after completion for any G-CW complex E constructed using cells $G/H_+ \wedge S^n$ for various proper subgroups H.

Now if G is finite, let V denote the reduced regular representation and let $S^{\infty V}$ be the union of the representation spheres S^{kV}. For a general compact Lie group G, we let $S^{\infty V}$ denote the union of the representation spheres S^V as V runs over the indexing spaces V such that $V^G = 0$ in a complete G-universe U.

Evidently ${S^{\infty V}}^H$ is contractible if H is a proper subgroup and ${S^{\infty V}}^G = S^0$. Thus $S^{\infty V}/S^0$ has no G-fixed points and may be constructed using cells $G/H_+ \wedge S^n$ for proper subgroups H. Thus, by the inductive hypothesis, $K_G^*(S^{\infty V}/S^0 \wedge \tilde{E}G) = 0$ after completion, and hence

$$K_G^*(S^{\infty V} \wedge \tilde{E}G) \cong K_G^*(S^0 \wedge \tilde{E}G) = K_G^*(\tilde{E}G)$$

after completion. But evidently the inclusion

$$S^{\infty V} = S^{\infty V} \wedge S^0 \longrightarrow S^{\infty V} \wedge \tilde{E}G$$

is an equivariant homotopy equivalence (consider the various fixed point sets). This proves a most convenient reduction: it is enough to prove that $K_G^*(S^{\infty V}) = 0$ after completion.

In fact, it is easy to see that $K_G^*(S^{\infty V}) = 0$ after completion. When G is finite, one just notes that (ignoring $\lim^1$ problems again)

$$K_G^*(S^{\infty V}) = \lim_k K_G^*(S^{kV}) = \lim_k (K_G^*(S^0), \lambda(V)) = 0$$

because $\lambda(V) \in I$. Indeed the inverse limit has the effect of making the element $\lambda(V)$ invertible, and if $IM = M$ then $M_I^{\wedge} = 0$. The argument in the general compact Lie case is only a little more elaborate.

To make this proof honest, we must address the two important properties that we used without justification: (a) that completed K-theory takes cofiberings to exact sequences and (b) that the K-theories of certain infinite complexes are the inverse limits of the K-theories of their finite subcomplexes. In other words the points that we skated over were the linked problems of the inexactness of completion and the nonvanishing of $\lim^1$ terms.

Now, since $R(G)$ is Noetherian, completion *is* exact on finitely generated modules, and the K groups of finite complexes are finitely generated. Accordingly, one route is to arrange the formalities so as to only discuss finite complexes: this is the method of pro-groups, as in the original approach of Atiyah. It is elementary and widely useful. Instead of considering the single group $K^*_G(X)$ we consider the inverse system of groups $K^*_G(X_\alpha)$ as X_α runs over the finite subcomplexes of X.

We do not need to know much about pro-groups. A pro-group is just an inverse system of Abelian groups. There is a natural way to define morphisms, and the resulting category is Abelian. The fundamental technical advantage of working in the category of pro-groups is that, in this category, the inverse limit functor is exact. For any Abelian group valued functor h on G-CW complexes or spectra, we define the associated pro-group valued functor $\mathbf{h}$ by letting $\mathbf{h}(X)$ be the inverse system $\{h(X_\alpha)\}$, where X_α runs over the finite subcomplexes of X.

As long as all K-theory is interpreted as pro-group valued, the argument just given is honest. The conclusion of the argument is that, for a finite G-CW complex X, $\pi : EG_+ \wedge X \longrightarrow X$ induces an isomorphism of I-completed pro-group valued K-theory. Here the I-completion of a pro-$R(G)$-module $\mathbf{M} = \{M_\alpha\}$ is just the inverse system $\{M_\alpha / I^r M_\alpha\}$. When $\mathbf{M}$ is a constant system, such as $\mathbf{K}^*_G(S^0)$, this is just an inverse system of epimorphisms and has zero $\lim^1$. It follows from the isomorphism of pro-groups that $\lim^1$ is also zero for the progroup $K^*_G(EG_+ \wedge X)$, and hence the group $K^*_G(EG_+ \wedge X)$ is the inverse limit of the K-theories of the skeleta of $EG_+ \wedge X$. We may thus simply pass to inverse limits to obtain the conclusion of Theorem 3.1 as originally stated for ordinary rather than pro-$R(G)$-modules.

There is an alternative way to be honest: we could accept the inexactness and adapt the usual methods for discussing it by derived functors. In fact we shall later see how to realize the construction of left derived functors of completion geometrically. This approach leads compellingly to consideration of completions of K_G-module spectra and to the consideration of homology. We invite the interested reader to turn to Chapter XXIV (especially Section 7).

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6. The generalization to families

The above statements and proofs for the universal free G-space EG and the augmentation ideal I carry over with the given proofs to theorems about the universal $\mathscr{F}$-free space $E\mathscr{F}$ and the ideal

$$I\mathscr{F} = \bigcap_{H\in\mathscr{F}} ker\{res_H^G : R(G) \longrightarrow R(H)\}.$$

The only difference is that for most families $\mathscr{F}$ there is no reduction of $K_G(E\mathscr{F})$ to the nonequivariant K-theory of some other space. Note that, by the injectivity of (2.1), if $\mathscr{F}$ includes all cyclic subgroups then $I\mathscr{F} = 0$.

THEOREM 6.1. For any family $\mathscr{F}$ and any finite G-CW-complex X the projection map $E\mathscr{F} \longrightarrow *$ induces completion, so that

$$K_G^*(E\mathscr{F}_+ \wedge X) \cong K_G^*(X)^\wedge_{I\mathscr{F}}.$$

In particular

$$K_G^0(E\mathscr{F}_+) \cong R(G)^\wedge_{I\mathscr{F}} \quad \text{and} \quad K_G^1(E\mathscr{F}_+) = 0.$$

Two useful consequences of these generalizations are that K-theory is detected on finite subgroups and that isomorphisms are detected by cyclic groups.

THEOREM 6.2 (MCCLURE). (a) If X is a finite G-CW-complex and $x \in K_G(X)$ restricts to zero in $K_H(X)$ for all finite subgroups H of G then $x = 0$.
(b) If $f : X \longrightarrow Y$ is a map of finite G-CW-complexes that induces an isomorphism $K_C(Y) \longrightarrow K_C(X)$ for all finite cyclic subgroups C then $f^* : K_G(Y) \longrightarrow K_G(X)$ is also an isomorphism.

Thinking about characters, one might be tempted to believe that finite subgroups could be replaced by finite cyclic subgroups in (a), but that is false.

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CHAPTER XV

An introduction to equivariant cobordism

by S. R. Costenoble

1. A review of nonequivariant cobordism

We start with a brief summary of nonequivariant cobordism.

We define a sequence of groups $\mathscr{N}_0$, $\mathscr{N}_1$, $\mathscr{N}_2$, ... as follows: We say that two smooth closed k-dimensional manifolds M_1 and M_2 are *cobordant* if there is a smooth $(k+1)$-dimensional manifold W (the *cobordism*) such that $\partial W \cong M_1 \coprod M_2$; this is an equivalence relation, and $\mathscr{N}_k$ is the set of cobordism classes of k-dimensional manifolds. We make this into an abelian group with addition being disjoint union. The 0 element is the class of the empty manifold $\emptyset$; a manifold is cobordant to $\emptyset$ if it bounds. Every manifold is its own inverse, since $M \coprod M$ bounds $M \times I$. We can make the graded group $\mathscr{N}_*$ into a ring by using cartesian product as multiplication. This ring has been calculated: $\mathscr{N}_* \cong \mathbb{Z}/2[x_k \,|\, k \neq 2^i - 1]$. We'll say more about how we attack this calculation in a moment. This is the *unoriented bordism ring*, due to Thom.

Thom also considered the variant in which the manifolds are oriented. In this case, the cobordism is also required to be oriented, and the boundary ∂W is oriented so that its orientation, together with the inward normal into W, gives the restriction of the orientation of W to ∂W. The effect is that, if M is a closed oriented manifold, then $\partial(M \times I) = M \coprod (-M)$ where $-M$ denotes M with its orientation reversed. This makes $-M$ the negative of M in the resulting *oriented bordism ring* Ω_*. This ring is more complicated than $\mathscr{N}_*$, having both a torsion-free part (calculated by Thom) and a torsion part, consisting entirely of elements of order 2 (calculated by Milnor and Wall).

There are many other variants of these rings, including *unitary bordism*, $\mathscr{U}_*$, which uses "stably almost complex" manifolds; M is such a manifold if there is given an embedding $M \subset \mathbb{R}^n$ and a complex structure on the normal bundle to this embedding. The calculation is $\mathscr{U}_* \cong \mathbb{Z}[z_{2k}]$. This and other variants are discussed in Stong.

These rings are actually coefficient rings of certain homology theories, the *bordism theories* (there is a nice convention, due to Atiyah, that we use the name bordism for the homology theory, and the name cobordism for the related cohomology theory). If X is a space, we define the group $\mathscr{N}_k(X)$ to be the set of bordism classes of maps $M \longrightarrow X$, where M is a k-dimensional smooth closed manifold and the map is continuous. Cobordisms must also map into X, and the restriction of the map to the boundary must agree with the given maps on the k-manifolds. Defining the relative groups $\mathscr{N}_*(X, A)$ is a little trickier. We consider maps $(M, \partial M) \longrightarrow (X, A)$. Such a map is cobordant to $(N, \partial N) \longrightarrow (X, A)$ if there exists a triple $(W, \partial_0 W, \partial_1 W)$, where $\partial W = \partial_0 W \cup \partial_1 W$, the intersection $\partial_0 W \cap \partial_1 W$ is the common boundary $\partial(\partial_0 W) = \partial(\partial_1 W)$, and $\partial_0 W \cong M \coprod N$, together with a map $(W, \partial_1 W) \longrightarrow (X, A)$ that restricts to the given maps on $\partial_0 W$. (This makes the most sense if you draw a picture.) It's useful to think of W as having a "corner" at $\partial_0 W \cap \partial_1 W$; otherwise you have to use resmoothings to get an equivalence relation. It is now a pretty geometric exercise to show that there is a long exact sequence

$$\cdots \longrightarrow \mathscr{N}_k(A) \longrightarrow \mathscr{N}_k(X) \longrightarrow \mathscr{N}_k(X, A) \longrightarrow \mathscr{N}_{k-1}(A) \longrightarrow \cdots$$

where the "boundary map" is precisely taking the boundary. There are oriented, unitary, and other variants of this homology theory.

Calculation of these groups is possible largely because we know the representing spectra for these theories. Let TO (the *Thom prespectrum*) be the prespectrum whose kth space is $TO(k)$, the Thom space of the universal k-plane bundle over $BO(k)$. It is an inclusion prespectrum and, applying the spectrification functor L to it, we obtain the *Thom spectrum MO*. Its homotopy groups are given by

$$\pi_k(MO) = \operatorname{colim}_q \pi_{q+k}(TO(q)).$$

Then $\mathscr{N}_* \cong \pi_*(MO)$, and in fact MO represents unoriented bordism.

The proof goes like this: Given a k-dimensional manifold M, embed M in some $\mathbb{R}^{q+k}$ with normal bundle ν. The unit disk of this bundle is homeomorphic to a tubular neighborhood N of M in $\mathbb{R}^{q+k}$, and so there is a collapse map $c : S^{q+k} \longrightarrow$

$T\nu$ given by collapsing everything outside of N to the basepoint. There is also a classifying map $T\nu \longrightarrow TO(q)$, and the composite

$$S^{q+k} \longrightarrow T\nu \longrightarrow TO(q)$$

represents an element of $\pi_k(MO)$. Applying a similar construction to a cobordism gives a homotopy between the two maps obtained from cobordant manifolds. This construction, known as the *Pontrjagin-Thom construction*, describes the map $\mathscr{N}_k \longrightarrow \pi_k(MO)$.

The inverse map is constructed as follows: Given a map $f : S^{q+k} \longrightarrow TO(q)$, we may assume that f is transverse to the zero-section. The inverse image $M = f^{-1}(BO(q))$ is then a k-dimensional submanifold of S^{q+k} (provided that we use Grassmannian manifold approximations of classifying spaces), and the normal bundle to the embedding of M in S^{q+k} is the pullback of the universal bundle. Making a homotopy between two maps transverse provides a cobordism between the two manifolds obtained from the maps. One can now check that these two constructions are well-defined and inverse isomorphisms. The analysis of $\mathscr{N}_*(X, A)$ is almost identical.

In fact MO is a ring spectrum, and the Thom isomorphism just constructed is an isomorphism of rings. The product on MO is induced from the maps

$$TO(j) \wedge TO(k) \longrightarrow TO(j+k)$$

of Thom complexes arising from the classifying map of the external sum of the jth and kth universal bundle. This becomes clearer when one thinks in a coordinate-free way; in fact, it was inspection of Thom spectra that led to the description of the stable homotopy category that May gave in Chapter XII.

Now MO is a very tractable spectrum. To compute its homotopy we have available such tools as the Thom isomorphism, the Steenrod algebra (mod 2), and the Adams spectral sequence for the most sophisticated calculation. (Stong gives a calculation not using the spectral sequence.) The point is that we now have something concrete to work with, and adequate tools to do the job. For oriented bordism, we replace MO with MSO, which is constructed similarly except that we use the universal oriented bundles over the spaces $BSO(k)$. Here we use the fact that an orientation of a manifold is equivalent to an orientation of its normal bundle. Similarly, for unitary bordism we use the spectrum MU, constructed out of the universal unitary bundles.

The standard general reference is
R. E. Stong. Notes on Cobordism Theory. Princeton University Press. 1968.

2. Equivariant cobordism and Thom spectra

Now we take a compact Lie group G and try to generalize everything to the G-equivariant context. This generalization of nonequivariant bordism was first studied by Conner and Floyd. Using smooth G-manifolds throughout we can certainly copy the definition of cobordism to obtain the equivariant bordism groups $\mathscr{N}_*^G$ and, for pairs of G-spaces (X, A), the groups $\mathscr{N}_*^G(X, A)$. We shall concentrate on unoriented bordism. To define unitary bordism, we consider a unitary manifold to be a smooth G-manifold M together with an embedding of M in either V or $V \oplus \mathbb{R}$, where V is a complex representation of G, and a complex structure on the resulting normal bundle. The notion of an oriented G-manifold is complicated and still controversial, although for odd order groups it suffices to look at oriented manifolds with an action of G; the action of G automatically preserves the orientation.

It is also easy to generalize the Thom spectrum. Let $\mathscr{U}$ be a complete G-universe. In view of the description of the K-theory G-spectra in the previous chapter, it seems most natural to start with the universal n-plane bundles

$$\pi(V) : EO(|V|, V \oplus \mathscr{U}) \longrightarrow BO(|V|, V \oplus \mathscr{U})$$

for indexing spaces V in $\mathscr{U}$. Let $TO_G(V)$ be the Thom space of $\pi(V)$. For $V \subseteq W$, the pullback of $\pi(W)$ over the inclusion

$$BO(|V|, V \oplus \mathscr{U}) \longrightarrow BO(|W|, W \oplus \mathscr{U})$$

is the Whitney sum of $\pi(V)$ and the trivial bundle with fiber $W - V$. Its Thom space is $\Sigma^{W-V} TO_G(V)$, and the evident map of bundles induces an inclusion

$$\sigma : \Sigma^{W-V} TO_G(V) \longrightarrow TO_G(W).$$

This construction gives us an inclusion G-prespectrum TO_G. We define the real Thom G-spectrum to be its spectrification $MO_G = LTO_G$. Using complex representations throughout, we obtain the complex analogs TU_G and MU_G. This definition is essentially due to tom Dieck.

The interesting thing is that MO_G does not represent $\mathscr{N}_*^G$. It is easy to define a map $\mathscr{N}_*^G \longrightarrow \pi_*^G(MO_G) = MO_*^G$ using the Pontrjagin-Thom construction, but we cannot define an inverse. The problem is the failure of transversality in the equivariant context. As a simple example of this failure, consider the group $G = \mathbb{Z}/2$, let $M = *$ be a one-point G-set (a 0-dimensional manifold), let $N = \mathbb{R}$ with the nontrivial linear action of G, and let $Y = \{0\} \subset N$. Let $f : M \longrightarrow N$

be the only G-map that can be defined: it takes M to Y. Clearly f cannot be made transverse to Y, since it is homotopic only to itself. This simple example is paradigmatic. In general, given manifolds M and $Y \subset N$ and a map $f : M \longrightarrow N$, if f fails to be homotopic to a map transverse to Y it is because of the presence in the normal bundle to Y of a nontrivial representation of G that cannot be mapped onto by the representations available in the tangent bundle of M. Wasserman provided conditions under which we can get transversality. If G is a product of a torus and a finite group, he gives a sufficient condition for transversality that amounts to saying that, where needed, we will always have in M a nontrivial representation mapping onto the nontrivial representation we see in the normal bundle to Y. Others have given obstruction theories to transversality, for example Petrie and Waner and myself.

Using Wasserman's condition, it is possible (for one of his G) to construct the G-spectrum that *does* represent $\mathscr{N}_*^G$. Again, let $\mathscr{U}$ be a complete G-universe. We can construct a G-prespectrum to_G with associated G-spectrum mo_G by letting V run through the indexing spaces in our complete universe $\mathscr{U}$ as before, but replacing $\mathscr{U}$ by its G-fixed point space $\mathscr{U}^G \cong \mathbb{R}^\infty$ in the bundles we start with. That is, we start with the G-bundles

$$EO(|V|, V \oplus \mathscr{U}^G) \longrightarrow BO(|V|, V \oplus \mathscr{U}^G)$$

for indexing spaces V in $\mathscr{U}$. Again, restricting attention to complex representations, we obtain the complex analogs tu_G and mu_G. The fact that there are so few nontrivial representations present in the bundle $EO(|V|, V \oplus \mathscr{U}^G)$ allows us to use Wasserman's transversality results to show that mo_G represents $\mathscr{N}_*^G$. The inclusion $\mathscr{U}^G \longrightarrow \mathscr{U}$ induces a map

$$mo_G \longrightarrow MO_G$$

that represents the map $\mathscr{N}_*^G \longrightarrow MO_*^G$ that we originally hoped was an isomorphism.

On the other hand, there is also a geometric interpretation of MO_*^G. Using either transversality arguments or a clever argument due to Bröcker and Hook that works for all compact Lie groups, one can show that

$$MO_k^G(X, A) \cong \operatorname{colim}_V \mathscr{N}_{k+|V|}^G((X, A) \times (D(V), S(V))).$$

Here the maps in the colimit are given by multiplying manifolds by disks of representations, smoothing corners as necessary. We interpret this in the simplest case as follows. A class in $MO_k^G \cong \operatorname{colim}_V \mathscr{N}_{k+|V|}^G(D(V), S(V))$ is represented by a manifold $(M, \partial M)$ together with a map $(M, \partial M) \longrightarrow (D(V), S(V))$. This map is equivalent in the colimit to $(M \times D(W), \partial(M \times D(W))) \longrightarrow (D(V \oplus W), S(V \oplus W))$ together with the original map crossed with the identity on $D(W)$. We call the equivalence class of such a manifold over the disk of a representation a *stable manifold*. Its (virtual) dimension is $\dim M - \dim V$. We can then interpret MO^G_k as the group of cobordism classes of stable manifolds of dimension k. A similar interpretation works for $MO^G_k(X, A)$.

With this interpretation we can see clearly one of the differences between $\mathcal{N}^G_*$ and MO^G_*. If V is a representation of G with no trivial summands, then there is a stable manifold represented by $* \longrightarrow D(V)$, the inclusion of the origin. This represents a nontrivial element $\chi(V) \in MO^G_{-n}$ where $n = |V|$. This element is called the *Euler class* of V. Tom Dieck showed the nontriviality of these elements and we'll give a version of the argument below; note that if V had a trivial summand, then $* \longrightarrow D(V)$ would be homotopic to a map into $S(V)$, so that $\chi(V) = 0$. On the other hand, $\mathcal{N}^G_*$ has no nontrivial elements in negative dimensions, by definition.

Here is another, related difference: Stable bordism is periodic in a sense. If V is any representation of G, then, by the definition of MO_G, $MO_G(V) \cong MO_G(|V|)$; the point is that $MO_G(V)$ really depends only on $|V|$. This gives an equivalence $\Sigma^V MO_G \simeq \Sigma^n MO_G$ if $n = |V|$, or

$$MO_G \simeq \Sigma^{V-n} MO_G.$$

One way of defining an explicit equivalence is to start by classifying the bundle $V \longrightarrow *$ and so obtain an associated map of Thom complexes (a *Thom class*)

$$S^V \longrightarrow TO_G(\mathbb{R}^n) \subset MO_G(\mathbb{R}^n).$$

This is adjoint to a map $\mu(V) : S^{V-n} = \Sigma^\infty_n S^V \longrightarrow MO_G$. Reversing the roles of V and $\mathbb{R}^n$, we obtain an analogous map $S^{n-V} \longrightarrow MO_G$. It is not hard to check that these are inverse units in the $RO(G)$-graded ring MO^G_*. The required equivalence is the evident composite

$$S^{V-n} \wedge MO_G \longrightarrow MO_G \wedge MO_G \longrightarrow MO_G.$$

In homology, this gives isomorphisms of MO^G_*-modules

$$MO^G_*(\Sigma^{|V|} X) \cong MO^G_*(\Sigma^V X)$$

and

$$MO^G_k(X) \cong MO^G_{k+n}(\Sigma^V X)$$

for all k. This is really a special case of a Thom isomorphism that holds for every bundle. The *Thom class* of a bundle ξ is the element in cobordism represented by the map of Thom complexes $T\xi \longrightarrow TO_G(|\xi|) \subset MO_G(|\xi|)$ induced by the classifying map of ξ. Another consequence of the isomorphisms above is that $MO^G_V(X) \cong MO^G_n(X)$, so that the $RO(G)$-graded groups that we get are no different from the groups in integer grading. We can think of this as a periodicity given by multiplication by the unit $\mu(V)$. It should also be clear that, if $|V| = m$ and $|W| = n$, then the composite isomorphism

$$MO^G_k(X) \cong MO^G_{k+m}(\Sigma^V X) \cong MO^G_{k+m+n}(\Sigma^{V\oplus W} X)$$

agrees with the isomorphism $MO^G_k(X) \cong MO^G_{k+m+n}(\Sigma^{V\oplus W} X)$ associated with the representation $V \oplus W$.

We record one further consequence of all this. Consider the inclusion $e : S^0 \longrightarrow S^V$, where $|V| = n$. This induces a map

$$MO^G_{k+n}(X) \longrightarrow MO^G_{k+n}(\Sigma^V X) \cong MO^G_k(X).$$

It is easy to see geometrically that this is given by multiplication by the stable manifold $* \longrightarrow D(V)$, the inclusion of the origin, which represents $\chi(V) \in MO^G_{-n}$. The similar map in cobordism,

$$MO^k_G(X) \cong MO^{k+n}_G(\Sigma^V X) \longrightarrow MO^{k+n}_G(X)$$

is also given by multiplication by $\chi(V) \in MO^n_G$, as we can see by representing $\chi(V)$ by the stable map

$$S^0 \longrightarrow S^V \longrightarrow \Sigma^V MO_G \simeq \Sigma^n MO_G.$$

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3. Computations: the use of families

For computations, we start with the fact that $\mathscr{N}^G_*(X)$ is a module over $\mathscr{N}_*$ (the nonequivariant bordism ring, which we know) by cartesian product. The question is then its structure as a module. We'll take a look at the main computational techniques and at some of the simpler known results.

The main computational technique was introduced by Conner and Floyd. Recall that a *family of subgroups* of G is a collection of subgroups closed under conjugacy and taking of subgroups (in short, under subconjugacy). If $\mathscr{F}$ is such a family, we define an *$\mathscr{F}$-manifold* to be a smooth G-manifold all of whose isotropy groups are in $\mathscr{F}$. If we restrict our attention to closed $\mathscr{F}$-manifolds and cobordisms that are also $\mathscr{F}$-manifolds, we get the groups $\mathscr{N}_*^G[\mathscr{F}]$ of cobordism classes of manifolds with restricted isotropy. Similarly, we can consider the bordism theory $\mathscr{N}_*^G[\mathscr{F}](X,A)$. Now there is a relative version of this as well. Suppose that $\mathscr{F}' \subset \mathscr{F}$. An *$(\mathscr{F},\mathscr{F}')$-manifold* is a manifold $(M,\partial M)$ where M is an $\mathscr{F}$-manifold and ∂M is an $\mathscr{F}'$-manifold (possibly empty, of course). To define cobordism of such manifolds, we must resort to manifolds with multipart boundaries, or manifolds with corners. Precisely, $(M,\partial M)$ is cobordant to $(N,\partial N)$ if there is a manifold $(W,\partial_0 W,\partial_1 W)$ such that W is an $\mathscr{F}$-manifold, $\partial_1 W$ is an $\mathscr{F}'$-manifold, and $\partial_0 W = M \coprod N$, where as usual $\partial W = \partial_0 W \cup \partial_1 W$ and $\partial_0 W \cap \partial_1 W$ is the common boundary of $\partial_0 W$ and $\partial_1 W$. With this definition we can form the relative bordism groups $\mathscr{N}_*^G[\mathscr{F},\mathscr{F}']$. Of course, there is also an associated bordism theory, although to describe the relative groups of that theory requires manifolds with 2-part boundaries, and cobordisms with 3-part boundaries!

From a homotopy theoretic point of view it's interesting to notice that $\mathscr{N}_*^G[\mathscr{F}] \cong \mathscr{N}_*^G(E\mathscr{F})$, since a manifold over $E\mathscr{F}$ must be an $\mathscr{F}$-manifold, and any $\mathscr{F}$-manifold has a unique homotopy class of maps into $E\mathscr{F}$. Similarly, $\mathscr{N}_*^G[\mathscr{F}](X) \cong \mathscr{N}_*^G(X \times E\mathscr{F})$, and so on. For the purposes of computation, it is usually more fruitful to think in terms of manifolds with restricted isotropy, however. Notice that this gives us an easy way to define $MO_*^G[\mathscr{F}]$: it is $MO_*^G(E\mathscr{F})$. We can also interpret this in terms of stable manifolds with restricted isotropy.

As a first illustration of the use of families, we give the promised proof of the nontriviality of Euler classes.

LEMMA 3.1. Let G be a compact Lie group and V be a representation of G without trivial summands. Then $\chi(V) \neq 0$ in MO_{-n}^G, where $n = |V|$.

PROOF. Let $\mathscr{A}$ be the family of all subgroups, and let $\mathscr{P}$ be the family of proper subgroups. Consider the map $MO_*^G \longrightarrow MO_*^G[\mathscr{A},\mathscr{P}]$. We claim that the image of $\chi(V)$ is invertible in $MO_*^G[\mathscr{A},\mathscr{P}]$ (which is nonzero), so that $\chi(V) \neq 0$. Thinking in terms of stable manifolds, $\chi(V) = [* \longrightarrow D(V)]$. Its inverse is

$D(V) \longrightarrow *$, which lives in the group $MO_*^G[\mathscr{A}, \mathscr{P}]$ because $\partial D(V) = S(V)$ has no fixed points. It's slightly tricky to show that the product, which is represented by $D(V) \longrightarrow * \longrightarrow D(V)$, is cobordant to the identity $D(V) \longrightarrow D(V)$, as we have to change the interpretation of the boundary $S(V)$ of the source from being the "$\mathscr{P}$-manifold part" to being the "maps into $S(V)$ part". However, a little cleverness with $D(V) \times I$ does the trick. $\square$

Returning to our general discussion of the use of families, note that, for a pair of families $(\mathscr{F}, \mathscr{F}')$, there is a long exact sequence

$$\cdots \longrightarrow \mathscr{N}_k^G[\mathscr{F}'] \longrightarrow \mathscr{N}_k^G[\mathscr{F}] \longrightarrow \mathscr{N}_k^G[\mathscr{F}, \mathscr{F}'] \longrightarrow \mathscr{N}_{k-1}^G[\mathscr{F}'] \longrightarrow \cdots,$$

where the boundary map is given by taking boundaries. (This is of course the same as the long exact sequence associated with the pair of spaces $(E\mathscr{F}, E\mathscr{F}')$.) We would like to use this exact sequence to calculate $\mathscr{N}_*^G$ inductively. To set this up a little more systematically, suppose that we have a sequence $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ of families of subgroups whose union is the family of all subgroups. If we can calculate $\mathscr{N}_k^G[\mathscr{F}_0]$ and each relative term $\mathscr{N}_k^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$, we may be able to calculate every $\mathscr{N}_k^G[\mathscr{F}_p]$ and ultimately $\mathscr{N}_*^G$. We can also introduce the machinery of spectral sequences here: The long exact sequences give us an exact couple

$$\begin{array}{ccc} \mathscr{N}_*^G[\mathscr{F}_{p-1}] & \xrightarrow{\qquad\qquad} & \mathscr{N}_*^G[\mathscr{F}_p] \\ & \nwarrow \qquad \swarrow & \\ & \mathscr{N}_*^G[\mathscr{F}_p, \mathscr{F}_{p-1}] & \end{array}$$

and hence a spectral sequence with $E^1_{p,q} = \mathscr{N}_q^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$ that converges to $\mathscr{N}_*^G$.

This would all be academic if not for the fact that $\mathscr{N}_*^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$ is often computable. Let us start off with the base of the induction: $\mathscr{N}_*^G[\{e\}, \emptyset] = \mathscr{N}_k^G[\{e\}]$. This is the bordism group of free closed G-manifolds. Now, if M is a free G-manifold, then M/G is also a manifold, of dimension $\dim M - \dim G$. There is a unique homotopy class of G-maps $M \longrightarrow EG$, which passes to quotients to give a map $M/G \longrightarrow BG$. Moreover, given the map $M/G \longrightarrow BG$ we can recover the original manifold M, since it is the pullback in the following diagram:

$$\begin{array}{ccc} M & \longrightarrow & EG \\ \downarrow & & \downarrow \\ M/G & \longrightarrow & BG. \end{array}$$

This applies equally well to manifolds with or without boundary, so it applies to cobordisms as well. This establishes the isomorphism

$$\mathscr{N}_k^G[\{e\}] \cong \mathscr{N}_{k-\dim G}(BG).$$

Now the bordism of a classifying space may or may not be easy to compute, but at least this is a nonequivariant problem.

The inductive step can also be reduced to a nonequivariant calculation. Suppose that G is finite or Abelian for convenience. We say that $\mathscr{F}$ and $\mathscr{F}'$ are adjacent if $\mathscr{F} = \mathscr{F}' \cup (H)$ for a single conjugacy class of subgroups (H), and it suffices to restrict attention to such an adjacent pair. Suppose that $(M, \partial M)$ is an $(\mathscr{F}, \mathscr{F}')$-manifold. Let $M^{(H)}$ denote the set of points in M with isotropy groups in (H); $M^{(H)}$ lies in the interior of M, since ∂M is an $\mathscr{F}'$-manifold, and $M^{(H)} = \cup_{K\in(H)} M^K$ is a union of closed submanifolds of M. Moreover, these submanifolds are pairwise disjoint, since (H) is maximal in $\mathscr{F}$. Therefore $M^{(H)}$ is a closed G-invariant submanifold in the interior of M, isomorphic to $G \times_{NH} M^H$. (Here is where it is convenient to have G finite or Abelian.) Thus $M^{(H)}$ has a G-invariant closed tubular neighborhood in M, call it N. Here is the key step: *$(M, \partial M)$ is cobordant to $(N, \partial N)$ as an $(\mathscr{F}, \mathscr{F}')$-manifold.* The cobordism is provided by $M \times I$ with corners smoothed (this is easiest to see in a picture).

As usual, let $WH = NH/H$. Now $(N, \partial N)$ is determined by the free WH-manifold M^H and the NH-vector bundle over it which is its normal bundle. Since WH acts freely on the base, each fiber is a representation of H with no trivial summands and decomposes into a sum of multiples of irreducible representations. This also decomposes the whole bundle: Suppose that the nontrivial irreducible representations of H are V_1, V_2, $\ldots$. Then $\nu = \oplus \nu_i$, where each fiber of each ν_i is a sum of copies of V_i. Clearly ν_i is completely determined by the free WH-bundle $\mathrm{Hom}_G(V_i, \nu_i)$, which has fibers $\mathbb{F}^n$ where $\mathbb{F}$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending on V_i. Notice, however, that the NH-action on ν induces certain isomorphisms among the ν_i: If V_i and V_j are conjugate representations under the action of NH, then ν_i and ν_j must be isomorphic.

The upshot of all of this is that $\mathscr{N}_k^G[\mathscr{F}, \mathscr{F}']$ is isomorphic to the group obtained in the following way. Suppose that the dimension of V_i is d_i and that $\mathrm{Hom}_G(V_i, V_i) = \mathbb{F}_i$, where $\mathbb{F}_i = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Consider free WH-manifolds M, together with a sequence of WH-bundles ξ_1, ξ_2, $\cdots$ over M, one for each V_i, the group of ξ_i being $O(\mathbb{F}_i, n_i)$ (i.e., $O(n_i)$, $U(n_i)$, or $Sp(n_i)$). If V_i and V_j are conjugate under the action of NH, then we insist that ξ_i and ξ_j be isomorphic.

The dimension of $(M;\xi_1,\xi_2,\cdots)$ is $\dim M+\sum n_i d_i$; that is, this should equal k. Now define $(M;\xi_1,\xi_2,\cdots)$ to be cobordant to $(N;\zeta_1,\zeta_2,\cdots)$ if there exists some $(W;\theta_1,\theta_2,\cdots)$ such that $\partial W=M\coprod N$ and the restriction of θ_i to ∂W is $\xi_i\coprod\zeta_i$. It should be reasonably clear from this description that we have an isomorphism

$$\mathscr{N}_k^G[\mathscr{F},\mathscr{F}']\cong\sum_{j+\sum n_i d_i=k}\mathscr{N}_j^{WH}(EWH\times(\times_i BO(\mathbb{F}_i,n_i)))$$

where WH acts on $\times_i BO(\mathbb{F}_i,n_i)$ via its permutation of the representations of H. One more step and this becomes a nonequivariant problem: We take the quotient by WH, which we can do because the argument $EWH\times(\times_i BO(\mathbb{F}_i,n_i))$ is free (this being just like the case $\mathscr{N}_*^G[\{e\}]$ above). This gives

$$(3.2)\qquad \mathscr{N}_k^G[\mathscr{F},\mathscr{F}']\cong\sum_{\dim WH+j+\sum n_i d_i=k}\mathscr{N}_j(EWH\times_{WH}(\times_i BO(\mathbb{F}_i,n_i))).$$

Notice that, if G is Abelian or if WH acts trivially on the representations of H for some other reason, then the argument is $BWH\times(\times_i BO(\mathbb{F}_i,n_i)))$.

P. E. Conner and E. E. Floyd. Differentiable periodic maps. Academic Press, Inc. 1964.

4. Special cases: odd order groups and $\mathbb{Z}/2$

If G is a finite group of odd order, then the differentials in the spectral sequence for $\mathscr{N}_*^G$ all vanish, and $\mathscr{N}_*^G$ is the direct sum over (H) of the groups displayed in (3.2). This is actually a consequence of a very general splitting result that will be explained in XVII§6. The point is that $\mathscr{N}_*^G$ is a $\mathbb{Z}/2$-vector space and, away from the order of the group, the Burnside ring $A(G)$ splits as a direct sum of copies of $\mathbb{Z}[1/|G|]$, one for each conjugacy class of subgroups of G. This induces splittings in all modules over the Burnside ring, including all $RO(G)$-graded homology theories (that is, those homology theories represented by spectra indexed on complete universes). The moral of the story is that, away from the order of the group, equivariant topology generally reduces to nonequivariant topology.

This observation can also be used to show that the spectra mo_G and MO_G split as products of Eilenberg-MacLane spectra, just as in the nonequivariant case. Remember that this depends on G having odd order.

Conner and Floyd computed the additive structure of $\mathscr{N}_*^{\mathbb{Z}/2}$, and Alexander computed its multiplicative structure. There is a split short exact sequence

$$0\longrightarrow\mathscr{N}_k^{\mathbb{Z}/2}\longrightarrow\oplus_{0\le n\le k}\mathscr{N}_{k-n}(BO(n))\longrightarrow\mathscr{N}_{k-1}(B\mathbb{Z}/2)\longrightarrow 0,$$

which is part of the long exact sequence of the pair $(\{\mathbb{Z}/2, e\}, \{e\})$. The first map is given by restriction to $\mathbb{Z}/2$-fixed points and the normal bundles to these. The second map is given by taking the unit sphere of a bundle, then taking the quotient by the antipodal map (a free $\mathbb{Z}/2$-action) and classifying the resulting $\mathbb{Z}/2$-bundle. This map is the only nontrivial differential in the spectral sequence. Now

$$\oplus_{0\leq n\leq k}\mathscr{N}_{k-n}(BO(n)) \cong \mathscr{N}_*[x_1, x_2, \cdots],$$

where $x_k \in \mathscr{N}_{k-1}(BO(1))$ is the class of the canonical line bundle over $\mathbb{R}P^{k-1}$. On the other hand,

$$\mathscr{N}_*(B\mathbb{Z}/2) \cong \mathscr{N}_*\{r_0, r_1, r_2, \cdots\}$$

is the free $\mathscr{N}_*$-module generated by $\{r_k\}$, where r_k is the class of $\mathbb{R}P^k \longrightarrow B\mathbb{Z}/2$. The splitting is the obvious one: it sends r_k to x_{k+1}. In fact, the x_k all live in the summand $\mathscr{N}_*(B\mathbb{Z}/2) = \mathscr{N}_*(BO(1))$, and the splitting is simply the inclusion of this summand. It follows that $\mathscr{N}_*^{\mathbb{Z}/2}$ is a free module over $\mathscr{N}_*$, and one can write down explicit generators. Alexander writes down explicit multiplicative generators.

A similar calculation can be done for $MO_*^{\mathbb{Z}/2}$. The short exact sequence is then

$$0 \longrightarrow MO_k^{\mathbb{Z}/2} \longrightarrow \oplus_n\mathscr{N}_{k-n}(BO) \longrightarrow \mathscr{N}_{k-1}(B\mathbb{Z}/2) \longrightarrow 0,$$

where now k and n range over the integers, positive and negative, and the sum in the middle is infinite. In fact,

$$\oplus_n\mathscr{N}_{*-n}(BO) \cong \mathscr{N}_*[x_1^{-1}, x_1, x_2, \cdots],$$

where the x_i are the images of the elements of the same name from the geometric case. Here x_1^{-1} is the image of χ_L, where L is the nontrivial irreducible representation of $\mathbb{Z}/2$.

It is natural to ask whether or not $mo_{\mathbb{Z}/2}$ and $MO_{\mathbb{Z}/2}$ are products of Eilenberg-MacLane $\mathbb{Z}/2$-spectra, as in the case of odd order groups. I showed that the answer turns out to be no.

J. C. Alexander. The bordism ring of manifolds with involution. Proc. Amer. Math. Soc. 31(1972), 536-542.

P. E. Conner and E. E. Floyd. Differentiable periodic maps. Academic Press, Inc. 1964.

S. Costenoble. The structure of some equivariant Thom spectra. Trans. Amer. Math. Soc. 315(1989), 231-254.

CHAPTER XVI

Spectra and G-spectra; change of groups; duality

In this and the following three chapters, we return to the development of features of the equivariant stable homotopy category. The basic reference is [LMS], and specific citations are given at the ends of sections.

1. Fixed point spectra and orbit spectra

Much of the most interesting work in equivariant algebraic topology involves the connection between equivariant constructions and nonequivariant topics of current interest. We here explain the basic facts concerning the relationships between G-spectra and spectra and between equivariant and nonequivariant cohomology theories.

We restrict attention to a complete G-universe U and we write $RO(G)$ for $RO(G;U)$. Given the details of the previous chapter, we shall be more informal about the $RO(G)$-grading from now on. In particular, we shall allow ourselves to write $E^\alpha_G(X)$ for $\alpha \in RO(G)$, ignoring the fact that, for rigor, we must first fix a presentation of α as a formal difference $V \ominus W$. We write S^α instead of $S^{V\ominus W}$ and, for G-spectra X and E, we write

(1.1) $$E^G_\alpha(X) = [S^\alpha, E \wedge X]_G$$

and

(1.2) $$E^\alpha_G(X) = [S^{-\alpha} \wedge X, E]_G = [S^{-\alpha}, F(X,E)]_G.$$

To relate this to nonequivariant theories, let $i : U^G \longrightarrow U$ be the inclusion of the fixed point universe. Recall that we have the forgetful functor

$$i^* : G\mathscr{S}U \longrightarrow G\mathscr{S}U^G$$

obtained by forgetting the indexing G-spaces with non-trivial G-action. The "underlying nonequivariant spectrum" of E is i^*E with its action by G ignored. Recall too that i^* has a left adjoint

$$i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$$

that builds in non-trivial representations. Explicitly, for a naive G-prespectrum D and an indexing G-space V,

$$(i_*D)(V) = D(V^G) \wedge S^{V-V^G}.$$

For a naive G-spectrum D, $i_*D = Li_*\ell D$, as usual. These change of universe functors play a subtle and critical role in relating equivariant and nonequivariant phenomena. Since, with G-actions ignored, the universes are isomorphic, the following result is intuitively obvious.

LEMMA 1.3. *For $D \in G\mathscr{S}U^G$, the unit G-map $\eta : D \longrightarrow i^*i_*D$ of the (i_*, i^*) adjunction is a nonequivariant equivalence. For $E \in G\mathscr{S}U$, the counit G-map $\varepsilon : i_*i^*E \longrightarrow E$ is a nonequivariant equivalence.*

We define the fixed point spectrum D^G of a naive G-spectrum D by passing to fixed points spacewise, $D^G(V) = (DV)^G$. This functor is right adjoint to the forgetful functor from naive G-spectra to spectra:

$$(1.4) \quad G\mathscr{S}U^G(C, D) \cong \mathscr{S}U^G(C, D^G) \quad \text{for} \quad C \in \mathscr{S}U^G \quad \text{and} \quad D \in G\mathscr{S}U^G.$$

It is essential that G act trivially on the universe to obtain well-defined structural homeomorphisms on D^G. For $E \in G\mathscr{S}U$, we define $E^G = (i^*E)^G$. Composing the (i_*, i^*)-adjunction with (1.4), we obtain

$$(1.5) \quad G\mathscr{S}U(i_*C, E) \cong \mathscr{S}U^G(C, E^G) \quad \text{for} \quad C \in \mathscr{S}U^G \quad \text{and} \quad D \in G\mathscr{S}U^G.$$

The sphere G-spectra $G/H_+ \wedge S^n$ in $G\mathscr{S}U$ are obtained by applying i_* to the corresponding sphere G-spectra in $G\mathscr{S}U^G$. When we restrict (1.1) and (1.2) to integer gradings and take $H = G$, we see that (1.5) implies

$$(1.6) \qquad E^G_n(X) \cong \pi_n((E \wedge X)^G)$$

and

$$(1.7) \qquad E_G^n(X) \cong \pi_{-n}(F(X, E)^G).$$

As in the second isomorphism, naive G-spectra D represent $\mathbb{Z}$-graded cohomology theories on naive G-spectra or on G-spaces. In contrast, as we have already noted in XIII§3, we cannot represent interesting homology theories on G-spaces

X in the form $\pi_*((D \wedge X)^G)$ for a naive G-spectrum D: here smash products commute with fixed points, hence such theories vanish on X/X^G. For genuine G-spectra, there is a well-behaved natural map

$$E^G \wedge (E')^G \longrightarrow (E \wedge E')^G, \tag{1.8}$$

but, even when E' is replaced by a G-space, it is not an equivalence. In Section 3, we shall define a different G-fixed point functor that does commute with smash products.

Orbit spectra D/G of naive G-spectra are constructed by first passing to orbits spacewise on the prespectrum level and then applying the functor L from prespectra to spectra. Here $(\Sigma^\infty X)/G \cong \Sigma^\infty(X/G)$. The orbit functor is left adjoint to the forgetful functor to spectra:

$$\text{(1.9)}\quad \mathscr{S}U^G(D/G, C) \cong G\mathscr{S}U^G(D, C) \quad \text{for} \quad C \in \mathscr{S}U^G \quad \text{and} \quad D \in G\mathscr{S}U^G.$$

For a genuine G-spectrum E, it is tempting to define E/G to be $L((i^*E)/G)$, but this appears to be an entirely useless construction. For free actions, we will shortly give a substitute.

[LMS, especially I§3]

2. Split G-spectra and free G-spectra

The calculation of the equivariant cohomology of free G-spectra in terms of the nonequivariant cohomology of orbit spectra is fundamental to the passage back and forth between equivariant and nonequivariant phenomena. This requires the subtle and important notion of a "split G-spectrum".

Definition 2.1. A naive G-spectrum D is said to be split if there is a nonequivariant map of spectra $\zeta : D \longrightarrow D^G$ whose composite with the inclusion of D^G in D is homotopic to the identity map. A genuine G-spectrum E is said to be split if i^*E is split.

The K-theory G-spectra K_G and KO_G are split. Intuitively, the splitting is obtained by giving nonequivariant bundles trivial G-action. The cobordism spectra MO_G and MU_G are also split. The Eilenberg-MacLane G-spectrum HM associated to a Mackey functor M is split if and only if the canonical map $M(G/G) \longrightarrow M(G/e)$ is a split epimorphism; this implies that G acts trivially on $M(G/e)$, which is usually not the case. The suspension G-spectrum $\Sigma^\infty X$ of a G-space X is split if and only if X is stably a retract up to homotopy of X^G, which again is usually not the case. In particular, however, the sphere G-spectrum $S = \Sigma^\infty S^0$ is split. The following consequence of Lemma 1.3 gives more examples.

LEMMA 2.2. If $D \in G\mathscr{S}U^G$ is split, then $i_*D \in G\mathscr{S}U$ is also split.

The notion of a split G-spectrum is defined in nonequivariant terms, but it admits the following equivariant interpretation.

LEMMA 2.3. If E is a G-spectrum with underlying nonequivariant spectrum D, then E is split if and only if there is a map of G-spectra $i_*D \longrightarrow E$ that is a nonequivariant equivalence.

Recall that a based G-space is said to be free if it is free away from its G-fixed basepoint. A G-spectrum, either naive or genuine, is said to be free if it is equivalent to a G-CW spectrum built up out of free cells $G_+ \wedge CS^n$. The functors $\Sigma^\infty : \mathscr{T} \longrightarrow G\mathscr{S}U^G$ and $i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$ carry free G-spaces to free naive G-spectra and free naive G-spectra to free G-spectra. In all three categories, X is homotopy equivalent to a free object if and only if the canonical G-map $EG_+ \wedge X \longrightarrow X$ is an equivalence. A free G-spectrum E is equivalent to i_*D for a free naive G-spectrum D, unique up to equivalence; the orbit spectrum D/G is the substitute for E/G that we alluded to above. A useful mnemonic slogan is that "free G-spectra live in the trivial universe". Note, however, that we cannot take D = i*E: this is not a free G-spectrum. For example, $\Sigma^\infty G_+ \in G\mathscr{S}U^G$ clearly satisfies $(\Sigma^\infty G_+)^G = *$, but we shall see later that $i_*\Sigma^\infty G_+$, which is the genuine suspension G-spectrum $\Sigma^\infty G_+ \in G\mathscr{S}U$, satisfies $(i^*\Sigma^\infty G_+)^G = S$.

THEOREM 2.4. If E is a split G-spectrum and X is a free naive G-spectrum, then there are natural isomorphisms

$$E_n^G(i_*X) \cong E_n((\Sigma^{Ad(G)}X)/G) \quad \text{and} \quad E_G^n(i_*X) \cong E^n(X/G),$$

where $Ad(G)$ is the adjoint representation of G and E_* and E^* denote the theories represented by the underlying nonequivariant spectrum of E.

The cohomology isomorphism holds by inductive reduction to the case $X = G_+$ and use of Lemma 2.3. The homology isomorphism is quite subtle and depends on a dimension-shifting transfer isomorphism that we shall say more about later. This result is an essential starting point for the approach to generalized Tate cohomology theory that we shall present later.

In analogy with (1.8), there is a well-behaved natural map

$$\Sigma^\infty(X^G) \longrightarrow (\Sigma^\infty X)^G, \tag{2.5}$$

but it is not an equivalence.

[LMS, especially II.1.8, II.2.8, II.2.12, II.8.4]

3. Geometric fixed point spectra

There is a "geometric fixed-point functor"

$$\Phi^G : G\mathscr{S}U \longrightarrow \mathscr{S}U^G$$

that enjoys the properties

$$\Sigma^\infty(X^G) \simeq \Phi^G(\Sigma^\infty X) \tag{3.1}$$

and

$$\Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E'). \tag{3.2}$$

To construct it, recall the definition of $\tilde{E}\mathscr{F}$ for a family $\mathscr{F}$ from V.2.8 and set

$$\Phi^G E = (E \wedge \tilde{E}\mathscr{P})^G, \tag{3.3}$$

where $\mathscr{P}$ is the family of all proper subgroups of G. Here $E \wedge \tilde{E}\mathscr{P}$ is H-trivial for all $H \in \mathscr{P}$.

The name "geometric fixed point spectrum" comes from an equivalent description of the functor Φ^G. There is an intuitive "spacewise G-fixed point functor" Φ^G from G-prespectra indexed on U to prespectra indexed on U^G. To be precise about this, we index G-prespectra on an indexing sequence $\{V_i\}$, so that $V_i \subset V_{i+1}$ and $U = \cup V_i$, and index prespectra on the indexing sequence $\left\{V_i^G\right\}$. Here we use indexing sequences to avoid ambiguities resulting from the fact that different indexing spaces in U can have the same G-fixed point space. For a G-prespectrum $D = \{DV_i\}$, the prespectrum $\Phi^G D$ is given by $(\Phi^G D)(V_i) = (DV_i)^G$, with structural maps $\Sigma^{V_{i+1}^G - V_i^G}(DV_i)^G \longrightarrow (DV_{i+1})^G$ obtained from those of D by passage to G-fixed points. We are interested in homotopical properties of this construction, and when applying it to spectra regarded as prespectra, we must first apply the cylinder functor K and CW approximation functor Γ discussed in XII§9. The relationship between the resulting construction and the spectrum-level construction (3.3) is as follows. Remember that ℓ denotes the forgetful functor from spectra to prespectra and L denotes its left adjoint.

Theorem 3.4. For Σ-cofibrant G-prespectra D, there is a natural weak equivalence of spectra

$$\Phi^G LD \longrightarrow L\Phi^G D.$$

For G-CW spectra E, there is a natural weak equivalence of spectra

$$\Phi^G E \longrightarrow L\Phi^G K\Gamma\ell E.$$

It is not hard to deduce the isomorphisms (3.1) and (3.2) from this prespectrum level description of Φ^G.

[LMS, II§9]

4. Change of groups and the Wirthmüller isomorphism

In the previous sections, we discussed the relationship between G-spectra and e-spectra, where we write e both for the identity element and the trivial subgroup of G. We must consider other subgroups and quotient groups of G. First, consider a subgroup H. Since any representation of NH extends to a representation of G and since a WH-representation is just an H-fixed NH-representation, the H-fixed point space U^H of our given complete G-universe U is a complete WH-universe. We define

$$E^H = (i^*E)^H, \quad i : U^H \subset U. \tag{4.1}$$

This gives a functor $G\mathscr{S}U \longrightarrow (WH)\mathscr{S}U^H$. Of course, we can also define E^H as a spectrum in $\mathscr{S}U^G$. The forgetful functor associated to the inclusion $U^G \longrightarrow U^H$ carries the first version of E^H to the second, and we use the same notation for both. For $D \in (NH)\mathscr{S}U^H$, the orbit spectrum D/H is also a WH-spectrum.

Exactly as on the space level in I§1, we have induced and coinduced G-spectra generated by an H-spectrum $D \in H\mathscr{S}U$. These are denoted by

$$G \ltimes_H D \quad \text{and} \quad F_H[G, D).$$

The "twisted" notation $\ltimes$ is used because there is a little twist in the definitions to take account of the action of G on indexing spaces. As on the space level, these functors are left and right adjoint to the forgetful functor $G\mathscr{S}U \longrightarrow H\mathscr{S}U$: for $D \in H\mathscr{S}U$ and $E \in G\mathscr{S}U$, we have

$$G\mathscr{S}U(G \ltimes_H D, E) \cong H\mathscr{S}U(D, E) \tag{4.2}$$

and

$$H\mathscr{S}U(E, D) \cong G\mathscr{S}U(E, F_H[G, D)). \tag{4.3}$$

Again, as on the space level, for $E \in G\mathscr{S}U$ we have

$$G \ltimes_H E \cong (G/H)_+ \wedge E \tag{4.4}$$

and

$$F_H[G, E) \cong F(G/H_+, E). \tag{4.5}$$

As promised earlier, we can now deduce as in (1.6) that

$$\pi_n^H(E) \equiv [G/H_+ \wedge S^n, E]_G \cong [S^n, E]_H \cong \pi_n(E^H). \tag{4.6}$$

In cohomology, the isomorphism (4.2) gives

$$E_G^*(G \ltimes_H D) \cong E_H^*(D). \tag{4.7}$$

We shall not go into detail, but we can interpret this in terms of $RO(G)$ and $RO(H)$ graded theories via the evident functor $\mathscr{R}O(G) \longrightarrow \mathscr{R}O(H)$. The isomorphism (4.3) does not have such a convenient interpretation as it stands. However, there is a fundamental change of groups result — called the Wirthmüller isomorphism — which in its most conceptual form is given by a calculation of the functor $F_H[G, D)$. It leads to the following homological complement of (4.7). Let $L(H)$ be the tangent H-representation at the identity coset of G/H. Then

$$E_*^G(G \ltimes_H D) \cong E_*^H(\Sigma^{L(H)} D). \tag{4.8}$$

THEOREM 4.9 (GENERALIZED WIRTHMÜLLER ISOMORPHISM). For H-spectra D, there is a natural equivalence of G-spectra

$$F_H[G, \Sigma^{L(H)} D) \longrightarrow G \ltimes_H D.$$

Therefore, for G-spectra E,

$$[E, \Sigma^{L(H)} D]_H \cong [E, G \ltimes_H D]_G.$$

The last isomorphism complements the isomorphism from (4.2):

$$[G \ltimes_H D, E]_G \cong [D, E]_H. \tag{4.10}$$

We deduce (4.8) by replacing E in (4.9) by a sphere, replacing D by $E \wedge D$, and using the generalization

$$G \ltimes_H (D \wedge E) \cong (G \ltimes_H D) \wedge E$$

of (4.4).

[LMS, II§§2-4]
K. Wirthmüller. Equivariant homology and duality. Manuscripta Math. 11(1974), 373-390.

5. Quotient groups and the Adams isomorphism

Let N be a normal subgroup of G with quotient group J. In practice, one is often thinking of a quotient map $NH \longrightarrow WH$ rather than $G \longrightarrow J$. There is an analog of the Wirthmüller isomorphism — called the Adams isomorphism — that compares orbit and fixed-point spectra. It involves the change of universe functors associated to the inclusion $i : U^N \longrightarrow U$ and requires restriction to N-free G-spectra. We note first that the fixed point and orbit functors $G\mathscr{S}U^N \longrightarrow J\mathscr{S}U^N$ are right and left adjoint to the evident pullback functor from J-spectra to G-spectra: for $D \in J\mathscr{S}U^N$ and $E \in G\mathscr{S}U^N$,

$$G\mathscr{S}U^N(D, E) \cong J\mathscr{S}U^N(D, E^N) \tag{5.1}$$

and

$$J\mathscr{S}U^N(E/N, D) \cong G\mathscr{S}U^N(E, D). \tag{5.2}$$

Here we suppress notation for the pullback functor $J\mathscr{S}U^N \longrightarrow G\mathscr{S}U^N$. An N-free G-spectrum E indexed on U is equivalent to i_*D for an N-free G-spectrum D indexed on U^N, and D is unique up to equivalence. Thus our slogan that "free G-spectra live in the trivial universe" generalizes to the slogan that "N-free G-spectra live in the N-fixed universe". This gives force to the following version of (5.2). It compares maps of J-spectra indexed on U^N with maps of G-spectra indexed on U.

THEOREM 5.3. Let $J = G/N$. For N-free G-spectra E indexed on U^N and J-spectra D indexed on U^N,

$$[E/N, D]_J \cong [i_*E, i_*D]_G.$$

The conjugation action of G on N gives rise to an action of G on the tangent space of N at e; we call this representation $Ad(N)$, or $Ad(N; G)$. The following result complements the previous one, but is very much deeper. When $N = G$, it is the heart of the proof of the homology isomorphism of Theorem 2.4. We shall later describe the dimension-shifting transfer that is the basic ingredient in its proof.

THEOREM 5.4 (GENERALIZED ADAMS ISOMORPHISM). Let $J = G/N$. For N-free G-spectra $E \in G\mathscr{S}U^N$, there is a natural equivalence of J-spectra

$$E/N \longrightarrow (\Sigma^{-Ad(N)} i_*E)^N.$$

Therefore, for $D \in J\mathscr{S}U^N$,

$$[D, E/N]_J \cong [i_*D, \Sigma^{-Ad(N)} i_*E]_G.$$

This result is another of the essential starting points for the approach to generalized Tate cohomology that we will present later. The last two results cry out for general homological and cohomological interpretations, like those of Theorem 2.4. Looking back at Lemma 2.3, we see that what is needed for this are analogs of the underlying nonequivariant spectrum and of the characterization of split G-spectra that make sense for quotient groups J. What is so special about the trivial group is just that it is naturally both a subgroup and a quotient group of G.

The language of families is helpful here. Let $\mathscr{F}$ be a family. We say that a G-spectrum E is $\mathscr{F}$-free, or is an $\mathscr{F}$-spectrum, if E is equivalent to a G-CW spectrum all of whose cells are of orbit type in $\mathscr{F}$. Thus free G-spectra are $\{e\}$-free. We say that a map $f : D \longrightarrow E$ is an $\mathscr{F}$-equivalence if $f^H : D^H \longrightarrow E^H$ is an equivalence for all $H \in \mathscr{F}$ or, equivalently by the Whitehead theorem, if f is an H-equivalence for all $H \in \mathscr{F}$.

Returning to our normal subgroup N, let $\mathscr{F}(N) = \mathscr{F}(N;G)$ be the family of subgroups of G that intersect N in the trivial group. Thus an $\mathscr{F}(N)$-spectrum is an N-free G-spectrum. We have seen these families before, in our study of equivariant bundles. We can now state precise generalizations of Lemma 2.3 and Theorem 2.4. Fix spectra

$$D \in J\mathscr{S}U^N \quad \text{and} \quad E \in G\mathscr{S}U.$$

LEMMA 5.5. *A G-map $\xi : i_*D \longrightarrow E$ is an $\mathscr{F}(N)$-equivalence if and only if the composite of the adjoint $D \longrightarrow (i^*E)^N$ of ξ and the inclusion $(i^*E)^N \longrightarrow i^*E$ is an $\mathscr{F}(N)$-equivalence.*

THEOREM 5.6. *Assume given an $\mathscr{F}(N)$-equivalence $i_*D \longrightarrow E$. For any N-free G-spectrum $X \in G\mathscr{S}U^N$,*

$$E_*^G(\Sigma^{-Ad(N)}(i_*X)) \cong D_*^J(X/N) \quad \text{and} \quad E_G^*(i_*X) \cong D_J^*(X/N).$$

Given E, when do we have an appropriate D? We often have theories that are defined on the category of all compact Lie groups, or on a suitable sub-category. When such theories satisfy appropriate naturality axioms, the theory E_J associated to J will necessarily bear the appropriate relationship to the theory E_G associated to G. We shall not go into detail here. One assumes that the homomorphisms $\alpha : H \longrightarrow G$ in one's category induce maps of H-spectra $\xi_\alpha : \alpha^*E_G \longrightarrow E_H$ in a functorial way, where some bookkeeping with universes is needed to make sense of α^*, and one assumes that ξ_α is an H-equivalence if α is an inclusion. For each

$H \in \mathscr{F}(N)$, the quotient map $q : G \longrightarrow J$ restricts to an isomorphism from H to its image K. If the five visible maps,

$$H \subset G,\ K \subset J,\ q : G \longrightarrow J,\ q : H \longrightarrow K, \text{ and } q^{-1} : K \longrightarrow H,$$

are in one's category, one can deduce that $\xi_q : q^*E_J = i_*E_J \longrightarrow E_G$ is an $\mathscr{F}(N)$-equivalence. This is not too surprising in view of Lemma 2.3, but it is a bit subtle: there are examples where all axioms are satisfied, except that q^{-1} is not in the category, and the conclusion fails because ξ_q is not an H-equivalence. However, this does work for equivariant K-theory and the stable forms of equivariant cobordism, generalizing the arguments used to prove that these theories split. For K-theory, the Bott isomorphisms are suitably natural, by the specification of the Bott elements in terms of exterior powers. For cobordism, we shall explain in XXV§5 that MO_G and MU_G arise from functors, called "global $\mathscr{I}_*$ functors with smash product", that are defined on all compact Lie groups and their representations and take values in spaces with group actions. All theories with such a concrete geometric source are defined with suitable naturality on all compact Lie groups G.

J. F. Adams. Prerequisites (on equivariant theory) for Carlsson's lecture. Springer Lecture Notes in Mathematics Vol. 1051, 1984, 483-532.
[LMS, II§§8-9]

6. The construction of G/N-spectra from G-spectra

A different line of thought leads to a construction of J-spectra from G-spectra, $J = G/N$, that is a direct generalization of the geometric fixed point construction $\Phi^G E$. The appropriate analog of $\mathscr{P}$ is the family $\mathscr{F}[N]$ of those subgroups of G that do not contain N. Note that this is a family since N is normal. For a spectrum E in $G\mathscr{S}U$, we define

$$\Phi^N E \equiv (E \wedge \tilde{E}\mathscr{F}[N])^N. \tag{6.1}$$

We have the expected generalizations of (3.1) and (3.2): for a G-space X,

$$\Sigma^\infty(X^N) \simeq \Phi^N(\Sigma^\infty X) \tag{6.2}$$

and, for G-spectra E and E',

$$\Phi^N(E) \wedge \Phi^N(E') \simeq \Phi^N(E \wedge E'). \tag{6.3}$$

We can define $\Phi^H E$ for a not necessarily normal subgroup H by regarding E as an NH-spectrum. Although the Whitehead theorem appears naturally as a statement about homotopy groups and thus about the genuine fixed point functors characterized by the standard adjunctions, it is worth observing that it implies a version in terms of these Φ-fixed point spectra.

THEOREM 6.4. A map $f : E \longrightarrow E'$ of G-spectra is an equivalence if and only if each $\Phi^H f : \Phi^H E \longrightarrow \Phi^H E'$ is a nonequivariant equivalence.

Note that, for any family $\mathscr{F}$ and any G-spectra E and E',

$$[E \wedge E\mathscr{F}_+, E' \wedge \tilde{E}\mathscr{F}]_G = 0$$

since $E\mathscr{F}$ only has cells of orbit type G/H and $\tilde{E}\mathscr{F}$ is H-contractible for such H. Therefore the canonical G-map $E \longrightarrow E \wedge \tilde{E}\mathscr{F}$ induces an isomorphism

$$[E \wedge \tilde{E}\mathscr{F}, E' \wedge \tilde{E}\mathscr{F}]_G \cong [E, E' \wedge \tilde{E}\mathscr{F}]_G. \tag{6.5}$$

In the case of $\mathscr{F}[N]$, $E \longrightarrow E \wedge \tilde{E}\mathscr{F}[N]$ is an equivalence if and only if E is concentrated over N, in the sense that E is H-contractible if H does not contain N. Maps into such G-spectra determine and are determined by the J-maps obtained by passage to Φ^N-fixed point spectra. In fact, the stable category of J-spectra is equivalent to the full subcategory of the stable category of G-spectra consisting of the G-spectra concentrated over N.

THEOREM 6.6. For J-spectra $D \in J\mathscr{S}U^N$ and G-spectra $E \in G\mathscr{S}U$ concentrated over N, there is a natural isomorphism

$$[D, E^N]_J \cong [i_*D \wedge \tilde{E}\mathscr{F}[N], E]_G.$$

For J-spectra D and D', the functor $i_*(\cdot) \wedge \tilde{E}\mathscr{F}[N]$ induces a natural isomorphism

$$[D, D']_J \cong [i_*D \wedge \tilde{E}\mathscr{F}[N], i_*D \wedge \tilde{E}\mathscr{F}[N]]_G.$$

For general G-spectra E and E', the functor $\Phi^N(\cdot)$ induces a natural isomorphism

$$[\Phi^N E, \Phi^N E']_J \cong [E, E' \wedge \tilde{E}\mathscr{F}[N]]_G.$$

PROOF. The first isomorphism is a consequence of (5.1) and (6.5). The other two isomorphisms follow once one shows that the unit

$$D \longrightarrow (i_*D \wedge \tilde{E}\mathscr{F}[N])^N = \Phi^N(i_*D)$$

and counit

$$(i_*E^N) \wedge \tilde{E}\mathscr{F}[N] \longrightarrow E$$

of the adjunction are equivalences. One proves this by use of a spacewise N-fixed point functor, also denoted Φ^N, from G-prespectra to J-prespectra. This functor is defined exactly as was the spacewise G-fixed point functor in Section 3. It satisfies $\Phi^N(i_*D) = D$, and it commutes with smash products. The following generalization of Theorem 3.4, which shows that the prespectrum level functor Φ^N induces a functor equivalent to Φ^N on the spectrum level, leads to the conclusion. □

THEOREM 6.7. For Σ-cofibrant G-prespectra D, there is a natural weak equivalence of J-spectra

$$\Phi^N LD \longrightarrow L\Phi^N D.$$

For G-CW spectra E, there is a natural weak equivalence of J-spectra

$$\Phi^N E \longrightarrow L\Phi^N K\Gamma\ell E.$$

As an illuminating example of the use of $RO(G)$-grading to allow calculational descriptions invisible to the $\mathbb{Z}$-graded part of a theory, we record how to compute the cohomology theory represented by $\Phi^N(E)$ in terms of the cohomology theory represented by E. This uses the Euler classes of representations, which appear ubiquitously in equivariant theory. For a representation V, we define $e(V) \in E_G^V(S^0)$ to be the image of $1 \in E_G^0(S^0) \cong E_G^V(S^V)$ under e^*, where $e : S^0 \longrightarrow S^V$ sends the basepoint to the point at ∞ and the non-basepoint to 0.

PROPOSITION 6.8. Let E be a ring G-spectrum. For a finite J-CW spectrum X, $(\Phi^N E)_J^*(X)$ is the localization of $E_G^*(X)$ obtained by inverting the Euler classes of all representations V such that $V^N = \{0\}$.

PROOF. By (6.3), $\Phi^N(E)$ inherits a ring structure from E. In interpreting the grading, we regard representations of J as representations of G by pullback. A check of fixed points, using the cofibrations $S(V) \longrightarrow B(V) \longrightarrow S^V$, shows that we obtain a model for $\tilde{E}\mathscr{F}[N]$ by taking the colimit of the spaces S^V as V ranges over the representations of G such that $V^N \cong \{0\}$. This leads to a colimit description of $(\Phi^N E)_J^*(X)$ that coincides algebraically with the cited localization. □

With motivation from the last few results, the unfortunate alternative notation $E_J = \Phi^N(E_G)$ was used in [LMS] and elsewhere. This is a red herring from the point of view of Theorem 5.6, and it is ambiguous on two accounts. First, the J-spectrum $\Phi^N(E_G)$ depends vitally on the extension $J = G/N$ and not just on the group J. Second, in classical examples, the spectrum "E_J" will generally not agree with the preassigned spectrum with the same notation. For example, the subquotient J-spectrum "K_J" associated to the K-theory G-spectrum K_G is not the K-theory J-spectrum K_J. However, if S_G is the sphere G-spectrum, then the subquotient J-spectrum S_J is the sphere J-spectrum. We shall see that this easy fact plays a key conceptual role in Carlsson's proof of the Segal conjecture.

[LMS, II§9]

7. Spanier-Whitehead duality

We can develop abstract duality theory in any symmetric monoidal category, such as $\bar{h}G\mathscr{S}$ for our fixed complete G-universe U. While the elegant approach is to start from the abstract context, we shall specialize to $\bar{h}G\mathscr{S}$ from the start since we wish to emphasize equivariant phenomena. Define the dual of a G-spectrum X to be $DX = F(X,S)$. There is a natural map

$$\nu : F(X,Y)\wedge Z \longrightarrow F(X, Y\wedge Z). \tag{7.1}$$

Using the unit isomorphism, it specializes to

$$\nu : (DX)\wedge X \longrightarrow F(X,X). \tag{7.2}$$

The adjoint of the unit isomorphism $S\wedge X \longrightarrow X$ gives a natural map $\eta : S \longrightarrow F(X,X)$. We say that X is "strongly dualizable" if there is a coevaluation map $\eta : S \longrightarrow X\wedge(DX)$ such that the following diagram commutes, where γ is the commutativity isomorphism.

$$\begin{array}{ccc} S & \xrightarrow{\eta} & X\wedge(DX) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ F(X,X) & \xleftarrow[\nu]{} & (DX)\wedge X \end{array} \tag{7.3}$$

It is a categorical implication of the definition that the map ν of (7.1) is an isomorphism if either X or Z is strongly dualizable, and there are various other such formal consequences, such as $X\cong DD(X)$ when X is strongly dualizable. In particular, if X is strongly dualizable, then the map ν of (3.2) is an isomorphism. Conversely, if the map ν of (7.2) is an isomorphism, then X is strongly dualizable since the coevaluation map η can and must be defined to be the composite $\gamma\nu^{-1}\eta$ in (7.3). Note that we have an evaluation map $\varepsilon : DX\wedge X \longrightarrow S$ for any X.

THEOREM 7.4. A G-CW spectrum is strongly dualizable if and only if it is equivalent to a wedge summand of a finite G-CW spectrum.

PROOF. The evaluation map of X induces a natural map

$$\varepsilon_{\#} : [Y, Z\wedge DX]_G \longrightarrow [Y\wedge X, Z]_G \tag{$*$}$$

via $\varepsilon_{\#}(f) = (\mathrm{Id} \wedge \varepsilon)(f \wedge \mathrm{Id})$, and X is strongly dualizable if and only if $\varepsilon_{\#}$ is an isomorphism for all Y and Z. The Wirthmüller isomorphism implies that $D(\Sigma^\infty G/H_+)$ is equivalent to $G \ltimes_H S^{-L(H)}$, and diagram chases show that it also implies that $\varepsilon_{\#}$ is an isomorphism. Actually, this duality on orbits is the heart of the Wirthmüller isomorphism, and we shall explain it in direct geometric terms in the next section. If X is strongly dualizable, then so is ΣX. The cofiber of a map between strongly dualizable G-spectra is strongly dualizable since both sides of (*) turn cofibrations in X into long exact sequences. By induction on the number of cells, a finite G-CW spectrum is strongly dualizable, and it is formal that a wedge summand of a strongly dualizable G-spectrum is strongly dualizable. For the converse, which was conjectured in [LMS] and proven by Greenlees (unpublished), let X be a strongly dualizable G-CW spectrum with coevaluation map η. Then η factors through $A \wedge DX$ for some finite subcomplex A of X, the following diagram commutes, and its bottom composite is the identity:

$$\begin{array}{ccccc}
 & \nearrow & A \wedge (DX) \wedge X & \xrightarrow{\mathrm{Id} \wedge \varepsilon} & A \wedge S \cong A \\
 & & \downarrow & & \downarrow \\
X \cong S \wedge X & \xrightarrow[\eta \wedge \mathrm{Id}]{} & X \wedge (DX) \wedge X & \xrightarrow[\mathrm{Id} \wedge \varepsilon]{} & X \wedge S \cong X.
\end{array}$$

Therefore X is a retract up to homotopy and thus a wedge summand up to homotopy of A. $\square$

In contrast to the nonequivariant case, wedge summands of finite G-CW spectra need not be equivalent to finite G-CW spectra.

Corollary 7.5 (Spanier-Whitehead duality). If X is a wedge summand of a finite G-CW spectrum and E is any G-spectrum, then

$$\nu : DX \wedge E \longrightarrow F(X, E)$$

is an isomorphism in $\bar{h}G\mathscr{S}$. Therefore, for any representation α,

$$E^G_\alpha(DX) \cong E_G^{-\alpha}(X).$$

So far, we have concentrated on the naturally given dual DX. However, it is important to identify the homotopy types of duals concretely, as we did in the case of orbits. There are a number of equivalent criteria. The most basic one goes as follows. Suppose given G-spectra X and Y and maps

$$\varepsilon : Y \wedge X \longrightarrow S \quad \text{and} \quad \eta : S \longrightarrow X \wedge Y$$

such that the composites

$$X \cong S \wedge X \xrightarrow{\eta\wedge \mathrm{Id}} X \wedge Y \wedge X \xrightarrow{\mathrm{Id}\wedge\varepsilon} X \wedge X \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{Id}\wedge\eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon\wedge \mathrm{Id}} Y \wedge S \cong Y$$

are the respective identity maps. Then the adjoint $\tilde{\varepsilon} : Y \longrightarrow DX$ of ε is an equivalence and X is strongly dualizable with coevaluation map $(\mathrm{Id}\wedge\tilde{\varepsilon})\eta$. It is important to note that the maps η and ε that display the duality are not unique — much of the literature on duality is quite sloppy.

This criterion admits a homological interpretation, but we will not go into that here. It entails a reinterpretation in terms of the slant products relating homology and cohomology that we defined in XIII§5, and it works in the same way equivariantly as nonequivariantly.

[LMS, III§§1-3]

8. V-duality of G-spaces and Atiyah duality

There is a concrete space level version of the duality criterion just given. To describe it, let X and Y be G-spaces and let V be a representation of G. Suppose given G-maps

$$\varepsilon : Y \wedge X \longrightarrow S^V \quad \text{and} \quad \eta : S^V \longrightarrow X \wedge Y$$

such that the following diagrams are stably homotopy commutative, where $\sigma : S^V \longrightarrow S^V$ is the sign map, $\sigma(v) = -v$, and the γ are transpositions.

$$\begin{array}{ccc} S^V \wedge X & \xrightarrow{\eta\wedge \mathrm{Id}} & X \wedge Y \wedge X \\ & {\scriptstyle\gamma}\searrow & \downarrow{\scriptstyle \mathrm{Id}\wedge\varepsilon} \\ & & X \wedge S^V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y \wedge S^V & \xrightarrow{\mathrm{Id}\wedge\eta} & Y \wedge X \wedge Y \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle \varepsilon\wedge \mathrm{Id}} \\ S^V \wedge Y & \xrightarrow[\sigma\wedge \mathrm{Id}]{} & S^V \wedge Y. \end{array}$$

On application of the functor Σ_V^∞, we find that $\Sigma^\infty X$ and $\Sigma_V^\infty Y$ are strongly dualizable and dual to one another by our spectrum level criterion.

For reasonable X and Y, say finite G-CW complexes, or, more generally, compact G-ENR's (ENR = Euclidean neighborhood retract), we can use the space level equivariant suspension and Whitehead theorems to prove that a pair of G-maps (ε, η) displays a V-duality between X and Y, as above, if and only if the fixed point pair (ε^H, η^H) displays an $n(H)$-duality between X^H and Y^H for each $H \subset G$, where $n(H) = dim(V^H)$.

If X is a compact G-ENR, then X embeds as a retract of an open set of a G-representation V. One can use elementary space level methods to construct an explicit V-duality between X_+ and the unreduced mapping cone $V \cup C(V-X)$. For a G-cofibration $A \longrightarrow X$, there is a relative version that constructs a V-duality between $X \cup CA$ and $(V-A) \cup C(V-X)$. The argument specializes to give an equivariant version of the Atiyah duality theorem, via precise duality maps. Recall that the Thom complex of a vector bundle is obtained by fiberwise one-point compactification followed by identification of the points at infinity. When the base space is compact, this is just the one-point compactification of the total space.

THEOREM 8.1 (ATIYAH DUALITY). If M is a smooth closed G-manifold embedded in a representation V with normal bundle ν, then M_+ is V-dual to the Thom complex $T\nu$. If M is a smooth compact G-manifold with boundary ∂M, $V = V' \oplus \mathbb{R}$, and $(M, \partial M)$ is properly embedded in $(V' \times [0,\infty), V' \times \{0\})$ with normal bundles ν' of ∂M in V' and ν of M in V, then $M/\partial M$ is V-dual to $T\nu$, M_+ is V-dual to $T\nu/T\nu'$, and the cofibration sequence

$$T\nu' \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'$$

is V-dual to the cofibration sequence

$$\Sigma(\partial M)_+ \longleftarrow M/\partial M \longleftarrow M_+ \longleftarrow (\partial M)_+.$$

We display the duality maps explicitly in the closed case. By the equivariant tubular neighborhood theorem, we may extend the embedding of M in V to an embedding of the normal bundle ν and apply the Pontrjagin-Thom construction to obtain a map $t : S^V \longrightarrow T\nu$. The diagonal map of the total space of ν induces the Thom diagonal $\Delta : T\nu \longrightarrow M_+ \wedge T\nu$. The map η is just $\Delta \circ t$. The map ε is equally explicit but a bit more complicated to describe. Let $s : M \longrightarrow \nu$ be the zero section. The composite of $\Delta : M \longrightarrow M \times M$ and $s \times \mathrm{Id} : M \times M \longrightarrow \nu \times M$ is an embedding with trivial normal bundle. The Pontrjagin-Thom construction gives a map $t : T\nu \wedge M_+ \longrightarrow M_+ \wedge S^V$. Let $\xi : M_+ \longrightarrow S^0$ collapse all of M to the non-basepoint. The map ε is just $(\xi \wedge \mathrm{Id}) \circ t$. This explicit construction implies that the maps $\xi : M_+ \longrightarrow S^0$ and $t : S^V \longrightarrow T\nu$ are dual to one another.

Let us specialize this discussion to orbits G/H (compare IX.3.4). Recall that $L = L(H)$ is the tangent H-representation at the identity coset of G/H. We have

$$\tau = G \times_H L(H) \quad \text{and} \quad T\tau = G_+ \wedge_H S^{L(H)}.$$

If G/H is embedded in V with normal bundle ν, then $\nu \oplus \tau$ is the trivial bundle $G/H \times V$. Let W be the orthogonal complement to $L(H)$ in the fiber over the identity coset, so that $V = L \oplus W$ as an H-space. Since G/H_+ is V-dual to $T\nu$, $\Sigma^\infty G/H_+$ is dual to $\Sigma^\infty_V T\nu$. Since $S^W \wedge S^{-V} \simeq S^{-L}$ as H-spectra, we find that $\Sigma^\infty_V T\nu \simeq G \ltimes_H S^{-L}$.

[LMS, III§§3-5]

9. Poincaré duality

Returning to general smooth G-manifolds, we can deduce an equivariant version of the Poincaré duality theorem by combining Spanier-Whitehead duality, Atiyah duality, and the Thom isomorphism.

Definition 9.1. Let E be a ring G-spectrum and let ξ be an n-plane G-bundle over a G-space X. An E-orientation of ξ is an element $\mu \in E^\alpha_G(T\xi)$ for some $\alpha \in RO(G)$ of virtual dimension n such that, for each inclusion $i : G/H \longrightarrow X$, the restriction of μ to the Thom complex of the pullback $i^*\xi$ is a generator of the free $E^*_H(S^0)$-module $E^*_G(Ti^*\xi)$.

Here $i^*\xi$ has the form $G \times_H W$ for some representation W of H and $Ti^*\xi = G_+ \wedge S^W$ has cohomology $E^*_G(Ti^*\xi) \cong E^*_H(S^W) \cong E^{*-w}_H(S^0)$. Thus the definition makes sense, but it is limited in scope. If X is G-connected, then there is an obvious preferred choice for α, namely the fiber representation V at any fixed point of X: each W will then be isomorphic to V regarded as a representation of H. In general, however, there is no preferred choice for α and the existence of an orientation implies restrictions on the coefficients $E^*_H(S^0)$: there must be units in degree $\alpha - w \in RO(H)$. If $\alpha \neq w$, this forces a certain amount of periodicity in the theory. There is a great deal of further work, largely unpublished, by Costenoble, Waner, Kriz, and myself in the area of orientation theory and Poincaré duality, but the full story is not yet in place. Where it applies, the present definition does have the expected consequences.

Theorem 9.2 (Thom isomorphism). Let $\mu \in E^\alpha_G(T\xi)$ be an orientation of the G-vector bundle ξ over X. Then

$$\cup\mu : E^\beta_G(X_+) \longrightarrow E^{\alpha+\beta}_G(T\xi)$$

is an isomorphism for all β.

There is also a relative version. Specializing to oriented manifolds, we obtain the Poincaré duality theorem as an immediate consequence. Observe first that, for bundles ξ and η over X, the diagonal map of X induces a canonical map

$$T(\xi \oplus \eta) \longrightarrow T(\xi \times \eta) \cong T\xi \wedge T\eta.$$

There results a pairing

$$(*) \qquad E^{\alpha}_{G}(T\xi) \otimes E^{\beta}_{G}(T\eta) \longrightarrow E^{\alpha+\beta}_{G}(T(\xi \oplus \eta)).$$

We say that a smooth compact G-manifold M is E-oriented if its tangent bundle τ is oriented, say via $\mu \in E^{\alpha}_{G}(T\tau)$. In view of our discussion above, this makes most sense when M is a V-manifold and we take α to be V. If M has boundary, the smooth boundary collar theorem shows that the normal bundle of ∂M in M is trivial, and we deduce that an orientation of M determines an orientation $\partial\mu$ of ∂M in degree $\alpha - 1$ such that, under the pairing (*), the product of $\partial\mu$ and the canonical orientation $\iota \in E^{1}_{G}(\Sigma(\partial M)_{+})$ of the normal bundle is the restriction of μ to $T(\tau|\partial M)$. Similarly, if M is embedded in V, then μ determines an orientation ω of the normal bundle ν such that the product of μ and ω is the canonical orientation of the trivial bundle in $E^{v}_{G}(\Sigma^{V} M_{+})$.

Definition 9.3 (Poincaré duality). If M is a closed E-oriented smooth G-manifold with orientation $\mu \in E^{\alpha}_{G}(T\tau)$, then the composite

$$D : E^{\beta}_{G}(M_{+}) \longrightarrow E^{V-\alpha+\beta}_{G}(T\nu) \longrightarrow E^{G}_{\alpha-\beta}(M)$$

of the Thom and Spanier-Whitehead duality isomorphisms is the Poincaré duality isomorphism; the element $[M] = D(1)$ in $E^{G}_{\alpha}(M)$ is called the fundamental class associated to the orientation. If M is a compact E-oriented smooth G-manifold with boundary, then the analogous composites

$$D : E^{\beta}_{G}(M_{+}) \longrightarrow E^{V-\alpha+\beta}_{G}(T\nu) \longrightarrow E^{G}_{\alpha-\beta}(M, \partial M)$$

and

$$D : E^{\beta}_{G}(M, \partial M) \longrightarrow E^{V-\alpha+\beta}_{G}(T\nu, T(\nu|\partial M)) \longrightarrow E^{G}_{\alpha-\beta}(M)$$

are called the relative Poincaré duality isomorphisms. With the Poincaré duality isomorphism for ∂M, they specify an isomorphism from the cohomology long exact sequence to the homology long exact sequence of $(M, \partial M)$. Here the element $[M] = D(1)$ in $E^{G}_{\alpha}(M, \partial M)$ is called the fundamental class associated to the orientation.

One can check that these isomorphisms are given by capping with the fundamental class, as one would expect.

S. R. Costenoble, J. P. May, and S. Waner. Equivariant orientation theory. Preprint.
S. R. Costenoble and S. Waner. Equivariant Poincaré duality. Michigan Math. J. 39(1992).
[LMS, III§6]

CHAPTER XVII

The Burnside ring

The basic references are tom Dieck and [LMS]; some specific citations will be given.

[tD] T. tom Dieck. Transformation groups and representation theory. Springer Lecture Notes in Mathematics. Vol. 766. 1979.

1. Generalized Euler characteristics and transfer maps

There are general categorical notions of Euler characteristic and trace maps that encompass a variety of phenomena in both algebra and topology. We again specialize directly to the stable category $\bar{h}G\mathscr{S}$. Let X be a strongly dualizable G-spectrum with coevaluation map $\eta : S \longrightarrow X \wedge DX$ and define the "Euler characteristic" $\chi(X)$ to be the composite

$$(1.1) \qquad \chi(X) : S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\varepsilon} S.$$

For a G-space X, we write $\chi(X) = \chi(\Sigma^\infty X_+)$; for a based G-space X, we write $\tilde{\chi}(X) = \chi(\Sigma^\infty X)$. We shall shortly define the Burnside ring $A(G)$ in terms of these Euler characteristics, and we shall see that it is isomorphic to $\pi_0^G(S)$, the zeroth stable homotopy group of G-spheres. Thus, via the unit isomorphism $S \wedge E \simeq E$, $A(G)$ acts on all G-spectra E and thus on all homotopy, homology, and cohomology groups of all G-spectra. Its algebraic analysis is central to a variety of calculations in equivariant stable homotopy theory.

Before getting to this, we give a closely related conceptual version of transfer maps. Assume given a diagonal map $\Delta : X \longrightarrow X \wedge X$. We are thinking of X as $\Sigma^\infty F_+$ for, say, a compact G-ENR F. We define the "transfer map" $\tau = \tau(X)$:

$S \longrightarrow X$ to be the following composite:

(1.2)

$$\tau : S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\mathrm{Id}\wedge\Delta} DX \wedge X \wedge X \xrightarrow{\varepsilon\wedge\mathrm{Id}} S \wedge X \simeq X.$$

We shall later call these "pretransfer maps". When applied fiberwise in a suitable fashion, they will give rise to the transfer maps of bundles, which provide a crucial calculational device in both nonequivariant and equivariant cohomology theory.

These simple conceptual definitions lead to easy proofs of the basic properties of these fundamentally important maps. For example, to specify the relation between them, assume given a map $\xi = \xi(X) : X \longrightarrow S$ such that $(\mathrm{Id} \wedge \xi) \circ \Delta : X \longrightarrow X \wedge S$ is the unit isomophism. We are thinking of $\Sigma^\infty \xi$, where $\xi : F_+ \longrightarrow S^0$ is the evident collapse map. In the bundle context, the following immediate consequence of the definitions will determine the behavior of the composite of projection and transfer.

(1.3) The composite $\xi(X) \circ \tau(X) : S \longrightarrow S$ is equal to $\chi(X)$.

There are many other obvious properties with useful consequences.

Before getting to more of these, we assure the reader that if M is a smooth closed G-manifold embedded in a representation V, then application of the functor Σ_V^∞ to the explicit geometric transfer map

$$\tau(M) : S^V \longrightarrow \Sigma^V M_+$$

constructed in IX.3.1 does in fact give the same map as the transfer $\tau : S \longrightarrow S \wedge M_+$ of (1.2). By (1.3), it follows that the Euler characteristic $\chi(M)$ above is obtained by applying Σ_V^∞ to the Euler characteristic $\chi(M) : S^V \longrightarrow S^V$ of IX.3.2. One way to see this is to work out the description of the transfer map τ of (1.2) in the homotopical context of duality for G-ENR's and then specialize to manifolds as in XVI§8.

We shall return later to transfer maps, but we restrict attention to Euler characteristics here. We note first that, via a little Lie group theory, (1.5) leads to a calculation of the nonequivariant Euler characteristics $\chi((G/H)^K)$ for subgroups H and K. The key point is that, since $L(H)^H$ is the tangent space at the identity element of WH, WH is infinite if and only if $L(H)$ contains a trivial representation, in which case $e : S^0 \longrightarrow S^{L(H)}$ is null homotopic as an H-map.

Lemma 1.4. If WH is infinite, then $\chi(G/H) = 0$ and $\chi((G/H)^K) = 0$ for all K. If WH is finite and G/H embeds in V, then the degree of $f^K : S^{V^K} \longrightarrow S^{V^K}$ is the cardinality of the finite set $(G/H)^K$ for each K such that WK is finite.

This gains force from the next few results, which show how to compute $\chi(X)$ in terms of the $\chi(G/H)$ for any strongly dualizable X.

LEMMA 1.5. Let X and Y be strongly dualizable G-spectra.

(i) $\chi(X) = \chi(Y)$ if X is G-equivalent to Y.
(ii) $\chi(*)$ is the trivial map and $\chi(S)$ is the identity map.
(iii) $\chi(X \vee Y) = \chi(X) + \chi(Y)$ and $\chi(X \wedge Y) = \chi(X)\chi(Y)$.
(iv) $\chi(\Sigma^n X) = (-1)^n \chi(X)$.

A direct cofibration sequence argument from the definition of $\chi(X)$ gives the following much more substantial additivity relation.

THEOREM 1.6. For a G-map $f : X \longrightarrow Y$, $\chi(Cf) = \chi(Y) - \chi(X)$.

By induction on the number of cells, this gives the promised calculation of $\chi(X)$ in terms of the $\chi(G/H)$.

THEOREM 1.7. Let X be a finite G-CW spectrum, and let $\nu(H, n)$ be the number of n-cells of orbit type G/H in X. Then

$$\chi(X) = \sum_n \sum_{(H)} (-1)^n \nu(H, n) \chi(G/H).$$

Taking G to be the trivial group, we see from this formula that the Euler characteristic defined by (1.1) specializes to the classical nonequivariant Euler characteristic. The formula is written in terms of a chosen cell decomposition. On the space level, there is a canonical formula for $\chi(X)$ for any compact G-ENR X, namely

$$\chi(X) = \sum_{(H)} \chi(X_{(H)}/G)\chi(G/H). \tag{1.8}$$

Here $X_{(H)} = \{x | (G_x) = (H)\}$ and $\chi(X_{(H)}/G)$ is the sum of the "internal Euler characteristics" $\chi(M) = \chi(\bar{M}) - \chi(\partial M)$ of the path components M of $X_{(H)}$; $\bar{M}$ is the closure of M in X/G and $\partial M = \bar{M} - M$.

Define a homomorphism $d_H : \pi_0^G(S) \longrightarrow \mathbb{Z}$ by letting

$$d_H(x) = deg(f^H), \text{ where } f : S^V \longrightarrow S^V \text{ represents } x. \tag{1.9}$$

In view of XVI.6.2, $\Phi^H S$ is a nonequivariant sphere spectrum, and we can write this more conceptually as

$$d_H(x) = deg(\Phi^H(x)). \tag{1.10}$$

For a compact G-ENR X, we can deduce from (1.10) and standard properties of nonequivariant Euler characteristics that

$$d_H(\chi(X)) = \chi(X^H). \tag{1.11}$$

Similarly, for a finite G-CW spectrum X, we can deduce that

$$d_H(\chi(X)) = \chi(\Phi^H X). \tag{1.12}$$

Note that nothing like this can be true for the genuine fixed points of G-spectra: X^H is virtually never a finite CW-spectrum.

Formula (1.11) shows how the equivariant Euler characteristics of compact G-ENR's determine the nonequivariant Euler characteristics of their fixed point spaces. Conversely, by the following obstruction theoretic observation, the equivariant Euler characteristic is determined by nonequivariant Euler characteristics on fixed point spaces.

Proposition 1.13. Let V be a complex representation of G and let f and f' be G-maps $S^V \longrightarrow S^V$. Then $f \simeq f'$ if and only if $deg(f^H) = deg(f'^H)$ for all H such that WH is finite. Therefore, for compact G-ENR's X and Y, $\chi(X) = \chi(Y)$ if and only if $\chi(X^H) = \chi(Y^H)$ for all such H.

The integers $\chi(X^H)$ as H varies are restricted by congruences. For example, for a finite p-group, we saw in our study of Smith theory that $\chi(X^G) \equiv \chi(X) \bmod p$. More general congruences can be derived by use of the Bott isomorphism in equivariant K-theory.

Proposition 1.14. Let V be a complex representation of G and let f be a G-map $S^V \longrightarrow S^V$. If WH is finite, then

$$\sum [NH : NH \cap NK]\mu(K/H) deg(f^K) \equiv 0 \bmod |WH|,$$

where the sum runs over the H-conjugacy classes of groups K such that $H \subset K \subset NH$ and K/H is cyclic and where $\mu(K/H)$ is the number of generators of K/H. Therefore, for a compact G-ENR X,

$$\sum [NH : NH \cap NK]\mu(K/H)\chi(X^K) \equiv 0 \bmod |WH|.$$

Observe that this is really a result about the WH-maps f^H and is thus a result about finite group actions.

[tD, 5.1–5.4]
[LMS, III§§7-8 and V§1]

2. The Burnside ring $A(G)$ and the zero stem $\pi_0^G(S)$

For a finite group G, the Burnside ring $A(G)$ is the Grothendieck ring associated to the set of isomorphism classes of finite G-sets, with sum and product given by the disjoint union and Cartesian product of G-sets. There are ring homomorphisms $\phi_H : A(G) \longrightarrow \mathbb{Z}$ that send a finite G-set S to the cardinality of S^H. The product over conjugacy classes (H) gives a monomorphism $\phi : A(G) \longrightarrow C(G)$, where $C(G)$ is the product of a copy of $\mathbb{Z}$ for each (H). The image of ϕ is precisely the subring of tuples (n_H) of integers that satisfy the congruences

$$\sum [NH : NH \cap NK]\mu(K/H)n_K \equiv 0 \bmod |WH|.$$

It is an insight of Segal that $A(G)$ is isomorphic to $\pi_0^G(S)$.

The generalization of this insight to compact Lie groups is due to tom Dieck. We define $A(G)$ to be the set of equivalence classes of compact G-ENR's under the equivalence relation $X \approx Y$ if $\chi(X) = \chi(Y)$ in $\pi_0^G(S)$. Disjoint union and Cartesian product give a sum and product that make $A(G)$ into a ring; Cartesian product with a compact ENR K with trivial action and $\chi(K) = -1$ gives additive inverses. We can define $A(G)$ equally well in terms of finite G-CW complexes or finite G-CW spectra. However defined, the results of the previous section imply that, additively, $A(G)$ is the free Abelian group with a basis element $[G/H]$ for each conjugacy class (H) such that WH is finite. It is immediate that taking Euler characteristics specifies a monomorphism of rings $\chi : A(G) \longrightarrow \pi_0^G(S)$. We define

$$\phi_H = d_H \circ \chi : A(G) \longrightarrow \mathbb{Z}.$$

Then, by (1.11), $\phi_H([X]) = \chi(X^H)$ for a compact G-ENR X.

To define the appropriate version of $C(G)$ for compact Lie groups G we need a little topological algebra. We let $\mathscr{C}G$ be the set of closed subgroups of G and $\mathscr{F}G$ be the subset of those H such that WH is finite. Let ΓG and ΦG be the sets of conjugacy classes of subgroups in $\mathscr{C}G$ and $\mathscr{F}G$, respectively. The set ΓG is countable. The set ΦG is finite if and only if WT acts trivially on the maximal torus T. The set of orders of the finite groups $|WG/W_0G|$ has a finite bound.

There is a Hausdorff metric on $\mathscr{C}G$ that measures the distance between subgroups, and $\mathscr{F}G$ is a closed subspace of $\mathscr{C}G$. The conjugation action of G is continuous. With the orbit space topology, ΓG and ΦG are totally disconnected compact metric spaces. Recall that "totally disconnected" means that every singleton set $\{x\}$ is a component: the non-empty connected subspaces are points. It follows that ΦG has a neighborhood basis consisting of open and closed subsets S. Such a set is specified by a characteristic map $\zeta : \Phi G \longrightarrow S^0$ that send points in S to 1 and points not in S to -1. The proofs of many statements about $A(G)$ combine use of characteristic functions with compactness arguments.

Give $\mathbb{Z}$ the discrete topology and define $C(G)$ to be the ring of continuous (= locally constant) functions $\Phi G \longrightarrow \mathbb{Z}$. Since ΦG is compact, such a function takes finitely many values. The degree function $d(f) : \Phi G \longrightarrow \mathbb{Z}$ specified by $d(f)(H) = deg(f^H)$ for a G-map $f : S^V \longrightarrow S^V$ is continuous, hence there results a ring homomorphism $d : \pi_0^G(S) \longrightarrow C(G)$, and we define $\phi = d\chi : A(G) \longrightarrow C(G)$. Thus we have the following commutative diagram of rings:

$$\begin{array}{ccc} A(G) & \xrightarrow{\ \chi\ } & \pi_0^G(S) \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle d} & \\ & C(G). & \end{array}$$

THEOREM 2.1. The homomorphism χ is an isomorphism. The homomorphisms ϕ and d are monomorphisms. For $H \in \Phi G$, there is a unique element $\gamma_H \in C(G)$ such that $|WH|\gamma_H = \phi([G/H])$, and $C(G)$ is the free Abelian group generated by these elements γ_H. A map $\nu : \Phi G \longrightarrow \mathbb{Z}$ is in the image of ϕ if and only if, for each $H \in \Phi G$,

$$\sum [NH : NH \cap NK]\mu(K/H)\nu_K \equiv 0 \mod \ |WH|.$$

Moreover, there is an integer q such that $q(C(G)/A(G)) = 0$, and $q = |G|$ if G is finite.

The index of summation is that specified in Proposition 1.14, which shows that only maps ν that satisfy the congruences can be in the image of ϕ. We know by Proposition 1.13 that d and therefore ϕ is a monomorphism. It is not hard to prove the rest by inductive integrality arguments starting from rational linear combinations, provided that one knows a priori that the rationalization of ϕ is an isomorphism; we shall say something about why this is true shortly.

[tD, 5.5-5.6]
[LMS, V§2]

3. Prime ideals of the Burnside ring

Calculational understanding of the equivariant stable category depends on understanding of the algebraic properties of $A(G)$. For example, suppose given an idempotent $e \in A(G)$. Then $eA(G)$ is the localization of $A(G)$ at the ideal generated by e. For a G-spectrum X, define eX to be the telescope of iterates of

$e : X \longrightarrow X$. Then

$$\underline{\pi}_*(eX) = e\underline{\pi}_*(X).$$

Visibly, the canonical map $X \longrightarrow eX \vee (1-e)X$ induces an isomorphism of homotopy groups and is thus an equivalence. Therefore splittings of $A(G)$ in terms of sums of orthogonal idempotents determine splittings of the entire stable category $\bar{h}G\mathscr{S}$.

The first thing to say about $A(G)$ is that it is Noetherian if and only if the set ΦG is finite. For this reason, $A(G)$ is a much less familiar kind of ring for general compact Lie groups than it is for finite groups.

To understand the structure of any commutative ring A, one must understand its spectrum Spec(A) of prime ideals. In the case of $A(G)$, it is clear that every prime ideal pulls back from a prime ideal of $C(G)$. We define

$$q(H,p) = \{\alpha | \phi_H(\alpha) \equiv 0 \bmod p\}, \tag{3.1}$$

where p is a prime or $p = 0$. Although these are defined for all H, they are redundant when WH is infinite. There are further redundancies. We shall be precise about this since the basic sources — [tD] and [LMS] — require supplementation from a later note by Bauer and myself. The only proper inclusions of prime ideals are of the form $q(H,0) \subset q(H,p)$, hence $A(G)$ has Krull dimension one. For a given prime ideal q, we wish to describe $\{H | q = q(H,p)\}$. This is easy if $p = 0$.

Proposition 3.2. Let $q = q(H,0)$ for a subgroup H of G.

(i) If $H \lhd J$ and J/T is a torus, then $q = q(J,0)$.
(ii) There is a unique conjugacy class (K) in ΦG such that $q = q(K,0)$; up to conjugation, $H \lhd K$ and K/H is a torus.
(iii) If $H \in \Phi G$ and $J \in \Phi G$, then $q(H,0) = q(J,0)$ if and only if $(H) = (J)$.

Fix a prime p. We say that a group G is "p-perfect" if it has no non-trivial quotient p-groups. For $H \subset G$, let H'_p be the maximal p-perfect subgroup of H; explicitly, H'_p is the inverse image in H of the maximal p-perfect subgroup of the finite group H/H_0. Then define $H_p \subset NH'_p$ to be the inverse image of a maximal torus in WH'_p; H_p is again p-perfect, but now WH_p is finite. This last fact is crucial; it will lead to some interesting new results further on.

Theorem 3.3. Let $q = q(H,p)$ for a subgroup H of G and a prime p.

(i) If $H \lhd J$ and J/T is an extension of a torus by a finite p-group, then $q = q(J,p)$; if $H \in \Phi G$ and $|WH| \equiv 0 \bmod p$, then there exists $J \in \Phi G$ such that $H \lhd J$ and J/H is a finite p-group.

(ii) There is a unique conjugacy class (K) in ΦG such that $q = q(K,p)$ and $|WK|$ is prime to p; if $H \in \Phi G$ and H is p-perfect, then, up to conjugation, $H \lhd K$ and K/H is a finite p-group.

(iii) $K_p = K'_p$, and K_p is the unique normal p-perfect subgroup of K whose quotient is a finite p-group.

(iv) K_p is maximal in $\{J | q(J,p) = q \text{ and } J \text{ is } p-\text{perfect}\}$, and this property characterizes K_p up to conjugacy.

(v) $(H_p) = (K_p)$, hence $q(H,p) = q(J,p)$ if and only if $(H_p) = (J_p)$.

(vi) If $H \subset K_p$ and H is p-perfect, then $HT = K_p$, where T is the identity component of the center of K_p.

It is natural to let H^p denote the subgroup K of part (ii). If G is finite, we conclude that $q(J,p) = q$ if and only if $(H_p) \leq (J) \leq (H^p)$. For general compact Lie groups, the situation is more complicated and the following seemingly innocuous, but non-trivial, corollary of the theorem was left as an open question in [LMS].

Corollary 3.4. If $H \subset J \subset K$ and $q(H,p) = q(K,p)$, then $q(J,p) = q(K,p)$.

S. Bauer and J. P. May. Maximal ideals in the Burnside ring of a compact Lie group. Proc. Amer. Math. Soc. 102(1988), 684-686.
[tD, 5.7]
[LMS, V§3]

4. Idempotent elements of the Burnside ring

One reason that understanding the prime ideal spectrum of a commutative ring A is so important is the close relationship that it bears to idempotents. A decomposition of the identity element of A as a sum of othogonal idempotents determines and is determined by a partition of $\mathrm{Spec}(A)$ as a disjoint union of non-empty open subsets. In particular, $\mathrm{Spec}(A)$ is connected if and only if 0 and 1 are the only idempotents of A. This motivates us to compute the set $\pi\,\mathrm{Spec}(A(G))$ of components of $A(G)$; we topologize this set as a quotient space of $\mathrm{Spec}(A(G))$. However, there is a key subtlety here that was missed in [LMS]: while the components of any space are closed, they need not be open (unless the space is locally connected). In particular, since $\pi\,\mathrm{Spec}(A(G))$ is not discrete, the components of $\mathrm{Spec}(A(G))$ need not be open, and they therefore do not determine idempotents in general.

A compact Lie group G is perfect if it is equal to the closure of its commutator subgroup. It is solvable if it is an extension of a torus by a finite solvable group. Let $\mathscr{P}G$ denote the subspace of $\mathscr{C}G$ consisting of the perfect subgroups and let

ΠG be its orbit space of conjugacy classes; ΠG is countable, but it is usually not finite unless G is finite.

Any compact Lie group G has a minimal normal subgroup G_a such that G/G_a is solvable, and G_a is perfect. Passage from G to G_a is a continuous function $\mathscr{C}G \longrightarrow \mathscr{C}G$, $\mathscr{P}G$ is a closed subspace of $\mathscr{C}G$, and ΠG is a closed subspace of ΓG and is thus a totally disconnected compact metric space. There is a finite normal sequence connecting G_a to G each of whose subquotients is either a torus or a cyclic group of prime order. Via the results above, this implies that, for a given H, all prime ideals $q(H,p)$ are in the same component of $\mathrm{Spec}(A(G))$ as H_a. This leads to the following result.

PROPOSITION 4.1. Define $\beta : \Pi G \longrightarrow \pi\,\mathrm{Spec}(A(G))$ by letting $\beta(L)$ be the component that contains $q(L,0)$. Then β is a homeomorphism.

In particular, G is solvable if and only if $A(G)$ contains no non-trivial idempotents. For example, the Feit-Thompson theorem that an odd order finite group G is solvable is equivalent to the statement that $A(G)$ has no non-trivial idempotents. (Several people have tried to use this fact as the starting point of a topological proof of the Feit-Thompson theorem, but without success.)

A key point in the proof, and in the proofs of the rest of the results of this section, is that, for a subring R of $\mathbb{Q}$, the function

$$q : \Phi G \times \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(A(G) \otimes R)$$

is a continuous closed surjection. This is deduced from the fact that

$$q : \Phi G \times \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(C(G) \otimes R)$$

is a homeomorphism. In turn, the latter holds by an argument that depends solely on the fact that ΦG is a totally disconnected compact Hausdorff space.

If L is a perfect subgroup of G that is not a limit of perfect subgroups, then the component of $\beta(L)$ in $\mathrm{Spec}(A(G)$ is open and L determines an idempotent e_L in $A(G)$. Even when G is finite, it is non-trivial to write e_L in the standard basis $\{[G/H] | (H) \in \Phi G\}$, and such a formula has not yet been worked out for general compact Lie groups. Nevertheless one can prove the following theorem. Observe that the trivial subgroup of G is perfect; we here denote it by 1.

THEOREM 4.2. Let L be a perfect subgroup of G that is not a limit of perfect subgroups. Then there is an idempotent $e_L = e_L^G$ in $A(G)$ that is characterized by

$$\phi_H(e_L) = 1 \text{ if } (H_a) = (L) \quad \text{and} \quad \phi_H(e_L) = 0 \text{ if } (H_a) \neq (L).$$

Restriction from G to NL and passage to L-fixed points induce ring isomorphisms

$$e_L^G A(G) \longrightarrow e_L^{NL} A(NL) \longrightarrow e_1^{WL} A(WL).$$

[tD, 5.11]
[LMS, V§4]

5. Localizations of the Burnside ring

Let $A(G)_p$ denote the localization of $A(G)$ at a prime p and let $A(G)_0$ denote the rationalization of $A(G)$. We shall describe these localizations and the localizations of $A(G)$ at its prime ideals $q(H,p)$. We shall also explain the analysis of idempotents in $A(G)_p$, which is parallel to the analysis of idempotents in $A(G)$ just given but, in the full generality of compact Lie groups, is less well understood.

We begin with $A(G)_0$. Let $\mathbb{Z}_H$ denote $\mathbb{Z}$ regarded as an $A(G)$-module via $\phi_H : A(G) \longrightarrow \mathbb{Z}$.

PROPOSITION 5.1. Let $(H) \in \Phi G$.

(i) The localization of $A(G)$ at $q(H,0)$ is the canonical homomorphism

$$A(G) \longrightarrow (A(G)/q(H,0))_0 \cong \mathbb{Q}.$$

(ii) $\phi_H : A(G) \longrightarrow \mathbb{Z}_H$ induces an isomorphism of localizations at $q(H,0)$.
(iii) $\phi : A(G) \longrightarrow C(G)$ induces an isomorphism of rationalizations.

COROLLARY 5.2. Rationalization $A(G) \longrightarrow A(G)_0 \cong C(G)_0$ is the inclusion of $A(G)$ in its total quotient ring, and $\phi : A(G) \longrightarrow C(G)$ is the inclusion of $A(G)$ in its integral closure in $C(G)_0$.

Here (i) makes essential use of the compactness of ΦG, and (i) implies (ii). To prove (iii) — which we needed to prove Theorem 2.1 — we can now exploit the fact that a map of rings is an isomorphism if it induces a homeomorphism on passage to Spec and an isomorphism upon localization at corresponding prime ideals. If G is finite, then $A(G)_0$ is just a finite product of copies of $\mathbb{Q}$. For general compact Lie groups G, $A(G)_0$ is a type of ring unfamiliar to topologists but familiar in other branches of mathematics under the name of an "absolutely flat" or "von Neumann regular" ring. One characterization of such a commutative ring is that all of its modules are flat; another, obviously satisfied by $A(G)_0$, is that the localization of A at any maximal ideal P is A/P. For any such ring A, Spec(A) is a totally disconnected compact Hausdorff space, and an ideal is finitely generated if and only if it is generated by a single idempotent element.

PROPOSITION 5.3. Let p be a prime and let $(H) \in \Phi G$.

(i) The localization of $A(G)$ at $q(H,p)$ is the canonical homomorphism

$$A(G) \longrightarrow (A(G)/I(H,p))_p;$$

here $I(H,p) = \cap q(J,0)$, where the intersection runs over

$$\Phi(G;H,p) \equiv \{(J) | (J) \in \Phi G \text{ and } q(J,p) = q(H,p)\}.$$

(ii) The ring homomorphism

$$\prod \phi_J : A(G) \longrightarrow \prod \mathbb{Z}_J$$

is a monomorphism, where the product runs over $(J) \in \Phi(G;H,p)$.

The following statement only appears in the literature for finite groups. The general case relies on the full strength of Theorem 3.3, and the line of proof is the same as that of Theorem 3.6. The essential point is the analog of Proposition 3.5, and the essential point for this is the following assertion, which is trivially true for finite groups but has not yet been investigated for general compact Lie groups.

CONJECTURE 5.4. The function $\mathscr{C}G \longrightarrow \mathscr{C}G$ that sends H to H_p is continuous.

THEOREM 5.5. Let L be a p-perfect subgroup of G that is maximal in the set of p-perfect subgroups H such that $q(H,p) = q(L,p)$ and is not a limit of such p-perfect subgroups. If Conjecture 3.10 holds, then there is an idempotent $e_L = e_L^G$ in $A(G)_p$ that is characterized by

$$\phi_H(e_L) = 1 \text{ if } (H_p) = (L) \text{ and } \phi_H(e_L) = 0 \text{ if } (H_p) \neq (L).$$

Restriction from G to NL and passage to L-fixed points induce ring isomorphisms

$$e_L^G A(G)_p \longrightarrow e_L^{NL} A(NL)_p \longrightarrow e_1^{WL} A(WL)_p.$$

Moreover, $e_L^G A(G)_p$ is isomorphic to the localization of $A(G)$ at $q(L,p)$. If G is finite, then

$$A(G)_p \cong \prod_{(L)} e_L^G A(G)_p.$$

Taking L to be any group in ΦG that is not a limit of groups in ΦL and taking H_0 to be H, we see that the statement is true when $p = 0$. Of course, in the general compact Lie case, $A(G)_p$ is no longer the product of the $e_L^G A(G)_p$. However, it seems possible that, by suitable arguments to handle limit groups L, $A(G)_p$ can be described sheaf theoretically in terms of these localizations. The point is that $A(G)_p$ has the unusual property that it is isomorphic to the ring of global sections of its structural sheaf over its maximal ideal spectrum. (Any commutative ring A is isomorphic to the ring of global sections of its structural sheaf over Spec(A).)

[tD, 7.8]
[LMS, V§5]

6. Localization of equivariant homology and cohomology

The results of the previous section imply algebraic decomposition and reduction theorems for the calculation of equivariant homology and cohomology theories. We shall go into some detail since, in the compact Lie case, the results of [LMS] require clarification. When G is finite, we shall obtain a natural reduction of the computation of homology and cohomology theories localized at a prime p to their calculation in terms of appropriate associated theories for subquotient p-groups of G. It is interesting that although the proof of this reduction makes heavy use of idempotents of $A(G)_p$, there is no reference to $A(G)$ in the description that one finally ends up with. We shall use this reduction in our proof of the generalized Segal conjecture.

Recall the geometric fixed point functors Φ^H from XVI§§3, 6. In view of (1.12), it should seem natural that this and not the genuine fixed point functor on G-spectra appears in the following results.

Theorem 6.1. Let L be a perfect subgroup of G that is not a limit of perfect subgroups. For G-spectra X and Y, there are natural isomorphisms

$$[X, e_L^G Y]_G \longrightarrow [X, e_L^{NL} Y]_{NL} \longrightarrow [\Phi^H X, e_1^{WL} \Phi^H Y]_{WL}.$$

We prefer to state the homological consequences in terms of G-spaces, but it applies just as well to Φ-fixed points of G-spectra.

Corollary 6.2. Let E be a G-spectrum and X be a G-space. For $\alpha \in RO(G)$, let $\beta = r_{NL}^G \alpha \in RO(NL)$ and $\gamma = \beta^L \in RO(WL)$. Then there are natural isomorphisms

$$e_L^G E_\alpha^G(X) \longrightarrow e_L^{NL} E_\beta^{NL}(X) \longrightarrow e_1^{WL} E_\gamma^{WL}(XL)$$

and

$$e_L^G E_G^\alpha(X) \longrightarrow e_L^{NL} E_{NL}^\beta(X) \longrightarrow e_1^{WL} E_{WL}^\gamma(XL),$$

where E_*^{NL} and E_{NL}^* denote the theories that are represented by E regarded as an NL-spectrum and E_*^{WL} and E_{WL}^* denote the theories that are represented by $\Phi^L E$.

Write X_p for the localization of a G-spectrum at a prime p. It can be constructed as the telescope of countably many iterates of $p : X \longrightarrow X$, and its properties are as one would expect from the G-space level.

THEOREM 6.3. Let L be a p-perfect subgroup of G that is maximal in the set of p-perfect subgroups H of G such that $q(H,p) = q(L,p)$ and is not a limit of such p-perfect subgroups. If G is finite, or if Conjecture 3.10 holds, then, for G-spectra X and Y, there are natural isomorphisms

$$[X, e_L^G Y_p]_G \longrightarrow [X, e_L^{NL} Y_p]_{NL} \longrightarrow [\Phi^H X, e_1^{WL}\Phi^H Y_p]_{WL}.$$

When $p = 0$, the statement holds for $L \in \Phi G$ if L is not a limit of groups in ΦG.

Here $\Phi^H(Y_p) \simeq (\Phi^H Y)_p$. We again state the homological version only for G-spaces, although it also applies to G-spectra and Φ-fixed points. There is a further isomorphism here that does not come from Theorem 4.3. We shall discuss it after stating the corollary.

COROLLARY 6.4. Let E be a G-spectrum and X be a G-space. With L as in Theorem 4.3, let VL be a p-Sylow subgroup of the finite group WL. For $\alpha \in RO(G)$, let $\beta = r_{NL}^G\alpha \in RO(NL)$, $\gamma = \beta^L \in RO(WL)$, and $\delta = r_{VL}^{WL}\gamma \in RO(VL)$. Then there are natural isomorphisms

$$e_L^G E_\alpha^G(X)_p \longrightarrow e_L^{NL} E_\beta^{NL}(X)_p \longrightarrow e_1^{WL} E_\gamma^{WL}(XL)_p \longrightarrow E_\delta^{VL}(X^L)_p^{inv}$$

and, assuming that X is a finite G-CW complex,

$$e_L^G E^\alpha_G(X)_p \longrightarrow e_L^{NL} E^\beta_{NL}(X)_p \longrightarrow e_1^{WL} E^\gamma_{WL}(XL)_p \longrightarrow E^\delta_{VL}(X^L)_p^{inv},$$

where E_*^{NL} and E^*_{NL} denote the theories represented by E regarded as an NL-spectrum, E_*^{WL} and E^*_{WL} denote the theories represented by $\Phi^L E$, and E_*^{VL} and E^*_{VL} denote the theories represented by $\Phi^L E$ regarded as a VL-spectrum. Therefore, if G is finite, then

$$E_*^G(X)_p \cong \prod_{(L)} E_*^{VL}(X^L)_p^{inv}$$

and, if X is a finite G-CW complex,

$$E^*_G(X)_p \cong \prod_{(L)} E^*_{VL}(X^L)_p^{inv}.$$

When $p = 0$, the statement holds with VL taken as the trivial group.

The ideas in XIII§1 are needed to be precise about the grading. Of course, there is no problem of interpretation for the $\mathbb{Z}$-graded part of the theories. For finite groups, this gives the promised calculation of the localization of equivariant homology and cohomology theories at p in terms of homology and cohomology theories that are associated to subquotient p-groups; in the case of rationalization, a better result will be described later. For general compact Lie groups, such a calculation may follow from the fact that one can reconstruct any module over

$A(G)_p$ as the module of global sections of its structural sheaf over the maximal ideal spectrum of $A(G)_p$. Intuitively, the idea is that the space of maximal ideals should carry the relevant Lie group theory; theories associated to subquotient p-groups should carry the algebraic topology.

We must still explain the "inv" notation and the final isomorphisms that appear in the corollary. These come from a typical application of the general concept of induction in the context of Mackey functors. We shall say more about this later, but we prefer to explain the idea without formalism here.

Let G be a finite group with p-Sylow subgroup K. We are thinking of WL and VL. For G-spectra X and Y, we define $([X,Y]_p^K)^{inv}$ to be the equalizer (= difference kernel) of the maps

$$[G/K_+ \wedge X, Y]_p^G \longrightarrow [G/K_+ \wedge G/K_+ \wedge X, Y]_p^G$$

induced by the two projections $G/K_+ \wedge G/K_+ \longrightarrow G/K_+$. Here we are using the notational convention

$$[X,Y]^G = [X,Y]_G.$$

For a G-spectrum E, we define $E_*^K(X)_p^{inv}$ by replacing X by sphere spectra and replacing Y by $E \wedge X$. We define $E_K^*(X)_p^{inv}$ by replacing X by its smash product with sphere spectra and replacing Y by E. The final isomorphisms of Corollary 3.4 are special cases of the following result; there we must restrict to finite X in cohomology because it is only for finite X that localized spectra represent algebraic localizations of cohomology groups.

PROPOSITION 6.5. If G is a finite group with p-Sylow subgroup K, then, for any G-spectra X and Y, the projection $G/K_+ \wedge X \longrightarrow X$ induces an isomorphism

$$[X,Y]_p^G \longrightarrow ([X,Y]_p^K)^{inv}.$$

Actually, the relevant induction argument works to prove more generally that the analogous map

$$[X,Y]_{q(K,p)}^G \longrightarrow ([X,Y]_{q(K,p)}^K)^{inv}$$

is an isomorphism, where G is a compact Lie group and $(K) \in \Phi G$. The idea is that we have a complex

$$0 \longrightarrow [X,Y]^G \xrightarrow{d^0} [G/K_+ \wedge X, Y]^G \xrightarrow{d^1} [G/K_+ \wedge G/K_+ \wedge X, Y]^G \xrightarrow{d^2} \cdots,$$

where d^n is the alternating sum of the evident projection maps. When localized at $q(K,p)$, this complex acquires the contracting homotopy that is specified by $s^n = [G/K]^{-1}\tau^*$. Here, for any X, τ means

$$\tau \wedge \mathrm{Id} : X \cong S \wedge X \longrightarrow (G/K)_+ \wedge X,$$

where $\tau : S \longrightarrow (G/K)_+$ is the transfer map discussed in Section 1. It is immediate from (1.3) that the composite of τ and the projection $\xi : G/K_+ \longrightarrow S$ is the Euler characteristic $\chi(G/K) : S \longrightarrow S$. This implies that $\tau^*\xi^*$ is multiplication by $[G/K]$. The essential point is that $[G/K]$ becomes a unit in $A(G)_{q(K,p)}$. In the context of the proposition, the localization of $[X,Y]^K$ at $q(K,p)$ is the same as its localization at p.

[tD, Ch 7]
[LMS, V§6]

CHAPTER XVIII

Transfer maps in equivariant bundle theory

The basic reference is [LMS]; specific citations are given at the ends of sections.

1. The transfer and a dimension-shifting variant

Transfer maps provide one of the main calculational tools in equivariant stable homotopy theory. We have given a first definition in XVII§1. We shall here refer to the "transfer map" there as a pretransfer. It will provide the map of fibers for the transfer maps of bundles, in a sense that we now make precise. We place ourselves in the context of VII§1, where we considered equivariant bundle theory. Thus we assume given an extension of compact Lie groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Fix a complete Γ-universe U and note that U^Π is a complete G-universe. Let Y be a Π-free Γ-spectrum indexed on U^Π and let $B = Y/\Pi$. We are thinking of Y as $\Sigma^\infty X_+$ for a Π-free Γ-space X, but it changes nothing to work with spectra. In fact, this has some advantages. For example, relative bundles can be treated in terms of spectrum level cofibers, obviating complications that would arise if we restricted to spaces. Fix a compact Γ-ENR F. We could take F to be a spectrum as well, but we desist.

We have the orbit spectrum $E = Y \wedge_\Pi F_+$, which we think of as the total G-spectrum of a G-bundle with base G-spectrum B. Write $\pi : E \longrightarrow B$ for the map induced by the projection $F_+ \longrightarrow S^0$. Since F is a compact G-ENR, we have the stable pretransfer Γ-map $\tau(F) : S^0 \longrightarrow F_+$ of XVII§1; we have omitted notation for the suspension Γ-spectrum functor, and we shall continue to do so, but it is essential to remember that $\tau(F)$ is a map of genuine Γ-spectra indexed on U. As we discussed in XVI§5, Π-free Γ-spectra live in the Π-trivial Γ-universe U^{Π}. On maps, this gives that the inclusion $i : U^{\Pi} \longrightarrow U$ induces an isomorphism

$$i_* : [Y, Y \wedge F_+]_\Gamma \longrightarrow [i_*Y, i_*(Y \wedge F_+)]_\Gamma \cong [i_*Y, i_*Y \wedge F_+]_\Gamma.$$

DEFINITION 1.1. Let $\tilde{\tau} : Y \longrightarrow Y \wedge F_+$ be the Γ-map indexed on U^{Π} such that

$$i_*(\tilde{\tau}) = \mathrm{Id} \wedge \tau(F) : i_*Y \longrightarrow i_*Y \wedge F_+.$$

Define the transfer

$$\tau = \tau(\pi) : B = Y/\Pi \longrightarrow Y \wedge_\Pi F_+ = E$$

to be the map of G-spectra indexed on U^{Π} that is obtained from $\tilde{\tau}$ by passage to orbits over Π.

When $G = e$, this gives the nonequivariant transfer; specialization to this case results in no significant simplification. Note that there is no finiteness condition on the base spectrum B.

The definition admits many variants. When we describe its properties, we shall often use implicitly that it does not require a complete Γ-universe, only a universe into which F can be embedded, so that duality applies.

We can apply the same construction to maps other than $\tau(F)$. We illustrate this by constructing the map that gives the generalized Adams isomorphism of XVI.5.4. Since the construction is a little intricate and will not be used in the rest of the chapter, the reader may prefer to skip ahead. The cited Adams isomorphism is a natural equivalence of G/N-spectra

$$E/N \longrightarrow (\Sigma^{-Ad(N)} i_*E)^N,$$

where N is a normal subgroup of G and E is an N-free G-spectrum indexed on the fixed points of a complete G-universe. By adjunction, such a map is determined by a "dimension-shifting transfer G-map"

$$i_*(E/N) \longrightarrow \Sigma^{-Ad(N)} i_*E.$$

We proceed to construct this map.

CONSTRUCTION 1.2. Let N be a normal subgroup of G and write Π for N considered together with its conjugation action c by G. Let Γ be the semi-direct product $G \times_c \Pi$. We then have the quotient map $\varepsilon : \Gamma \longrightarrow G$. We also have a twisted quotient map $\theta : \Gamma \longrightarrow G$, $\theta(g, n) = gn$, that restricts to the identity

$\Pi \longrightarrow N$. Let X be an N-free G-space and let θ^*X denote X regarded as a Γ-space via θ; then θ^*X is Π-free. It is easy to check that we have G-homeomorphisms

$$X \cong \theta^*X \times_\Pi N \quad \text{and} \quad X/G \cong \theta^*X \times_\Pi pt.$$

This tells us how to view X as a Π-free Γ-space, placing us in the context of Definition 1.1. Here, however, we really need the spectrum level generalization. Let E be an N-free G-spectrum indexed on $(U^\Pi)^N$, where U is a complete Γ-universe. Let $i : (U^\Pi)^N \longrightarrow U^\Pi$ be the inclusion and let $Y = i_*\theta^*E$. Then Y is a Π-free Γ-spectrum indexed on U^Π, and there are natural isomorphisms of G-spectra

$$i_*E \cong Y \wedge_\Pi N_+ \quad \text{and} \quad i_*(E/N) \cong Y/\Pi.$$

The relevant "pretransfer" in the present context is a map

$$t : S \longrightarrow \Sigma^{-Ad(N)} N_+$$

of Γ-spectra indexed on U. The tangent bundle of $N = \Gamma/G$ is the trivial bundle $N \times Ad(N)$, where Γ acts on $Ad(N)$ by pullback along ε. Embed N in a Γ-representation V and let W be the resulting representation $V - Ad(N)$ of Γ. Embedding a normal tube and taking the Pontrjagin-Thom construction, we obtain a Γ-map

$$S^V \longrightarrow \Gamma_+ \wedge_G S^W \cong N_+ \wedge S^W.$$

We obtain the pretransfer t by applying the suspension spectrum functor and then desuspending by V. We are now in a position to apply the construction of Definition 1.1. Letting j denote the inclusion of U^Π in U to avoid confusion with i, observe that

$$j_*(Y \wedge \Sigma^{-Ad(N)} N_+) \cong j_*(\Sigma^{-Ad(N)}(Y \wedge N_+)).$$

Thus, smashing Y with t, pulling back to the universe U^Π, and passing to orbits over Π, we obtain the desired transfer map

$$i_*(E/N) \cong Y/\Pi \longrightarrow \Sigma^{-Ad(N)}(Y \wedge_\Pi N_+) \cong \Sigma^{-Ad(N)} i_*E.$$

[LMS, II§7 and IV§3]

2. Basic properties of transfer maps

Now return to the context of Definition 1.1. While we shall not go into detail, the transfer can be axiomatized by the basic properties that we list in the following omnibus theorem. They are all derived from corresponding statements about pretransfer maps. By far the most substantial of these properties is (v), which is proven by a fairly elaborate exercise in diagram chasing of cofiber sequences in the context of Spanier-Whitehead duality.

THEOREM 2.1. The transfer satisfies the following properties.

(i) *Naturality.* The transfer is natural with respect to maps $f : Y \longrightarrow Y'$ of Π-free Γ-spectra.

(ii) *Stability.* For a representation V of G regarded by pullback as a representation of Γ, $\Sigma^V\tau$ coincides with the transfer

$$\tau : \Sigma^V(Y/\Pi) \cong (\Sigma^V Y)/\Pi \longrightarrow (\Sigma^V Y) \wedge_\Pi F_+ \cong \Sigma^V(Y \wedge_\Pi F_+).$$

(iii) *Normalization.* With $F = pt$, the transfer associated to the identity map is the identity map.

(iv) *Fiber invariance.* The following diagram commutes for an equivalence $\phi : F \longrightarrow F'$ of compact Γ-ENR's:

$$\begin{array}{ccc} & Y/\Pi & \\ {}^{\tau}\swarrow & & \searrow^{\tau} \\ Y \wedge_\Pi F_+ & \xrightarrow[\mathrm{Id}\wedge\phi]{} & Y \wedge_\Pi F'_+. \end{array}$$

(v) *Additivity on fibers.* Let F be the pushout of a Γ-cofibration $F_0 \longrightarrow F_1$ and a Γ-map $F_0 \longrightarrow F_2$, where the F_k are compact Γ-ENR's. Let τ_k be the transfer associated to $Y \wedge_\Pi (F_k)_+ \longrightarrow Y/\Pi$ and let $j_k : Y \wedge_\Pi (F_k)_+ \longrightarrow Y \wedge_\Pi F_+$ be induced by the canonical map $F_k \longrightarrow F$. Then

$$\tau = j_1\tau_1 + j_2\tau_2 - j_0\tau_0.$$

(vi) *Change of groups.* Assume given an inclusion of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Theta & \longrightarrow & \Lambda & \longrightarrow & H & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Then the following diagram commutes for a Θ-free Λ-spectrum Y indexed on U^{Π} regarded as a Λ-universe:

$$\begin{array}{ccc} G \ltimes_H (Y/\Theta) & \xrightarrow{\mathrm{Id} \ltimes \tau} & G \ltimes_H (Y \wedge_\Theta F_+) \\ \cong \downarrow & & \downarrow \cong \\ (\Gamma \ltimes_\Lambda Y)/\Pi \xrightarrow[\tau]{} (\Gamma \ltimes_\Lambda Y) \wedge_\Pi F_+ & \xrightarrow[\cong]{} & (\Gamma \ltimes_\Lambda (Y \wedge F_+))/\Pi. \end{array}$$

Modulo a fair amount of extra bookkeeping to make sense of it, part (vi) remains true if we require only the homomorphism $H \longrightarrow G$ in our given map of extensions to be an inclusion. There is also a change of groups property that holds for a map of extensions in which $\Theta \longrightarrow \Pi$ is the identity but the other two maps are unrestricted. Such properties are useful and important, but we shall not go into more detail here. Rather, we single out a particular example of the kind of information that they imply. Let $H \subset G$ and consider the bundles

$$G/H \longrightarrow pt \quad \text{and} \quad BH = EG \times_H (G/H) \longrightarrow BG$$

and the collapse maps $\varepsilon : EG_+ \longrightarrow S^0$ and $\varepsilon : EH_+ \longrightarrow S^0$.

PROPOSITION 2.2. Let E be a split G-spectrum. Then the following diagram commutes:

$$\begin{array}{ccccc} E^*_H(S^0) & \xrightarrow{\varepsilon^*} & E^*_H(EH_+) & \xrightarrow{\cong} & E^*(BH_+) \\ \tau^* \downarrow & & \downarrow \tau^* & & \downarrow \tau^* \\ E^*_G(S^0) & \xrightarrow[\varepsilon^*]{} & E^*_G(EG_+) & \xrightarrow[\cong]{} & E^*(BG_+). \end{array}$$

Here E^* is the theory represented by the underlying nonequivariant spectrum of E. For example, if E represents complex equivariant K-theory, then the transfer map on the left is induction $R(H) \longrightarrow R(G)$ and the transfer map on the right is the nonequivariant one. The horizontal maps become isomorphisms upon completion at augmentation ideals, by the Atiyah-Segal completion theorem.

[LMS, IV§§3–4]

3. Smash products and Euler characteristics

The transfer commutes with smash products, and a special case of this implies a basic formula in terms of Euler characteristics for the evaluation of the composite

$\xi \circ \tau$ for a G-bundle ξ. The commutation with smash products takes several forms. For an external form, we assume given extensions,

$$1 \longrightarrow \Pi_i \longrightarrow \Gamma_i \longrightarrow G_i \longrightarrow 1$$

and complete Γ_i-universes U_i for $i = 1$ and $i = 2$.

THEOREM 3.1. The following diagram of $(G_1 \times G_2)$-spectra indexed on the universe $(U_1)^{\Pi_1} \oplus (U_2)^{\Pi_2}$ commutes for Π_i-free Γ_i-spectra Y_i and finite Γ_i-spaces F_i:

$$\begin{array}{ccc}
(Y_1/\Pi_1) \wedge (Y_2/\Pi_2) & \xrightarrow{\tau \wedge \tau} & (Y_1 \wedge_{\Pi_1} F_{1+}) \wedge (Y_2 \wedge_{\Pi_2} F_{2+}) \\
\cong \downarrow & & \downarrow \cong \\
(Y_1 \wedge Y_2)/(\Pi_1 \times \Pi_2) & \xrightarrow[\tau]{} & (Y_1 \wedge Y_2) \wedge_{\Pi_1 \times \Pi_2} (F_1 \times F_2)_+ .
\end{array}$$

When $G = G_1 = G_2$, we can use change of groups to internalize this result. Modulo a certain amount of detail to make sense of things, we see in this case that the diagram of the previous theorem can be interpreted as a commutative diagram of G-spectra. Either specializing this result or just inspecting definitions, we obtain the following useful observation. We revert to the notations of Definition 1.1, so that U is a Γ-complete universe.

COROLLARY 3.2. Let Y be a Π-free Γ-spectrum indexed on U^{Π}, F be a compact Γ-ENR, and E be a G-spectrum indexed on U^{Π}. Then the following diagram commutes:

$$\begin{array}{ccc}
(Y \wedge E)/\Pi & \xrightarrow{\tau} & (Y \wedge E) \wedge_{\Pi} F_+ \\
\cong \downarrow & & \downarrow \cong \\
(Y/\Pi) \wedge E & \xrightarrow[\tau \wedge \mathrm{id}]{} & (Y \wedge_{\Pi} F_+) \wedge E .
\end{array}$$

In the presence of suitable diagonal maps, this leads to homological formulas involving cup and cap products. While more general results are valid and useful, we shall restrict attention to the case of a given space-level bundle. Here the previous corollary and diagram chases give the following result.

COROLLARY 3.3. Let X be a Π-free Γ-space and F be a compact Γ-ENR. Then the following diagram commutes, where we have written Δ for various maps induced from the diagonal maps of X and F.

$$\begin{array}{ccccc}
(X/\Pi)_+ & \xrightarrow{\tau} & (X\times_\Pi F)_+ & \xleftarrow{\tau} & (X/\Pi)_+ \\
\downarrow{\scriptstyle\Delta} & & & & \downarrow{\scriptstyle\Delta} \\
(X/\Pi)_+\wedge(X/\Pi)_+ & & \downarrow{\scriptstyle\Delta} & & (X/\Pi)_+\wedge(X/\Pi)_+ \\
\downarrow{\scriptstyle\tau\wedge\mathrm{id}} & & & & \downarrow{\scriptstyle\mathrm{id}\wedge\tau} \\
(X\times_\Pi F)_+\wedge(X/\Pi)_+ & \xleftarrow[\mathrm{id}\wedge\xi]{} & (X\times_\Pi F)_+\wedge(X\times_\Pi F)_+ & \xrightarrow[\xi\wedge\mathrm{id}]{} & (X/\Pi)_+\wedge(X\times_\Pi F)_+.
\end{array}$$

Retaining the hypotheses of the corollary and constructing cup and cap products as in XIII§5, we easily deduce the following formulas relating the maps induced on homology and cohomology by the maps Δ, τ, and ξ displayed in its diagram.

Proposition 3.4. The following formulas hold, where E is a ring G-spectrum.

(i) $\tau^*(w)\cup y = \tau^*(w\cup\xi^*(y))$ for $w\in E^*_G(X\times_\Pi F)$ and $y\in E^*_G(Y/\Pi)$

(ii) $x\cup\tau^*(z) = \tau^*(\xi^*(x)\cup z)$ for $x\in E^*_G(X/\Pi)$ and $z\in E^*_G(Y\wedge_\Pi F_+)$

(iii) $y\cap\tau^*(x) = \xi_*(\tau_*(y)\cap w)$ for $y\in E^*_G(Y/\Pi)$ and $w\in E^*_G(X\times_\Pi F)$

(iv) $\tau_*(y)\cap\xi^*(x) = \tau_*(y\cap x)$ for $y\in E^*_G(Y/\Pi)$ and $x\in E^*_G(X/\Pi)$

Define the Euler characteristic of the bundle $\xi : X\times_\Pi F \longrightarrow X$ to be

$$\chi(\xi) = \tau^*(1)\in E^0_G(X/\Pi). \tag{3.5}$$

Taking $w = 1$ in the first equation above, we obtain the following conclusion.

Corollary 3.6. The composite

$$E^*_G(X/\Pi_+) \xrightarrow{\xi^*} E^*_G(X\times_\Pi F)_+ \xrightarrow{\tau^*} E^*_G(X/\Pi_+)$$

is multiplication by $\chi(\xi)$.

In many applications of the transfer, one wants to use this by proving that $\chi(\xi)$ is a unit and deducing that $E^*_G(X/\Pi_+)$ is a direct summand of $E^*_G(X\times_\Pi F)_+$. When $\chi(\xi)$ is or is not a unit is not thoroughly understood. The strategy for studying the problem is to relate $\chi(\xi)$ to the Euler characteristic

$$\chi(F) = \xi^*(\tau(F))\in\pi^0_\Gamma(S).$$

We need a bit of language in order to state the basic result along these lines.

If $X/\Pi = G/H$, then $X = \Gamma/\Lambda$ for some Λ such that $\Lambda\cap\Pi = e$. The composite $\Lambda\subset\Gamma\longrightarrow G$ maps Λ isomorphically onto H. Inverting this isomorphism, we obtain a homomorphism $\alpha : H \cong \Lambda \subset \Gamma$. For a general Π-free Γ-space X and an orbit $G/H \subset X/\Pi$, the pullback bundle over G/H gives rise to such a homomorphism $\alpha : H \longrightarrow \Gamma$, which we call the fiber representation of X at G/H. Write α^*F for F regarded as an H-space by pullback along α.

Theorem 3.7. Let X be a Π-free Γ-space and F be a Γ-space. Let $B = X/\Pi$ and consider the bundle $\xi : X \times_\Pi F \longrightarrow B$. For a ring G-spectrum E, the Euler characteristic $\chi(\xi) \in E^0_G(B_+)$ is a unit if any of the following conditions hold.

(i) $\chi(\alpha^*F) \in E^0_H(S)$ is a unit for each fiber representation $\alpha : H \longrightarrow \Gamma$ of X.
(ii) B is G-connected with basepoint $*$ and $\chi(\alpha^*F) \in E^0_G(S)$ is a unit, where $\alpha : G \longrightarrow F$ is the fiber representation of X at $*$.
(iii) B is G-free and the nonequivariant Euler characteristic $\chi(F) \in E^0_e(S)$ is a unit.

Nonequivariantly, with $G = e$, the connectivity hypothesis of (ii) is inconsequential, but it is a serious limitation in the equivariant case and one must in general fall back on (i). The following implication is frequently used.

Theorem 3.8. If G is a finite p-group and $\xi : Y \longrightarrow B$ is a finite G-cover whose fiber F has cardinality prime to p, then the composite map

$$\Sigma^\infty B_+ \xrightarrow{\tau} \Sigma^\infty Y_+ \xrightarrow{\xi} \Sigma^\infty B_+$$

become an equivalence upon localization at p.

[LMS, IV§5]

4. The double coset formula and its applications

This section summarizes results of Feshback that are generalized and given simpler proofs in [LMS]. Basically, they are consequences of the additivity on fibers of transfer maps. That result leads to decomposition theorems for the computation of the transfer associated to any stable bundle $\xi : Y \wedge_\Pi F_+ \longrightarrow Y/\Pi$, and we state these first. Since we must keep track of varying orbits, we write

$$\xi(\Lambda, \Gamma) : Y \wedge_\Pi (\Gamma/\Lambda)_+ \longrightarrow Y/\Pi$$

for the stable bundle associated to a Π-free Γ-spectrum Y and the Γ-space Γ/Λ, and we write $\tau(\Lambda, \Gamma)$ for the associated transfer map.

THEOREM 4.1. Let F be a finite Γ-CW complex and let

$$j_i : \Gamma/\Lambda_i \subset \Gamma/\Lambda_i \times D^{n_i} \longrightarrow F$$

be the composite of the inclusion of an orbit and the ith characteristic map for some enumeration of the cells of F. Then, for any Π-free Γ-spectrum Y,

$$\tau = \sum_i (-1)^{n_i} j_i \tau(\Lambda_i, \Gamma) : Y/\Pi \longrightarrow Y \wedge_\Pi F_+.$$

There is a more invariant decomposition that applies to a general compact Γ-ENR F. For $\Lambda \subset \Gamma$, we let $F^{(\Lambda)}$ be the subspace of points whose isotropy groups are conjugate to Λ. A path component M of the orbit space $F^{(\Lambda)}/\Gamma$ is called an orbit type component of F/Γ. If $\bar{M}$ is the closure of M in F/Γ and $\partial M = \bar{M} - M$, we defined the (nonequivariant) internal Euler characteristic $\chi(M)$ to be the reduced Euler characteristic of the based space $\bar{M}/\partial M$.

THEOREM 4.2. Let F be a compact Γ-ENR and let

$$j_M : \Gamma/\Lambda \subset M \subset F$$

be the inclusion of an orbit in the orbit type component M. Then, for any Π-free Γ-spectrum Y,

$$\tau = \sum_M \chi(M) j_M \tau(\Lambda, \Gamma) : Y/\Pi \longrightarrow Y \wedge_\Pi F_+.$$

While it is possible to deduce a double coset formula in something close to our full generality, we shall simplify the bookkeeping by restricting to the case when $\Gamma = G \times \Pi$, which is the case of greatest importance in the applications. Recall that a principal (G, Π)-bundle is the same thing as a Π-free $(G \times \Pi)$-space and let Y be a Π-free $(G \times \Pi)$-spectrum indexed on U^Π, where U is a complete $(G \times \Pi)$-universe. For a subgroup Λ of Π, we have the stable (G, Π)-bundle

$$\xi(\Lambda, \Pi) : Y/\Lambda \cong Y \wedge_\Pi (\Pi/\Lambda)_+ \longrightarrow Y/\Pi$$

with associated transfer map $\tau(\Lambda, \Pi)$.

THEOREM 4.3 (DOUBLE COSET FORMULA). Let Λ and Φ be subgroups of Π and let $\Lambda\backslash\Pi/\Phi$ be the double coset space regarded as the space of orbits under Λ of Π/Φ. Let $\{m\}$ be a set of representatives in Π for the orbit type component manifolds M of $\Lambda\backslash\Pi/\Phi$ and let $\chi(M)$ be the internal Euler characteristic of M in $\Lambda\backslash\Pi/\Phi$. Then, for any Π-free $(G\times\Pi)$-spectrum Y, the composite

$$Y/\Lambda \xrightarrow{\xi} Y/\Pi \xrightarrow{\tau} Y/\Phi$$

is the sum over M of $\chi(M)$ times the composite

$$Y/\Lambda \xrightarrow{\tau} Y/\Phi^m\cap\Lambda \xrightarrow{\xi} Y/\Phi^m \xrightarrow{c_m} Y/\Phi.$$

Here $\Phi^m = m\Phi m^{-1}$ and c_m is induced by the left Π-map $\Pi/\Phi^m \longrightarrow \Pi/\Phi$ given by right multiplication by m. In symbols,

$$\tau(\Phi,\Pi)\xi(\Lambda,\Pi) = \sum_M \chi(M)\, c_m \circ \xi(\Phi^m\cap\Lambda, \Phi^M)\circ\tau(\Phi^m\cap\Lambda,\Lambda).$$

PROOF. The composite

$$\Lambda/\Phi^m\cap\Lambda \xrightarrow{\subset} \Pi/\Phi^m \xrightarrow{c_m} \Pi/\Phi$$

is a homeomorphism onto the double coset $\Lambda\, m\,\Phi$. Modulo a little diagram chasing and the use of change of groups, the conclusion follows directly from the previous theorem applied to $\xi(\Lambda,\Pi)$. □

If Φ has finite index in Π, then M is the point $\Lambda\, m\,\Phi$ and $\chi(M)=1$. Here the formula is of the same form as the classical double coset formula in the cohomology of groups. Observe that the formula depends only on the structure of the fibers and has the same form equivariantly as in the nonequivariant case $G=e$ (which is the case originally proven by Feshback, at least over compact base spaces).

The theorem is most commonly used for the study of classifying spaces, with $Y=\Sigma^\infty E(G,\Pi)_+$. Here $E(G,\Pi)/\Phi$ is a classifying G-space for principal (G,Φ)-bundles and the result takes the following form.

COROLLARY 4.4. The composite

$$\Sigma^\infty B(G,\Lambda)_+ \xrightarrow{\xi} \Sigma^\infty B(G,\Pi)_+ \xrightarrow{\tau} \Sigma^\infty B(G,\Phi)_+$$

is the sum over M of $\chi(M)$ times the composite

$$\Sigma^\infty B(G,\Lambda)_+ \xrightarrow{\tau} \Sigma^\infty B(G,\Phi^m\cap\Lambda)_+ \xrightarrow{\xi} \Sigma^\infty B(G,\Phi^m)_+ \xrightarrow{c_m} \Sigma^\infty B(G,\Phi)_+.$$

Of course, the formula is very complicated in general. However, many terms simplify or disappear in special cases. For example, if the group $W\Phi = N\Phi/\Pi$ is infinite, then the transfer $\tau(\Phi, \Pi)$ is trivial. This observation and a little bookkeeping, lead to the following examples where the formula reduces to something manageable.

COROLLARY 4.5. Let Y be any Π-free $(G \times \Pi)$-spectrum.

(i) If N is the normalizer of a maximal torus T in Π, then

$$\tau(N, \Pi)\xi(T, \Pi) = \xi(T, N) : Y/T \longrightarrow Y/N.$$

(ii) If T is a maximal torus in Π, then

$$\tau(T, \Pi)\xi(T, \Pi) = \sum c_m : Y/T \longrightarrow Y/T,$$

where the sum ranges over a set of coset representatives for the Weyl group $W = WT$ of Π.

(iii) If Λ is normal and of finite index in Π, then

$$\tau(\Lambda, \Pi)\xi(\Lambda, \Pi) = \sum c_m : Y/\Lambda \longrightarrow Y/\Lambda,$$

where the sum runs over a set of coset representatives for Π/Λ.

Typically, the double coset formula is applied to the computation of $E^*_G(Y/\Pi)$ in terms of $E^*_G(Y/\Phi)$ for a subgroup Φ. Here it is used in combination with the Euler characteristic formula of Corollary 3.6 and the unit criteria of Theorem 3.7. We need a definition to state the conclusions.

DEFINITION 4.6. An element $x \in E^*_G(Y/\Phi)$ is said to be stable if

$$\xi(\Phi \cap \Phi^m, \Phi)^*(x) = \xi(\Phi \cap \Phi^m, \Phi^m)^* c^*_m(x)$$

for all $m \in \Pi$. Let $E^*_G(Y/\Phi)^S$ denote the set of stable elements and observe that $\operatorname{Im} \xi(\Phi, \Pi)^* \subset E^*_G(Y/\Phi)^S$ since $\xi(\Phi, \Pi) \circ c_m = \xi(\Phi^m, \Pi)$.

The double coset and Euler characteristic formulas have the following direct implication.

THEOREM 4.7. Let X be a Π-free $(G\times\Pi)$-space and let E be a ring G-spectrum. Let $\Phi \subset \Pi$ and consider $\xi = \xi(\Phi, \Pi)$. If $\chi(\xi) \in E^0_G(X/\Pi_+)$ is a unit, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/\Phi_+)^S$$

is an isomorphism.

Unfortunately, only the first criterion of Theorem 3.7 applies to equivariant classifying spaces, and more work needs to be done on this. However, we have the following application of its last two criteria, and the nonequivariant case $G = e$ gives considerable information about nonequivariant characteristic classes.

THEOREM 4.8. Let X be a Π-free $(G\times\Pi)$-space and let E be a ring G-spectrum. Assume further that X/Π is either G-connected with trivial fiber representation $G \longrightarrow \Pi$ at any fixed point or G-free.

(i) If N is the normalizer of a maximal torus in Π, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/N_+)^S$$

is an isomorphism.

(ii) If $N(p)$ is the inverse image in the normalizer of a maximal torus T of a p-Sylow subgroup of the Weyl group $W = WT$ and E is p-local, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/N(p)_+)^S$$

is an isomorphism.

(iii) If T is a maximal torus in Π and E is local away from the order of the Weyl group $W = WT$, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/T_+)^W$$

is an isomorphism.

(iv) If Φ is normal and of finite index in Π and E is local away from $|\Pi/\Phi|$, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/\Phi_+)^{\Pi/\Phi}$$

is an isomorphism.

It is essential here that we are looking at theories represented by local spectra and not at theories obtained by algebraically localizing theories represented by general spectra. The point is that if F is the localization of a spectrum E at a set of primes T, then $F^*_G(X)$ is usually not isomorphic to $E^*_G(X)\otimes \mathbb{Z}_T$ unless X is a finite G-CW complex. The proof of the unit criteria makes use of the wedge axiom, which is not satisfied by the algebraically localized theories.

M. Feshbach. The transfer and compact Lie groups. Trans. Amer. Math. Soc. 251(1979), 139-169.
[LMS, IV§6]

5. Transitivity of the transfer

While a transitivity relation can be formulated and proven in our original general context of extensions of compact Lie groups, we shall content ourselves with its statement in the classical context of products $G \times \Pi$. We suppose given compact Lie groups G, Π, and Φ and a complete $(G \times \Pi \times \Phi)$-universe U'. Then $U = (U')^{\Phi}$ is a complete $(G \times \Pi)$-universe and $U^{\Pi} = (U')^{\Pi \times \Phi}$ is a complete G-universe.

We shall consider transitivity for stable bundles that are built up from bundles of fibers. Let P be a Φ-free finite $(\Pi \times \Phi)$-CW complex with orbit space $K = P/\Phi$ and let J be any finite Φ-CW complex. Let $F = P \times_{\Phi} J$. The resulting bundle $\zeta : F \longrightarrow K$ is to be our bundle of fibers. Here F and K are finite Π-CW complexes and ζ is a (Π, Φ)-bundle with fiber J. By pullback, we may regard ζ as a $(G \times \Pi, \Phi)$-bundle. With these hypotheses, we have a transitivity relation for pretransfers that leads to a transitivity relation for stable G-bundles. It is proven by using additivity and naturality to reduce to the case when P is an orbit and then using a change of groups argument.

THEOREM 5.1. The following diagram of $(G \times \Pi \times \Phi)$-spectra commutes:

$$\begin{array}{ccc} & S & \\ {}^{\tau(K)}\swarrow & & \searrow^{\tau(F)} \\ \Sigma^{\infty} K_{+} & \xrightarrow[\tau(\zeta)]{} & \Sigma^{\infty} F_{+}. \end{array}$$

THEOREM 5.2. Let Y be a Π-free $(G \times \Pi)$-spectrum indexed on U^{Π}. Observe that the G-map $\mathrm{id} \wedge_{\Pi} \zeta : D \wedge_{\Pi} F_{+} \longrightarrow D \wedge_{\Pi} K_{+}$ is a stable $(G, \Pi \times \Phi)$-bundle with fiber J and consider the following commutative diagram of stable G-bundles:

$$\begin{array}{ccc} Y \wedge_{\Pi} F_{+} & \xrightarrow{\mathrm{id} \wedge_{\Pi} \zeta} & Y \wedge_{\Pi} K_{+} \\ {}_{\xi}\searrow & & \swarrow_{\xi'} \\ & Y/\Pi. & \end{array}$$

The following diagram of G-spectra commutes:

$$\begin{array}{ccc} & Y/\Pi & \\ {}^{\tau(\xi')}\swarrow & & \searrow^{\tau(\xi)} \\ Y \wedge_{\Pi} K_{+} & \xrightarrow[\tau(\mathrm{id} \wedge_{\Pi} \zeta)]{} & Y \wedge_{\Pi} F_{+}. \end{array}$$

The special case $P = \Pi$ is of particular interest. It gives transitivity for the diagram of transfers associated to the commutative diagram

$$\begin{array}{ccc} Y \wedge_{\Pi} (\Pi \times_{\Phi} J)_+ & \xrightarrow{\cong} & Y \wedge_{\Phi} J_+ \\ \downarrow & & \downarrow \\ Y/\Pi & \xleftarrow[\xi(\Phi,\Pi)]{} & Y/\Phi. \end{array}$$

[LMS, IV§7]

CHAPTER XIX

Stable homotopy and Mackey functors

1. The splitting of equivariant stable homotopy groups

One can reprove the isomorphism $A(G) \cong \pi_0^G(S)$ by means of the following important splitting theorem for the stable homotopy groups of G-spaces in terms of nonequivariant stable homotopy groups. When G is finite, we shall see that this result provides a bridge connecting the equivariant and non-equivariant versions of the Segal conjecture. Recall that $Ad(G)$ denotes the adjoint representation of G. Remember that our homology theories, including π_*, are understood to be reduced.

THEOREM 1.1. For based G-spaces Y, there is a natural isomorphism

$$\pi_*^G(Y) \cong \sum_{(H)\in\Gamma G} \pi_*(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H).$$

Observe that the sum ranges over all conjugacy classes, not just the conjugacy classes $(H) \in \Phi G$. However, WH is finite if and only if $Ad(WH) = 0$, and $EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H$ is connected if $Ad(WH) \neq 0$.

COROLLARY 1.2. For based G-spaces Y, there is a natural isomorphism

$$\pi_0^G(Y) \cong \sum_{(H)\in\Phi G} H_0(WH; \pi_0(Y^H)).$$

With $Y = S^0$, this is consistent with the statement that $A(G)$ is $\mathbb{Z}$-free on the basis $\{[G/H] \,|\, (H) \in \Phi G\}$. We shall come back to this point in the discussion of Mackey functors in Section 3. Theorem 1.1 implies a description of the G-fixed point spectra of equivariant suspension spectra.

THEOREM 1.3. For based G-spaces Y, there is a natural equivalence

$$(\Sigma^\infty Y)^G \simeq \bigvee_{(H)\in \Gamma G} \Sigma^\infty(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H).$$

Here the suspension spectrum functors are $\Sigma^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U$ on the left and $\Sigma^\infty : \mathscr{T} \longrightarrow \mathscr{S}U^G$ on the right, where U is a fixed complete G-universe. Actually, the most efficient proof seems to be to first write down an explicit map

$$\theta = \sum \theta_H : \sum \pi_*(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H) \longrightarrow \pi_*^G(Y)$$

of homology theories in Y and use it to prove Theorem 1.1 and then write down an explicit map

$$\xi = \sum \xi_H : \bigvee \Sigma^\infty(EWH_+ \wedge_W H\Sigma^{Ad(WH)} Y^H) \longrightarrow (\Sigma^\infty Y)^G$$

of spectra and prove by a diagram chase that the map induced on homotopy groups by the wedge summand ξ_H is the same as the map induced by the summand θ_H. We shall first write down these maps and then say a little about the proofs.

Since the definitions of our maps proceed one H at a time, we abbreviate notation by writing:

$$N = NH, \quad W = WH, \quad E = EWH, \quad \text{and} \quad A = Ad(WH).$$

We let L be the tangent N-representation at the identity coset of G/N. A Lie theoretic argument shows that $(G/N)^H$ is a single point, and this implies that $L^H = \{0\}$. Now θ_H is defined by the following commutative diagram:

$$\begin{array}{ccccc}
\pi_*(E_+ \wedge_W \Sigma^A Y^H) & \xrightarrow{\ \alpha\ } & \pi_*^W(E_+ \wedge Y^H) & \xrightarrow{\ \lambda\ } & \pi_*^N(\Sigma^L(E_+ \wedge Y)) \\
\Big\downarrow{\scriptstyle \theta_H} & & & & \Big\downarrow{\scriptstyle \omega} \\
\pi_*^G(Y) & \xleftarrow[(\rho\wedge \mathrm{Id})_*]{} & \pi_*^G((G\times_N E)_+ \wedge Y)) & \xleftarrow[\zeta_*]{} & \pi_*^G(G_+ \wedge_N (E_+ \wedge Y)).
\end{array}$$

Here α is an instance of the Adams isomorphism of XVI.5.4, ω is an instance of the Wirthmüller isomorphism of XVI.4.9, ζ_* is induced by a canonical isomorphism of G-spectra, $\rho : (G \times_N E)_+ \longrightarrow S^0$ is the collapse map, and λ is the composite of the map $\pi_*^W \longrightarrow \pi_*^N$ obtained by regarding W-maps as H-fixed N-maps and the map induced by the inclusion of fixed point spaces

$$E_+ \wedge Y^H = (\Sigma^L(E + \wedge Y))^H \longrightarrow \Sigma^L(E_+ \wedge Y).$$

Why is the sum θ of the θ_H an isomorphismΓ Clearly θ is a map of homology theories in Y. Recall the spaces $E(\mathscr{F}', \mathscr{F})$ defined in V.4.6 for inclusions of families

$\mathscr{F} \subset \mathscr{F}'$. For a homology theory E_* on G-spaces (or G-spectra), we define the associated homology theory concentrated between $\mathscr{F}$ and $\mathscr{F}'$ by

$$E[\mathscr{F}', \mathscr{F}]_*(X) = E_*(X \wedge E(\mathscr{F}', \mathscr{F})).$$

We say that $(\mathscr{F}', \mathscr{F})$ is an adjacent pair if $\mathscr{F}' - \mathscr{F}$ consists of a single conjugacy class of subgroups. One can check, using an easy transfinite induction argument in the compact Lie case, that a map of homology theories is an isomorphism if the associated maps of homology theories concentrated between adjacent families are all isomorphisms.

Returning to θ, consider an adjacent pair of families with $\mathscr{F}' - \mathscr{F} = (H)$. We find easily that $EWJ_+ \wedge E(\mathscr{F}', \mathscr{F})$ is WJ-contractible unless $(H) = (J)$. Therefore, when we concentrate our theories between $\mathscr{F}$ and $\mathscr{F}'$, all of the summands of the domain of θ vanish except the domain of θ_H. It remains to prove that θ_H is an isomorphism when Y is replaced by $Y \wedge E(\mathscr{F}', \mathscr{F})$. We claim that each of the maps in the diagram defining θ_H is then an isomorphism, and three of the five are always isomorphisms. It is easy to see that $(G \times_N E)^H = E^H$, which is a contractible space. Since $E(\mathscr{F}', \mathscr{F})^J$ is contractible unless $(J) = (H)$, the Whitehead theorem implies that $\rho \wedge \mathrm{Id}$ is a G-homotopy equivalence.

It only remains to consider λ. Passage to H-fixed points on representative maps gives a homomorphism

$$\phi : \pi_*^N(\Sigma^L(E_+ \wedge Y \wedge E(\mathscr{F}', \mathscr{F})) \longrightarrow \pi_*^W(E_+ \wedge Y^H \wedge E(\mathscr{F}', \mathscr{F})^H)$$

such that $\phi \circ \lambda = \mathrm{Id}$. It suffices to show that ϕ is an isomorphism. As an N-space, $E(\mathscr{F}', \mathscr{F})$ is $E(\mathscr{F}'|N, \mathscr{F}|N)$. While $(\mathscr{F}'|N, \mathscr{F}|N)$ need not be an adjacent pair, $\mathscr{F}'|N - \mathscr{F}|N$ is the disjoint union of N-conjugacy classes (K), where the K are G-conjugate to H. It follows that $E(\mathscr{F}'|N, \mathscr{F}|N)$ is N-equivalent to a wedge of spaces $E(\mathscr{E}', \mathscr{E})$, where each $(\mathscr{E}', \mathscr{E})$ is an adjacent pair with $\mathscr{E}' - \mathscr{E} = (K)$ for some such K. However, it is easy to see that $E_+ \wedge E(\mathscr{E}', \mathscr{E})$ is N-contractible unless the N-conjugacy classes (H) and (K) are equal. Thus only the wedge summand $E(\mathscr{E}', \mathscr{E})$ with $\mathscr{E}' - \mathscr{E} = (H)$ contributes to the source and target of ϕ. Here $(H) = \{H\}$ since H is normal in N. A check of fixed points shows that $E(\mathscr{E}', \mathscr{E})^H$ is W-equivalent to E_+.

We now claim more generally that

$$\phi : \pi_*^N(Y \wedge E(\mathscr{E}', \mathscr{E})) \longrightarrow \pi_*^W(Y^H \wedge E(\mathscr{E}', \mathscr{E})^H) = \pi_*^W(Y^H \wedge E_+)$$

is an isomorphism for any N-CW complex Y. Writing out both sides as colimits of space level homotopy classes of maps, we see that it suffices to check that

$$\phi : [X, Y \wedge E(\mathscr{E}', \mathscr{E})]_N \longrightarrow [X^H, Y^H \wedge E_+]_W$$

is a bijection for any N-CW complex X. By easy cofibration sequence arguments, we may assume that all isotropy groups of X (except at its basepoint) are in $\mathscr{E}' - \mathscr{E} = \{H\}$. This uses the fact that the set $X_{\mathscr{E}}$ of points of X with isotropy group not in $\mathscr{E}$ is a subcomplex: we first show that X can be replaced by $X/X_{\mathscr{E}'}$, which has isotropy groups in $\mathscr{E}'$, and we then show that this new X can be replaced by $X_{\mathscr{E}}$, which has isotropy groups in $\mathscr{E}' - \mathscr{E}$. Under this assumption, $X = X^H$ and the conclusion is obvious.

Retaining our abbreviated notations, we next describe the map

$$\xi_H : \Sigma^\infty(E_+ \wedge_W \Sigma^A Y^H) \longrightarrow (\Sigma^\infty Y)^G.$$

This is a map of spectra indexed on U^G, and it suffices to describe its adjoint map of G-spectra indexed on U:

$$\tilde{\xi}_H : \Sigma^\infty(E_+ \wedge_W \Sigma^A Y^H) \longrightarrow \Sigma^\infty Y.$$

Here we regard $E_+ \wedge_W \Sigma^A Y^H$ as a G-trivial G-space, and the relevant suspension spectrum functor is $\Sigma^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U$ on both left and right. Suppressing notation for Σ^∞, implicitly using certain commutation relations between Σ^∞ and other functors, and abbreviating notation by setting $Z = E_+ \wedge_W \Sigma^A Y^H$, we define $\tilde{\xi}_H$ to be the composite displayed in the following commutative diagram:

$$\begin{array}{ccccccc}
Z & \xrightarrow{\tau\wedge \mathrm{Id}} & & & G/N_+ \wedge Z & \xrightarrow{\cong} & G_+ \wedge_N Z \\
\downarrow{\scriptstyle \xi_H} & & & & & & \downarrow{\scriptstyle \mathrm{Id}\wedge\tau} \\
Y & \xleftarrow[\rho\wedge \mathrm{Id}]{} & (G \times NE)_+ \wedge Y & \xleftarrow[\zeta]{} & G_+ \wedge_N (E_+ \wedge Y) & \xleftarrow[\supset]{} & G_+ \wedge_N (E_+ \wedge Y^H)
\end{array}$$

On the top line, τ is the transfer stable G-map $S^0 \longrightarrow G/N_+$ of IX.3.4 (or XVII.1.2). At the right, $\tau : E_+ \wedge_W \Sigma^A Y^H \longrightarrow E_+ \wedge Y^H$ is the stable N-map obtained by applying $i_* : W\mathscr{S}U^H \longrightarrow N\mathscr{S}U$, $i : U^H \longrightarrow U$, to the dimension-shifting transfer W-map of XVIII.1.2 that is at the heart of the Adams isomorphism that appears in the definition of θ_H. A diagram chase shows that the map on homotopy groups induced by ξ_H coincides with θ_H, and the wedge sum of the ξ_H is therefore an equivalence.

T. tom Dieck. Orbittypen und äquivariante Homologie. I. Arch. Math. 23(1972), 307-317.

T. tom Dieck. Orbittypen und äquivariante Homologie. II. Arch. Math. 26(1975), 650-662.
[LMS, V§§10-11]

2. Generalizations of the splitting theorems

We here formulate generalizations of Theorems 1.1 and 1.3 that are important in the study of generalized versions of the Segal conjecture. The essential ideas are the same as those just sketched, but transfer maps of bundles enter into the picture and the bookkeeping needed to define the relevant maps and prove that the relevant diagrams commute is quite complicated. We place ourselves in the context in which we studied generalized equivariant bundles in VII§1. Thus let Π be a normal subgroup of a compact Lie group Γ with quotient group G. Let $E(\Pi;\Gamma)$ be the universal $(\Pi;\Gamma)$-bundle of VII.2.1. Let $Ad(\Pi;\Gamma)$ denote the adjoint representation of Γ derived from Π; it is the tangent space of Π at e with the action of Γ induced by the conjugation action of Γ on Π. We regard G-spaces as Γ-spaces by pullback. For based Γ-spaces X and Y, we write

$$\{X, Y\}_n^{\Gamma} = [\Sigma^n \Sigma^\infty X, \Sigma^\infty Y]_\Gamma$$

for integers n. With these notations, we have the following results.

THEOREM 2.1. Let X be a based G-space and Y be a based Γ-space. Assume either that X is a finite G-CW complex or that Π is finite. Then $\{X, Y\}_*^{\Gamma}$ is naturally isomorphic to the direct sum over the Γ-conjugacy classes of subgroups Λ of Π of the groups

$$\{X, E(W_\Pi\Lambda; W_\Gamma\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda;W_\Gamma\Lambda)} Y^\Lambda\}_*^{W_\Gamma\Lambda/W_\Pi\Lambda}.$$

Here the quotient homomorphism $\Gamma \longrightarrow G$ induces an inclusion of $W_\Gamma\Lambda/W_\Pi\Lambda$ in G and so fixes an action of this group on X. Of course, when G is finite, the adjoint representations in the theorem are all zero. If we set $\Pi = \Gamma$ (and rename it G), then the theorem reduces to a mild generalization of Theorem 1.1. When Π is finite, the specified sum satisfies the wedge axiom. In general, the sum is infinite and we must restrict to finite G-CW complexes X.

THEOREM 2.2. For based Γ-spaces Y, there is a natural equivalence of G-spectra from $(\Sigma^\infty Y)^\Pi$ to the wedge over the Γ-conjugacy classes of subgroups Λ of Π of the suspension spectra of the G-spaces

$$G_+ \wedge_{W_\Gamma\Lambda/W_\Pi\Lambda} (E(W_\Pi\Lambda; W_\Gamma\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda;W_\Gamma\Lambda)} Y^\Lambda).$$

Here the suspension spectrum functor applied to Y is $\Sigma^\infty : \Gamma\mathscr{T} \longrightarrow \Gamma\mathscr{S}U$ and that applied to the wedge summands is $\Sigma^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U^\Pi$, where U is a complete Γ-universe.

[LMS, V§§10-11]

3. Equivalent definitions of Mackey functors

In IX§4, we defined a Mackey functor to be an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$, and we have observed that the Burnside category $\mathscr{B}_G$ is just the full subcategory of the stable category whose objects are the orbit spectra $\Sigma^\infty G/H_+$, but with objects denoted G/H. This is the appropriate definition of a Mackey functor for general compact Lie groups, but we show here that it is equivalent to an older, and purely algebraic, definition when G is finite. We first describe the maps in $\mathscr{B}_G$. As observed in IX§4, their composition is hard to describe in general. However, for finite groups G, there is a conceptual algebraic description. In fact, in this case there is an extensive literature on the algebraic theory of Mackey functors, and we shall say just enough to be able to explain the important idea of induction theorems in the next section.

When we specialize the diagram-chasing needed for the proofs in Section 1 to the calculation of $\pi_0^G(Y)$, we arrive at the following simple conclusion. Recall Corollary 1.2.

Proposition 3.1. For any based G-space Y, $\pi_0^G(Y)$ is the free Abelian group generated by the following composites, where (H) runs over ΦG and y runs over a representative point in Y^H of each non-basepoint component of Y^H/WH:

$$S \xrightarrow{\tau} \Sigma^\infty G/H_+ \xrightarrow{\Sigma^\infty \tilde{y}} \Sigma^\infty Y;$$

here τ is the transfer and $\tilde{y} : G/H_+ \longrightarrow Y$ is the based G-map such that $\tilde{y}(eH) = y$.

There is a useful conceptual reformulation of this calculation. Since we are interested in orbits G/H, we switch to unbased G-spaces.

Corollary 3.2. Let X be an unbased G-space. For $H \subset G$, the group

$$\pi_0^H(X_+) = [\Sigma^\infty G/H_+, \Sigma^\infty X_+]_G$$

is isomorphic to the free Abelian group generated by the equivalence classes of diagrams of space level G-maps

$$G/H \xleftarrow{\phi} G/K \xrightarrow{\chi} X,$$

where $K \subset H$ and W_HK is finite. Here (ϕ, χ) is equivalent to (ϕ', χ') if there is a G-homeomorphism $\xi : G/K \longrightarrow G/K'$ such that the following diagram is G-homotopy commutative:

$$\begin{array}{ccccc} & & G/K & & \\ & {}^{\phi}\swarrow & & \searrow^{\chi} & \\ G/H & & \downarrow{\scriptstyle \xi} & & X \\ & {}_{\phi'}\nwarrow & & \nearrow_{\chi'} & \\ & & G/K' & & \end{array}$$

We are thinking of ϕ as the corresponding transfer map $\Sigma^\infty G/H_+ \longrightarrow \Sigma^\infty G/K_+$, namely $G \ltimes_H (\tau)$, where $\tau : S^0 \longrightarrow \Sigma^\infty H/K_+$ is the transfer H-map.

This result specializes to give a good description of the maps of $\mathscr{B}_G$. In principle, their composition can be described in terms of a double coset formula, but this is quite hard to compute with in general. However, when G is finite, it admits an attractive conceptual reformulation.

To see this, let $\hat{\mathscr{B}}_G$ be the category whose objects are the finite G-sets and whose morphisms are the stable G-maps $X_+ \longrightarrow Y_+$. That is, up to an abbreviated notation for objects, $\hat{\mathscr{B}}_G$ is the full subcategory of the stable category whose objects are the $\Sigma^\infty X_+$ for finite G-sets X. Clearly $\mathscr{B}_G$ embeds as a full subcategory of $\hat{\mathscr{B}}_G$, and every object of $\hat{\mathscr{B}}_G$ is a disjoint union of objects of $\mathscr{B}_G$. We easily find that maps in $\hat{\mathscr{B}}_G$ can be described as equivalence classes $[\phi, \chi]$ of pairs (ϕ, χ), exactly as in the previous corollary, but now the composite of maps

$$V \xleftarrow{\ \phi\ } W \xrightarrow{\ \chi\ } X \quad \text{and} \quad X \xleftarrow{\ \psi\ } Y \xrightarrow{\ \omega\ } Z$$

can be specified as the equivalence class of the diagram

$$\begin{array}{ccccccc} & & & P & & & \\ & & \swarrow & & \searrow & & \\ & W & & & & Y & \\ {}^{\phi}\swarrow & & \searrow^{\xi} & & {}^{\psi}\swarrow & & \searrow^{\omega} \\ V & & & X & & & Z, \end{array}$$

where the top square is a pullback. This gives a complete description of $\hat{\mathscr{B}}_G$ in purely algebraic terms, with disjoint unions thought of as direct sums. It is important, and obvious, that this category is abstractly self-dual. Moreover, the duality isomorphism is given topologically by Spanier-Whitehead duality on orbits.

Since an additive functor necessarily preserves any finite direct sums in its domain, it is clear that an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$ determines and is determined by an additive contravariant functor $\hat{\mathscr{B}}_G \longrightarrow \mathscr{A}b$. In turn, as a matter of algebra, an additive contravariant functor $\hat{\mathscr{B}}_G \longrightarrow \mathscr{A}b$ determines and is determined by a Mackey functor in the classical algebraic sense. Precisely, such a Mackey functor M consists of a contravariant functor M^* and a covariant functor M_* from finite G-sets to Abelian groups. These functors have the same object function, denoted M, and M converts disjoint unions to direct sums. Writing $M^*\alpha = \alpha^*$ and $M_*\alpha = \alpha_*$, it is required that $\alpha^* \circ \beta_* = \delta_* \circ \gamma^*$ for pullback diagrams of finite G-sets

$$\begin{array}{ccc} P & \xrightarrow{\delta} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ Y & \xrightarrow[\beta]{} & Z. \end{array}$$

For an additive contravariant functor $M : \hat{\mathscr{B}}_G \longrightarrow \mathscr{A}b$, the maps $M[\phi, 1]$ and $M[1, \chi]$ specify the covariant and contravariant parts ϕ^* and χ_* of the corresponding algebraic Mackey functor, and conversely.

[LMS, V§9]

4. Induction theorems

Assuming that G is finite, and working with the algebraic notion of a Mackey functor just defined, we now consider the problem of computing $M(*)$, where $* = G/G$, in terms of the $M(G/H)$ for proper subgroups H. For a finite G-set X, let X^n be the product of n copies of X and let $\pi_i : X^{n+1} \longrightarrow X^n$ be the projection that omits the ith variable. We then have the chain complex

$$(*) \qquad 0 \longrightarrow M(*) \longrightarrow M(X) \longrightarrow M(X^2) \longrightarrow \cdots,$$

where the differential $d^n : M(X^n) \longrightarrow M(X^{n+1})$ is the alternating sum of the maps $(\pi_i)^*$, $0 \leq i \leq n$. Let $M(X)^{inv}$ be the kernel of d^1, namely the equalizer of $(\pi_0)^*$ and $(\pi_1)^*$. We are interested in determining when the resulting map $M(*) \longrightarrow M(X)^{inv}$ is an isomorphism. Of course, this will surely hold if the sequence (*) is exact. We have already seen an instance of this kind of argument in XVII§6.

When is (*) exactΓ Let M_X be the Mackey functor that sends a finite G-set Y to $M(X \times Y)$, and similarly for maps. The projections $\pi : X \times Y \longrightarrow Y$ induce a map of Mackey functors $\theta^X : M \longrightarrow M_X$. We say that M is "X-injective" if θ^X is a split monomorphism. If θ^X is split by $\psi : M_X \longrightarrow M$, so that $\psi \circ \theta_X = \mathrm{Id}$, then the homomorphisms

$$\psi(X^n) : M(X \times X^n) \longrightarrow M(X^{n+1})$$

specify a contracting homotopy for (*). Therefore (*) is exact if M is X-injective.

When is M X-injectiveΓ To obtain a good criterion, we must first specify the notion of a pairing $\mu : L \times M \longrightarrow N$ of Mackey functors. This consists of maps $\mu : L(X) \otimes M(X) \longrightarrow N(X)$ for finite G-sets X such that the evident covariant and contravariant functoriality diagrams and the following Frobenius diagram commute for a G-map $f : X \longrightarrow Y$.

$$\begin{array}{ccccc} L(X)\otimes M(Y) & & \xrightarrow{f_*\otimes \mathrm{Id}} & & L(Y)\otimes M(Y)\\ {\scriptstyle \mathrm{Id}\otimes f^*}\downarrow & & & & \downarrow{\scriptstyle\mu}\\ L(X)\otimes M(X) & \xrightarrow[\mu]{} & N(X) & \xrightarrow[f_*]{} & N(Y). \end{array}$$

A Green functor is a Mackey functor R together with a pairing μ that makes each $R(X)$ a commutative and associative unital ring, the maps f^* being required to preserve units and thus to be ring homomorphisms. The notion of a module M over a Green functor R is defined in the evident way. With these notions, one can prove the following very useful general fact.

Proposition 4.1. If R is a Green functor and the projection $X \longrightarrow *$ induces an epimorphism $R(X) \longrightarrow R(*)$, then every R-module M is X-injective. Therefore $M(*) \cong M(X)^{inv}$ for every R-module M.

For a Green functor R, there is a unique minimal set $\{(H)\}$ of conjugacy classes of subgroups of G such that $R(\coprod G/H) \longrightarrow R(*)$ is an epimorphism; this set is called the "defect set" of R. By an "induction theorem", we understand an identification of the defect set of a Green functor. For example, the complex representation rings $R(H)$ are the values on G/H of a Green functor $\underline{R}$, and the "Brauer induction theorem" states that the set of products $P \times C$ of a p-group P and a cyclic group C in G contains a defect set of $\underline{R}$. We will shortly give another example, one that we will use later to reduce the generalized Segal conjecture to the case of finite p-groups.

We must first explain the relationship of Burnside rings to Mackey functors. For a finite G-set X, we have a Grothendieck ring $\underline{A}(X)$ of isomorphism classes of G-sets over X. The multiplication is obtained by pulling Cartesian products back along the diagonal of the base G-set X. When $X = *$, this is the Burnside ring $A(G)$. More generally, a G-set $\alpha : T \longrightarrow G/H$ over G/H determines and is determined by the H-set $\alpha^{-1}(eH)$, and it follows that $\underline{A}(G/H) \cong A(H)$. A G-map $f : X \longrightarrow Y$ determines $f^* : \underline{A}(Y) \longrightarrow \underline{A}(X)$ by pullback along f, and it determines $f_* : \underline{A}(X) \longrightarrow \underline{A}(Y)$ by composition with f. In more down to earth terms, if $f : G/H \longrightarrow G/K$ is the G-map induced by an inclusion $H \subset K$, then $f^* : A(K) \longrightarrow A(H)$ sends a K-set to the same set regarded as an H-set and $f_* : A(H) \longrightarrow A(K)$ sends an H-set X to the K-set $K \times_H X$; we call f_* induction. This gives the Burnside Green functor $\underline{A}$.

Any Mackey functor M is an $\underline{A}$-module via the pairings

$$\underline{A}(X) \otimes M(X) \longrightarrow M(X)$$

that send $\alpha \otimes m$, $\alpha : Y \longrightarrow X$, to $\alpha_*\alpha^*(m)$. Therefore, by pullback along the ring map $A(G) = \underline{A}(*) \longrightarrow \underline{A}(X)$, each $M(X)$ is an $A(G)$-module. Any Green functor R has a unit map of Green functors $\eta : \underline{A} \longrightarrow R$ that sends $\alpha : Y \longrightarrow X$ to $\alpha_*\alpha^*(1)$. Thus we see that the Burnside Green functor plays a universal role.

Observe that we can localize Mackey functors termwise at any multiplicative subset S of $A(G)$. We can complete Mackey functors that are termwise finitely $A(G)$-generated at any ideal $I \subset A(G)$. We wish to establish an induction theorem applicable to such localized and completed Mackey functors. This amounts to determination of the defect set of $S^{-1}\underline{A}\hat{_I}$.

It is useful to use a little commutative algebra. The following observation is standard algebra, but its relevance to the present question was noticed in work of Adams, Haeberly, Jackowski, and myself and its extension by Haeberly. We shall state it for pro-modules — which are just inverse systems of modules — but only actual modules need be considered at the moment. Its pro-module version will be used in the proof of the generalized Segal conjecture in XX§§2, 3. Localizations of completions of pro-modules $\mathbf{M} = \{M_\alpha\}$ are understood to be inverse systems

$$S^{-1}\mathbf{M}\hat{_I} = \{S^{-1}M_\alpha / I^r M_\alpha\}.$$

Lemma 4.2. Let $\mathbf{M}$ be a pro-finitely generated module over a commutative ring A, let S be a multiplicative subset of A, and let I be an ideal of A. Then $S^{-1}\mathbf{M}\hat{_I}$

is pro-zero if and only if $(S_P)^{-1}\mathbf{M}_{\hat{P}}$ is pro-zero for every prime ideal P such that $P \cap S = \emptyset$ and $P \supset I$, where S_P is the multiplicative subset $A - P$.

For a prime ideal P of $A(G)$, we let $K(P)$ be a maximal element of the set $\{H | P = q(H, p)\}$. We have discussed these subgroups in XVII§3.

Lemma 4.3. $\{(K(P))\}$ is the defect set of the Green functor $(S_P)^{-1}\underline{A}$.

Proof. Essentially this result was observed, in less fancy language, at the end of XVII§6. The subgroup $K = K(P)$ is characterized by $P = q(K, p)$ and $|WK| \not\equiv 0 \bmod p$. (We allow $p = 0$.) The composite

$$A(G) \longrightarrow A(K) \longrightarrow A(G)$$

of restriction and induction is multiplication by $[G/K]$. Since this element of $A(G)$ maps to a unit in $A(G)_{q(K,p)}$, the displayed composite becomes an isomorphism upon localization at $q(K, p)$. □

Proposition 4.4. Let S be a multiplicative subset of $A(G)$ and let I be an ideal of $A(G)$. Then the defect set of the Green functor $S^{-1}\underline{A}_{\hat{I}}$ is

$$\{(K(P)) | P \cap S = \emptyset \text{ and } P \supset I\}.$$

Proof. The statement means that the sum of transfer maps

$$\sum S^{-1}A(K(P))_{\hat{I}} \longrightarrow S^{-1}A(G)_{\hat{I}}$$

is an epimorphism, and Lemmas 4.2 and 4.3 imply that its cokernel is zero. □

The starting point for arguments like this was the following result of McClure and myself, which is the special case when $S = \{1\}$ and I is the augmentation ideal (alias $q(e, 0)$). If $P = q(e, p)$, then $K(P)$ is a p-Sylow subgroup of G.

Corollary 4.5. If I is the augmentation ideal of $A(G)$, then the defect set of the Green functor $\underline{A}_{\hat{I}}$ is the set of p-Sylow subgroups of G.

This will be applied in conjunction with the following observation.

Lemma 4.6. Let M be a Mackey functor over a finite p-group G and let $\pi^* : M(*) \longrightarrow M(G)$ be induced by the projection $G \longrightarrow *$. Then the p-adic and I-adic topologies coincide on $\mathrm{Ker}(\pi^*)$.

PROOF. Since multiplication by $[G]$ is the composite $\pi_*\pi^*$, $[G]\,\mathrm{Ker}(\pi^*) = 0$. Since $[G] - |G| \in I$, $|G|\,\mathrm{Ker}(\pi^*) \subset I\,\mathrm{Ker}(\pi^*)$. If $H \neq e$, then $\phi_H([G/K] - |G/K|)$ is divisible by p because $G/K - (G/K)^H$ is a disjoint union of non-trivial H-orbits. Therefore $\phi(I) \subset pC(G)$. Let $|G| = p^n$. Since $|G|C(G) \subset \phi(A(G))$, we see that $\phi(I^{n+1}) \subset p\phi(I)$ and thus $I^{n+1} \subset pI$. The conclusion follows. □

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5. Splittings of rational G-spectra for finite groups G

We here analyze the rational equivariant stable category for finite groups G. The essential point is that any rational G-spectrum splits as a product of Eilenberg-MacLane G-spectra $K(M,n) = \Sigma^n HM$.

THEOREM 5.1. Let G be finite. Then, for rational G-spectra X, there is a natural equivalence $X \longrightarrow \prod K(\underline{\pi}_n(X), n)$.

There is something to prove here since the counterexamples of Triantafillou discussed in III§3 show that, unless G is cyclic of prime power order, the conclusion is false for naive G-spectra. A counterexample of Haeberly shows that the conclusion is also false for genuine G-spectra when G is the circle group, the rationalization of KU_G furnishing a counterexample. Greenlees has recently studied what does happen for general compact Lie groups.

The proof of Theorem 5.1 depends on two facts, one algebraic and one topological. We assume that G is finite in the rest of this section.

PROPOSITION 5.2. In the Abelian category of rational Mackey functors, all objects are projective and injective.

The analog for coefficient systems is false, and so is the analog for compact Lie groups. The following result is easy for finite groups and false for compact Lie groups.

PROPOSITION 5.3. For $H \subset G$ and $n \neq 0$, $\underline{\pi}_n(G/H_+) \otimes \mathbb{Q} = 0$.

Let $\mathscr{M} = \mathscr{M}_G$ denote the Abelian category of Mackey functors over G. For G-spectra X and Y, there is an evident natural map

$$\theta : [X, Y]_G \longrightarrow \prod \mathrm{Hom}_{\mathscr{M}}(\underline{\pi}_n(X), \underline{\pi}_n(Y)).$$

Let Y be rational. By the previous result and the Yoneda lemma, θ is an isomorphism when $X = \Sigma^\infty G/H_+$ for any H. Throwing in suspensions, we can extend θ to a graded map

$$\theta : Y^q_G(X) = [X, Y]^q_G = [\Sigma^{-q}X, Y]_G \longrightarrow \prod \mathrm{Hom}_{\mathscr{M}}(\underline{\pi}_n(\Sigma^{-q}X), \underline{\pi}_n(Y)).$$

It is still an isomorphism when X is an orbit. Of course, we obtain the same groups if we replace X and the Mackey functors $\underline{\pi}_n(\Sigma^{-q}X)$ by their rationalizations. Since the Mackey functors $\underline{\pi}_n(Y)$ are injective, the right hand side is a cohomology theory on G-spectra X. Clearly θ is a map of cohomology theories and this already proves the following result. With $Y = \prod K(\underline{\pi}_n(X), n)$, Theorem 5.1 is an easy consequence.

THEOREM 5.4. If Y is rational, then θ is a natural isomorphism.

This classifies rational G-spectra, and we next classify maps between them. Recall that $\phi \otimes \mathbb{Q} : A(G) \otimes \mathbb{Q} \longrightarrow C(G) \otimes \mathbb{Q}$ is an isomorphism and that $C(G) \otimes \mathbb{Q}$ is the product of a copy of $\mathbb{Q}$ for each conjugacy class (H). There results a complete set of orthogonal idempotents $e_H = e^G_H$ in $A(G) \otimes \mathbb{Q}$. Multiplication by the e_H induces splittings of $A(G) \otimes \mathbb{Q}$-modules, rational Mackey functors, and rational G-spectra, and we have the commutation relation

$$\underline{\pi}_n(e_H X) \cong e_H \underline{\pi}_n(X).$$

In all three settings, there are no non-zero maps $e_H X \longrightarrow e_J Y$ unless H is conjugate to J. This gives refinements of Theorems 5.1 and 5.4.

THEOREM 5.5. For rational G-spectra X, there are natural equivalences

$$X \simeq \bigvee e_H X \simeq \prod K(e_H \underline{\pi}_n(X), n).$$

THEOREM 5.6. For rational G-spectra X and Y, there are natural isomorphisms

$$[X, Y]_G \cong \sum [e_H X, e_H Y]_G \cong \sum \prod \mathrm{Hom}_{\mathscr{M}}(e_H \underline{\pi}_n(X), e_H \underline{\pi}_n(Y)).$$

Moreover, if $V_{n,H}(X) = (e_H \underline{\pi}_n(X))(G/H) \subset \pi_n(X^H)$, then

$$\mathrm{Hom}_{\mathscr{M}}(e_H \underline{\pi}_n(X), e_H \underline{\pi}_n(Y)) \cong \mathrm{Hom}_{WH}(V_{n,H}(X), V_{n,H}(Y)).$$

Thus the computation of maps between rational G-spectra reduces to the computation of maps between functorially associated modules over subquotient groups. The last statement of the theorem is a special case of the following algebraic result.

THEOREM 5.7. For rational Mackey functors M and N, there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{M}}(e_H M, e_H N) \cong \mathrm{Hom}_{WH}(V_H(M), V_H(N)),$$

where $V_H(M)$ is the $\mathbb{Q}[WH]$-module $(e_H M)(G/H) \subset M(G/H)$.

The proof of Proposition 5.2 is based on the following consequence of the fact that $V_H(N)$ is a projective and injective $\mathbb{Q}[WH]$-module.

LEMMA 5.8. If the conclusion of Theorem 5.7 holds for all N and for a given M and H, then $e_H M$ is projective; if the conclusion holds for all M and for a given N and H, then $e_H N$ is injective.

Now let $\mathscr{M}_{\mathbb{Q}}$ be the category of rational Mackey functors over G. Let $\mathscr{Q}[G]$ be the category of $\mathbb{Q}[G]$-modules. Fix $H \subset G$. Then there are functors

$$U_H : \mathscr{M}_{\mathbb{Q}} \longrightarrow \mathscr{Q}[WH] \quad \text{and} \quad F_H : \mathscr{Q}[WH] \longrightarrow \mathscr{M}_{\mathbb{Q}}.$$

Explicitly,

$$U_H M = M(G/H) \quad \text{and} \quad (F_H V)(G/K) = (\mathbb{Q}[(G/K)^H] \otimes V)^{WH}.$$

These functors are both left and right adjoint to each other if we replace $\mathscr{M}_{\mathbb{Q}}$ by its full subcategory $\mathscr{M}_{\mathbb{Q}}/H$ of those Mackey functors M such that $M(G/J) = 0$ for all proper subconjugates J of H. Since $(F_H V)(G/K) = 0$ unless H is subconjugate to K, $F_H V$ is in $\mathscr{M}_{\mathbb{Q}}/H$.

PROOFS OF PROPOSITION 5.2 AND THEOREM 5.7. One easily proves both of these results when $M = F_H V$ by use of the adjunctions and idempotents. Even integrally, every Mackey functor M is built up by successive extensions from Mackey functors of the form $F_H V$. Rationally, the extensions split by the projectivity of the $F_H V$. Therefore any rational Mackey functor M is a direct sum of Mackey functors of the form $F_H V$ for varying H and V. □

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CHAPTER XX

The Segal conjecture

1. The statement in terms of completions of G-spectra

There are many ways to think about the Segal conjecture and its generalizations. Historically, the original source of the conjecture was just the obvious analogy between K-theory and stable cohomotopy. According to the Atiyah-Segal completion theorem, the K-theory of the classifying space of a compact Lie group G is isomorphic to the completion of the representation ring $R(G)$ at its augmentation ideal. Here $R(G)$ is $K^0_G(S^0)$, and $K^1_G(S^0) = K^1(BG_+) = 0$. The Burnside ring $A(G)$ is $\pi^0_G(S^0)$, and it is natural to guess that the stable cohomotopy of BG is isomorphic to the completion of $\pi^*_G(S^0)$ at the augmentation ideal I of $A(G)$. This guess is the Segal conjecture. It is not true for compact Lie groups in general, but it turns out to be correct for finite groups G. We shall restrict ourselves to finite groups throughout our discussion. A survey of what is known about the Segal conjecture for compact Lie groups has been given by Lee and Minami.

Here we are thinking about $\mathbb{Z}$-graded theories, and that is the right way to think about the proof. However, one can also think about the result in purely equivariant terms, and the conclusion then improves to a result about G-spectra and thus about $RO(G)$-graded cohomology theories. To see this, let's at first generalize and consider any G-spectrum k_G. We have the projection $EG_+ \longrightarrow S^0$, and it induces a G-map

$$\varepsilon : k_G = F(S^0, k_G) \longrightarrow F(EG_+, k_G). \tag{1.1}$$

We think of ε as a kind of geometric completion of k_G.

It is natural to think about such completions more generally. Let $\mathscr{F}$ be a family of subgroups of G. We have the projection $E\mathscr{F}_+ \longrightarrow S^0$, and we have the induced

G-map

$$\varepsilon : k_G = F(S^0, k_G) \longrightarrow F(E\mathscr{F}_+, k_G). \tag{1.2}$$

We think of ε as the geometric completion of k_G at $\mathscr{F}$.

We want to compare this with an algebraic completion. The family $\mathscr{F}$ determines an ideal $I\mathscr{F}$ of $A(G)$, namely

$$I\mathscr{F} = \bigcap_{H \in \mathscr{F}} \operatorname{Ker}(A(G) \longrightarrow A(H)). \tag{1.3}$$

Just as $I = I\{e\} = q(e, 0)$, by definition, it turns out algebraically that

$$I\mathscr{F} = \bigcap_{H \in \mathscr{F}} q(H, 0). \tag{1.4}$$

Since $A(G)$ plays the same role in equivariant theory that $\mathbb{Z}$ plays in nonequivariant theory, it is natural to introduce completions of G-spectra at ideals of the Burnside ring. This is quite easy to do. For an element α of $A(G)$, define $S_G[\alpha^{-1}]$, the localization of the sphere G-spectrum S_G at α, to be the telescope of countably many iterates of $\alpha : S_G \longrightarrow S_G$. Then let $K(\alpha)$ be the fiber of the canonical map $S_G \longrightarrow S_G[\alpha^{-1}]$. For an ideal I generated by a set $\{\alpha_1, \cdots, \alpha_n\}$, define

$$K(I) = K(\alpha_1) \wedge \cdots \wedge K(\alpha_n). \tag{1.5}$$

It turns out that, up to equivalence, $K(I)$ is independent of the choice of generators of I. Now define

$$(k_G)\hat{_I} = F(K(I), k_G). \tag{1.6}$$

By construction, $K(I)$ comes with a canonical map $\zeta : K(I) \longrightarrow S_G$, and there results a map

$$\gamma : k_G \longrightarrow (k_G)\hat{_I}. \tag{1.7}$$

We call γ the completion of k_G at the ideal I. For those who know about such things, we remark that completion at I is just Bousfield localization at $K(I)$. We shall later use "brave new algebra" to generalize this construction.

Now specialize to $I = I\mathscr{F}$ for a family $\mathscr{F}$. For $\alpha \in I\mathscr{F}$, $\alpha : S_G \longrightarrow S_G$ is null homotopic as an H-map for any $H \in \mathscr{F}$. Therefore $S_G[\alpha^{-1}]$ is H-contractible, $K(I\mathscr{F})$ is H-equivalent to S_G, and the cofiber of ζ is H-contractible. This implies that there is a unique G-map

$$\xi : \Sigma^\infty E\mathscr{F}_+ \longrightarrow K(I\mathscr{F}) \tag{1.8}$$

over S_G. There results a canonical map of G-spectra

$$\xi^* : F(K(I\mathscr{F}), kG) \longrightarrow F(E\mathscr{F}_+, k_G). \tag{1.9}$$

We view this as a comparison map relating the algebraic to the geometric completion of k_G at $\mathscr{F}$.

One can ask for which G-spectra k_G the map ξ^* is an equivalence. We can now state what I find to be the most beautiful version of the Segal conjecture. Recall that $D(E) = F(E, S_G)$.

THEOREM 1.10. For every family $\mathscr{F}$, the map

$$\xi^* : (S_G)\hat{}_{I\mathscr{F}} = D(K(I\mathscr{F})) \longrightarrow D(E\mathscr{F}_+)$$

is an equivalence of G-spectra.

Parenthetically, one can also pass to smash products rather than function spectra from the map ξ, obtaining

$$\xi_* : k_G \wedge E\mathscr{F}_+ \longrightarrow k_G \wedge K(I\mathscr{F}_+). \tag{1.11}$$

One can ask for which G-spectra k_G this map is an equivalence. A standard argument shows that ξ^* is an equivalence if k_G is a ring spectrum and ξ_* is an equivalence. Once we introduce Tate theory, we will be able to give a remarkable partial converse. The point to make here is that ξ_* is an equivalence for K_G, as we shall explain in XXIV§7, but is certainly not an equivalence for S_G. That would be incompatible with the splitting of $(S_G)^G$ in XIX§1. Our original analogy will only take us so far.

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2. A calculational reformulation

What does Theorem 1.10 say calculationallyΓ To give an answer, we go back to our algebraic completions. The I-adic completion functor is neither left nor right exact in general, and it has left derived functors L_i^I. Because $A(G)$ has Krull dimension one, these vanish for $i > 1$. In precise analogy with the calculation of the homotopy groups of p-adic completions of spaces, we find that, for any G-spectrum X, there is a natural short exact sequence

$$0 \longrightarrow L^I_1(\underline{\pi}_{q-1}(X)) \longrightarrow \underline{\pi}_q(X\hat{_I}) \longrightarrow L^I_0(\underline{\pi}_q(X)) \longrightarrow 0, \tag{2.1}$$

where we apply our derived functors to Mackey functors termwise. Thinking cohomologically, for any G-spectra X and k_G, there are natural short exact seqences

$$0 \longrightarrow L^I_1((k^{\nu+1}_G(X)) \longrightarrow ((k_G)\hat{_I})^\nu_G(X) \longrightarrow L^I_0(k^\nu_G(X)) \longrightarrow 0. \tag{2.2}$$

As a matter of algebra, the L^I_i admit the following descriptions, which closely parallels the algebra we summarized when we discussed completions at p in II§4. Abbreviate $A = A(G)$ and consider an A-module M. Then we have the following natural short exact sequences.

$$0 \longrightarrow \lim{}^1 \operatorname{Tor}^A_1(A/I^r, M) \longrightarrow L^I_0(M) \longrightarrow M\hat{_I} \longrightarrow 0. \tag{2.3}$$

$$0 \longrightarrow \lim{}^1 \operatorname{Tor}^A_2(A/I^r, M) \longrightarrow L^I_1(M) \longrightarrow \lim \operatorname{Tor}^A_1(A/I^r, M) \longrightarrow 0. \tag{2.4}$$

There is interesting algebra in the passage from the topological definition of completion to the algebraic interpretation (2.1). Briefly, there are "local homology groups" $H^I_i(M)$ analogous to Grothendieck's local cohomology groups. Our topological construction mimics the algebraic definition of the $H^I_i(M)$, and, as a matter of algebra, $L^I_i(M) \cong H^I_i(M)$. This leads to alternative procedures for calculation, but begins to take us far from the Segal conjecture. We shall return to the relevant algebra in Chapter XXIV.

The last two formulas show that, if M is finitely generated, then $L^I_0(M) \cong M\hat{_I}$ and $L^I_1(M) = 0$. When a G-spectrum k_G is bounded below and of finite type, in the sense that each of its homotopy groups is finitely generated, we can construct a model for $(k_G)\hat{_I}$ and study its properties by induction up a Postnikov tower, exactly as we studied p-completion in II§5. As there, we find that a map from k_G to an "I-complete spectrum" that induces I-adic completion on all homotopy groups is equivalent to the I-completion of k_G. Moreover, a sufficient condition for a bounded below spectrum to be I-complete is that its homotopy groups are finitely generated modules over $A(G)\hat{_I}$.

We deduce from XIX.1.1 that S_G is of finite type. Thus the I-adic completions of its homotopy groups are bounded below and of finite type over $A(G)\hat{_I}$. A little diagram chase now shows that the following theorem will imply Theorem 1.10.

THEOREM 2.5. The map $\varepsilon : S_G \longrightarrow D(E\mathscr{F}_+)$ induces an isomorphism

$$\underline{\pi}_*(S_G)\hat{_{I\mathscr{F}}} \longrightarrow \underline{\pi}_*(D(E\mathscr{F}_+)).$$

There is an immediate problem here. A priori, we do not know anything about the homotopy groups of $D(E\mathscr{F}_+)$, which, on the face of it, need be neither bounded below nor of finite type. There is a $\lim^1$ exact sequence for their calculation in terms of the duals of the skeleta of $E\mathscr{F}_+$. To prove that the $\lim^1$ terms vanish, and to make sure that we are always working with finitely generated $A(G)$-modules, we work with pro-groups and only pass to actual inverse limits at the very end. We have already said nearly all that we need to say about this in XIV§5. Recall that, for any Abelian group valued functor h on G-CW complexes or spectra, we define the associated pro-group valued functor $\mathbf{h}$ by letting $\mathbf{h}(X)$ be the inverse system $\{h(X_\alpha)\}$, where X_α runs over the finite subcomplexes of X. Our functors take values in finitely generated $A(G)$-modules. For an ideal I in $A(G)$ and such a pro-module $\mathbf{M} = \{M_\alpha\}$, $\mathbf{M}\hat{_I}$ is the inverse system $\{M_\alpha / I^r M_\alpha\}$. For a multiplicative subset S, $S^{-1}M = \{S^{-1}M_\alpha\}$.

We define pro-Mackey functors just as we defined Mackey functors, but changing the target category from groups to pro-groups. Now Theorem 2.5 will follow from its pro-Mackey functor version.

THEOREM 2.6. The map $\varepsilon : S_G \longrightarrow D(E\mathscr{F}_+)$ induces an isomorphism

$$\underline{\boldsymbol{\pi}}_*(S_G)\hat{_{I\mathscr{F}}} \longrightarrow \underline{\boldsymbol{\pi}}_*(D(E\mathscr{F}_+)).$$

The point is that the pro-groups on the left certainly satisfy the Mittag-Leffler condition guaranteeing the vanishing of $\lim^1$ terms, hence the $\lim^1$ terms for the calculation of $\underline{\pi}_*(D(E\mathscr{F}_+))$ vanish and we obtain the isomorphism of Theorem 2.5 on passage to limits. We now go back to something we omitted: making sense of the induced map in Theorem 2.6. For a finite G-CW complex X such that X^H is empty for $H \notin \mathscr{F}$, we find by induction on the number of cells and the very definition of $I\mathscr{F}$ that $\underline{\pi}_*(D(X_+))$ is annihilated by some power of $I\mathscr{F}$. This implies that the canonical pro-map

$$\underline{\boldsymbol{\pi}}_*(D(X_+)) \longrightarrow \underline{\boldsymbol{\pi}}_*(D(X_+))\hat{_{I\mathscr{F}}}$$

is an isomorphism. Applying this to the finite subcomplexes of $E\mathscr{F}$, we see that the right side in Theorem 2.6 is $I\mathscr{F}$-adically complete. Thus the displayed map makes sense.

J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. Topology 27(1988), 7-21.
J. P. C. Greenlees and J. P. May. Derived functors of I-adic completion and local homology. J. Algebra 149(1992), 438-453.

3. A generalization and the reduction to p-groups

Now we change our point of view once more, thinking about individual pro-homotopy groups rather than Mackey functors. Using a little algebra to check that the ideal in $A(H)$ generated by the image of $I\mathscr{F}$ under restriction has the same radical as $I(\mathscr{F}|H)$, we see that the Hth term of the map in Theorem 2.6 is

$$\boldsymbol{\pi}_H^*(S^0)\hat{}_{I(\mathscr{F}|H)} \longrightarrow \boldsymbol{\pi}_H^*(E(\mathscr{F}|H)_+).$$

We may as well proceed by induction on the order of G, so that we may assume this map to be an isomorphism for all proper subgroups. In any case, Theorem 2.6 can be restated as follows.

THEOREM 3.1. The map $E\mathscr{F} \longrightarrow *$ induces an isomorphism

$$\boldsymbol{\pi}_G^*(S^0)\hat{}_{I\mathscr{F}} \longrightarrow \boldsymbol{\pi}_G^*(E\mathscr{F}_+).$$

Now $E\mathscr{F} \longrightarrow *$ is obviously an example of an $\mathscr{F}$-equivalence, that is, a map that induces an equivalence on H-fixed points for $H \in \mathscr{F}$. We are really proving an invariance theorem:

An $\mathscr{F}$-equivalence $f : X \longrightarrow Y$ induces an isomorphism $\boldsymbol{\pi}_G^*(f)\hat{}_{I\mathscr{F}}$.

We can place this in a more general framework. Given a set $\mathscr{H}$ of subgroups of G, closed under conjugacy, we say that a cohomology theory is $\mathscr{H}$-invariant if it carries $\mathscr{H}$-equivalences to isomorphisms. We say that a G-space X is $\mathscr{H}$-contractible if X^H is contractible for $H \in \mathscr{H}$. By an immediate cofiber sequence argument, a theory is $\mathscr{H}$-invariant if and only if it vanishes on $\mathscr{H}$-contractible spaces. It is not difficult to show that, for any cohomology theory h^*, there is a unique minimal class $\mathscr{H}$ such that h^* is $\mathscr{H}$-invariant: determination of this class gives a best possible invariance theorem for h^*. Given an ideal I and a collection $\mathscr{H}$, we can try to obtain such a theorem for the theory $\boldsymbol{\pi}_G^*(\cdot)\hat{}_I$.

Answers to such questions in the context of localizations rather than completions have a long history and demonstrated value, but there one usually assumes that $\mathscr{H}$ is closed under passage to larger rather than smaller subgroups. For such a "cofamily" $\mathscr{H}$, we have the $\mathscr{H}$-fixed point subcomplex $X^{\mathscr{H}} = \{x | G_x \in \mathscr{H}\}$;

the inclusion $i : X^{\mathscr{H}} \longrightarrow X$ is an $\mathscr{H}$-equivalence. A cohomology theory is $\mathscr{H}$-invariant if and only it carries all such inclusions i to isomorphisms.

It seems eminently reasonable to ask about localizations and completions together. We can now state the following generalization of Theorem 3.1. Define the support of a prime ideal P in $A(G)$ to be the conjugacy class (L) such that P is in the image of $\mathrm{Spec}(A(L))$ but is not in the image of $\mathrm{Spec}(A(K))$ for any subgroup K of L. We know what the supports are: (H) for $q(H,0)$ and (H_p) for $q(H,p)$.

THEOREM 3.2. For any multiplicative subset S and ideal I, the cohomology theory $S^{-1}\boldsymbol{\pi}^*_G(\cdot)\hat{_I}$ is $\mathscr{H}$-invariant, where

$$\mathscr{H} = \bigcup \{\mathrm{Supp}(P) | P \cap S = \emptyset \;\text{ and }\; P \supset I\}.$$

With $S = \emptyset$ and $I = I\mathscr{F}$, Theorem 3.1 follows once one checks that the resulting $\mathscr{H}$ is contained in $\mathscr{F}$. In fact it equals $\mathscr{F}$ since the primes that contain $I\mathscr{F}$ are all of the $q(H,p)$ with $H \in \mathscr{F}$, and this allows $p = 0$. It looks as if we have made our work harder with this generalization but in fact, precisely because we have introduced localization, which we have already studied in some detail, the general theorem quickly reduces to a very special case.

In fact, by XIX.4.2, it is enough to show that $(S_P)^{-1}\boldsymbol{\pi}^*_G(X)\hat{_P} = 0$ if X^L is contractible for $L \in \mathrm{Supp}(P)$, where $S_P = A - P$. By XVII.5.5, there is an idempotent $e^G_L \in A(G)_p$ such that $(S_P)^{-1}A(G) = e^G_L A(G)_p$. Remembering that the Φ-fixed point functor satisfies $\Phi^H S_G = S_H$, we see that, for any finite G-CW complex X, XVII.6.4 specializes to give the chain of isomorphisms

$$e^G_L \pi^n_G(X)_p \longrightarrow e^{NL}_L \pi^n_{NL}(X)_p \longrightarrow e^{WL}_1 \pi^n_{WL}(X^L)_p \longrightarrow \pi^n_{VL}(X^L)^{inv}_p$$

where VL is a p-Sylow subgroup of WL. The transfer argument used to prove the last isomorphism gives further that $\pi^n_{VL}(X^L)^{inv}_p$ is naturally a direct summand in $\pi^n_{VL}(X^L)_p$. Passing to pro-modules, we conclude that $(S_P)^{-1}\boldsymbol{\pi}^*_G(X)\hat{_P}$ is a direct summand in $\boldsymbol{\pi}^*_{VL}(X^L)\hat{_p}$. Therefore Theorem 3.2 is implied by the following special case.

THEOREM 3.3. The theory $\boldsymbol{\pi}^*_G(\cdot)\hat{_p}$ is e-invariant for any finite p-group G. That is, it vanishes on nonequivariantly contractible G-spaces.

This is Carlsson's theorem, and we will discuss its proof in the next section. In the case of the augmentation ideal there is a shortcut to the reduction to p-groups and p-adic completion: it is immediate from XIX.4.5 and XIX.4.6. Let us say a word about the nonequivariant interpretation of the Segal conjecture in this case. Since S_G is a split G-spectrum, we can conclude that

$$\pi^*_G(S^0)\hat{}_I \cong \pi^*_G(EG_+) \cong \pi^*(BG_+). \tag{3.4}$$

Of course, the cohomotopy groups on the left lie in non-positive degrees and are just the homotopy groups reindexed. By XIX.1.1,

$$\pi^G_*(S^0) = \sum_{(H)} \pi_*(BWH_+). \tag{3.5}$$

The left side is a ring, but virtually nothing seems to be known about the multiplicative structure on the right. Nor is much known about the $A(G)$-module structure. Of course, the last problem disappears upon completion in the case of p-groups, by XIX.4.6.

J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. Topology 27(1988), 7-21.

4. The proof of the Segal conjecture for finite p-groups

There are two basic strategies. One is to use (3.5) and a nonequivariant interpretation of the completion map to reduce to a nonequivariant problem. For elementary p-groups, the ideas that we discussed in the context of the Sullivan conjecture can equally well be used to prove the Segal conjecture, and Lannes has an unpublished nonequivariant argument that handles general p-groups.

The other is to use equivariant techniques, which is the method used by Carlsson. Historically, Lin first proved the Segal conjecture for $\mathbb{Z}/2$, Gunawardena for $\mathbb{Z}/p$, p odd, and Adams, Gunawardena, and Miller for general elementary Abelian p-groups, all using nonequivariant techniques and the Adams spectral sequence. Carlsson's theorem reduced the case of general finite p-groups to the case of elementary Abelian p-groups. His ideas also led to a substantial simplification of the proof in the elementary Abelian case, as was first observed by Caruso, Priddy, and myself. For this reason, the full original proof of Adams, Gunawardena, and Miller was never published. Since I have nothing to add to the exposition that Caruso, Priddy, and I gave, which includes complete details of a variant of Carlsson's proof of the reduction to elementary Abelian p-groups, I will give an outline that may gain clarity by the subtraction of most of the technical details.

We assume throughout that G is a finite p-group. We begin with a general G-spectrum k_G, and we will work with the bitheory

$$k_G^q(X;Y) = k_{-q}^G(X;Y)$$

on spaces X and Y. It can be defined as the cohomology of X with coefficients in the spectrum $Y \wedge k_G$. The following easy first reduction of Carlsson is a key step. It holds for both represented and pro-group valued theories. Let $\mathscr{P}$ be the family of proper subgroups of G.

LEMMA 4.1. Assume that k_H^* is e-invariant for all $H \in \mathscr{P}$. Then k_G^* is e-invariant if and only if $k_G^*(\tilde{E}\mathscr{P}) = 0$.

PROOF. Let X be e-contractible. We must show that $k_G^*(X) = 0$ if $k_G^*(\tilde{E}\mathscr{P}) = 0$. Write $Y = \tilde{E}\mathscr{P}$. Then $Y^G = S^0$ and Y is H-contractible for $H \in \mathscr{P}$. Let $Z = Y/S^0$. We have the cofiber sequence

$$X \longrightarrow X \wedge Y \longrightarrow X \wedge Z.$$

We claim that $k_G^*(W \wedge Y) = 0$ for any G-CW complex W and that $k_G^*(X \wedge Z) = 0$ for any G-CW complex Z such that $Z^G = *$. The first claim holds by hypothesis on orbit types G/G and holds trivially on orbit types G/H with $H \in \mathscr{P}$. The second claim holds on orbits by the induction hypothesis. The general cases of both claims follow. □

The cofiber sequence $EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$ gives rise to a long exact sequence

$$(4.2) \quad \longrightarrow k_G^q(Y; EG_+) \longrightarrow k_G^q(Y) \longrightarrow k_G^q(Y; \tilde{E}G) \xrightarrow{\delta} k_G^{q+1}(Y; EG_+) \longrightarrow .$$

The $\tilde{E}G$ terms carry the singular part of the problem; the EG_+ terms carry the free part.

Let us agree once and for all that all of our theories are to be understood as pro-group valued and completed at p, since that is the form of the theorem we need to prove. We must show that $\pi_G^*(Y) = 0$. However, studying more general theories allows a punch line in the elementary Abelian case: there the map δ in (4.2) is proven to be an isomorphism by comparison with a theory for which the analogue of the Segal conjecture holds trivially.

For a normal subgroup K of H with quotient group J write $k_{H/K}^* = k_J^*$ for the theory represented by $\Phi^K(k_H)$, where k_H denotes k_G regarded as an H-spectrum. We pointed out the ambiguity of the notation k_J^* at the end of XVI§6, but we also observed there that the notation π_J^* is correct and unambiguous. As we shall