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SMALL H SPACES RELATED TO MOORE SPACES

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IN this paper we will discuss some quite simple H spaces with a number of remarkable properties. These properties make the spaces useful in the factorization of loop spaces on the one hand [23] and pivotal in understanding the unstable development of v_2 periodic homotopy on the other [12].

1. NOTATION AND SUMMARY OF RESULTS

Throughout this paper, all spaces will be localized at a prime $p \geq 5$. Let us write $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$ for the double suspension map between spheres and $p^r: S^{2n-1} \to S^{2n-1}$ for a map of degree p^r. Let us write $P^{2n+1}(p^r) = S^{2n} \cup_{p^r} e^{2n+1}$ for the $\mathbf{Z}_{p^r}$ Moore space.† In [4–6], the authors study the homotopy of $P^{2n+1}(p^r)$ when $p > 2$ by decomposing the loop space $\Omega P^{2n+1}(p^r)$ as a weak infinite product. As a remarkable corollary, they construct a map $\pi_n: \Omega^2 S^{2n+1} \to S^{2n-1}$ so that the diagram:

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, "\Omega^2(p^r)"] \arrow[dr, "\pi_n"] & \Omega^2 S^{2n+1} \\ S^{2n-1} \arrow[u, "E^2"] \arrow[r, "p^r"] & S^{2n-1} \arrow[u, "E^2"'] \end{tikzcd} \tag{1A}$$

commutes up to homotopy. From this they quite easily settled the exponent question for the odd torsion in the homotopy groups of spheres.

In this paper we will construct decompositions of certain loop spaces, from which we will obtain H spaces with certain universal properties. Our main result is:

THEOREM A.‡ *There is an H fibration sequence*:

$$\cdots \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T^{2n-1}(p^r) \longrightarrow \Omega S^{2n+1},$$

where π_n fits into diagram (1A). *In particular, π_n is an H map and the fiber of π_n is a loop space. Furthermore, $H_*(T^{2n-1}(p^r); \mathbf{Z}_p)$ is a free commutative associative algebra on generators u and v of dimensions $2n-1$ and $2n$, respectively, with $\beta^{(r)} v = u$, where $\beta^{(r)}$ is the rth Bockstein.*

† Throughout this paper we will write $\mathbf{Z}_{p^r}$ and, when $r = 1$, $\mathbf{Z}_p$ to denote cyclic groups of orders p^r and p, respectively.

‡ Theorem 5.2 in the sequel.

The possibility of the existence of a fibering such as the one in Theorem A was first considered in [6]. The spaces $T^{2n-1}(p^r)$ and the fibering were first constructed in [3]. Thus the thrust of Theorem A is the H space structure.

The second feature of these spaces is the following universal property.†

THEOREM B.‡ *Suppose X is an H space of finite type and $p^{r+i-1}\pi_{2np^i-1}(X;\mathbf{Z}_{p^{r+i}}) = 0$ for all $i \geq 1$. Then any map $\varphi: P^{2n}(p^r) \to X$ has an extension over $T^{2n-1}(p^r)$*

$$\begin{tikzcd} P^{2n}(p^r) \arrow[r, "\varphi"] \arrow[d] & X \\ T^{2n-1}(p^r) \arrow[ur, dashed, "\hat{\varphi}"'] & \end{tikzcd}$$

The spaces $T^{2n-1}(p^r)$ are constructed as the direct limit of spaces $T_k^{2n-1}(p^r)$. Each of these enjoys a universal property similar to the above. Furthermore, we construct a sequence of co-H spaces $G_k^{2n}(p^r)$ which interact in the following way:§

(A) *Both G_k and T_k are atomic (except T_0 when $n = r = 1$),*
(B) *G_k is a retract of ΣT_k,*
(C) *T_k is a retract of ΩG_k.*

The existence of the spaces $T^{2n-1}(p^r)$ was predicted in [10]; we shall see that they fit into an EHP spectral sequence for the Moore space spectrum. The thrust of [10] is that although the James–Hopf invariant and the suspension do not fit into long exact sequences for Moore spaces (or any spaces other than spheres), except in a limited range of dimensions, there does exist a sequence of spaces $T^n\{p^r\}$ together with "suspensions" $E: T^n\{p^r\} \to \Omega T^{n+1}\{p^r\}$ so that

(a) the resulting spectrum is equivalent to the $\mathbf{Z}_{p^r}$ Moore space spectrum,
(b) there is a long exact "EHP sequence" in which the third term is the homotopy of a space T^i for some i.

The spaces T^i are constructed from the Moore space $P^{i+1}(p^r)$ by killing off irrelevant homotopy.¶ In particular, $T^{2n}(p^r) = S^{2n+1}\{p^r\}$ and the spaces $T^{2n-1}(p^r)$ are from Theorem A.

For technical reasons (see Conjecture 5.3), we will replace the spaces $T^n(p^r)$ by $\hat{T}^n(p^r)$, defined as

$$\hat{T}^n(p^r) = \begin{cases} BW_k & \text{if } n = 2kp - 1 \text{ and } r = 1 \\ \Omega T^n(p^r) & \text{otherwise} \end{cases}$$

where BW_k is the classifying space for the double suspension (see [8]).

THEOREM C. *There are H fibrations:*

$$\hat{T}^{2n}(p^r) \xrightarrow{E} \Omega\hat{T}^{2n+1}(p^r) \xrightarrow{H} \Omega\hat{T}^{2np+q+1}(p)$$
$$\hat{T}^{2n-1}(p^r) \xrightarrow{E} \Omega\hat{T}^{2n}(p^r) \xrightarrow{H} \Omega\hat{T}^{2np-1}(p)$$

† We expect that this result can be improved.
‡ This is the case $k = \infty$ of Theorem 4.7 in the sequel.
§ See 3.7, 3.8, and 3.10.
¶ For example, at odd primes, it is convenient to build the sphere spectrum with the spaces S^{2n+1} and $\hat{S}^{2n} = S^{2n} \cup \cdots \cup e^{2n(p-1)} \subset J(S^{2n})$—the subspace of the James construction of words of length less than p.

and the resulting spectrum $\{\hat{T}^n\}$ *satisfies*

$$E^{-1}\{\hat{T}^n\} \cong \{T^n\} \cong S^0 \cup_{p^r} e^1.$$

These fibrations yield EHP sequences similar in form to the EHP sequences for spheres, and they work in a similar way. In case $r = 1$, one can endeavor for a self-referential inductive calculation, and this data can be used for calculation in the cases where $r > 1$. These EHP sequences have been used in the unstable calculation of v_2 periodicity [12].

The spaces $T^{2n-1}(p^r)$ are actually the limit of a sequence of spaces $T_k^{2n-1}(p^r)$ constructed in [3].† Along the way we show that each of these is an atomic H space, and construct corresponding co-H spaces G_k which we think of as generalizations of the Moore space $P^{2n+1}(p^r) = G_0$.

We will fix the positive integers r and n throughout our inductive construction (Sections 2 and 3) and often supress them from the notation. The integer k will occur as an index of induction and occur as a subscript (D_k, T_k, G_k, etc.). Throughout this paper we will be constructing various maps and taking the homotopy fiber or homotopy cofiber of such maps. In a surprising number of cases, these spaces turn out to be wedges of Moore spaces. We will write $\mathscr{W}_r^{r+k}$ for the collection of all spaces which have the homotopy type of simply connected finite type wedges of mod p^s Moore spaces for $r \leq s \leq r + k$.

Most of our spaces will have a fixed connectivity and infinite dimension. With the exception of the Moore space (as above) we will use a superscript to indicate the dimension of the bottom cell (e.g., $D_0^{2n} = G_0^{2n} = P^{2n+1}(p^r)$). For most purposes, n is fixed, and if it will not lead to confusion, we will leave it out of the notation.

The organization of the paper is as follows. In Section 2 we will summarize most of what we need from [2]. Section 3 will be devoted to the inductive argument and its many consequences. Section 4 is devoted to universal properties, and Section 5 to applications to the Moore space spectrum. Finally, we have relegated a number of results of a general nature to an Appendix. These results require diverse and independent lines of reasoning, and we did not want them to interrupt the flow of our main lines of argument.

2. REVIEW OF T_k AND RELATED SPACES

In this section we will summarize the results in [3]. These results generalize those of [4–6] and we begin by recalling the latter results using the notation of [3]. Let us fix a prime $p \geq 5$ and integers $n, r \geq 1$.

Cohen, Moore and Neisendorfer produce a decomposition

$$\Omega D_0 \simeq T_0 \times \Omega W_0$$

in their seminal work [4] where $D_0 = P^{2n+1}(p^r)$ and $W_0 \in \mathscr{W}_r^r$. The interesting part for our purpose is thus the space T_0. It comes with a fibration

$$\cdots \longrightarrow S^{2n-1} \times V_0 \longrightarrow T_0 \longrightarrow \Omega S^{2n+1}$$

where $V_0 = \prod_{j>0} S^{2np^j-1}\{p^{r+1}\}$.

One of the goals of [3] is to eliminate the factor V_0 and produce a fibration

$$\cdots \longrightarrow S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}.$$

† The space we are denoting $T_k^{2n-1}(p^r)$ was denoted $T_k^n\{p^r\}$ or merely T_k in [3]; the notation $T^{2n}(p^r)$ was not used there.

The homology classes in T_0 that come from V_0 can be removed by forming mapping cones. Consider the composition

$$P^{2np-1}(p) \xrightarrow{\omega} P^{2np-1}(p^{r+1}) \longrightarrow S^{2np-1}\{p^{r+1}\} \longrightarrow V_0 \longrightarrow T_0 \longrightarrow \Omega D_0 \qquad (2A)$$

where ω corresponds to the coefficient homomorphism $\mathbf{Z}_p \subset \mathbf{Z}_{p^{r+1}}$. Let D_1 be the mapping cone of the adjoint to this composition:

$$D_1 = D_0 \cup CP^{2np} = S^{2n} \cup_{p^r} e^{2n+1} \cup e^{2np} \cup_p e^{2np+1}.$$

The first element of order p^{r+1} in $\pi_*(D_0)$ is consequently reduced to have order p^r in $\pi_*(D_1)$. The pinch map $D_0 \to S^{2n+1}$ clearly extends over D_1. The first author then analyzes the fibering

$$\cdots \longrightarrow F_1 \longrightarrow D_1 \longrightarrow S^{2n+1}$$

as in [5]. The process is iterated and the author obtains, for each $k \geq 0$, a commutative diagram of fibrations:

$$\begin{tikzcd}
S^{2n-1} \times V_k \arrow[r] \arrow[d, "*"] & T_k \arrow[r] \arrow[d] & \Omega S^{2n-1} \arrow[d] \\
W_k \arrow[r, "="] \arrow[d] & W_k \arrow[r] \arrow[d] & * \arrow[d] \\
F_k \arrow[r] & D_k \arrow[r] & S^{2n+1}
\end{tikzcd} \qquad (2B)$$

where $V_k = \prod_{j>k} S^{2np^j-1}\{p^{r+k+1}\}$ and $W_k \in \mathscr{W}_r^{r+k}$. The connectivity of V_k increases with k and D_k is obtained from D_{k-1} by "coning off" a $\mathbf{Z}_p$ Moore space in such a way to truncate the first element in $\pi_*(D_{k-1})$ of order p^{r+k} so that its order is p^{r+k-1} as in (2A). Thus

$$D_k = (S^{2n} \cup_{p^r} e^{2n+1}) \cup \cdots \cup (e^{2np^k} \cup_p e^{2np^k+1}).$$

The analysis leading to (2B) is lengthy and complex. Various facts are developed along the way. Two important facts are summarized in the following lemma.

LEMMA 2.1. [3] *The spaces D_k satisfy:*

(a) $\Sigma^2 \Omega D_k \in \mathscr{W}_r^{r+k}$,

(b) $\Sigma(\Omega D_k \wedge \Omega D_k) \in \mathscr{W}_r^{r+k}$.

The proof of (a) is found in the section on the construction of ψ_{r+k}^k in Section 10. Part (b) is Theorem 12.4(xvi).

We will now recall some algebraic constructions from [3]. These will be used at one point in the main inductive argument (Theorem 3.1). A free associative differential graded algebra (dga) $A(X)$ is called an Adams–Hilton model for a space X if there is a quasi-isomorphism† $A(X) \to C_*(\Omega X)$. Such a model is not unique, but any two such are quasi-isomorphic. If $f: X \to Y$ is a map of spaces, a dga homomorphism $\varphi_f: A(X) \to A(Y)$ is called an Adams–Hilton model for f if the obvious diagram commutes up to dga homotopy. See [1; Section 8] for details.

† That is, a dga homomorphism which induces a homology isomorphism.

Frequently Adams–Hilton models are chosen so as to satisfy one or both of two useful properties. First, the generators may be chosen in 1–1 correspondence with the cells in some known CW structure for X. Second, $A(X)$ can be the universal enveloping algebra (denoted U) of some free differential graded Lie algebra (dgL).

In [3, Chap. 3] free differential graded Lie algebras $\tilde{M}^k$ and N^k are introduced so that $U(\tilde{M}^k)$ is an Adams–Hilton model for D_k. $U(N^k)$ will be seen to be an Adams–Hilton model for a space G_k constructed in the next section. Furthermore, a dgL monomorphism $h^k: N^k \to \tilde{M}^k$ is constructed in [3]. $U(h^k)$ will be seen to be an Adams–Hilton model for a map $e_k: G_k \to D_k$ constructed in the next section.

A non-free dgL, M^k, is introduced in [3] such that $U(M^k)$ is quasi-isomorphic to $U(\tilde{M}^k)$. Furthermore, we have the following.

LEMMA 2.2. *Let* $k \geq 0$. *The* dga $U(M^k)$ *has the following properties*:

(a) $UM^k = \mathbf{Z}_{(p)}\langle b, a_0, \ldots, a_k\rangle / J$, *where* $\mathbf{Z}_{(p)}\langle b, a_0, \ldots, a_k\rangle$ *is the tensor algebra with* $\dim b = 2n$, $\dim a_i = 2np^i - 1$, *and* J *is a certain ideal.*

(b) *There are cycles* $c_i \in UM^k$ *with* $c_i \equiv a_i \bmod$ *decomposables for* $0 \leq i \leq k$.

(c) $\beta^{(r+s)} b_s = -c_s$, *where* $b_s = b^{p^s}$ *and* $\beta^{(i)}$ *denotes the ith Bockstein homomorphism.*

(d) *There is a homomorphism* $\phi: UM_+^k \otimes \mathbf{Z}_p \to \mathbf{Z}_p\{b_i, c_i\}$ *(the vector space spanned by* b_i *and* c_i *for* $0 \leq i \leq k$*) such that* $\phi(a_i) = c_i$, $\phi(b_i) = b_i$ *and* ϕ *is zero on odd dimensional decomposable elements.*

3. CONSTRUCTION OF G_k AND e_k

This section contains our central argument, which is a construction by induction on k of co-H spaces $G_k = G_k^{2n}$. These spaces are improvements on the spaces D_k (in that they are co-H spaces) and have the simultaneous properties:

$$T_k \text{ is a retract of } \Omega G_k;$$

$$G_k \text{ is a retract of } \Sigma T_k.$$

We will construct G_k so that $G_k \supset G_{k-1}$ together with compatible maps $e_k: G_k \to D_k$, where D_k is the space described in Section 2. The cellular structure of G_k is

$$G_k = (S^{2n} \cup_{p^r} e^{2n+1}) \cup \cdots \cup (e^{2np^k} \cup_{p^{r+k}} e^{2np^k+1}) \tag{3A}$$

and e_k will induce an epimorphism in integral homology.

THEOREM 3.1. *For each* $k \geq 0$ *there are co-H spaces* G_k *and maps* $e_k: G_k \to D_k$ *such that*:

(a) *The composite* $G_k \xrightarrow{s} \Sigma\Omega G_k \xrightarrow{\Sigma\Omega e_k} \Sigma\Omega D_k$ *has a left homotopy inverse.*

(b) $\Omega D_k \simeq T_k \times \Omega W_k$, *where* W_k *is the space appearing in* (2B). *In particular,* T_k *is an H space.*

(c) UN^k *is an Adams–Hilton model for* G_k *and* $U(h^k)$ *is an Adams–Hilton model for* e_k.

(d) e_k *induces a splitting*:

$$\Sigma\Omega G_k \simeq \Sigma\Omega D_k \vee J_k$$

with $J_k \in \mathscr{W}_r^{r+k}$.

(e) $\Sigma\Omega G_k / G_k \in \mathscr{W}_r^{r+k}$.

(f) $\Omega D_k / \Omega G_k \in \mathscr{W}_r^{r+k}$.

(g) $\Sigma(\Omega G_k \wedge \Omega G_k) \in \mathscr{W}_r^{r+k}$

(h) $\Omega G_k \simeq \Omega D_k \times \Omega Y_k$, *where* Y_k *is the fiber of* e_k.

Proof. As mentioned, the proof will be by induction on k. When $k = 0$, let $G_0 = D_0 = P^{2n+1}(p^r)$, and take $e_0 = 1$. Properties (a)–(h) are either straightforward or proved in [4]. Now assume $k \geq 1$ and (a)–(h) are true for $k - 1$. To construct G_k for $k \geq 1$, recall that D_k was constructed as the cofiber of a composition:

$$P^{2np^k}(p) \xrightarrow{\omega} P^{2np^k}(p^{r+k}) \xrightarrow{\hat{\wp}_{p^k}} E_{k-1} \xrightarrow{\lambda_{k-1}\eta_{k-1}} D_{k-1}$$

where ω is the coefficient map corresponding to the inclusion $\mathbf{Z}_p \subset \mathbf{Z}_{p^{r+k}}$ ([3, 9.1]). We now construct a commutative diagram:

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[r, "p^{r+k-1}"] \arrow[d, "\rho"'] & P^{2np^k}(p^{r+k}) \arrow[r, "y"] \arrow[dr, "\lambda_{k-1}\eta_{k-1}\hat{\wp}_{p^k}"] & G_{k-1} \arrow[d, "e_{k-1}"] \\
P^{2np^k}(p) \arrow[r, "="] & P^{2np^k}(p) \arrow[u, "\omega"] \arrow[r] & D_{k-1}
\end{tikzcd} \tag{3B}$$

where y is a lifting of $\lambda_{k-1}\eta_{k-1}\hat{\wp}_{p^k}$ which exists by property (h) of the induction. We obtain G_k and e_k by taking mapping cones of the horizontal composites:

$$\begin{tikzcd}
P^{2np^k} \arrow[r, "p^{r+k-1}y"] \arrow[d, "\rho"'] & G_{k-1} \arrow[r] \arrow[d, "e_{k-1}"'] & G_k \arrow[d, "e_k"'] \\
P^{2np^k} \arrow[r] & D_{k-1} \arrow[r] & D_k
\end{tikzcd} \tag{3C}$$

Thus G_k is the mapping cone of $p^{r+k-1}y$. Clearly G_k is a CW complex of the type described in (3A). To show that G_k is a co-H space we apply Lemma A1 of the Appendix. We show that $c(p^{r+k-1}y) = p^{r+k-1}c(y) = 0$. In fact, we assert that

$$p^{r+k-1}[P^{2np^k}(p^{r+k}), \Sigma\Omega G_{k-1} \wedge \Omega G_{k-1}] = 0.$$

By part (g) of the inductive hypothesis $\Sigma\Omega G_{k-1} \wedge \Omega G_{k-1}$ is a wedge of mod p^s Moore spaces for various s with $r \leq s \leq r + k - 1$. The lowest dimensional homotopy class in this wedge of order $> p^{r+k-1}$ occurs in the homotopy of a mod p^{r+k-1} Moore space. The first class in the homology of ΩG_{k-1} of order p^{r+k-1} occurs in dimension $2np^{k-1} - 1$. Consequently the first such homology class in $\Sigma\Omega G_{k-1} \wedge \Omega G_{k-1}$ occurs in dimension $4np^{k-1} - 1$. Thus an element in the homotopy of $\Sigma\Omega G_{k-1} \wedge \Omega G_{k-1}$ of order $> p^{r+k-1}$ must lie in dimension $\geq 4np^{k-1}$. By [5], the first element of order p^{r+k} in $\pi_*(P^{2m+1}(p^{r+k-1}))$ is in dimension $2mp - 1$ and the first such element in $\pi_*(P^{2m}(p^{r+k-1}))$ is in dimension $4mp - 2p - 1$. These dimensions are large enough that we may conclude that there are no elements of order p^{r+k} in $\pi_s(\Sigma\Omega G_{k-1} \wedge \Omega G_{k-1})$ for $s \leq 2np^k$. Thus $p^{r+k-1}[P^{2np^k}(p^{r+k}), \Sigma\Omega G_{k-1} \wedge \Omega G_{k-1}] = 0$ as asserted.

We now prove parts (a) and (b) for k. These will follow from Theorem 14.9 of [A1] once we establish the existence of a map $Z \longrightarrow \Sigma\Omega D_k$ with certain homological properties. Consider the diagram:

$$\begin{tikzcd}
G_k \arrow[r, "s"] \arrow[dr, "="'] & \Sigma\Omega G_k \arrow[r, "\Sigma\Omega e_k"] \arrow[d, "\mathrm{ev}"] & \Sigma\Omega D_k \arrow[d, "\mathrm{ev}"] \\
 & G_k \arrow[r, "e_k"] & D_k
\end{tikzcd}$$

in which ev denotes the evaluation map. Let $Z = G_k$ and $f: Z \longrightarrow \Sigma\Omega D_k$ be the upper composite. Choose generators $u_i \in H_{2np^i}(G_k; \mathbf{Z}_p)$ and $v_i \in H_{2np^i+1}(G_k; \mathbf{Z}_p)$ with $\beta^{(r+i)}v_i = u_i$. Since $(e_k)_*(u_i) \neq 0$, $f_*(u_i) \neq 0$. Since $(\mathrm{ev})_*$ is zero on decomposables, $f_*(u_i)$ is indecomposable. By Lemma 2.2(a), $f_*(u_i) \equiv \lambda a_i$ modulo decomposables for some unit λ which we can absorb in the choice of u_i. By Lemma 2.2(b) and (d), $\phi(f_*(u_i)) = c_i$. By 2.2(b), $\phi f_*(v_i) = b_i$. We have thus satisfied the hypothesis of 14.9 of [3]. Parts (a) and (b) follow immediately.

We now prove part (c). At this point we refer the reader to the discussion in Section 1 and to Chapter 3 of [3]. Our inductive hypothesis tells us that $U(N^{k-1})$ and $U(h^{k-1})$ are Adams–Hilton models for G_{k-1} and e_{k-1}. Since G_k is obtained from G_{k-1} by attaching two cells, the Adams–Hilton model for G_k, $A(G_k)$ can be assumed to be $A(G_{k-1})$ with two more generators freely adjoined. That is, $A(G_k) = U(N^{k-1}) \amalg \mathbf{Z}_{(p)}\langle u_k, v_k\rangle$ with $|u_k| = 2np^k - 1$ and $|v_k| = 2np^k$. $U(N^k)$ has this form as an algebra, so it suffices to show that the differential in $A(G_k)$ can be chosen to satisfy

$$d(u_k) = p^{-1}d(w'_{p^k}) \quad\text{and}\quad d(v_k) = -p^{r+k}u_k + p^{r+k-1}d(w'_{p^k}) \tag{3D}$$

with the notation w'_{p^k} as in [3, Proof of 3.8]. To prove this, it suffices to show that an Adams–Hilton model for the attaching map $p^{r+k-1}y$ of (3B) and (3C) can be given by

$$A(p^{r+k-1}y)(u) = p^{-1}d(w'_{p^k}), \qquad A(p^{r+k-1}y)(v) = p^{r+k-1}w'_{pk} \tag{3E}$$

where we are writing $A(P^{2np^k}(p^{r+k}))$ as $\mathbf{Z}_{(p)}\langle u, v\rangle$ with $dv = p^{r+k}u$. The formula (3E) in connection with (3C) also shows that $A(e_k)$ can be chosen to be an extension of $A(e_{k-1})$ satisfying $A(e_k)(u_k) = \tilde{u}_k$ and $A(e_k)(v_k) = p^{r+k-1}\tilde{v}_k$, where $\tilde{u}_k$ and $\tilde{v}_k$ denote generators of $A(D_k)$ corresponding to the two top cells of D_k attached by the bottom row of (3C). This extension is precisely $U(h^k)$. We thus focus our attention entirely on proving (3E).

To compute $A(p^{r+k-1}y)$ we first determine as much as possible about $A(y)$. We exploit the homotopy in the upper right triangle of (3B), which tells us that the dga homomorphisms $A(e_{k-1})A(y) = U(h^{k-1})A(y)$ and $A(\lambda_{k-1}\eta_{k-1}\hat{y}_{p^k})$ are homotopic. Applying [3, 6.2] we conclude that there exist elements $\hat{u}', \hat{v}' \in U\tilde{M}^{k-1}$ satisfying:

$$u(h^{k-1})A(y)(u) = p^{-r-k}d(\tilde{w}'_{p^k}) + d(\tilde{u}')$$
$$u(h^{k-1})A(y)(v) = \tilde{w}_{p^k} + p^{r+k}\tilde{u}' + d(\tilde{v}').$$

Now set $u' = A(y)(a) - p^{-r-k}d(w'_{p^k}) \in U(N^{k-1})$, and observe that $U(h^{k-1})(u') = d(\tilde{u}')$; it follows that $U(h^{k-1})d(u') = 0$. Since $U(h^{k-1})$ is a monomorphism, u' is a cycle in $U(N^{k-1})$. Similarly, put $v' = A(y)(v) - w'_{p^k}$, so that $U(h^{k-1})(v') = p^{r+k}\tilde{u}' + d(\tilde{v}')$ and $d(v') = p^{r+k}u'$. Thus our model $A(y)$ is given by

$$A(y)(u) = p^{-r-k}d(w'_{p^k}) + u'$$
$$A(y)(v) = w'_{p^k} + v'.$$

Clearly, then, one model for the map $p^{r+k-1}y$ is given by

$$A(p^{r+k-1}y)(u) = p^{-1}d(w'_{p^k}) + p^{r+k-1}u', \quad A(p^{r+k-1}y)(v) = p^{r+k-1}w'_{p^k} + p^{r+k-1}v'.$$

Now since u' is a cycle of $U(N^{k-1})$, and $p^{r+k-1}\tilde{H}_*(U(N^{k-1})) = \mathbf{0}$ by induction, $p^{r+k-1}u'$ is a boundary. Let us write $p^{r+k-1}u' = d(u'')$. Using [3, 6.2] again, we may replace our model $A(p^{r+k-1}y)$ by a homotopic dga homomorphism so that it satisfies

$$A(p^{r+k-1}y)(u) = p^{-1}d(w'_{p^k}), \quad A(p^{r+k-1}y)(v) = p^{r+k-1}w'_{p^k} + p^{r+k-1}v' - p^{r+k}u''.$$

Since $d(v' - pu'') = p^{r+k}u' - p\cdot p^{r+k-1}u' = \mathbf{0}$, $v - pu''$ is a cycle in $U(N^{k-1})$ and we may write $p^{r+k-1}(v' - pu'') = d(v'')$ for some $v'' \in U(N^{k-1})$. Again applying [3, 6.2], we see that $A(p^{r+k-1}y)$ is homotopic to the dga homomorphism given by (3E). This completes the proof of part (c).

Now consider the diagram of cofibrations:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & G_k \arrow[r, "="] \arrow[d, "s"] & G_k \arrow[d, "f"] \\ \Omega D_k/\Omega G_k \arrow[r] \arrow[d, "="] & \Sigma\Omega G_k \arrow[r, "\Sigma\Omega e_k"] \arrow[d] & \Sigma\Omega D_k \arrow[d, "f'"] \\ \Omega D_k/\Omega G_k \arrow[r, "f'''"] & \Sigma\Omega G_k/G_k \arrow[r, "f''"] & Y_k' \end{tikzcd} \tag{3F}$$

By (a), f_* is a split monomorphism in integral homology. Thus f_*' is a split epimorphism. By (c), $(\Omega e_k)_*$ coincides with $U(g^k h^k)_*$, which is a split epimorphism by [3, 4.10b]. Thus $(\Sigma\Omega e_k)_*$ is a split epimorphism in integral homology. It follows that $(f'')_*$ is a split epimorphism and we deduce the following.

LEMMA 3.2. *$(f''')_*$ is a split monomorphism in integral homology.*

We now begin a subsidiary induction which uses Lemma 3.2. We will construct, by induction on m, a 2-connected space $X_m \in \mathscr{W}_r^{r+k}$ and a map $\varphi_m : \Sigma\Omega G_k \to X_m$ such that the sum

$$\Sigma\Omega G_k \xrightarrow{\Sigma\Omega e_k + \varphi_m} \Sigma\Omega D_k \vee X_m$$

is an isomorphism in integral homology in dimensions $\leq m$. This is clearly possible when $m \leq 2n + 1$ with $X_m = *$. Suppose we have such a map for a given value of m. Since X_m is 2-connected, $X_m = \Sigma X_m'$ with $X_m' \in \mathscr{W}_r^{r+k}$. Consider the composite

$$\begin{aligned} \Sigma(\Omega G_k \wedge \Omega G_k) &\longrightarrow (\Sigma\Omega D_k \vee X_m) \wedge \Omega G_k \\ &\xrightarrow{\simeq} (\Omega D_k \vee X_m') \wedge \Sigma\Omega G_k \\ &\longrightarrow (\Omega D_k \vee X_m') \wedge (\Sigma\Omega D_k \vee X_m). \end{aligned}$$

Let $W' = (\Omega D_k \vee X_m') \wedge (\Sigma\Omega D_k \vee X_m)$. By the Künneth theorem, both maps in this composite are homology isomorphisms in dimensions $\leq m + 2n - 1$. By Lemma 2.1, $W' \in W_r^{r+k}$ and is 2-connected. Now by Lemma A2 of the Appendix, the composition

$$\Sigma\Omega G_k/G_k \xrightarrow{\gamma} \Sigma(\Omega G_k \wedge \Omega G_k) \longrightarrow W'$$

induces a split monomorphism in homology in dimensions $\leq m + 2n - 1$ where γ is a right inverse to $\mu\delta$. (See Lemma A2 for notation.) By Lemma A3 of the Appendix, there is a retract W'' of W' such that the composite

$$\pi' : \Sigma\Omega G_k/G_k \longrightarrow W' \longrightarrow W''$$

induces an isomorphism in integral homology in dimensions $\leq m + 2n - 1$. By Theorem 3.1 and Lemma A3 again, we obtain a retract W''' of W'' and a commutative diagram

$$\begin{tikzcd} \Omega D_k/\Omega G_k \arrow[r, "f'''"] \arrow[d, "\pi'''"] & \Sigma\Omega G_k/G_k \arrow[d, "\pi'"] \\ W''' & W'' \arrow[l, "\pi''"] \end{tikzcd}$$

in which the vertical maps induce isomorphisms in dimensions $\leq m + 2n - 1$. Since f''' factors as

$$\Omega D_k/\Omega G_k \xrightarrow{j} \Sigma\Omega G_k \xrightarrow{i} \Sigma\Omega G_k/G_k$$

we have a commutative diagram of cofibration sequences:

$$\begin{tikzcd} \Omega D_k/\Omega G_k \arrow[r, "j"] \arrow[d, "\pi'''"] & \Sigma\Omega G_k \arrow[r] \arrow[d, "c"] & \Sigma\Omega D_k \arrow[d, "="] \\ W''' \arrow[r] & \Sigma\Omega D_k \vee W''' \arrow[r] & \Sigma\Omega D_k \end{tikzcd} \tag{3G}$$

where $c = \Sigma\Omega e_k + \pi''\pi' i$. Since π''' induces an isomorphism in integral homology in dimensions $\leq m + 2n - 1$, the same holds for c. Since $m + 2n - 1 \geq m + 1$, we may take X_{m+1} to be W''' and φ_{m+1} to be $\pi''\pi' i$. This completes the induction. Putting $J_k = \varinjlim X_m$ we obtain (d). To prove (e), take a limit of the maps $\Sigma\Omega G_k/G_k \xrightarrow{\pi'} W''$ as m increases. Then (f) follows from Lemma 3.2 and (e). Use Lemma 2.1(b) and (d) to prove (g).

Finally we use (d) to conclude that $\Sigma\Omega G_k \longrightarrow \Sigma\Omega D_k$ has a right homotopy inverse. We then apply Lemma A4 of the Appendix to the fibering $\Omega G_k \longrightarrow \Omega D_k \longrightarrow Y_k$ to obtain a splitting $\Omega G_k \simeq \Omega D_k \times \Omega Y_k$. This completes the proof of Theorem 3.1. □

Having constructed the spaces G_k, we now proceed to derive many of the consequent relationships and conclusions. Numerous spaces will be seen to be the wedges of Moore spaces. In particular, it is easy to see that the map

$$\Sigma\Omega Y_k \longrightarrow \Sigma\Omega G_k \longrightarrow \Sigma\Omega G_k/\Sigma\Omega D_k = J_k$$

is a split monomorphism in integral homology, and hence $\Sigma\Omega Y_k \in \mathscr{W}_r^{r+k}$ by Lemma A3 of the Appendix. In fact, a much stronger result holds.

PROPOSITION 3.3. $Y_k \in \mathscr{W}_r^{r+k}$.

The proof is based on the following result which is of interest in its own right.

LEMMA 3.4. *There is a cofibration sequence*: $C_k \longrightarrow G_k \longrightarrow D_k$ *with*

$$C_k = \bigvee_{i=1}^{k} P^{2np^i+1}(p^{r+i-1}) \in \mathscr{W}_r^{r+k-1}.$$

Proof. When $k = 0$, $C_0 \approx *$ and the lemma is obviously true; suppose $k \geq 1$, and that the lemma holds at $k - 1$. Consider the diagram

$$\begin{tikzcd} P^{2np^k}(p^{r+k-1}) \arrow[r, "0"] \arrow[d, "\omega"] & G_{k-1} \arrow[r] \arrow[d] & C_k \arrow[d, dashed] \\ P^{2np^k}(p^{r+k}) \arrow[r, "p^{r+k-1}y"] \arrow[d, "\rho"] & G_{k-1} \arrow[r] \arrow[d, "e_{k-1}"] & G_k \arrow[d, "e_k"] \\ P^{2np^k}(p) \arrow[r] & D_{k-1} \arrow[r] & D_k \end{tikzcd} \tag{3H}$$

The lower two rows of (3H) are the cofibrations that define G_k, D_k, and e_k (See (3C)). The top row of (3H) is a cofibration which defines C_k. By our induction hypothesis, the upper middle vertical arrow of (3H) exists, and the upper left square of (3H) commutes because p^{r+k-1} times any map out of $P^{2np^k}(p^{r+k-1})$ is null. Because the first two columns of (3H) are cofibrations, we deduce that the dotted arrow of (3H) exists and that the right-hand column of (3H) is also a cofibration. □

Proof of Proposition 3.3. Apply Corollary A7 of the Appendix with $X = G_k$ and $A = C_k$ using 3.1(h). It follows that $Y_k \simeq C_k \wedge (\Omega D_k)^+ \in \mathscr{W}_r^{r+k-1}$ by 2.1(a). □

We wish to establish a diagram analogous to (2B) for the spaces G_k. We will first define spaces R_k by the pull-back diagram:

$$\begin{tikzcd} T_k \arrow[r, "="] \arrow[d] & T_k \arrow[d] \\ R_k \arrow[r] \arrow[d, "c_k"'] & W_k \arrow[d] \\ G_k \arrow[r, dashed, "e_k"] & D_k \end{tikzcd} \tag{3I}$$

Since T_k is a retract of ΩG_k, the inclusion $T_k \longrightarrow R_k$ is null homotopic. By 3.4 we may apply Lemma A6 of the Appendix and conclude that $W_k \simeq R_k \cup C(C_k \wedge T_k^+)$. Since Ωe_k has a right homotopy inverse and W_k is a suspension, the projection $W_k \longrightarrow D_k$ lifts to G_k and hence W_k is a retract of R_k. Thus the cofibration sequence

$$C_k \wedge T_k^+ \longrightarrow R_k \longrightarrow W_k$$

splits and we have the following.

PROPOSITION 3.5. $R_k \simeq W_k \vee (C_k \wedge T_k^+) \in \mathscr{W}_r^{r+k}$.

Now define $\bar{F}_k$ to be the fiber of the projection $G_k \xrightarrow{e_k} D_k \longrightarrow S^{2n+1}$. Then from (3I) and (2B) we obtain a commutative diagram of fibrations:

$$\begin{tikzcd} S^{2n-1} \times V_k \arrow[r] \arrow[d] & T_k \arrow[r] \arrow[d] & \Omega S^{2n+1} \arrow[d] \\ R_k \arrow[r, "="] \arrow[d] & R_k \arrow[r] \arrow[d] & * \arrow[d] \\ \bar{F}_k \arrow[r] & G_k \arrow[r] & S^{2n+1} \end{tikzcd} \tag{3J}$$

and hence the following corollary.

COROLLARY 3.6. *We have* $\Omega G_k \simeq \Omega R_k \times T_k$ *and* $\Omega \bar{F}_k \simeq \Omega R_k \times S^{2n-1} \times V_k$. *Furthermore,* $p^{\max(r+k+1,n-1)}\pi_t(\bar{F}_k) = 0$ *when* $t > 2n-1$ *and* $p^{2r+k+1}\pi_*(G_k) = 0$.

Proof. The splittings follow directly from (3J). Clearly $p^{r+k+1}\pi_*(R_k) = 0$, $p^{n-1}\pi_t(S^{2n-1}) = 0$, if $t > 2n-1$, $p^{r+k+1}\pi_*(V_k) = 0$ and by [3, 14.10], $p^{2r+k+1}\pi_*(T_k) = 0$. □

We now describe some important structure maps involving the spaces T_k and G_k. Let $h_k : \Omega G_k \longrightarrow T_k$ be the connecting map from the fibering (3J) and choose g_k to be an *arbitrary*† right homotopy inverse.

PROPOSITION 3.7. *There are maps* $f_k : G_k \longrightarrow \Sigma T_k$ *such that the composites*

$$G_k \xrightarrow{f_k} \Sigma T_k \xrightarrow{g_k^*} G_k$$

$$T_k \xrightarrow{g_k} \Omega G_k \xrightarrow{h_k} T_k$$

are homotopic to the identity, where g_k^* *denotes the adjoint of* g_k.

To prove Proposition 3.7 we will first establish the following lemma.

LEMMA 3.8. G_k *is atomic for* $0 \leq k \leq \infty$.

Proof. We wish to apply Lemma A5 of the appendix, so we need to show that $p^{r+k-1}y \neq 0$ (see (3B)). Choose a map $\bar{y} : P^{2np^k}(p^{r+k}) \longrightarrow \bar{F}_{k-1}$ so that the diagram

$$\begin{tikzcd}
P^{2np^k}(p^{r+k}) \arrow[dr, "\bar{y}"] \arrow[drr, "y"] \arrow[ddr, "\eta_{k-1}\hat{y}_{p^k}"'] & & \\
& \bar{F}_{k-1} \arrow[r] \arrow[d] & G_{k-1} \arrow[d, "e_{k-1}"] \\
& F_{k-1} \arrow[r, "\lambda_{k-1}"] & D_{k-1}
\end{tikzcd} \tag{3K}$$

commutes up to homotopy. This is possible since the right-hand square is a pullback diagram. Let $y' : P^{2np^k-1}(p^{r+k}) \rightarrow \Omega F_{k-1}$ be the adjoint of $\eta_{k-1}\hat{y}_{p^k}$. By the definition of δ_{d-1}(see [3, 13CD]), y' is homotopic to the composite

$$P^{2np^k-1}(p^{r+k}) \subset S^{2np^k-1}\{r^{r+k}\} \subset V_{k-1} \xrightarrow{\delta_{k-1}} \Omega E_{k-1} \xrightarrow{\Omega\eta_{k-1}} \Omega F_{k-1}.$$

Now consider the homotopy commutative diagram:

$$\begin{tikzcd}
P^{2np^k-1}(p^{r+k}) \arrow[r, "y'"] \arrow[d] & \Omega F_{k-1} \arrow[r, "\Omega\lambda_{k-1}"] \arrow[d, "\varepsilon_{k-1}"] & \Omega D_{k-1} \arrow[d] \\
S^{2np^k-1}\{p^{r+k}\} \arrow[r] & S^{2n-1} \times V_{k-1} \arrow[r, "\partial_{k-1}"] & T_{k-1}
\end{tikzcd} \tag{3L}$$

Let $\bar{u}, \bar{v}$ generate $\tilde{H}_*(P^{2np^k}(p^{r+k}); \mathbf{Z}_p)$ with $\beta^{(r+k)}(\bar{v}) = \bar{u}$. Let $\bar{u}$ and $\bar{v}$ be the images of $\bar{u}$ and $\bar{v}$ in $H_*(T_{k-1}; \mathbf{Z}_p)$. Clearly $\bar{u}$ and $\bar{v}$ are nonzero since $H_*(T_{k-1}; \mathbf{Z}_p) \simeq im(\partial_{k-1})_* \otimes H_*(\Omega S^{2n+1}; \mathbf{Z}_p)$, and $\beta^{(n+k)}(\bar{v}) \equiv \bar{u}$(mod lower Bocksteins). However, it is easy to see that no element of $H_*(T_{k-1}; \mathbf{Z}_p)$ other than $\bar{v}$ can have a Bockstein on it equal to $\bar{u}$. It follows by comparing the homotopy and homology Bockstein spectral sequences that the composite from (3L), $P^{2np^k-1}(p^{r+k}) \rightarrow T_{k-1}$, has order p^{r+k} (see [4, 7.3]). Thus $(\Omega\lambda_{k-1})y'$ and hence $e_{k-1}y$ and finally y all have order p^{r+k}. □

As we shall see later, the spaces T_k are also atomic, except when $k = 0$, $r = 1$, and $n = 1$.

† There is certainly some choice for g_k which consequently effects the choice of f_k. Thus the triple (f_k, g_k, h_k) is not determined by the structure of G_k or T_k.

Proof of Proposition 3.7. We need to establish the existence of the map $f_k: G_k \longrightarrow \Sigma T_k$ such that $g_k^* f_k \sim 1$. Choose an arbitrary co-H structure $s: G_k \longrightarrow \Sigma\Omega G_k$ and consider the composition

$$G_k \xrightarrow{s} \Sigma\Omega G_k \xrightarrow{\Sigma h_k} \Sigma T_k.$$

This may not serve as f_k in general. Consider the composition

$$G_k \xrightarrow{s} \Sigma\Omega G_k \xrightarrow{\Sigma h_k} \Sigma T_k \xrightarrow{g_k^*} G_k.$$

We can easily see that each map in this composition induces an isomorphism in H_{2n}. Since G_k is atomic, the composition is a homotopy equivalence. Choose a homotopy inverse $e: G_k \longrightarrow G_k$ to this composition and define f_k as the composition

$$G_k \xrightarrow{e} G_k \xrightarrow{s} \Sigma\Omega G_k \xrightarrow{\Sigma h_k} \Sigma T_k.$$

This completes the proof. □

At this point we have used an arbitrary co-H structure. We know that such a structure exists by Theorem 3.1. The construction in Theorem 3.1 is so encumbered by choice that it is not useful in specifying a co-H structure. It would be desirable, for example, to know whether there is a co-associative co-H structure on G_k. The existence of f_k allows us to partially specify a co-H structure. It is not hard to show that any co-associative co-H structure is of this form. We have the following.

PROPOSITION 3.9. *There is a co-H structure* $v_k: G_k \longrightarrow \Sigma\Omega G_k$ *and an H structure*† $\mu_k: \Omega\Sigma T_k \longrightarrow T_k$ *so that the diagrams*

$$\begin{tikzcd} & G_k \arrow[dr] & \\ G_k \arrow[ur, "="] \arrow[rr, "v_k"] \arrow[dr, "f_k"'] & & \Sigma\Omega G_k \\ & \Sigma T_k \arrow[ur, "\Sigma g_k"'] & \end{tikzcd} \qquad \begin{tikzcd} & T_k \arrow[dr, "="] & \\ \Omega\Sigma T_k \arrow[ur, "\iota"] \arrow[rr, "\mu_k"] \arrow[dr, "\Omega g_k^*"'] & & T_k \\ & \Sigma G_k \arrow[ur, "\Sigma g_k"'] & \end{tikzcd}$$

commute, and the composites

$$G_k \xrightarrow{v_k} \Sigma\Omega G_k \xrightarrow{\Sigma h_k} \Sigma T_k \xrightarrow{g_k^*} G_k$$

$$T_k \xrightarrow{g_k} \Omega G_k \xrightarrow{\Omega f_k} \Omega\Sigma T_k \xrightarrow{\mu_k} T_k$$

are the identity.

We conjecture that g_∞ may be chosen so that μ_∞ is an H map and f_k can be chosen so that v_k is a co-H map (i.e. v_k is co-associative).

Proof of Proposition 3.9. Define v_k and μ_k by the lower triangles; $\Omega g_k^* \circ \iota = g_k$ and $\mathrm{ev} \circ \Sigma g_k = g_k^*$ so the upper triangles commute. The composites are clearly the identity by Proposition 3.7. □

PROPOSITION 3.10. *Either T_k is atomic or $k = 0$ and $n = r = 1$.*

Proof. We will follow the method of [4, 4.1] where it is proved that T_0 is atomic except when $n = r = 1$. Since the Serre spectral sequence for the $\mathbf{Z}_p$ homology of the fibering

† We may define a multiplication $\tilde{\mu}_k: T_k \times T_k \longrightarrow T_k$ by the composition $T_k \times T_k \longrightarrow \Omega\Sigma T_k \xrightarrow{\mu_k} T_k$, but the correspondence $\mu_k \to \tilde{\mu}_k$ is not one to one. If μ_k is an H map, T_k is homotopy associative and μ_k is determined by $\tilde{\mu}_k$.

$S^{2n-1}\times V_k \xrightarrow{\iota} T_k \xrightarrow{\pi} \Omega S^{2n+1}$ collapses, we may apply Lemma A8 of the Appendix to conclude that each primitive element $x\in H_*(T_k;\mathbf{Z}_p)$ satisfies either $x=\iota_*(x')$ or $\pi_*(x)\neq 0$.

Now let $g: T_k \longrightarrow T_k$ be a map such that $g_*: H_{2n-1}(T_k:\mathbf{Z}_p)\longrightarrow H_{2n-1}(T_k;\mathbf{Z}_p)$ is an isomorphism. Let T be the telescope of g and $h: T_k\longrightarrow T$ the natural map. Then h_* is onto. Let K be the kernel of h_* and l be the least dimension in which $K_l\neq 0$. K_l is contained in the primitives. Since T_k is a retract of ΩG_k by Theorem 3.1(b) and $\Sigma^2\Omega G_k\in\mathscr{W}_r^{r+k}$ by Lemma 2.1(a) and Theorem 3.1(d), the term E^{r+k+1} of the Bockstein spectral sequence for $\tilde{H}_*(T_k;\mathbf{Z}_p)$ is zero. Choose $x\neq 0\in K_l$. Suppose $\pi_*(x)=0$ so $x\in \operatorname{im}\iota_*$. Since $\beta^{(i)}=0$ in $H_*(S^{2n-1}\times V_k;\mathbf{Z}_p)$ for $i\leq r+k$, $\beta^{(i)}x=0$ for $i\leq r+k$ and hence $x=\beta^{(i)}y$ for some i, $r\leq i\leq r+k$.

Since $\beta^{(i)}y\neq 0$ with $i\leq r+k, y\notin \operatorname{im}\iota_*$. We claim that y can be chosen to be primitive (and hence $\pi_* y\neq 0$). This is clear if $n>1$ since K_{l+1} is contained in the primitives. If $n=1$, $\bar{\psi}(K_{l+1})\subset K_l\otimes u+u\otimes K_l$, where $u\in H_1(T_k;\mathbf{Z}_p)$. If y is not primitive, $\psi(y)=y\otimes 1+(-1)^l u\otimes x'+x'\otimes u+1\otimes y$ for some $x'\in K_l$. Now let $y'=y-x'u$. Then y' is primitive and $\beta^{(i)}y'=x$ since $\beta^{(i)}u=0$ for all i and $\beta^{(i)}x'\in K_{l-1}=0$. We conclude that either $\pi_*(x)\neq 0$ or there exists y with $\pi_*(y)\neq 0$ (where $x\in K_l$ and $y\in K_{l+1}$). Since the homotopy fiber of h is $l-1$ connected, x or y is represented by a map $\phi: P^t(p^i)\longrightarrow T_k$ (where $t=l$ or $l+1$), $r\leq i\leq r+k$, and $\pi\phi$ is nonzero in homology.

Thus we have a map $\pi\phi: P^t(p^i)\to\Omega S^{2n+1}$ which is nonzero in homology. As in [4, 4.3] we conclude that $n=r=i=1$ and $t=2p$. Finally we wish to conclude that $k=0$. In dimensions less than $2p^{k+1}-2$, $H^*(T_k;\mathbf{Z}_p)\simeq\bigotimes_{i=0}^k\mathbf{Z}_p[v_i,2p^i]/(v_i^p)\otimes\Lambda(u,1)$ and $\beta^{(i)}uv_0^{p-1}\cdots v_{i-1}^{p-1}=v_i$ by an integral cohomology calculation. Since $g^*(u)=\lambda u$ for some unit $\lambda\in\mathbf{Z}_p$, we easily see that g^* is an isomorphism in dimensions $<2p^{k+1}-2$. Thus, g_* is an isomorphism in dimensions $<2p^{k+1}-3$ and $K_l=0$ for $l<2p^{k+1}-3$. Since $K_{2p}\neq 0$, $k=0$.

4. UNIVERSAL PROPERTIES

In this section we examine the universal properties that the space G_k and T_k enjoy. These properties could be considerably strengthened if we knew that G_k is co-associative or T is associative; we will make note of the potential alterations in the sequel. We reintroduce the superscript (G_k^{2n}, T_k^{2n-1}) to indicate the dimension of the bottom cell.

PROPOSITION 4.1. *Suppose X is an H space, $\varphi_{k-1}: G_{k-1}^{2n}\longrightarrow\Sigma X$ is a map and $K\subset G_{k-1}^{2n}$ is a skeleton of G_{k-1}^{2n} such that $\varphi_{k-1}|_K$ is a co-H map. Suppose $p^{r+k-1}\pi_{2np^k-1}(X;\mathbf{Z}_{p^{r+k}})=0$. Then there is a map $\varphi_k: G_k^{2n}\longrightarrow\Sigma X$ with $\varphi_k|_K\sim\varphi_{k-1}|_K$.*

Note. We shall see in the proof that if the co-H structure maps $s_k: G_k\longrightarrow\Sigma\Omega G_k$ can be chosen to be co-H maps (i.e. G_k is co-associative) then φ_k can be chosen to extend φ_{k-1}.

Proof of Proposition 4.1. We begin by replacing φ_{k-1} with the composite

$$G_{k-1}^{2n}\xrightarrow{s_{k-1}}\Sigma\Omega G_{k-1}^{2n}\xrightarrow{\Sigma\Omega\varphi_{k-1}}\Sigma\Omega\Sigma X\xrightarrow{\Sigma\mu}\Sigma X$$

which we label $\bar{\varphi}_{k-1}$. Since the inclusion $K\subset G_{k-1}^{2n}$ is a co-H map $\bar{\varphi}_{k-1}|_K\sim(\Sigma\mu)(\Sigma i)\varphi_{k-1}|_K\sim\varphi_{k-1}|_K$, where $i: X\to\Omega\Sigma X$ is the standard inclusion. We will extend $\bar{\varphi}_{k-1}$ to a map φ_k. Consider the diagram:

$$\begin{tikzcd}
G_{k-1}^{2n} \arrow[r, "s_{k-1}"] & \Sigma\Omega G_{k-1}^{2n} \arrow[r, "\Sigma\Omega\varphi_{k-1}"] & \Sigma\Omega\Sigma X \arrow[r, "\Sigma\mu"] & \Sigma X \\
P^{2np^k}(p^{r+k}) \arrow[u, "\alpha = p^{r+k-1}y"] \arrow[r, "\Sigma\iota"] & \Sigma\Omega P^{2np^k}(p^{r+k}) \arrow[u, "\Sigma\Omega\alpha"'] \arrow[urr, "\Sigma(\mu\circ\Omega(\varphi_{k-1}\circ\alpha))"'] & &
\end{tikzcd}$$

The composite $\Sigma\Omega(\varphi_{k-1}\circ\alpha)\circ\Sigma\iota: P^{2np^k}(p^{r+k}) \longrightarrow \Sigma\Omega\Sigma X$ is the suspension of the adjoint of $\varphi_{k-1}\circ\alpha$. Hence the entire lower composite is the suspension of $\mu\circ\Omega\varphi_{k-1}\circ\alpha^*$ which is divisible by p^{r+k-1}. It follows that the lower composite is inessential and $\bar{\varphi}_{k-1}$ extends over the mapping cone of α which is G_k^{2n}. □

COROLLARY 4.2. *Let $0 \le s < m \le \infty$. Suppose that X is an H space of finite type and that $p^{r+k-1}\pi_{2np^k-1}(X; \mathbf{Z}_{p^{r+k}}) = 0$, when $s < k \le m$. Let $\varphi: P^{2np^s}(p^{r+s}) \longrightarrow X$. Then there is a map $\varphi_m: G_m^{2n} \longrightarrow \Sigma X$ such that the diagram*

$$\begin{tikzcd} G_s^{2n} \arrow[r] \arrow[d] & G_s^{2n}/G_{s-1}^{2n} \simeq P^{2np^s+1}(p^{r+s}) \arrow[d, "\Sigma\varphi"] \\ G_m^{2n} \arrow[r, "\varphi_m"] & \Sigma X \end{tikzcd}$$

commutes up to homotopy.

Proof. Since G_{s-1}^{2n} is a sub co-H space of G_s^{2n}, the composite $G_s^{2n} \longrightarrow G_s^{2n}/G_{s-1}^{2n}$ is a co-H map with the induced co-H structure on $G_s^{2n}/G_{s-1}^{2n} \simeq P^{2np^s+1}(p^{r+s})$. This space, however, has a unique co-H structure. Thus the composite

$$G_s^{2n} \longrightarrow G_s^{2n}/G_{s-1}^{2n} \simeq P^{2np^s+1}(p^{r+s}) \xrightarrow{\Sigma\varphi} \Sigma X$$

is a co-H map. Apply Proposition 4.1 successively to construct maps $\varphi_k: G_k^{2n} \longrightarrow \Sigma X$ for each k, $m \ge k > s$ extending φ. We now consider the case $m = \infty$. Since X has finite type, the set of homotopy classes of extensions φ_k of φ is finite for each k. Call φ_k infinitely extendible if there is an extension φ_l of φ_k for each $l \ge k$.

By induction on k each infinitely extendible map φ_k has an infinitely extendible extension φ_{k+1}; for if not, since there are only finitely many extensions of φ_k to φ_{k+1}, φ_k would not be infinitely extendible. Thus we may choose a compatible sequence φ_k of infinitely extendible maps and hence their limit φ_∞.

Our first application of these results is to the uniqueness of the spaces G_k and T_k. In [3] the spaces D_k and T_k are defined and depend on choices exercised in the discussion. For example, the map $\rho_k: D_k \longrightarrow S^{2n+1}\{p^r\}$ is a completely arbitrary extension of $\rho_0: D_0 \longrightarrow S^{2n+1}\{p^r\}$. Suppose some other choices are made leading to the construction of spaces $\bar{D}_k$ and $\bar{T}_k$ and finally $\bar{G}_k$ satisfying all the properties of D_k, T_k, and G_k.

PROPOSITION 4.3. *$G_k^{2n} \simeq \bar{G}_k^{2n}$ and $T_k^{2n-1} \simeq \bar{T}_k^{2n-1}$.*

For the proof of this we need the following lemma.

LEMMA 4.4. *$p^{r+1}\pi_*(T_\infty^{2n-1}) = 0$ and $p^r\pi_{2np-1}(T_\infty^{2n-1}) = 0$.*

Proof. The first part follows from [3, Lemma 15.2] which is obtained from the fibration sequence

$$W_n \longrightarrow T_\infty^{2n-1} \longrightarrow \Omega S^{2n+1}\{p^r\},$$

where W_n is the fiber of the double suspension

$$W_n \longrightarrow S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}.$$

However, $\pi_{2np-1}(W_n) = 0$ so the second part follows as well. □

Proof of Proposition 4.3. For $i \leq k$, $\pi_{2np^i-1}(T_k) \cong \pi_{2np^i-1}(T_\infty)$ so, by the lemma, $p^{r+i-1}\pi_{2np^i-1}(T_k) = 0$ for $1 \leq i \leq k$. Consequently we may apply Corollary 4.2 to obtain a map $\varphi_k : \bar{G}_k^{2n} \longrightarrow \Sigma T_k^{2n-1}$ which induces an isomorphism in H_{2n}. Consequently the composite

$$\bar{T}_k \xrightarrow{\bar{g}_k} \Omega\bar{G}_k \xrightarrow{\Omega\varphi_k} \Omega\Sigma T_k \xrightarrow{\mu_k} T_k$$

induces an isomorphism in π_{2n-1}. Reversing the roles of the two constructions, we obtain a similar map $T_k \longrightarrow \bar{T}_k$. Since both spaces are atomic, both composites are homotopy equivalences and hence each map is a homotopy equivalence as well. Thus $T_k \simeq \bar{T}_k$. Similarly we construct a homotopy equivalence

$$G_k \xrightarrow{f_k} \Sigma T_k \xrightarrow{\simeq} \Sigma\bar{T}_k \xrightarrow{\bar{g}_k^*} \bar{G}_k$$

using Lemma 3.8. □

The next result gives an affirmative answer to question 15.7 of [3].

PROPOSITION 4.5. *There is a map*

$$T_\infty^{2n-1}(p^r) \xrightarrow{H} T_\infty^{2np-1}(p^{r+1})$$

which induces a monomorphism in cohomology with $\mathbf{Z}_p$ *coefficients.*

Proof. In fact we will construct a map

$$H_k : T_k^{2n-1}(p^r) \longrightarrow T_{k-1}^{2np-1}(p^{r+1})$$

which induces an isomorphism in H_{2np-1} when $1 \leq k \leq \infty$. In the case $k = \infty$ this implies that $H = H_\infty$ induces a monomorphism in cohomology with $\mathbf{Z}_p$ coefficients because of the Bockstein and cup product structures of these spaces.

We begin by constructing a map $\bar{\varphi}_k : G_k^{2n}(p^r) \longrightarrow G_{k-1}^{2np}(p^{r+1})$ inducing an isomorphism in H_{2np}. This is accomplished by applying Corollary 4.2 with $X = T_\infty^{2np-1}(p^{r+1})$ and $s = 1$. Using Lemma 4.4 we extend the composite

$$G_1^{2n}(p^r) \longrightarrow P^{2np+1}(p^{r+1}) \longrightarrow \Sigma T_\infty^{2np-1}(p^{r+1})$$

to $G_\infty^{2n}(p^r)$. Now define $\bar{\varphi}$ to be the composite

$$G_\infty^{2n}(p^r) \longrightarrow \Sigma T_\infty^{2np-1}(p^{r+1}) \xrightarrow{g^*} G_\infty^{2np}(p^{r+1}).$$

By cellular approximation and restriction we obtain

$$G_k^{2n}(p^r) \xrightarrow{\bar{\varphi}_k} G_{k-1}^{2np}(p^{r+1}).$$

Let θ be the composite

$$\Sigma T_k^{2n-1}(p^r) \xrightarrow{g_k^*} G_k^{2n}(p^r) \xrightarrow{\bar{\varphi}_k} G_{k-1}^{2np}(p^{r+1}) \xrightarrow{f_{k-1}} \Sigma T_{k-1}^{2np-1}(p^{r+1}).$$

Then θ induces an isomorphism in H_{2np}. Now any map $\theta : \Sigma A \longrightarrow \Sigma B$ can be factored as

$$\Sigma A \xrightarrow{\Sigma\theta^*} \Sigma\Omega\Sigma B \xrightarrow{\mathrm{ev}} \Sigma B.$$

Thus if θ induces a monomorphism in H_j, θ^* induces a monomorphism in H_{j-1}. In our case we conclude that

$$\theta^* : T_k^{2n-1}(p^r) \longrightarrow \Omega\Sigma T_{k-1}^{2np-1}(p^{r+1})$$

induces a monomorphism in H_{2np-1}. Since both groups are isomorphic to $\mathbf{Z}_{p^{r+1}}$, θ^* and hence the composite

$$T_k^{2n-1}(p^r) \xrightarrow{'\theta^*} \Omega\Sigma T_{k-1}^{2np-1}(p^{r+1}) \xrightarrow{\mu} T_{k-1}^{2np-1}(p^{r+1})$$

both induce an isomorphism in H_{2np-1}. Let H_k be this composite. □

COROLLARY 4.6. *There is a map*

$$\varphi : G_k^{2n}(p^r) \longrightarrow G_{k-1}^{2np}(p^{r+1})$$

which induces a monomorphism in cohomology for $1 \le k \le \infty$, *and a cofibration sequence*

$$G_{k-1}^{2n}(p^r) \longrightarrow G_{k+s}^{2n}(p^r) \longrightarrow G_s^{2np^k}(p^{r+k}). \tag{4A}$$

Proof. Define φ as the composite

$$G_k^{2n}(p^r) \xrightarrow{f_k} \Sigma T_k^{2n-1}(p^r) \xrightarrow{\Sigma H_k} \Sigma T_{k-1}^{2np-1}(p^{r+1}) \xrightarrow{g_{k-1}^*} G_{k-1}^{2np}(p^{r+1}).$$

The cofibration sequence follows immediately if $k = 1$. The general case is obtained by iterating φ. □

Our last result in this section gives a universal property for the spaces T_k.

PROPOSITION 4.7. *Let X be an H space of finite type,* $\varphi : P^{2n}(p^r) \longrightarrow X$ *and suppose that* $p^{r+i-1}\pi_{2np^i-1}(X; \mathbf{Z}_{p^{r+i}}) = 0$ *for* $1 \le i \le k$. *Then there is an extension*:

$$\begin{array}{ccc} P^{2n}(p^r) & \xrightarrow{\varphi} & X \\ \downarrow & \nearrow_{\hat{\varphi}} & \\ T_k^{2n-1}(p^r) & & \end{array}$$

Proof. Apply Corollary 4.2 to obtain $\varphi_k : G_k^{2n} \longrightarrow \Sigma X$ extending $\Sigma\varphi$. Then define $\hat{\varphi}$ as the composite

$$T_k^{2n-1}(p^r) \xrightarrow{g_k} \Omega G_k^{2n} \xrightarrow{\Omega\varphi_k} \Omega\Sigma X \xrightarrow{\mu} X.$$

In general we cannot expect $\hat{\varphi}$ to be an H map or to be unique. (The hypotheses are satisfied, for example, by $X = \Omega G_\infty^{2n}$. Clearly there is more than one section $T_k^{2n-1} \to \Omega G_k^{2n}$.) We conjecture that if X is homotopy associative and homotopy commutative, there will be a unique H map $\hat{\varphi}$—at least in case $k = \infty$. □

5. APPLICATION TO SPECTRA

In this section we discuss some applications of the spaces $T_\infty^{2n-1}(p^r)$. In [10], the existence of these spaces was predicted. Various considerations of the way stable homotopy is built out of unstable homotopy suggested that there should be atomic $(n-1)$ connected H spaces $T^n(p^r)$ which fit together in a spectrum equivalent to the Moore space spectrum $S^0 \cup_{p^r} e^1$. That this is true will now be seen to be the consequence of the results in the previous sections.

In this section we will write $T^{2n-1}(p^r)$ for $T_\infty^{2n-1}(p^r)$ and ignore the case $k < \infty$. We will write $T^{2n}(p^r)$ for $S^{2n+1}\{p^r\}$. This defines $T^n(p^r)$ for $n \ge 0$. The order p^r will be fixed throughout this section and suppressed from the notation.

PROPOSITION 5.1. *There are "suspension maps" $E: T^n \longrightarrow \Omega T^{n+1}$ which are H maps. Furthermore, there are commutative diagrams of H fibration sequences:*

$$
\begin{tikzcd}
 & \Omega S^{2n-1} \arrow[r, "="] \arrow[d] & \Omega S^{2n-1} \arrow[d] \\
W_n \arrow[r] \arrow[d, "="] & T^{2n-2} \arrow[r, "E"] \arrow[d] & \Omega T^{2n-1} \arrow[d] \\
W_n \arrow[r] & S^{2n-1} \arrow[r, "E^2"] & \Omega^2 S^{2n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
W_n \arrow[r] \arrow[d, "="] & S^{2n-1} \arrow[r, "E^2"] \arrow[d] & \Omega^2 S^{2n+1} \arrow[d] \\
W_n \arrow[r] & \Omega T^{2n-1} \arrow[r, "E"] \arrow[d] & \Omega T^{2n} \arrow[d] \\
 & \Omega^2 S^{2n+1} \arrow[r, "="] & \Omega^2 S^{2n+1}
\end{tikzcd}
$$

where E^2 is the usual double suspension for spheres.

Proof. Since T^{2n-1} is an H space, ΩT^{2n-1} is a homotopy commutative homotopy associative H space. Thus by [9, 3.4] each map $\phi: P^{2n-1}(p^r) \longrightarrow \Omega T^{2n-1}$ has a unique extension to an H map $E = \hat{\phi}: T^{2n-2} \longrightarrow \Omega T^{2n-1}$. Since $\Omega^2 S^{2n+1}$ is also homotopy commutative and homotopy associative, the uniqueness assertion in [4, 3.4] implies that the square

$$
\begin{tikzcd}
T^{2n-2} \arrow[r, "E"] \arrow[d] & \Omega T^{2n-1} \arrow[d] \\
S^{2n-1} \arrow[r, "E^2"] & \Omega^2 S^{2n+1}
\end{tikzcd}
$$

homotopy commutes. The induced self-map on ΩS^{2n-1} is homotopic to the identity since there is a unique H map $\Omega S^{2n-1} \longrightarrow \Omega T^{2n-1}$ extending a given map $S^{2n-2} \longrightarrow \Omega T^{2n-1}$. It follows that the horizontal fibers are homotopy equivalent.

The case of the right-hand diagram is slightly more complicated. We define E as the composite

$$T^{2n-1} \xrightarrow{g} \Omega G \xrightarrow{\Omega e} \Omega D \xrightarrow{\Omega \rho} \Omega T^{2n}.$$

The diagram clearly commutes up to homotopy. It remains to show that E is an H map. By Corollary 3.6, $\Omega G \simeq \Omega R \times T$. According to [3, 15O], the composite

$$W \longrightarrow D \xrightarrow{\rho} T^{2n}$$

is null homotopic. From diagram (3F) it follows that

$$R \xrightarrow{c} G \xrightarrow{d} T^{2n}$$

is null homotopic where $d = \rho e$. Now consider the homotopy commutative diagram:

$$
\begin{tikzcd}
\Omega G \arrow[r, "="] \arrow[d] \arrow[dd, bend right=60, "h"'] & \Omega G \arrow[r, "\Omega d"] & \Omega T^{2n} \\
T^{2n-1} \times \Omega R \arrow[r, "g \times \Omega c"] \arrow[d, "\pi_1"] & \Omega G \times \Omega G \arrow[r, "\Omega d \times \Omega d"] \arrow[u, "\mu"] & \Omega T^{2n} \times \Omega T^{2n} \arrow[u, "\mu"] \\
T^{2n-1} \arrow[r, "g"] \arrow[rr, bend right, "E"'] & \Omega G \arrow[r, "\Omega d"] & \Omega T^{2n} \arrow[u, "\iota_1"] \arrow[uu, bend right=60, "="']
\end{tikzcd}
$$

Now insert this into the following diagram:

$$\begin{tikzcd}
T^{2n-1} \times T^{2n-1} \arrow[r, "g \times g"] \arrow[d, "g \times g"'] \arrow[dd, bend right=60, "E \times E"'] & \Omega G \times \Omega G \arrow[r, "\mu"] \arrow[d, phantom, "="] & \Omega G \arrow[r, "h"] \arrow[d, "="] & T^{2n-1} \arrow[d, "g"] \arrow[dd, bend left=60, "E"] \\
\Omega G \times \Omega G \arrow[rr, "\mu"] \arrow[d, "\Omega d \times \Omega d"'] & & \Omega G \arrow[dr, "\Omega d"'] & \Omega G \arrow[d, "\Omega d"] \\
\Omega T^{2n} \times \Omega T^{2n} \arrow[rrr, "\mu"] & & & \Omega T^{2n}
\end{tikzcd}$$

and we conclude that E is an H map. □

As an immediate consequence we have proved the following.

THEOREM 5.2. *There is an H fibration sequence:*

$$\cdots \longrightarrow \Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T^{2n-1} \longrightarrow \Omega S^{2n+1}$$

where the H map π_n fits into a commutative diagram:

$$\begin{tikzcd}
\Omega^2 S^{2n+1} \arrow[r, "\Omega^2(p^r)"] \arrow[dr, "\pi_n"] & \Omega^2 S^{2n+1} \\
S^{2n-1} \arrow[u, "E^2"] \arrow[r, "p^r"] & S^{2n-1} \arrow[u, "E^2"']
\end{tikzcd} \tag{5A}$$

In particular, π_n is an H map and the fiber of π_n is a loop space. Furthermore, $H_*(T^{2n-1}; \mathbf{Z}_p)$ is a free commutative associative algebra on generators u and v of dimensions $2n - 1$ and $2n$, respectively, and $\beta^{(r)}v = u$, where $\beta^{(r)}$ is the rth Bockstein. □

The fibrations in Proposition 5.1 are best understood in terms of the following.

REFLEXITIVITY CONJECTURE 5.3. $BW_n \simeq \Omega T^{2np-1}(p)$.

There is, however, a somewhat artificial way of circumventing our lack of knowledge on this point. Define spaces $\hat{T}^n(p^r)$ as follows:

$$\hat{T}^n(p) = \begin{cases} \Omega T^n(p^r) & \text{if } n \neq 2kp - 1 \text{ or } r \neq 1 \\ BW_k & \text{if } n = 2kp - 1 \text{ and } r = 1. \end{cases}$$

Then we have the following theorem.

THEOREM 5.4. *There are H fibrations*

$$\cdots \xrightarrow{P} \hat{T}^{2n}(p^r) \xrightarrow{E} \Omega \hat{T}^{2n+1}(p^r) \xrightarrow{H} \Omega \hat{T}^{2np+q+1}(p)$$
$$\cdots \xrightarrow{P} \hat{T}^{2n-1}(p^r) \xrightarrow{E} \Omega \hat{T}^{2n}(p^r) \xrightarrow{H} \Omega \hat{T}^{2np-1}(p)$$

and the resulting spectrum $\{\hat{T}^n\}$ satisfies

$$E^{-1}\{\hat{T}^n\} \simeq \{T^n\} \simeq S^0 \cup_{p^r} e^1.$$

Corresponding to this filtration there is an EHP spectral sequence as usual.

One should notice the formal analogy between the EHP fibrations above and the classical EHP fibrations

$$S^{2n-1} \xrightarrow{E} \Omega\hat{S}^{2n} \xrightarrow{H} \Omega S^{2np-1}$$
$$\hat{S}^{2n} \xrightarrow{E} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1}.$$

The general theory suggests that for a spectrum of type $V(m)$, when it exists and is an associative ring spectrum, there will be a similar unstable development with q replaced by $q_{m+1} = 2(p^{m+1} - 1)$. Thus the sphere spectrum corresponds to $V(-1)$.

Finally, we wish to state the following.

UNIVERSALITY CONJECTURE 5.5. (a) *T^n is a homotopy commutative and homotopy associative H space.*
(b) *Suppose X is a homotopy commutative, homotopy associative H space and $\phi: P^{n+1}(p^r) \longrightarrow X$. Then there is a unique H map $\hat{\phi}: T^n(p^r) \longrightarrow X$ extending ϕ.*

In the case that n is even, the universality conjecture is true. See [11, 3.4] and [13, 0.2]. One consequence of the universality conjecture is that the p^rth power map on T^n is null homotopic—a strengthening of [3, Conjecture 15.3] in the odd case and [13, 0.2] in the even case. This follows immediately since the p^rth power map and the constant map are H maps: $T^n \longrightarrow T^n$ extending a null homotopic map: $P^{n+1}(p^r) \longrightarrow T^n$.

APPENDIX

In this section we will state and prove some results of a general nature which have been used throughout. Many of these results have independent interest.

Our first result is due to Genea [7].

LEMMA A1. (a) *A space G is a co-H space iff the evaluation map* $\mathrm{ev}: \Sigma\Omega G \longrightarrow G$ *has a right homotopy inverse* $s: G \longrightarrow \Sigma\Omega G$.

(b) *If G is a co-H space and $f: \Sigma W \longrightarrow G$, there is a well-defined obstruction $c(f) \in [\Sigma W, \Sigma(\Omega G \wedge \Omega G)]$ such that f is a co-H map iff $c(f) = 0$. Furthermore, if $\alpha: V \longrightarrow W$, $c(f \circ \Sigma\alpha) = c(f) \circ \Sigma\alpha$. In particular, $c(mf) = mc(f)$ for any integer m if W is a co-H space.*

(c) *In case $f: \Sigma W \longrightarrow G$ is a co-H map, there is a co-H space structure on $G \cup_f C\Sigma W$ compatible with the structure on G.*

Proof. Part (a) follows from the pull back diagram of fibrations:

$$\begin{array}{ccc} \Sigma\Omega G \wedge \Omega G & \longrightarrow & \Sigma\Omega G \wedge \Omega G \\ \downarrow{\scriptstyle\delta} & & \downarrow \\ \Sigma\Omega G & \xrightarrow{e_1 + e_2} & G \vee G \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow \\ G & \xrightarrow{\Delta} & G \times G \end{array} \tag{1}$$

where Δ is the diagonal and e_i is the composition of ev with the injection in the ith factor.

If $f:\Sigma W \longrightarrow G$, the difference between the two composites in the diagram

$$\begin{tikzcd} \Sigma W \arrow[r, "f"] \arrow[d] & G \arrow[d] \\ \Sigma W \vee \Sigma W \arrow[r, "f \vee f"] & G \vee G \end{tikzcd}$$

factors through the fiber of the inclusion $G \vee G \subset G \times G$. This factorization is unique since $\Omega\Sigma(\Omega G \wedge \Omega G) \longrightarrow \Omega(G \vee G)$ has a left homotopy inverse. This defines $c(f)$ and proves part (b). For part (c), we define the co-H structure by the cofibration sequence

$$\begin{tikzcd} \Sigma W \vee \Sigma W \arrow[r] & G \vee G \arrow[r] & C_f \vee C_f \\ \Sigma W \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & C_f \arrow[u] \end{tikzcd}$$

where $C_f = G \cup_f C\Sigma W$ is the mapping cone. Here care is taken to use a primitive homotopy for the left-hand square. See [7] for details. □

LEMMA A2. *Suppose G is a simply connected co-H space. Then the composition*

$$\Sigma(\Omega G \wedge \Omega G) \xrightarrow{\delta} \Sigma\Omega G \xrightarrow{\mu} \Sigma\Omega G/G$$

has a right homotopy inverse where δ is the map in diagram (1).

Proof. Suppose we are given a fibration

$$F \xrightarrow{\delta'} E \xrightarrow{\pi} B$$

with E a simply connected co-H space. Suppose, in addition, we are given a section $s: B \longrightarrow E$. Then we obtain an equivalence $E \simeq B \vee C_s$ from the diagram of cofibration sequences:

$$\begin{tikzcd} B \arrow[r, "s"] \arrow[d, "="'] & E \arrow[r, "\mu'"] \arrow[d, "\pi + \mu'"] & C_s \arrow[d, "="] \\ B \arrow[r] & B \vee C_s \arrow[r] & C_s \end{tikzcd}$$

where μ' is the projection onto the mapping cone C_s of s, and $\pi + \mu'$ is defined via the co-H structure on E. A homotopy inverse for $\pi + \mu'$ gives a map $\varphi: C_s \longrightarrow E$ with $\pi\varphi \sim *$ and $\mu'\varphi \sim 1$. This defines a right inverse γ to $\mu'\delta'$ as follows:

$$\begin{tikzcd} & F \arrow[d, "\delta'"] & \\ C_s \arrow[r, "\varphi"] \arrow[ur, "\gamma"] & E \arrow[r, "\mu'"] \arrow[d, "\pi"] & C_s \\ & B & \end{tikzcd}$$

Apply this to the fibration $\Sigma(\Omega G \wedge \Omega G) \xrightarrow{\delta} \Sigma\Omega G \xrightarrow{\text{ev}} G$.

LEMMA A3. *Suppose X is simply connected, $Y \in \mathscr{W}_r^{r+k}$, and $f: X \longrightarrow Y$ induces a split monomorphism in integral homology in dimensions $\leq n$. Then there is a retract Y' of Y such that the composition $X \longrightarrow Y \longrightarrow Y'$ induces isomorphisms in integral homology in dimensions $\leq n$.*

Proof. Choose a minimal set of generators for $\operatorname{im} f_*$ in dimensions $\leq n$. Since $Y \in \mathscr{W}_r^{r+k}$, $h: \pi_*(Y) \longrightarrow H_*(Y)$ is a split epimorphism. Choose corresponding generators, of the same order, for $\pi_*(Y)$. Let Y' be the wedge of the corresponding Moore spaces, and $i: Y' \longrightarrow Y$ be the natural map. Then $\operatorname{im} i_* = \operatorname{im} f_*$. By making a corresponding construction for the complementary summand in dimensions $\leq n$ and all of $H_*(Y)$ in dimensions $>n$, we obtain an equivalence $Y \simeq Y' \vee Y''$; thus Y' is a retract of Y and the composite $X \xrightarrow{f} Y \longrightarrow Y'$ is an isomorphism in dimensions $\leq n$. □

LEMMA A4. *Suppose $F \xrightarrow{i} E \xrightarrow{\pi} B$ is a fibration with E and B simply connected and F of finite type. Suppose that there is a right inverse $\varphi: \Sigma E \longrightarrow \Sigma F$ for Σi. Then $F \simeq E \times \Omega B$.*

Proof. We begin by considering the diagram of fibration sequences:

$$\begin{array}{ccc} \Omega B & \xrightarrow{=} & \Omega B \\ \downarrow & & \downarrow \\ F \times \Omega B & \xrightarrow{a} & F \\ \downarrow & & \downarrow \\ F & \xrightarrow{i} & E \end{array}$$

where a is the action map from the original fibration. Let k be a field and $I_* \subset H_*(F{:}\ k)$ be the desuspended image of φ_*. Then $i_*(I_*) = H_*(E; k)$ so the submodule $I_* \otimes H_*(\Omega B; k)$ of the E^2 term of the Serre spectral sequence for the left-hand fibration maps onto the corresponding E^2 term for the right-hand fibration. Since the left-hand spectral sequence collapses, the right-hand one does as well, and $a_*(I_* \otimes H_*(\Omega B; k)) = H_*(F; k)$.

Now construct a map $\theta: \Sigma(E \times \Omega B) \to \Sigma F$ by

$$\Sigma(E \times \Omega B) \simeq \Sigma E \wedge (\Omega B)^+ \vee \Sigma\Omega B \xrightarrow{\varphi \wedge 1 \vee 1} \Sigma F \wedge (\Omega B)^+ \vee \Sigma\Omega B \simeq \Sigma(F \times \Omega B) \xrightarrow{\Sigma a} \Sigma F.$$

Clearly $\theta_*(\xi \otimes \eta) = a_*(\varphi_*(\xi) \otimes \eta)$. It follows that θ_* is onto. However, since the right-hand spectral sequence collapses, $H_*(F; k) \cong H_*(E; k) \otimes H_*(\Omega B; k)$. Since both groups involved are finite-dimensional vector spaces of the same dimension, θ_* is an isomorphism. Since this is true for each field k, θ is a homotopy equivalence; i.e. $\Sigma(E \times \Omega B) \simeq \Sigma F$. From this we see that $\Sigma\Omega B \longrightarrow \Sigma F$ has a left homotopy inverse $\gamma: \Sigma F \longrightarrow \Sigma\Omega B$, as in [5, Lemma 1.6], we construct a left inverse for $\Omega B \longrightarrow F$ as the adjoint of $\Sigma F \xrightarrow{\gamma} \Sigma\Omega B \xrightarrow{\text{ev}} B$, and hence an equivalence $F \simeq E \times \Omega B$. □

LEMMA A5. *Let $X_0 \subseteq X_1 \subseteq \cdots$ be a nested sequence of spaces where $X_i = X_{i-1} \cup_{\alpha_i} CP_i$ where $P_i = P^{n_i}(p^{k_i})$ is a Moore space, $n_i > n_{i-1} + 1$, and $X_0 = \Sigma P_0$. Suppose α_i is essential for each $i > 0$. Then $X = \bigcup X_i$ is atomic.*

Proof. Suppose $f: X \longrightarrow X$ is cellular and $f_i = f|_{X_i}: X_i \longrightarrow X_i$ is an equivalence for $i < m$. Consider the diagram

$$\begin{tikzcd} P_m \arrow[r, "\alpha_m"] \arrow[d, "f_{m-1}"] & X_{m-1} \arrow[r] \arrow[d, "f_{m-1}"] & X_m \arrow[d, "f_m"] \\ P_m \arrow[r, "\alpha_m"] & X_{m-1} \arrow[r] & X_m \end{tikzcd} \tag{2}$$

The map $\tilde{f}_{m-1}$ exists since for X simply connected, the rows of (2) are fibration sequences in dimensions $\leq n_m$. Now $\tilde{f}_{m-1}$ induces multiplication by λ_m in π_{n_i-1} for some $\lambda_m \in \mathbf{Z}$. Since f_{m-1} is an equivalence, $\lambda_m \alpha_m$ has the same order as α_m. Since α_m is essential, λ_m is prime to p; thus $\tilde{f}_{m-1}$ is an equivalence. It follows that f_m is an equivalence. □

LEMMA A6. *Suppose that we have a pull back diagram of fibrations:*

$$\begin{tikzcd} F \arrow[r, "="] \arrow[d, "i"] & F \arrow[d, "i'"] \\ E \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & X \cup CA \end{tikzcd}$$

and i is null homotopic. Let $\theta: A \times F \to E$ be a trivialization of the pull back to A. Choose $\theta' \sim \theta$ with $\theta'(* \times F) = *$. Then $E' \cong E \cup_{\theta'} C(A \wedge F^+)$.

Proof. By [8, 1b] we know that E' has the same homotopy type as $E \cup_\theta (CA) \times F$, where CA is the unreduced cone on A. Clearly $E \cup_\theta (CA) \times F \simeq E \cup_{\theta'} (C^*A) \times F$ where C^* is the reduced cone. However, there is a homeomorphism

$$E \cup_{\theta'} (C^*A) \times F \equiv E \cup_{\theta'} C^*(A \wedge F^+)$$

so we are done. □

We also have the following corollary.

COROLLARY A7. *Let $A \subset X$ be simply connected and suppose $\Omega X \longrightarrow \Omega(X \cup CA)$ has a right homotopy inverse. Let F be the homotopy fiber of the inclusion $X \longrightarrow X \cup A$. Then $F \simeq A \wedge (\Omega(X \cup CA))^+$.*

Proof. Apply A6 to the diagram:

$$\begin{tikzcd} \Omega(X \cup CA) \arrow[r, "="] \arrow[d, "*"] & \Omega(X \cup CA) \arrow[d] \\ F \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & X \cup CA \end{tikzcd}$$

to conclude that the cofiber of $A \wedge (\Omega(X \cup CA))^+ \longrightarrow F$ is contractable. This map consequently induces an isomorphism on homology. Since A is simply connected $\pi_1(X) = \pi_1(X \cup CA)$ so $\pi_1(F) = 0$. Since

$$A \wedge (\Omega(X \cup CA))^+ \simeq [A \times \Omega(X \cup CA)] \cup C(\Omega(X \cup CA))$$

this space is also simply connected and consequently the above map is a homotopy equivalence. □

LEMMA A8. *Suppose $F \longrightarrow E \longrightarrow B$ is a fibering such that the Serre spectral sequence for the $\mathbf{Z}_p$ homology collapses; i.e., $E^2_{s,t} \simeq E^\infty_{s,t}$. Then there is an exact sequence:*

$$0 \longrightarrow PH_*(F; \mathbf{Z}_p) \longrightarrow PH_*(E; \mathbf{Z}_p) \longrightarrow PH_*(B; \mathbf{Z}_p)$$

where PH_* is the submodule of primitive elements in H_*.

Proof. The Serre spectral sequence is a spectra sequence of coalgebras and

$$PE^2_{s,t} = \begin{cases} PH_s(B; \mathbf{Z}_p) & \text{if } t = 0 \\ PH_t(F; \mathbf{Z}_p) & \text{if } s = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now write $H_k(E; \mathbf{Z}_p) = F_k \supset F_{k-1} \supset \cdots \supset F_0$ with $F_i/F_{i-1} = E^\infty_{i,k-i} = E^2_{i,k-i}$. Suppose $x \in F_i$ is primitive. Then so is $\bar{x} \in F_i/F_{i-1}$. Thus $\bar{x} = 0$ or $i = 0$ or $i = k$. Now suppose $x \in PH_*(E; \mathbf{Z}_p)$ and $\pi_*(x) = 0$. Then $x \in F_{k-1}$ and hence $x \in F_0$; i.e., $x = i_*(y)$. Since i_* is a monomorphism, y is primitive which proves exactness in the middle. Clearly $i_*(PH_*(F; \mathbf{Z}_p)) \subset PH_*(E; \mathbf{Z}_p)$ so we are done. □

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