SOME CALCULATIONS WITH MILNOR HYPERSURFACES AND AN APPLICATION TO GINZBURG'S SYMPLECTIC BORDISM RING

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Introduction

In this note we present some formulæ in complex bordism associated to *Milnor hypersurfaces* which we then apply to prove the surjectivity of a certain homomorphism

$$\sigma\colon \Omega_*^{\mathrm{Sym}} \longrightarrow MU_*(\mathbb{C}\mathrm{P}^\infty)$$

whose domain is the *symplectic bordism ring* of V. L. Ginzburg [2]. As Ginzburg proved σ to be injective, this establishes it to be an isomorphism, a result first proved by J. Morava [3] by a more topological argument. We also make some observations on the associated homology theory. For the benefit of topologists, we remark that the notion of *manifold with symplectic structure* is distinct from that of manifold with normal reduction to some compact symplectic group $Sp(N) \subset U(N)$. The resulting bordism ring Ω_*^{Sym} may not be the homotopy ring of a Thom complex since there is apparently no transversality theory for such symplectic manifolds; the more familiar symplectic bordism ring is of course the homotopy of the Thom spectrum *MSp*.

For background notation and basic notions of complex bordism we refer to Adams [1].

This material came about as a result of correspondence with Jack Morava on his preprint [3] and is part of joint work with him. I would like to thank the Isaac Newton Institute for support whilst this note was written, also Charles Thomas for discussions on symplectic and contact manifolds.

1. The complex bordism of Milnor hypersurfaces

We will us the notations

$$X \underset{MU}{+} Y = \mathrm{F}^{MU}(X, Y)$$

for the formal group sum of X and Y, and $[-1]_{MU}(X)$ for the formal group inverse of X.

We recall that for $r, s \geqslant 0$, the Milnor hypersurface $H_{r,s}$ is the degree 1 hypersurface dual to the 1st Chern class of the line bundle $\eta_r^* \otimes \eta_s^* \longrightarrow \mathbb{C}\mathrm{P}^r \times \mathbb{C}\mathrm{P}^s$. This is an algebraic submanifold which is a Kähler manifold whose symplectic form $\omega_{r,s}$ is obtained by restricting that of $\mathbb{C}\mathrm{P}^r \times \mathbb{C}\mathrm{P}^s$. Such a hypersurface gives rise to a map $H_{r,s} \longrightarrow \mathbb{C}\mathrm{P}^\infty$ and hence a bordism class

$$h_{r,s} = [H_{r,s} \longrightarrow \mathbb{C}\mathrm{P}^\infty] \in MU_{2(r+s-1)}(\mathbb{C}\mathrm{P}^\infty).$$

We will use the generating function

$$h(X, Y) = \sum_{r,s \geqslant 0} h_{r,s} X^r Y^s.$$

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We will also consider the bordism class of $H_{r,s}$ as a stably complex manifold, $[H_{r,s}] \in MU_{2(r+s-1)}$. Let $[\mathbb{C}\mathrm{P}^n] \in MU_{2n}$ denote the bordism class of $\mathbb{C}\mathrm{P}^n$ and

$$\mathbb{C}\mathrm{P}(T) = \sum_{n\geqslant 0} [\mathbb{C}\mathrm{P}^n] T^n.$$

Let

$$\mathrm{cp}_n = [\mathbb{C}\mathrm{P}^n \xrightarrow{[-1]} \mathbb{C}\mathrm{P}^\infty] \in MU_{2n}(\mathbb{C}\mathrm{P}^\infty)$$

denote the bordism class of the map $[-1]\colon \mathbb{C}\mathrm{P}^n \longrightarrow \mathbb{C}\mathrm{P}^\infty$ classifying the dual canonical line bundle η_n^*, and

$$\mathrm{cp}(T) = \sum_{n\geqslant 0} \mathrm{cp}_n T^n.$$

If we choose orientations according to the discussion in the Appendix (rather than following Adams [1]), then cp_n is the image of the MU-homology fundamental class $(\mathbb{C}\mathrm{P}^n)_{MU}$ in $MU_*(\mathbb{C}\mathrm{P}^\infty)$ under the homomorphism $[-1]_*$ induced by $[-1]$. Let β_n^{MU} ($n \geqslant 0$) be the standard MU_*-module generator for $MU_*(\mathbb{C}\mathrm{P}^\infty)$ and

$$\beta^{MU}(T) = \sum_{n\geqslant 0} \beta_n^{MU} T^n.$$

Let $\tilde\beta_n^{MU}$ be defined by the generating function

$$\tilde\beta^{MU}(T) = \sum_{n\geqslant 0} \tilde\beta_n^{MU} T^n = \beta^{MU}([-1]_{MU}(T)).$$

Then the $\tilde\beta_n^{MU}$ ($n \geqslant 0$) form an MU_*-basis for $MU_*(\mathbb{C}\mathrm{P}^\infty)$ is characterised by

$$c_1^{MU}(\eta^*) \cap \tilde\beta_n^{MU} = \tilde\beta_{n-1}^{MU}$$

for $n \geqslant 1$. Notice that we also have

$$\tilde\beta_n^{MU} = (-1)^n \beta_n^{MU} + (MU_*\text{-linear combination of } \beta_i^{MU} \text{ with } i < n). \tag{1.1}$$

Theorem 1.1. *The series $h(X,Y)$ is given by*

$$h(X,Y) = \mathbb{C}\mathrm{P}(X)\mathbb{C}\mathrm{P}(Y)(X \underset{MU}{+} Y)\tilde\beta(X \underset{MU}{+} Y).$$

Proof. By definition, the MU-homology fundamental class of $H_{r,s}$ maps to

$$c_1^{MU}(\eta_r^* \otimes \eta_s^*) \cap (\mathbb{C}\mathrm{P}^r \times \mathbb{C}\mathrm{P}^s)_{MU} \in MU_{2(r+s-1)}(\mathbb{C}\mathrm{P}^r \times \mathbb{C}\mathrm{P}^s).$$

This fundamental class can be mapped to the element $h_{r,s}$ of $MU_*(\mathbb{C}\mathrm{P}^\infty)$ using the classifying map for the line bundle $\eta_r^* \otimes \eta_s^*$. In this way we obtain

$$\begin{aligned} h(X,Y) &= \mu_*\left(c_1^{MU}(\eta^* \otimes \eta^*) \cap (\mathrm{cp}(X) \otimes \mathrm{cp}(Y))\right) \\ &= \mu_*\left(c_1^{MU}(\eta^* \otimes \eta^*) \cap \sum_{r,s\geqslant 0} \mathrm{cp}_r \otimes \mathrm{cp}_s X^r Y^s\right) \end{aligned}$$

where the $\mu\colon \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \longrightarrow \mathbb{C}\mathrm{P}^\infty$ denotes the product in $\mathbb{C}\mathrm{P}^\infty$. From the Equation (4.1) of the Appendix we have

$$\mathrm{cp}(T) = \mathbb{C}\mathrm{P}(T)\tilde\beta^{MU}(T). \tag{1.2}$$

and hence

$$\mathrm{cp}_n = \sum_{0\leqslant k\leqslant n} [\mathbb{C}\mathrm{P}^k] \tilde\beta_{n-k}^{MU}, \tag{1.3}$$

From [4] we have the identities

$$\beta^{MU}(X)\beta^{MU}(Y) = \beta^{MU}(\mathrm{F}^{MU}(X,Y)), \tag{1.4}$$

$$\beta^{MU}(X)^{-1} = \beta^{MU}(\mathrm{F}^{MU}([-1]_M UX), \tag{1.5}$$

from which we deduce

$$\mu_* \left(\beta^{MU}(X) \otimes \beta^{MU}(Y)\right) = \beta^{MU}(X \underset{MU}{+} Y). \tag{1.6}$$

Combining these results we obtain

$$\begin{aligned}
h(X,Y) &= \mathbb{C}\mathrm{P}(X)\mathbb{C}\mathrm{P}(Y)\mu_* \left(c_1^{MU}(\eta^* \otimes \eta^*) \cap \tilde{\beta}^{MU}(X) \otimes \tilde{\beta}^{MU}(Y)\right) \\
&= \mathbb{C}\mathrm{P}(X)\mathbb{C}\mathrm{P}(Y)\mu_* \left(\mathrm{F}^{MU}(c_1^{MU}(\eta^*) \otimes 1, 1 \otimes c_1^{MU}(\eta^*)) \cap \tilde{\beta}^{MU}(X) \otimes \tilde{\beta}^{MU}(Y)\right) \\
&= \mathbb{C}\mathrm{P}(X)\mathbb{C}\mathrm{P}(Y)(X \underset{MU}{+} Y)\tilde{\beta}^{MU}(X \underset{MU}{+} Y),
\end{aligned}$$

where the last line follows from an easy formal series calculation. □

We can make some immediate deductions from this result. First we can determine the coefficient of $\tilde{\beta}^{MU}_{r+s-1}$ in the expansion

$$h_{r,s} = C_{r,s,0}\tilde{\beta}^{MU}_{r+s-1} + C_{r,s,1}\tilde{\beta}^{MU}_{r+s-2} + \cdots + C_{r,s,r+s-2}\tilde{\beta}^{MU}_1 + [H_{r,s}].$$

We have

$$C_{r,s,0} = \binom{r+s}{r}. \tag{1.7}$$

This can be seen by ignoring all terms in the expansion for $h(X,Y)$ involving elements of MU_{2n} for $n > 0$.

Second we can determine the ordinary homology Hurewicz image $\underline{h}[H_{r,s}]$ modulo decomposables in $H_*(MU,\mathbb{Z})$. To do this we ignore all the terms involving $\tilde{\beta}^{MU}_n$ for $n > 0$, and work in $H_*(MU,\mathbb{Z})$ which contains MU_* as the image of the monomorphic Hurewicz homomorphism

$$\underline{h}\colon MU = \pi_*(MU) \longrightarrow H_*(MU,\mathbb{Z}).$$

Working modulo decomposables in $H_*(MU;\mathbb{Z})$, we have

$$\begin{aligned}
\sum_{r,s\geqslant 0} [H_{r,s}]X^rY^s &\equiv \mathbb{C}\mathrm{P}(X)\mathbb{C}\mathrm{P}(Y)(X \underset{MU}{+} Y) \\
&\equiv \mathbb{C}\mathrm{P}(X)\mathbb{C}\mathrm{P}(Y)\left(X + Y - \sum_{k>0} \frac{[\mathbb{C}\mathrm{P}^k]}{(k+1)}\left((X+Y)^{k+1} - X^{k+1} - Y^{k+1}\right)\right) \\
&\equiv (\mathbb{C}\mathrm{P}(X) + \mathbb{C}\mathrm{P}(Y))(X+Y) - \sum_{k>0} \frac{[\mathbb{C}\mathrm{P}^k]}{(k+1)}\left((X+Y)^{k+1} - X^{k+1} - Y^{k+1}\right) \\
&\equiv \mathbb{C}\mathrm{P}(X)X + \mathbb{C}\mathrm{P}(Y)Y - \sum_{r,s>1} \binom{r+s}{r}\frac{[\mathbb{C}\mathrm{P}^{r+s-1}]}{(r+s)}X^rY^s,
\end{aligned}$$

and hence,

$$\underline{h}[H_{r,s}] \equiv \begin{cases} -\dbinom{r+s}{r}\dfrac{[\mathbb{C}\mathrm{P}^{r+s-1}]}{(r+s)} & \text{if } r,s>1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Thus for example,

$$h_{1,n} = (n+1)\tilde{\beta}^{MU}_n + \cdots + [H_{1,n}]. \tag{1.9}$$

and in MU_*,

(1.10) $$[H_{1,n}] \equiv 0 \pmod{\text{decomposables}}.$$

Using Milnor's Criterion for polynomial generators of MU_*, we may deduce the following result which should be compared that of [1].

Proposition 1.2. *For each $n > 0$, there is a $\mathbb{Z}$-linear combination of the elements*

$$[H_{r,n+1-r}] \quad (1 \leqslant r \leqslant n-1)$$

which provides a polynomial generator for MU_*.

2. Determination of Ginzburg's symplectic bordism ring

In this section we give a proof of a result of J. Morava's [3] which completes the programme begun by V. L. Ginzburg [2] to compute the *symplectic bordism ring* Ω_*^{Sym}.

In [2], Ginzburg defines a homomorphism of graded rings

$$\sigma\colon \Omega_*^{\mathrm{Sym}} \longrightarrow MU_*(\mathbb{C}\mathrm{P}^\infty);$$
$$[M^n, \omega] \longmapsto [M^n \xrightarrow{f_\omega} \mathbb{C}\mathrm{P}^\infty],$$

where for the manifold with symplectic form ω, f_ω denotes the classifying map for the complex line bundle $\lambda_\omega \longrightarrow M^n$ whose 1st Chern class is the cohomology class represented by the 2-form $\omega/2\pi\mathrm{i}$. He shows that σ is injective and we will prove that it is also surjective, thus establishing the following result of [3].

Theorem 2.1. *The map σ is an isomorphism.*

Proof. It suffices to show that for each prime p, the localization

$$\sigma_{(p)}\colon (\Omega_{2n}^{\mathrm{Sym}})_{(p)} \longrightarrow MU_{2n}(\mathbb{C}\mathrm{P}^\infty)_{(p)}$$

is an isomorphism. We will establish this by induction on n.

Consider the elements cp_1 and $\mathrm{X} = [\mathbb{T} \xrightarrow{\omega_\mathbb{T}} \mathbb{C}\mathrm{P}^\infty]$ in $\operatorname{im}\sigma$, where $(\mathbb{T}, \omega_\mathbb{T})$ is any complex elliptic curve with its canonical symplectic form. The case $n = 1$ follows as in [3], since $[\mathbb{C}\mathrm{P}^1] = \mathrm{cp}_1 - \mathrm{X}$.

Now suppose that $(\operatorname{im}\sigma_{2k})_{(p)} = MU_{2k}(\mathbb{C}\mathrm{P}^\infty)_{(p)}$ whenever $k < n$. We have three distinct cases.

Case A: $p \nmid (n+1)$;
Case B: $n + 1 = p$;
Case C: $p|(n+1) > p$.

By Proposition 1.2, there is an element

$$q_n = Q_n + \sum_{0 \leqslant j \leqslant n-1} C_j \beta_{n-j}^{MU} \in \operatorname{im}\sigma_{2n},$$

where Q_n is a polynomial generator for MU_* and $C_j \in MU_{2j}$. By the induction hypothesis, each of the terms $C_j\beta_{n-j}^{MU}$ with $1 \leqslant j \leqslant (n-1)$ is in $\operatorname{im}\sigma$, hence the element $q'_n = Q_n + C_0\beta_n^{MU}$ lies in this image too. We will deal with each of these cases separately.

Case A: The element $h_{1,n}$ has the form

$$h_{1,n} \equiv (n+1)\tilde{\beta}_n^{MU} \pmod{\operatorname{im}\sigma},$$

by Equations (1.8) and (1.9) together with the inductive hypothesis. But then working p-locally, we can subtract a suitable multiple of $h_{1,n}$ from q'_n to obtain an element of the form Q_n modulo decomposables in $\operatorname{im}\sigma_{(p)}$, and this is still a polynomial generator for $(MU_*)_{(p)}$.

Case B: The element X^{p-1} lies in $\operatorname{im}\sigma$ and it is easily checked that

$$\mathrm{X}^{p-1} = (p-1)!\beta_{p-1}^{MU} + Y_1\beta_{p-2}^{MU} + \cdots + Y_{p-2}\beta_1^{MU}$$

for suitable $Y_k \in MU_{2k}$. As $(p-1)!$ is a p-local unit, we can subtract a multiple of X^{p-1} from q'_{p-1} to see that Q_{p-1} lies in $\operatorname{im}\sigma_{(p)}$.

Case C: The element cp_n lies in $\operatorname{im}\sigma$, and by Milnor's Criterion, $[\mathbb{C}\mathrm{P}^n]$ (mod decomposables) is divisible by p. Hence, by Equation (1.3), we see that some multiple $u\mathrm{cp}_n$ together with suitable elements elements of form $A_k\beta^{MU}_{n-k}$, $A_k \in MU_{2k}$, can be subtracted from q'_n to give an element of form

$$Q_n + u[\mathbb{C}\mathrm{P}^n] \in \operatorname{im}\sigma_{(p)},$$

which is a polynomial generator for $(MU_*)_{(p)}$.

Hence in all three cases we obtain a polynomial generator for $(MU_*)_{(p)}$ of degree $2n$ which lies in $\operatorname{im}\sigma_{(p)}$. By Equations (1.3) and (1.1) we also have

$$\beta^{MU}_n \equiv \pm\mathrm{cp}_n \pmod{\operatorname{im}\sigma_{(p)}},$$

so since $\mathrm{cp}_n \in \operatorname{im}\sigma$, we have the inductive step. □

3. The homology theory defined by symplectic bordism

Given a reasonable space X, we can consider a mapping of a symplectic manifold (M^n,ω) into X, $f\colon (M^n,\omega) \longrightarrow X$. There is a notion of bordism for such mappings, and on passage to bordism classes we obtain a $\mathbb{Z}$-graded functor $\Omega^{\mathrm{Sym}}_*(X)$, which is a homology theory when thought of as a functor in the variable X. A standard argument shows that this functor has the form

$$\Omega^{\mathrm{Sym}}_*(X) \cong (MU\wedge \mathbb{C}\mathrm{P}^\infty_+)_*(X) = MU_*(\mathbb{C}\mathrm{P}^\infty_+ \wedge X)$$

since for the case of X being a point, this is correct. There is a dual cohomology theory for which

$$(\Omega^{\mathrm{Sym}})^*(X) \cong (MU\wedge \mathbb{C}\mathrm{P}^\infty_+)^*(X) = [X, MU\wedge \mathbb{C}\mathrm{P}^\infty_+]^*.$$

These theories are multiplicative, with obvious geometric interpretations of products making use of the ring spectrum MU together with the fact that $\mathbb{C}\mathrm{P}^\infty$ is an H-space. At the level of the representing spectra, we have a map of ring spectra

$$MU \simeq MU\wedge S^0 \longrightarrow MU\wedge \mathbb{C}\mathrm{P}^\infty_+,$$

where we use the inclusion of a point in $\mathbb{C}\mathrm{P}^\infty$ to generate the map $S^0 \longrightarrow \mathbb{C}\mathrm{P}^\infty$. We remark that this homology theory can be defined using the fact that $MU_*(\mathbb{C}\mathrm{P}^\infty)$ is a free MU_*-module, and therefore

$$MU_*(\mathbb{C}\mathrm{P}^\infty) \underset{MU_*}{\otimes} MU_*(X)$$

is a homology theory as a functor of the space X, and similarly for the cohomology theory.

Now the cohomology theory $(MU\wedge \mathbb{C}\mathrm{P}^\infty_+)^*(\ )$ is *complex oriented* in the sense of [1]. Indeed, as we shall see, there are two very natural orientations for the universal line bundle $\eta \longrightarrow \mathbb{C}\mathrm{P}^\infty$. The first comes from the natural orientation $x^{MU}\colon \mathbb{C}\mathrm{P}^\infty \simeq MU(1) \longrightarrow \Sigma^2 MU$ followed by the map of ring spectra $MU \longrightarrow MU\wedge \mathbb{C}\mathrm{P}^\infty_+$ mentioned above. The second orientation arises as the map

$$y^{\mathrm{Sym}}\colon \mathbb{C}\mathrm{P}^\infty \simeq MU(1) \xrightarrow{\tilde\Delta} MU(1)\wedge \mathbb{C}\mathrm{P}^\infty_+ \xrightarrow{x^{MU}\wedge \mathrm{Id}} \Sigma^2 MU\wedge \mathbb{C}\mathrm{P}^\infty_+,$$

where $\tilde\Delta\colon MU(1) \longrightarrow MU(1)\wedge \mathbb{C}\mathrm{P}^\infty_+$ is the diagonal which sends a point V in the fibre over $P \in \mathbb{C}\mathrm{P}^\infty$ to the equivalence class of the pair (V,P), and the basepoint in the domain to the basepoint in the codomain. For the space $\mathbb{C}\mathrm{P}^\infty$ we have the following descriptions of the cohomology ring.

$$(\Omega^{\mathrm{Sym}})^*(\mathbb{C}\mathrm{P}^\infty) \cong MU_*(\mathbb{C}\mathrm{P}^\infty)[[x^{MU}]] = MU_*(\mathbb{C}\mathrm{P}^\infty)[[y^{\mathrm{Sym}}]]. \tag{3.1}$$

Proposition 3.1. *In the ring $(\Omega^{\mathrm{Sym}})^*(\mathbb{C}\mathrm{P}^\infty)$ we have the relation*

$$y^{\mathrm{Sym}} = \beta^{MU}(x^{MU})x^{MU}.$$

Proof. We calculate using the identity

$$[X, MU \wedge Y]^* \cong \operatorname{Hom}_{MU_*}(MU_*(X), MU_*(Y))$$

from [1], which holds for any space X with $MU_*(X)$ being MU_*-projective. Then for $X = Y = \mathbb{C}\mathrm{P}^\infty$ we can represent $(x^{MU})^m$ by the homomorphism

$$(x^{MU})^m_*\colon MU_*(\mathbb{C}\mathrm{P}^\infty) \longrightarrow MU_*(\mathbb{C}\mathrm{P}^\infty);$$

$$\beta_n^{MU} \longmapsto \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

On the other hand, the map $\tilde{\Delta}$ induces

$$\tilde{\Delta}_*\colon MU_*(MU(1)) \longrightarrow MU_*(MU(1) \wedge \mathbb{C}\mathrm{P}^\infty_+) \cong MU_*(MU(1)) \underset{MU_*}{\otimes} MU_*(\mathbb{C}\mathrm{P}^\infty);$$

$$\beta_n^{MU} \longmapsto \sum_{1 \leqslant k \leqslant n} \beta_k^{MU} \otimes \beta_{n-k}^{MU}.$$

Hence, y^{Sym} is represented by the homomorphism

$$y_*^{\mathrm{Sym}}\colon \beta_n^{MU} \longmapsto \beta_{n-1}^{MU}$$

Using Equation (3.2), it is now easy to see that we have

$$y_*^{\mathrm{Sym}} = \sum_{1 \leqslant m} \beta_{m-1}^{MU} (x^{MU})^m_*$$

from which the desired result follows. □

Related to these orientations are two formal group laws, one of which is the image of the universal one on MU_*, the other less familiar and given by

$$\mathrm{F}^{\mathrm{Sym}}(X, Y) = \beta^{MU}(\mathrm{F}^{MU}(X, Y))\mathrm{F}^{MU}(X, Y). \tag{3.3}$$

Of course the power series $\beta^{MU}(X)X$ provides a strict isomorphism

$$\mathrm{F}^{MU} \longrightarrow \mathrm{F}^{\mathrm{Sym}}.$$

The orientation y^{Sym} appears to be strongly connected with symplectic (co)bordism. For example, using a symplectic analogue Quillen's interpretation of cobordism classes in X as manifolds mapped into X (see [5]), we can interpret the orientation y^{Sym} restricted to $\mathbb{C}\mathrm{P}^n$ as the cobordism class of the inclusion map $j_n\colon (\mathbb{C}\mathrm{P}^{n-1}, \omega) \longrightarrow \mathbb{C}\mathrm{P}^n$ where the domain has the standard symplectic structure on $\mathbb{C}\mathrm{P}^{n-1}$. We record the following fact about the logarithm of $\mathrm{F}^{\mathrm{Sym}}$. Let $\mathrm{cp}_m^{(d)} \in MU_{2m}(\mathbb{C}\mathrm{P}^\infty)$ denote the bordism class of the map $[d]\colon \mathbb{C}\mathrm{P}^m \longrightarrow \mathbb{C}\mathrm{P}^\infty$ classifying the line bundle $\eta_m^d \longrightarrow \mathbb{C}\mathrm{P}^m$.

Proposition 3.2. *The logarithm of the formal group law* $\mathrm{F}^{\mathrm{Sym}}$ *is the series*

$$\log^{\mathrm{Sym}} X = \sum_{n \geqslant 1} \frac{\mathrm{cp}_{n-1}^{(-n)}}{n} X^n.$$

Proof. By Proposition 3.1 we have

$$\exp^{\mathrm{Sym}} Z = \beta^{MU}(\exp^{MU} Z) \exp^{MU} Z$$

where $\exp^F$ denotes the inverse of the logarithm $\log^F$ of a formal group law (here of course we are forced to work over the rational algebra $MU_*(\mathbb{C}\mathrm{P}^\infty) \otimes \mathbb{Q}$). By Lagrange Inversion (see [1]) we obtain for the coefficients of logarithm series

$$\log^{\mathrm{Sym}} X = \sum_{n \geqslant 1} \lambda_n^{\mathrm{Sym}} X^n$$

the identity

$$\lambda_n^{\mathrm{Sym}} = \frac{1}{n}\left[\frac{\beta^{MU}(\exp^{MU} Z)^{-n}}{(\exp^{MU} Z)^n}\right]_{Z^{-1}}.$$

Using standard residue calculus arguments together with the formal identity

$$\frac{\mathrm{d}}{\mathrm{d}W}\log^{MU} W = \mathbb{C}\mathrm{P}(W),$$

we see that this agrees with

$$\frac{1}{n}\left[\mathbb{C}\mathrm{P}(W)\beta^{MU}(W)^{-n}W^{-n}\right]_{W^{-1}} = \frac{1}{n}\left[\mathbb{C}\mathrm{P}(W)\beta^{MU}([-n]_{MU}W)W^{-n}\right]_{W^{-1}},$$

where $[d]_{MU}$ denotes the d-series for the formal group law F^{MU} and we use by a generalization of Equation (1.4),

$$\beta^{MU}(X)^d = \beta^{MU}([d]_{MU}X) \quad (d \in \mathbb{Z}). \tag{3.4}$$

Now let us investigate the bordism classes $\mathrm{cp}_m^{(d)}$. These are captured in the generating function

$$\sum_{m\geqslant 1}\frac{\mathrm{cp}_{m-1}^{(d)}}{m}X^m = \mathbb{C}\mathrm{P}(X)\beta^{MU}([d]_{MU}X) = \mathbb{C}\mathrm{P}(X)\beta^{MU}(X)^d.$$

obtained form Equation (3.4).

Taking $m = n$ and $d = -n$, we obtain

$$\mathrm{cp}_{n-1}^{(-n)} = n\lambda_n^{\mathrm{Sym}}$$

as desired. □

Of course, the logarithm for F^{MU} is

$$\log^{MU} X = \sum_{n\geqslant 1}\frac{[\mathbb{C}\mathrm{P}^{n-1}]}{n}X^n.$$

The stable cooperation algebra for this theory is

$$\begin{aligned}(MU\wedge\mathbb{C}\mathrm{P}^\infty_+)_*(MU\wedge\mathbb{C}\mathrm{P}^\infty_+) &= MU_*(\mathbb{C}\mathrm{P}^\infty)\underset{MU_*}{\otimes} MU_*MU \underset{MU_*}{\otimes} MU_*(\mathbb{C}\mathrm{P}^\infty)\\ &= MU_*(\mathbb{C}\mathrm{P}^\infty)\underset{MU_*}{\otimes}[B_k : k\geqslant 1]\underset{MU_*}{\otimes} MU_*(\mathbb{C}\mathrm{P}^\infty),\end{aligned} \tag{3.5}$$

where we use the two distinct units

$$MU_* \xrightarrow{\eta_{\mathrm{L}}} MU_*MU \xleftarrow{\eta_{\mathrm{R}}} MU_*$$

to define the two-sided tensor products; note that these units extend to ring homomorphisms

$$MU_*(\mathbb{C}\mathrm{P}^\infty) \xrightarrow{\eta_{\mathrm{L}}} (MU\wedge\mathbb{C}\mathrm{P}^\infty_+)_*(MU\wedge\mathbb{C}\mathrm{P}^\infty_+) \xleftarrow{\eta_{\mathrm{R}}} MU_*(\mathbb{C}\mathrm{P}^\infty).$$

The element $B_k \in MU_{2k}MU = MU_{2k}(MU)$ is the standard generator of [1] pushed into the ring $(MU\wedge\mathbb{C}\mathrm{P}^\infty_+)_*(MU\wedge\mathbb{C}\mathrm{P}^\infty_+)$ by using the map $MU \longrightarrow MU\wedge\mathbb{C}\mathrm{P}_+$. Let β_k^{L} and β_k^{R} denote the images of β_k^{Sym} under the left and right units. As a (left) algebra over $MU_*(\mathbb{C}\mathrm{P}^\infty)$ we have

$$(MU\wedge\mathbb{C}\mathrm{P}^\infty_+)_*(MU\wedge\mathbb{C}\mathrm{P}^\infty_+) = MU_*(\mathbb{C}\mathrm{P}^\infty)[B_k, \beta_k^{\mathrm{R}} : k\geqslant 1]/(\text{relations}), \tag{3.6}$$

where the relations are those satisfied by the β_k^{MU} in $MU_*(\mathbb{C}\mathrm{P}^\infty)$ which can be encoded in the generating function identity Equation (1.4).

There is another set of generators $B_k^{\mathrm{Sym}} \in (MU\wedge\mathbb{C}\mathrm{P}^\infty_+)_*(MU\wedge\mathbb{C}\mathrm{P}^\infty_+)$ which arise from the formal group law $\mathrm{F}^{\mathrm{Sym}}$. The two series $B(T) = \sum_{k\geqslant 0} B_kT^k$ and $B^{\mathrm{Sym}}(T) = \sum_{k\geqslant 0} B_k^{\mathrm{Sym}}T^k$ satisfy

$$B^{\mathrm{Sym}}(\beta^{\mathrm{L}}(B(T))B(T)) = \beta^{\mathrm{R}}(B(T))B(T). \tag{3.7}$$

Dually, we can describe stable operations as elements of

$$(MU \wedge \mathbb{C}\mathrm{P}^\infty_+)^*(MU \wedge \mathbb{C}\mathrm{P}^\infty_+) \cong \mathrm{Hom}_{MU_*(\mathbb{C}\mathrm{P})}((MU \wedge \mathbb{C}\mathrm{P}^\infty_+)_*(MU \wedge \mathbb{C}\mathrm{P}^\infty_+), MU_*(\mathbb{C}\mathrm{P}^\infty)).$$

Thus for example, we have elements S^{Sym}_n ($n \geqslant 1$) defined by requiring that they be derivations over $MU_*(\mathbb{C}\mathrm{P}^\infty)$ and satisfy

$$S^{\mathrm{Sym}}_n(B^{\mathrm{Sym}}_m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{else.} \end{cases} \tag{3.8}$$

$$S^{\mathrm{Sym}}_n(\beta^{\mathrm{R}}_m) = 0 \tag{3.9}$$

There are also elements with similar property except that the rôles of B^{Sym}_k and β^{R}_k are interchanged in Equation (3.8).

4. Appendix: Orientations in complex projective spaces

In this Appendix, we discuss the choice of (complex) orientations in projective spaces. The need for this arises from the fact that the choices of orientation and almost complex structure for $\mathbb{C}\mathrm{P}^n$ made in [1] (a reference widely used by topologists), although convenient for topological purposes, are not the natural ones in algebraic geometry. One reason why this point is usually ignored is that these different choices give rise to the same complex bordism class! We will elucidate these points and describe the conventions used in this work.

Let V denote a complex vector space of dimension $(n + 1)$. Then as usual we define the projective space of V by

$$\mathbb{C}\mathrm{P}(V) = (V - \{0\})/\mathbb{C}^\times,$$

which has the structure of complex analytic manifold since the (left) action of $\mathbb{C}^\times$ on $V_0 = V - \{0\}$ is analytic and free. Of course, we may view $\mathbb{C}\mathrm{P}(V)$ as the set of lines in V. There is a tautological holomorphic line bundle $\eta_V \longrightarrow \mathbb{C}\mathrm{P}(V)$ for which the fibre over $[x] \in \mathbb{C}\mathrm{P}(V)$ is given by

$$(\eta_V)_{[x]} = \{zx : z \in \mathbb{C}\}.$$

This bundle is also given up to isomorphism as

$$V_0 \underset{\mathbb{C}^\times}{\times} \mathbb{C}_\chi \longrightarrow \mathbb{C}\mathrm{P}(V),$$

where $\mathbb{C}_\chi$ denotes $\mathbb{C}$ with the action of $\mathbb{C}^\times$ by the character $\chi\colon \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ for which

$$\chi(z) = z^{-1}.$$

The problem with this bundle is that although it is holomorphic it has no global holomorphic sections. On the other hand, its dual $\eta^*_V \longrightarrow \mathbb{C}\mathrm{P}(V)$ can be described up to isomorphism as

$$V_0 \underset{\mathbb{C}^\times}{\times} \mathbb{C} \longrightarrow \mathbb{C}\mathrm{P}(V),$$

where this time we take the natural action of $\mathbb{C}^\times$ on $\mathbb{C}$. This bundle is also holomorphic and admits many holomorphic sections. Indeed, we have

$$\Gamma_{\mathrm{hol}}(\mathbb{C}\mathrm{P}(V), \eta^*_V) \cong V^*,$$

where under this isomorphism, a linear form $\sigma \in V^*$ corresponds to the section s for which

$$s([x])(zx) = \sigma(zx).$$

More generally, setting

$$\eta^k_V = \begin{cases} \eta_V^{\otimes k} & \text{if } k > 0, \\ (\eta^*_V)^{\otimes -k} & \text{if } k \leqslant 0, \end{cases}$$

we have

$$\Gamma_{\mathrm{hol}}(\mathbb{C}\mathrm{P}(V), \eta_V^k) = \begin{cases} 0 & \text{if } k > 0, \\ \mathrm{Sym}^{(-k)}(V^*) & \text{if } k \leqslant 0, \end{cases}$$

where $\mathrm{Sym}^m(W)$ denotes the mth symmetric power of the vector space W. In algebraic and complex analytic geometry, the bundle η_V^k is usually denoted $\mathcal{O}_{\mathbb{C}\mathrm{P}(V)}(-k)$; for example, $\mathcal{O}_{\mathbb{C}\mathrm{P}(V)} = \mathcal{O}_{\mathbb{C}\mathrm{P}(V)}(0)$ is the structure sheaf and agrees with the trivial bundle. It is a standard result that the holomorphic tangent bundle of $\mathbb{C}\mathrm{P}(V)$ satisfies

$$\mathrm{T}\mathbb{C}\mathrm{P}(V) + \mathcal{O}_{\mathbb{C}\mathrm{P}(V)} = (n+1)\mathcal{O}_{\mathbb{C}\mathrm{P}(V)}(1),$$

hence stably we get

$$\mathrm{T}\mathbb{C}\mathrm{P}(V) \simeq (n+1)\eta_V^*,$$

which should be compared with [1] where the identification

$$\mathrm{T}\mathbb{C}\mathrm{P}(V) \simeq (n+1)\eta_V,$$

is given.

To choose an orientation for $\mathbb{C}\mathrm{P}(V)$ in ordinary cohomology $H^*(\mathbb{C}\mathrm{P}(V); R)$, we take the dual of

$$c_1(\eta_V^*)^n = (-1)^n c_1(\eta_V)^n$$

under the duality isomorphism

$$H^*(\mathbb{C}\mathrm{P}(V); R) = \mathrm{Hom}_R(H_*(\mathbb{C}\mathrm{P}(V); R), R).$$

Note that [1] chooses the dual of $c_1(\eta_V)^n$.

In fact, Adams' choices can be 'explained' as those obtained as follows. On fixing a basis $v_1, \ldots, v_{n+1}$ for V, together with the dual basis $v_1^*, \ldots, v_{n+1}^*$ of V^*, the standard isomorphism $V \xrightarrow{\cong} V^*$ given by

$$\sum_i c_i v_i \longleftrightarrow \sum_i c_i v_i^*$$

induces a (real) analytic diffeomorphism $\mathbb{C}\mathrm{P}(V) \cong \mathbb{C}\mathrm{P}(V^*)$ under which the bundles η_V and η_{V^*} correspond. Moreover, it is easily seen that our above choice for orientation on $\mathbb{C}\mathrm{P}(V^*)$ corresponds to Adams' choice on $\mathbb{C}\mathrm{P}(V)$ under this diffeomorphism. A calculation of Chern numbers in ordinary cohomology also shows that the bordism classes $[\mathbb{C}\mathrm{P}(V)], [\mathbb{C}\mathrm{P}(V^*)] \in MU_{2n}$ agree, hence Adams' choices for $\mathbb{C}\mathrm{P}(V)$ also lead to this same class.

In complex cobordism $MU^*(\ )$, the difference between this approach and that of Adams becomes more pronounced. If we follow Adams and set $x^{MU} = c_1^{MU}(\eta_V)$, then the fundamental class $(\mathbb{C}\mathrm{P}(V))_{MU}$ for $\mathbb{C}\mathrm{P}(V)$, based on our choice of orientation and dual to $c_1^{MU}(\eta_V^*)^n$, satisfies

$$(\mathbb{C}\mathrm{P}(V))_{MU} = c_1^{MU}(\eta_V^*) \cap (\mathbb{C}\mathrm{P}(V+\mathbb{C}))_{MU} \in MU_{2n}(\mathbb{C}\mathrm{P}(V)),$$

since the section $z_{\mathbb{C}} \in \Gamma_{\mathrm{hol}}(\mathbb{C}\mathrm{P}(V+\mathbb{C}), \eta_{V+\mathbb{C}}^*)$ which is determined by

$$z_{\mathbb{C}}(v,t) = t$$

vanishes on $\mathbb{C}\mathrm{P}(V) \subset \mathbb{C}\mathrm{P}(V+\mathbb{C})$. It is easy to see that the fundamental class for $\mathbb{C}\mathrm{P}^n = \mathbb{C}\mathrm{P}(\mathbb{C}^{n+1})$ can be read off from the generating function

$$\sum_n (\mathbb{C}\mathrm{P}^n) T^n = \mathbb{C}\mathrm{P}(T)\beta^{MU}([-1]_{MU}T) \tag{4.1}$$

rather than $\mathbb{C}\mathrm{P}(T)\beta^{MU}(T)$. If we apply the map classifying η_n^* to $(\mathbb{C}\mathrm{P}^n)_{MU}$, the dual to the canonical line bundle η_n, we obtain the class $\mathrm{cp}_n \in MU_{2n}(\mathbb{C}\mathrm{P}^\infty)$ which is exactly the same element as Adams obtains for the image of his choice of fundamental class $[\mathbb{C}\mathrm{P}^n]_{MU}$ under the map classifying η_n.

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