $z \neq 0$, $i(z)$ is the generator of $H_2(V)$ and so is $i(\theta(z))$ and therefore all the chosen generators of $H_2(\partial V)$ except z are in the kernel of i(recall that we chose a w^2-basis) and we are ready to continue with the general procedure. If $z = 0$ but $b(y_1, y_1) \neq 0$, then we work as above with $Y = gp\{y_1\}$. This time $\ker(i)$ is the sum of H and a subgroup of Y and $\partial t(y_1) = (\theta_1/2)\,(x_1 - \theta(x_1))$. Again Lemma 4.3 may be used and it will replace $i(y_1)$ by a generator of the same order as y_1, i mapping all other generators to zero. Similarly if $z = 0$, $b(y_1, y_1) = 0$ but $w^2(e_1) \neq 0$, $i(e_1)$ may be replaced by an element of infinite order without disturbing the other elements of the basis, and we are ready to continue as before.

Thus in all cases we shall obtain a cobordism W between X and M satisfying the hypotheses of Lemma 4.4, with θ being the isomorphism between $H_2(X)$ and $H_2(M)$ given in the hypotheses of Theorem 2.2. So there is an h-cobordism W' between X and M inducing the same isomorphism. But by Theorem A, W' is diffeomorphic to $X \times I$. This induces an orientation preserving diffeomorphism f of X with M and clearly the isomorphism of second homology group induced by f is θ.

DOWNING COLLEGE, CAMBRIDGE AND
CORNELL UNIVERSITY

REFERENCES

1. S. I. ADYAN, *Algorithmic insolubility of the problem of determining certain properties of groups*, Dokl. Akad. Nauk. S.S.S.R., 103 (1955), 533–535.
2. W. BROWDER, Cap products and Poincaré duality, mimeographed note, Cambridge Topology Symposium, 1964.

3. J. CERF, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France, 89 (1961), 227–380.
4. A. DOLD, *Erzeugende der Thomschen Algebra* $\mathfrak{N}$, Math. Zeit. 65 (1956), 25–35.
5. A. HAEFLIGER, *Plongements différentiables de variétés dans variétés,* Comment. Math. Helv., 36 (1961), 47–82.
6. P. J. HILTON and S. WYLIE, Homology Theory, Cambridge University Press, 1960.
7. M. W. HIRSCH, *Immersions of manifolds,* Trans. Amer. Math. Soc., 93 (1959) 242–276.
8. I. KAPLANSKY, Infinite abelian groups, University of Michigan, 1954.
9. M. A. KERVAIRE and J. W. MILNOR, *Groups of homotopy spheres*: I, Ann. of Math., 77 (1963), 504–537.
10. A. A. MARKOV, The insolubility of the problem of homoeomorphy, Proc. Int. Congress of Math. 1958, Cambridge 1960.
11. J. W. MILNOR, Differentiable manifolds which are homotopy spheres, (mimeographed), Princeton University, 1959.
12. ———, A procedure for killing the homotopy groups of differentiable manifolds, Symposia in Pure Mathematics III: Differential Geometry. Amer. Math. Soc., 1961.
13. ———, Lectures on characteristic classes, (mimeographed), Princeton University, 1957.
14. S. P. NOVIKOV, *Some properties of* $(4k + 2)$*-dimensional manifolds,* Soviet Mathematics, 4 (1963) 1768–1772.
15. G. F. PAECHTER, *On the groups* $\pi_r(V_{m,n})$: I, Quart. J. Math. (Oxford), 7 (1956), 249–268.
16. H. SEIFERT and W. Threllfall, Lehrbuch der Topologie, Leipzig, 1934.
17. S. SMALE, *Generalized Poincaré's conjecture in dimensions greater than four,* Ann. of Math., 74 (1961), 391–406.
18. ———, *On the structure of manifolds,* Amer. J. Math., 84 (1962), 387–399.
19. ———, *On the structure of* 5*-manifolds,* Ann. of Math., 75 (1962), 38–46.
20. J. R. STALLINGS, *Polyhedral homotopy spheres,* Bull. Amer. Math. Soc., 66 (1960), 485–488.
21. N. E. STEENROD, The Topology of Fibre Bundles, Princeton, 1951.
22. R. THOM, *Quelques propriétés globales des variétés différentiables,* Comment. Math. Helv., 28 (1954), 17–86.
23. C. T. C. WALL, *Killing the middle homotopy group of odd dimensional manifolds,* Trans. Amer. Math. Soc., 103 (1962), 421–433.
24. ———, Differential topology seminar notes, (mimeographed), Cambridge University 1962–1964.
25. ———, *Diffeomorphisms of* 4*-manifolds,* J. London Math. Soc., 39 (1964), 131–140.
26. ———, *Classification of* $(n - 1)$*-connected* $(2n + 1)$*-manifolds,* to appear.
27. A. H. WALLACE, *Modifications and cobounding manifolds*: I, Canad. J. Math., 12 (1960), 503–528.
28. ———, *Modifications and cobounding manifolds*: II, J. Math. and Mech., 10 (1961), 773–809.
29. G. W. WHITEHEAD, *On spaces with vanishing low-dimensional homotopy groups,* Proc. Nat. Acad. Sci., U.S.A., 34 (1948), 207–211.
30. H. WHITNEY, *The self-intersections of a smooth n-manifold in 2n-space,* Ann. of Math., 45 (1944), 220–246.
31. E. C. ZEEMAN, *The generalised Poincaré conjecture,* Bull. Amer. Math. Soc., 67 (1961), 270.

(Received May 20, 1963)
(Revised March 4, 1965)