Journal of Pure and Applied Algebra 216 (2012) 2361–2376

Contents lists available at SciVerse ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Brauer groups for commutative S-algebras

Andrew Baker [a,*], Birgit Richter [b], Markus Szymik [c]

[a] School of Mathematics & Statistics, University of Glasgow, Glasgow G12 8QW, Scotland, United Kingdom
[b] Fachbereich Mathematik der Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
[c] Mathematisches Institut, Heinrich-Heine-Universität Düsseldorf, Universitätsstrasse 1, 40225 Düsseldorf, Germany

ARTICLE INFO

Article history:
Received 6 May 2011
Received in revised form 4 February 2012
Available online 30 March 2012
Communicated by C.A. Weibel

To the memory of Gorô Azumaya, ∗ 26th of February 1920, † 8th of July 2010

MSC:
Primary: 55P43
Secondary: 11R52; 11R32

ABSTRACT

We investigate a notion of Azumaya algebras in the context of structured ring spectra and give a definition of Brauer groups. We investigate their Galois theoretic properties, and discuss examples of Azumaya algebras arising from Galois descent and cyclic algebras. We construct examples that are related to topological Hochschild cohomology of group ring spectra and we present a $K(n)$-local variant of the notion of Brauer groups.

Introduction

The investigation of Brauer groups of commutative S-algebras is one aspect of the attempt to understand arithmetic properties of structured ring spectra.

In classical algebraic settings, Brauer groups are defined in terms of Azumaya algebras over fields or more generally over commutative rings [4,3,32] and are closely involved in Galois theoretic considerations. In this paper, we discuss some ideas on Brauer groups for commutative S-algebras and in Section 3, we investigate their behaviour with respect to Galois extensions of commutative S-algebras in the sense of John Rognes [29]. In earlier work, the first author and Andrey Lazarev discussed notions of Azumaya algebras [5, sections 2, 4], but these appear to be technically problematic: there faithfulness of the underlying module spectra was not required, but many of the standard constructions with Azumaya algebras rely on this property. Also, the link with other definitions (for instance [35,20]) works only under this faithfulness assumption.

Since our work was begun, other people have carried out work on Azumaya algebras and Brauer groups in contexts related to ours. Niles Johnson [20] discusses Azumaya objects for general closed autonomous symmetric monoidal bicategories, and proves a comparison result [20, proposition 5.4] which compares our definition of Azumaya algebras with his. In [20, theorem 1.5], he also shows that the derived Brauer group of a commutative ring in the sense of Bertrand Toën agrees with our Brauer group of the corresponding Eilenberg–Mac Lane ring spectrum.

In [34, definition 4.6], the third author extends our current work to construct a Brauer space for commutative S-algebras such that the fundamental group of that space agrees with our Brauer group. The approaches in [35,2,16] give descriptions of Brauer groups in terms of étale cohomology groups in the derived context and the context of ring spectra, respectively.

∗ Corresponding author.
E-mail addresses: a.baker@maths.gla.ac.uk (A. Baker), richter@math.uni-hamburg.de (B. Richter), szymik@math.uni-duesseldorf.de (M. Szymik).
URLs: http://www.maths.gla.ac.uk/~ajb (A. Baker), http://www.math.uni-hamburg.de/home/richter/ (B. Richter), http://www.math.uni-duesseldorf.de/Personen/indiv/Szymik (M. Szymik).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2012.03.001

We present our definition of topological Azumaya algebras in Section 1 and show that such algebras are always homotopically central (in the sense of Definition 1.2) and separable, and also that the Azumaya property is preserved under base change.

In Section 2 we define Brauer groups of commutative S-algebras and in Section 3 we prove a version of Galois descent for topological Azumaya algebras. This is applied in Section 4 where we explain how the classical theory of cyclic algebras can be extended to the context of commutative S-algebras.

In the case of Eilenberg–Mac Lane spectra we show in Section 5 under some assumptions on the ring R that an extension $HR \longrightarrow HA$ is topologically Azumaya if and only if the extension of commutative rings $R \longrightarrow A$ is an algebraic Azumaya extension. Furthermore, using recent work of Bertrand Toën [35], we can deduce that the Brauer group $Br(H\boldsymbol{k})$ is trivial if $\boldsymbol{k}$ is an algebraically or separably closed field.

Classically, the centre of an associative algebra A over a commutative ring R can be described as endomorphisms of A in the category of modules over the enveloping algebra $A^e = A \otimes_R A^o$. For structured ring spectra, the direct analogue of this definition does not yield a homotopy invariant notion. Instead one has to replace A by a cofibrant object in the category of module spectra over the enveloping algebra spectrum, so the centre of an associative R-algebra spectrum A is given by the topological Hochschild cohomology spectrum $THH_R(A, A)$. This spectrum is not strictly commutative in general, but due to the affirmatively solved Deligne conjecture [26] it is an E_2-spectrum. There are however exceptions and in Section 7 we discuss some examples arising from group ring spectra and their homotopy fixed point spectra.

In Section 8 we offer a variant of the construction of Brauer groups in the $K(n)$-local context where it appears that technical difficulties are minimized and we discuss some examples related to $EO_2 = L_{K(2)}TMF$ in Section 9. In Section 10 we describe a non-trivial element in the $K(n)$-local Brauer group of the $K(n)$-local sphere.

1. Azumaya algebras over commutative S-algebras

Throughout, let R be a cofibrant commutative S-algebra. We work in the categories of R-modules, $\mathscr{M}_R$, and associative R-algebras, $\mathscr{A}_R$ and for definiteness we choose the framework of [15]. Following [6,29], we will say that an R-module W is *faithful* if for an R-module X, $W \wedge_R X \simeq *$ implies that $X \simeq *$.

We recall some ideas from [5]. If A is an R-algebra, we denote by A^o the R algebra whose underlying R-module is A but whose multiplication is reversed. The topological Hochschild cohomology spectrum of A (over R) is

$$THH_R(A) = THH_R(A, A) = F_{A\wedge_R A^o}(\widetilde{A}, \widetilde{A}),$$

where $\widetilde{A}$ is a cofibrant replacement for A in the category of left $A \wedge_R A^o$-modules $\mathscr{M}_{A\wedge_R A^o}$. We write $\eta\colon R \longrightarrow THH_R(A)$ for the canonical map into the R-algebra $THH_R(A)$; we also write $\mu\colon A \wedge_R A^o \longrightarrow F_R(A, A)$ for the R-algebra map induced by the left and right actions of A and A^o on A.

Definition 1.1. Let A be an R-algebra. Then A is a *weak* (*topological*) *Azumaya algebra over R* if and only if the first two of the following conditions hold, while A is a (*topological*) *Azumaya algebra over R* if and only if all three of them hold.

(1) A is a dualizable R-module.
(2) $\mu\colon A \wedge_R A^o \longrightarrow F_R(A, A)$ is a weak equivalence.
(3) A is faithful as an R-module.

Note that this definition of Azumaya algebras over R differs from that in [5] since we demand faithfulness of A over R and not just A-locality of R as an R-module.

If T is an ordinary commutative ring with unit and if B is an associative T-algebra, then the centre of B can be identified with the endomorphisms of B as an $B \otimes_T B^o$-module. Therefore $THH_R(A)$ can be viewed as a homotopy invariant version of the centre of A.

Definition 1.2. An R-algebra A is said to be *homotopically central* if the canonical map $\eta\colon R \longrightarrow THH_R(A)$ is a weak equivalence.

For the following we recall a special case of the Morita theory developed in [5, section 1]. For a topological Azumaya algebra A over R we consider the category of left modules over the endomorphism spectrum $F_R(A, A)$, $\mathscr{M}_{F_R(A,A)}$ and we take a cofibrant replacement $\overline{A}$ of A in this category. The functor

$$F\colon \mathscr{M}_R \longrightarrow \mathscr{M}_{F_R(A,A)}$$

that sends X to $X \wedge_R \overline{A}$ has an adjoint

$$G\colon \mathscr{M}_{F_R(A,A)} \longrightarrow \mathscr{M}_R$$

with $G(Y) = F_{F_R(A,A)}(\overline{A}, Y)$. Then [5, theorem 1.2] implies that this adjoint pair of functors passes to an adjoint pair of equivalences between the corresponding derived categories

$$\begin{tikzcd} \mathscr{D}_R \arrow[r, shift left, "\widetilde{F}"] & \mathscr{D}_{F_R(A,A)} \arrow[l, shift left, "G"] \end{tikzcd}$$

and as a direct consequence we obtain the following result.

Proposition 1.3 ([5, proposition 2.3]). *Every topological Azumaya algebra A over R is homotopically central.*

By proposition 2.3 and definition 2.1 of [5], we also see that any topological Azumaya algebra A over R is dualizable as an $A \wedge_R A^o$-module and $A \wedge_R A^o$ is A-local as a left module over itself.

In classical algebra, Azumaya algebras are in particular separable. Using Morita theory we can deduce the analogous statement for topological Azumaya algebras. Here an R-algebra is *separable* in the sense of [29, definition 9.1.1] if the multiplication $m\colon A \wedge_R A \longrightarrow A$ has a section in the derived category of left $A \wedge_R A^o$-modules, $\mathscr{D}_{A\wedge_R A^o}$.

Proposition 1.4. *Let A be a topological Azumaya R-algebra. Then A is separable.*

Proof. By the remark following [29, definition 9.1.1], it suffices to prove that the induced map

$$m_*\colon THH_R(A, A \wedge_R A) \longrightarrow THH_R(A, A)$$

is surjective on $\pi_0(-)$. Denote by $\widetilde{A}$ a cofibrant replacement of A in the category of $A \wedge_R A^o$-modules. Morita equivalence yields the two weak equivalences

$$\widetilde{G} \circ \widetilde{F}(R) \simeq THH_R(A, A),$$
$$\widetilde{G} \circ \widetilde{F}(A) \simeq THH_R(A, A \wedge_R A).$$

The functoriality of $\widetilde{G} \circ \widetilde{F}$ ensures that the unit $\eta\colon R \longrightarrow A$ induces a map $\widetilde{G} \circ \widetilde{F}(\eta)$ with

$$R \xrightarrow{\simeq} \widetilde{G} \circ \widetilde{F}(R) \xrightarrow{\widetilde{G}\circ\widetilde{F}(\eta)} \widetilde{G} \circ \widetilde{F}(A) \xleftarrow{\simeq} A.$$

This is given by sending the coefficient module of THH, $\widetilde{A} \simeq R \wedge_R \widetilde{A} \simeq R \wedge_R A$, to $A \wedge_R A \simeq A \wedge_R \widetilde{A}$ using η. Therefore

$$\pi_0(m_*) \circ \pi_0(\widetilde{G} \circ \widetilde{F}(\eta)) = \mathrm{id},$$

and so $\pi_0(m_*)$ is surjective. □

We now describe the behaviour of Azumaya algebras under base change.

Proposition 1.5. *Let A, B, C be R-algebras.*

(1) *If A is an Azumaya algebra over R and if C is a commutative R-algebra, then $A \wedge_R C$ is an Azumaya algebra over C.*
(2) *Conversely, let C be a commutative R-algebra such that C is dualizable and faithful as an R-module. If $A \wedge_R C$ is an Azumaya algebra over C, then A is an Azumaya algebra over R.*
(3) *If A and B are Azumaya algebras over R, then $A \wedge_R B$ is also Azumaya over R.*

Proof. If A is an Azumaya algebra over R, then it is formal to verify that $A \wedge_R C$ is dualizable and faithful over C (compare [29, 4.3.3, 6.2.3]). It remains to show that

$$\mu_{A\wedge_R C}\colon (A \wedge_R C) \wedge_C (A \wedge_R C)^o \longrightarrow F_C(A \wedge_R C, A \wedge_R C)$$

is a weak equivalence. Note that since the multiplication in $A \wedge_R C$ is defined componentwise,

$$(A \wedge_R C)^o = A^o \wedge_R C^o.$$

The diagram

$$\begin{tikzcd}
(A \wedge_R C) \wedge_C (A \wedge_R C)^o \arrow[rr, "\mu_{A\wedge_R C}"] \arrow[d, "\simeq"'] & & F_C(A \wedge_R C, A \wedge_R C) \arrow[d, "\simeq"] \\
A \wedge_R A^o \wedge_R C \arrow[dr, "\mu_A \wedge_R C"] & & F_R(A, A \wedge_R C) \\
& F_R(A, A) \wedge_R C \arrow[ur, "\nu"] &
\end{tikzcd} \tag{1.1}$$

commutes. Here $\nu\colon F_R(A, A) \wedge_R C \longrightarrow F_R(A, A \wedge_R C)$ denotes the duality map. As A is Azumaya over R we know that ν and μ_A are equivalences, and thus we obtain that the top map is an equivalence as well.

For the converse we assume that $A \wedge_R C$ is Azumaya over C and C is faithful and dualizable as an R-module. If M is an R-module, then $A \wedge_R M \simeq *$ implies that

$$(A \wedge_R C) \wedge_R M \simeq (A \wedge_R C) \wedge_C (C \wedge_R M) \simeq *.$$

Also, the faithfulness of $A \wedge_R C$ over C ensures that $C \wedge_R M \simeq *$. But as we assumed that C is faithful over R, we can conclude that M was trivial.

The fact that A is dualizable over R follows from [29, lemma 6.2.4]. Making use of diagram Eq. (1.1) we see that μ_A is also a weak equivalence.

The proof of the third claim is straightforward. □

Later we will consider Azumaya algebras in a Bousfield local setting. Let L be a cofibrant R-module.

Definition 1.6. An L-local R-algebra A is an (L-*local*) *Azumaya algebra* if

(1) A is a dualizable L-local R-module.
(2) The natural morphism of R-algebras $A \wedge_R A^{\mathrm{o}} \longrightarrow F_R(A, A)$ is an L-local equivalence.
(3) A is faithful as an L-local R-module.

Here dualizability as an L-local R-module means dualizability in the derived category of L-local R-modules. This is a symmetric monoidal category with the L-localization of the smash product over R as the symmetric monoidal product, so the definition of dualizability from [14] applies.

2. Brauer groups

Now suppose that M is a dualizable R-module as discussed in [29,6]; a more detailed discussion of dualizability can be found in [14]. Let $\mathcal{E}_R(M) = F_R(M, M)$ be its endomorphism R-algebra. Then there is a weak equivalence

$$\mathcal{E}_R(M) \simeq F_R(M, R) \wedge_R M. \tag{2.1}$$

In order to identify endomorphism spectra of faithful and dualizable R-modules as trivial Azumaya algebras we need the following auxiliary result.

Lemma 2.1. *Let M be a dualizable R-module.*

(1) *If M is a faithful R-module, then the dual $F_R(M, R)$ is also faithful.*
(2) *If M is L-local with respect to a cofibrant R-module L, then $F_R(M, R)$ is L-local.*

Proof. (1) Dualizability of M implies that the composition

$$M \simeq R \wedge_R M \xrightarrow{\delta\wedge\mathrm{id}} M \wedge_R F_R(M, R) \wedge_R M \xrightarrow{\mathrm{id}\wedge\varepsilon} M \wedge_R R \simeq M$$

is the identity on M. Here $\delta\colon R \longrightarrow M \wedge_R F_R(M, R)$ is the counit, and $\varepsilon\colon F_R(M, R) \wedge_R M \longrightarrow R$ is the evaluation map. Now if N is an R-module for which $F_R(M, R) \wedge_R N \simeq *$, then the identity of $M \wedge_R N$ factors through the trivial map, hence $N \simeq *$ by faithfulness of M.
(2) A similar argument with the functor $F_R(W, -)$ shows that if $L \wedge_R W \simeq *$, then the identity map on $F_R(W, F_R(M, R))$ factors through

$$F_R(W, F_R(M, R) \wedge_R M \wedge_R F_R(M, R)) \simeq F_R(W \wedge_R M \wedge_R M, M) \simeq *. \quad \square$$

It was shown in [5, proposition 2.11] that if M is a dualizable, cofibrant R-module, then $\mathcal{E}_R(M)$ is a weak topological Azumaya algebra in the sense of [5, definition 2.1].

Proposition 2.2. *If M is a faithful, dualizable, cofibrant R-module, then (a cofibrant replacement of) $\mathcal{E}_R(M)$ is an Azumaya R-algebra.*

Proof. As $\mathcal{E}_R(M)$ is a weak Azumaya algebra, it suffices to show that $\mathcal{E}_R(M)$ is a faithful R-module. Dualizability of M ensures that

$$\mathcal{E}_R(M) \simeq F_R(M, R) \wedge_R M,$$

and this is a smash product of two faithful R-modules which is also faithful. $\square$

This result shows that we can take the R-algebras of the form $\mathcal{E}_R(M)$ with M faithful, dualizable and cofibrant, to be trivial Azumaya algebras when defining a topological version of a Brauer group which we now do.

First we note that every Azumaya algebra is weakly equivalent to a retract of a cell R-module, so the following construction yields a *set* of equivalence classes. Define $Az(R)$ to be the collection of all Azumaya algebras. Now we introduce our version of the Brauer equivalence relation $\approx$ on $Az(R)$.

Definition 2.3. Let R be a cofibrant commutative S-algebra. If $A_1, A_2 \in Az(R)$, then $A_1 \approx A_2$ if and only if there are faithful, dualizable, cofibrant R-modules M_1, M_2 for which

$$A_1 \wedge_R F_R(M_1, M_1) \simeq A_2 \wedge_R F_R(M_2, M_2)$$

as R-algebras. We denote the set of equivalence classes of these by $Br(R)$ and we use the notation $[A]$ for the equivalence class of an R-Azumaya algebra A.

Theorem 2.4. *The set $Br(R)$ is an abelian group with multiplication induced by the smash product $\wedge_R$. Furthermore, Br is a functor from the category of commutative S-algebras to abelian groups.*

Proof. The details involve routine modifications of the approach used in the case of Brauer groups of commutative rings in [3, theorem 5.2].

Functoriality for morphisms of commutative S-algebras $R \longrightarrow R'$ is achieved by sending an R-algebra A to the R'-algebra $R' \wedge_R A$. $\square$

Remark 2.5. Johnson's work [20, lemma 5.7] shows that two Azumaya R-algebras are Brauer equivalent in the sense of Definition 2.3 if and only if they are Eilenberg–Watts equivalent in the sense of [20, definition 1.1]. As a consequence any Azumaya algebra that is Brauer equivalent to R is weakly equivalent to $F_R(M, M)$ for some dualizable faithful cofibrant R-module spectrum M.

For a cofibrant R-module L, we can similarly define the sets of L-local Azumaya algebras $Az_L(R)$ and the associated L-local Brauer group $Br_L(R)$.

In order to relate Azumaya algebras to Galois theory, we require the following notions modelled on algebraic analogues.

Definition 2.6. Let $R \longrightarrow R'$ be an extension of commutative S-algebras. Then the Azumaya algebra $R \longrightarrow A$ is *split* by $R \longrightarrow R'$ (or just by R') if $R' \wedge_R A \approx R'$, or equivalently if $[A] \in Ker(Br(R) \longrightarrow Br(R'))$. We define the *relative Brauer group*

$$Br(R'/R) = Ker(Br(R) \longrightarrow Br(R')).$$

Similarly we can define a *relative L-local Brauer group*

$$Br_L(R'/R) = Ker(Br_L(R) \longrightarrow Br_L(R')).$$

In practise, we will use this when $R \longrightarrow R'$ is a faithful G-Galois extension for some finite group G.

3. Galois extensions and Azumaya algebras

Consider a map of commutative S-algebras $A \longrightarrow B$, which we often denote by B/A. If A is cofibrant as a commutative S-algebra, B is cofibrant as a commutative A-algebra, and if G is a finite group which acts on B by morphisms of commutative A-algebras, then following Rognes [29], then we call B/A a *G-Galois extension* if the canonical maps $i\colon A \longrightarrow B^{hG}$ and $h\colon B \wedge_A B \longrightarrow F(G_+, B)$ are weak equivalences.

In addition to these conditions, we will assume that B is faithful as an A-module spectrum. This is a further restriction as there are examples of Galois extensions which are not faithful. The following example is due to Wieland (see [30]).

Remark 3.1. Let p be a prime. Then the $\mathbb{Z}/p$-Galois extension

$$F(B\mathbb{Z}/p_+, H\mathbb{F}_p) \longrightarrow F(E\mathbb{Z}/p_+, H\mathbb{F}_p) \simeq H\mathbb{F}_p$$

is not faithful. To its eyes the $\mathbb{Z}/p$-Tate spectrum of $H\mathbb{F}_p$ appears trivial, but it is not.

Let $B\langle G\rangle$ be the twisted group algebra over B, *i.e.*, the A-algebra whose underlying A-module is $B \wedge G_+$ and whose multiplication is the composition $\widetilde{\mu}$

$$\begin{tikzcd}
B \wedge G_+ \wedge B \wedge G_+ \arrow[r, "\mathrm{id}\wedge\Delta\wedge\mathrm{id}"] \arrow[ddrr, bend right, "\widetilde{\mu}"'] & B \wedge G_+ \wedge G_+ \wedge B \wedge G_+ \arrow[r, "\mathrm{id}\wedge\nu\wedge\mathrm{id}"] & B \wedge G_+ \wedge B \wedge G_+ \arrow[d, "(2{,}3)"] \\
& & B \wedge B \wedge G_+ \wedge G_+ \arrow[d, "\mu_B\wedge\mu_G"] \\
& & B \wedge G_+
\end{tikzcd}$$

where Δ is the diagonal, ν denotes the G-action on B, μ_B is the multiplication of B and μ_G the multiplication in G. Then $\widetilde{\mu}$ factors through $(B \wedge G_+) \wedge_A (B \wedge G_+)$ and turns $B\langle G\rangle$ into an A-algebra. Note that $B\langle G\rangle$ is an associative algebra but in general it lacks commutativity. More precisely, we know that the morphism $j\colon B\langle G\rangle \longrightarrow F_A(B, B)$ is a weak equivalence of A-algebras for every G-Galois extension $A \longrightarrow B$. In particular, $B\langle G\rangle$ gives rise to a trivial element in the Brauer group of A.

Lemma 3.2. *Let B/A be a faithful G-Galois extension and let M be a $B\langle G\rangle$-module which is of the form $B \wedge_A N$ for some A-module N, where the $B\langle G\rangle$-module structure is given by the B-factor of $B \wedge_A N$. Then there is a weak equivalence of A-modules $N \simeq M^{hG}$.*

Proof. Consider $B \wedge_A M = B \wedge_A B \wedge_A N$. As B is G-Galois over A, the latter term is equivalent to $F(G_+, B) \wedge_A N$ and this in turn is equivalent to $F(G_+, B \wedge_A N)$ because G_+ is finite. As B is dualizable over A, the homotopy fixed point spectrum $(B \wedge_A M)^{hG}$ is equivalent to $B \wedge_A M^{hG}$.

There is a chain of equivalences of B-modules

$$B \wedge_A N \xrightarrow{\simeq} F(G_+, B \wedge_A N)^{hG} \xrightarrow{\simeq} (B \wedge_A B \wedge_A N)^{hG} = (B \wedge_A M)^{hG} \xleftarrow{\simeq} B \wedge_A M^{hG},$$

and the result follows by faithfulness of B over A. $\square$

The following two results give analogues of Galois descent of algebraic Azumaya algebras as in [32, proposition 6.11].

Proposition 3.3. *Suppose that C is an Azumaya algebra over B for which the natural morphism $B \wedge_A C^{hG} \longrightarrow C$ is a weak equivalence of $B\langle G\rangle$-modules. Then C^{hG} is also an Azumaya algebra over A.*

Proof. We know from [29, lemma 6.2.4] that the A-algebra C^{hG} is dualizable as an A-module.

As C is Azumaya over B, we know that $C \wedge_B C^{\mathrm{o}} \simeq F_B(C, C)$. Also, dualizability of C^{hG} over A guarantees that

$$\begin{aligned} B \wedge_A F_A(C^{hG}, C^{hG}) &\simeq F_A(C^{hG}, B \wedge_A C^{hG}) \\ &\cong F_B(B \wedge_A C^{hG}, B \wedge_A C^{hG}) \\ &\simeq F_B(C, C) \simeq C \wedge_B C^{\mathrm{o}}, \end{aligned}$$

and so

$$\begin{aligned} C \wedge_B C^{\mathrm{o}} &\simeq (B \wedge_A C^{hG}) \wedge_B (B \wedge_A (C^{hG})^{\mathrm{o}}) \\ &\simeq B \wedge_A (C^{hG} \wedge_A (C^{hG})^{\mathrm{o}}). \end{aligned}$$

As B is faithful over A, this shows that

$$C^{hG} \wedge_A (C^{hG})^{\mathrm{o}} \simeq F_A(C^{hG}, C^{hG}).$$

Since C is faithful as a B-module and B is faithful as an A-module, we know that C is faithful as an A-module. Assume that for an A-module M we have $C^{hG} \wedge_A M \simeq *$. This is the case if and only if

$$B \wedge_A C^{hG} \wedge_A M \simeq C \wedge_A M \simeq *$$

because B is a faithful A-module. Now faithfulness of C over A implies that C^{hG} is also faithful over A. $\square$

Suppose that B/A is a faithful G-Galois extension in the sense of Rognes [29], where G is a finite group. Now let $H \lhd K \leqslant G$ so that B/B^{hH} is a faithful H-Galois extension, K acts on B^{hH} by B^{hK}-algebra maps and $B^{hK} \longrightarrow B^{hH}$ is a faithful K/H-Galois extension, in particular,

$$B^{hK} \simeq (B^{hH})^{h(K/H)}. \tag{3.1}$$

By [29, lemma 6.1.2(b)], the twisted group ring $B\langle H\rangle \simeq F_{B^{hH}}(B, B)$ is an Azumaya algebra over B^{hH}, and K acts on $B\langle H\rangle$ by extending the action on B by conjugation on H, so we will write $B\langle H_c\rangle$ to emphasize this.

If $K = Q \ltimes H$ is a semi-direct product or H is abelian, the quotient $Q = K/H$ acts by conjugation on H.

Note that as in algebra, there is an isomorphism of $A[K]$-modules

$$A[K] \cong \prod_K A.$$

The algebraic version of this isomorphism is given by

$$\sum_{k\in K} a_k k \leftrightarrow (a_{k^{-1}})_{k\in K}$$

and we will use the topological analogue of this.

Our next result is based on [32, proposition 6.11(b)].

Proposition 3.4. *Suppose that $K = Q \ltimes H$ is a semi-direct product, or that H is abelian. Then the B^{hK}-algebra $B\langle H_c\rangle^{hQ}$ is Azumaya, and*

$$B^{hH} \wedge_{B^{hK}} B\langle H_c\rangle^{hQ} \simeq B\langle H_c\rangle.$$

Hence the Azumaya algebra $B\langle H_c\rangle^{hQ}$ over B^{hK} is split by B^{hH}.

Proof. Note that we can assume that $G = K$ and $B^{hK} = A$. Making use of a faithful base change, it suffices to assume that B is the trivial K-Galois extension, $B = \prod_K A$.

There are isomorphisms of $A[K]$-modules

$$\begin{aligned} B\langle H_c\rangle &\cong {}_{\mathrm{diag}}\left(\prod_K A \wedge_A A[H_c]\right) \\ &\cong {}_{\mathrm{left}}\left(\prod_K A \wedge_A A[H]\right) \\ &\cong {}_{\mathrm{left}}(A[K] \wedge_A A[H]) \\ &\cong {}_{\mathrm{left}}(A[K \times H]), \end{aligned} \tag{3.2}$$

where ${}_{\mathrm{diag}}(-)$ and ${}_{\mathrm{left}}(-)$ indicate the diagonal and left K-actions respectively, the second isomorphism is the standard equivariant shear map similar to the map sh of [29, section 3.5], and $K \times H$ is viewed as a K-set through the action on the left hand factor. As a Q-set, K decomposes into free orbits indexed on H. On taking Q-homotopy fixed points we obtain an equivalence of A-modules

$$B\langle H_c\rangle^{hQ} \cong A[H \times H]. \tag{3.3}$$

There is a map of A-modules

$$B^{hH} \xrightarrow{\text{unit}} B^{hH} \wedge_A B\langle H_c\rangle^{hQ} \longrightarrow B\langle H_c\rangle$$

which is also a map of $B^{hH}\langle Q\rangle$-modules. Applying $\pi_*(-)$ and working algebraically with $\pi_*(A)$-modules, using [32, proposition 6.11(b)] it follows that we have an isomorphism

$$\pi_*(B^{hH} \wedge_A B\langle H_c\rangle^{hQ}) \cong \pi_*(B\langle H_c\rangle),$$

and therefore a weak equivalence

$$B^{hH} \wedge_A B\langle H_c\rangle^{hQ} \xrightarrow{\simeq} B\langle H_c\rangle$$

of $B^{hH}\langle Q\rangle$-modules. Now Proposition 3.3 shows that $B\langle H_c\rangle^{hQ}$ is Azumaya over B^{hK}. $\square$

4. Cyclic algebras

In this section, we will assume that $K \to L$ is a faithful Galois extension of commutative S-algebras, and that the Galois group $G = Gal(L/K)$ is generated by an element σ of order n, say. In particular, this group has to be cyclic. The choice of a generator corresponds to an isomorphism $\mathbb{Z}/n \cong G$, whose inverse can be thought of as a (primitive) character of G. Last but not least, we also need a strict unit u in K. For the time being, this just means that there is an action of the group $\mathbb{Z}$ of integers on the spectrum K via maps of K-modules. We will extend this action to L without change of notation. The strictness of u is needed in order to ensure that relations hold on the nose and this in turn is necessary to obtain a strictly associative algebra extension.

Cyclic K-algebras will be defined here via Galois descent from matrix algebras over L. As a model for the matrix algebra we use

$$M_n(L) = \bigvee_{i,j=1}^{n} L_{i,j},$$

with all $L_{i,j} = L$ and multiplication given on summands

$$L_{i,j} \wedge_K L_{j,k} \longrightarrow L_{i,k}$$

by the multiplication in L. One could also work with the endomorphism K-algebra spectrum $F_K(\bigvee_n L, \bigvee_n L)$, but this mixes covariant and contravariant behaviour in $\bigvee_n L$ and that is inconvenient for the explicit formulae that we need.

The cyclic group $\mathbb{Z}/n$ acts on the L-algebra $M_n(L)$ component-wise, *i.e.*, the generator acts as $\sigma\colon L_{i,j} \longrightarrow L_{i,j}$ on each summand. The multiplication and unit $L \longrightarrow M_n(L)$ are equivariant, and we have equivalences

$$M_n(L)^{h\mathbb{Z}/n} \simeq M_n(L^{h\mathbb{Z}/n}) \simeq M_n(K)$$

of K-algebras. Something possibly more interesting happens when we twist this action with the chosen unit u: we define a self-map on $M_n(L)$ as the composition

$$L_{i,j} \xrightarrow{\quad\text{id}\quad} L_{i+1,j+1} \xrightarrow{\quad u^{\delta_{i,n}-\delta_{j,n}}\quad} L_{i+1,j+1} \xrightarrow{\quad\sigma\quad} L_{i+1,j+1}, \tag{4.1}$$

where the indices $i+1$ and $j+1$ are read modulo n, and δ is the Kronecker symbol.

Lemma 4.1. *The above self-map generates a $\mathbb{Z}/n$-action on $M_n(L)$ as an associative K-algebra.*

Proof. This follows as in algebra using the fact that

$$u \wedge_K L = L \wedge_K u\colon L \wedge_K L \longrightarrow L \wedge_K L$$

since u is a unit in K. This guarantees that the K-algebra multiplication on L behaves well with respect to the twisted action. Together with the naturality and symmetry of the fold map this proves the claim. $\square$

Definition 4.2. The *cyclic K-algebra*

$$A(L, \sigma, u) = M_n(L)^{h\mathbb{Z}/n},$$

associated with L, σ, and u is obtained from the matrix algebra $M_n(L)$ with the twisted $\mathbb{Z}/n$-action by passage to homotopy fixed points.

The following result shows that the cyclic K-algebra $A(L, \sigma, u)$ defines a class in the relative Brauer subgroup $Br(L/K)$ of $Br(K)$.

Theorem 4.3. *The cyclic algebra $A(L, \sigma, u)$ is an Azumaya algebra over K which splits over L.*

Proof. Proposition 3.3 above says that if we have an Azumaya algebra B over L with a compatible G-action such that the natural morphism

$$L \wedge_K B^{hG} \longrightarrow B$$

is an equivalence of $L\langle G\rangle$-modules, then $A = B^{hG}$ is also an Azumaya algebra over K. We want to apply this here to the situation $B = M_n(L)$ and $G \cong \mathbb{Z}/n$.

Since L is a dualizable K-module,

$$L \wedge_K M_n(L)^{hG} \simeq (L \wedge_K M_n(L))^{hG},$$

where G acts only on the right hand factor in $L \wedge_K M_n(L)$.

As L is a G-Galois extension of K, we have $L \wedge_K L \simeq Map(G_+, L)$, and therefore

$$L \wedge_K M_n(L) \cong M_n(L \wedge_K L) \simeq M_n(Map(G_+, L)) \simeq Map(G_+, M_n(L)).$$

As the latter is equivariantly equivalent to $L[G]$, we see that

$$L \wedge_K M_n(L)^{hG} \simeq M_n(L)$$

and this yields the result. $\square$

Of course, it may happen that a cyclic K-algebra $A(L, \sigma, u)$ represents the trivial element in the Brauer group $Br(K)$ of K. This depends very much on the chosen unit u, for example. One way to prove non-triviality is to compute the homotopy groups of $A(L, \sigma, u)$, and to compare the result with the homotopy groups of the representatives of the elementary Azumaya algebras.

For this and other reasons, it is useful to know that one may calculate the homotopy groups of the cyclic algebra $A(L, \sigma, u)$ by means of the homotopy fixed point spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{H}^s(\mathbb{Z}/n, \pi_t M_n(L)) \Longrightarrow \pi_{t-s}(M_n(L)^{h\mathbb{Z}/n}). \tag{4.2}$$

The action on $\pi_t M_n(L) \cong M_n(\pi_t L)$ is generated by whatever the twisted action Eq. (4.1) induces in homotopy:

$$a_{i,j} \longmapsto \sigma_* u_*^{\delta_{i,n}-\delta_{j,n}}(a_{i+1,j+1}).$$

But, if we identify $(\pi_t L)^{\oplus n}$ with the first row (or column) of $\pi_t M_n(L)$, we easily find

$$\pi_t M_n(L) \cong \mathbb{Z}[\mathbb{Z}/n] \otimes_{\mathbb{Z}} (\pi_t L)^{\oplus n},$$

which shows that $\pi_t M_n(L)$ is an induced $\mathbb{Z}/n$-module. Therefore the E_2-term of the homotopy fixed point spectral sequence Eq. (4.2) vanishes above the 0-line, which shows

$$\pi_t A(L, \sigma, u) \cong (\pi_t L)^{\oplus n},$$

additively. In fact, this determines the underlying homotopy type.

Theorem 4.4. *If $n = 2$, and if the unit u has order 2 in the sense that it comes from a $\mathbb{Z}/2$-action, then the class of $A(L, \sigma, u)$ has order at most 2 in the Brauer group of $Br(K)$.*

Proof. We prove that there is a $\mathbb{Z}/2$-equivariant equivalence of K-algebras between $M_2(L)$ and $M_2(L)^{\mathrm{o}}$. As in algebra, this equivalence is given by the transposition of matrices, which we have to model by the permutation $L_{i,j} \longrightarrow L_{j,i}$. Then the same proof as in algebra shows that this is a map of associative K-algebras. In order to show that the action passes to the homotopy fixed points we have to prove that it is compatible with the twisted action of Eq. (4.1) that we impose on $M_2(L)$. But this is trivial except for the u-action, where we have to use that $u = u^{-1}$ is a second root of unity. $\square$

5. Azumaya algebras over Eilenberg–Mac Lane spectra

In this section we consider the case of Azumaya algebras over the Eilenberg–Mac Lane spectrum of a commutative ring. In [35], Toën introduces the algebraic notion of a *derived Azumaya algebra* over a commutative ring as a special case of the more general notion for simplicial rings. First we explain how the topological and algebraic notions are related.

In [15, section IV.2], an equivalence of categories

$$\Psi\colon \mathscr{D}_{HR} \longrightarrow \mathscr{D}_R \tag{5.1}$$

is constructed, where Ψ is defined on a CW HR-module M to be the cellular chain complex $C_*(M)$. By [15, proposition IV.2.5], for CW HR-modules M, N there are isomorphisms of chain complexes of R-modules

$$C_*(M \wedge_{HR} N) \cong C_*(M) \otimes_R C_*(N),$$
$$C_*(F_{HR}(M, N)) \cong Hom_R(C_*(M), C_*(N)).$$

The inverse functor $\Phi = \Psi^{-1}$ also preserves the monoidal structure, thus we have an equivalence of symmetric monoidal categories.

Following Toën [35], see remark 1.2, and [20, theorem 1.5] we find that an Azumaya algebra A over HR, corresponds to a derived Azumaya algebra over R. Note that as we are working with associative (but not commutative) HR-algebras, a cofibrant HR-algebra is a retract of a cell HR-module relative to HR by [15, theorem VII.6.2].

We get the following correspondence whose assumptions are satisfied when R is a principal ideal domain for instance.

Proposition 5.1. *Let R be a commutative ring such that for any finitely presented R-module M with $Tor_k^R(M, M) = 0$ for $k > 0$ we can deduce that M is flat over R.*

Let T be an R-algebra. Then the HR-algebra HT is a topological Azumaya algebra if and only if T is an algebraic Azumaya R-algebra.

Proof. One direction is easy to see: if $R \longrightarrow T$ is an algebraic Azumaya extension, then $HR \longrightarrow HT$ is topologically Azumaya without any additional assumptions on R.

For the converse, from [15, theorem IV.2.1] we have

$$\pi_n(HT \wedge_{HR} HT^o) = Tor_n^R(T, T^o), \tag{5.2}$$
$$\pi_n(F_{HR}(HT, HT)) = Ext_R^{-n}(T, T). \tag{5.3}$$

Because $Tor_s^R = 0 = Ext_R^s$ when $s < 0$, the Azumaya condition

$$\mu\colon HT \wedge_{HR} HT^o \xrightarrow{\simeq} F_{HR}(HT, HT)$$

implies that for $n \neq 0$,

$$\pi_n(HT \wedge_{HR} HT^o) = Tor_n^R(T, T^o) = 0 = Ext_R^n(T, T) = \pi_n(F_{HR}(HT, HT)). \tag{5.4}$$

In particular,

$$T \otimes_R T^o = \pi_0(HT \wedge_{HR} HT^o) \cong \pi_0(F_{HR}(HT, HT)) = Hom_R(T, T). \tag{5.5}$$

According to [35, remark 1.2], the R-module T is finitely presented and flat by assumption, therefore it is finitely generated and projective by the corollary to [25, theorem 7.12].

For faithfulness, suppose that M is a non-trivial R-module. Since HT is a faithful HR-module, $HT \wedge_{HR} HM \not\simeq *$. Flatness of T over R together with [15, theorem IV.2.1] yields the isomorphisms

$$\pi_*(HT \wedge_{HR} HM) \cong \pi_0(HT \wedge_{HR} HM) \cong T \otimes_R M,$$

and therefore $T \otimes_R M$ is not trivial. □

Proposition 5.2. *For any commutative ring with unit R there is a natural homomorphism*

$$H\colon Br(R) \longrightarrow Br(HR)$$

induced by the functor which sends a ring to its Eilenberg–Mac Lane spectrum.

Proof. Let $[A]$ be an element of $Br(R)$, then Proposition 5.1 identifies HA as an HR-Azumaya algebra. If $[A] = 0$, *i.e.*, if there is a finitely generated faithful projective R-module M with $A \cong Hom_R(M, M)$, then

$$HA \simeq HHom_R(M, M) \simeq F_{HR}(HM, HM)$$

and therefore HA is trivial in $Br(HR)$. □

Remark 5.3. Toën shows that the Brauer group of derived Azumaya algebras over R is parametrized by $H^2_{\text{ét}}(R, \mathbb{G}_m) \times H^1_{\text{ét}}(R, \mathbb{Z})$ whereas the ordinary Brauer group of R, $Br(R)$, corresponds to the torsion part in $H^2_{\text{ét}}(R, \mathbb{G}_m)$. Combining this with the comparison result of Johnson [20, theorem 1.5] yields that the above homomorphism H is injective for any R corresponding to $H^2_{\text{ét}}(R, \mathbb{G}_m)_{\text{tor}} \subseteq H^2_{\text{ét}}(R, \mathbb{G}_m)$.

The situation is drastically different if we consider arbitrary HR-algebra spectra A. For instance, for every R, every R-module spectrum $\Sigma^n HR$ is faithful and dualizable, and therefore $F_{HR}(HR \vee \Sigma^n HR, HR \vee \Sigma^n HR)$ is a trivial topological Azumaya HR-algebra whose homotopy groups spread over positive and negative degrees. This indicates that the Eilenberg–Mac Lane functor of Proposition 5.2 will not induce an isomorphism in general. The relationship with étale cohomology groups in [35] in fact shows that there are derived Azumaya algebras that are not Brauer equivalent to an ordinary Azumaya algebra. Toën describes a concrete example in [35, section 4] originating in an example by Mumford.

We will discuss the case of $H\boldsymbol{k}$ for a field $\boldsymbol{k}$. If A is Azumaya over $H\boldsymbol{k}$, then as A is dualizable over $H\boldsymbol{k}$ we know that the homotopy groups of A are concentrated in finitely many degrees, say $\pi_r(A) \neq 0$ only when $-m \leqslant r \leqslant n$ for some $m, n \geqslant 0$. As $\boldsymbol{k}$ is a field, we have

$$\pi_*(A \wedge_{H\boldsymbol{k}} A^{\mathrm{o}}) \cong \pi_*(A) \otimes_{\boldsymbol{k}} \pi_*(A)^{\mathrm{o}}.$$

Using the fact that μ induces an isomorphism, we can deduce that $n = m$ because otherwise the kernel of $\pi_*(\mu)$ would be non-trivial.

A derived Azumaya algebra over the field $\boldsymbol{k}$ is a differential graded $\boldsymbol{k}$-algebra B_* whose underlying chain complex is a compact generator of the derived category of chain complexes of $\boldsymbol{k}$-vector spaces $\mathscr{D}_{\boldsymbol{k}}$ such that the natural map

$$\mu_{B_*} : B_* \otimes_{\boldsymbol{k}} B_*^{\mathrm{o}} \longrightarrow Hom_{\boldsymbol{k}}(B_*, B_*)$$

is an isomorphism in $\mathscr{D}_{\boldsymbol{k}}$. Here $B_* \otimes_{\boldsymbol{k}} B_*^{\mathrm{o}}$ agrees with the derived tensor product because we are working over a field, and similarly, $Hom_{\boldsymbol{k}}(B_*, B_*)$ is the graded $\boldsymbol{k}$-vector space of derived endomorphisms of B_*. Now we can relate topological $H\boldsymbol{k}$-Azumaya algebras to derived Azumaya algebras over $\boldsymbol{k}$.

Proposition 5.4. *If A is a topological Azumaya algebra over $H\boldsymbol{k}$, then $\pi_*(A)$ is a derived Azumaya algebra over $\boldsymbol{k}$.*

Proof. As A is dualizable over $H\boldsymbol{k}$, its homotopy groups build a finite dimensional graded $\boldsymbol{k}$-vector space and hence $\pi_*(A)$ is a compact generator of $\mathscr{D}_{\boldsymbol{k}}$. The weak equivalence

$$\mu : A \wedge_{H\boldsymbol{k}} A^{\mathrm{o}} \longrightarrow F_{H\boldsymbol{k}}(A, A)$$

yields isomorphisms

$$\mu_{\pi_*(A)} : \pi_*(A) \otimes_{\boldsymbol{k}} \pi_*(A)^{\mathrm{o}} \cong \pi_*(A \wedge_{H\boldsymbol{k}} A^{\mathrm{o}}) \cong \pi_* F_{H\boldsymbol{k}}(A, A) \cong Hom_{\boldsymbol{k}}(A_*, A_*)$$

and so $\pi_*(A)$ is a derived Azumaya algebra over $\boldsymbol{k}$. □

Using Proposition 5.4 together with Toën's results of [35, section 1] we obtain the following.

Theorem 5.5. *For any algebraically closed field $\boldsymbol{k}$, the Brauer group of $H\boldsymbol{k}$ is trivial.*

Proof. Let A be a derived Azumaya algebra over $\boldsymbol{k}$. We know from [35, corollary 1.11] that every derived Azumaya algebra over an algebraically closed field $\boldsymbol{k}$, in particular $\pi_*(A)$, is quasi-isomorphic to a graded $\boldsymbol{k}$-vector space $Hom_{\boldsymbol{k}}(V, V)$ for some finite dimensional graded $\boldsymbol{k}$-vector space V.

Let

$$M = HV = \bigvee_{i=1}^{n} \Sigma^{m_i} H\boldsymbol{k}$$

be the $H\boldsymbol{k}$-module spectrum such that $\pi_* M \cong V$ as graded $\boldsymbol{k}$-vector spaces. Then A is weakly equivalent to $F_{H\boldsymbol{k}}(A, A)$ since there are isomorphisms

$$\pi_*(A) \cong Hom_{\boldsymbol{k}}(V, V) \cong \pi_*(F_{H\boldsymbol{k}}(M, M)).$$

Therefore $[A]$ is trivial in the Brauer group $Br(H\boldsymbol{k})$. □

Remark 5.6. Using [35, corollary 1.15] one can extend the result to obtain the triviality of the Brauer group $Br(H\boldsymbol{k})$ for any separably closed field $\boldsymbol{k}$.

6. Realizability of algebraic Azumaya extensions

Using Angeltveit's obstruction theory [1, theorem 3.5], we can import algebraic Azumaya algebra extensions into topology. Let R be a commutative S-algebra and let $\pi_0 R \longrightarrow A_0$ be an algebraic Azumaya extension. Then

$$A_* := \pi_* R \otimes_{\pi_0 R} A_0$$

is a projective module over $R_* = \pi_* R$ and there is an R-module spectrum A' with $\pi_*(A') \cong A_*$ which can be built as a mapping telescope of an idempotent corresponding to viewing A_* as a direct summand of a free R_*-module. The methods of [6] carry over to give a homotopy associative R-ring spectrum A that realizes A_* as the homotopy ring $\pi_* A$.

Angeltveit's obstruction theory [1] then yields the following.

Theorem 6.1. *There is a unique A_∞ R-algebra structure on A,* i.e., *there is a unique rigidification $r(A)$ of A to an associative R-algebra. The resulting extension $R \longrightarrow r(A)$ is an Azumaya algebra.*

Proof. The existence of the A_∞ structure on A is given by [1, theorem 3.5], because $\pi_*(A \wedge_R A^o)$ is separable over A_* and hence the possible obstructions to an A_∞-structure on A (which live in Hochschild cohomology groups of $\pi_*(A \wedge_R A^o)$ over A_*) are trivial. The possibility of rigidification follows from [15, II.4]. Uniqueness also follows from the vanishing of all higher Hochschild cohomology groups.

As A_0 is finitely generated projective and faithful over $\pi_0 R$, $r(A)$ is dualizable and faithful as an R-module spectrum. The Azumaya condition

$$\mu : A_0 \otimes_{\pi_0 R} A_0^o \cong Hom_{\pi_0 R}(A_0, A_0)$$

for A_0 guarantees that the μ-map

$$\mu : r(A) \wedge_R r(A)^o \longrightarrow F_R(r(A), r(A))$$

is a weak equivalence. □

Corollary 6.2. *There is a natural group homomorphism*

$$r : Br(\pi_0 R) \longrightarrow Br(R); \quad [A] \mapsto [r(A)].$$

This result implies Proposition 5.2 but reaches further. For instance in the presence of enough roots of one, we can build generalized quaternionic extensions of ring spectra or consider cyclic extensions. Note, however, that in many cases $Br(\pi_0(R)) = 0$, for instance if $\pi_0(R)$ is isomorphic to a finite field, $\mathbb{Z}$ (see [18]) or $\mathbb{Z}[A]$ for some finite abelian group A (see [21]). We learned from David Gepner that the Brauer group for a connective commutative S-algebra R can be described via the second étale cohomology group of $\pi_0(R)$ with coefficients in the units and the first étale cohomology group of $\pi_0(R)$ with coefficients in $\mathbb{Z}$ via a short exact sequence.

John Rognes drew our attention to the non-trivial examples of Brauer groups in [31].

Proposition 6.3. *There is a quaternionic extension of the sphere spectrum with* 2 *inverted that is not trivial in the Brauer group and hence*

$$Br(S[1/2]) \neq 0.$$

Proof. The Brauer group of $\mathbb{Z}[1/2]$ is isomorphic to $\mathbb{Z}/2$, see [31, 2.8]. Extension of scalars to $\mathbb{R}$ has to send the generator of $Br(\mathbb{Z}[1/2])$ to the generator of $Br(\mathbb{R}) \cong \mathbb{Z}/2$ and hence this extension is equivalent to the $\mathbb{R}$-algebra of quaternions $\mathbb{H}$. Thus we know that a representative of the generator of $Br(\mathbb{Z}[1/2])$ is given by the class of the subring of localized Hurwitz quaternions $\mathbb{Z}[1/2](i, j, k) \subseteq \mathbb{H}$ with $i^2 = j^2 = k^2 = -1$, $k = ij = -ji$, which is the quaternionic extension of $\mathbb{Z}[1/2]$. We can realize this extension topologically as an Azumaya algebra H over $S[1/2]$.

Using base-change to $H\mathbb{R}$ we get the following commutative diagram.

$$\begin{tikzcd}
Br(S[1/2]) \arrow[r] & Br(H\mathbb{R}) \\
Br(\mathbb{Z}[1/2]) \arrow[r] \arrow[u] & Br(\mathbb{R}) \arrow[u]
\end{tikzcd}.$$

As the map from $Br(\mathbb{R})$ to $Br(H\mathbb{R})$ is injective (compare Remark 5.3), the image of the class of $\mathbb{Z}[1/2](i, j, k)$ in $Br(S[1/2])$ cannot be trivial. □

Remark 6.4. As the Brauer groups of $\mathbb{Z}[1/p]$ and $\mathbb{Z}_{(p)}$ are non-trivial for odd primes as well (see [27] and [37, p.145]), the above result can be used to obtain that other Brauer groups of connective commutative ring spectra are non-trivial. In particular, the Brauer groups of the corresponding localized spheres are non-trivial.

7. Topological Hochschild cohomology of group rings

We will consider Azumaya algebra extensions that arise as follows. For a finite discrete group G and a commutative S-algebra A, we consider the group A-algebra spectrum $A[G] = A \wedge G_+$. Note, that if G is not abelian, then $A[G]$ is not commutative. We want to identify the extension $THH_A(A[G]) \longrightarrow A[G]$ as an Azumaya extension in good cases.

For ordinary commutative rings R and groups G, DeMeyer and Janusz describe in [12] conditions on R and G which ensure that $R[G]$ is an Azumaya algebra over its centre. First, we document a well-known identification of topological Hochschild cohomology of group rings, see for instance [24, 4.2.10]. This can we viewed as a topological version of Mac Lane's isomorphisms [23, 7.4.2].

Lemma 7.1. *For A and G as above we have*

$$THH_A(A[G], A[G]) \simeq (A[G]^c)^{hG} = F_G(EG_+, A[G]^c).$$

Here $A[G]^c$ denotes the naive G-spectrum $A[G]$, where G acts by conjugation on G.

Proof. Topological Hochschild cohomology of $A[G]$ can be described as the totalization of the cosimplicial spectrum that has

$$F_A(A[G]^q, A[G]) \cong F(G_+^q, A[G])$$

as q-cosimplices [26]. First, we mimic the identification that is used in the Mac Lane isomorphism for usual Hochschild cohomology in order to identify this cosimplicial spectrum with the one that has $F(G^q, A[G]^c)$ as q-cosimplices. In algebra this identification is given by $f \mapsto f'$ where

$$f'(g_1, \dots, g_q) = f(g_1, \dots, g_q)g_q^{-1} \dots g_1^{-1}.$$

An analogous identification works on spectrum level. The coface maps in the cosimplicial structure in $F(G^\bullet, A[G]^c)$ are given by

$$\begin{aligned} d_0(f)(g_1, \dots, g_q) &= g_1 f(g_2, \dots, g_q)g_1^{-1}, \\ d_i(f)(g_1, \dots, g_q) &= f(g_1, \dots, g_i g_{i+1}, \dots, g_q), \quad (0 < i < q) \\ d_q(f)(g_1, \dots, g_q &= f(g_1, \dots, g_{q-1}). \end{aligned}$$

Consider the simplicial model of EG with q-simplices G^{q+1}, with diagonal G-action, and where the i-th face map in EG is given by omitting the i-th group element. We can write the homotopy fixed point spectrum $F_G(EG_+, A[G]^c)$ as

$$F_G(EG_+, A[G]^c) \cong Tot([q] \mapsto F_G(G^{q+1}, A[G]^c)).$$

Let $\varphi\colon F(G^\bullet, A[G]^c) \longrightarrow F_G(EG_+, A[G]^c)$ be the map that we can describe symbolically as

$$(\varphi f)(g_0, \dots, g_q) = g_0 f(g_0^{-1}g_1, \dots, g_{q-1}^{-1}gq)g_0^{-1}.$$

It is then straightforward to check that φ in fact respects the cosimplicial structure. □

Now fix a prime p. Let $\boldsymbol{k}$ be an algebraically closed field of characteristic p and let $H\boldsymbol{k}$ be the corresponding Eilenberg–Mac Lane spectrum realized as a commutative S-algebra. We also adopt the notation of [8]. Thus E_n is the Lubin–Tate spectrum associated with the prime p and the Honda formal group of height n and E_n^{nr} is its maximal unramified Galois extension. These commutative S-algebras have 'residue fields' in the sense of [6,7], namely K_n and K_n^{nr} respectively, and these are algebras over E_n and E_n^{nr} respectively, but only homotopy commutative when $p \neq 2$ and not even that when $p = 2$.

Theorem 7.2. *Let G be a non-trivial finite discrete group whose order is not divisible by p. Suppose that A is either $H\boldsymbol{k}$ or E_n^{nr}.*

(1) *If G is abelian, then $(A[G]^c)^{hG} \longrightarrow A[G]$ and the trivial extension $\mathrm{id}\colon A[G] \longrightarrow A[G]$ are equivalent.*
(2) *If G is non-abelian, then $A[G]$ is a non-trivial $(A[G])^{hG}$-Azumaya algebra.*

Proof. In all cases, we will consider the homotopy fixed point spectral sequence

$$\mathrm{E}_2^{s,t} = H^{-s}(G; A_t[G]^c) \Longrightarrow \pi_{s+t}((A[G]^c)^{hG}).$$

If p does not divide the order of the group G, then this spectral sequence collapses and the only surviving non-trivial terms are the G-invariants

$$\mathrm{E}_2^{0,t} = (A_t[G]^c)^G$$

which can be identified with the centre of the group ring $Z(A_*[G])$. In particular, $\pi_*((A[G]^c)^{hG})$ is a graded commutative A_*-algebra.

If G is abelian, then the conjugation action is trivial and as p does not divide $|G|$ we obtain

$$(A[G]^c)^{hG} = F(BG_+, A[G]) \simeq A[G],$$

so we have the trivial Azumaya extension. If G is not abelian, then the centre of the group ring $A_*[G]$ is a proper subring of $A_*[G]$.

For $A = H\boldsymbol{k}$ we can use Artin–Wedderburn theory to obtain a splitting of the semisimple ring $\boldsymbol{k}[G]$ into a product of matrix algebras over the algebraically closed field $\boldsymbol{k}$,

$$\boldsymbol{k}[G] \cong \prod_{i=1}^r M_{m_i}(\boldsymbol{k}),$$

where r agrees with the number of conjugacy classes in G. Thus the centre of $\boldsymbol{k}[G]$ is a product of copies of $\boldsymbol{k}$ and is therefore an étale $\boldsymbol{k}$-algebra. By the obstruction theory of Robinson or Goerss–Hopkins [28,17], there is a unique E_∞ $H\boldsymbol{k}$-algebra spectrum that is weakly equivalent to $(A[G]^c)^{hG}$. By abuse of notation we denote the corresponding commutative $H\boldsymbol{k}$-algebra by $(A[G]^c)^{hG}$.

We have to describe $A[G]$ as an associative $(A[G]^c)^{hG}$-algebra. For this we use [1, theorem 3.5] again. Starting with our commutative model of $(A[G]^c)^{hG}$ we can build a homotopy associative ring spectrum B with $\pi_*(B) \cong A_*[G]$, and as G is finite and discrete this extension is of the form

$$\pi_*(B) \cong \pi_*(A[G]^c)^{hG} \otimes_{\pi_0(A[G]^c)^{hG}} B_0,$$

with $\pi_0(A[G]^c)^{hG} \longrightarrow B_0$ being algebraically Azumaya. Thus we can apply Theorem 6.1 to see that there is an associative $(A[G]^c)^{hG}$-algebra B which models $A[G]$ and such that B is Azumaya over $(A[G]^c)^{hG}$.

For E_n^{nr} we pass to the residue field K_n^{nr}. The homotopy fixed point spectral sequence gives

$$\begin{aligned}\pi_*((E_n^{\mathrm{nr}}[G]^c)^{hG}) &\cong Z((E_n^{\mathrm{nr}})_*[G]) \\ &\cong Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G])[u^{\pm 1}].\end{aligned}$$

Reducing modulo the maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$ gives the homotopy groups of the G-homotopy fixed points of $K_n^{\mathrm{nr}}[G]$ with respect to the conjugation action, $Z(\overline{\mathbb{F}}_p[G])[u^{\pm 1}]$ and again we can identify this term as $\prod_{i=1}^r \overline{\mathbb{F}}_p$ where r denotes the number of conjugacy classes in G. The idempotents that give rise to these splittings can be lifted to idempotents for $Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G])$ and $W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G]$ and therefore these two algebras also split into products with r factors:

$$\begin{aligned}W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G] &\cong \prod_{i=1}^r B_i, \\ Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G]) &\cong \prod_{i=1}^r C_i,\end{aligned}$$

where

$$B_i/\mathfrak{m} \cong M_{m_i}(\overline{\mathbb{F}}_p),$$

while for $1 \leqslant i \leqslant r$, the C_i are commutative and satisfy

$$C_i/\mathfrak{m}C_i \cong \overline{\mathbb{F}}_p.$$

Additively we know that $Z(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G])$ is the free module on the conjugacy classes and so we can conclude that $(E_n^{\mathrm{nr}}[G]^c)^{hG}$ is weakly equivalent to $\prod_{i=1}^r E_n^{\mathrm{nr}}$ and the latter spectrum can be modelled by a commutative E_n^{nr}-algebra spectrum and $E_n^{\mathrm{nr}}[G]$ is dualizable over $\prod_{i=1}^r E_n^{\mathrm{nr}}$.

Artin–Wedderburn theory gives a semisimple decomposition

$$\overline{\mathbb{F}}_p[G] \cong \prod_{i=1}^r M_{d_i}(\overline{\mathbb{F}}_p),$$

and the centre $Z(\overline{\mathbb{F}}_p[G])$ can be identified with the product of the centres of the matrix ring factors. There are associated central idempotents of $\overline{\mathbb{F}}_p[G]$ accomplishing this splitting. By the theory of idempotent lifting described in [22, section 21] for example, these idempotents lift to give an associated splitting

$$W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G] \cong \prod_{i=1}^r M_{d_i}(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]]),$$

and again the centre of $W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]][G]$ can be identified with the product of the centres of the matrix factors. Notice that $M_{d_i}(W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]])$ is Azumaya over $W\overline{\mathbb{F}}_p[[u_1,\ldots,u_{n-1}]]$. The rest of the proof involves realizing the central idempotents as morphisms of S-algebras, but this is well known to be possible since the projections are Bousfield localizations, see [33]. □

8. Azumaya algebras over Lubin–Tate spectra

From now on we will use E to denote E_n, E_n^{nr} or any commutative Galois extension of E_n obtained as a homotopy fixed point algebra $E = (E_n^{\mathrm{nr}})^{h\Gamma}$ for some closed normal subgroup $\Gamma \lhd Gal(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n})$. Similarly, K will denote the corresponding residue field of E, so when $E = E_n$ or E_n^{nr} we have $K = K_n$ or K_n^{nr}.

We will work with dualizable K-local E-modules. By [8, section 7] we know that such modules are retracts of finite cell E-modules. If $W \in \mathscr{M}_{E,K}$, then since $\pi_*(K \wedge_E W)$ is a graded vector space over the graded field $K_* = \pi_*(K)$, it follows that

$$K \wedge_E W \simeq L_K \bigvee_i \Sigma^{d(i)} K,$$

where the right hand wedge is non-trivial if and only if W is non-trivial in $\mathscr{D}_{E,K}$. In particular, if W is dualizable this wedge is finite and

$$K \wedge_E W \simeq \bigvee_i \Sigma^{d(i)} K$$

since W is K-local. For any $X \in \mathscr{M}_{E,K}$,

$$K \wedge_E (W \wedge_E X) \simeq L_K \bigvee_i \Sigma^{d(i)} K \wedge_E X,$$

so $W \wedge_E X$ is trivial in $\mathscr{D}_{E,K}$ if and only if both of W and X are trivial in $\mathscr{D}_{E,K}$. Thus every E-module W which is non-trivial as an element of $\mathscr{D}_{E,K}$ is faithful and cofibrant as a K-local E-module; furthermore, every $X \in \mathscr{M}_{E,K}$ is W-local.

By [1], there are many examples of K-local Azumaya algebras over E which have K as their underlying ring spectrum. These examples have no analogue in the algebraic context since they are not projective E-modules, nor do they split over suitable Galois extensions. Instead we focus on split examples. A good source of these can be found in the situation of [29, section 5.4.3], based on work of Devinatz and Hopkins [13] and we will discuss these in Section 9.

For background ideas on Azumaya algebras graded on a finite abelian group, we follow [10] which generalizes work of Wall [36] and others. We will only consider the case where the grading group is $\mathbb{Z}/2$ with the non-trivial symmetric bilinear map $\mathbb{Z}/2 \times \mathbb{Z}/2 \longrightarrow \{\pm 1\}$ determining the relevant signs, however in periodic topological contexts it may also prove useful to modify the grading to other finite quotient groups lying between $\mathbb{Z}$ and $\mathbb{Z}/2$, and the above reference should provide appropriate generality for such algebra.

Over a field $\boldsymbol{k}$, an (ungraded) Azumaya algebra A is a central simple algebra, so by Wedderburn's theorem, there is an isomorphism of $\boldsymbol{k}$-algebras

$$A \cong M_r(D),$$

where D is a central division algebra over $\boldsymbol{k}$. If $d = \dim_{\boldsymbol{k}} D$, then

$$\dim_{\boldsymbol{k}} A = (rd)^2,$$

so $\dim_{\boldsymbol{k}} A$ is a square. In the graded case, such restrictions do not always apply, and this has consequences for the topological situation.

Theorem 8.1. *Suppose that p is an odd prime and let A be a K-local Azumaya algebra over E. Then $\pi_*(K \wedge_E A)$ is an Azumaya algebra over K_*.*

Proof. The ring K_* is a 2-periodic graded field which we will view as $\mathbb{Z}/2$-graded, and $\pi_*(K \wedge_E A)$ will also be viewed as a $\mathbb{Z}/2$-graded K_*-algebra.

We have isomorphisms of K_*-algebras

$$\begin{aligned}
\pi_*(K \wedge_E A) \otimes_{K_*} \pi_*(K \wedge_E A)^{o} &\cong \pi_*(K \wedge_E A) \otimes_{K_*} \pi_*(K^{o} \wedge_E A^{o}) \\
&\cong \pi_*(K \wedge_E (A \wedge_E A^{o})) \\
&\cong \pi_*(K \wedge_E F_E(A, A)).
\end{aligned}$$

Since A and K are strongly dualizable, using results of [15] we have

$$K \wedge_E F_E(A, A) \simeq F_K(K \wedge_E A, K \wedge_E A),$$

so the universal coefficient spectral sequence over K yields

$$\pi_*(K \wedge_E F_E(A, A)) \cong End_{K_*}(\pi_*(K \wedge_E A)).$$

Therefore $\pi_*(K \wedge_E A)$ is a K_*-Azumaya algebra. $\square$

Corollary 8.2. *If $\pi_*(K \wedge_E A)$ is concentrated in even degrees then its dimension is a square,* i.e., *for some natural number m,*

$$\dim_{K_*} \pi_*(K \wedge_E A) = m^2.$$

In fact we have

Proposition 8.3. *If $\pi_*(K \wedge_E A)$ is concentrated in even degrees then $\pi_*(A)$ is a $\mathbb{Z}/2$-graded algebra Azumaya algebra over E_0. In particular, as an E-module A is equivalent to a wedge of m^2 copies of E, where*

$$m^2 = \dim_{K_*} \pi_*(K \wedge_E A) = rank_{E_*} \pi_*(A).$$

Proof. By [8] (see section 7 and the proof of theorem 5.1), the E_*-module $\pi_*(A)$ is finitely generated, free and concentrated in even degrees, hence

$$\pi_*(A) \otimes_{E_*} \pi_*(A)^{o} \cong \pi_*(A \wedge_E A^{o}) \cong \pi_*(F_E(A, A)) \cong Hom_{E_*}(\pi_*(A), \pi_*(A)),$$

where the last isomorphism follows from the collapsing of the universal coefficient spectral sequence. $\square$

Recall that $Az_K(E)$ is the collection of all cofibrant K-local topological Azumaya algebras over E. The Brauer equivalence relation $\approx$ on $Az_K(E)$ is then given as follows:

- If $A, B \in Az_K(E)$, then $A \approx B$ if and only if there are faithful, dualizable, cofibrant E-modules U, V for which there is an equivalence in the derived category of K-local E-algebras

$$A \wedge_E F_E(U, U) \simeq B \wedge_E F_E(V, V).$$

The set of equivalence classes of $\approx$ is $Br_K(E)$; this is indeed a set since every dualizable K-local E-module is a retract of a finite cell E-module.

9. Some examples of K_n-local Azumaya algebras

We now recall Proposition 3.4. By work of Devinatz and Hopkins [13], and subsequently Davis [11], as explained in [29, theorem 5.4.4], for each pair of closed subgroups

$$H \leqslant G \leqslant \mathbb{G}_n = Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathbb{S}_n$$

of the Morava stabilizer group, there is an associated pair of homotopy fixed point spectra $E^{hG} \longrightarrow E^{hH}$, and if $H \lhd G$ then this is a K-local G/H-Galois extension. In particular, when $H \leqslant \mathbb{G}_n$ is finite, $E^{hH} \longrightarrow E$ is a K-local H-Galois extension.

A particularly interesting source of examples is provided by taking G to be a maximal finite subgroup of $\mathbb{G}_n$. If p is odd and $n = (p-1)k$ with $p \nmid k$, or $p = 2$ and $n = 2k$ with k odd, then such maximal subgroups are unique up to conjugation and then the homotopy fixed point spectrum E^{hG} is denoted EO_n. For $p = 3$ Behrens [9, remark 1.7.3] gives an argument for the identification of EO_2 with the $K(2)$-localization of the spectrum of topological modular forms, TMF. This can be adapted to $p = 2$. We proceed with an example, studied in [29, section 5.4.3].

Example 9.1. At the prime $p = 2$, the group $\mathbb{G}_2$ has a maximal finite subgroup G_{48} of order 48 which is isomorphic to a semi-direct product of the group HQ_{24} of order 24, which consists of the units in the ring of Hurwitz quaternions, with the Galois group $Gal(\mathbb{F}_4/\mathbb{F}_2)$ of order 2. Therefore E_2/EO_2 is a G_{48}-Galois extension. Applying Proposition 3.4, we see that the splitting of the group G_{48} yields an Azumaya algebra

$$(E_2 \langle HQ_{24} \rangle)^{hC_2}$$

over EO_2. There are two more examples like this: Of the 15 conjugacy classes of subgroups of G_{48}, there are 7 which are normal: 1, C_2, C_4, Q_8, $Syl_2(G_{48})$, HQ_{24}, and G_{48}. The first and last of them are uninteresting here, as $(E_2)^{hG_{48}} \simeq EO_2$ and $E_2 \langle G_{48} \rangle \simeq F_{EO_2}(E_2, E_2)$ are trivial in the Brauer group of EO_2. The second and third do not split the group, but the other three do. The last of these yields the Azumaya algebra displayed above, but the other two give rise to further examples

$$(E_2 \langle Q_8 \rangle)^{hC_6} \quad (E_2 \langle Syl_2(G_{48}) \rangle)^{hC_3}$$

of Azumaya algebras over EO_2.

10. The Brauer group of the $K(n)$-local sphere

In this section we discuss the $K(n)$-local Brauer group of the $K(n)$-local sphere $L_{K(n)}S$.

Theorem 10.1. *Suppose that $p > 2$ and $n > 1$. Then the $K(n)$-local Brauer group of $L_{K(n)}S$ is non-trivial.*

Proof. As usual, let us write $q = p^n$. The cyclic group $C = \mathbb{F}_q^\times$ of order $q - 1$ consists of roots of unity, and the Galois group $G = Gal(\mathbb{F}_q/\mathbb{F}_p)$ is cyclic of order n, generated by the Frobenius. As $n \neq 1$, the Galois group is non-trivial, and its Galois action gives rise to an extension $F = G \ltimes C$.

Let $\mathbb{G}_n$ denote again the n-th extended Morava stabilizer group. We refer to section 2.3 and the appendix of [19] for the following facts. The reduction of the determinant gives rise to a surjection $\mathbb{G}_n \to F$, a splitting of which is induced by the Teichmüller character $\mathbb{F}_q^\times \longrightarrow \mathbb{W}\mathbb{F}_q^\times$. Consequently, if we write N for the kernel of that surjection, then there is an isomorphism $\mathbb{G}_n \cong F \ltimes N$.

We will now invoke Proposition 3.4 in order to get an Azumaya algebra

$$A = (E_n^{hN} \langle C \rangle)^{hG}$$

over $L_{K(n)}S$. That result also implies that the image of $[A]$ automatically maps to zero in the local Brauer group of E_n^{hN}. In particular, it vanishes in $Br_{K(n)}(E_n)$ itself. It remains to show that $[A] \neq 0$ in the local Brauer group of $L_{K(n)}S$. In particular, it suffices to prove that its image in $Br_{K(n)}(E_n^{hF})$ is non-zero. That image is equivalent to $(E_n \langle C \rangle)^{hG}$.

We assume on the contrary that there were an equivalence between our example $(E_n \langle C \rangle)^{hG}$ and $F_{E_n^{hF}}(W, W)$ for some faithful, dualizable, cofibrant E_n^{hF}-module W. We get a contradiction by looking at the centres of $\pi_0 \otimes \mathbb{Q}$ for both algebras.

The centre of $\pi_0(F_{E_n^{hF}}(W, W)) \otimes \mathbb{Q}$ is just

$$\pi_0(E_n^{hF}) \otimes \mathbb{Q} \cong \pi_0(E_n \otimes \mathbb{Q})^F.$$

As the group of roots of unity C acts only on the grading, and the Galois group G acts only on the coefficients, this is isomorphic to $\mathbb{Q}_p[\![u_1, \ldots, u_{n-1}]\!]$. However, as p is odd, the G-action on $E_n \langle C \rangle$ not only leaves the summand corresponding to the root 1 in C invariant, but also the one corresponding to the unique element -1 of order 2 in C. As a consequence, $\pi_0((E_n \langle C \rangle)^{hG}) \otimes \mathbb{Q}$ contains more than $\mathbb{Q}_p[\![u_1, \ldots, u_{n-1}]\!]$. $\square$

Summarizing, we know that localizations of the sphere (at an ordinary prime or at Morava-K-theory) possess non-trivial Brauer groups, compare Remark 6.4. However, we conjectured that the Brauer group of the (global) sphere spectrum is trivial. Antieau–Gepner's [2], Gepner–Lawson's [16] and Toën's [35] results that relate Brauer groups of commutative S-algebras to étale cohomology groups, allow to prove this conjecture, so

$$Br(S) = 0.$$

Acknowledgements

The first author thanks the Fachbereich Mathematik der Universität Hamburg for its hospitality. The second author was supported by the Glasgow Mathematical Journal Learning and Research Support Fund and he thanks the Mathematics Department of the University of Glasgow for its hospitality.

References

[1] Vigleik Angeltveit, Topological Hochschild homology and cohomology of A_∞ ring spectra, Geom. Topol. 12 (2008) 987–1032.
[2] Benjamin Antieau, David Gepner, Brauer groups and étale cohomology in derived algebraic geometry (in preparation).
[3] Maurice Auslander, Oscar Goldman, The Brauer group of a commutative ring, Trans. Amer. Math. Soc. 97 (1960) 367–409.
[4] Gorô Azumaya, On maximally central algebras, Nagoya Math. J. 2 (1951) 119–150.
[5] Andrew Baker, Andrey Lazarev, Topological Hochschild cohomology and generalized Morita equivalence, Algebr. Geom. Topol. 4 (2004) 623–645.
[6] Andrew Baker, Birgit Richter, Realizability of algebraic Galois extensions by strictly commutative ring spectra, Trans. Amer. Math. Soc. 359 (2007) 827–857.
[7] Andrew Baker, Birgit Richter, Invertible modules for commutative S-algebras with residue fields, Manuscripta Math. 118 (2005) 99–119.
[8] Andrew Baker, Birgit Richter, Galois extensions of Lubin-Tate spectra, Homology, Homotopy Appl. 10 (2008) 27–43.
[9] Mark Behrens, A modular description of the $K(2)$-local sphere at the prime 3, Topology 45 (2006) 343–402.
[10] Lindsay N. Childs, Gerald Garfinkel, Morris Orzech, The Brauer group of graded Azumaya algebras, Trans. Amer. Math. Soc. 175 (1973) 299–326.
[11] Daniel G. Davis, Iterated homotopy fixed points for the Lubin-Tate spectrum (with an appendix by Daniel G. Davis and Ben Wieland), Topology Appl. 156 (2009) 2881–2898.
[12] Frank R. DeMeyer, Gerald J. Janusz, Group rings which are Azumaya algebras, Trans. Amer. Math. Soc. 279 (1983) 389–395.
[13] Ethan S. Devinatz, Michael J. Hopkins, Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups, Topology 43 (2004) 1–47.
[14] Albrecht Dold, Dieter Puppe, Duality, trace, and transfer, In: Proceedings of the International Conference on Geometric Topology (Warsaw, 1978) (1980) 81–102.
[15] Anthony Elmendorf, Igor Kriz, Michael Mandell, J. Peter May, Rings, modules, and algebras in stable homotopy theory, Math. Surveys Monogr. 47 (1999).
[16] David Gepner, Tyler Lawson, Brauer groups and Galois cohomology of commutative ring spectra (in preparation).
[17] Paul G. Goerss, Michael J. Hopkins, Moduli spaces of commutative ring spectra, in: Structured Ring Spectra, in: London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, 2004, pp. 151–200.
[18] Alexander Grothendieck, Le groupe de Brauer. III. Exemples et compléments, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam; Masson, Paris, 1968, pp. 88–188.
[19] Hans-Werner Henn, On finite resolutions of $K(n)$-local spheres, in: Elliptic Cohomology, in: London Math. Soc. Lecture Note Ser., vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 122–169.
[20] Niles Johnson, Azumaya objects in bicategories, arxiv:1005.4878.
[21] Max-Albert Knus, Manuel Ojanguren, A Mayer-Vietoris sequence for the Brauer group, J. Pure Appl. Algebra 5 (1974) 345–360.
[22] Tsit Yuen Lam, A First Course in Noncommutative Rings, 2nd edition, in: Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001.
[23] Jean-Louis Loday, Cyclic homology, 2nd edition, in: Grundlehren der Mathematischen Wissenschaften, vol. 301, Springer-Verlag, Berlin, 1998.
[24] Eric J. Malm, String topology and the Hochschild cohomology of the based loop space, arXiv:1103.6198.
[25] Hideyuki Matsumura, Commutative ring theory, Cambridge University Press, 1986.
[26] James E. McClure, Jeffrey H. Smith, A solution of Deligne's Hochschild cohomology conjecture, in: Recent Progress in Homotopy Theory (Baltimore, MD, 2000), in: Contemp. Math., vol 293, Amer. Math. Soc, 2002, pp. 153–193.
[27] Morris Orzech, Charles Small, The Brauer group of commutative rings, in: Lecture Notes in Pure and Applied Mathematics, vol. 11, Marcel Dekker, Inc, New York, 1975.
[28] Alan Robinson, Gamma homology, Lie representations and E_∞ multiplications, Invent. Math. 152 (2003) 331–348.
[29] John Rognes, Galois extensions of structured ring spectra, Mem. Amer. Math. Soc. 192 (898) (2008) 1–97.
[30] John Rognes, A Galois extension that is not faithful, http://folk.uio.no/rognes/papers/unfaithful.pdf.
[31] John Rognes, Charles Weibel, Two-primary algebraic K-theory of rings of integers in number fields, J. Amer. Math. Soc. 13 (2000) 1–54.
[32] David J. Saltman, Lectures on division algebras, CBMS Regional Conference Series in Mathematics 94, published by American Mathematical Society, on behalf of Conference Board of the Mathematical Sciences (1999).
[33] Roland Schwänzl, Rainer M. Vogt, Friedhelm Waldhausen, Adjoining roots of unity to E_∞ ring spectra in good cases – a remark, Contemp. Mathematicians 239 (1999) 245–249.
[34] Markus Szymik, Brauer spaces for commutative rings and structured ring spectra, arxiv:1110.2956.
[35] Bertrand Toën, Derived Azumaya algebras and generators for twisted derived categories, Invent. Math. arXiv:1002.2599, in press (http://dx.doi.org/10.1007/s00222-011-0372-1).
[36] Charles Terence Clegg Wall, Graded Brauer groups, J. Reine Angew. Math. 213 (1963/1964) 187–199.
[37] Charles Weibel, Algebraic K-theory of rings of integers in local and global fields, in: Handbook of K-Theory, vol 1, Springer, Berlin, 2005, pp. 139–190.