Brauer groups for commutative S-algebras

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To the memory of Gorô Azumaya, + 26th of February 1920, † 8th of July 2010

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\textbf{ABSTRACT}

We investigate a notion of Azumaya algebras in the context of structured ring spectra and give a definition of Brauer groups. We investigate their Galois theoretic properties, and discuss examples of Azumaya algebras arising from Galois descent and cyclic algebras. We construct examples that are related to topological Hochschild cohomology of group ring spectra and we present a $K(n)$-local variant of the notion of Brauer groups.

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\textbf{Introduction}

The investigation of Brauer groups of commutative S-algebras is one aspect of the attempt to understand arithmetic properties of structured ring spectra.

In classical algebraic settings, Brauer groups are defined in terms of Azumaya algebras over fields or more generally over commutative rings \cite{4,3,32} and are closely involved in Galois theoretic considerations. In this paper, we discuss some ideas on Brauer groups for commutative $S$-algebras and in Section 3, we investigate their behaviour with respect to Galois extensions of commutative $S$-algebras in the sense of John Rognes \cite{29}. In earlier work, the first author and Andrey Lazarev discussed notions of Azumaya algebras \cite[sections 2, 4]{5}, but these appear to be technically problematic: there faithfulness of the underlying module spectra was not required, but many of the standard constructions with Azumaya algebras rely on this property. Also, the link with other definitions (for instance \cite{35,20}) works only under this faithfulness assumption.

Since our work was begun, other people have carried out work on Azumaya algebras and Brauer groups in contexts related to ours. Niles Johnson \cite{20} discusses Azumaya objects for general closed autonomous symmetric monoidal bicategories, and proves a comparison result \cite[proposition 5.4]{20} which compares our definition of Azumaya algebras with his. In \cite[theorem 1.5]{20}, he also shows that the derived Brauer group of a commutative ring in the sense of Bertrand Toën agrees with our Brauer group of the corresponding Eilenberg–Mac Lane ring spectrum.

In \cite[definition 4.6]{34}, the third author extends our current work to construct a Brauer space for commutative $S$-algebras such that the fundamental group of that space agrees with our Brauer group. The approaches in \cite{35,2,16} give descriptions of Brauer groups in terms of étale cohomology groups in the derived context and the context of ring spectra, respectively.

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We present our definition of topological Azumaya algebras in Section 1 and show that such algebras are always homotopically central (in the sense of Definition 1.2) and separable, and also that the Azumaya property is preserved under base change.

In Section 2 we define Brauer groups of commutative S-algebras and in Section 3 we prove a version of Galois descent for topological Azumaya algebras. This is applied in Section 4 where we explain how the classical theory of cyclic algebras can be extended to the context of commutative S-algebras.

In the case of Eilenberg–Mac Lane spectra we show in Section 5 under some assumptions on the ring R that an extension $HR \longrightarrow HA$ is topologically Azumaya if and only if the extension of commutative rings $R \longrightarrow A$ is an algebraic Azumaya extension. Furthermore, using recent work of Bertrand Toën [35], we can deduce that the Brauer group $Br(Hk)$ is trivial if $k$ is an algebraically or separably closed field.

Classically, the centre of an associative algebra $A$ over a commutative ring $R$ can be described as endomorphisms of $A$ in the category of modules over the enveloping algebra $A^e = A \otimes_R A^o$. For structured ring spectra, the direct analogue of this definition does not yield a homotopy invariant notion. Instead one has to replace $A$ by a cofibrant object in the category of module spectra over the enveloping algebra spectrum, so the centre of an associative $R$-algebra spectrum $A$ is given by the topological Hochschild cohomology spectrum $THH_R(A, A)$. This spectrum is not strictly commutative in general, but due to the affirmatively solved Deligne conjecture [26] it is an $E_2$-spectrum. There are however exceptions and in Section 7 we discuss some examples arising from group ring spectra and their homotopy fixed point spectra.

In Section 8 we offer a variant of the construction of Brauer groups in the $K(n)$-local context where it appears that technical difficulties are minimized and we discuss some examples related to $EO_2 = k_{K(2)}TMF$ in Section 8. In Section 10 we describe a non-trivial element in the $K(n)$-local Brauer group of the $K(n)$-local sphere.

1. Azumaya algebras over commutative $S$-algebras

Throughout, let $R$ be a cofibrant commutative $S$-algebra. We work in the categories of $R$-modules, $\mathcal{M}_R$, and associative $R$-algebras, $\mathcal{A}_R$ and for definiteness we choose the framework of [15]. Following [6,29], we will say that an $R$-module $W$ is faithful if for an $R$-module $X$, $W \wedge_R X \simeq \ast$ implies that $X \simeq \ast$.

We recall some ideas from [5]. If $A$ is an $R$-algebra, we denote by $A^o$ the $R$-algebra whose underlying $R$-module is $A$ but whose multiplication is reversed. The topological Hochschild cohomology spectrum of $A$ (over $R$) is

$$THH_R(A) = THH_R(A, A) = F_{A \wedge_R A^o}(\tilde{A}, \tilde{A}),$$

where $\tilde{A}$ is a cofibrant replacement for $A$ in the category of left $A \wedge_R A^o$-modules. $\mathcal{M}_R(A^o)$. We write $\eta: R \longrightarrow THH_R(A)$ for the canonical map into the $R$-algebra $THH_R(A)$; we also write $\mu: A \wedge R A^o \longrightarrow F_R(A, A)$ for the $R$-algebra map induced by the left and right actions of $A$ and $A^o$ on $A$.

**Definition 1.1.** Let $A$ be an $R$-algebra. Then $A$ is a weak (topological) Azumaya algebra over $R$ if and only if the first two of the following conditions hold, while $A$ is a (topological) Azumaya algebra over $R$ if and only if all three of them hold.

1. $A$ is a dualizable $R$-module.
2. $\mu: A \wedge R A^o \longrightarrow F_R(A, A)$ is a weak equivalence.
3. $A$ is faithful as an $R$-module.

Note that this definition of Azumaya algebras over $R$ differs from that in [5] since we demand faithfulness of $A$ over $R$ and not just $A$-locality of $R$ as an $R$-module.

If $T$ is an ordinary commutative ring with unit and if $B$ is an associative $T$-algebra, then the centre of $B$ can be identified with the endomorphisms of $B$ as an $B \otimes_T B^o$-module. Therefore $THH_T(A)$ can be viewed as a homotopy invariant version of the centre of $A$.

**Definition 1.2.** An $R$-algebra $A$ is said to be homotopically central if the canonical map $\eta: R \longrightarrow THH_R(A)$ is a weak equivalence.

For the following we recall a special case of the Morita theory developed in [5, section 1]. For a topological Azumaya algebra $A$ over $R$ we consider the category of left modules over the endomorphism spectrum $F_R(A, A)$. $\mathcal{M}_R(A, A)$ and we take a cofibrant replacement $\tilde{A}$ of $A$ in this category. The functor

$$F: \mathcal{M}_R \longrightarrow \mathcal{M}_{F_R(A, A)}$$

that sends $X$ to $X \wedge_R \tilde{A}$ has an adjoint

$$G: \mathcal{M}_{F_R(A, A)} \longrightarrow \mathcal{M}_R$$

with $G(Y) = F_{F_R(A, A)}(\tilde{A}, Y)$. Then [5, theorem 1.2] implies that this adjoint pair of functors passes to an adjoint pair of equivalences between the corresponding derived categories

$$\mathcal{D}_{F_R(A, A)} \cong \mathcal{D}_{F_R(A, A)}$$

and as a direct consequence we obtain the following result.

**Proposition 1.3** ([5, proposition 2.3]). Every topological Azumaya algebra $A$ over $R$ is homotopically central.
By proposition 2.3 and definition 2.1 of [5], we also see that any topological Azumaya algebra $A$ over $R$ is dualizable as an $A \wedge_R A^0$-module and $A \wedge_R A^0$ is $A$-local as a left module over itself.

In classical algebra, Azumaya algebras are in particular separable. Using Morita theory we can deduce the analogous statement for topological Azumaya algebras. Here an $R$-algebra is separable in the sense of [29, definition 9.1.1] if the multiplication $m: A \wedge_R A \to A$ has a section in the derived category of left $A \wedge_R A^0$-modules, $\mathcal{D}_{A \wedge_R A^0}$.

**Proposition 1.4.** Let $A$ be a topological Azumaya $R$-algebra. Then $A$ is separable.

**Proof.** By the remark following [29, definition 9.1.1], it suffices to prove that the induced map

$$m_*: THH_R(A, A \wedge_R A) \to THH_R(A, A)$$

is surjective on $\pi_0(-)$. Denote by $\widetilde{A}$ a cofibrant replacement of $A$ in the category of $A \wedge_R A^0$-modules. Morita equivalence yields the two weak equivalences

$$\widetilde{G} \circ \tilde{F}(R) \simeq THH_R(A, A),$$

$$\widetilde{G} \circ \tilde{F}(A) \simeq THH_R(A, A \wedge_R A).$$

The functoriality of $\widetilde{G} \circ \tilde{F}$ ensures that the unit $\eta: R \to A$ induces a map $\widetilde{G} \circ \tilde{F}(\eta)$ with

$$R \xrightarrow{\eta} \widetilde{G} \circ \tilde{F}(R) \xrightarrow{\widetilde{G} \circ \tilde{F}(\eta)} \widetilde{G} \circ \tilde{F}(A) \simeq A.$$

This is given by sending the coefficient module of $THH, \widetilde{A} \simeq R \wedge_R \widetilde{A} \simeq R \wedge_R A$, to $A \wedge_R A \simeq A \wedge_R \widetilde{A}$ using $\eta$. Therefore

$$\pi_0(m_*) \circ \pi_0(\widetilde{G} \circ \tilde{F}(\eta)) = \text{id},$$

and so $\pi_0(m_*)$ is surjective. □

We now describe the behaviour of Azumaya algebras under base change.

**Proposition 1.5.** Let $A, B, C$ be $R$-algebras.

1. If $A$ is an Azumaya algebra over $R$ and if $C$ is a commutative $R$-algebra, then $A \wedge_R C$ is an Azumaya algebra over $C$.
2. Conversely, let $C$ be a commutative $R$-algebra such that $C$ is dualizable and faithful as an $R$-module. If $A \wedge_R C$ is an Azumaya algebra over $C$, then $A$ is an Azumaya algebra over $R$.
3. If $A$ and $B$ are Azumaya algebras over $R$, then $A \wedge_R B$ is also Azumaya over $R$.

**Proof.** If $A$ is an Azumaya algebra over $R$, then it is formal to verify that $A \wedge_R C$ is dualizable and faithful over $C$ (compare [29, 4.3.3, 6.2.3]). It remains to show that

$$\mu_{A \wedge_R C}: (A \wedge_R C) \wedge_C (A \wedge_R C)^0 \to F_C(A \wedge_R C, A \wedge_R C)$$

is a weak equivalence. Note that since the multiplication in $A \wedge_R C$ is defined componentwise,

$$(A \wedge_R C)^0 = A^0 \wedge_R C^0.$$  

The diagram

$$
\begin{array}{ccc}
(A \wedge_R C) \wedge_C (A \wedge_R C)^0 & \xrightarrow{\mu_{A \wedge_R C}} & F_C(A \wedge_R C, A \wedge_R C) \\
| & \downarrow{\simeq} & | \\
A \wedge_R A^0 \wedge_R C & \xrightarrow{\mu_{A \wedge_R C}} & F_R(A, A \wedge_R C) \\
\end{array}
$$

(1.1)

commutes. Here $v : F_R(A, A) \wedge_R C \to F_R(A, A \wedge_R C)$ denotes the duality map. As $A$ is Azumaya over $R$ we know that $v$ and $\mu_A$ are equivalences, and thus we obtain that the top map is an equivalence as well.

For the converse we assume that $A \wedge_R C$ is Azumaya over $C$ and $C$ is faithful and dualizable as an $R$-module. If $M$ is an $R$-module, then $A \wedge_R M \simeq *$ implies that

$$(A \wedge_R C) \wedge_R M \simeq (A \wedge_R C) \wedge_C (C \wedge_R M) \simeq *.$$ 

Also, the faithfulness of $A \wedge_R C$ over $C$ ensures that $C \wedge_R M \simeq *$. But as we assumed that $C$ is faithful over $R$, we can conclude that $M$ was trivial.

The fact that $A$ is dualizable over $R$ follows from [29, lemma 6.2.4]. Making use of diagram Eq. (1.1) we see that $\mu_A$ is also a weak equivalence.

The proof of the third claim is straightforward. □
Later we will consider Azumaya algebras in a Bousfield local setting. Let \( L \) be a cofibrant \( R \)-module.

**Definition 1.6.** An \( L \)-local \( R \)-algebra \( A \) is an \((L \text{-local})\) Azumaya algebra if

1. \( A \) is a dualizable \( L \)-local \( R \)-module.
2. The natural morphism of \( R \)-algebras \( A \wedge_R A^0 \to F_R(A, A) \) is an \( L \)-local equivalence.
3. \( A \) is faithful as an \( L \)-local \( R \)-module.

Here dualizability as an \( L \)-local \( R \)-module means dualizability in the derived category of \( L \)-local \( R \)-modules. This is a symmetric monoidal category with the \( L \)-localization of the smash product over \( R \) as the symmetric monoidal product, so the definition of dualizability from [14] applies.

2. **Brauer groups**

Now suppose that \( M \) is a dualizable \( R \)-module as discussed in [29,6]; a more detailed discussion of dualizability can be found in [14]. Let \( \xi_R(M) = F_R(M, M) \) be its endomorphism \( R \)-algebra. Then there is a weak equivalence

\[
\xi_R(M) \simeq F_R(M, R) \wedge_R M.
\]

In order to identify endomorphism spectra of faithful and dualizable \( R \)-modules as trivial Azumaya algebras we need the following auxiliary result.

**Lemma 2.1.** Let \( M \) be a dualizable \( R \)-module.

1. If \( M \) is a faithful \( R \)-module, then the dual \( F_R(M, R) \) is also faithful.
2. If \( M \) is \( L \)-local with respect to a cofibrant \( R \)-module \( L \), then \( F_R(M, R) \) is \( L \)-local.

**Proof.**

(1) Dualizability of \( M \) implies that the composition

\[
M \simeq R \wedge_R M \xrightarrow{\delta \wedge \text{id}} M \wedge_R F_R(M, R) \wedge_R M \xrightarrow{\text{id} \wedge \varepsilon} M \wedge_R R \simeq M
\]

is the identity on \( M \). Here \( \delta : R \to M \wedge_R F_R(M, R) \) is the counit, and \( \varepsilon : F_R(M, R) \wedge_R M \to R \) is the evaluation map. Now if \( N \) is an \( R \)-module for which \( F_R(M, R) \wedge_R N \simeq * \), then the identity of \( M \wedge_R N \) factors through the trivial map, hence \( N \simeq * \) by faithfulness of \( M \).

(2) A similar argument with the functor \( F_R(W, -) \) shows that if \( L \wedge_R W \simeq * \), then the identity map on \( F_R(W, F_R(M, R)) \) factors through

\[
F_R(W, F_R(M, R)) \wedge_R M \simeq F_R(W \wedge_R M, M) \simeq *.
\]

It was shown in [5, proposition 2.11] that if \( M \) is a dualizable, cofibrant \( R \)-module, then \( \xi_R(M) \) is a weak topological Azumaya algebra in the sense of [5, definition 2.1].

**Proposition 2.2.** If \( M \) is a faithful, dualizable, cofibrant \( R \)-module, then (a cofibrant replacement of) \( \xi_R(M) \) is an Azumaya \( R \)-algebra.

**Proof.** As \( \xi_R(M) \) is a weak Azumaya algebra, it suffices to show that \( \xi_R(M) \) is a faithful \( R \)-module. Dualizability of \( M \) ensures that

\[
\xi_R(M) \simeq F_R(M, R) \wedge_R M,
\]

and this is a smash product of two faithful \( R \)-modules which is also faithful. \( \square \)

This result shows that we can take the \( R \)-algebras of the form \( \xi_R(M) \) with \( M \) faithful, dualizable and cofibrant, to be trivial Azumaya algebras when defining a topological version of a Brauer group which we now do.

First we note that every Azumaya algebra is weakly equivalent to a retract of a cell \( R \)-module, so the following construction yields a set of equivalence classes. Define \( \text{Az}(R) \) to be the collection of all Azumaya algebras. Now we introduce our version of the Brauer equivalence relation \( \simeq \) on \( \text{Az}(R) \).

**Definition 2.3.** Let \( R \) be a cofibrant commutative \( S \)-algebra. If \( A_1, A_2 \in \text{Az}(R) \), then \( A_1 \simeq A_2 \) if and only if there are faithful, dualizable, cofibrant \( R \)-modules \( M_1, M_2 \) for which

\[
A_1 \wedge_R F_R(M_1, M_1) \simeq A_2 \wedge_R F_R(M_2, M_2)
\]

as \( R \)-algebras. We denote the set of equivalence classes of these by \( \text{Br}(R) \) and we use the notation \([A]\) for the equivalence class of an \( R \)-Azumaya algebra \( A \).

**Theorem 2.4.** The set \( \text{Br}(R) \) is an abelian group with multiplication induced by the smash product \( \wedge_R \). Furthermore, \( \text{Br} \) is a functor from the category of commutative \( S \)-algebras to abelian groups.
3. Galois extensions and Azumaya algebras

Consider a map of commutative $S$-algebras $A \rightarrow B$, which we often denote by $B/A$. If $A$ is cofibrant as a commutative $S$-algebra, $B$ is cofibrant as a commutative $A$-algebra, and if $G$ is a finite group which acts on $B$ by morphisms of commutative $A$-algebras, then following Rognes [29], then we call $B/A$ a $G$-Galois extension if the canonical maps $i: A \rightarrow B^hG$ and $h: B \land_A B \rightarrow F_h(G_+, B)$ are weak equivalences.

In addition to these conditions, we will assume that $B$ is faithful as an $A$-module spectrum. This is a further restriction as there are examples of Galois extensions which are not faithful. The following example is due to Wieland (see [30]).

**Remark 3.1.** Let $p$ be a prime. Then the $\mathbb{Z}/p$-Galois extension

$$F(B\mathbb{Z}/p, H^F_p) \rightarrow F(E\mathbb{Z}/p, H^F_p) \simeq H^F_p$$

is not faithful. To its eyes the $\mathbb{Z}/p$-Tate spectrum of $H^F_p$ appears trivial, but it is not.

Let $B \langle G \rangle$ be the twisted group algebra over $B$, i.e., the $A$-algebra whose underlying $A$-module is $B \land G_+$ and whose multiplication is the composition $\tilde{\mu}$

$$B \land G_+ \land B \land G_+ \xrightarrow{id \land A \land id} B \land G_+ \land G_+ \land B \land G_+ \xrightarrow{id \land \mu \land id} B \land G_+ \land B \land G_+$$

where $\Delta$ is the diagonal, $\nu$ denotes the $G$-action on $B$, $\mu_B$ is the multiplication of $B$ and $\mu_G$ the multiplication in $G$. Then $\tilde{\mu}$ factors through $(B \land G_+) \land_A (B \land G_+)$ and turns $B \langle G \rangle$ into an $A$-algebra. Note that $B \langle G \rangle$ is an associative algebra but in general it lacks commutativity. More precisely, we know that the morphism $j: B \langle G \rangle \rightarrow F_h(G_+, B)$ is a weak equivalence of $A$-algebras for every $G$-Galois extension $A \rightarrow B$. In particular, $B \langle G \rangle$ gives rise to a trivial element in the Brauer group of $A$.

**Lemma 3.2.** Let $B/A$ be a faithful $G$-Galois extension and let $M$ be a $B \langle G \rangle$-module which is of the form $B \land_A N$ for some $A$-module $N$, where the $B \langle G \rangle$-module structure is given by the $B$-factor of $B \land_A N$. Then there is a weak equivalence of $A$-modules $N \simeq M^hG$.

**Proof.** Consider $B \land_A M = B \land_A B \land_A N$. As $B$ is a $G$-Galois over $A$, the latter term is equivalent to $F(G_+, B \land_A N)$ and this in turn is equivalent to $F(G_+, B \land_A N)$ because $G_+$ is finite. As $B$ is dualizable over $A$, the homotopy fixed point spectrum $(B \land_A M)^{hG}$ is equivalent to $B \land_A M^{hG}$.

There is a chain of equivalences of $B$-modules

$$B \land_A N \xrightarrow{\sim} F(G_+, B \land_A N)^{hG} \xrightarrow{\sim} (B \land_A B \land_A N)^{hG} = (B \land_A M)^{hG} \xrightarrow{\sim} B \land_A M^{hG},$$

and the result follows by faithfulness of $B$ over $A$. □
The following two results give analogues of Galois descent of algebraic Azumaya algebras as in [32, proposition 6.11].

**Proposition 3.3.** Suppose that $C$ is an Azumaya algebra over $B$ for which the natural morphism $B \otimes_A C^hG \rightarrow C$ is a weak equivalence of $B$ ($G$)-modules. Then $C^hG$ is also an Azumaya algebra over $A$.

**Proof.** We know from [29, lemma 6.2.4] that the $A$-algebra $C^hG$ is dualizable as an $A$-module.

As $C$ is Azumaya over $B$, we know that $C \otimes_B C^o \simeq F_B(C, C)$. Also, dualizability of $C^hG$ over $A$ guarantees that

$$B \otimes_A F_A(C^{hG}, C^{hG}) \simeq F_B(B \otimes_A C^{hG}, B \otimes_A C^{hG})$$

$$\simeq F_B(C, C) \simeq C \otimes_B C^o,$$

and so

$$C \otimes_B C^o \simeq (B \otimes_A C^{hG}) \otimes_B (B \otimes_A (C^{hG})^o)$$

$$\simeq B \otimes_A (C^{hG} \otimes_A (C^{hG})^o).$$

As $B$ is faithful over $A$, this shows that

$$C^{hG} \otimes_A (C^{hG})^o \simeq F_A(C^{hG}, C^{hG}).$$

Since $C$ is faithful as a $B$-module and $B$ is faithful as an $A$-module, we know that $C$ is faithful as an $A$-module. Assume that for an $A$-module $M$ we have $C^{hG} \otimes_A M \simeq \ast$. This is the case if and only if

$$B \otimes_A C^{hG} \otimes_A M \simeq C \otimes_A M \simeq \ast$$

because $B$ is a faithful $A$-module. Now faithfulness of $C$ over $A$ implies that $C^{hG}$ is also faithful over $A$. □

Suppose that $B/A$ is a faithful $G$-Galois extension in the sense of Rognes [29], where $G$ is a finite group. Now let $H < K < G$ so that $B/B^{hH}$ is a faithful $H$-Galois extension, $K$ acts on $B^{hH}$ by $B^{hH}$-algebra maps and $B^{hK} \rightarrow B^{hH}$ is a faithful $K/H$-Galois extension, in particular,

$$B^{hK} \simeq (B^{hH})_{h(K/H)}.$$ (3.1)

By [29, lemma 6.1.2(b)], the twisted group ring $B \langle H \rangle \simeq F_{B^{hH}}(B, B)$ is an Azumaya algebra over $B^{hH}$, and $K$ acts on $B \langle H \rangle$ by extending the action on $B$ by conjugation on $H$, so we will write $B \langle H \rangle$ to emphasize this.

If $K = Q \times H$ is a semi-direct product or $H$ is abelian, the quotient $Q = K/H$ acts by conjugation on $H$.

Note that as in algebra, there is an isomorphism of $A[K]$-modules

$$A[K] \cong \prod_K A.$$ (3.2)

The algebraic version of this isomorphism is given by

$$\sum_{k \in K} a_k \leftrightarrow (a_{k^{-1}})_{k \in K}$$

and we will use the topological analogue of this.

Our next result is based on [32, proposition 6.11(b)].

**Proposition 3.4.** Suppose that $K = Q \times H$ is a semi-direct product, or that $H$ is abelian. Then the $B^{hK}$-algebra $B \langle H \rangle_{hQ}$ is Azumaya, and

$$B^{hH} \otimes_{B^{hK}} B \langle H \rangle_{hQ} \simeq B \langle H \rangle.$$ (3.3)

Hence the Azumaya algebra $B \langle H \rangle_{hQ}$ over $B^{hK}$ is split by $B^{hH}$.

**Proof.** Note that we can assume that $G = K$ and $B^{hK} = A$. Making use of a faithful base change, it suffices to assume that $B$ is the trivial $K$-Galois extension, $B = \prod_K A$.

There are isomorphisms of $A[K]$-modules

$$B \langle H \rangle \cong \text{diag} \left( \prod_K A \otimes_A A[H_k] \right)$$

$$\cong \text{left} \left( \prod_K A \otimes_A A[H] \right)$$

$$\cong \text{left}(A[K] \otimes_A A[H])$$

$$\cong \text{left}(A[K \times H]),$$ (3.4)
where \(\text{diag}(\cdot)\) and \(\text{left}(\cdot)\) indicate the diagonal and left \(K\)-actions respectively, the second isomorphism is the standard equivariant sheaf map similar to the map \(s\) of [29, section 3.5], and \(K \times H\) is viewed as a \(K\)-set through the action on the left hand factor. As a \(Q\)-set, \(K\) decomposes into free orbits indexed on \(H\). On taking \(Q\)-homotopy fixed points we obtain an equivalence of \(A\)-modules
\[
B\langle H_c \rangle^{hQ} \cong A[H \times H].
\]

(3.3)

There is a map of \(A\)-modules
\[
B^{hH} \xrightarrow{\text{unit}} B^{hH} \wedge A B\langle H_c \rangle^{hQ} \longrightarrow B\langle H_c \rangle
\]
which is also a map of \(B^{hH}(Q)\)-modules. Applying \(\pi_*\)(\(\cdot\)) and working algebraically with \(\pi_*\)(\(A\))-modules, using [32, proposition 6.11(b)] it follows that we have an isomorphism
\[
\pi_*\bigl(B^{hH} \wedge A B\langle H_c \rangle^{hQ}\bigr) \cong \pi_*\bigl(B\langle H_c \rangle\bigr),
\]
and therefore a weak equivalence
\[
B^{hH} \wedge A B\langle H_c \rangle^{hQ} \xrightarrow{\sim} B\langle H_c \rangle
\]
of \(B^{hH}(Q)\)-modules. Now Proposition 3.3 shows that \(B\langle H_c \rangle^{hQ}\) is Azumaya over \(B^{hK}\). □

4. Cyclic algebras

In this section, we will assume that \(K \rightarrow L\) is a faithful Galois extension of commutative \(S\)-algebras, and that the Galois group \(G = \text{Gal}(L/K)\) is generated by an element \(\sigma\) of order \(n\), say. In particular, this group has to be cyclic. The choice of generator corresponds to an isomorphism \(\mathbb{Z}/n \cong G\), whose inverse can be thought of as a (primitive) character of \(G\). Last but not least, we also need a strict unit \(u\) in \(K\). For the time being, this just means that there is an action of the group \(\mathbb{Z}\) of integers on the spectrum \(K\) via maps of \(K\)-modules. We will extend this action to \(L\) without change of notation. The strictness of \(u\) is needed in order to ensure that relations hold on the nose and this in turn is necessary to obtain a strictly associative algebra extension.

Cyclic \(K\)-algebras will be defined here via Galois descent from matrix algebras over \(L\). As a model for the matrix algebra we use
\[
M_n(L) = \bigvee_{i,j=1}^n L_{i,j},
\]
with all \(L_{i,j} = L\) and multiplication given on summands
\[
L_{i,j} \wedge_K L_{j,k} \longrightarrow L_{i,k}
\]
by the multiplication in \(L\). One could also work with the endomorphism \(K\)-algebra spectrum \(F_K(\bigvee_n L, \bigvee_n L)\), but this mixes covariant and contravariant behaviour in \(\bigvee_n L\) and that is inconvenient for the explicit formulae that we need.

The cyclic group \(\mathbb{Z}/n\) acts on the \(L\)-algebra \(M_n(L)\) component-wise, i.e., the generator acts as \(\sigma : L_{i,j} \longrightarrow L_{i,j}\) on each summand. The multiplication and unit \(L \longrightarrow M_n(L)\) are equivariant, and we have equivalences
\[
M_n(L)^{h\mathbb{Z}/n} \simeq M_n(L^{h\mathbb{Z}/n}) \simeq M_n(K)
\]
of \(K\)-algebras. Something possibly more interesting happens when we twist this action with the chosen unit \(u\): we define a self-map on \(M_n(L)\) as the composition
\[
L_{i,j} \xrightarrow{id} L_{i+1,j+1} \xrightarrow{u^{h\alpha - h\beta}} L_{i+1,j+1} \xrightarrow{\sigma} L_{i+1,j+1}.
\]

(4.1)

where the indices \(i+1\) and \(j+1\) are read modulo \(n\), and \(\delta\) is the Kronecker symbol.

Lemma 4.1. The above self-map generates a \(\mathbb{Z}/n\)-action on \(M_n(L)\) as an associative \(K\)-algebra.

Proof. This follows as in algebra using the fact that
\[
u \wedge_K L = L \wedge_K u : L \wedge_K L \longrightarrow L \wedge_K L
\]
since \(u\) is a unit in \(K\). This guarantees that the \(K\)-algebra multiplication on \(L\) behaves well with respect to the twisted action. Together with the naturality and symmetry of the fold map this proves the claim. □

Definition 4.2. The cyclic \(K\)-algebra
\[
A(L, \sigma, u) = M_n(L)^{h\mathbb{Z}/n},
\]
associated with \(L\), \(\sigma\), and \(u\) is obtained from the matrix algebra \(M_n(L)\) with the twisted \(\mathbb{Z}/n\)-action by passage to homotopy fixed points.
The following result shows that the cyclic $K$-algebra $A(L, \sigma, u)$ defines a class in the relative Brauer subgroup $Br(L/K)$ of $Br(K)$.

**Theorem 4.3.** The cyclic algebra $A(L, \sigma, u)$ is an Azumaya algebra over $K$ which splits over $L$.

**Proof.** Proposition 3.3 above says that if we have an Azumaya algebra $B$ over $L$ with a compatible $G$-action such that the natural morphism

$$L \otimes_K B^G \longrightarrow B$$

is an equivalence of $L \langle G \rangle$-modules, then $A = B^G$ is also an Azumaya algebra over $K$. We want to apply this here to the situation $B = M_\sigma(L)$ and $G \cong \mathbb{Z}/n$.

Since $L$ is a dualizable $K$-module,

$$L \otimes_K M_\sigma(L)^G \cong (L \otimes_K M_\sigma(L))^G,$$

where $G$ acts only on the right hand factor in $L \otimes_K M_\sigma(L)$.

As $L$ is a $G$-Galois extension of $K$, we have $L \otimes_K L \cong \text{Map}(G_+, L)$, and therefore

$$L \otimes_K M_\sigma(L) \cong M_\sigma(L \otimes_K L) \cong M_\sigma(\text{Map}(G_+, L)) \cong \text{Map}(G_+, M_\sigma(L)).$$

As the latter is equivariantly equivalent to $L[G]$, we see that

$$L \otimes_K M_\sigma(L)^G \cong M_\sigma(L)$$

and this yields the result. □

Of course, it may happen that a cyclic $K$-algebra $A(L, \sigma, u)$ represents the trivial element in the Brauer group $Br(K)$ of $K$. This depends very much on the chosen unit $u$, for example. One way to prove non-triviality is to compute the homotopy groups of $A(L, \sigma, u)$, and to compare the result with the homotopy groups of the representatives of the elementary Azumaya algebras.

For this and other reasons, it is useful to know that one may calculate the homotopy groups of the cyclic algebra $A(L, \sigma, u)$ by means of the homotopy fixed point spectral sequence

$$E_2^{s,t} = H^t(\mathbb{Z}/n, \pi_s M_\sigma(L)) \Longrightarrow \pi_{t-s}(M_\sigma(L)^{\mathbb{Z}/n}). \tag{4.2}$$

The action on $\pi_t M_\sigma(L) \cong \pi_t M_\sigma(L)$ is generated by whatever the twisted action Eq. (4.1) induces in homotopy:

$$a_{i,j} \mapsto \sigma^u \cup^{\mathbb{Z}/n} \cup^{\mathbb{Z}/n} (a_{i+1,j+1}).$$

But, if we identify $(\pi_t M_\sigma(L))^{\otimes n}$ with the first row (or column) of $\pi_t M_\sigma(L)$, we easily find

$$\pi_t M_\sigma(L) \cong \mathbb{Z}[\mathbb{Z}/n] \otimes_{\mathbb{Z}} (\pi_t M_\sigma(L))^{\otimes n},$$

which shows that $\pi_t M_\sigma(L)$ is an induced $\mathbb{Z}/n$-module. Therefore the $E_2$-term of the homotopy fixed point spectral sequence Eq. (4.2) vanishes above the 0-line, which shows

$$\pi_t A(L, \sigma, u) \cong (\pi_t M_\sigma(L))^{\otimes n},$$

additively. In fact, this determines the underlying homotopy type.

**Theorem 4.4.** If $n = 2$, and if the unit $u$ has order 2 in the sense that it comes from a $\mathbb{Z}/2$-action, then the class of $A(L, \sigma, u)$ has order at most 2 in the Brauer group of $Br(K)$.

**Proof.** We prove that there is a $\mathbb{Z}/2$-equivariant equivalence of $K$-algebras between $M_2(L)$ and $M_2(L)^{\sigma}$. As in algebra, this equivalence is given by the transposition of matrices, which we have to model by the permutation $L_{i,j} \longrightarrow L_{j,i}$. Then the same proof as in algebra shows that this is a map of associative $K$-algebras. In order to show that the action passes to the homotopy fixed points we have to prove that it is compatible with the twisted action of Eq. (4.1) that we impose on $M_2(L)$. But this is trivial except for the $u$-action, where we have to use that $u = u^{-1}$ is a second root of unity. □

5. Azumaya algebras over Eilenberg–Mac Lane spectra

In this section we consider the case of Azumaya algebras over the Eilenberg–Mac Lane spectrum of a commutative ring. In [35], Toën introduces the algebraic notion of a derived Azumaya algebra over a commutative ring as a special case of the more general notion for simplicial rings. First we explain how the topological and algebraic notions are related.
In [15, section IV.2], an equivalence of categories
\[ \Psi : \mathcal{D}_{HR} \longrightarrow \mathcal{D}_R \] (5.1)
is constructed, where \( \Psi \) is defined on a CW HR-module \( M \) to be the cellular chain complex \( C_*(M) \). By [15, proposition IV.2.5], for CW HR-modules \( M, N \) there are isomorphisms of chain complexes of \( R \)-modules
\[
C_* (M \wedge_{HR} N) \cong C_* (M) \otimes_R C_* (N), \\
C_* (F_{HR}(M, N)) \cong \text{Hom}_R(C_*(M), C_*(N)).
\]
The inverse functor \( \Phi = \Psi^{-1} \) also preserves the monoidal structure, thus we have an equivalence of symmetric monoidal categories.

Following Toën [35], see remark 1.2, and [20, theorem 1.5] we find that an Azumaya algebra \( A \) over \( HR \), corresponds to a derived Azumaya algebra over \( R \). Note that as we are working with associative (but not commutative) \( HR \)-algebras, a cofibrant \( HR \)-algebra is a retract of a cell \( HR \)-module relative to \( HR \) by [15, theorem VII.6.2].

We get the following correspondence whose assumptions are satisfied when \( R \) is a principal ideal domain for instance.

**Proposition 5.1.** Let \( R \) be a commutative ring such that for any finitely presented \( R \)-module \( M \) with \( \text{Tor}^R_k(M, M) = 0 \) for \( k > 0 \) we can deduce that \( M \) is flat over \( R \).

Let \( T \) be an \( R \)-algebra. Then the \( HR \)-algebra \( HT \) is a topological Azumaya algebra if and only if \( T \) is an algebraic Azumaya \( R \)-algebra.

**Proof.** One direction is easy to see: if \( R \longrightarrow T \) is an algebraic Azumaya extension, then \( HR \longrightarrow HT \) is topologically Azumaya without any additional assumptions on \( R \).

For the converse, from [15, theorem IV.2.1] we have
\[
\pi_n(HT \wedge_{HR} HT^o) = \text{Tor}^R_n(T, T^o), \\
\pi_n(F_{HR}(HT, HT)) = \text{Ext}^n_R(T, T).
\]
Because \( \text{Tor}^R_s = 0 = \text{Ext}^s_R \) when \( s < 0 \), the Azumaya condition
\[
\mu : HT \wedge_{HR} HT^o \xrightarrow{\cong} F_{HR}(HT, HT)
\]
implies that for \( n \neq 0 \),
\[
\pi_n(HT \wedge_{HR} HT^o) = \text{Tor}^R_n(T, T^o) = 0 = \text{Ext}^n_R(T, T) = \pi_n(F_{HR}(HT, HT)).
\]
In particular,
\[
T \otimes_R T^o = \pi_0(HT \wedge_{HR} HT^o) \cong \pi_0(F_{HR}(HT, HT)) = \text{Hom}_R(T, T).
\]
According to [35, remark 1.2], the \( R \)-module \( T \) is finitely presented and flat by assumption, therefore it is finitely generated and projective by the corollary to [25, theorem 7.12].

For faithfulness, suppose that \( M \) is a non-trivial \( R \)-module. Since \( HT \) is a faithful \( HR \)-module, \( HT \wedge_{HR} HM \ncong \ast \). Flatness of \( T \) over \( R \) together with [15, theorem IV.2.1] yields the isomorphisms
\[
\pi_* (HT \wedge_{HR} HM) \cong \pi_0(HT \wedge_{HR} HM) \cong T \otimes_R M,
\]
and therefore \( T \otimes_R M \) is not trivial. \( \Box \)

**Proposition 5.2.** For any commutative ring with unit \( R \) there is a natural homomorphism
\[
H : \text{Br}(R) \longrightarrow \text{Br}(HR)
\]
induced by the functor which sends a ring to its Eilenberg–Mac Lane spectrum.

**Proof.** Let \([A]\) be an element of \( \text{Br}(R) \), then **Proposition 5.1** identifies \( HA \) as an \( HR \)-Azumaya algebra. If \([A] = 0 \), i.e., if there is a finitely generated faithful projective \( R \)-module \( M \) with \( A \cong \text{Hom}_R(M, M) \), then
\[
HA \cong H\text{Hom}_R(M, M) \simeq F_{HR}(HM, HM)
\]
and therefore \( HA \) is trivial in \( \text{Br}(HR) \). \( \Box \)

**Remark 5.3.** Toën shows that the Brauer group of derived Azumaya algebras over \( R \) is parametrized by \( H^2_{\text{et}}(R, \mathbb{G}_m) \times H^1_{\text{et}}(R, \mathbb{Z}) \) whereas the ordinary Brauer group of \( R, \text{Br}(R) \), corresponds to the torsion part in \( H^2_{\text{et}}(R, \mathbb{G}_m) \). Combining this with the comparison result of Johnson [20, theorem 1.5] yields that the above homomorphism \( H \) is injective for any \( R \) corresponding to \( H^2_{\text{et}}(R, \mathbb{G}_m)_{\text{tor}} \subseteq H^2_{\text{et}}(R, \mathbb{G}_m) \).
The situation is drastically different if we consider arbitrary $HR$-algebra spectra $A$. For instance, for every $R$, every $R$-module spectrum $\Sigma^n HR$ is faithful and dualizable, and therefore $F_{\Sigma^n HR}(HR \vee \Sigma^n HR, HR \vee \Sigma^n HR)$ is a trivial topological Azumaya $HR$-algebra whose homotopy groups spread over positive and negative degrees. This indicates that the Eilenberg–Mac Lane functor of Proposition 5.2 will not induce an isomorphism in general. The relationship with étale cohomology groups in [35] in fact shows that there are derived Azumaya algebras that are not Brauer equivalent to an ordinary Azumaya algebra. Toën describes a concrete example in [35, section 4] originating in an example by Mumford.

We will discuss the case of $HK$ for a field $k$. If $A$ is Azumaya over $HK$, then as $A$ is dualizable over $HK$ we know that the homotopy groups of $A$ are concentrated in finitely many degrees, say $\pi_r(A) \neq 0$ only when $-m \leq r \leq n$ for some $m, n \geq 0$. As $k$ is a field, we have

$$\pi_*(A \wedge_{HK} k^0) \cong \pi_*(A) \otimes_k \pi_*(A)^o.$$  

Using the fact that $\mu$ induces an isomorphism, we can deduce that $n = m$ because otherwise the kernel of $\pi_*(\mu)$ would be non-trivial.

A derived Azumaya algebra over the field $k$ is a differential graded $k$-algebra $B_*$ whose underlying chain complex is a compact generator of the derived category of chain complexes of $k$-vector spaces $\mathcal{D}_k$ such that the natural map

$$\mu_{B_*} : B_* \otimes_k B_*^0 \longrightarrow \text{Hom}_k(B_*, B_*)$$

is an isomorphism in $\mathcal{D}_k$. Here $B_* \otimes_k B_*^0$ agrees with the derived tensor product because we are working over a field, and similarly, $\text{Hom}_k(B_*, B_*)$ is the graded $k$-vector space of derived endomorphisms of $B_*$. Now we can relate topological $HK$-Azumaya algebras to derived Azumaya algebras over $k$.

**Proposition 5.4.** If $A$ is a topological Azumaya algebra over $HK$, then $\pi_*(A)$ is a derived Azumaya algebra over $k$.

**Proof.** As $A$ is dualizable over $HK$, its homotopy groups build a finite dimensional graded $k$-vector space and hence $\pi_*(A)$ is a compact generator of $\mathcal{D}_k$. The weak equivalence

$$\mu : A \wedge_{HK} k^0 \longrightarrow F_{\Sigma^n HR}(A, A)$$

yields isomorphisms

$$\mu_{\pi_*(A)} : \pi_*(A) \otimes_k \pi_*(A)^o \cong \pi_*(A \wedge_{HK} k^0) \cong \pi_* F_{\Sigma^n HR}(A, A) \cong \text{Hom}_k(A_*, A_*)$$

and so $\pi_*(A)$ is a derived Azumaya algebra over $k$. $\square$

Using Proposition 5.4 together with Toën’s results of [35, section 1] we obtain the following.

**Theorem 5.5.** For any algebraically closed field $k$, the Brauer group of $HK$ is trivial.

**Proof.** Let $A$ be a derived Azumaya algebra over $k$. We know from [35, corollary 1.11] that every derived Azumaya algebra over an algebraically closed field $k$, in particular $\pi_*(A)$, is quasi-isomorphic to a graded $k$-vector space $\text{Hom}_k(V, V)$ for some finite dimensional graded $k$-vector space $V$.

Let

$$M = HV = \bigvee_{i=1}^n \Sigma^m HK$$

be the $HK$-module spectrum such that $\pi_* M \cong V$ as graded $k$-vector spaces. Then $A$ is weakly equivalent to $F_{\Sigma^m HR}(A, A)$ since there are isomorphisms

$$\pi_*(A) \cong \text{Hom}_k(V, V) \cong \pi_*(F_{\Sigma^m HR}(M, M)).$$

Therefore $[A]$ is trivial in the Brauer group $Br(HK)$. $\square$

**Remark 5.6.** Using [35, corollary 1.15] one can extend the result to obtain the triviality of the Brauer group $Br(HK)$ for any separably closed field $k$.

### 6. Realizability of algebraic Azumaya extensions

Using Angeltveit’s obstruction theory [1, theorem 3.5], we can import algebraic Azumaya algebra extensions into topology. Let $R$ be a commutative $S$-algebra and let $\pi_0 R \longrightarrow A_0$ be an algebraic Azumaya extension. Then

$$A_* := \pi_* R \otimes_{\pi_0 R} A_0$$

is a projective module over $R_* = \pi_* R$ and there is an $R$-module spectrum $A’$ with $\pi_*(A’) \cong A_*$ which can be built as a mapping telescope of an idempotent corresponding to viewing $A_*$ as a direct summand of a free $R_*$-module. The methods of [6] carry over to give a homotopy associative $R$-ring spectrum $A$ that realizes $A_*$ as the homotopy ring $\pi_* A$.

Angeltveit’s obstruction theory [1] then yields the following.

**Theorem 6.1.** There is a unique $A_*$-$R$-algebra structure on $A$, i.e., there is a unique rigidification $r(A)$ of $A$ to an associative $R$-algebra. The resulting extension $R \longrightarrow r(A)$ is an Azumaya algebra.
The existence of the $A_{\infty}$ structure on $A$ is given by [1, theorem 3.5], because $\pi_\ast(A \wedge_{R} A^0)$ is separable over $A$, and hence the possible obstructions to an $A_{\infty}$-structure on $A$ (which live in Hochschild cohomology groups of $\pi_\ast(A \wedge_{R} A^0)$ over $A$) are trivial. The possibility of rigidification follows from [15, II.4]. Uniqueness also follows from the vanishing of all higher Hochschild cohomology groups.

As $A_0$ is finitely generated projective and faithful over $\pi_0R$, $r(A)$ is dualizable and faithful as an $R$-module spectrum. The Azumaya condition

$$\mu : A_0 \otimes_{\pi_0R} A_0^0 \cong \text{Hom}_{\pi_0R}(A_0, A_0)$$

for $A_0$ guarantees that the $\mu$-map

$$\mu : r(A) \wedge_{R} r(A)^0 \to F_{\emptyset}(r(A), r(A))$$

is a weak equivalence. □

**Corollary 6.2.** There is a natural group homomorphism

$$r : Br(\pi_0R) \to Br(R) ; \ [A] \mapsto [r(A)].$$

This result implies Proposition 5.2 but reaches further. For instance in the presence of enough roots of one, we can build generalized quaternionic extensions of ring spectra or consider cyclic extensions. Note, however, that in many cases $Br(\pi_0(R)) = 0$, for instance if $\pi_0(R)$ is isomorphic to a finite field, $\mathbb{Z}$ (see [17]) or $\mathbb{Z}[A]$ for some finite abelian group $A$ (see [21]). We learned from David Gepner that the Brauer group for a connective commutative $S$-algebra $R$ can be described via the second étale cohomology group of $\pi_0(R)$ with coefficients in the units and the first étale cohomology group of $\pi_0(R)$ with coefficients in $\mathbb{Z}$ via a short exact sequence.

John Rognes drew our attention to the non-trivial examples of Brauer groups in [31].

**Proposition 6.3.** There is a quaternionic extension of the sphere spectrum with 2 inverted that is not trivial in the Brauer group and hence

$$Br(S[1/2]) \neq 0.$$ 

**Proof.** The Brauer group of $\mathbb{Z}[1/2]$ is isomorphic to $\mathbb{Z}/2$, see [31, 2.8]. Extension of scalars to $\mathbb{R}$ has to send the generator of $Br(\mathbb{Z}[1/2])$ to the generator of $Br(\mathbb{R}) \cong \mathbb{Z}/2$ and hence this extension is equivalent to the $\mathbb{R}$-algebra of quaternions $\mathbb{H}$. Thus we know that a representative of the generator of $Br(\mathbb{Z}[1/2])$ is given by the class of the subring of localized Hurwitz quaternions $\mathbb{Z}[1/2](i, j, k) \subseteq \mathbb{H}$ with $i^2 = j^2 = k^2 = -1, j = kj = k^2i = -1$. This is the quaternion extension of $\mathbb{Z}[1/2]$. We can realize this extension topologically as an Azumaya algebra $H$ over $S[1/2]$.

Using base-change to $H\mathbb{R}$ we get the following commutative diagram.

$$\xymatrix{ Br(S[1/2]) \ar[r] \ar[d] & Br(H\mathbb{R}) \ar[d] \\
Br(\mathbb{Z}[1/2]) \ar[r] & Br(\mathbb{R}) }$$

As the map from $Br(\mathbb{R})$ to $Br(H\mathbb{R})$ is injective (compare Remark 5.3), the image of the class of $\mathbb{Z}[1/2](i, j, k)$ cannot be trivial. □

**Remark 6.4.** As the Brauer groups of $\mathbb{Z}[1/p]$ and $\mathbb{Z}_{(p)}$ are non-trivial for odd primes as well [see [27] and [37, p.145]], the above result can be used to obtain that other Brauer groups of connective commutative ring spectra are non-trivial. In particular, the Brauer groups of the corresponding localized spheres are non-trivial.

7. Topological Hochschild cohomology of group rings

We will consider Azumaya algebra extensions that arise as follows. For a finite discrete group $G$ and a commutative $S$-algebra $A$, we consider the group $A$-algebra spectrum $A[G] = A \wedge_{G}$. Note, that if $G$ is not abelian, then $A[G]$ is not commutative. We want to identify the extension $THH_A(A[G]) \to A[G]$ as an Azumaya extension in good cases.

For ordinary commutative rings $R$ and groups $G$, DeMeyer and Janusz describe in [12] conditions on $R$ and $G$ which ensure that $R[G]$ is an Azumaya algebra over its centre. First, we document a well-known identification of topological Hochschild cohomology of group rings, see for instance [24, 4.2.10]. This can be viewed as a topological version of Mac Lane’s isomorphisms [23, 7.4.2].

**Lemma 7.1.** For $A$ and $G$ as above we have


Proof. Topological Hochschild cohomology of $A[G]$ can be described as the totalization of the cosimplicial spectrum that has


as $q$-cosimplices [26]. First, we mimic the identification that is used in the Mac Lane isomorphism for usual Hochschild cohomology in order to identify this cosimplicial spectrum with the one that has $F(G^q, A[G]^q)$ as $q$-cosimplices. In algebra this identification is given by $f \mapsto f'$ where

$$f'(g_1, \ldots, g_q) = f(g_1, \ldots, g_q)g_q^{-1} \cdots g_1^{-1}.$$ 

An analogous identification works on spectrum level. The coface maps in the cosimplicial structure in $F(G^q, A[G]^q)$ are given by

$$d_0(f)(g_1, \ldots, g_q) = g_1 f(g_2, \ldots, g_q)g_1^{-1},$$

$$d_i(f)(g_1, \ldots, g_q) = f(g_1, \ldots, g_{i-1}, g_i, \ldots, g_q), \quad (0 < i < q)$$

$$d_q(f)(g_1, \ldots, g_q) = f(g_1, \ldots, g_{q-1}).$$

Consider the simplicial model of $EG$ with $q$-simplices $G^{q+1}$, with diagonal $G$-action, and where the $i$-th face map in $EG$ is given by omitting the $i$-th group element. We can write the homotopy fixed point spectrum $F_C(EG^+, A[G]^q)$ as

$$F_C(EG^+, A[G]^q) \cong \text{Tot}(\{q \mapsto F_C(G^{q+1}, A[G]^q)\}).$$

Let $\phi : (G^q, A[G]^q) \mapsto F_C(EG^+, A[G]^q)$ be the map that we can describe symbolically as

$$\phi(f)(g_0, \ldots, g_q) = g_0 f(g_q^{-1}g_1, \ldots, g_{q-1}g_q)g_0^{-1}.$$ 

It is then straightforward to check that $\phi$ in fact respects the cosimplicial structure. \hfill $\square$

Now fix a prime $p$. Let $k$ be an algebraically closed field of characteristic $p$ and let $Hk$ be the corresponding Eilenberg–Mac Lane spectrum realized as a commutative $S$-algebra. We also adopt the notation of [8]. Thus $E_n$ is the Lubin–Tate spectrum associated with the prime $p$ and the Honda formal group of height $n$ and $E_n^\text{ur}$ is its maximal unramified Galois extension. These commutative $S$-algebras have `residue fields' in the sense of [6,7], namely $K_n$ and $K_n^\text{ur}$ respectively, and these are algebras over $E_n$ and $E_n^\text{ur}$ respectively, but only homotopy commutative when $p \neq 2$ and not even that when $p = 2$.

Theorem 7.2. Let $G$ be a non-trivial finite discrete group whose order is not divisible by $p$. Suppose that $A$ is either $Hk$ or $E_n^\text{ur}$.


2. If $G$ is non-abelian, then $A[G]$ is a non-trivial $(A[G])^{hG}$-Azumaya algebra.

Proof. In all cases, we will consider the homotopy fixed point spectral sequence

$$E_2^{q,i} = H^q(G; A[G]^i) \Longrightarrow \pi_{q+i}((A[G]^q)^{hG}).$$

If $p$ does not divide the order of the group $G$, then this spectral sequence collapses and the only surviving non-trivial terms are the $G$-invariants

$$E_2^{q,i} = (A[G]^i)^G$$

which can be identified with the centre of the group ring $Z(A_n[G])$. In particular, $\pi_{\ast}((A[G]^q)^{hG})$ is a graded commutative $A_n$-algebra.

If $G$ is abelian, then the conjugation action is trivial and as $p$ does not divide $|G|$ we obtain


so we have the trivial Azumaya extension. If $G$ is not abelian, then the centre of the group ring $A_n[G]$ is a proper subring of $A_n[G]$.

For $A = Hk$ we can use Artin–Wedderburn theory to obtain a splitting of the semisimple ring $k[G]$ into a product of matrix algebras over the algebraically closed field $k$,

$$k[G] \cong \prod_{i=1}^r M_{n_i}(k),$$

where $r$ agrees with the number of conjugacy classes in $G$. Thus the centre of $k[G]$ is a product of copies of $k$ and is therefore an étale $k$-algebra. By the obstruction theory of Robinson or Goerss–Hopkins [28,17], there is a unique $E_{\infty}$ $Hk$-algebra spectrum that is weakly equivalent to $(A[G]^q)^{hG}$. By abuse of notation we denote the corresponding commutative $Hk$-algebra by $(A[G]^q)^{hG}$. 
We have to describe $A[G]$ as an associative $(A[G])^{\text{hg}}$-algebra. For this we use [1, theorem 3.5] again. Starting with our commutative model of $(A[G])^{\text{hg}}$ we can build a homotopy associative ring spectrum $B$ with $\pi_*(B) \cong A_*[G]$, and as $G$ is finite and discrete this extension is of the form
\[ \pi_*(B) \cong \pi_*(A[G])^{\text{hg}} \otimes_{\pi_*(A[G])^{\text{hg}}} B_0, \]
with $\pi_0(A[G])^{\text{hg}} \longrightarrow B_0$ being algebraically Azumaya. Thus we can apply Theorem 6.1 to see that there is an associative $(A[G])^{\text{hg}}$-algebra $B$ which models $A[G]$ and such that $B$ is Azumaya over $(A[G])^{\text{hg}}$.

For $E_n$ we pass to the residue field $K_n$. The homotopy fixed point spectral sequence gives
\[ \pi_*(((E_n^{nr}[G])^{\text{hg}}) \cong Z((E_n^{nr})_n[G])) \cong Z(WF_p[u_1, \ldots, u_{n-1}])[u^\pm 1]. \]
Reducing modulo the maximal ideal $m = (p, u_1, \ldots, u_{n-1})$ gives the homotopy groups of the $G$-homotopy fixed points of $K_n^{nr}[G]$ with respect to the conjugation action, $Z(WF_p[G])[u^\pm 1]$ and again we can identify this term as $\prod_{i=1}^r G$ where $r$ denotes the number of conjugacy classes in $G$. The idempotents that give rise to these splittings can be lifted to idempotents for $Z(WF_p[u_1, \ldots, u_{n-1}])[G]$ and $WF_p[u_1, \ldots, u_{n-1}][G]$ and therefore these two algebras also split into products with $r$ factors:
\[ WF_p[u_1, \ldots, u_{n-1}][G] \cong \prod_{i=1}^r B_i, \]
\[ Z(WF_p[u_1, \ldots, u_{n-1}])[G] \cong \prod_{i=1}^r C_i, \]
where $B_i/m \cong M_i(F_p)$, while for $1 \leq i \leq r$, the $C_i$ are commutative and satisfy $C_i/mC_i \cong F_p$.

Additively we know that $Z(WF_p[u_1, \ldots, u_{n-1}])[G])$ is the free module on the conjugacy classes and so we can conclude that $(E_n^{nr}[G])^{\text{hg}}$ is weakly equivalent to $\prod_{i=1}^r E_n^{nr}$ and the latter spectrum can be modelled by a commutative $E_n^{nr}$-algebra spectrum and $E_n^{nr}[G]$ is dualizable over $\prod_{i=1}^r E_n^{nr}$.

Artin–Wedderburn theory gives a semisimple decomposition
\[ \mathbb{F}_p[G] \cong \prod_{i=1}^r M_{d_i}(\mathbb{F}_p), \]
and the centre $Z(\mathbb{F}_p[G])$ can be identified with the product of the centres of the matrix ring factors. There are associated central idempotents of $\mathbb{F}_p[G]$ accomplishing this splitting. By the theory of idempotent lifting described in [22, section 21] for example, these idempotents lift to give an associated splitting
\[ WF_p[u_1, \ldots, u_{n-1}][G] \cong \prod_{i=1}^r M_{d_i}(WF_p[u_1, \ldots, u_{n-1}]), \]
and again the centre of $WF_p[u_1, \ldots, u_{n-1}][G]$ can be identified with the product of the centres of the matrix factors. Notice that $M_{d_i}(WF_p[u_1, \ldots, u_{n-1}])$ is Azumaya over $WF_p[u_1, \ldots, u_{n-1}]$. The rest of the proof involves realizing the central idempotents as morphisms of $S$-algebras, but this is well known to be possible since the projections are Bousfield localizations, see [33].

8. Azumaya algebras over Lubin–Tate spectra

From now on we will use $E$ to denote $E_n$, $E_n^{nr}$ or any commutative Galois extension of $E_n$ obtained as a homotopy fixed point algebra $E = (E_n^{nr})^{\text{hg}}$ for some closed normal subgroup $\Gamma < \text{Gal}(\mathbb{F}_p/\mathbb{F}_p^\nu)$. Similarly, $K$ will denote the corresponding residue field of $E$, so when $E = E_n$ or $E_n^{nr}$ we have $K = K_n$ or $K_n^{nr}$.

We will work with dualizable $K$-local $E$-modules. By [8, section 7] we know that such modules are retracts of finite cell $E$-modules. If $W \in \mathcal{M}_{E,K}$, then since $\pi_n(K \wedge_E W)$ is a graded vector space over the graded field $K_n = \pi_n(K)$, it follows that
\[ K \wedge_E W \cong K \bigvee_{i} \Sigma^{d(i)} K, \]
where the right hand wedge is non-trivial if and only if $W$ is non-trivial in $\mathcal{M}_{E,K}$. In particular, if $W$ is dualizable this wedge is finite and
\[ K \wedge_E W \cong \bigvee_{i} \Sigma^{d(i)} K \]
since $W$ is $K$-local. For any $X \in \mathcal{M}_{E,K}$,
\[
K \wedge_{E} (W \wedge_{E} X) \simeq L_{k} \bigvee_{i} \Sigma^{d(i)} K \wedge_{E} X,
\]
so $W \wedge_{E} X$ is trivial in $\mathcal{S}_{E,K}$ if and only if both of $W$ and $X$ are trivial in $\mathcal{S}_{E,K}$. Thus every $E$-module $W$ which is non-trivial as an element of $\mathcal{S}_{E,K}$ is faithful and cofibrant as a $K$-local $E$-module; furthermore, every $X \in \mathcal{M}_{E,K}$ is $W$-local.

By [1], there are many examples of $K$-local Azumaya algebras over $E$ which have $K$ as their underlying ring spectrum. These examples have no analogue in the algebraic context since they are not projective $E$-modules, nor do they split over suitable Galois extensions. Instead we focus on split examples. A good source of these can be found in the situation of [29, section 5.4.3], based on work of Devinatz and Hopkins [13] and we will discuss these in Section 9.

For background ideas on Azumaya algebras graded on a finite abelian group, we follow [10] which generalizes work of Wall [36] and others. We will only consider the case where the grading group is $\mathbb{Z}/2$ with the non-trivial symmetric bilinear map $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \{\pm 1\}$ determining the relevant signs, however in periodic topological contexts it may also prove useful to modify the grading to other finite quotient groups lying between $\mathbb{Z}$ and $\mathbb{Z}/2$, and the above reference should provide appropriate generality for such algebra.

Over a field $k$, an (ungraded) Azumaya algebra $A$ is a central simple algebra, so by Wedderburn’s theorem, there is an isomorphism of $k$-algebras
\[
A \simeq M_{r}(D),
\]
where $D$ is a central division algebra over $k$. If $d = \dim_{k} D$, then
\[
\dim_{k} A = (rd)^{2},
\]
so $\dim_{k} A$ is a square. In the graded case, such restrictions do not always apply, and this has consequences for the topological situation.

Theorem 8.1. Suppose that $p$ is an odd prime and let $A$ be a $K$-local Azumaya algebra over $E$. Then $\pi_{s}(K \wedge_{E} A)$ is an Azumaya algebra over $K_{s}$.

Proof. The ring $K_{s}$ is a 2-periodic graded field which we will view as $\mathbb{Z}/2$-graded, and $\pi_{s}(K \wedge_{E} A)$ will also be viewed as a $\mathbb{Z}/2$-graded $K_{s}$-algebra.

We have isomorphisms of $K_{s}$-algebras
\[
\pi_{s}(K \wedge_{E} A) \otimes_{K_{s}} \pi_{s}(K \wedge_{E} A) \cong \pi_{s}(K \wedge_{E} A) \otimes_{K_{s}} \pi_{s}(K \wedge_{E} A) \cong \pi_{s}(K \wedge_{E} (A \wedge_{E} A)) \cong \pi_{s}(K \wedge_{E} F_{E}(A, A)).
\]

Since $A$ and $K$ are strongly dualizable, using results of [15] we have
\[
K \wedge_{E} F_{E}(A, A) \simeq F_{K}(K \wedge_{E} A, K \wedge_{E} A),
\]
so the universal coefficient spectral sequence over $K$ yields
\[
\pi_{s}(K \wedge_{E} F_{E}(A, A)) \cong \text{End}_{K_{s}}(\pi_{s}(K \wedge_{E} A)).
\]

Therefore $\pi_{s}(K \wedge_{E} A)$ is a $K_{s}$-Azumaya algebra. \hfill $\Box$

Corollary 8.2. If $\pi_{s}(K \wedge_{E} A)$ is concentrated in even degrees then its dimension is a square, i.e., for some natural number $m$, $\dim_{K_{s}} \pi_{s}(K \wedge_{E} A) = m^{2}$.

In fact we have

Proposition 8.3. If $\pi_{s}(K \wedge_{E} A)$ is concentrated in even degrees then $\pi_{s}(A)$ is a $\mathbb{Z}/2$-graded algebra Azumaya algebra over $E_{0}$. In particular, as an $E$-module $\pi_{s}(A)$ is finitely generated, free and concentrated in even degrees, hence
\[
\pi_{s}(A) \otimes_{E_{0}} \pi_{s}(A) \cong \pi_{s}(A \wedge_{E} A) \cong \pi_{s}(F_{E}(A, A)) \cong \text{Hom}_{E_{0}}(\pi_{s}(A), \pi_{s}(A)),
\]
where the last isomorphism follows from the collapsing of the universal coefficient spectral sequence. \hfill $\Box$

Recall that $A_{E_{0}}(E)$ is the collection of all cofibrant $K$-local topological Azumaya algebras over $E$. The Brauer equivalence relation $\sim$ on $A_{E_{0}}(E)$ is then given as follows:

- If $A, B \in A_{E_{0}}(E)$, then $A \sim B$ if and only if there are faithful, dualizable, cofibrant $E$-modules $U, V$ for which there is an equivalence in the derived category of $K$-local $E$-algebras
\[
A \wedge_{E} F_{E}(U, U) \simeq B \wedge_{E} F_{E}(V, V).
\]

The set of equivalence classes of $\sim$ is $Br_{E}(E)$; this is indeed a set since every dualizable $K$-local $E$-module is a retract of a finite cell $E$-module.
9. Some examples of $K_n$-local Azumaya algebras

We now recall Proposition 3.4. By work of Devinatz and Hopkins [13], and subsequently Davis [11], as explained in [29, theorem 5.4.4], for each pair of closed subgroups

$$H \leq G \leq \mathbb{G}_n = \text{Gal}(\mathbb{F}_p/\mathbb{F}_p) \times S_n$$

of the Morava stabilizer group, there is an associated pair of homotopy fixed point spectra $E^{hG} \rightarrow E^{hH}$, and if $H \leq G$ then this is a $K$-local $G/H$-Galois extension. In particular, when $H \leq \mathbb{G}_n$ is finite, $E^{hG} \rightarrow E$ is a $K$-local $H$-Galois extension.

A particularly interesting source of examples is provided by taking $G$ to be a maximal finite subgroup of $\mathbb{G}_n$. If $p$ is odd and $n = (p - 1)k$ with $p \nmid k$, or $p = 2$ and $n = 2k$ with $k$ odd, then such maximal subgroups are unique up to conjugation and then the homotopy fixed point spectrum $E^{hG}$ is denoted $EO_n$. For $p = 3$ Behrens [9, remark 1.7.3] gives an argument for the identification of $EO_2$ with the $K(2)$-localization of the spectrum of topological modular forms, $\text{TMF}$. This can be adapted to $p = 2$. We proceed with an example, studied in [29, section 5.4.3].

Example 9.1. At the prime $p = 2$, the group $\mathbb{G}_2$ has a maximal finite subgroup $G_{48}$ of order 48 which is isomorphic to a semi-direct product of the group $\mathbb{H}Q_8$ of order 24, which consists of the units in the ring of Hurwitz quaternions, with the Galois group $\text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ of order 2. Therefore $E_2/EO_2$ is a $G_{48}$-Galois extension. Applying Proposition 3.4, we see that the splitting of the group $G_{48}$ yields an Azumaya algebra

$$(E_2 \langle H_{48} \rangle)^{h\mathbb{G}_2}$$

over $EO_2$. There are two more examples like this: Of the 15 conjugacy classes of subgroups of $G_{48}$, there are 7 which are normal: $C_2, C_4, Q_8, Syl_2(G_{48}), H_{24}$, and $G_{48}$. The first and last of them are uninteresting here, as $(E_2)^{h\mathbb{G}_2} \simeq EO_2$ and $E_2 \langle G_{48} \rangle \simeq \mathbb{F}_{10}^\times$ is trivial in the Brauer group of $EO_2$. The second and third do not split the group, but the other three do. The last of these yields the Azumaya algebra displayed above, but the other two give rise to further examples

$$(E_2 \langle Q_8 \rangle)^{h\mathbb{G}_6} \quad (E_2 \langle Syl_2(G_{48}) \rangle)^{h\mathbb{G}_3}$$

of Azumaya algebras over $EO_2$.

10. The Brauer group of the $K(n)$-local sphere

In this section we discuss the $K(n)$-local Brauer group of the $K(n)$-local sphere $L_{K(n)}S$.

Theorem 10.1. Suppose that $p > 2$ and $n > 1$. Then the $K(n)$-local Brauer group of $L_{K(n)}S$ is non-trivial.

Proof. As usual, let us write $q = p^n$. The cyclic group $C = \mathbb{F}_q^\times$ of order $q - 1$ consists of roots of unity, and the Galois group $G = \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is cyclic of order $n$, generated by the Frobenius. As $n \neq 1$, the Galois group is non-trivial, and its Galois action gives rise to an extension $F = G \rtimes C$.

Let $\mathbb{G}_n$ denote again the $n$-th extended Morava stabilizer group. We refer to section 2.3 and the appendix of [19] for the following facts. The reduction of the determinant gives rise to a surjection $\mathbb{G}_n \rightarrow F$, a splitting of which is induced by the Teichmüller character $\mathbb{F}_q^\times \rightarrow \mathbb{F}_p^\times$. Consequently, if we write $N$ for the kernel of that surjection, then there is an isomorphism $\mathbb{G}_n \cong F \rtimes N$.

We will now invoke Proposition 3.4 in order to get an Azumaya algebra

$$A = (E_n^{hF}(\langle C \rangle))^{h\mathbb{G}_n}$$

over $L_{K(n)}S$. That result also implies that the image of $[A]$ automatically maps to zero in the local Brauer group of $E_n^{hF}$. In particular, it vanishes in $Br_{K(n)}(E_n)$ itself. It remains to show that $[A] \neq 0$ in the local Brauer group of $L_{K(n)}S$. In particular, it suffices to prove that its image in $Br_{K(n)}(E_n^{hF})$ is non-zero. That image is equivalent to $(E_n(\langle C \rangle))^{h\mathbb{G}_n}$.

We assume on the contrary that there were an equivalence between our example $(E_n(\langle C \rangle))^{h\mathbb{G}_n}$ and $F_{E_n}^{hF}(W, W)$ for some faithful, dualizable, cofibrant $E_n^{hF}$-module $W$. We get a contradiction by looking at the centres of $\pi_0 \otimes \mathbb{Q}$ for both algebras.

The centre of $\pi_0(F_{E_n}^{hF}(W, W)) \otimes \mathbb{Q}$ is just

$$\pi_0(E_n^{hF}) \otimes \mathbb{Q} \cong \pi_0(E_n \otimes \mathbb{Q})^F.$$
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