Thus, when it holds, the localization theorem for A implies a calculation of both $M_*(EG_+ \wedge_G X)$ and $M^*(EG_+ \wedge_G X)$ for all split A-modules M and all based G-spaces X.

We must still define the algebraic construction whose brave new counterpart is given by our completion functors. Returning to the algebraic context of Section 1, we want to define a suitable dual to local cohomology. Since local cohomology is obtained as $H^*(K \otimes M)$ for a suitable complex K, we expect to have to take $H_*(\mathrm{Hom}(K, M))$. However this will be badly behaved unless we first replace K by a complex of projective R-modules. Thus we choose an R-free complex $PK^\bullet(I)$ and a homology isomorphism $PK^\bullet(I) \longrightarrow K^\bullet(I)$. Since both complexes consist of flat modules we could equally well have used $PK^\bullet(I)$ in the definition of local cohomology. For finitely generated ideals $I = (\alpha_1, \cdots, \alpha_n)$, we take tensor products and define $PK^\bullet(I) = PK^\bullet(\alpha_1) \otimes \ldots \otimes PK^\bullet(\alpha_n)$; independence of generators follows from that of $K^\bullet(I)$.

We may then define local homology by

$$H_*^I(R; M) = H_*(\mathrm{Hom}(PK^\bullet(I), M)). \tag{5.6}$$

We often omit R from the notation. Because we chose a projective complex we obtain a third quadrant univeral coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s(H_I^{-t}(R), M) \Rightarrow H^I_{-t-s}(R; M)$$

with differentials $d_r : E_r^{s,t} \to E_r^{s+t,t-r+1}$ that relates local cohomology to local homology.

It is not hard to check from the definition that if R is Noetherian and M is either free or finitely generated, then $H_0^I(R; M) \cong M_I^\wedge$, and one may also prove that in these cases the higher local homology groups are zero. It follows that $H_*^I(R; M)$ calculates the left derived functors of the (not necessarily right exact) I-adic completion functor. In fact, this holds under weaker hypotheses on R than that it be Noetherian.

Returning to our topological context, it is now clear that if R is a commutative S_G-algebra and I is a finitely generated ideal in R_*^G, then the completion functor $M_I^\wedge$ on R-modules is the brave new analogue of local homology: we have the spectral sequence (5.2).

J.P.C. Greenlees and J.P. May. Derived functors of I-adic completion and local homology. J. Algebra **149** (1992), 438-453.

6. A proof and generalization of the localization theorem

To prove systematically that the map κ_A of (4.7) is a weak equivalence we need to know that when we restrict the map κ of (4.4) to a subgroup H, we obtain an analogous map of H-spectra. Write I_H for the augmentation ideal $\mathrm{Ker}(\mathrm{res}^H_1 \subset R^H_*)$. Even for cohomotopy it is not true that $\mathrm{res}(I_G) = I_H$, but in that case they do have the same radical. To give a general result, we must assume that this holds.

ASSUMPTION 6.1. For all subgroups $H \subseteq G$

$$\sqrt{\mathrm{res}(I_G)} = \sqrt{I_H}.$$

For theories such as cohomotopy and K-theory, where we understand all of the primes of R^G_*, this is easy to verify. Note that both (4.2) and (6.1) are assumptions on R that have nothing to do with A. We need an assumption that relates R^G_* to A^G_*. Let $J = J_G$ be the augmentation ideal in A^G_*. The unit $R \longrightarrow A$ induces a homomorphism of rings $R^G_* \longrightarrow A^G_*$ that is compatible with restrictions to subgroups, hence we have an inclusion of ideals $I \cdot A^G_* \subseteq J$.

ASSUMPTION 6.2. The augmentation ideals of R^*_G and A^*_G are related by

$$\sqrt{I \cdot A^G_*} = \sqrt{J}.$$

Recall from (4.8) that $A^{G,R}_*(M) = \pi^G_*(M \wedge_R A)$. The final ingredient of our proof will be the existence of Thom isomorphisms

$$A^{G,R}_*(S^V \wedge M) \cong A^{G,R}_*(S^{|V|} \wedge M) \tag{6.3}$$

of $A^{G,R}_*$-modules for all complex representations V and R-modules M. For example, with $A = R$, homotopical bordism and K-theory have such Thom isomorphisms. Cohomotopy does not, and that is why our proof (and the theorem) fail in that case.

THEOREM 6.4 (LOCALIZATION). If A is an R-ring spectrum such that, for all subgroups H of G, the theories $A^{H,R}_*(\cdot)$ admit Thom isomorphisms and if assumptions (4.2), (6.1), and (6.2) hold for G and for all of its subgroups, then the localization theorem holds for A.

PROOF. We have observed that the cofiber of κ is equivalent to $\tilde{E}G \wedge K(I)$. We must prove that $\tilde{E}G \wedge K(I) \wedge_R A \simeq *$. We proceed by induction on the size of the group. By Assumption (6.1) and Lemma 4.3, we see that

$$(\tilde{E}G \wedge K(I_G))|_H \simeq \tilde{E}H \wedge K(I_H).$$

Thus our inductive assumption implies that

$$G/H_+ \wedge \tilde{E}G \wedge K(I) \wedge_R A \simeq *$$

for all proper subgroups $H \subset G$. Arguing exactly as in Carlsson's first reduction, XX.4.1, of the Segal conjecture for finite p-groups, we find that it suffices to prove that $\tilde{E}\mathscr{P} \wedge K(I) \wedge_R A \simeq *$. Indeed, $(\tilde{E}\mathscr{P})^G = S^0$ and $\tilde{E}\mathscr{P}/S^0$ can be constructed from cells $G/H_+ \wedge S^n$ with H proper. Therefore

$$(\tilde{E}\mathscr{P}/S^0) \wedge \tilde{E}G \wedge K(I) \wedge_R A \simeq *$$

and thus

$$\tilde{E}G \wedge K(I) \wedge_R A \simeq \tilde{E}\mathscr{P} \wedge \tilde{E}G \wedge K(I) \wedge_R A.$$

However, the map $\tilde{E}\mathscr{P} \longrightarrow \tilde{E}\mathscr{P} \wedge \tilde{E}G$ induced by the map $S^0 \longrightarrow \tilde{E}G$ is a G-equivalence by a check of fixed point spaces.

Now, if G is finite, consider the reduced regular representation V. As we observed in the proof of the Segal conjecture, $S^{\infty V} = \operatorname{colim} S^{kV}$ is a model for $\tilde{E}\mathscr{P}$ since $V^H \neq 0$ if H is proper and $V^G = 0$. For a general compact Lie group G, we write $S^{\infty V}$ for the colimit of the spheres S^V, where V runs over a suitably large set of representations V such that $V^G = \{0\}$, for example all such V that are contained in a complete G-universe U. Again, $S^{\infty V}$ is a model for $\tilde{E}\mathscr{P}$.

At this point we must recall how Thom isomorphisms give rise to Euler classes $\chi(V) \in A^{G,R}_{-|V|}$. Indeed the inclusion $e : S^0 \longrightarrow S^V$ and the Thom isomorphism give a natural map of $A^{G,R}_*$-modules

$$A^{G,R}_*(X) \xrightarrow{e_*} A^{G,R}_*(S^V \wedge X) \cong A^{G,R}_*(S^{|V|} \wedge X) \cong A^{G,R}_{*-|V|}(X),$$

and this map is given by multiplication by $\chi(V)$. Thus, for finite G,

$$\begin{aligned} A^{G,R}_*(S^{\infty V} \wedge K(I)) &= \operatorname{colim}_k A^{G,R}_*(S^{kV} \wedge K(I)) \\ &= \operatorname{colim}_k (A^{G,R}_*(K(I)), \chi(V)) \\ &= A^{G,R}_*(K(I))[\chi(V)^{-1}]. \end{aligned}$$

Here $\chi(V)$ is in J since e is nonequivariantly null homotopic. Therefore, using Assumption 6.2 and Remark 3.3, we see that

$$H^*_I(R^G_*; N)[\chi(V)^{-1}] \cong H^*_J(A^G_*; N)\left[\chi(V)^{-1}\right] = 0$$

for any A^G_*-module N. From the spectral sequence (3.2), we deduce that

$$A^{G,R}_*(S^{\infty V} \wedge K(I)) = 0.$$

A little elaboration of the argument gives the same conclusion when G is a general compact Lie group. Since $S^{\infty V}$ is H-equivariantly contractible for all proper subgroups H, this shows that $S^{\infty V} \wedge K(I) \wedge_R A \simeq *$, as required. □

There is a substantial generalization of the theorem that admits virtually the same proof. Recall from V.4.6 that, for a family $\mathscr{F}$, we have the cofiber sequence

$$E\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{F}.$$

We discussed family versions of the Atiyah-Segal completion theorem in XIV§6 and of the Segal conjecture in XX§§1-3. As in those cases, we define

$$I\mathscr{F} = \cap_{H \in \mathscr{F}} \operatorname{Ker}(\operatorname{res}^G_H : R^G_* \longrightarrow R^H_*).$$

Arguing exactly as above, we obtain a map

$$\kappa : E\mathscr{F}_+ \wedge R \longrightarrow K(I\mathscr{F}). \tag{6.5}$$

DEFINITION 6.6. The '$\mathscr{F}$-localization theorem' holds for an R-ring spectrum A if

$$\kappa_A = \kappa \wedge \mathrm{id} : E\mathscr{F}_+ \wedge A = E\mathscr{F}_+ \wedge R \wedge_R A \longrightarrow K(I\mathscr{F}) \wedge_R A$$

is a weak equivalence of R-modules, that is, if it is an isomorphism in $G\mathscr{D}_R$.

We combine and record the evident analogs of Lemmas 4.10 and 5.4.

LEMMA 6.7. If the $\mathscr{F}$-localization theorem holds for the R-ring spectrum A, then the maps

$$E\mathscr{F}_+ \wedge M \longrightarrow \Gamma_{I\mathscr{F}}(M), \qquad \tilde{E}\mathscr{F} \wedge M \longrightarrow M[I\mathscr{F}^{-1}],$$

and

$$M^\wedge_{I\mathscr{F}} = F_R(K(I\mathscr{F}), M) \longrightarrow F_R(E\mathscr{F}_+ \wedge R, M) \cong F(E\mathscr{F}_+, M)$$

are isomorphisms in $G\mathscr{D}_R$ for all A-modules M.

A family $\mathscr{F}$ in G restricts to a family $\mathscr{F}|_H = \{K | K \in \mathscr{F} \text{ and } K \subset H\}$, and Assumptions 4.2, 6.1, and 6.2 admit evident analogs for $I\mathscr{F}$.

THEOREM 6.8 (F-LOCALIZATION). If A is an R-ring spectrum such that, for all subgroups H of G, the theories $A^{R|_H}_*(\cdot)$ admit Thom isomorphisms and if, for a given family $\mathscr{F}$, the $\mathscr{F}$ versions of assumptions (4.2), (6.1), and (6.2) hold for G and for all of its subgroups, then the $\mathscr{F}$-localization theorem holds for A.

PROOF. Here we must prove that $\tilde{E}\mathscr{F} \wedge K(I\mathscr{F}) \wedge_R A \simeq *$, and we assume that $G \notin \mathscr{F}$ since otherwise $\tilde{E}\mathscr{F} \simeq *$. As in the proof of the localization theorem, since the evident map $\tilde{E}\mathscr{P} \longrightarrow \tilde{E}\mathscr{P} \wedge \tilde{E}\mathscr{F}$ is a G-equivalence, the problem reduces inductively to showing that $\tilde{E}\mathscr{P} \wedge K(I\mathscr{F}) \wedge_R A \simeq *$. We take $S^{\infty V}$ as our model for $\tilde{E}\mathscr{P}$ and see that, since $V^H = \{O\}$ for all $H \neq G$, $\chi(V) \in J\mathscr{F}$. The rest is the same as in the proof of the localization theorem. □

REMARK 6.9. It is perhaps of philosophical interest to note that the localization theorem is true for all R that satisfy (4.2) and (6.1) *provided that we work with $RO(G)$-graded rings*. Indeed the proof is the same except that instead of using the integer graded element $\chi(V) \in R^G_{-|V|}$ we must use $e(V) = e_*(1) \in R^G_{-V}$. The conclusion is only that there are spectral sequences

$$H^*_I(R^G_*) \Rightarrow R^G_*(EG_+)$$

and so forth, where $RO(G)$-grading of R^G_* is understood. In practice this theorem is not useful because the $RO(G)$-graded coefficient ring is hard to compute and is usually of even greater Krull dimension than the integer graded coefficent ring R^G_*. The Thom isomorphisms allow us to translate the $RO(G)$-graded augmentation ideal into its integer graded counterpart.

7. The application to K-theory

We can apply the $\mathscr{F}$-localization theorem to complex and real periodic equivariant K-theory in two quite different ways. The essential point is that Bott periodicity clearly gives the Thom isomorphisms necessary for both applications (see XIV§3). Unfortunately, for entirely different reasons, both applications are at present limited to finite groups.

First, we recall from XXII.6.13 that, for finite groups G, complex and real equivariant K-theory are known to be represented by commutative S_G-algebras. In view of Bott periodicity, we may restrict attention to the (complex or real) representation ring of G regarded as the subring of degree zero elements of K^G_* or KO^G_* (compare Remark 3.3), and our complete understanding of these rings makes verification of the $\mathscr{F}$ versions of (4.2) and (6.1) straightforward. In fact, these verifications work for arbitrary compact Lie groups G. The following theorem would hold in that generality if only we knew that K_G and KO_G were represented by commutative S_G-algebras in general. For this reason, although the completion theorem is known for all compact Lie groups, the localization theorem is only known for finite groups. The problem is that, at this writing, equivariant infinite loop space theory has not yet been developed for compact Lie groups of equivariance.

THEOREM 7.1. Let G be finite. Then, for every family $\mathscr{F}$, the $\mathscr{F}$-localization theorem holds for K_G regarded as a K_G-algebra, and similarly for KO_G.

Second, we have the first author's original version of the $\mathscr{F}$-localization theorem for K-theory. For that version, we regard K_G and KO_G as S_G-ring spectra. Here we may restrict attention to the Burnside ring of G regarded as the subring of degree zero elements of $\pi_*^G(S_G)$. Again, when G is finite, our complete understanding of $A(G)$ makes verification of the $\mathscr{F}$ versions of (4.2) and (6.1) straightforward, and we observed in and after XXI.5.3 that the $\mathscr{F}$ version of (6.2) holds. Note, however, that $A(G)$ is not Noetherian for general compact Lie groups, so that (4.2) and (6.1) are not available to us in that generality. Moreover, $A(G)$ and $R(G)$ are not closely enough related for (6.2) to hold. For example, the augmentation ideal of $A(G)$ is zero when G is a torus.

THEOREM 7.2. Let G be finite. Then, for every family $\mathscr{F}$, the $\mathscr{F}$-localization theorem holds for K_G regarded as an S_G-ring spectrum, and similarly for KO_G.

In the standard case $\mathscr{F} = \{e\}$, we explained in XXI§5 how Tate theory allows us to process the conclusions of the theorems to give an explicit computation of $K_*(BG)$; see XXI.5.4. The following references give further computational information. A comment on the relative generality of the two theorems is in order. The first only gives information about K_G-modules of the brave new sort, whereas the second gives information about K_G-module spectra of the classical sort. However, a remarkable result of Wolbert shows that the nonequivariant implications are the same: every classical K-module spectrum is weakly equivalent to the underlying spectrum of a brave new K-module.

J. P. C. Greenlees. K homology of universal spaces and local cohomology of the representation ring. Topology 32(1993), 295-308.

J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoir American Math. Soc. No. 543. 1995.

J. Wolbert. Toward an algebraic classification of module spectra. Preprint, 1995. University of Chicago. (Part of 1996 PhD thesis in preparation.)

8. Local Tate cohomology

When the $\mathscr{F}$-localization theorem holds, it implies good algebraic behaviour of the $\mathscr{F}$-Tate spectrum. We here explain what such good behaviour is by defining the algebraic ideal to which the Tate spectrum aspires: the local Tate cohomology groups of a module. We proceed by strict analogy with the construction of the topological $\mathscr{F}$-Tate spectrum,

$$t_{\mathscr{F}}(k) = F(E\mathscr{F}_+, k) \wedge \tilde{E}\mathscr{F}.$$

Thus, again working in the algebraic context of Section 1, we define the local Tate cohomology groups to be

$$\hat{H}^*_I(R; M) = H^*(\mathrm{Hom}(PK^\bullet(I), M) \otimes P\check{C}^\bullet(I)). \tag{8.1}$$

Here $P\check{C}^\bullet(I)$ is the projective Čech complex, which is defined by the algebraic fiber sequence

$$PK^\bullet(I) \longrightarrow R \longrightarrow P\check{C}^\bullet(I) \tag{8.2}$$

of chain complexes. There results a local Tate spectral sequence of the form

$$E_2^{*,*} = \check{H}^*_I(H^I_*(R; M)) \Rightarrow \hat{H}^*_I(R; M).$$

In favorable cases this starts with the Čech cohomology of the derived functors of I-adic completion.

The usefulness of the definition becomes apparent from the form that periodicity takes in this manifestation of Tate theory. It turns out that unexpectedly many elements of R induce isomorphisms of the R-module $\hat{H}^*_I(R; M)$. It is simplest to state this formally when R has Krull dimension 1.

Theorem 8.3 (Rationality). If R is Noetherian and of Krull dimension 1, then multiplication by any non-zero divisor of R is an isomorphism on $\hat{H}^*_I(R; M)$.

The Burnside ring $A(G)$ and the representation ring $R(G)$ of a finite group G are one dimensional Noetherian rings of particular topological interest.

Corollary 8.4. Let G be finite. For any ideal I of $A(G)$ and any $A(G)$-module M, $\hat{H}^*_I(A(G); M)$ is a rational vector space.

Corollary 8.5. Let G be finite. For any ideal I of $R(G)$ and any $R(G)$-module M, $\hat{H}^*_I(R(G); M)$ is a rational vector space.

Returning to our S_G-algebra R and its modules M, we define the 'I-local Tate spectrum' of M for a finitely generated ideal $I \subset R^G_*$ by

$$t_I(M) = F_R(K(I), M) \wedge_R \check{C}(I). \tag{8.6}$$

It is then immediate that there is a spectral sequence

$$E_2^{s,t} = \hat{H}_I^s(R_G^*; M_G^*)^t \Rightarrow \pi_{-s-t}^G(t_I(M)). \tag{8.7}$$

In particular, we may draw topological corollaries from Corollaries 8.4 and 8.5.

COROLLARY 8.8. Let G be finite. For any ideal I in $A(G) = \pi_0^G(S_G)$ and any G-spectrum E, $t_I(E)$ is a rational G-spectrum.

COROLLARY 8.9. Let G be finite. For any ideal I in $R(G) = \pi_0^G(K_G)$ and any K_G-module M, $t_I(M)$ is a rational G-spectrum.

Now assume the $\mathscr{F}$ version of (4.2). Let A be an R-ring spectrum and consider the diagram

$$\begin{array}{ccccc} E\mathscr{F}_+ \wedge A & \longrightarrow & S^0 \wedge A & \longrightarrow & \tilde{E}\mathscr{F} \wedge A \\ \downarrow{\scriptstyle \kappa_A} & & \| & & \downarrow{\scriptstyle \tilde{\kappa}_A} \\ K(I\mathscr{F}) \wedge_R A & \longrightarrow & A & \longrightarrow & \check{C}(I\mathscr{F}) \wedge_R A. \end{array}$$

If the $\mathscr{F}$-localization theorem holds for A, then κ_A and $\tilde{\kappa}_A$ are weak equivalences of R-modules. We may read off remarkable implications for the Tate spectrum $t_{\mathscr{F}}(M)$ of any A-module spectrum M. If κ_A is a weak equivalence, this $\mathscr{F}$-Tate spectrum is equivalent to the $I\mathscr{F}$-local Tate spectrum: a manipulation of isotropy groups is equivalent to a manipulation of ideals in brave new commutative algebra.

THEOREM 8.10. If the $\mathscr{F}$-localization theorem holds for the R-ring spectrum A, then the $\mathscr{F}$-Tate and $I\mathscr{F}$-local Tate spectra of any A-module spectrum M are equivalent:

$$t_{\mathscr{F}}(M) \simeq t_{I\mathscr{F}}(M).$$

PROOF. Since $F_R(X, M)$ is an A-module for any R-module X, Lemma 6.7 implies that all maps in the following diagram are weak equivalences of R-modules:

$$\begin{array}{ccc}
 & & t_{\mathscr{F}}(M) \\
 & & \| \\
F_R(K(I\mathscr{F}), M) \wedge_R R \wedge \tilde{E}\mathscr{F} & \longrightarrow & F_R(E\mathscr{F}_+ \wedge R, M) \wedge_R R \wedge \tilde{E}\mathscr{F} \\
\downarrow & & \downarrow \\
F_R(K(I\mathscr{F}), M) \wedge_R \check{C}(I\mathscr{F}) & \longrightarrow & F_R(E\mathscr{F}_+ \wedge R, M) \wedge_R \check{C}(I\mathscr{F}). \\
\| & & \\
t_{I\mathscr{F}}(M) & & \square
\end{array}$$

Theorem 8.3 gives a striking consequence.

COROLLARY 8.11. Assume that R^G_* is Noetherian of dimension 1 and $\mathbb{Z}$-torsion free. If the $\mathscr{F}$-localization theorem holds for an R-ring spectrum A, then the $\mathscr{F}$-Tate spectrum $t_{\mathscr{F}}(M)$ is rational for any A-module M.

REMARK 8.12. Upon restriction to the Burnside ring $A(G) = \pi^G_0(S_G)$, the corollary applies to $R = S_G$. In this case it has a converse: if the completion theorem holds for A and $t_{\mathscr{F}}(A)$ is rational, then the localization theorem holds for A. The proof (which is in our memoir on Tate cohomology) uses easy formal arguments and the fact that $\kappa : E\mathscr{F}_+ \wedge S_G \longrightarrow K(I\mathscr{F})$ is a rational equivalence.

We should comment on analogues of Corollary 8.11 in the higher dimensional case. The essence of Theorem 8.10 is that if the localization theorem holds for A, then the Tate spectrum of an A-module M is algebraic and is therefore dominated by the behaviour of the local Tate cohomology groups $\hat{H}^*_I(R^*_G; M^*_G)$ via the spectral sequence (8.7). Now these groups are modules over the ring $\hat{H}^*_I(R^*_G)$, so an understanding of the prime ideal spectrum of this ring is fundamental. For example, the first author's proof of the Rationality Theorem shows that analogues of it hold under appropriate hypotheses on spec(R^*_G).

These comments are relevant to the discussion of XXI§6. As noted there, we know that applying the Tate construction to spectra of type $E(n)$, on which v_n is invertible, forces v_{n-1} to be invertible (in a suitable completion). One guesses that this can be explained in terms of the subvariety of Spec$(E(n)^*_G)$ defined by v_{n-1} and its intersection with that of I. Unfortunately our ignorance of $E(n)^*_G$ prevents us from justifying this intuition.

J. P. C. Greenlees. Tate cohomology in commutative algebra. J. Pure and Applied Algebra 94(1994), 59-83.

J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoir American Math. Soc. No. 543. 1995.

CHAPTER XXV

Localization and completion in complex bordism

by J. P. C. Greenlees and J. P. May

1. The localization theorem for stable complex bordism

There is a large literature that is concerned with the calculation of homology and cohomology groups $M_*(BG)$ and $M^*(BG)$ for MU-module spectra M, such as MU itself, K, BP, $K(n)$, $E(n)$, and so forth. Here G is a compact Lie group, in practice a finite group or a finite extension of a torus. The results do not appear to fall into a common pattern.

Nevertheless, there is a localization and completion theorem for stable complex bordism, and this shows that all such calculations must fit into a single general pattern dominated by the structure of the equivariant bordism ring MU^G_*. Indeed, as we showed in XXIII§5, there is a general procedure for constructing an equivariant version M_G of any nonequivariant MU-module M. Since M_G is split with underlying nonequivariant MU-module M, the theorem applies to the calculation of $M_*(BG_+)$ and $M^*(BG_+)$ for all such M. This is not, at present, calculationally useful since rather little is known about MU^G_*. Nevertheless, the theorem gives an intriguing new relation between equivariant and nonequivariant algebraic topology.

While the basic philosophy behind the theorem is the same as for the localization theorem XXIV.6.4, that result does not apply because its basic algebraic assumptions, XXIV.4.2 and 6.1, do not hold. In particular, since the augmentation ideal of MU^G_* is certainly not finitely generated and presumably not radically finitely generated, it is not even clear what we mean by the localization theorem, and different techniques are needed for its proof. Let $J = J_G$ denote the augmentation ideal of MU_*^G (with integer grading understood). For finitely generated subideals I of J, we can perform all of the topological constructions discussed in the previous chapter.

Theorem 1.1. Let G be finite or a finite extension of a torus. Then, for any sufficiently large finitely generated ideal $I \subset J$, $\kappa : EG_+ \wedge MU_G \to K(I)$ is an equivalence.

It is reasonable to define $K(J)$ to be $K(I)$ for any sufficiently large I and to define $\Gamma_J(M_G)$ and $(M_G)^\wedge_J$ similarly. The theorem implies that these MU_G-modules are independent of the choice of I.

Consequences are drawn exactly as they were for the localization theorem in Sections 4 and 5 of the previous chapter. In particular,

$$EG_+ \wedge M_G \to \Gamma_J(M_G) \quad \text{and} \quad (M_G)^\wedge_J \to F(EG_+, M_G)$$

are equivalences for any MU_G-module M_G.

The fact that the theorem holds for a finite extension of a torus and thus for the normalizer of a maximal torus in an arbitrary compact Lie group strongly suggests that the following generalization should be true, but we have not succeeded in finding a proof.

Conjecture 1.2. The theorem remains true for any compact Lie group G.

Most of this chapter is taken from the following paper, which gives full details. The last section discusses an earlier completion "theorem" for MU_*^G when G is a compact Abelian Lie group. While it may be true, we have only been able to obtain a complete proof in special cases.

J. P. C. Greenlees and J. P. May. Localization and completion theorems for MU-module spectra. Preprint, 1995.

2. An outline of the proof

We shall emphasize the general strategy. Let G be a compact Lie group and let S_G be the sphere G-spectrum. We assume given a commutative S_G-algebra R_G with underlying nonequivariant commutative S-algebra R. As in the localization theorem, we shall assume that the theory R_*^G has Thom isomorphisms

$$R_*^G(S^V \wedge X) \cong R_*^G(S^{|V|} \wedge X) \tag{2.1}$$

for complex representations V and G-spectra X. More precisely, we shall assume this for all subgroups $H \subseteq G$, and we shall later impose a certain naturality condition on these Thom isomorphisms. We have already seen in XV§2 that MU_G has such Thom isomorphisms. As in the proof of XXIV.6.4, the Thom isomorphism gives rise to an Euler class $\chi(V) \in R^G_{|-V|}$. Let J_H be the augmentation ideal $\mathrm{Ker}(\mathrm{res}^H_1 : R^H_* \longrightarrow R_*)$; remember that $J = J_G$.

DEFINITION 2.2. Assume that R^H_* has Thom isomorphisms for all $H \subseteq G$. Let I be a finitely generated subideal of J and, for $H \subseteq G$, let $r^G_H(I)$ denote the resulting subideal $\mathrm{res}^G_H(I) \cdot R^H_*$ of J_H. We say that I is sufficiently large at H if there is a non-zero complex representation V of H such that $V^H = 0$ and the Euler class $\chi(V) \in R^H_*$ is in the radical $\sqrt{r^G_H(I)}$. We say that the ideal I is sufficiently large if it is sufficiently large at all $H \subseteq G$.

We have the canonical map of R_G-modules

$$\kappa : EG_+ \wedge R_G \longrightarrow K(I),$$

and our goal is to prove that it is an equivalence. The essential point of our strategy is the following result, which reduces the problem to the construction of a sufficiently large finitely generated subideal I of J.

THEOREM 2.3. Assume that R^H_* has Thom isomorphisms for all $H \subseteq G$. If I is a sufficiently large finitely generated subideal of J, then

$$\kappa : EG_+ \wedge R_G \longrightarrow K(I)$$

is an equivalence.

PROOF. The cofiber of κ is equivalent to $\tilde{E}G \wedge K(I)$, and we must prove that this is contractible. Using the transitivity of restriction maps to see that $r^G_H(I)$ is a large enough subideal of R^H_*, we see that the hypotheses of the theorem are inherited by any subgroup. Therefore we may assume inductively that the theorem holds for $H \in \mathscr{P}$. Observing that

$$(\tilde{E}G \wedge K(I))|_H = \tilde{E}H \wedge K(r^G_H(I))$$

for $H \subseteq G$, we see that our definition of a sufficiently large ideal provides exactly what is needed to allow us to obtain the conclusion by parroting the proof the localization theorem XXIV.6.4. $\square$

Thus our problem is to prove that there is a large enough finitely generated ideal I. One's first instinct is to take I to be generated by finitely many well chosen Euler classes. While that does work in some cases, we usually need to add in other elements, and we shall do so by exploiting norm, or "multiplicative transfer", maps. We explain the strategy before stating what it means for a theory to have such norm maps.

We assume from now on that G is a toral group, namely an extension

$$1 \longrightarrow T \longrightarrow G \longrightarrow F \longrightarrow 1,$$

where T is a torus and F is a finite group.

THEOREM 2.4. If G is toral and the R_*^H for $H \subseteq G$ admit norm maps and Thom isomorphisms, then J contains a sufficiently large finitely generated subideal.

The proof of the theorem depends on two lemmas. As usual, we write

$$\mathrm{res}_H^G : R(G) \longrightarrow R(H)$$

for the restriction homomorphism. When H has finite index in G, we write

$$\mathrm{ind}_H^G : R(H) \longrightarrow R(G)$$

for the induction homomorphism. Recall that $\mathrm{ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

LEMMA 2.5. There are finitely many non-zero complex representations $V_1, \cdots, V_s$ of T such that T acts freely on the product of the unit spheres of the representations

$$\mathrm{res}_T^G \, \mathrm{ind}_T^G \, V_i.$$

While this is not obvious, its proof requires only elementary Lie theory and does not depend on the use of norm maps. We shall say no more about it since it is irrelevant when G is finite.

LEMMA 2.6. Let F' be a subgroup of F with inverse image G' in G. There is an element $\xi(F')$ of J such that

$$\mathrm{res}_{G'}^G(\xi(F')) = \chi(V')^{w'},$$

where V' is the reduced regular complex representation of F' regarded by pullback as a representation of G' and w' is the order of $WG' = NG'/G'$.

We shall turn to the proof of this in the next section, but we first show how these lemmas imply Theorem 2.4.

PROOF OF THEOREM 2.4. We claim that the ideal

$$I = (\chi(\operatorname{ind}_T^G V_1), \cdots, \chi(\operatorname{ind}_T^G V_s)) + (\xi(F')|F' \subseteq F)$$

is sufficiently large.

If H is a subgroup of G that intersects T non-trivially, then, by Lemma 2.5, $(\operatorname{res}_T^G \operatorname{ind}_T^G V_i)^{H\cap T} = \{0\}$ for some i and therefore $(\operatorname{ind}_T^G V_i)^H = \{0\}$. Since

$$\chi(\operatorname{res}_H^G \operatorname{ind}_T^G V_i) = \operatorname{res}_H^G(\chi(\operatorname{ind}_T^G V_i)) \in r_H^G(I),$$

this shows that I is sufficiently large at H in this case.

If H is a subgroup of G that intersects T trivially, as is always the case when G is finite, then H maps isomorphically to its image F' in F. If G' is the inverse image of F' in G and V' is the reduced regular complex representation of F' regarded as a representation of G', then $\operatorname{res}_H^{G'}(V')$ is the reduced regular complex representation of H and $(\operatorname{res}_H^{G'}(V'))^H = 0$. By Lemma 2.6, we have $\operatorname{res}_{G'}^G(\xi(F')) = \chi(V')^{w'}$ and therefore

$$\chi(\operatorname{res}_H^{G'}(V'))^{w'} = \operatorname{res}_H^{G'}(\chi(V')^{w'}) = \operatorname{res}_H^{G'} \operatorname{res}_{G'}^G(\xi(F')) = \operatorname{res}_H^G(\xi(F')) \in r_H^G(I).$$

This shows that I is sufficiently large at H in this case. $\square$

3. The norm map and its properties

We must still explain the proof of Lemma 2.6, and to do so we must explain our hypothesis that R_*^G has norm maps. We shall give a rather crude definition that prescribes exactly what we shall use in the proof. The crux of the matter is a double coset formula, and we need some notations in order to state it. For $g \in G$ and $H \subseteq G$, let ${}^gH = gHg^{-1}$ and let $c_g : {}^gH \longrightarrow H$ be the conjugation isomorphism. For a based H-space X, we have a natural isomorphism

$$c_g : R_*^H(X) \longrightarrow R_*^{{}^gH}({}^gX),$$

where ${}^gX$ denotes X regarded as a ${}^gH$-space by pullback along c_g. We also have a natural restriction homomorphism

$$\operatorname{res}_H^G : R_*^G(X) \longrightarrow R_*^H(X).$$

DEFINITION 3.1. We say that R_*^G has norm maps if, for a subgroup H of finite index n in G and an element $y \in R_{-r}^H$, where $r \geq 0$ is even, there is an element

$$\overline{\operatorname{norm}}_H^G(1+y) \in \sum_{i=0}^{n} R_{-ri}^G$$

that satisfies the following properties; here $1 = 1_H \in R_0^H$ denotes the identity element.

(i) $\overline{\mathrm{norm}}_G^G(1+y) = 1+y$.
(ii) $\overline{\mathrm{norm}}_H^G(1) = 1$.
(iii) [The double coset formula]

$$res_K^G\, \overline{\mathrm{norm}}_H^G\,(1+y) = \prod_g \overline{\mathrm{norm}}_{{}^gH\cap K}^K\ \mathrm{res}_{{}^gH\cap K}^{{}^gH}\ c_g(1+y),$$

where K is any subgroup of G and $\{g\}$ runs through a set of double coset representatives for $K\backslash G/H$.

Proof of Lemma 2.6. Since the restriction of the reduced regular representation of F' to any proper subgroup contains a trivial representation, the restriction of $\chi(V') \in R_*^{G'}$ to a subgroup that maps to a proper subgroup of F' is zero. In $R_*^{G'}$, the double coset formula gives

$$\mathrm{res}_{G'}^G\ \overline{\mathrm{norm}}_{G'}^G\,(1+\chi(V')) = \prod_g \overline{\mathrm{norm}}_{{}^gG'\cap G'}^{G'}\ \mathrm{res}_{{}^gG'\cap G'}^{{}^gG'}\ c_g(1+\chi(V')), \tag{3.2}$$

where g runs through a set of double coset representatives for $G'\backslash G/G'$. We require that our Thom isomorphisms be natural with respect to conjugation in the sense that their Euler classes satisfy $c_g(\chi(V)) = \chi({}^gV)$, where ${}^gV$ is the pullback of V along c_g. In particular, this gives that

$$c_g(1+\chi(V')) = 1+\chi({}^gV').$$

Here ${}^gV'$ is the reduced regular representation of ${}^gG'$. Clearly ${}^gG'\cap G'$ is the inverse image in G of ${}^gF'\cap F'$. If ${}^gF'\cap F'$ is a proper subgroup of F', then the restriction of $\chi(V')$ to ${}^gG'\cap G'$ is zero. Therefore all terms in the product on the right side of (3.2) are 1 except for those that are indexed on elements $g\in NG'$. There is one such g for each element of $WG' = NG'/G'$, and the term in the product that is indexed by each such g is just $1+\chi(V')$. Therefore (3.2) reduces to

$$\mathrm{res}_{G'}^G\ \overline{\mathrm{norm}}_{G'}^G(1+\chi(V')) = (1+\chi(V'))^{w'}. \tag{3.3}$$

If V' has real dimension r, then the summand of $(1+\chi(V'))^{w'}$ in degree rw' is $\chi(V')^{w'}$. Since $\mathrm{res}_{G'}^G$ preserves the grading, we may take $\xi(F')$ to be the summand of degree rw' in $\overline{\mathrm{norm}}_{G'}^G(1+\chi(V'))$. $\square$

4. The idea behind the construction of norm maps

We give an intuitive idea of the construction here, but we need some preliminaries to establish the context. Let H be a subgroup of finite index n in a compact Lie group G. The norm map is intimately related to $\mathrm{ind}_H^G : RO(H) \longrightarrow RO(G)$, and we begin with a description of induction that suggests an action of G on the nth smash power X^n of any based H-space X. Recall that the wreath product $\Sigma_n \int H$ is the set $\Sigma_n \times H^n$ with the product

$$(\sigma, h_1, \dots, h_n)(\tau, h'_1, \dots, h'_n) = (\sigma\tau, h_{\tau 1}h'_1, \cdots, h_{\tau n}h'_n).$$

Choose coset representatives $t_1, \dots, t_n$ for H in G and define the "monomial representation"

$$\alpha : G \longrightarrow \Sigma_n \int H$$

by the formula

$$\alpha(\gamma) = (\sigma(\gamma), h_1(\gamma), \dots, h_n(\gamma)),$$

where $\sigma(\gamma)$ and $h_i(\gamma)$ are defined implicitly by the formula

$$\gamma t_i = t_{\sigma(\gamma)(i)} h_i(\gamma).$$

LEMMA 4.1. The map α is a homomorphism of groups. If α' is defined with respect to a second choice of coset representatives $\{t'_i\}$, then α and α' differ by a conjugation in $\Sigma_n \int H$.

The homomorphism α is implicitly central to induction as the following lemma explains. Write $\alpha^* W$ for a representation W of $\Sigma_n \int H$ regarded as a representation of G by pullback along α.

LEMMA 4.2. Let V be a representation of H. Then the sum nV of n copies of V is a representation of $\Sigma_n \int H$ with action given by

$$(\sigma, h_1, \dots, h_n)(v_1, \dots, v_n) = (h_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(n)} v_{\sigma^{-1}(n)}),$$

and $\alpha^*(nV)$ is isomorphic to the induced representation $\mathrm{ind}_H^G V = \mathbb{R}[G] \otimes_{\mathbb{R}[H]} V$.

LEMMA 4.3. If X is a based H-space, then the smash power X^n is a $(\Sigma_n \int H)$-space with action given by

$$(\sigma, h_1, \dots, h_n)(x_1 \wedge \dots \wedge x_n) = h_{\sigma^{-1}(1)} x_{\sigma^{-1}(1)} \wedge \dots \wedge h_{\sigma^{-1}(n)} x_{\sigma^{-1}(n)}.$$

For a based $\Sigma_n \int H$-space Y, such as $Y = X^n$ for a based H-space X, write $\alpha^* Y$ for Y regarded as a G-space by pullback along α. Note in particular that $\alpha^*((S^V)^n) \cong S^{\mathrm{ind}_H^G V}$ for an H-representation V.

To begin the construction of $\overline{norm}_H^G$, one constructs a natural function

$$\text{norm}_H^G : R_0^H(X) \longrightarrow R_0^G(\alpha^* X^n). \tag{4.4}$$

The norm map $\overline{\text{norm}}_H^G$ of Definition 3.1 is then obtained by taking X to be the wedge $S^0 \vee S^r$, studying the decomposition of X^n into wedge summands of G-spaces described in terms of smash powers of spheres and thus of representations, and using Thom isomorphisms to translate the result to integer gradings. We shall say no more about this step here. The properties of $\overline{\text{norm}}_H^G$ are deduced from the following properties of norm_H^G.

$$\text{norm}_G^G \quad \text{is the identity function.} \tag{4.5}$$

$$\text{norm}_H^G(1_H) = 1_G, \quad \text{where } 1_H \in R_0^H(S^0) \text{ is the identity element.} \tag{4.6}$$

$$\text{norm}_H^G(xy) = \text{norm}_H^G(x)\,\text{norm}_H^G(y) \;\; \text{if } x \in R_0^H(X) \text{ and } y \in R_0^H(Y). \tag{4.7}$$

Here the product xy on the left is defined by use of the evident map

$$R_0^H(X) \otimes R_0^H(Y) \longrightarrow R_0^H(X \wedge Y) \tag{4.8}$$

and similarly on the right, where we must also use the isomorphism

$$R_0^G(X^n \wedge Y^n) \cong R_0^G((X \wedge Y)^n).$$

The most important property is the double coset formula

$$\text{res}_K^G \; \text{norm}_H^G(x) = \prod_g \text{norm}_{{}^gH \cap K}^K \; \text{res}_{{}^gH \cap K}^{{}^gH} \; c_g(x), \tag{4.9}$$

where K is any subgroup of G and $\{g\}$ runs through a set of double coset representatives for $K \backslash G/H$. Here, if ${}^gH \cap K$ has index $n(g)$ in ${}^gH$, then $n = \sum n(g)$ and the product on the right is defined by use of the evident map

$$\bigotimes_g R_0^K(X^{n(g)}) \longrightarrow R_0^K(X^n). \tag{4.10}$$

An element of $R_0^H(X)$ is represented by an H-map $x : S_G \longrightarrow R_G \wedge X$. There is no difficulty in using the product on R_G to produce an H-map

$$S_G \cong (S_G)^n \xrightarrow{x^n} (R_G \wedge X)^n \cong (R_G)^n \wedge X^n \longrightarrow R_G \wedge X^n. \tag{4.11}$$

The essential point of the construction is to do this in such a way as to produce a G-map: this will be $\mathrm{norm}_H^G(x)$. This is the basic idea, but carrying it out entails several difficulties. Of course, since our group actions involve permutations of smash powers, we must be working in the brave new world of associative and commutative smash products, with an associative and commutative multiplication on R_G. Our first instinct is to interpret the smash powers in (4.11) in terms of $\wedge_S$. Certainly the maps in (4.11) are then both H-maps and Σ_n-maps. However, the H-action on $(R_G)^n$ does not come by pullback along the diagonal of an H^n-action, so that $\Sigma_n \int H$ need not act on $(R_G)^n$. This is only to be expected since $(R_G)^n$ is indexed on the original complete G-universe U on which R_G is indexed, not on a complete $\Sigma_n \int H$-universe. Since our G-actions come by restriction of actions of wreath products $\Sigma_n \int H$, it is essential to bring $(\Sigma_n \int H)$-spectra into the picture. External smash products seem more reasonable than $\wedge_S$ for this purpose since the external smash power $(R_G)^n$ is indexed on the complete $\Sigma_n \int H$-universe U^n.

5. Global $\mathcal{I}_*$-functors with smash product

The solution to the difficulties that we have indicated is to work with a restricted kind of commutative S_G-algebra, namely one that arises from a global $\mathscr{I}_*$-functor with smash product, abbreviated $\mathscr{G}\mathscr{I}_*$-FSP. Unlike general commutative S_G-algebras, these have structure given directly in terms of external smash products, as is needed to make sense of (4.11).

The notion of an $\mathscr{I}_*$-FSP was introduced by May, Quinn, and Ray around 1973, under the ugly name of an $\mathscr{I}_*$-prefunctor. (The name "functor with smash product" was introduced much later by Bökstedt, who rediscovered essentially the same concept.) While $\mathscr{I}_*$-FSP's were originally defined nonequivariantly, the definition transcribes directly to one in which a given compact Lie group G acts on everything in sight. The adjective "global" means that we allow G to range through all compact Lie groups G, functorially with respect to homomorphisms of compact Lie groups. We let $\mathscr{G}$ denote the category of compact Lie groups and their homomorphisms.

Definition 5.1. Define the global category $\mathscr{G}\mathscr{T}$ of equivariant based spaces to have objects (G, X), where G is a compact Lie group and X is a based G-space. The morphisms are the pairs

$$(\alpha, f) : (G, X) \longrightarrow (G', X')$$

where $\alpha : G \longrightarrow G'$ is a homomorphism of Lie groups and $f : X \longrightarrow X'$ is an α-equivariant map, in the sense that $f(gx) = \alpha(g)f(x)$ for all $x \in X$ and $g \in G$. Topologize the set of maps $(G, X) \longrightarrow (G', X')$ as a subspace of the evident product of mapping spaces and observe that composition is continuous.

DEFINITION 5.2. Define the global category $\mathscr{G}\mathscr{I}_*$ of finite dimensional equivariant complex inner product spaces to have objects (G, V), where G is a compact Lie group and V is a finite dimensional inner product space with an action of G through linear isometries. The morphisms are the pairs

$$(\alpha, f) : (G, V) \longrightarrow (G', V')$$

where $\alpha : G \longrightarrow G'$ is a homomorphism and $f : V \longrightarrow V'$ is an α-equivariant linear isomorphism.

The definitions work equally well with real inner product spaces; we restrict attention to complex inner product spaces for convenience in our present application. Each morphism (α, f) in $\mathscr{G}\mathscr{I}_*$ factors as a composite

$$(G, V) \xrightarrow{(\mathrm{id}, f)} (G, W) \xrightarrow{(\alpha, \mathrm{id})} (H, W),$$

where G acts through α on W. We have a similar factorization of morphisms in $\mathscr{G}\mathscr{T}$. We also have forgetful functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}$ and $\mathscr{G}\mathscr{T} \longrightarrow \mathscr{G}$. We shall be interested in functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over $\mathscr{G}$, that is, functors that preserve the group coordinate. For example, one-point compactification of inner product spaces gives such a functor, which we shall denote by $S^{\bullet}$. As in this example, the space coordinate of our functors will be the identity on morphisms of the form (α, id).

DEFINITION 5.3. A $\mathscr{G}\mathscr{I}_*$-functor is a continuous functor $T : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over $\mathscr{G}$, written (G, TV) on objects (G, V), such that

$$T(\alpha, \mathrm{id}) = (\alpha, \mathrm{id}) : (G, TW) \longrightarrow (H, TW)$$

for a representation W of H and a homomorphism $\alpha : G \longrightarrow H$.

The following observation is the germ of the definition of the norm map.

LEMMA 5.4. Let $A = \mathrm{Aut}(G, V)$ be the group of automorphisms of (G, V) in the category $\mathscr{G}\mathscr{I}_*$. For any $\mathscr{G}\mathscr{I}_*$-functor T, the group $A \ltimes G$ acts on the space TV.

Define the direct sum functor $\oplus : \mathcal{G}\mathcal{I}_* \times \mathcal{G}\mathcal{I}_* \longrightarrow \mathcal{G}\mathcal{I}_*$ by

$$(G, V) \oplus (H, W) = (G \times H, V \oplus W).$$

Define the smash product functor $\wedge : \mathcal{G}\mathcal{T} \times \mathcal{G}\mathcal{T} \longrightarrow \mathcal{G}\mathcal{T}$ by

$$(G, X) \wedge (H, Y) = (G \times H, X \wedge Y).$$

These functors lie over the functor $\times : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$.

DEFINITION 5.5. A $\mathcal{G}\mathcal{I}_*$-FSP is a $\mathcal{G}\mathcal{I}_*$-functor together with a continuous natural unit transformation $\eta : S^\bullet \longrightarrow T$ of functors $\mathcal{G}\mathcal{I}_* \longrightarrow \mathcal{G}\mathcal{T}$ and a continuous natural product transformation $\omega : T \wedge T \longrightarrow T \circ \oplus$ of functors $\mathcal{G}\mathcal{I}_* \times \mathcal{G}\mathcal{I}_* \longrightarrow \mathcal{G}\mathcal{T}$ such that the composite

$$TV \cong TV \wedge S^0 \xrightarrow{\mathrm{id} \wedge \eta} TV \wedge T(0) \xrightarrow{\omega} T(V \oplus 0) \cong TV$$

is the identity map and the following unity, associativity, and commutativity diagrams commute:

$$\begin{array}{ccc} S^V \wedge S^W & \xrightarrow{\eta \wedge \eta} & TV \wedge TW \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \omega} \\ S^{V \oplus W} & \xrightarrow[\eta]{} & T(V \oplus W), \end{array}$$

$$\begin{array}{ccc} TV \wedge TW \wedge TZ & \xrightarrow{\omega \wedge \mathrm{id}} & T(V \oplus W) \wedge TZ \\ {\scriptstyle \mathrm{id} \wedge \omega}\downarrow & & \downarrow{\scriptstyle \omega} \\ TV \wedge T(W \oplus Z) & \xrightarrow[\omega]{} & T(V \oplus W \oplus Z), \end{array}$$

and

$$\begin{array}{ccc} TV \wedge TW & \xrightarrow{\omega} & T(V \oplus W) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle T(\tau)} \\ TW \wedge TV & \xrightarrow[\omega]{} & T(W \oplus V). \end{array}$$

Actually, this is the notion of a commutative $\mathcal{G}\mathcal{I}_*$-FSP; for the more general non-commutative notion, the commutativity diagram must be replaced by a weaker centrality of unit diagram. Observe that the space coordinate of each map $T(\alpha, f)$ is necessarily a homeomorphism since $(\alpha, f) = (\alpha, \mathrm{id}) \circ (\mathrm{id}, f)$ and f is an isomorphism. Spheres and Thom complexes give naturally occurring examples.

Example 5.6. The sphere functor $S^\bullet$ is a $\mathcal{G}\mathcal{I}_*$-FSP with unit given by the identity maps of the S^V and product given by the isomorphisms $S^V \wedge S^W \cong S^{V \oplus W}$. For any $\mathcal{G}\mathcal{I}_*$-FSP T, the unit $\eta : S^\bullet \longrightarrow T$ is a map of $\mathcal{G}\mathcal{I}_*$-FSP's.

Example 5.7. Let $\dim V = n$ and, as in XV§2, define TV to be the one-point compactification of the canonical n-plane bundle EV over the Grassmann manifold $Gr_n(V \oplus V)$. An action of G on V induces an action of G that makes EV a G-bundle and TV a based G-space. Take $V = V \oplus \{0\}$ as a canonical basepoint in $Gr_n(V \oplus V)$. The inclusion of the fiber over the basepoint induces a map $\eta : S^V \longrightarrow TV$. The canonical bundle map $EV \oplus EW \longrightarrow E(V \oplus W)$ induces a map $\omega : TV \wedge TW \longrightarrow T(V \oplus W)$. With the evident definition of T on morphisms, T is a $\mathcal{G}\mathcal{I}_*$-functor.

It is useful to regard a $\mathcal{G}\mathcal{I}_*$-FSP as a $\mathcal{G}\mathcal{I}_*$-prespectrum with additional structure.

Definition 5.8. A $\mathcal{G}\mathcal{I}_*$-prespectrum is a $\mathcal{G}\mathcal{I}_*$-functor $T : \mathcal{G}\mathcal{I}_* \longrightarrow \mathcal{G}\mathcal{T}$ together with a continuous natural transformation $\sigma : T \wedge S^\bullet \longrightarrow T \circ \oplus$ of functors $\mathcal{G}\mathcal{I}_* \times \mathcal{G}\mathcal{I}_* \longrightarrow \mathcal{T}$ such that the composites

$$TV \cong TV \wedge S^0 \xrightarrow{\sigma} T(V \oplus 0) \cong TV$$

are identity maps and each of the following diagrams commutes:

$$\begin{array}{ccc} TV \wedge S^W \wedge S^Z & \xrightarrow{\sigma \wedge \mathrm{id}} & T(V \oplus W) \wedge S^Z \\ \cong \downarrow & & \downarrow \sigma \\ TV \wedge S^{W \oplus Z} & \xrightarrow[\sigma]{} & T(V \oplus W \oplus Z). \end{array}$$

Lemma 5.9. If T is a $\mathcal{G}\mathcal{I}_*$-FSP, then T is a $\mathcal{G}\mathcal{I}_*$-prespectrum with respect to the composite maps

$$\sigma : TV \wedge S^W \xrightarrow{\mathrm{id} \wedge \eta} TV \wedge TW \xrightarrow{\omega} T(V \oplus W).$$

It is evident that a $\mathcal{G}\mathcal{I}_*$-prespectrum restricts to a G-prespectrum indexed on U for every G and U.

Notations 5.10. Let $T_{G,U}$ denote the G-prespectrum indexed on U associated to a $\mathcal{G}\mathcal{I}_*$-FSP T. Write $R_{G,U}$ for the spectrum $LT_{G,U}$ associated to $T_{G,U}$.

There is a notion of an $\mathscr{L}$-prespectrum, due to May, Quinn, and Ray, and $T_{G,U}$ is an example. The essential point is that if $f : U^j \longrightarrow U$ is a linear isometry and V_i are indexing spaces in U, then we have maps

$$\xi_j(f) : TV_1 \wedge \cdots TV_j \xrightarrow{\omega} T(V_1 \oplus \cdots \oplus V_j) \xrightarrow{Tf} Tf(V_1 \oplus \cdots \oplus V_j). \tag{5.11}$$

The notion of an $\mathscr{L}$-prespectrum was first defined in terms of just such maps. It was later redefined more conceptually in [LMS] in terms of maps

$$\xi_j : \mathscr{L}(j) \ltimes E^j \longrightarrow E \tag{5.12}$$

induced by the $\xi_j(f)$. It was shown in the cited sources that the spectrification functor L converts $\mathscr{L}$-prespectra to $\mathscr{L}$-spectra. We conclude that, for every G and every G-universe U, $R_{G,U}$ is an $\mathscr{L}$-spectrum and thus an E_∞ ring G-spectrum when U is complete. Of course, the $\mathscr{L}$-spectrum $R_{G,U}$ determines the weakly equivalent commutative $S_{G,U}$-algebra $S_{G,U} \ltimes_{\mathscr{L}} R_{G,U}$.

M. Bökstedt. Topological Hochschild homology. Preprint, 1990.
J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Volume 577. 1977.

6. The definition of the norm map

We have the following crucial observation about $\mathscr{G}\mathscr{I}_*$-FSP's.

PROPOSITION 6.1. Let T be a $\mathscr{G}\mathscr{I}_*$-FSP. For an H-representation V, $(TV)^n$ and $T(V^n)$ are $\Sigma_n \int H$-spaces and the map

$$\omega : (TV)^n \longrightarrow T(V^n)$$

is $(\Sigma_n \int H)$-equivariant. If U is an H-universe, then U^n is a $(\Sigma_n \int H)$-universe and the maps ω define a map of $(\Sigma_n \int H)$-prespectra indexed on U^n

$$\omega : (T_{H,U})^n \longrightarrow T_{\Sigma_n \int H, U^n},$$

where $(T_{H,U})^n$ is the nth external smash power of $T_{H,U}$. If $T = S^\bullet$, then ω is an isomorphism of prespectra.

This allows us to define the norm maps we require. Recall Notations 5.10.

DEFINITION 6.2. Let T be a $\mathscr{G}\mathscr{I}_*$-FSP, let X be a based H-space, and let U be a complete H-universe. An element $x \in R_0^H(X)$ is given by a map of H-spectra $x : S_{H,U} \longrightarrow R_{H,U} \wedge X$. Let G act on U^n through $\alpha : G \longrightarrow \Sigma_n \int H$, observe that the G-universe U^n is then complete, and define the norm of x to be the element of $R_0^G(\alpha^* X^n)$ given by the composite map of G-spectra indexed on U^n displayed in the commutative diagram:

$$\begin{array}{ccccc} S_{G,U^n} & \xrightarrow{\omega^{-1}} & (S_{H,U})^n & \xrightarrow{x^n} & (R_{H,U} \wedge X)^n \\ \downarrow{\scriptstyle \mathrm{norm}_H^G(x)} & & & & \downarrow{\scriptstyle \cong} \\ R_{G,U^n} \wedge X^n & & \xleftarrow[\omega \wedge \mathrm{id}]{} & & (R_{H,U})^n \wedge X^n. \end{array} \tag{6.3}$$

Strictly speaking, if we start with H-spectra defined in fixed complete H-universes U_H for all H, then we must choose an isomorphism $U_G \cong U_H^n$ to transfer the norm to a map of spectra indexed on U_G, but it is more convenient to derive formulas from the definition as given. From here, all of the properties of the norm except the double coset formula are easy consequences of the definition. The proof of the double coset formula is in principle straightforward diagram chasing from the definitions, but it requires precise combinatorial understanding of double cosets and some fairly elaborate bookkeeping. It is noteworthy that the formula is actually derived from a precise equality of the point set level maps that represent the two sides of the formula.

7. The splitting of MU_G as an algebra

In the context of $\mathscr{G}\mathscr{I}_*$-FSP's, we can complete an unfinished piece of business, namely an indication of the proof that MU_G is split as an algebra in the sense of XXIII.5.8. This was at the heart of our assertion that MU-modules M naturally give rise to split MU_G-modules M_G. In fact, the result we need applies to the S_G-algebra associated to any $\mathscr{G}\mathscr{I}_*$-FSP T, and we adopt Notations 5.10.

We need a preliminary observation. If $f : U \longrightarrow U'$ is a linear isometry, we have maps $Tf : TV \longrightarrow T(fV)$ for indexing spaces $V \subset U$. These specify a map of prespectra $T_{G,U} \longrightarrow f^* T_{G,U'}$ indexed on U and thus, by adjunction, a map $f_* T_{G,U} \longrightarrow T_{G,U'}$ of prespectra indexed on U'. On passage to spectra, these glue together to define a map

$$\xi : \mathscr{I}(U, U') \ltimes R_{G,U} \longrightarrow R_{G,U'}. \tag{7.1}$$

Moreover, this map factors over coequalizers to give a map of $\mathbb{L}'$-spectra

$$\xi : I_U^{U'} R_{G,U} = \mathscr{I}(U, U') \ltimes_{\mathscr{I}(U,U)} R_{G,U} \longrightarrow R_{G,U'}. \tag{7.2}$$

PROPOSITION 7.3. Consider

$$R' = S_{e,U^G} \wedge_{\mathscr{L}} R_{e,U^G} \quad \text{and} \quad R_G = S_{G,U} \wedge_{\mathscr{L}} R_{G,U}$$

(where the subscripts $\mathscr{L}$ refer respectively to U^G and to U) and let $\gamma : R \longrightarrow R'$ be a q-cofibrant approximation of the commutative S-algebra R'. Then the commutative S_G-algebra R_G is split as an algebra with underlying nonequivariant S-algebra R.

PROOF. It suffices to to construct a map $\eta' : I_{U^G}^U R' \longrightarrow R_G$ of S_G-algebras that is a nonequivariant equivalence of spectra, since we can then precompose it with $I_{U^G}^U \gamma$ to obtain a map $\eta : I_{U^G}^U R \longrightarrow R_G$ of S_G-algebras that is a nonequivariant equivalence. In fact, we shall construct a map η' that is actually an isomorphism. Replace U and U' by U^G and U in (7.2). It is not hard to check from the definition of a $\mathscr{G}\mathscr{I}_*$-FSP that

$$R_{e,U^G} = R_{G,U^G} \quad \text{and} \quad R_{G,U}^{\#} = R_{e,U^{\#}}, \tag{7.4}$$

where the superscript $\#$ denotes that we are forgetting actions by G. That is, R_{G,U^G} is R_{e,U^G} regarded as a G-trivial G-spectrum indexed on the G-trivial universe U^G, and $R_{G,U}$ regarded as a nonequivariant spectrum indexed on $U^{\#}$ is $R_{e,U^{\#}}$. The first equality in (7.4) allows us to specialize the map ξ to obtain a map of E_∞ ring spectra

$$\xi : I_{U^G}^U R_{e,U^G} = \mathscr{I}(U^G, U) \ltimes_{I(U^G,U^G)} R_{G,U^G} \longrightarrow R_{G,U}. \tag{7.5}$$

The second equality allows us to identify the target of the underlying map $\xi^{\#}$ of nonequivariant spectra with $R_{e,U^{\#}}$, and it is not hard to check that $\xi^{\#}$ is actually an isomorphism of spectra. We obtain the required map η' on passage to S_G-algebras, using from XXIII.4.5 that we have an isomorphism of S_G-algebras

$$I_{U^G}^U R' \cong S_G \wedge_{\mathscr{L}} I_{U^G}^U R_{e,U^G}. \quad \square$$

J. P. May. Equivariant and nonequivariant module spectra. Preprint, 1995.

8. Löffler's completion conjecture

While computations of MU_*^G are in general out of reach, they are more manageable for compact Abelian Lie groups. Moreover, in this case $MU^*(BG_+)$ is well understood due to work of Landweber and others. Early on in the study of stable complex cobordism, Löffler stated the following assertion as a theorem, although details of proof never appeared.

CONJECTURE 8.1 (LÖFFLER). If G is a compact Abelian Lie group, then

$$(MU_G^*)^\wedge_J \cong MU^*(BG_+).$$

When this holds, it combines with our topological result to force the following algebraic conclusion. A direct proof would be out of reach.

COROLLARY 8.2. If G is a compact Abelian Lie group such that the conjecture holds and I is a sufficiently large ideal in MU_*^G, then

$$H_0^I(MU_G^*) \cong ((MU_G)^\wedge_I)^*_G \cong (MU_G^*)^\wedge_I$$

and

$$H_p^I(MU_G^*) = 0 \quad \text{if} \quad p \neq 0.$$

We do not know whether or not the conjecture holds in general, but it does hold in many cases, as we shall explain in the rest of this section. We also indicate the flaw in the argument sketched by Löffler. We are indebted to Comezaña for details, and our proofs rely on results that he will prove in the next chapter. In particular, the following result is XXVI.5.3; it is stated by Löffler, but no proof appears in the literature.

THEOREM 8.3. For a compact Abelian Lie group G, $MU_*^G$is a free MU^*-module on even degree generators.

Since MU_G is a split G-spectrum, the projection $EG \longrightarrow *$ induces a natural map

$$\alpha : MU_G^*(X) \longrightarrow MU_G^*(EG_+ \wedge X) \cong MU^*(EG_+ \wedge_G X).$$

We shall mainly concern ourselves with the case $X = S^0$ relevant to Conjecture 8.1. We may take EG to be a G-CW complex with finite skeleta, and there results a model for BG as a CW complex with finite skeleta BG^n. We shall need the following result of Landweber.

PROPOSITION 8.4 (LANDWEBER). For a compact Lie group G and a finite G-CW complex X, the natural map $MU^*(EG_+ \wedge_G X) \longrightarrow \lim MU^*(EG^n_+ \wedge_G X)$ is an isomorphism.

The vanishing of $\lim^1$ terms here is analogous to part of the Atiyah-Segal completion theorem. In fact, in view of the Conner-Floyd isomorphism

$$K^*(X) \cong MU^*(X) \otimes_{MU^*} K^*$$

for finite X, the result for MU can be deduced from its counterpart for K. Some power J^q of the augmentation ideal of MU_*^G annihilates $MU_G^*(X)$ for any finite free G-CW complex X, by the usual induction on the number of cells, and we conclude that $MU_G^*(EG_+) \cong MU^*(BG_+)$ is J-adically complete. Therefore α gives rise to a natural map

$$MU_G^*(X)_J^\wedge \longrightarrow MU^*(EG_+ \wedge_G X)$$

on finite G-CW complexes X.

A basic tool in the study of this map is the Gysin sequence

(8.5)

$$\cdots \longrightarrow MU_G^{q-2d}(X) \xrightarrow{\chi(V)} MU_G^q(X) \longrightarrow MU_G^q(X \wedge SV_+) \rightarrow MU_G^{q-2d+1}(X) \longrightarrow \cdots,$$

where V is a complex G-module of complex dimension d and we write SV and DV for the unit sphere and unit disc of V. Noting that DV is G-contractible and DV/SV is equivalent to S^V, we can obtain this directly from the long exact sequence of the pair (DV, SV) by use of the Thom isomorphism

$$MU_G^{q-2d}(X) \longrightarrow MU^q(X \wedge S^V).$$

LEMMA 8.6. Conjecture 8.1 holds when $G = S^1$.

PROOF. Let $V = \mathbb{C}$ with the standard action of S^1. Since $SV = S^1$, we have $MU_{S^1}^*(SV_+) \cong MU^*$, which of course is concentrated in even degrees. Therefore the Gysin sequence for V, with $X = S^0$, breaks up into short exact sequences and multiplication by $\chi(V)$ is a monomorphism on $MU_{S^1}^*$. By the multiplicativity of Euler classes, $\chi(nV) = \chi(V)^n$. Thus multiplication by $\chi(nV)$ is also a monomorphism and the Gysin sequence of nV breaks up into short exact sequences

$$0 \longrightarrow MU_{S^1}^{2q-2nd} \xrightarrow{\chi(V)^n} MU_{S^1}^{2q} \longrightarrow MU_{S^1}^{2q}(S(nV)_+) \longrightarrow 0.$$

Since S^1 acts freely on SV, the union $S(\infty V)$ of the $S(nV)$ is a model for ES^1. On passage to limits, there results an isomorphism

$$(MU_{S^1}^*)_{(\chi(V))}^\wedge \cong MU_{S^1}^*(S(\infty V)_+) \cong MU^*(BS_+^1).$$

It is immediate from the Gysin sequence that $J_{S^1} = (\chi(V))$, and the result follows. □

Clearly the proof implies the standard calculation $MU^*(BS^1) \cong MU^*[[c]]$, where $c \in MU^2(BS^1)$ is the image of the Euler class.

The steps of the argument generalize to give the following two results.

LEMMA 8.7. For any compact Abelian Lie group G,

$$(MU^*_{G\times S^1})^{\wedge}_{(\chi(V))} \cong MU^*_G(BS^1_+) \cong MU^*_G[[c]].$$

PROOF. Here we regard $V = \mathbb{C}$ as a representation of $G \times S^1$, with G acting trivially, and we note that $S(V) \cong (G\times S^1)/G$, so that $MU^*_{G\times S^1}(S(V)_+) \cong MU^*_G$. The rest of the proof is as in Lemma 8.6. □

LEMMA 8.8. Let $T = T^r$ be a torus, let $V_q = \mathbb{C}$ with T acting through its projection to the qth factor, and let $\chi_q = \chi(V_q)$. Then $J_T = (\chi_1, \cdots, \chi_r)$.

PROOF. Clearly J_T annihilates $MU^*_T(S(V_1)_+ \wedge \cdots \wedge S(V_r)_+) \cong MU^*$. By an easy inductive use of Gysin sequences, we find that, for $1 \le q \le r$,

$$MU^*_T(S(V_1)_+ \wedge \cdots \wedge S(V_q)_+) \cong MU^*_T/(\chi_1, \cdots, \chi_q)MU^*_T.$$

The rest of the proof is as in Lemma 8.6. □

We put the previous two lemmas together to obtain Conjecture 8.1 for tori.

PROPOSITION 8.9. Conjecture 8.1 holds when G is a torus.

PROOF. Write $G = T \times S^1$ and assume inductively that the conclusion holds for T. Letting c_q be the image of χ_q, we find that

$$\begin{aligned}(MU^*_G)^{\wedge}_{J_G} \cong (MU^*_G)^{\wedge\ \wedge}_{J_T\, J_{S^1}} \cong (MU^*_G)^{\wedge\ \wedge}_{J_{S^1}\, J_T} \cong (MU^*_T[[c_r]])^{\wedge}_{J_T} \\ \cong (MU^*_T)^{\wedge}_{J_T}[[c_r]] \cong MU^*[[c_1, \cdots, c_r]] \cong MU^*(BG_+),\end{aligned}$$

the first equality being an evident identification of a double limit with a single one. □

We would like to deduce the general case of Conjecture 8.1 from the case of a torus. Thus, for the rest of the section, we consider a group $G = C_1 \times \cdots \times C_r$, where each C_q is either S^1 or a subgroup of S^1. This fixes an embedding of G in the torus $T = T^r$, and of course every compact Abelian Lie group can be written in this form. We have the following pair of lemmas, the first of which follows from the known calculation of $MU^*(BG_+)$; see for example the second paper of Landweber below.

LEMMA 8.10. The restriction map $MU^*(BT_+) \longrightarrow MU^*(BG_+)$ is an epimorphism. In particular, $MU^*(BG_+)$ is concentrated in even degrees.

LEMMA 8.11. The restriction map $MU_T^* \longrightarrow MU_G^*$ is an epimorphism. In particular, J_T maps epimorphically onto J_G and the completion of an MU_*^G-module at J_G is isomorphic to its completion at J_T.

PROOF. It suffices to prove that each restriction map

$$MU^*_{T^q\times C_{q+1}\times\cdots\times C_r} \longrightarrow MU^*_{T^{q-1}\times C_q\times C_{q+1}\times\cdots\times C_r}$$

is an epimorphism. Let C_q be cyclic of order $k(q)$. Let $V_q = \mathbb{C}$ regarded as a T-module with all factors of S^1 acting trivially except the qth, which acts via its $k(q)$th power map. Restricting V_q to a representation of $T^q \times C_{q+1} \times \cdots \times C_r$, we see that its unit sphere can be identified with the quotient group

$$(T^q \times C_{q+1} \times \cdots \times C_r)/(T^{q-1} \times C_q \times C_{q+1} \times \cdots \times C_r).$$

With $X = S^0$ and $G = T^q \times C_{q+1} \times \cdots \times C_r$, the Gysin sequence of $\chi(V_q)$ breaks up into short exact sequences that give the conclusion. $\square$

Now consider the following commutative diagram:

$$\begin{array}{ccc} (MU_T^*)^\wedge_{J_T} & \longrightarrow & MU^*(BT_+) \\ \downarrow & & \downarrow \\ (MU_G^*)^\wedge_{J_G} & \longrightarrow & MU^*(BG_+). \end{array} \tag{8.12}$$

The top horizontal arrow is an isomorphism and both vertical arrows are epimorphisms. Thus Conjecture 8.1 will hold if the following conjecture holds.

CONJECTURE 8.13. The map $(MU_G^*)^\wedge_{J_G} \longrightarrow MU^*(BG)$ is a monomorphism.

LEMMA 8.14. Conjecture 8.1 holds if G is a finite cyclic group.

PROOF. We embed G in S^1 and consider the standard representation $V = \mathbb{C}$ of S^1 as a representation of G. Again, $S(\infty V)$ is a model for EG. With $X = S^0$, the Gysin sequence (8.5) breaks up into four term exact sequences. Here we cannot conclude that multiplication by $\chi(V)$ is a monomorphism: its kernel is the image in MU_G^* of the odd degree elements of $MU_G^*(S(V)_+)$. However, in even degrees, the Gysin sequences of the representations nV give isomorphisms

$$MU_G^*/\chi(V)^n MU_G^* \cong MU_G^{2*}(S(nV)_+).$$

Therefore $(MU_G^*)^\wedge_{(\chi(V))}$ maps isomorphically onto $MU^{2*}(BG_+)$. This proves Conjecture 8.13; indeed, since $MU^*(BG_+)$ is concentrated in even degrees, it proves Conjecture 8.1 directly. □

Löffler asserts without proof that the general case of Conjecture 8.13 follows by the methods above. However, although $MU^*(BG_+)$ is concentrated in even degrees, the intended inductive proof may founder over the presence of odd degree elements in Gysin sequences, and we do not know whether or not the conjecture is true in general.

P. E. Conner and E. E. Floyd. The relation of corbordism to K-theories. Springer Lecture Notes in Mathematics Vol. 28. 1966.

P. S. Landweber. Elements of infinite filtration in complex cobordism. Math. Scand. 30(1972), 223-226.

P. S. Landweber. Cobordism and classifying spaces. Proc. Symp. Pure Math. Vol. 22., 1971, pp125-129.

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CHAPTER XXVI

Some calculations in complex equivariant bordism

by G. Comezaña

In this chapter we shall explain some basic results about the homology and cohomology theories represented by the spectrum MU_G. These theories arise from stabilized bordism groups of G-manifolds carrying a certain "complex structure"; exactly what this means is something we feel is not adequately discussed in the literature. Since the chapter includes a substantial amount of well-known information, as well as some new material and proofs of results claimed without proof elsewhere, we make no claims to originality except where noted. The author would like to thank Steven Costenoble for discussions and insights that have thrown a great deal of light on the subject matter.

1. Notations and terminology

G will stand throughout for a compact (and, in most cases, Abelian) Lie group, and subgroups of a such a group will be assumed to be closed. All manifolds considered will be compact and smooth, and all group actions smooth. If (X, A) and (Y, B) are pairs of G-spaces, we will use the notation $(X, A) \times (Y, B)$ for the pair $(X \times Y, (X \times B) \cup (Y \times A))$. Homology and cohomology theories on G-spaces will be reduced.

G-vector bundles over a G-space will be assumed to carry an inner product (which will be hermitian if the bundle is complex). Unless explicit mention to the contrary is made, representations will be understood to be finite-dimensional and $\mathbb{R}$-linear. Depending on the context, we shall sometimes think of V as a G-vector bundle over a point. If ξ is a G-bundle, $|\xi|$ will stand for its real dimension, $S(\xi)$

for its unit sphere, $D(\xi)$ for its unit disk, and $T(\xi)$ for its Thom space. If V is a representation of G, S^V will denote its one-point compactification. The trivial G-vector bundle over a G-space X with fiber V will be denoted ε_V.

We define the V-suspension $\Sigma^V X$ of a based G-space X to be $X \wedge S^V$; thus if ε_V is the trivial G-vector bundle over X with fiber V, then $T(\varepsilon_V) = \Sigma^V X$. We define the V-suspension $\Sigma^V(X, A)$ of a pair of spaces to be $(X, A) \times (DV, SV)$. In both cases, Σ^V is a functor; if V is a subrepresentation of W with orthogonal complement $W - V$, the inclusion induces a natural transformation $\sigma^{W-V} : \Sigma^V \longrightarrow \Sigma^W$.

2. Stably almost complex structures and bordism

When G is the trivial group, a *stably almost complex structure* on a compact smooth manifold M is an element $[\xi] \in \tilde{K}(M)$, which goes to the class $[\nu M]$ of the stable normal bundle under the map

$$\tilde{K}(M) \longrightarrow \widetilde{KO}(M).$$

It is, of course, essentially equivalent to define this with $[\tau M]$ replacing $[\nu M]$, since these classes are additive inverses in $\widetilde{KO}(M)$.

The following definition gives the obvious equivariant generalization of this.

DEFINITION 2.1. If $[\xi] \in \tilde{K}_G(M)$ is a lift of $[\nu M] \in \widetilde{KO}_G(M)$ under the natural map, we call the pair $(M, [\xi])$ a normally almost complex G-manifold.

We will use the notation $M_{[\xi]}$ when necessary, but we will drop $[\xi]$ whenever there is no risk of confusion.

The bordism theory of these objects, denoted mu_*^G, is the "complex analog" of the unoriented theory mo_*^G discussed in Chapter XV. If V is a *complex* G-module and $(M, \partial M)_{[\xi]}$ is a G-manifold with a stably almost complex structure, then its V-suspension becomes a G-manifold after "straightening the angles", and $[\xi] - [\varepsilon_V]$ is a complex structure on $\Sigma^V(M, \partial M)$. This gives rise to a *suspension homomorphism*

$$\sigma^V : mu_*^G(X, A) \longrightarrow mu_{*+|V|}^G(\Sigma^V(X, A)),$$

which sends the class of a map $(M, \partial M) \longrightarrow (X, A)$ to the class of its suspension. Due to the failure of G-transversality, both the suspension homomorphisms and the Pontrjagin-Thom map are generally not bijective.

We construct a stabilized version of this theory as follows. Let $\mathscr{U}$ be an infinite-dimensional complex G-module equipped with a hermitian inner product whose underlying $\mathbb{R}$-linear structure is that of a complete G-universe. Define

$$MU_*^G(X,A) = \operatorname{colim}_V mu_*^G(\Sigma^V(X,A)),$$

where V ranges over all finite-dimensional complex G-subspaces of $\mathscr{U}$ and the colimit is taken over all suspension maps induced by inclusions. We should perhaps point out that MU_*^G is not a connective theory unless G is trivial. The advantage of this new theory over mu_*^G is that the bad behavior of the Pontrjagin-Thom map is corrected, and the maps induced by suspension by complex G-modules are isomorphisms by construction. This should be interpreted as a form of periodicity. Homology or cohomology theories with this property are often referred to in the literature as *complex-stable.* Other examples of such theories include equivariant complex K-theory, its associated Borel construction, etc. Complex-stability isomorphisms should not be confused with suspension isomorphisms of the form

$$\Sigma^V : h_*^G(X,A) \longrightarrow h_{*+[V]}^G(\Sigma^V(X,A)),$$

which are part of the structure of all $RO(G)$-graded homology theories.

MU_*^G or, more precisely, its dual cohomology theory was first constructed by tom Dieck in terms of a G-prespectrum TU_G, bearing the same relationship to complex Grassmanians as the G-prespectrum TO_G discussed in XV§2, does to real ones. An argument of Bröcker and Hook for unoriented bordism readily adapts to the complex case to show the equivalence of the two approaches. In what follows, we shall focus on the spectrification MU_G of TU_G. As with any representable equivariant homology theory, MU_*^G can be extended to an $RO(G)$-graded homology theory, but we shall concern ourselves only with integer gradings. We point out, however, that complex-stable theories are always $RO(G)$-gradable.

A key feature of MU_G, proven in XXV§7, is the fact that it is a split G-spectrum; this may be seen geometrically as a consequence of the fact that the augmentation map $MU_*^G \longrightarrow MU_*$, given on representatives by neglect of structure, can be split by regarding non-equivariant stably almost complex manifolds as G-manifolds with trivial action. The splitting makes $MU_*^G = MU_*^G(S^0)$ a module over the ring MU_*.

The multiplicative structure of the ring G-spectrum MU_G can be interpreted geometrically as coming from the fact that the class of normally stably almost complex manifolds is closed under finite products. The complex-stability isomorphisms are well-behaved with respect to the multiplicative structure: in cohomology, we have a commutative diagram

$$\begin{array}{ccc} MU_G^*(X) \otimes MU_G^*(Y) & \longrightarrow & MU_G^*(X \wedge Y) \\ \Big\downarrow{\scriptstyle \sigma^V \otimes \sigma^W} & & \Big\downarrow{\scriptstyle \sigma^{V \oplus W}} \\ MU_G^{*+|V|}(\Sigma^V X) \otimes MU_G^{*+|W|}(\Sigma^W Y) & \longrightarrow & MU_G^{*+|V|+|W|}(\Sigma^{V\oplus W} X \wedge Y) \end{array}$$

for all based G-spaces X and Y and complex G-modules V and W. In general, for a multiplicative cohomology theory, commutativity of a diagram of the form above is assumed as part of the definition of complex-stability. K_G^* is another example of a multiplicative complex-stable cohomology theory, as is the Borel construction on any such theory.

The role of MU_G in the equivariant world is analogous to that of MU in classical homotopy theory, for its associated cohomology theory has a privileged position among those which are multiplicative, complex-stable, and have natural Thom classes (for complex G-vector bundles). We record the axiomatic definition of such theories.

DEFINITION 2.2. A G-equivariant multiplicative cohomology theory h_G^* is said to have natural Thom classes for complex G-vector bundles if for every such bundle ξ of complex dimension n over a pointed G-space X there exists a class $\tau_\xi \in h_G^{2n}(T(\xi))$, with the following three properties:

(1) *Naturality:* If $f : Y \longrightarrow X$ is a pointed G-map, then $\tau_{f^*\xi} = f^*(\tau_\xi)$.

(2) *Multiplicativity:* If ξ and η are complex G-vector bundles over X, then

$$\tau_{\xi\oplus\eta} = \tau_\xi \times \tau_\eta \in h_G^{|\xi|+|\eta|}(T(\xi \oplus \eta)).$$

(3) *Normalization:* If V is a complex G-module, then $\tau_V = \sigma^V(1)$.

The following result, which admits a quite formal proof (given for example by Okonek) explains the universal role played by MU_G.

PROPOSITION 2.3. If h_G^* is a multiplicative, complex-stable, cohomology theory with natural Thom classes for complex G-bundles, then there is a unique natural transformation $MU_G^*(\bullet) \longrightarrow h_G^*(\bullet)$ of multiplicative cohomology theories that takes Thom classes to Thom classes.

Returning to homology, for a complex G-bundle ξ of complex dimension k, the Thom class of ξ gives rise to a Thom isomorphism

$$\tau : MU_*^G(T(\xi)) \longrightarrow MU_{*-2k}^G(B(\xi)_+),$$

and similarly in cohomology. This isomorphism is constructed in the same way as in the nonequivariant case (see e.g. [LMS]), without using any feature of MU_*^G other than the existence and formal properties of Thom classes. However, in this special case, its *inverse* has a rather pleasant geometric interpretation: if $f : M \longrightarrow B(\xi)$ represents an element in $mu_n^G(B(\xi)_+)$, the map $\overline{f}$ in the pullback diagram

$$\begin{array}{ccc} E(f^*\xi) & \xrightarrow{\overline{f}} & E(\xi) \\ {\scriptstyle f^*\xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ M & \xrightarrow[f]{} & B(\xi) \end{array}$$

represents an element in $mu_{n+2k}^G(T(\xi))$. This procedure allows the construction of a homomorphism which stabilizes to the inverse of the Thom isomorphism. See Bröcker and Hook for the details of a treatment of the Thom isomorphism (in the unoriented case) that uses this interpretation.

T. Bröcker and E.C. Hook. Stable equivariant bordism. Mathematische Zeitschrift 129(1972), pp. 269–277.

T. tom Dieck. Bordism of G-manifolds and integrality theorems. Topology 9(1970), pp. 345–358.

C. Okonek. Der Conner-Floyd-Isomorphismus für Abelsche Gruppen. Mathematische Zeitschrift 179(1982), pp. 201–212.

3. Tangential structures

Unfortunately, both mu_*^G and MU_*^G are rather intractable from the computational point of view. In order to address this difficulty, we shall introduce a new bordism theory, much more amenable to calculation, whose stabilization is also MU_G^*.

Consider the following variant of reduced K-theory: if X is a G-space, instead of taking the quotient by the subgroup generated by all trivial complex G-bundles, take the quotient by the subgroup generated by those trivial bundles of the form $\mathbb{C}^n \times X$, where G acts trivially on $\mathbb{C}^n$. We denote the group so obtained as $\check{K}_G$; there is an analogous construction in the real case, which we denote $\check{KO}_G$.

Definition 3.1. A tangentially stably almost complex manifold is a smooth manifold equipped with a lift of the class $[\tau M] \in \check{KO}_G(M)$ to $\check{K}_G(M)$.

We shall refer to the bordism theory of these manifolds as *tangential complex bordism*, denoted $\Omega_*^{U,G}$.

We warn the reader that nowhere in the literature is the distinction between the complex bordism theories $\Omega_*^{U,G}$ and mu_*^G made clear. This is not mere pedantry on our part, as our next result will show. It was pointed out to the author by Costenoble that this result does not hold for normally stably almost complex G-manifolds.

PROPOSITION 3.2. If M is a tangentially stably almost complex G-manifold and $H \subseteq G$ is a closed normal subgroup, then the G-tubular neighborhood around M^H has a complex structure.

We stress the fact that no stabilization is necessary to get a complex structure on the tubular neighborhood; this lies at the heart of the calculations we shall carry out later in the chapter.

PROOF. The first thing to observe is that $\tau(M^H) = (\tau M|_{M^H})^H$ as real vector bundles. If ξ is the restriction to M^H of a complex G-vector bundle over M that represents its tangential stably almost complex structure, and the underlying real G-vector bundle of ξ is $\tau M|_{M^H} \oplus \varepsilon_{\mathbb{R}^n}$, then $(\xi^H)^-$ is a complex G-vector bundle. We have

$$\xi = \xi^H \oplus (\xi^H)^- = (\tau M|_{M^H})^H \oplus \varepsilon_{\mathbb{R}^n} \oplus \nu(M^H, M).$$

This gives the desired structure. □

We next explore the relation between mu_*^G and $\Omega_*^{U,G}$. There is a commutative square

$$\begin{array}{ccc} \check{K}_G(X) & \longrightarrow & \check{KO}_G(X) \\ \downarrow & & \downarrow \\ \widetilde{K}_G(X) & \longrightarrow & \widetilde{KO}_G(X) \end{array}$$

that yields a natural transformation of homology theories $\phi : mu_*^G \longrightarrow \Omega_*^{U,G}$. Just as we did with mu_*^G, we may stabilize $\Omega_*^{U,G}$ with respect to suspensions by finite-dimensional complex subrepresentations of a complete G-universe, obtaining a new complex-stable homology theory which we shall provisionally denote $\check{MU}_*^G$. The map ϕ stabilizes to a natural transformation $\Phi : \check{MU}_*^G \longrightarrow MU_*^G$. The following result was first proved by the author and Costenoble by a different argument and is central to the results of this chapter.

THEOREM 3.3. Φ is an isomorphism of homology theories.

We shall need the following standard result.

LEMMA 3.4. *(Change of groups isomorphism)* If $H \subseteq G$ is a closed subgroup of codimension j, then for all H-spaces X there is an isomorphism

$$mu_*^H(X_+) \xrightarrow{\cong} mu_{*+j}^G((G \times_H X)_+)$$

induced by application of the functor $G \times_H (\bullet))$ to representatives of bordism classes of maps, and similarly for pairs. The analogous result holds for $\Omega_*^{U,G}$ and MU_*^G.

SKETCH PROOF. If we apply the functor $G \times_H (\bullet)$ to a map $f : M \longrightarrow X$ that represents an element of $mu_n^H(X_+)$, we obtain an element of $mu_{n+j}^G((G \times_H X)_+)$. Conversely, if $g : N \longrightarrow G \times_H X$ represents an element of $mu_{n+j}^G((G \times_H X)_+)$ and if $\pi : G \times_H X \longrightarrow X$ is the evident H-map, we set $M = (\pi g)^{-1}(eH)$ and see that M is an H-manifold such that $N = G \times_H M$ and the restriction of g to M represents an element of $mu_n^H(X_+)$. $\square$

PROOF OF THEOREM 3.3. We show first that the theorem is true for $G = SU(2k+1)$ and then extend the result to the general case by a change of groups argument.

We recall a few standard facts about representations of special unitary groups (e.g., from Bröcker and tom Dieck). Let M be the complex $SU(2k+1)$-module such that $M = \mathbb{C}^{2k+1}$ with the action of $SU(2k+1)$ given by matrix multiplication and let $\Lambda^i = \Lambda^i M$. Then $R(SU(2k+1))$ is the polynomial algebra over $\mathbb{Z}$ on the representations Λ^i, $1 \leq i \leq 2k$, all of which are irreducible and of complex type. Furthermore, $\Lambda^{2k-i+1} = \overline{\Lambda^i}$. This implies that any irreducible real representation of $SU(2k+1)$ is either trivial or admits a complex structure. To see this, let W be a non-trivial irreducible real $SU(2k+1)$-module. Suppose first that $W \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible. Since the restriction to $\mathbb{R}$ of an irreducible complex representation of quaternionic type is irreducible, our assumptions imply that $W \otimes_{\mathbb{R}} \mathbb{C}$ is of real type and of the form $V \otimes_{\mathbb{C}} \overline{V}$, where V is a monomial in the Λ^i, $1 \leq i \leq k$. We have

$$(\overline{V} \otimes_{\mathbb{C}} V) \otimes_{\mathbb{R}} \mathbb{C} \cong (2W) \otimes_{\mathbb{R}} \mathbb{C} \cong 2(V \otimes_{\mathbb{C}} \overline{V})$$

as complex representations. On the other hand, since $2W \cong V \otimes_{\mathbb{C}} \overline{V}$, we have isomorphisms of complex $SU(2k+1)$-modules

$$(2W) \otimes_{\mathbb{R}} \mathbb{C} \cong (V \otimes_{\mathbb{C}} \overline{V}) \otimes_{\mathbb{R}} \mathbb{C} \cong V \otimes_{\mathbb{C}} (\overline{V} \otimes_{\mathbb{R}} \mathbb{C})$$

and

$$V \otimes_{\mathbb{C}} (\overline{V} \otimes_{\mathbb{R}} \mathbb{C}) \cong (V \otimes_{\mathbb{C}} V) \oplus (V \otimes_{\mathbb{C}} \overline{V})$$

(because $\overline{V} \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$). So it follows that

$$2(V \otimes_{\mathbb{C}} \overline{V}) \cong (V \otimes_{\mathbb{C}} V) \oplus (V \otimes_{\mathbb{C}} \overline{V}),$$

which is absurd in view of the structure of $RSU(2k+1)$. Thus W must be reducible and so it is either of the form $V_1 \oplus V_1$, for an irreducible complex V_1 of quaternionic type, or $V_1 \oplus \overline{V_1}$, for an irreducible complex V_1 of complex type. The first possibility is ruled out by the fact that all self-conjugate irreducible complex representations of $SU(2k+1)$ are of real type. So we must have

$$2W \cong V_1 \oplus \overline{V}_1 \cong 2V$$

as real representations, and therefore, using the uniqueness of isotypical decompositions, we may conclude that $W \cong V$ as real representations.

Now let X be a $SU(2k+1)$-space and consider a map representing an element in $MU_*^G(X)$. By complex-stability, there is no loss of generality in assuming that our map is of the form $f : M \longrightarrow X$, where $\tau M \oplus \epsilon_V \cong \xi$, V is a real representation, and ξ is a complex $SU(2k+1)$-vector bundle. By the remark above, $V = W \oplus \mathbb{R}^k$ for a complex representation W. Then $\Sigma^W(M, \partial M)$ is a tangentially stably almost complex manifold and the class of $\Sigma^W f$ is in the image of ϕ. It follows that Φ is surjective. A similar argument applied to bordisms shows that Φ is injective.

To obtain the general case, observe that any compact Lie group embeds in $U(2k)$, and $U(2k)$ embeds in $SU(2k+1)$ (via the map that sends $A \in U(2k)$ to $(det\ A)^{-1} \cdot 1_{\mathbb{R}} \oplus A$), and apply Lemma 3.4. □

T. Bröcker and T. tom Dieck. representations of compact Lie groups. Springer. 1985.
C. Okonek. Der Conner-Floyd-Isomorphismus für Abelsche Gruppen. Mathematische Zeitschrift 179(1982), pp. 201–212.

4. Calculational tools

For the remainder of the chapter, all Lie groups we consider will be Abelian.

There is a long list of names associated to the calculation of $\Omega_*^{U,G}(S^0)$ for different classes of compact Lie groups: Landweber (cyclic groups), Stong (Abelian p-groups), Ossa (finite Abelian groups), Löffler (Abelian groups), Lazarov (groups of order pq for distinct primes p and q), and Rowlett (extensions of a cyclic group by a cyclic group of relatively prime order). All of these authors rely on the study of fixed point sets by various subgroups, together with their normal bundles, through the use of bordism theories with suitable restrictions on isotropy subgroups.

The main calculational tool is the use of families of subgroups, which works in exactly the same fashion as was discussed in the real case in XV§3. Recall that, for a family $\mathscr{F}$, an $\mathscr{F}$-space is a G-space all of whose isotropy subgroups are in $\mathscr{F}$ and that we write $E\mathscr{F}$ for the universal $\mathscr{F}$-space. Recall too that, for a G-homology theory h_*^G and a pair of families $(\mathscr{F}, \mathscr{F}')$, $\mathscr{F}' \subseteq \mathscr{F}$, there is an associated homology theory $h_*^G[\mathscr{F}, \mathscr{F}']$, defined on pairs of G-spaces as

$$h_*^G[\mathscr{F}, \mathscr{F}'](X, A) = h_*^G(X \times E\mathscr{F}, (X \times \mathscr{F}') \cup (A \times E\mathscr{F})).$$

When $\mathscr{F}' = \emptyset$, we use the notation $h_*^G[\mathscr{F}]$. The theories $h_*^G[\mathscr{F}]$, $h_*^G[\mathscr{F}']$, and $h_*^G[\mathscr{F}, \mathscr{F}']$ fit into a long exact sequence. Of course, there is an analogous construction in cohomology.

In the special case of $\Omega_*^{U,G}$ (and similarly for other bordism theories), it is easy to see that $\Omega_*^{U,G}[\mathscr{F}, \mathscr{F}']$ has an alternative interpretation: it is the bordism theory of $(\mathscr{F}, \mathscr{F}')$- tangentially almost-complex manifolds, that is, compact, tangentially almost complex $\mathscr{F}$-manifolds with boundary, whose boundary is an $\mathscr{F}'$-manifold.

Definition 4.1. A pair of families $(\mathscr{F}, \mathscr{F}')$ of subgroups of G is called a neighboring pair differing by H if there is a subgroup H such that if $K \in \mathscr{F} - \mathscr{F}'$, then H is a subconjugate of K.

This notion was first used by Löffler, but the terminology is not standard. A special case is the more usual notion of an *adjacent* pair of families pair differing by H, which is a neighboring pair $(\mathscr{F}, \mathscr{F}')$ such that $\mathscr{F} - \mathscr{F}'$ consists of those subgroups conjugate to H.

The next proposition explains the importance of neighboring families. We introduce some terminology and notation to facilitate its discussion.

Given a subgroup H of an Abelian Lie group G, we choose a set $\mathscr{C}_{G,H}$ of finite dimensional complex G-modules whose restrictions to H form a non-redundant, complete set of irreducible, nontrivial complex H-modules. If $\mathbb{C}$ denotes the trivial irreducible representation, we let $\mathscr{C}^+_{G,H} = \mathscr{C}_{G,H} \cup \{\mathbb{C}\}$. For a nonnegative even integer k, we shall call an array of nonnegative integers $P = (p_V)_{V \in \mathscr{C}_{G,H}}$ a (G, H)-*partition* of k if

$$k = \sum_{V \in \mathscr{C}_{G,H}} 2p_V.$$

For such a partition P, we let

$$BU(P,G) = \prod_{V \in \mathscr{C}_{G,H}} BU(p_V, G).$$

We let $\mathscr{P}(k,G,H)$ denote the set of all (G,H)-partitions of k.

PROPOSITION 4.2. If $(\mathscr{F},\mathscr{F}')$ is a neighboring pair of families of subgroups of a compact Abelian Lie group G differing by a subgroup H, then

$$\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X,A) \cong \bigoplus_{\substack{0 \le 2k \le n \\ P \in \mathscr{P}(2k,G,H)}} \Omega_{n-2k}^{U,G/H}[\mathscr{F}/H]((X^H,A^H) \times BU(P,G/H)),$$

where $\mathscr{F}/H$ denotes the family of subgroups of G/H that is obtained by taking the quotient of each element of $\mathscr{F} - \mathscr{F}'$ by H.

SKETCH OF PROOF. For simplicity, we concentrate on the absolute case. Let $f : M \longrightarrow X$ represent an element in $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X_+)$ and let T be a (closed) G-tubular neighborhood of M^H. We may view T as the total space of the unit disc bundle of the normal bundle to M^H. We may also view T as an n-dimensional $\mathscr{F}$-manifold whose boundary is an $\mathscr{F}'$-manifold. Thus T represents an element of $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](S^0)$, and we see that $[f] = [f|_T]$ in $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X_+)$. Furthermore, $[f] = 0$ if and only if there is an H-trivial G-nullbordism of $f|_T$, equipped with a complex G-vector bundle whose unit disc bundle restricts to T on M^H. Observe that M^H breaks up into various components of constant even codimension. In other words, $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X_+)$ can be identified with the direct sum, with $2k$ ranging between 0 and n, of bordism of H-trivial $\mathscr{F}$-manifolds of dimension $n-2k$ equipped with a complex G-vector bundle of dimension k, containing no H-trivial summands. Note the twofold importance of Proposition 3.2: not only are we using that M^H is tangentially almost complex, but also that its tubular neighborhood carries a complex structure.

Consider the bundle-theoretic analog of the isotypical decomposition of a linear representation. For complex G-vector bundles E and F over a space X we may construct the vector bundle $\mathrm{Hom}_{\mathbb{C}}(E,F)$ whose fiber over $x \in X$ is $\mathrm{Hom}_{\mathbb{C}}(E_x,F_x)$; G acts on $\mathrm{Hom}_{\mathbb{C}}(E,F)$ by conjugation. If X is H-trivial, then $\mathrm{Hom}_H(E,F) = (\mathrm{Hom}_{\mathbb{C}}(E,F))^H$ is an H-trivial G-subbundle; if one regards X as a (G/H)-space, then $\mathrm{Hom}_H(E,F)$ becomes a (G/H)-vector bundle over X.

We apply this to $F = T$ and $E = \varepsilon_V$, where V is a complex G-module whose restriction to H is irreducible, thus obtaining a (G/H)-vector bundle which we call the V-*multiplicity* of E. The evaluation map

$$\bigoplus_{V \in \mathscr{C}^+_{G,H}} \mathrm{Hom}_H(\varepsilon_V, T) \otimes_{\mathbb{C}} \varepsilon_V \longrightarrow T$$

is a G-vector bundle isomorphism, and this decomposition into isotypical summands is unique. Note that in the special case we are considering, the multiplicity associated to the trivial representation is 0, so the sum really does run over $\mathscr{C}_{G,H}$.

T can therefore be identified with a direct sum of (G/H)-vector bundles over M^H, each corresponding to an irreducible complex representation of H, and M^H breaks into a disjoint union of components on which the dimension of each multiplicity remains constant; each of these components has therefore an associated (G,H)-partition, accounting for the summation over $\mathscr{P}(2k,G,H)$ in our formula. Clearly the bundle on the component associated to a (G,H)-partition P is classified by $BU(P, G/H)$. □

Similar methods allow us to prove the following standard result.

PROPOSITION 4.3. With the notation above, if H is a subgroup of an Abelian Lie group G, then

$$BU(n,G)^H \cong \coprod_{P \in \mathscr{P}(n,G,H)} \prod_{V \in \mathscr{C}^+_{G,H}} BU(p_V, G/H)$$

as H-trivial G-spaces.

PROOF. It suffices to observe that the right hand side classifies n-dimensional complex G-vector bundles over H-trivial G-spaces. □

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C. Lazarov. Actions of groups of order pq. Transactions of the Amer. Math. Soc. 173(1972), pp. 215–230.
P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. Manuscripta Mathematica 12(1974), 307–327.
E. Ossa, Unitary bordism of Abelian groups. Proceedings of the American Mathematical Society 33(1972), pp. 568–571.
R.J. Rowlett. Bordism of metacyclic group actions. Michigan Mathematical Journal 27(1980), pp. 223–233.
R. Stong. Complex and oriented equivariant bordism. in Topology of Manifolds (J.C. Cantrell and C.H. Edwards, editors). Markham, Chicago 1970.

5. Statements of the main results

We come now to a series of theorems, some old, some new, that are consequences of the previous results. In all of them, we consider a given compact Abelian Lie group G.

THEOREM 5.1 (LÖFFLER). If V is a complex G-module, and X is a disjoint union of pairs of G-spaces of the form

$$(DV, SV) \times \prod_{i=1}^{k} BU(n_i, G),$$

then $\Omega_*^{U,G}(X)$ is a free MU_*-module concentrated in even degrees.

THEOREM 5.2. With the same hypotheses on X, the map

$$\Omega_*^{U,G}(BU(n,G) \times X) \longrightarrow \Omega_*^{U,G}(BU(n+1,G) \times X)$$

induced by Whitney sum with the trivial bundle $\varepsilon_{\mathbb{C}}$ is a split monomorphism of MU_*-modules.

THEOREM 5.3. MU_*^G is a free MU_*-module concentrated in even degrees.

THEOREM 5.4. The stabilization map $\Omega_*^{U,G} \longrightarrow MU_*^G$ is a split monomorphism of MU_*-modules.

Theorem 5.3 is stated in the second paper of Löffler cited below, but there seems to be no proof in the literature. Ours is a refinement of the ideas in the proof of Theorem 5.1, which yields Theorem 5.4 as a by-product, and is entirely self-contained (that is, it does not depend on results on finite Abelian groups). Tom Dieck has used a completely different method to prove a weaker version of Theorem 5.4, for G cyclic of prime order, but to the best of our knowledge nothing of the sort has previously been claimed or proved at our level of generality. Theorem 5.2, which also seems to be new, is required in the course of the proof of Theorem 5.3 and is of independent interest.

In the light of these results, it is natural to conjecture, probably overoptimistically, that MU_*^G is free over MU_* and concentrated in even degrees for any compact Lie group G. We have succeeded in verifying this for a class of non-Abelian groups that includes $O(2)$ and the dihedral groups. The statement about the injectivity of the stabilization map also holds for these groups. We hope to extend these results to other classes of non-Abelian groups; details will appear elsewhere.

The results above should be proven in the given order, but, since the proofs have a large overlap, we shall deal with all of them simultaneously.

We shall proceed by induction on the number of "cyclic factors" of the group, where, for the purposes of this discussion, S^1 counts as a cyclic group. The argument in each case is as follows: the result is either trivial or well-known for the trivial group. Then, one shows that if the result is true for a compact Lie group G, it also holds for $G \times S^1$, and this in turn implies the same for $G \times \mathbb{Z}_n$.

T. tom Dieck. Bordism of G-manifolds and integrality theorems. Topology 9(1970), pp. 345–358.
P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. Manuscripta Mathematica 12(1974), 307–327.
P. Löffler. Equivariant unitary bordism and classifying spaces. Proceedings of the International Symposium on Topology and its Applications, Budva, Yugoslavia 1973, pp. 158–160.

6. Preliminary lemmas and families in $G \times S^1$

For brevity, the subgroups $\{1\} \times S^1 \subseteq G \times S^1$ and $\{1\} \times \mathbb{Z}_n \subseteq G \times \mathbb{Z}_n$ will be denoted S^1 and $\mathbb{Z}_n$, respectively.

We shall need to consider the following families of subgroups of $G \times S^1$:

$$\begin{aligned} \mathscr{F}_i &= \{H \subseteq G \times S^1 \mid |H \cap S^1| \leq i\} \\ \mathscr{F}_\infty &= \{H \subseteq G \times S^1 \mid H \cap S^1 \neq S^1\} \\ \mathscr{A} &= \{\text{all closed subgroups of } G \times S^1\} \end{aligned}$$

These give rise to the neighboring pairs $(\mathscr{F}_{i+1}, \mathscr{F}_i)$ (differing by $\mathbb{Z}_{i+1}$) and $(\mathscr{A}, \mathscr{F}_\infty)$ (differing by S^1). Observe that $\mathscr{F}_\infty$ is the union of its subfamilies $\mathscr{F}_i$.

Lemma 6.1. Let G be a compact Lie group and X be a pair of $(G \times S^1)$-spaces. Then

$$\Omega_*^{U, G \times S^1}(X \times S^1) \cong \Omega_{*-1}^{U,G}(X)$$

and

$$\Omega_*^{U, G \times S^1}((X \times S^1)/\mathbb{Z}_n) \cong \Omega_{*-1}^{U, G \times \mathbb{Z}_n}(X),$$

where $G \times S^1$ acts on S^1 and $S^1/\mathbb{Z}_n$ through the projection $G \times S^1 \longrightarrow S^1$; the same statement holds for the theories $mu_*^{G \times S^1}$ and $MU_*^{G \times S^1}$.

The proofs of these isomorphisms are easy verifications and will be omitted; see Löffler. We shall also need the following result of Conner and Smith.

Lemma 6.2. A graded, projective, bounded below MU_*-module is free.

LEMMA 6.3. Consider a diagram of projective modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0. \end{array}$$

If f_1 and f_3 (resp. f_2 and f_3) are split monomorphisms, so is f_2 (resp. f_1).

PROOF. Add a third row consisting of the cokernels of the f_i, which will be exact by the Snake Lemma. An easy diagram chase shows that the modules in the new row are projective, and therefore the conclusion follows. □

Note that we make no assumptions about compatibility of the splittings.

REMARK 6.4. If X is a pair of G-spaces of the kind appearing in the statement of Theorem 5.1 and H is a subgroup of G, then restricting the action to H yields an H-pair of the same kind. Moreover, by Proposition 4.3, X^H is a (G/H)-pair of the same type. This class of pairs of spaces is also closed under cartesian product with $BU(n,G)$ and with pairs of the form (DW, SW) for a complex G-module W.

P. E. Conner, L. Smith, *On the complex bordism of finite complexes*, Publications Mathématiques de l'IHES, no. 37 (1969), pp. 417–521.
P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. Manuscripta Mathematica 12(1974), 307–327.

7. On the families $\mathscr{F}_i$ in $G \times S^1$

In what follows, for a G-pair X and a homology theory h_*, ψ will designate a map of the form

$$\psi : h_*(BU(n,G) \times X) \longrightarrow h_*(BU(n+1,G) \times X)$$

that is induced by taking the Whitney sum of the universal complex G-bundle over $BU(n,G)$ and the trivial G-bundle $\varepsilon_{\mathbb{C}}$.

Suppose that all four theorems stated above have been proved for G. We shall deduce the following result in the case $G \times S^1$.

THEOREM 7.1. The following statements hold for each $i \geq 1$ and for $i = \infty$.

(1) $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$ is a free MU_*-module concentrated in *odd* degrees.

(2) The map

$$\psi : \Omega_*^{U,G\times S^1}[\mathscr{F}_i](BU(n,G\times S^1)\times X) \longrightarrow \Omega_*^{U,G\times S^1}[\mathscr{F}_i](BU(n+1,G\times S^1)\times X)$$

is a split monomorphism of MU_*-modules.

(3) If W is an irreducible complex $(G \times S^1)$-module, then

$$\sigma^W : \Omega_*^{U,G\times S^1}[\mathscr{F}_i](X) \longrightarrow \Omega_{*+2}^{U,G\times S^1}[\mathscr{F}_i]((DW, SW) \times X)$$

is a split monomorphism of MU_*-modules.

(4) The map $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X) \longrightarrow \Omega_*^{U,G\times S^1}(X)$ is zero.

PROOF. We first prove this for $i = 1$, making use of a suitable model for the space $E\mathscr{F}_1$. Let $(W_i)_{i\geq 1}$ be a sequence of irreducible complex $(G \times S^1)$-modules such that S^1 acts freely on their unit circles, and every isomorphism class of such $(G \times S^1)$-modules appears infinitely many times. Let $V_k = \bigoplus_{i=1}^k W_i$ and

$$SV_\infty = \text{colim}_k SV_k;$$

SV_∞ is the required space. Note also that this space embeds into the equivariantly contractible space

$$DV_\infty = \text{colim}_k DV_k.$$

Using Lemma 6.1 and our assumptions about G, we see that $\Omega_*^{U,G\times S^1}(SV_1 \times X)$ is a free MU_*-module concentrated in odd degrees, and that

$$\sigma^W : \Omega_*^{U,G\times S^1}(SV_1 \times X) \longrightarrow \Omega_*^{U,G\times S^1}((DW, SW) \times SV_1 \times X)$$

and

$$\Omega_*^{U,G\times S^1}(SV_1 \times BU(n, G \times S^1) \times X) \longrightarrow \Omega_*^{U,G\times S^1}(SV_1 \times BU(n+1, G \times S^1) \times X)$$

are split monomorphisms of MU_*-modules.

We calculate $\Omega_*^{U,G\times S^1}((SV_{k+1}, SV_k) \times X)$ using the homotopy equivalence

$$(SV_{k+1}, SV_k) \simeq (SW_{k+1} * SV_k, DW_{k+1} \times SV_k),$$

and the excisive inclusion

$$SW_{k+1} \times (DV_k, SV_k) \longrightarrow (SW_{k+1} * SV_k, DW_{k+1} \times SV_k).$$

The action of $G \times S^1$ on SW_{k+1} determines and is determined by a split group epimorphism $G \times S^1 \longrightarrow S^1$ with kernel $H \subseteq G \times S^1$, $H \cong G$. This implies that SW_{k+1} is $(G \times S^1)$-homeomorphic to $(G \times S^1)/H$. By a change of groups argument and the inductive hypothesis, we see that $\Omega_*^{G,U}((SV_{k+1}, SV_k) \times X)$ is free and concentrated in odd degrees and that the maps induced respectively by suspension by an irreducible complex G-module and by addition of the bundle $\varepsilon_{\mathbb{C}}$ are split monomorphisms of MU_*-modules.

The diagram with exact columns (in which j is odd)

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_j^{U,G\times S^1}(SV_k \times X) & \xrightarrow{\sigma^W} & \Omega_{j+2}^{U,G\times S^1}((DW,SW)\times SV_k \times X) \\
\downarrow & & \downarrow \\
\Omega_j^{U,G\times S^1}(SV_{k+1} \times X) & \xrightarrow{\sigma^W} & \Omega_{j+2}^{U,G\times S^1}((DW,SW)\times SV_{k+1} \times X) \\
\downarrow & & \downarrow \\
\Omega_j^{U,G\times S^1}((SV_{k+1}, SV_k) \times X) & \xrightarrow{\sigma^W} & \Omega_{j+2}^{U,G\times S^1}((DW,SW)\times (SV_{k+1}, SV_k) \times X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

and the results above show by induction that, for all $k \geq 1$, $\Omega_*^{U,G\times S^1}(SV_k \times X)$ is free and concentrated in odd degrees and that σ^W is a split monomorphism. An analogous diagram shows the same is true for the map ψ induced by adding $\varepsilon_{\mathbb{C}}$.

To complete the proofs of (1) – (3) when $i = 1$, it suffices to observe that each step in the colimit contributes a direct summand to SV_∞. To prove (4), let $f : M \longrightarrow X \times SV_\infty$ represent an element of $\Omega_*^{U,G\times S^1}[\mathscr{F}_1](X)$. Since S^1 acts freely on M and all actions on a circle are linear, $p : M \longrightarrow M/S^1$ is the unit circle bundle of a 1-dimensional complex G-bundle E (the complex structure is given by multiplication by $i \in S^1$). Obviously, the circle bundle bounds a disc bundle, whose total space is a complex $(G \times S^1)$-manifold W. Any point $x \in W$ can be written as ty, where $t \in [0,1]$ and $y \in M$, so f extends to an equivariant map $F : W \longrightarrow X \times DV_\infty$ defined as $F(ty) = tf(y)$, where the multiplication on the right hand side is given by the linear structure of DV_∞.

We prove the case $i \geq 1$ of Theorem 7.1 by induction on i. Observe first that the case $i = \infty$ will follow directly from the case of finite i since

$$E\mathscr{F}_\infty = \mathrm{colim}_i E\mathscr{F}_i.$$

Indeed, we shall see that each stage in the construction of $E\mathscr{F}_\infty$ as a colimit contributes a free direct summand to $\Omega_*^{U,G\times S^1}[\mathscr{F}_\infty](X)$ on which σ^W and ψ are split monomorphisms of MU_*-modules and the map to $\Omega_*^{U,G\times S^1}(X)$ is zero.

Applying Proposition 4.2 with (G, H) replaced by $(G\times S^1, \mathbb{Z}_{i+1})$ and noting that $(G \times S^1)/\mathbb{Z}_{i+1} \cong G \times S^1$ and that, under this isomorphism, the family $\mathscr{F}_{i+1}/\mathbb{Z}_{i+1}$ corresponds to the family $\mathscr{F}_1$, we find that

$$\Omega_n^{U,G\times S^1}[\mathscr{F}_{i+1}, \mathscr{F}_i](X) \cong \bigoplus_{\substack{0\le 2k\le n \\ P\in \mathscr{P}(2k, G\times S^1, \mathbb{Z}_{n+1})}} \Omega_{n-2k}^{U,G\times S^1}[\mathscr{F}_1](X^{\mathbb{Z}_{n+1}} \times BU(P, G \times S^1)).$$

Thus the case $i = 1$, combined with Remark 6.4, shows that the left-hand side is free and concentrated in *odd* degrees.

One then concludes, by using the long exact sequences of the pairs $[\mathscr{F}_{i+1}, \mathscr{F}_i]$, that for all i, $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$ is concentrated in *odd* degrees.

The diagrams with exact columns (in which j is odd)

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_j^{U,G}[\mathscr{F}_i](BU(n, G \times S^1) \times X) & \longrightarrow & \Omega_j^{U,G}[\mathscr{F}_i](BU(n+1, G \times S^1) \times X) \\
\downarrow & & \downarrow \\
\Omega_j^{U,G}[\mathscr{F}_{i+1}](BU(n, G \times S^1) \times X) & \longrightarrow & \Omega_j^{U,G}[\mathscr{F}_{i+1}](BU(n+1, G \times S^1) \times X) \\
\downarrow & & \downarrow \\
\Omega_j^{U,G}[\mathscr{F}_{i+1}, \mathscr{F}_i](BU(n, G \times S^1) \times X) & \longrightarrow & \Omega_j^{U,G}[\mathscr{F}_{i+1}, \mathscr{F}_i](BU(n+1, G \times S^1) \times X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

show that, for all i, $\Omega_*^{U,G}[\mathscr{F}_i](X)$ is a free MU_*-module and the map induced by addition of $\varepsilon_{\mathbb{C}}$ is a split monomorphism of MU_*-modules.

The study of the suspension map σ^W must be broken into two cases. Since W is an irreducible representation of $G \times S^1$, its fixed point space W^{S^1} is either W or $\{0\}$ and therefore either

(1) $W^{\mathbb{Z}_{i+1}} = W$ or
(2) $W^{\mathbb{Z}_{i+1}} = \{0\}$.

In the first case, the map

$$\sigma^W : \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_{i+1}, \mathscr{F}_i](X) \longrightarrow \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_{i+1}, \mathscr{F}_i]((DW, SW) \times X), \tag{7.2}$$

can be regarded via Proposition 4.2 as a direct sum of suspension maps

$$\Omega_{2l+1}^{U,G\times S^1}[\mathscr{F}_1](Y) \longrightarrow \Omega_{2l+3}^{U,G\times S^1}[\mathscr{F}_1]((DW,SW)\times Y),$$

where $Y = X^{\mathbb{Z}_{i+1}} \times BU(P, G\times S^1)$ for some partition P of $2(j-l)$ and we think of W as a representation of $G\times (S^1/\mathbb{Z}_{i+1}) \cong G\times S^1$. Thus it follows from the case $i=1$ that (7.2) is a split monomorphism of MU_*-modules in this case.

For the second case consider a $(G\times S^1, \mathbb{Z}_{i+1})$-partition $P = (p_V)_{V\in\mathscr{C}_{G\times S^1,\mathbb{Z}_{i+1}}}$ of an even integer k. Let $P' = (p'_V)_{V\in\mathscr{C}_{G\times S^1}}$ denote the $(G\times S^1, \mathbb{Z}_{i+1})$-partition of $k+2$ defined by

$$p'_V = \begin{cases} p_V + 1 & \text{if } V = W \\ p_V & \text{otherwise.}\end{cases}$$

Since $W^{\mathbb{Z}_{i+1}} = \{0\}$, Proposition 4.2 implies that the map (7.2) can be interpreted as a direct sum of maps of the form

$$\psi : \Omega_{2l+1}^{U,G\times S^1}[\mathscr{F}_1](X^{\mathbb{Z}_{i+1}} \times BU(P,G)) \longrightarrow \Omega_{2l+3}^{U,G\times S^1}[\mathscr{F}_1](X^{\mathbb{Z}_{i+1}} \times BU(P',G))$$

induced by addition of $\varepsilon_{\mathbb{C}}$ to the multiplicity bundle corresponding to the V in the decomposition. We know already that maps of this kind are split monomorphisms of MU_*-modules, and we conclude that (7.2) is always a split monomorphism of MU_*-modules.

Now the following diagram with exact columns implies inductively that, for all i, σ^W is a split monomorphism of MU_*-modules on $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_i](X) & \xrightarrow{\sigma^W} & \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_i]((DW,SW)\times X) \\
\downarrow & & \downarrow \\
\Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_{i+1}](X) & \xrightarrow{\sigma^W} & \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_{i+1}]((DW,SW)\times X) \\
\downarrow & & \downarrow \\
\Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_{i+1},\mathscr{F}_i](X) & \xrightarrow{\sigma^W} & \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_{i+1},\mathscr{F}_i]((DW,SW)\times X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Finally, to prove (4) of Theorem 7.1, let $f : M \longrightarrow X$ represent an element of $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$, $i > 1$, and suppose that we have already proved that

$$\Omega_*^{U,G\times S^1}[\mathscr{F}_j](X) \longrightarrow \Omega_*^{U,G\times S^1}(X)$$

is zero for all $j < i$. We shall construct a bordism with no isotropy restrictions from f to a map $f' : M' \longrightarrow X$ where M' is an $\mathscr{F}_{i-1}$-manifold. By the induction hypothesis, this will complete the proof.

Let us pause for a moment to explain informally how the bordism will be constructed. The idea is based on a standard technique in geometric topology known as "attaching handles". Any sphere S^k is the boundary of a disc D^{k+1}; if $S^k \subset N^n$ is embedded with trivial normal bundle in a manifold N and has a tubular neighborhood T, we can obtain a bordism of N to a new manifold by crossing N with the unit interval and pasting $D^{k+1} \times D^{n-k-1}$ (a *handle* with *core* D^k) to $N \times I$ by identifying $T \times \{1\}$ with $S^k \times D^{n-k-1}$. Our construction will be basically "attaching a generalized handle" to our manifold M. Instead of an embedded sphere, we shall use $M^{\mathbb{Z}_i}$, which bounds a manifold W; this will be the "core" of our "handle". The "handle" itself will be the total space of a disc bundle over W. The total space of its restriction to $M^{\mathbb{Z}_i}$ will be equivariantly diffeomorphic to a tubular neighborhood of $M^{\mathbb{Z}_i}$ in M, so we may take $M \times I$ and glue the "handle" in the obvious way, thus obtaining the desired bordism. Of course, all the required properties of the bordism have to be checked, and an extension of f to the bordism has to be constructed. We give the details next.

Consider a tubular neighborhood T of $M^{\mathbb{Z}_i}$, regarded as the total space of a disc bundle over $M^{\mathbb{Z}_i}$. We shall use the notation ST for the corresponding unit circle bundle, and T° for $T - ST$. We remark that $M - T^\circ$ and ST are $\mathscr{F}_{i-1}$-manifolds. When there is no danger of confusion, we shall make no notational distinction between a bundle and its total space.

Let λ denote a generator of $\mathbb{Z}_i \subset S^1 \subset \mathbb{C}$, and let V_k, $0 < k < i$, be 1-dimensional representations of $\mathbb{Z}_i$ such that λ acts by multiplication by λ^k. These form a complete, non-redundant set of nontrivial irreducible representations, and each of the V_k's obviously extends to $G \times S^1$ (an element $(g, s) \in G \times S^1$ acts by multiplication by s^k). We use these to obtain an isotypical decomposition of T. Let T_k denote the bundle $\mathrm{Hom}_{\mathbb{Z}_i}(\varepsilon_{V_k}, T)$.

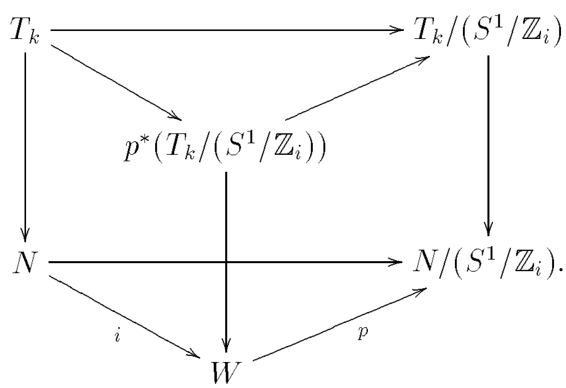
Since $M^{\mathbb{Z}_i}$ is $(S^1/\mathbb{Z}_i)$-free, our proof in the case $i = 1$ shows that $f|_{M^{\mathbb{Z}_i}}$ bounds a map $\tilde{f} : W \longrightarrow X$, where W is the total space of a $\mathbb{Z}_i$-trivial 1-dimensional

$(G \times S^1)$-disc bundle over $Z = M^{\mathbb{Z}_i}/(S^1/\mathbb{Z}_i)$ whose unit circle bundle is $M^{\mathbb{Z}_i}$.

Passage to orbits gives a pull-back diagram

$$\begin{array}{ccc} T_k & \longrightarrow & T_k/(S^1/\mathbb{Z}_i) \\ \downarrow & & \downarrow \\ N & \longrightarrow & N/(S^1/\mathbb{Z}_i), \end{array}$$

for each k, where the right vertical arrow is a G-disc bundle, which may also be thought of as a $(G \times (S^1/\mathbb{Z}_i))$-bundle with trivial $(S^1/\mathbb{Z}_i)$-action. This makes the diagram above a pull-back of $(G \times (S^1/\mathbb{Z}_i))$-vector bundles. Since the zero-section of this bundle can be identified with $Z = SW/(S^1/\mathbb{Z}_i)$, we have a diagram of $(G \times (S^1/\mathbb{Z}_i))$-bundles



Clearly the bundle $\hat{T} = \bigoplus_k p^*(T_k/(S^1/\mathbb{Z}_i)) \otimes \varepsilon_{V_k}$ extends T to W; we claim that its unit sphere bundle is an $\mathscr{F}_{i-1}$-manifold. To prove this, observe that

$$W - Z \cong M^{\mathbb{Z}_i} \times [0,1),$$

where $[0,1)$ has trivial action, and so $S\hat{T}|_{W-Z}$ is equivariantly homeomorphic to $S\hat{T}|_{W-Z} \times [0,1)$. Therefore, S^1-stabilizers of points in $S\hat{T} - ST$ not already present in ST can only appear in $S\hat{T}|_Z$, but since there is no component associated to the trivial representation (recall our remark in the course of the proof of Proposition 4.2) all these are *proper* subgroups of $\mathbb{Z}_i$, so the claim follows.

Let

$$M' \cong (M - T^\circ) \cup_{ST} S\hat{T};$$

by construction, this is an $\mathscr{F}_{i-1}$-manifold. Since $T \cup W$ is a $(G \times S^1)$-deformation retract of $\hat{T}$, there is a map $\hat{f} : W \longrightarrow X$ with $\hat{f}|_T = f|_T$ and $\hat{f}|_W = \tilde{f}$. We obtain a bordism by crossing M with the closed unit interval, pasting $\hat{T}$ to $M \times \{1\}$ along

$T \times \{1\}$, and extending f in the obvious way to a map F from the bordism into X. The maps $f' = F|_{M'}$ and f represent the same element in the bordism of X with no isotropy restrictions, as required. □

8. Passing from G to $G \times S^1$ and $G \times \mathbb{Z}_k$

To complete the proofs of our theorems, it suffices to prove the following result, in which we again assume that we have proven all of our theorems for G.

THEOREM 8.1. Let $C = S^1$ or $C = \mathbb{Z}_k$. The following statements hold.

(1) $\Omega_*^{U,G\times C}(X)$ is a free MU_*-module concentrated in *even* degrees.

(2) The map

$$\psi : \Omega_*^{U,G\times C}(BU(n, G \times S^1) \times X) \longrightarrow \Omega_*^{U,G\times C}(BU(n+1, G \times S^1) \times X)$$

is a split monomorphism of MU_*-modules.

(3) If W is an irreducible complex $(G \times C)$-module, then

$$\sigma^W : \Omega_*^{U,G\times C}(X) \longrightarrow \Omega_{*+2}^{U,G\times C}((DW, SW) \times X)$$

is a split monomorphism of MU_*-modules.

We first show that $\Omega_*^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$ is a free MU_*-module concentrated in even degrees and that σ^W and ψ here are split monomorphisms of MU_*-modules. By Proposition 4.2, we have

$$\Omega_n^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X) \cong \bigoplus_{\substack{0 \le 2k \le n \\ P \in \mathscr{P}(2k, G\times S^1, S^1)}} \Omega_{n-2k}^{U,G}(X^{S^1} \times BU(P, G)).$$

Thus, by the induction hypothesis, $\Omega_n^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$ is free over MU_* and concentrated in even degrees, and the maps ψ induced by addition of $\varepsilon_{\mathbb{C}}$ are split monomorphisms of MU_*-modules.

Theorem 7.1(4) implies that the long exact sequence of the pair $(\mathscr{A}, \mathscr{F}_\infty)$ breaks into short exact sequences. In particular, the map

$$\Omega_*^{U,G\times S^1}(X) \longrightarrow \Omega_*^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$$

is a monomorphism, hence $\Omega_*^{U,G\times S^1}(X)$ is concentrated in even degrees.

In order to study the effect of σ^W on $\Omega_n^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$, it is necessary to distinguish two cases:

(1) $W^{S^1} = W$ and
(2) $W^{S^1} = \{0\}$.

The analysis is similar to the one carried out in the previous section and will be omitted; it yields the expected conclusion: σ^W is a split monomorphism of MU_*-modules on $\Omega_n^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X)$.

The diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_{2j}^{U,G\times S^1}(X) & \xrightarrow{\sigma^W} & \Omega_{2j+2}^{U,G\times S^1}((DW,SW)\times X) \\
\downarrow & & \downarrow \\
\Omega_{2j}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X) & \xrightarrow{\sigma^W} & \Omega_{2j+2}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty]((DW,SW)\times X) \\
\downarrow & & \downarrow \\
\Omega_{2j-1}^{U,G\times S^1}[\mathscr{F}_\infty](X) & \xrightarrow{\sigma^W} & \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_\infty]((DW,SW)\times X) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

together with Lemmas 6.2 and 6.3 shows that $\Omega_*^{U,G\times S^1}(X)$ is projective, and therefore free, and that σ^W is a split monomorphism of MU_*-modules on $\Omega_*^{U,G\times S^1}(X)$. A similar diagram gives the corresponding conclusion for ψ.

This completes the proof of Theorem 8.1 for $C = S^1$, and it remains to deal with the case $C = \mathbb{Z}_k$. Let V denote the 1-dimensional complex representation of $G\times S^1$ on which G acts trivially and an element $e^{2\pi it}\in S^1$ acts by multiplication by $e^{2\pi itk}$. Since S^1 acts without fixed points on $SV\times X$, $\Omega_*^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](SV\times X) = 0$. Therefore, by the long exact sequence of the pair (DV,SV),

$$\Omega_*^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X) \longrightarrow \Omega_*^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty]((DV,SV)\times X)$$

is an isomorphism, and, by the long exact sequence of the pair $(\mathscr{A},\mathscr{F}_\infty)$,

$$\Omega_*^{U,G\times S^1}[\mathscr{F}_\infty](SV\times X) \longrightarrow \Omega_*^{U,G\times S^1}(SV\times X)$$

is an isomorphism.

By Theorem 7.1, we conclude that $\Omega_*^{U,G\times S^1}(SV\times X)$ is a free MU_*-module concentrated in odd degrees. This being so, the long exact sequence of the pair (DV,SV) breaks up into short exact sequences

$$0 \longrightarrow \Omega_{2j}^{U,G\times S^1}(X) \xrightarrow{\alpha} \Omega_{2j}^{U,G\times S^1}((DV,SV)\times X) \longrightarrow \Omega_{2j-1}^{U,G\times S^1}(SV\times X) \longrightarrow 0.$$

Since SV can be identified with $S^1/\mathbb{Z}_k$, we conclude from Lemma 3.4 that

$$\Omega_*^{U,G\times\mathbb{Z}_k}(X) \cong \text{coker } \alpha.$$

Now apply the Snake Lemma to the diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\Omega_{2j}^{U,G\times S^1}(X) & \xrightarrow{\alpha} & \Omega_{2j}^{U,G\times S^1}((DV,SV)\times X) \\
\downarrow & & \downarrow \\
\Omega_{2j}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X) & \xrightarrow{\cong} & \Omega_{2j}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty]((DV,SV)\times X) \\
\downarrow & & \downarrow \\
\Omega_{2j-1}^{U,G\times S^1}[\mathscr{F}_\infty](X) & \xrightarrow{\beta} & \Omega_{2j-1}^{U,G\times S^1}[\mathscr{F}_\infty]((DV,SV)\times X) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

Since α is a monomorphism and β is an epimorphism, we see that coker $\alpha \cong$ ker β. Since ker β is a free MU_*-module concentrated in odd degrees, $\Omega_*^{U,G\times\mathbb{Z}_k}(X)$ is free and concentrated in even degrees.

To show that σ^W is a split monomorphism, let $Y = (DW, SW)\times X$ and consider the maps

$$\alpha' : \Omega_{2j+2}^{U,G\times S^1}(Y) \longrightarrow \Omega_{2j+2}^{U,G\times S^1}((DV,SV)\times Y)$$

and

$$\beta' : \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_\infty](Y) \longrightarrow \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_\infty]((DV,SV)\times Y)$$

that fit into the diagram obtained from the previous one by raising all degrees by two and replacing X by Y. Then σ^W induces a map from the original diagram to the new diagram, and there results a commutative square

$$\begin{array}{ccc}
\text{coker } \alpha & \xrightarrow{\sigma^W} & \text{coker } \alpha' \\
\cong\downarrow & & \downarrow\cong \\
\text{ker } \beta & \xrightarrow[\sigma^W]{} & \text{ker } \beta'.
\end{array}$$

By Lemma 6.2, the bottom arrow is a split monomorphism of MU_*-modules, hence so is the top arrow. The proof that ψ is a split monomorphism is similar.

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