Our results on G-CW approximation of G-spaces and on cellular approximation of G-maps imply that these are well-defined functors on the category $\bar{h}G\mathscr{U}$. Similarly, we can approximate any pair (X, A) by a G-CW pair $(\Gamma X, \Gamma A)$. Less obviously, if $(X; A, B)$ is an excisive triad, so that X is the union of the interiors of A and B, we can approximate $(X; A, B)$ by a triad $(\Gamma X; \Gamma A, \Gamma B)$, where ΓX is the union of its subcomplexes ΓA and ΓB.

That is all there is to the construction of ordinary equivariant homology and cohomology groups satisfying the evident equivariant versions of the Eilenberg-Steenrod axioms.

G. Bredon. Equivariant cohomology theories. Springer Lecture Notes in Mathematics Vol 34. 1967.

S. Illman. Equivariant singular homology and cohomology. Memoirs Amer. Math. Soc. No. 156. 1975.

S. J. Willson. Equivariant homology theories on G-complexes. Trans. Amer. Math. Soc. 212(1975), 155-271.

5. Obstruction theory

Obstruction theory works exactly as it does nonequivariantly, and I'll just give a quick sketch. Fix $n \geq 1$. Recall that a connected space X is said to be n-simple if $\pi_1(X)$ is Abelian and acts trivially on $\pi_q(X)$ for $q \leq n$. Let (X, A) be a relative G-CW complex and let Y be a G-space such that Y^H is non-empty, connected, and n-simple if H occurs as an isotropy subgroup of $X \setminus A$. Let $f : X^n \cup A \longrightarrow Y$ be a G-map. We ask when f can be extended to X^{n+1}. Composing the attaching maps $G/H \times S^n \longrightarrow X$ of cells of $X \setminus A$ with f gives elements of $\pi_n(Y^H)$. These elements specify a well-defined cocycle

$$c_f \in C_G^{n+1}(X, A; \underline{\pi}_n(Y)),$$

and f extends to X^{n+1} if and only if $c_f = 0$. If f and f' are maps $X^n \cup A \longrightarrow Y$ and h is a homotopy rel A of the restrictions of f and f' to $X^{n-1} \cup A$, then f, f', and h together define a map

$$h(f, f') : (X \times I)^n \longrightarrow Y.$$

Applying $c_{h(f,f')}$ to cells $j \times I$, we obtain a deformation cochain

$$d_{f,f',h} \in C_G^n(X, A; \underline{\pi}_n(Y))$$

such that $\delta d_{f,f',h} = c_f - c_{f'}$. Moreover, given f and d, there exists f' that coincides with f on X^{n-1} and satisfies $d_{f,f'} = d$, where the constant homotopy h is understood. This gives the first part of the following result, and the second part is similar.

THEOREM 5.1. For $f : X^n \cup A \longrightarrow Y$, the restriction of f to $X^{n-1} \cup A$ extends to a map $X^{n+1} \cup A \longrightarrow Y$ if and only if $[c_f] = 0$ in $H^{n+1}_G(X, A; \underline{\pi}_n(Y))$. Given maps $f, f' : X^n \longrightarrow Y$ and a homotopy rel A of their restrictions to $X^{n-1} \cup A$, there is an obstruction in $H^n_G(X, A; \underline{\pi}_n(Y))$ that vanishes if and only if the restriction of the given homotopy to $X^{n-2} \cup A$ extends to a homotopy $f \simeq f'$ rel A.

G. Bredon. Equivariant cohomology theories. Springer Lecture Notes in Mathematics Vol 34. 1967.

6. Universal coefficient spectral sequences

While easy to define, Bredon cohomology is hard to compute. However, we do have universal coefficient spectral sequences, which we describe next.

Let W_0H be the component of the identity element of WH and define a coefficient system $\underline{J}_*(X)$ by

$$\underline{J}_*(X)(G/H) = \underline{H}_*(X^H/W_0H; \mathbb{Z}). \tag{6.1}$$

Thus $\underline{J}_*(X)$ coincides with the obvious coefficient system $\underline{H}_*(X)$ if G is discrete. We claim that, if G is a compact Lie group, then $\underline{J}_*(X)$ is the coefficient system that is obtained by taking the homology of $\underline{C}_*(X)$. The point is that a Lie theoretic argument shows that

$$\pi_0((G/K)^H) \cong (G/K)^H/W_0H.$$

We deduce that the filtration of X^H/W_0H induced by the filtration of X gives rise to the chain complex $\underline{C}_*(X)(G/H)$.

We can construct an injective resolution Q^* of the coefficient system M and form $\mathrm{Hom}_{\mathscr{G}}(\underline{C}_*(X), Q^*)$. This is a bicomplex with total differential the sum of the differentials induced by those of $\underline{C}_*(X)$ and of Q^*. It admits two filtrations. Using one of them, the differential on E_0 comes from the differential on Q^*, and $E_1^{p,q}$ is $\mathrm{Ext}^{p,q}_{\mathscr{G}}(\underline{C}_*(X), M)$. Since $\underline{C}_*(X)$ is projective, the higher Ext groups are zero, and E_1 reduces to $C^*_G(X; M)$. Thus $E_2 = E_\infty = H^*_G(X; M)$, and our bicomplex computes Bredon cohomology. Filtering the other way, the differential on E_0 comes from the differential on $\underline{C}_*(X)$, and we can identify E_2. Using a projective resolution of N, we obtain an analogous homology spectral sequence.

THEOREM 6.2. Let G be either discrete or a compact Lie group and let X be a G-CW complex. There are universal coefficient spectral sequences

$$E_2^{p,q} = \mathrm{Ext}_{\mathscr{G}}^{p,q}(\underline{J}_*(X), M) \Longrightarrow H_G^n(X; M)$$

and

$$E_{p,q}^2 = \mathrm{Tor}^{\mathscr{G}}_{p,q}(\underline{J}_*(X), N) \Longrightarrow H_n^G(X; N).$$

We should say something about change of groups and about products in cohomology, but it would take us too far afield to go into detail. For the first, we simply note that, for $H \subset G$, we can obtain H-coefficient systems from G-coefficient systems via the functor $\mathscr{H} \to \mathscr{G}$ that sends H/K to $G/K = G \times_H H/K$. For the second, we note that, for groups H and G, projections give a functor from the orbit category of $H \times G$ to the product of the orbit categories of H and of G, so that we can tensor an H-coefficient system and a G-coefficient system to obtain an $(H \times G)$-coefficient system. When $H = G$, we can then apply change of groups to the diagonal inclusion $G \subset G \times G$. The resulting pairings of coefficient systems allow us to define cup products exactly as in ordinary cohomology, using cellular approximations of the diagonal maps of G-CW complexes.

G. Bredon. Equivariant cohomology theories. Springer Lecture Notes in Mathematics Vol 34. 1967.

S. J. Willson. Equivariant homology theories on G-complexes. Trans. Amer. Math. Soc. 212(1975), 155-271.

CHAPTER II

Postnikov Systems, Localization, and Completion

1. Eilenberg-MacLane G-spaces and Postnikov systems

Let M be a coefficient system. An Eilenberg-Mac Lane G-space $K(M,n)$ is a G-space of the homotopy type of a G-CW complex such that

$$\underline{\pi}_q(K(M,n)) = \begin{cases} M & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

While our interest is in Abelian group-valued coefficient systems, we can allow M to be set-valued if $n = 0$ and group-valued if $n = 1$. I will give an explicit construction later. Ordinary cohomology theories are characterized by the usual axioms, and, by checking the axioms, it is easily verified that the reduced cohomology of based G-spaces X is represented in the form

$$\tilde{H}^n_G(X; M) \cong [X, K(M,n)]_G, \tag{1.1}$$

where homotopy classes of based maps (in $\bar{h}G\mathscr{T}$) are understood.

Recall that a connected space X is said to be simple if $\pi_1 A$ is Abelian and acts trivially on $\pi_n(X)$ for $n \geq 2$. More generally, a connected space X is said to be nilpotent if $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_n(X)$ for $n \geq 2$. A G-connected G-space X is said to be simple if each X^H is simple. A G-connected G-space X is said to be nilpotent if each X^H is nilpotent and, for each $n \geq 1$, the orders of nilpotency of the $\pi_1(X^H)$-groups $\pi_n(X^H)$ have a common bound. We shall restrict our sketch proofs to simple G-spaces, for simplicity, in the next few sections, but everything that we shall say about their Postnikov towers and about localization and completion generalizes readily to the case of nilpotent G-spaces. The only difference is that each homotopy group system must be built up in finitely many stages, rather than all at once.

A Postnikov system for a based simple G-space X consists of based G-maps

$$\alpha_n : X \longrightarrow X_n \quad \text{and} \quad p_{n+1} : X_{n+1} \longrightarrow X_n$$

for $n \geq 0$ such that X_0 is a point, α_n induces an isomorphism $\underline{\pi}_q(X) \longrightarrow \underline{\pi}_q(X_n)$ for $q \leq n$, $p_{n+1}\alpha_{n+1} = \alpha_n$, and p_{n+1} is the G-fibration induced from the path space fibration over a $K(\underline{\pi}_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \longrightarrow K(\underline{\pi}_{n+1}(X), n+2)$. It follows that $X_1 = K(\underline{\pi}_1(X), 1)$ and that $\underline{\pi}_q(X_n) = 0$ for $q > n$. Our requirement that Eilenberg-Mac Lane G-spaces have the homotopy types of G-CW complexes ensures that each X_n has the homotopy type of a G-CW complex. The maps α_n induce a weak equivalence $X \longrightarrow \lim X_n$, but the inverse limit generally will not have the homotopy type of a G-CW complex. Just as nonequivariantly, the k-invariants that specify the tower are to be regarded as cohomology classes

$$k^{n+2} \in H_G^{n+2}(X_n; \underline{\pi}_{n+1}(X)).$$

These classes together with the homotopy group systems $\underline{\pi}_n(X)$ specify the weak homotopy type of X. On passage to H-fixed points, a Postnikov system for X gives a Postnikov system for X^H. We outline the proof of the following standard result since there is no complete published proof and my favorite nonequivariant proof has also not been published. The result generalizes to nilpotent G-spaces.

THEOREM 1.2. *A simple G-space X of the homotopy type of a G-CW complex has a Postnikov tower.*

PROOF. Assume inductively that $\alpha_n : X \longrightarrow X_n$ has been constructed. By the homotopy excision theorem applied to fixed point spaces, we see that the cofiber $C(\alpha_n)$ is $(n+1)$-connected and satisfies

$$\underline{\pi}_{n+2}(C\alpha_n) = \underline{\pi}_{n+1}(X).$$

More precisely, the canonical map $F(\alpha_n) \longrightarrow \Omega C(\alpha_n)$ induces an isomorphism on $\underline{\pi}_q$ for $q \leq n+1$. We construct

$$j : C(\alpha_n) \longrightarrow K(\underline{\pi}_{n+1}(X), n+2)$$

by inductively attaching cells to $C(\alpha_n)$ to kill its higher homotopy groups. We take the composite of j and the inclusion $X_n \subset C(\alpha_n)$ to be the k-invariant k^{n+2}. By our definition of a Postnikov tower, X_{n+1} must be the homotopy fiber of k^{n+2}. Its points are pairs (ω, x) consisting of a path $\omega : I \longrightarrow K(\underline{\pi}_{n+1}(X), n+2)$ and a point $x \in X_n$ such that $\omega(0) = *$ and $\omega(1) = k^{n+2}(x)$. The map $p_{n+1} : X_{n+1} \longrightarrow X_n$ is given by $p_{n+1}(\omega, x) = x$, and the map $\alpha_{n+1} : X \longrightarrow X_{n+1}$ is given by $\alpha_{n+1}(x) = (\omega(x), \alpha_n(x))$, where $\omega(x)(t) = j(x, 1-t)$, $(x, 1-t)$ being a point on the cone $CX \subset C(\alpha_n)$. Clearly $p_{n+1}\alpha_{n+1} = \alpha_n$. It is evident that α_{n+1} induces an isomorphism on $\underline{\pi}_q$ for $q \leq n$, and a diagram chase shows that this also holds for $q = n+1$. $\square$

2. Summary: localizations of spaces

Nonequivariantly, localization at a prime p or at a set of primes T is a standard first step in homotopy theory. We quickly review some of the basic theory. Say that a map $f : X \longrightarrow Y$ is a T-cohomology isomorphism if

$$f^* : H^*(Y; A) \longrightarrow H^*(X; A)$$

is an isomorphism for all T-local Abelian groups A.

THEOREM 2.1. The following properties of a nilpotent space Z are equivalent. When they hold, Z is said to be T-local.

(a) Each $\pi_n(Z)$ is T-local.
(b) If $f : X \longrightarrow Y$ is a T-cohomology isomorphism, then $f^* : [Y, Z] \longrightarrow [X, Z]$ is a bijection.
(c) The integral homology of Z is T-local.

THEOREM 2.2. Let X be a nilpotent space. The following properties of a map $\lambda : X \longrightarrow X_T$ from X to a T-local space X_T are equivalent. There is one and, up to homotopy, only one such map λ. It is called the localization of X at T.

(a) $\lambda^* : [X_T, Z] \longrightarrow [X, Z]$ is a bijection for all T-local spaces Z.
(b) λ is a T-cohomology isomorphism.
(c) $\lambda_* : \pi_*(X) \longrightarrow \pi_*(X_T)$ is localization at T.
(d) $\lambda_* : H_*(X; \mathbb{Z}) \rightarrow H_*(X_T; \mathbb{Z})$ is localization at T.

A. K. Bousfield and D. M. Kan. Homotopy limits, completions, and localizations. Springer Lecture Notes in Mathematics Vol 304. 1972.
D. Sullivan. The genetics of homotopy theory and the Adams conjecture. Annals of Math. 100(1974), 1-79.

3. Localizations of G-spaces

Now let G be a compact Lie group. Say that a G-map $f : X \longrightarrow Y$ is a T-cohomology isomorphism if

$$f^* : H^*_G(Y; M) \longrightarrow H^*_G(X; M)$$

is an isomorphism for all T-local coefficient systems M.

THEOREM 3.1. The following properties of a nilpotent G-space Z are equivalent. When they hold, Z is said to be T-local.

(a) Each Z^H is T-local.
(b) If $f : X \longrightarrow Y$ is a T-cohomology isomorphism, then

$$f^* : [Y, Z]_G \longrightarrow [X, Z]_G$$

is a bijection.

THEOREM 3.2. Let X be a nilpotent G-space. The following properties of a map $\lambda : X \longrightarrow X_T$ from X to a T-local G-space X_T are equivalent. There is one and, up to homotopy, only one such map λ. It is called the localization of X at T.

(a) $\lambda^* : [X_T, Z] \longrightarrow [X, Z]$ is a bijection for all T-local G-spaces Z.
(b) λ is a T-cohomology isomorphism.
(c) Each $\lambda^H : X^H \longrightarrow (X_T)^H$ is localization at T.

PROOFS. We restrict attention to simple G-spaces. Assuming (a) in Theorem 3.1, we may replace Z by a weakly equivalent Postnikov tower and we may assume that the G-spaces X and Y given in (b) are G-CW complexes, so that we are dealing with actual homotopy classes of maps. Then (a) implies (b) by a word-for-word dualization of our proof of the Whitehead theorem. Conversely, (b) implies (a) since the specialization of (b) to T-cohomology isomorphisms of the form $G/H_+ \wedge f$, where $f : X \longrightarrow Y$ is a nonequivariant T-cohomology isomorphism, implies (b) of Theorem 2.1. In Theorem 3.2, (a) implies (b) by letting Z run through $K(M, n)$'s, and (b) implies (a) by Theorem 3.1. Let $\mathbb{Z}_T$ be the localization of $\mathbb{Z}$ at T. One sees that (c) implies (b) by applying the universal coefficient spectral sequence of I.6.2, taken with homology and coefficient systems tensored with $\mathbb{Z}_T$. The maps λ^H induce isomorphisms on homology with coefficients in $\mathbb{Z}_T$, and one can deduce (with some work in the general compact Lie case) that they therefore induce an isomorphism $\underline{J}_*(X; \mathbb{Z}_T) \longrightarrow \underline{J}_*(X_T; \mathbb{Z}_T)$. Since the universal property (a) implies uniqueness, to complete the proof we need only construct a map λ that satisfies (c). For this, we may assume that X is a Postnikov tower, and we localize its terms inductively by localizing k-invariants and comparing fibration sequences. The starting point is just the observation that the algebraic localization $M \longrightarrow M_T = M \otimes \mathbb{Z}_T$ of coefficient systems induces localization maps $\lambda : K(M,n) \longrightarrow K(M_T,n)$. The relevant diagram is:

$$\begin{tikzcd}
K(\underline{\pi}_{n+1}(X), n+1) \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & K(\underline{\pi}_{n+1}(X), n+2) \arrow[d] \\
K(\underline{\pi}_{n+1}(X)_T, n+1) \arrow[r] & (X_{n+1})_T \arrow[r] & (X_{n+1})_T \arrow[r] & K(\underline{\pi}_{n+1}(X)_T, n+2).
\end{tikzcd}$$

We construct the right square by localizing the k-invariant, we define $(X_{n+1})_T$ to be the fiber of the localized k-invariant, and we obtain $X_{n+1} \longrightarrow (X_{n+1})_T$ making the middle square commute and the left square homotopy commute by standard fiber sequence arguments. $\square$

J. P. May. The dual Whitehead theorems. London Math. Soc. Lecture Note Series Vol 86, 1983, 46-54.

J. P. May, J. McClure, and G. V. Triantafillou. Equivariant localization. Bull. London Math. Soc. 14(1982), 223-230.

4. Summary: completions of spaces

Completion at a prime p or at a set of primes T is another standard first step in homotopy theory. Since completion at T is the product of the completions at p for $p \in T$, we restrict to the case of a single prime. We first review some of the nonequivariant theory. The algebra we begin with is a preview of algebra to come later in our discussion of completions of G-spectra at ideals of the Burnside ring.

The p-adic completion functor, $\hat{A}_p = \lim(A/p^n A)$, is neither left nor right exact in general, and it has left derived functors L_0 and L_1. If

$$0 \longrightarrow F' \longrightarrow F \longrightarrow A \longrightarrow 0$$

is a free resolution of A, then $L_0 A$ and $L_1 A$ are the cokernel and kernel of $\hat{F'}_p \longrightarrow \hat{F}_p$, and there results a natural map $\eta : A \longrightarrow L_0 A$. The higher left derived functors are zero, and a short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

gives rise to a six term exact sequence

$$0 \longrightarrow L_1 A' \longrightarrow L_1 A \longrightarrow L_1 A'' \longrightarrow L_0 A' \longrightarrow L_0 A \longrightarrow L_0 A'' \longrightarrow 0.$$

If $L_1A = 0$, then we call $\eta : A \longrightarrow L_0A$ the "p-completion" of A. It must not to be confused with the p-adic completion. We say that A is "p-complete" if $L_1A = 0$ and η is an isomorphism. The groups L_0A, L_1A, and $\hat{A}_p$ are p-complete for any Abelian group A. While derived functors give the best conceptual descriptions of L_0A and L_1A, there are more easily calculable descriptions. Let $\mathbb{Z}/p^\infty$ be the colimit of the sequence of homomorphisms $p : \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n+1}$. Then $\mathbb{Z}/p^\infty \cong \mathbb{Z}[p^{-1}]/\mathbb{Z}$ and there are natural isomorphisms

$$L_0(A) \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \quad \text{and} \quad L_1(A) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, A).$$

There is also a natural short exact sequence

$$0 \longrightarrow \lim{}^1 \mathrm{Hom}(\mathbb{Z}/p^n, A) \longrightarrow L_0A \longrightarrow \hat{A}_p \longrightarrow 0.$$

In particular, $L_1A = 0$ and $L_0A \cong \hat{A}_p$ if the p-torsion of A is of bounded order.

Say that a map $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism if

$$f^* : H^*(Y; A) \longrightarrow H^*(X; A)$$

is an isomorphism for all p-complete Abelian groups A. This holds if and only if it holds for all $\mathbb{F}_p$-vector spaces A, and this in turn holds if and only if $f_* : H_*(X; \mathbb{F}_p) \longrightarrow H_*(Y; \mathbb{F}_p)$ is an isomorphism, where $\mathbb{F}_p$ is the field with p elements. While this homological characterization is essential to our proofs, we prefer to emphasize cohomology.

Theorem 4.1. The following properties of a nilpotent space Z are equivalent. When they hold, Z is said to be p-complete.

(a) Each $\pi_n(Z)$ is p-complete.
(b) If $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism, then $f^* : [Y, Z] \longrightarrow [X, Z]$ is a bijection.

Theorem 4.2. Let X be a nilpotent space. The following properties of a map $\gamma : X \longrightarrow \hat{X}_p$ from X to a p-complete space $\hat{X}_p$ are equivalent. There is one and, up to homotopy, only one such map γ. It is called the completion of X at p.

(a) $\gamma^* : [\hat{X}_p, Z] \longrightarrow [X, Z]$ is a bijection for all p-complete spaces Z.
(b) γ is a $\hat{p}$-cohomology isomorphism.

For $n \geq 1$, there is a natural and splittable short exact sequence

$$0 \longrightarrow L_0\pi_n(X) \longrightarrow \pi_n(\hat{X}_p) \longrightarrow L_1\pi_{n-1}(X) \longrightarrow 0.$$

If $L_1\pi_*(X) = 0$, then γ is also characterized by

(c) $\gamma_* : \pi_*(X) \longrightarrow \pi_*(\hat{X}_p)$ is completion at p.

A. K. Bousfield and D. M. Kan. Homotopy limits, completions, and localizations. Springer Lecture Notes in Mathematics, Vol. 304. 1972.
D. Sullivan. The genetics of homotopy theory and the Adams conjecture. Annals of Math. 100(1974), 1-79.

5. Completions of G-spaces

Now let G be a compact Lie group. Say that a G-map $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism if

$$f^* : H^*_G(Y; M) \longrightarrow H^*_G(X; M)$$

is an isomorphism for all p-complete coefficient systems M. This will hold if each $f^H : X^H \longrightarrow Y^H$ is a $\hat{p}$-cohomology isomorphism by another application of the universal coefficients spectral sequence, with a little work in the general compact Lie case to handle $\underline{J}_*(f)$.

THEOREM 5.1. The following properties of a nilpotent G-space Z are equivalent. When they hold, Z is said to be p-complete.

(a) Each Z^H is p-complete.
(b) If $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism, then $f^* : [Y, Z]_G \longrightarrow [X, Z]_G$ is a bijection.

THEOREM 5.2. Let X be a nilpotent G-space. The following properties of a map $\gamma : X \longrightarrow \hat{X}_p$ from X to a p-complete G-space $\hat{X}_p$ are equivalent. There is one and, up to homotopy, only one such map γ. It is called the completion of X at p.

(a) $\gamma^* : [\hat{X}_p, Z] \longrightarrow [X, Z]$ is a bijection for all p-complete G-spaces Z.
(b) γ is a $\hat{p}$-cohomology isomorphism.
(c) Each $\gamma^H : X^H \longrightarrow (\hat{X}_p)^H$ is completion at p.

For $n \geq 1$, there is a natural short exact sequence

$$0 \longrightarrow L_0\underline{\pi}_n(X) \longrightarrow \underline{\pi}_n(\hat{X}_p) \longrightarrow L_1\underline{\pi}_{n-1}(X) \longrightarrow 0.$$

PROOFS. The proofs are the same as those of Theorems 3.1 and 3.2, except that completions of Eilenberg-Mac Lane G-spaces are not Eilenberg-Mac Lane G-spaces in general. For a coefficient system M, $\eta : M \longrightarrow L_0M$ induces p-completions $K(M, n) \longrightarrow K(L_0M, n)$ whenever $L_1M = 0$. For the general case, let FM be the coefficient system obtained by applying the free Abelian group functor to M regarded as a set-valued functor. There results a natural epimorphism $FM \longrightarrow M$ of coefficient systems. Let $F'M$ be its kernel. Since L_1 vanishes on free modules, we can construct the completion of $K(M,n)$ at p via the following diagram of fibrations:

$$\begin{tikzcd}
K(FM,n) \arrow[r] \arrow[d] & K(M,n) \arrow[r] \arrow[d] & K(F'M,n+1) \arrow[r] \arrow[d] & K(FM,n+1) \arrow[d] \\
K(L_0FM,n) \arrow[r] & K(M,n)^\wedge_p \arrow[r] & K(L_0F'M,n+1) \arrow[r] & K(L_0FM,n+1).
\end{tikzcd}$$

That is, $K(M,n)^\wedge_p$ is the fiber of $K(L_0F'M,n+1) \longrightarrow K(L_0FM,n+1)$. It is complete since its homotopy group systems are complete. The map $K(M,n) \longrightarrow K(M,n)^\wedge_p$ is a $\hat{p}$-cohomology isomorphism because its fixed point maps are so, by the Serre spectral sequence. $\square$

J. P. May. Equivariant completion. Bull. London Math. Soc. 14(1982), 231-237.

CHAPTER III

Equivariant Rational Homotopy Theory

by Georgia Triantafillou

1. Summary: the theory of minimal models

Let G be a finite group. In this chapter, we summarize our work on the algebraicization of rational G-homotopy theory.

To simplify the statements we assume simply connected spaces throughout the chapter. The theory can be extended to the nilpotent case in a straightforward manner. We recall that by rationalizing a space X, we approximate it by a space X_0 the homotopy groups of which are equal to $\pi_*(X)\otimes\mathbb{Q}$, thus neglecting the torsion. The advantage of doing so is that rational homotopy theory is determined completely by algebraic invariants, as was shown by Quillen and later by Sullivan. Our theory is analogous to Sullivan's theory of minimal models, which we now review. For our purposes we prefer Sullivan's approach because of its computational advantage and its relation to geometry by use of differential forms.

The algebraic invariants that determine the rational homotopy type are certain algebras that we call DGA's. By definition a DGA is a graded, commutative, associative algebra with unit over the rationals, with differential $d : A^n \to A^{n+1}$ for $n \geq 0$. We say that A is *connected* if $H^0(A) = \mathbb{Q}$ and *simply connected* if, in addition, $H^1(A) = 0$. Again we assume that all DGA's in sight are connected and simply connected. A map of DGA's is said to be a *quasi-isomorphism* if it induces an isomorphism on cohomology.

Certain DGA's, the so called minimal ones, play a special role to be described below. A DGA $\mathcal{M}$ is said to be *minimal* if it is free and its differential is decomposable. Freeness means that $\mathcal{M}$ is the tensor product of a polynomial algebra generated by elements of even degree and an exterior algebra generated by elements of odd degree. Decomposability of the differential means that $d(\mathcal{M}) \subseteq \mathcal{M}^+ \wedge \mathcal{M}^+$, where $\mathcal{M}^+$ is the set of positive degree elements of $\mathcal{M}$.

There is an algebraic notion of homotopy between maps of DGA's that mirrors the topological notion. Let $\mathbb{Q}(t, dt)$ be the free DGA on two generators t and dt of degree 0 and 1 respectively with $d(t) = dt$.

Definition 1.1. Two morphisms $f, g : \mathcal{A} \to \mathcal{B}$ are homotopic if there is a map $H : \mathcal{A} \to \mathcal{B} \otimes \mathbb{Q}(t, dt)$ such that $e_0 \circ H = f$ and $e_1 \circ H = g$, where e_0 is the projection $t = 0, dt = 0$ and e_1 the projection $t = 1, dt = 0$.

The basic example of a DGA in the theory is the PL De Rham algebra $\mathcal{E}_X$ of a simplicial complex X, which is constructed as follows. Let

$$\sigma^n = \Delta^n = \{(t_0, t_1, \dots, t_n) | 0 \leq t_i \leq 1, \Sigma_{i=0}^n t_i = 1\}$$

be an n-simplex of X canonically embedded in $\mathbb{R}^{n+1}$. A polynomial form of degree p on σ^n is an expression

$$\sum_I f_I(t_0, \dots, t_n) dt_{i_1} \wedge \dots \wedge dt_{i_p},$$

where $I = \{i_1, \dots, i_p\}$ and f_I is a polynomial with coefficients in $\mathbb{Q}$. A global PL (piecewise linear) form on X is a collection of polynomial forms, one for each simplex of X, which coincide on common faces. The set of PL forms of X is the DGA $\mathcal{E}_X$. A version of the classical de Rham theorem holds, namely that

$$H^*(\mathcal{E}_X) = H^*(X; \mathbb{Q}).$$

We have the following facts.

Theorem 1.2. A quasi-isomorphism between minimal DGA's is an isomorphism.

Theorem 1.3. If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of DGA's and $\mathcal{M}$ is a minimal DGA, then $f_* : [\mathcal{M}, \mathcal{A}] \to [\mathcal{M}, \mathcal{B}]$ is an isomorphism.

Theorem 1.4. For any simply connected DGA $\mathcal{A}$ there is a minimal DGA $\mathcal{M}$ and a quasi-isomorphism $\rho : \mathcal{M} \to \mathcal{A}$. Moreover $\mathcal{M}$ is unique up to (non-canonical) isomorphism, namely if $\rho' : \mathcal{M}' \to \mathcal{A}$ is another quasi- isomorphism then there is an isomorphism $e : \mathcal{M} \to \mathcal{M}'$ such that $\rho' \circ e$ and ρ are homotopic.

Here $\mathcal{M}$ is said to be the *minimal model* of $\mathcal{A}$. The minimal model $\mathcal{M}_X$ of the PL de Rham algebra $\mathcal{E}_X$ of a simply connected space X is called the minimal model of X.

THEOREM 1.5. The correspondance $X \to \mathcal{M}_X$ induces a bijection between rational homotopy types of simplicial complexes on the one hand and isomorphism classes of minimal DGA's on the other.

More precisely, assuming X is a rational space, the homotopy groups $\pi_n(X)$ of X are isomorphic to $Q(\mathcal{M}_X)_n$, where $Q(\mathcal{M}) \equiv \mathcal{M}^+/\mathcal{M}^+ \wedge \mathcal{M}^+$ is the space of indecomposables of $\mathcal{M}$. The nth stage X_n of the Postnikov tower of X has $\mathcal{M}_X(n)$ as its minimal model, where $\mathcal{M}(n)$ denotes the subalgebra of $\mathcal{M}$ that is generated by the elements of degree $\leq n$. The k-invariant $k^{n+2} \in H^{n+2}(X_n, \pi_{n+1}(X))$, which can be represented as a map $\pi_{n+1}(X)^* \to H^{n+2}(X_n)$, is determined by the differential $d : Q(\mathcal{M}_X)_{n+1} \to H^{n+2}(\mathcal{M}_X(n))$. These properties enable the inductive construction of a rational space that realizes a given minimal algebra.

On the morphism level we have

THEOREM 1.6. If Y is a rational space then

$$[X, Y] \equiv [\mathcal{M}_Y, \mathcal{M}_X].$$

We warn here that the minimal model, though very useful computationally, is not a functor. In particular a map of spaces induces a map of the corresponding minimal models only up to homotopy.

D. Quillen. Rational Homotopy Theory, Ann of Math. 90(1968), 205-295.
D. Sullivan. Infinitesimal Computations in Topology, Publ. Math. IHES 47(1978), 269-332.

2. Equivariant minimal models

For finite groups G an analogous theory can be developed for G-rational homotopy types of G-simplicial complexes. For simplicity we assume throughout that the spaces X are G-connected and G-simply connected, which means that each fixed point space X^H is connected and simply connected; however, the theory works just as well for G-nilpotent spaces. In fact, by work of B. Fine, the theory can be extended in such a way that no fixed base point and no connectivity assumption on the fixed point sets are required.

Let Vec_G be the category of rational coefficient systems and Vec_G^* the category of covariant functors from $\mathcal{G}$ to rational vector spaces. Our invariants for determining

G-rational homotopy types are functors of a special type from $\mathcal{G}$ into DGA's, which we now describe.

DEFINITION 2.1. A system of DGA's is a covariant functor from $\mathcal{G}$ to simply connected DGA's such that the underlying functor in Vec_G^* is injective.

The injective objects of Vec_G^* or, equivalently, the projective rational coefficient systems can be characterized as follows.

THEOREM 2.2. (i) For $H \subseteq G$ and a WH-representation V, there is a projective coefficient system $\underline{V} \in Vec_G$ such that

$$\underline{V}(G/K) = \mathbb{Q}[(G/H)^K] \otimes_{\mathbb{Q}[WH]} V,$$

where the first factor is the vector space generated by the set $(G/H)^K$.
(ii) Every projective coefficient system is a direct sum of such $\underline{V}$'s.

The basic system of DGA's in the theory is the system of de Rham algebras $\mathcal{E}_{X^H}$ of the fixed point sets X^H of a G-simplicial complex X. We denote this system by $\underline{\mathcal{E}}_X$. It is crucial to realize that $\underline{\mathcal{E}}_X$ is injective and that this property is central to the theory. The injectivity of $\underline{\mathcal{E}}_X$ can be shown by utilising the splitting of X into its orbit types.

We note that $\mathcal{E}_X$ together with the induced G-action determine a minimal algebra equipped with a G-action. However there are in general many G-rational homotopy types of G-simplicial complexes that realize this minimal G-algebra. In order to have unique spacial realizations we need to take into account the algebraic data of all fixed point sets, which leads us to systems of DGA's.

Define the cohomology of a system $\mathcal{A}$ of DGA's with respect to a covariant coefficient system $N \in Vec_G^*$ to be the cohomology of the cochain complex $Hom_{\mathcal{G}}(N, \mathcal{A})$. An equivariant de Rham theorem follows by use of the universal coefficients spectral sequence.

THEOREM 2.3. For $M \in Vec_G$ with dual $M^* \in Vec_G^*$,

$$H_G^*(X; M) \equiv H_G^*(\underline{\mathcal{E}}_X; M^*).$$

The lack of functoriality of the minimal model of a space complicates the construction of equivariant minimal models. We cannot, for instance, define "the system of minimal models" $\mathcal{M}_{X^H}$ of the fixed point sets of a G-complex X. It turns out that the right definition of minimal models in the equivariant context is the following.

DEFINITION 2.4. A system $\mathcal{M}$ of DGA's is said to be minimal if

(i) each algebra $\mathcal{M}(G/H)$ is free commutative,
(ii) the DGA $\mathcal{M}(G/G)$ is minimal, and
(iii) the differential on each $\mathcal{M}(G/H)$ is decomposable when restricted to the intersection of the kernels of the maps $\mathcal{M}(G/H) \to \mathcal{M}(G/K)$ induced by proper inclusions $H \subset K$.

One can think of *(ii)* as an "initial condition" and of *(iii)* as the minimality condition that guarantees the uniqueness of equivariant minimal models. As in the nonequivariant case, minimal systems are classified by their cohomology.

THEOREM 2.5. A quasi-isomorphism between minimal systems of DGA's is an isomorphism.

Also, Theorems 1.3, 1.4, 1.5, and 1.6 have equivariant counterparts.

THEOREM 2.6. If $\mathcal{A}$ is a system of DGA's, then there is a quasi-isomorphism $f : \mathcal{M} \to \mathcal{A}$, where $\mathcal{M}$ is a minimal system. Moreover $\mathcal{M}$ is unique up to (non-canonical) isomorphism.

This result provides the existence of equivariant minimal models. Unlike the nonequivariant case the proof is rather involved and is based on a careful investigation of the universal coefficients spectral sequence. We define the *equivariant minimal model* $\mathcal{M}_X$ of a G-simplicial complex X to be the minimal system of DGA's that is quasi-isomorphic to the system of de Rham algebras $\underline{\mathcal{E}}_X$.

A notion of homotopy can be defined for systems of DGA's. If $\mathcal{A}$ is a system of DGA's we denote by $\mathcal{A} \otimes \mathbb{Q}(t, dt)$ the functor

$$\mathcal{A} \otimes \mathbb{Q}(t, dt)(G/H) = \mathcal{A}(G/H) \otimes \mathbb{Q}(t, dt).$$

It can be shown that this functor is injective and therefore it forms a system of DGA's. Homotopy of maps of systems of DGA's can now be defined in the obvious way suggested by the nonequivariant case. Let $[\mathcal{A}, \mathcal{B}]_G$ denote homotopy classes of maps of systems.

THEOREM 2.7. If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of systems of DGA's and $\mathcal{M}$ is a minimal system of DGA's, then

$$f^* : [\mathcal{M}, \mathcal{A}]_G \to [\mathcal{M}, \mathcal{B}]_G$$

is an isomorphism.

The equivariant minimal model determines the rational G-homotopy type of a G-space, namely

THEOREM 2.8. The correspondence $X \to \mathcal{M}_X$ induces a bijection between rational G-homotopy types of G-simplicial complexes on the one hand and isomorphism classes of minimal systems of DGA's on the other.

More precisely, there is a filtration of minimal subsystems of DGA's

$$\cdots \subseteq \mathcal{M}_X(n) \subseteq \mathcal{M}_X(n+1) \subseteq \cdots \subseteq \mathcal{M}_X$$

such that each stage is the equivariant minimal model of the equivariant Postnikov term X_n of the space X. The system of rational homotopy groups of the fixed point sets $\underline{\pi}_n(X) \otimes \mathbb{Q}$ and the rational equivariant k-invariants can also be read from the model $\mathcal{M}_X$. This makes the inductive construction of the Postnikov decomposition of the rationalization X_0 possible if the equivariant minimal model is given.

On the morphism level we have

THEOREM 2.9. If Y is a rational G-simplicial complex then there is a bijection

$$[X, Y] \equiv [\mathcal{M}_Y, \mathcal{M}_X].$$

G. Triantafillou. Äquivariante Rationale Homotopietheorie, Bonner Math. Schriften Vol. 110. 1978.

G. Triantafillou. Equivariant minimal models. Trans. Amer. Math. Soc. 274(1982), 509-532.

3. Rational equivariant Hopf spaces

In spite of the conceptual analogy of the equivariant theory to the nonequivariant one, the calculations in the equivariant case are much more subtle and can yield surprising results. We illustrate this by describing our work on rational Hopf G-spaces. It is a basic feature of nonequivariant homotopy theory that the rational Hopf spaces split as products of Eilenberg-Mac Lane spaces. The equivariant analogue is false. By a Hopf G-space we mean a based G-space X together with a G-map $X \times X \to X$ such that the base point is a two-sided unit for the product. Examples include Lie groups K with a G-action such that G is a finite subgroup of the inner automorphisms of K, and loop spaces $\Omega(X)$ of G-spaces based at a G-fixed point of X.

THEOREM 3.1. Let X be a G-connected rational Hopf G-space of finite type. If G is cyclic of prime power order, then X splits as a product of Eilenberg-Mac Lane G-spaces. If $G = \mathbb{Z}_p \times \mathbb{Z}_q$ for distinct primes p and q, then there are counterexamples to this statement.

Outline of proof: In this outline we suppress the technical part of the proof which is quite extensive. As in the nonequivariant case, the n^{th} term X_n of a Postnikov tower of X is a Hopf G-space. Moreover the k-invariant $k^{n+2} \in H^{n+2}(X_n; \underline{\pi}_{n+1}(X))$ is a primitive element. This means that

$$m^*(k^{n+2}) = (p_1)^*(k^{n+2}) + (p_2)^*(k^{n+2}),$$

in $H^{n+2}(X_n \times X_n; \underline{\pi}_{n+1}(X))$, where m is the product and the p_i are the projections.

The difference in the two cases $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_p \times \mathbb{Z}_q$ stems from the fact that rational coefficient systems for these groups have different projective dimensions. Indeed, systems for $\mathbb{Z}_{p^k}$ have projective dimension at most 1, whereas there are rational coefficient systems for $\mathbb{Z}_p \times \mathbb{Z}_q$ of projective dimension 2. Using this fact about $\mathbb{Z}_{p^k}$ we can compute inductively the equivariant minimal model of each Postnikov term X_n and its cohomology. In particular we show that all non-zero elements of $H^{n+2}_G(X_n; \underline{\pi}_{n+1}(X))$ are decomposable and therefore non-primitive.

In the case of $\mathbb{Z}_p \times \mathbb{Z}_q$ we construct counterexamples which are 2-stage Postnikov systems with primitive k-invariant. As in the nonequivariant case, if X has only two non-vanishing homotopy group systems, then the primitivity of the unique k-invariant is a sufficient condition for X to be a Hopf G-space. By construction, the two systems of homotopy groups $\underline{\pi}_n(X)$ and $\underline{\pi}_{n+1}(X)$ are as follows. The groups $\pi_n(X^H)$ are zero for all proper subgroups H and $\pi_n(X^{\mathbb{Z}_p \times \mathbb{Z}_q}) = \mathbb{Z}$. The groups $\pi_{n+1}(X^H)$ are zero for all nontrivial subgroups H and $\pi_{n+1}(X) = \mathbb{Z}$. The first coefficient system has projective dimension 2. This and the universal coefficients spectral sequence yields $H^{n+2}_G(X_n; \underline{\pi}_{n+1}(X)) = \mathbb{Z}$. Moreover all non-zero elements of this group are primitive. This gives an infinite choice of primitive k-invariants and therefore an infinite collection of rationally distinct Hopf G-spaces which do not split rationally into products of Eilenberg-Mac Lane G-spaces.

The counterexamples X constructed in the theorem are infinite loop G-spaces in the sense that there are G-spaces E_n and homotopy equivalences $E_n \to \Omega E_{n+1}$, with $X = E_0$. For the more sophisticated notion of infinite loop G-spaces where indexing over the representation ring of G is used, no such pathological behavior is possible.

As a final comment we mention that the theory of equivariant minimal models has been used by my collaborators and myself to obtain aplications of a more geometric nature, like the classification of a large class of G-manifolds up to finite ambiguity and the equivariant formality of G-Kähler manifolds.

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CHAPTER IV

Smith Theory

1. Smith theory via Bredon cohomology

We shall explain two approaches to the classical results of P.A. Smith. We begin with the statement. Let G be a finite p-group and let X be a finite dimensional G-CW complex such that $H^*(X;\mathbb{F}_p)$ is a finite dimensional vector space, where $\mathbb{F}_p$ denotes the field with p elements. All cohomology will have coefficients in $\mathbb{F}_p$ here.

THEOREM 1.1. If X is a mod p cohomology n-sphere, then X^G is empty or is a mod p cohomology m-sphere for some $m \leq n$. If p is odd, then $n-m$ is even and X^G is non-empty if n is even.

If H is a non-trivial normal subgroup of G, then $X^G = (X^H)^{G/H}$. By induction on the order of G, Theorem 1.1 will be true in general if it is true when $G = \mathbb{Z}/p$ is the cyclic group of order p. Our first proof is an almost trivial exercise in the use of Bredon cohomology. We restrict attention to $G = \mathbb{Z}/p$, but we do not assume that X is a mod p cohomology sphere until we put things together at the end. Observe that an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of coefficient systems give rise to a long exact sequence

(1.2)
$$\cdots \longrightarrow H^q_G(X;L) \longrightarrow H^q_G(X;M) \longrightarrow H^q_G(X;N) \longrightarrow H^{q+1}_G(X;L) \longrightarrow \cdots .$$

Let $FX = X/X^G$. The action of G on FX is free away from the basepoint. There are coefficient systems L, M, and N such that

$$H^q_G(X;L) \cong \tilde{H}^q(FX/G),$$
$$H^q_G(X;M) \cong H^q(X),$$

and

$$H^q_G(X;N) \cong H^q(X^G).$$

To determine L, M, and N, we need only calculate the right sides when $q = 0$ and X is an orbit, that is, $X = G$ or $X = *$. We find:

$$\begin{array}{rl} L(G) = \mathbb{F}_p & L(*) = 0 \\ M(G) = \mathbb{F}_p[G] & M(*) = \mathbb{F}_p \\ N(G) = 0 & N(*) = \mathbb{F}_p. \end{array}$$

Let I be the augmentation ideal of the group ring $\mathbb{F}_p[G]$, and let I^n denote both the n^{th} power of I and the coefficient system whose value on G is I^n and whose value on $*$ is zero. Then $I^{p-1} = L$. It is easy to check that we have exact sequences of coefficient systems

$$0 \longrightarrow I \longrightarrow M \longrightarrow L \oplus N \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow M \longrightarrow I \oplus N \longrightarrow 0.$$

These exact sequences coincide if $p = 2$. By (1.2), they give rise to long exact sequences

$$\cdots \longrightarrow H^q_G(X;I) \longrightarrow H^q(X) \longrightarrow \tilde{H}^q(FX/G) \oplus H^q(X^G) \longrightarrow H^{q+1}_G(X;I) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \tilde{H}^q(FX/G) \longrightarrow H^q(X) \longrightarrow H^q_G(X;I) \oplus H^q(X^G) \longrightarrow \tilde{H}^{q+1}(FX/G) \longrightarrow \cdots .$$

Define

$$a_q = \dim \tilde{H}^q(FX/G),\ \ \bar{a}_q = \dim H^q_G(X;I),\ \ b_q = \dim H^q(X),\ \ c_q = \dim H^q(X^G).$$

Note that $a_q = \bar{a}_q$ if $p = 2$. We read off the inequalities

$$a_q + c_q \leq b_q + \bar{a}_{q+1} \quad \text{and} \quad \bar{a}_q + c_q \leq b_q + a_{q+1}.$$

Iteratively, these imply the following inequality for $q \geq 0$ and $r \geq 0$.

(1.3) $$a_q + c_q + c_{q+1} + \cdots + c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1},$$

where r is odd if $p > 2$. In particular, with $q = 0$ and r large,

$$\sum c_q \leq \sum b_q. \tag{1.4}$$

Using the further short exact sequences

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow L \longrightarrow 0, 1 \leq n \leq p-1,$$

we can also read off the the Euler characteristic formula

$$\chi(X) = \chi(X^G) + p\tilde{\chi}(FX/G). \tag{1.5}$$

First proof of Theorem 1.1. Here $\sum b_q = 2$, hence $\sum c_q \leq 2$. The case $\sum c_q = 1$ is ruled out by the congruence $\chi(X) \equiv \chi(X^G) \mod p$; when $p > 2$, this congruence also implies that $n - m$ is even and that X^G is non-empty if n is even. Taking $q = n+1$ and r large in (1.3), we see that m cannot be greater than n. □

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2. Borel cohomology, localization, and Smith theory

Let EG be a free contractible G-space. For a G-space X, the Borel construction on X is the orbit space $EG \times_G X$ and the Borel homology and cohomology of X (with coefficients in an Abelian group A) are defined to be the nonequivariant homology and cohomology of this space. For reasons to be made clear later, the Borel construction is also called the "homotopy orbit space" and is sometimes denoted X_{hG}. People not focused on equivariant algebraic topology very often refer to Borel cohomology as "equivariant cohomology." We can relate it to Bredon cohomology in a simple way. Let $\underline{A}$ denote the constant coefficient system at A. Since the orbit spaces $(G/H)/G$ are points, we see immediately from the axioms that $H^*_G(X;\underline{A})$ is isomorphic to $H^*(X/G;A)$, and similarly in homology. Therefore

$$H_*(EG \times_G X; A) \cong H^q_G(EG \times X; \underline{A}) \quad \text{and} \quad H^*(EG \times_G X; A) \cong H^*_G(EG \times X; \underline{A}).$$

Observe that the Borel cohomology of a point is the cohomology of the classifying space $BG = EG/G$. In this section, we shall use the notation

$$H^*_G(X) = H^*(EG \times_G X),$$

standard in much of the literature.

Here we fix a prime p and understand mod p coefficients. If X is a based G-space, we let $\tilde{H}^*_G(X)$ be the kernel of $H^*_G(X) \longrightarrow H^*_G(*) = H^*(BG)$. Equivalently,

$$\tilde{H}^*_G(X) = \tilde{H}^*(EG_+ \wedge_G X).$$

Because G acts freely on EG, it acts freely on $EG \times X$. Therefore, by the Whitehead theorem, if $f : X \longrightarrow Y$ is a G-map between G-CW complexes that is a nonequivariant homotopy equivalence, then

$$1 \times f : EG \times X \longrightarrow EG \times Y$$

is a G-homotopy equivalence and therefore

$$1 \times_G f : EG \times_G X \longrightarrow EG \times_G Y$$

is a homotopy equivalence. At first sight, it seems unreasonable to expect $EG \times_G X$ to carry much information about X^G, but it does.

We now assume that G is an elementary Abelian p-group, $G = (\mathbb{Z}/p)^n$ for some n, and that X is a finite dimensional G-CW complex. We shall describe how to use Borel cohomology to determine the mod p cohomology of X^G as an algebra over the Steenrod algebra, and we shall sketch another proof of Theorem 1.1. Our starting point is the localization theorem.

Since $G = (\mathbb{Z}/p)^n$, $H^*(BG)$ is a polynomial algebra on n generators of degree one if $p = 2$ and is the tensor product of an exterior algebra on n generators of degree one and the polynomial algebra on their Bocksteins if $p > 2$. Let S be the multiplicative subset of $H^*(BG)$ generated by the non-zero elements of degree one if $p = 2$ and by the non-zero images of Bocksteins of degree two if $p > 2$.

Theorem 2.1 (Localization). For a finite dimensional G-CW complex X, the inclusion $i : X^G \longrightarrow X$ induces an isomorphism

$$i^* : S^{-1}H^*_G(X) \longrightarrow S^{-1}H^*_G(X^G).$$

Proof. Let $FX = X/X^G$. By the cofiber sequence $X^G_+ \longrightarrow X_+ \longrightarrow FX$, it suffices to show that $S^{-1}\tilde{H}^*_G(FX) = 0$. Here FX is a finite dimensional G-CW complex and $(FX)^G = *$. By induction up skeleta, it suffices to show that $S^{-1}\tilde{H}^*_G(Y) = 0$ when Y is a wedge of copies of $G/H_+ \wedge S^q$ for some $H \neq G$, and such a wedge can be rewritten as $Y = G/H_+ \wedge K$, where K is a wedge of copies of S^q. Since $EG \times_G (G/H) = EG/H$ is a model for BH, we see that $EG_+ \wedge_G Y = BH_+ \wedge K$. At least one element of S restricts to zero in $H^*(BH)$, and this implies that $S^{-1}\tilde{H}^*_G(Y) = 0$.

Localization theorems of this general sort appear ubiquitously in equivariant theory. As here, the proofs of such results reduce to the study of orbits by general nonsense arguments, and the specifics of the situation are then used to determine what happens on orbits. When $n = 1$, we can be a little more precise.

LEMMA 2.2. If $G = \mathbb{Z}/p$ and $\dim X = r$, then $i^* : H^q_G(X) \longrightarrow H^q_G(X^G)$ is an isomorphism for $q > r$.

PROOF. It suffices to show that $\tilde{H}^*_G(FX) = 0$ for $q > r$. Since FX is G-free away from its basepoint, the projection $EG_+ \longrightarrow S^0$ induces a G-homotopy equivalence $EG_+ \wedge FX \longrightarrow FX$ and therefore a homotopy equivalence $EG_+ \wedge_G FX \longrightarrow FX/G$. Obviously $\dim(FX/G) \leq r$. □

Since G acts trivially on X^G, $EG \times_G X^G = BG \times X^G$.

SECOND PROOF OF THEOREM 1.1. Take $G = \mathbb{Z}/p$ and let X be a mod p homology n-sphere. We assume that X^G is non-empty. The Serre spectral sequence of the bundle $EG \times_G X \longrightarrow BG$ converges from

$$H^*(G; H^*(X)) = H^*(BG) \otimes H^*(X)$$

to $H^*_G(X)$. Since a fixed point of X gives a section, $E_2 = E_\infty$. Therefore $\tilde{H}^*_G(X)$ is a free $H^*(BG)$-module on one generator of degree n and, in high degrees, this must be isomorphic to

$$\tilde{H}^*_G(X^G) = H^*(BG_+ \wedge X^G) = H^*(BG) \otimes \tilde{H}^*(X^G).$$

By a trivial dimension count, this can only happen if X^G is a mod p cohomology m-sphere for some m. Naturality arguments from the $H^*(BG)$-module structure show that m must be less than n and must be congruent to n mod 2 if $p > 2$. To see that X^G is non-empty if $p > 2$ and n is even, one assumes that X^G is empty and deduces from the multiplicative structure of the spectral sequence that X cannot be finite dimensional. □

Returning to the context of the localization theorem, one would like to retrieve $H^*(X^G)$ algebraically from $S^{-1}H^*_G(X)$. As a matter of algebra, $S^{-1}H^*_G(X)$ inherits a structure of algebra over the mod p Steenrod algebra A from $H^*_G(X)$. However, it no longer satisfies the instability conditions that are satisfied in the cohomology of spaces. For any A-module M, the subset of elements that do satisfy these conditions form a submodule $\mathrm{Un}(M)$. Obviously the localization map

$$H^*(BG) \otimes H^*(X^G) \cong H^*_G(X^G) \longrightarrow S^{-1}H^*_G(X^G) \cong S^{-1}H^*_G(X)$$

takes values in $\mathrm{Un}(S^{-1}H^*_G(X))$. By a purely algebraic analysis, using basic information about the Steenrod operations, Dwyer and Wilkerson proved the following remarkable result. (They assume that X is finite, but the argument still works when X is finite dimensional.)

THEOREM 2.3. For any elementary Abelian p-group G and any finite dimensional G-CW complex X,

$$H^*(BG) \otimes H^*(X^G) \longrightarrow \mathrm{Un}(S^{-1}H^*_G(X))$$

is an isomorphism of A-algebras and $H^*(BG)$-modules. Therefore

$$H^*(X^G) \cong \mathbb{F}_p \otimes_{H^*(BG)} \mathrm{Un}(S^{-1}H^*_G(X)).$$

We will come back to this point when we talk about the Sullivan conjecture.

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CHAPTER V

Categorical Constructions; Equivariant Applications

1. Coends and geometric realization

We pause to introduce some categorical and topological constructs that are used ubiquitously in both equivariant and nonequivariant homotopy theory. They will be needed in a number of later places. We are particularly interested in homotopy colimits. These are examples of geometric realizations of spaces, which in turn are examples of coends, which in turn are examples of coequalizers.

Let Λ be a small category and let $\mathscr{C}$ be a category that has all colimits. Write $\coprod$ for the categorical coproduct in $\mathscr{C}$. The coequalizer $C(f, f')$ of maps $f, f' : X \longrightarrow Y$ is a map $g : Y \longrightarrow C(f, f')$ such that $gf = gf'$ and g is universal with this property. It can be constructed as the pushout in the following diagram, where $\nabla = 1 + 1$ is the folding map:

$$\begin{tikzcd} X \coprod X \arrow[r, "f+f'"] \arrow[d, "\nabla"'] & Y \arrow[d, "g"] \\ X \arrow[r] & C(f, f'). \end{tikzcd}$$

Coends are categorical generalizations of tensor products. Given a functor $F : \Lambda^{op} \times \Lambda \longrightarrow \mathscr{C}$, the coend

$$\int^{\Lambda} F(n, n)$$

is defined to be the coequalizer of the maps

$$f, f' : \coprod_{\phi : m \rightarrow n} F(n, m) \longrightarrow \coprod F(n, n)$$

whose restrictions to the ϕth summand are

$$F(\phi, \mathrm{id}) : F(n,m) \longrightarrow F(m,m) \text{ and } F(Id,\phi) : F(n,m) \longrightarrow F(n,n),$$

respectively. It satisfies a universal property like that of tensor products. If the objects of $F(n,n)$ have points that can be written in the form of "tensors" $x \otimes y$, then the coend is obtained from the coproduct of the $F(n,n)$ by identifying $x\phi \otimes y$ with $x \otimes \phi y$ whenever this makes sense. Here ϕ is a map in Λ, contravariant actions are written from the right, and covariant actions are written from the left.

Dually, if $\mathscr{C}$ has limits, a functor $F : \Lambda^{op} \times \Lambda \longrightarrow \mathscr{C}$ has an end

$$\int^{\Lambda} F(n,n).$$

It is defined to be the equalizer, $E(f,f')$, of the maps

$$f, f' : \prod F(n) \longrightarrow \prod_{\phi : m \to n} F(m,n)$$

whose projections to the ϕth factor are

$$F(\mathrm{id}, \phi) : F(m,m) \longrightarrow F(m,n) \text{ and } F(\phi, \mathrm{id}) : F(n,n) \longrightarrow F(m,n).$$

Recall that a simplicial object in a category $\mathscr{C}$ is a contravariant functor $\triangle \longrightarrow \mathscr{C}$, where $\triangle$ is the category of sets $\underline{n} = \{0,1,2,\ldots,n\}$ and monotonic maps. Using the usual face and degeneracy maps, we obtain a covariant functor $\triangle_*: \triangle \longrightarrow \mathscr{U}$ that sends $\underline{n}$ to the standard topological n-simplex $\triangle_n$. For a simplicial space $X_* : \triangle \longrightarrow \mathscr{U}$, we have the product functor

$$X_* \times \triangle_* : \triangle^{op} \times \triangle \longrightarrow \mathscr{U}.$$

Define the geometric realization of X_* to be the coend

$$|X_*| = \int^{\triangle} X_n \times \triangle_n. \tag{1.1}$$

If X_* is a simplicial based space, so that all its face and degeneracy maps are basepoint preserving, then all points of each subspace $\{*\} \times \triangle_n$ are identified to the point $(*,1) \in X_0 \times \triangle_0$ in the construction of $|X_*|$, hence

$$|X_*| = \int^{\triangle} X_n \wedge (\triangle_n)_+. \tag{1.2}$$

If X_* is a simplicial G-space, then $|X_*|$ inherits a G-action such that

$$|X_*|^H = |X_*^H| \text{ for all } H \subset G. \tag{1.3}$$

S. Mac Lane. Categories for the Working Mathematician. Springer. 1976.
J. P. May. The Geometry of Iterated Loop Spaces (§11). Springer Lecture Notes Vol 271. 1972.

2. Homotopy colimits and limits

Let $\mathscr{D}$ be any small topological category. We understand $\mathscr{D}$ to have a discrete object set and to have spaces of maps $d \to d'$ such that composition is continuous. Let $B_n(\mathscr{D})$ be the set of n-tuples $\underline{f} = (f_1, \dots, f_n)$ of composable arrows of $\mathscr{D}$, depicted

$$d_0 \xleftarrow{f_1} d_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} d_n.$$

Here $B_0(\mathscr{D})$ is the set of objects of $\mathscr{D}$ and $B_n(\mathscr{D})$ is topologized as a subspace of the n-fold product of the total morphism space $\coprod \mathscr{D}(d,d')$. With zeroth and last face given by deleting the zeroth or last arrow of n-tuples $\underline{f}$ (or by taking the source or target of f_1 if $n = 1$) and with the remaining face and degeneracy operations given by composition or by insertion of identity maps in the appropriate position, $B_*(\mathscr{D})$ is a simplicial set called the nerve of $\mathscr{D}$. Its geometric realization is the classifying space $B\mathscr{D}$. If $\mathscr{D}$ has a single object d, then $G = \mathscr{D}(d,d)$ is a topological monoid (= associative Hopf space with unit) and $B\mathscr{D} = BG$ is its classifying space.

We can now define the two-sided categorical bar construction. It will specialize to give homotopy colimits. Let $T : \mathscr{D} \longrightarrow \mathscr{U}$ be a continuous contravariant functor. This means that each $T(d)$ is a space and each function $T : \mathscr{D}(d,d') \longrightarrow \mathscr{U}(T(d'), T(d))$ is continuous. Let $S : \mathscr{D} \longrightarrow \mathscr{U}$ be a continuous covariant functor. We define

$$B(T, \mathscr{D}, S) = |B_*(T, \mathscr{D}, S)|. \tag{2.1}$$

Here $B_*(T, \mathscr{D}, S)$ is the simplicial space whose set of n-simplices is

$$\{(t, \underline{f}, s) | t \in T(d_0), \underline{f} \in B_n(\mathscr{D}), \text{ and } s \in S(d_n)\},$$

topologized as a subspace of the product $(\coprod T(d)) \times (\coprod \mathscr{D}(d,d'))^n \times (\coprod S(d))$; $B_0(T, \mathscr{D}, S) = \coprod T(d) \times S(d)$. The zeroth and last face use the evaluation of the functors T or S; the remaining faces and the degeneracies are defined like those of $B_*\mathscr{D}$.

Since the coend of $T \times S : \mathscr{D}^{op} \times \mathscr{D} \longrightarrow \mathscr{U}$ is exactly the coequalizer of $d_0, d_1 : B_1(T, \mathscr{D}, S) \longrightarrow B_0(T, \mathscr{D}, S)$, we obtain a natural map

$$\epsilon : B(T, \mathscr{D}, S) \longrightarrow \int^{\mathscr{D}} T(d) \times S(d) \equiv T \otimes_{\mathscr{D}} S. \tag{2.2}$$

It is obtained by using iterated compositions to map$B_*(T,\mathscr{D},S)$to the constant simplicial space at the cited coend, which we denote by $T\otimes_{\mathscr{D}} S$.

Let $\mathscr{D}_e$ be the covariant functor represented by an object e of $\mathscr{D}$, so that $\mathscr{D}_e(d)=\mathscr{D}(e,d)$. Then ϵ reduces to a map

$$\varepsilon : B(T,\mathscr{D},\mathscr{D}_e)\longrightarrow T(e),$$

and this map is a homotopy equivalence. In fact, using the identity map of e, we obtain an inclusion $\eta : T(e)\longrightarrow B(T,\mathscr{D},\mathscr{D}_e)$ such that $\varepsilon\eta = 1$ and a simplicial deformation $\eta\varepsilon\simeq \mathrm{id}$. There is a left–right symmetric analogue.

If the functor S takes values in $G\mathscr{U}$, then $B_*(T,\mathscr{D},S)$ is a simplicial G-space and $B(T,\mathscr{D},S)$ is a G-space such that

$$B(T,\mathscr{D},S)^H = B(T,\mathscr{D},S^H). \tag{2.3}$$

We define the homotopy colimit of our covariant functor S by

$$\operatorname{Hocolim} S = B(*,\mathscr{D},S), \tag{2.4}$$

where $* : \mathscr{D}\longrightarrow\mathscr{U}$ is the trivial functor to a 1-point space. Here the coend on the right of (1.5) is exactly the ordinary colimit of S. Thus we have

$$\varepsilon : \operatorname{hocolim} S\longrightarrow \operatorname{colim} S. \tag{2.5}$$

When G is a group regarded as category with a single object and X is a (left) G-space regarded as a covariant functor, the homotopy colimit of X is the "homotopy orbit space" $EG\times_G X = X/hG$, and ε is the natural map $X/hG\longrightarrow X/G$.

Our preferred definition of homotopy limits is precisely dual. We have a cosimplicial space $C_*(T,\mathscr{D},S)$, the two-sided cobar construction. Its set of n-cosimplices is the product over all $\underline{f}\in B_n(\mathscr{D})$ of the spaces $T(d_0)\times S(d_n)$, topologized as a subspace of $\mathrm{Map}(B_n(\mathscr{D}),\coprod T(d)\times S(d'))$. The $\underline{f}$th coordinates of the cofaces and codegeneracies with target $C_n(T,\mathscr{D},S)$ are obtained by projecting onto the coordinate of their source that is indexed by the corresponding face or degeneracy applied to $\underline{f}$, except that, for the zeroth and last coface, we must compose with

$$T(f_1)\times\mathrm{id} : T(d_0)\times S(d_n)\longrightarrow T(d_1)\times S(d_n)$$

or

$$\mathrm{id}\times S(f_n) : T(d_0)\times S(d_n)\longrightarrow T(d_0)\times S(d_{n-1}).$$

We define the geometric realization, or totalization, "TotY_*" of a cosimplicial space Y_* to be the end

$$\mathrm{Tot}Y_* = \int_{\triangle} \mathrm{Map}(\triangle_n, Y_n). \tag{2.6}$$

Here we are using the evident functor $\triangle^{op} \times \triangle \longrightarrow \mathscr{U}$ that sends (m,n) to $\mathrm{Map}(\triangle_m, Y_n)$. If Y_* takes values in based spaces, we may rewrite this as

$$\mathrm{Tot}Y_* = \int_{\triangle} F((\triangle_n)_+, Y_n). \tag{2.7}$$

We then define

$$C(T, \mathscr{D}, S) = \mathrm{Tot}C_*(T, \mathscr{D}, S), \tag{2.8}$$

and we have a natural map

$$\eta : \int^{\mathscr{D}} T(d) \times S(d) \longrightarrow C(T, \mathscr{D}, S). \tag{2.9}$$

We define the homotopy limit of our contravariant functor $T : \mathscr{D} \longrightarrow \mathscr{U}$ to be

$$\mathrm{Holim}\, T = \mathrm{Tot}C_*(T, \mathscr{D}, *), \tag{2.10}$$

and we see that η specializes to give a natural map

$$\eta : \lim T \longrightarrow \mathrm{holim}\, T. \tag{2.11}$$

When G is a group regarded as a category with a single object and X is a (right) G-space regarded as a contravariant functor, the homotopy limit of X is the "homotopy fixed point space" of G-maps $EG \longrightarrow X$,

$$\mathrm{Map}_G(EG, X) = \mathrm{Map}(EG, X)^G = X^{hG},$$

and η is the natural map $X^G \longrightarrow X^{hG}$ that sends a fixed point to the constant function at that point. This map is the object of study of the Sullivan conjecture.

A. K. Bousfield and D. M. Kan. Homotopy limits, completions, and localizations. Springer Lecture Notes in Mathematics Vol 304. 1972.

J. P. May. Classifying spaces and fibrations (§12). Memoirs Amer. Math. Soc. No. 155, 1975.

3. Elmendorf's theorem on diagrams of fixed point spaces

Recall that $\mathscr{G}$ is the category of orbit spaces. We shall regard $\mathscr{G}$ as a topological category with a discrete set of objects. We write $[G/H]$ for a typical object, to avoid confusing it with the G-space G/H. The space of morphisms $[G/H] \longrightarrow [G/K]$ is the space of G-maps $G/H \longrightarrow G/K$, and this space may be identified with $(G/K)^H$. Define a $\mathscr{G}$-space to be a continuous contravariant functor $\mathscr{G} \longrightarrow \mathscr{U}$. A map of $\mathscr{G}$-spaces is a natural transformation, and we write $\mathscr{G}\mathscr{U}$ for the category of G-spaces. We shall compare this category with $G\mathscr{U}$. We have already observed that a G-space X gives a $\mathscr{G}$-space X^*, and we write

$$\Phi : G\mathscr{U} \longrightarrow \mathscr{G}\mathscr{U}$$

for the functor that sends X to X^*. We wish to determine how much information the functor Φ loses.

By the definition of $\underline{C}_*(X)$, it is clear that the ordinary homology and cohomology of X depend only on ΦX. If $T : \mathscr{G} \longrightarrow \mathscr{U}$ is a $\mathscr{G}$-space such that each $T(G/H)$ is a CW-complex and each $T(G/K) \longrightarrow T(G/H)$ is a cellular map, then we can define $H^*_G(T; M)$ exactly as we defined $H^*_G(X; M)$. Note, however, that unless G is discrete, X^H will not inherit a structure of a CW-complex from a G-CW complex X. Indeed, for compact Lie groups, we saw that it was not quite the functor X^* that was relevant to ordinary cohomology, but rather the functor that sends G/H to X^H/W_0H.

There is an obvious way that $\mathscr{G}$-spaces determine G-spaces.

Lemma 3.1. Define a functor $\Theta : \mathscr{G}\mathscr{U} \longrightarrow G\mathscr{U}$ by $\Theta T = T(G/e)$, with the G-maps $G/e \longrightarrow G/e$ inducing the action. Then Θ is left adjoint to Φ,

$$\mathscr{G}\mathscr{U}(T, \Phi X) \cong G\mathscr{U}(\Theta T, X).$$

Proof. Clearly $\Theta\Phi X = X$. The quotient map $G \longrightarrow G/H$ induces a map $\eta : T(G/H) \longrightarrow T(G/e)^H$, and these maps together specify a natural map $\eta : T \longrightarrow \Phi\Theta T$. Passage from $\phi : T \longrightarrow \Phi X$ to $\Theta\phi : \Theta T \longrightarrow X$ is a bijection whose inverse sends $f : \Theta T \longrightarrow X$ to $\Phi f \circ \eta$. $\square$

The following result of Elmendorf shows that $\mathscr{G}$-spaces determine G-spaces in a less obvious way. In fact, up to homotopy, any $\mathscr{G}$-space can be realized as the fixed point system of a G-space and, up to homotopy, the functor Φ has a right adjoint as well as a left adjoint. Note that we can form the product $T \times K$ of a $\mathscr{G}$-space

T and a space K by setting $(T \times K)(G/H) = T(G/H) \times K$. In particular, $T \times I$ is defined, and we have a notion of homotopy between maps of $\mathscr{G}$-spaces. Write $[T, T']_{\mathscr{G}}$ for the set of homotopy classes of maps $T \longrightarrow T'$.

THEOREM 3.2 (ELMENDORF). There is a functor $\Psi : \mathscr{G}\mathscr{U} \longrightarrow G\mathscr{U}$ and a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{id}$ such that each $\varepsilon : (\Psi T)^H \longrightarrow T(G/H)$ is a homotopy equivalence. If X has the homotopy type of a G-CW complex, then there is a natural bijection

$$[X, \Psi T]_G \cong [\Phi X, T]_{\mathscr{G}}.$$

PROOF. Let $S : \mathscr{G} \longrightarrow G\mathscr{U}$ be the covariant functor that sends the object $[G/H]$ to the G-space G/H. On morphisms, it is given by identity maps

$$\mathscr{G}([G/H], [G/K]) \longrightarrow G\mathscr{U}(G/H, G/K).$$

For a $\mathscr{G}$-space T, define ΨT to be the G-space $B(T, \mathscr{G}, S)$. We have

$$S^H[G/K] = (G/K)^H = G\mathscr{U}(G/H, G/K) = \mathscr{G}([G/H], [G/K]),$$

and (2.2) and (2.3) give homotopy equivalences $\varepsilon : (\Psi T)^H \longrightarrow T(G/H)$ that define a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{id}$. Clearly

$$\Theta\varepsilon : \Psi T = \Theta\Phi\Psi T \longrightarrow \Theta T$$

is a weak equivalence of G-spaces for any T. With $T = \Phi X$, this gives a weak equivalence $\Theta\varepsilon : \Psi\Phi X \longrightarrow X$. We can check that $\Psi\Phi X$ has the homotopy type of a G-CW complex if X does. Therefore $\Theta\varepsilon$ is an equivalence, and we choose a homotopy inverse $(\Theta\varepsilon)^{-1}$. Define

$$\alpha : [X, \Psi T]_G \longrightarrow [\Phi X, T]_{\mathscr{G}} \quad \text{and} \quad \beta : [\Phi X, T]_{\mathscr{G}} \longrightarrow [X, \Psi T]_G$$

by $\alpha(f) = \varepsilon \circ \Phi f$ and $\beta(\phi) = \Psi\phi \circ (\Theta\varepsilon)^{-1}$. Easy diagram chases show that $\alpha\beta(\phi) \simeq \phi$ and $\beta\alpha(f) \simeq (\Psi\varepsilon) \circ (\Theta\varepsilon)^{-1} \circ f$. Since $\Psi\varepsilon$ is a weak equivalence, the Whitehead theorem gives that $\beta\alpha$ is a bijection. It follows formally that α and β are inverse bijections. $\square$

A. D. Elmendorf. Systems of fixed point sets. Trans. Amer. Math. Soc. 277(1983), 275-284.

4. Eilenberg-Mac Lane G-spaces and universal $\mathcal{F}$-spaces

We give some important applications of this construction, starting with the construction of equivariant Eilenberg-Mac Lane spaces that we promised earlier.

EXAMPLE 4.1. Let B be the classifying space functor from topological monoids to spaces. It is product-preserving, and it therefore gives an Abelian topological group when applied to an Abelian topological group. If π is a discrete Abelian group, then the n-fold iterate $B^n\pi$ is a $K(\pi, n)$. A coefficient system $M : h\mathscr{G} \longrightarrow \mathscr{A}b$ may be regarded as a continuous functor $\mathscr{G} \longrightarrow \mathscr{U}$ (with discrete values). We may compose with B^n to obtain a $\mathscr{G}$-space $B^n \circ M$. In view of the equivalences $\varepsilon : \Psi(B^n \circ M)^H \longrightarrow K(M(G/H), n)$, $\Psi(B^n \circ M)$ is a $K(M, n)$. Theorem 3.2 gives a homotopical description of ordinary cohomology in terms of maps of $\mathscr{G}$-spaces:

$$\tilde{H}^n_G(X; M) \cong [X, K(M, n)]_G \cong [\Phi X, B^n \circ M]_{\mathscr{G}}.$$

In interpreting this, one must remember that the right side concerns homotopy classes of genuine natural transformations $\Phi X \longrightarrow B^n M$, and not just natural transformations in the homotopy category. The latter would be directly computable in terms of nonequivariant comology.

EXAMPLE 4.2. If M is a contravariant functor from $h\mathscr{G}$ to (not necessarily Abelian) groups, then we can regard $B \circ M$ as a $\mathscr{G}$-space and so obtain an Eilenberg-Mac Lane G-space $K(M, 1) = \Psi(B \circ M)$.

EXAMPLE 4.3. A set-valued functor M on $h\mathscr{G}$ is the same thing as a continuous set-valued functor on $\mathscr{G}$. Applying Ψ to such an M, we obtain an Eilenberg-Mac Lane G-space $K(M, 0)$. Its fixed point spaces $K(M, 0)^H$ are homotopy equivalent to the discrete spaces $M(G/H)$, but the G-space $K(M, 0)$ generally has non-trivial cohomology groups in arbitrarily high dimension. For set-valued coefficient systems M and M', let $Nat_{\mathscr{G}}(M, M')$ be the set of natural transformations $M \longrightarrow M'$. Then Theorem 3.2 and the discreteness of M give isomorphisms

$$[X, K(M, 0)]_G \cong [\Phi X, M]_{\mathscr{G}} \cong Nat_{\mathscr{G}}(\underline{\pi}_0(X), M). \tag{4.4}$$

This may seem frivolous at first sight, but in fact the spaces $K(M, 0)$ are central to equivariant homotopy theory. For example, we shall see later that the isomorphisms just given specialize to give a classification theorem for equivariant bundles — and to reprove the classical classification of nonequivariant bundles. The relevant $K(M, 0)$'s are special cases of those in the following basic definition.

DEFINITION 4.5. A family $\mathscr{F}$ in G is a set of subgroups of G that is closed under subconjugacy: if $H \in \mathscr{F}$ and $g^{-1}Kg \subset H$, then $K \in \mathscr{F}$. An $\mathscr{F}$-space is a G-space all of whose isotropy groups are in $\mathscr{F}$. Define a functor $\underline{\mathscr{F}} : h\mathscr{G} \longrightarrow Sets$ by sending G/H to the 1-point set if $H \in \mathscr{F}$ and to the empty set if $H \notin \mathscr{F}$. Define the universal $\mathscr{F}$-space $E\mathscr{F}$ to be $\Psi\underline{\mathscr{F}}$. It is universal in the sense that, for an $\mathscr{F}$-space X of the homotopy type of a G-CW complex, there is one and, up to homotopy, only one G-map $X \longrightarrow E\mathscr{F}$. Define the classifying space of the family $\mathscr{F}$ to be the orbit space $B\mathscr{F} = E\mathscr{F}/G$.

In thinking about this example, it should be remembered that there are no maps from a non-empty set to the empty set. In particular, there are no G-maps $X \longrightarrow E\mathscr{F}$ if X is not an $\mathscr{F}$-space. This also shows that the functor $\underline{\mathscr{F}}$ only makes sense if the given set $\mathscr{F}$ of subgroups of G is a family. We augment the definition with the following relative version. It will become very important later.

DEFINITION 4.6. For a subfamily $\mathscr{F}$ of a family $\mathscr{F}'$, define $E(\mathscr{F}', \mathscr{F})$ to be the cofiber of the based G-map (unique up to homotopy) $E\mathscr{F}_+ \longrightarrow E\mathscr{F}'_+$. Let $\mathscr{A}\ell\ell$ be the family of all subgroups of G, and let $\tilde{E}\mathscr{F} = E(\mathscr{A}\ell\ell, \mathscr{F})$. Since $E\mathscr{A}\ell\ell$ is G-contractible, $\tilde{E}\mathscr{F}$ is equivalent to the unreduced suspension of $E\mathscr{F}$ with one of the cone points as basepoint. The space $(\tilde{E}\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and is the two-point space S^0 if $H \notin \mathscr{F}$. For $\mathscr{F} \subset \mathscr{F}'$, the G-space $E(\mathscr{F}', \mathscr{F})$ is equivalent to $E\mathscr{F}'_+ \wedge \tilde{E}\mathscr{F}$.

The following observation will become valuable when we examine the structure of equivariant classifying spaces.

LEMMA 4.7. Let $\mathscr{F}$ be a family in G and H be a subgroup of G.

(a) Regarded as an H-space, $E\mathscr{F}$ is $E(\mathscr{F}|H)$, where

$$\mathscr{F}|H = \{K | K \in \mathscr{F} \quad \text{and} \quad K \subset H\}.$$

(b) If $H \in \mathscr{F}$, then, regarded as a WH-space, $(E\mathscr{F})^H$ is $E(\mathscr{F}^H)$, where

$$\mathscr{F}^H = \{L | L = K/H \text{ for some } K \in \mathscr{F} \text{ such that } H \subset K \subset NH\}.$$

The classical example is $\mathscr{F} = \{e\}$. An $\{e\}$-space X is a G-space all of whose isotropy groups are trivial. That is, X is a free G-space. Then $EG \equiv E\{e\}$ is exactly the standard example of a free contractible G-space, and the quotient map $\pi : EG \longrightarrow BG$ is a principal G-bundle. Given the result that pullbacks of bundles along homotopic maps are homotopic, we have already proven that π is universal.

Indeed, if $p : E \longrightarrow B$ is a principal G-bundle, we have a unique homotopy class of G-maps $\tilde{f} : E \longrightarrow EG$. The map $f : B \longrightarrow BG$ that is obtained by passage to orbits from $\tilde{f}$ is the classifying map of p. Certainly p is equivalent to the bundle obtained by pulling π back along f.

When G is discrete, the ordinary homology and cohomology of the G-spaces $E\mathscr{F}$ admit descriptions as Ext groups, generalizing the classical identification of the homology and cohomology of groups with the homology and cohomology of $K(\pi, 1)$'s. This can be seen from the projectivity of the cellular chains $\underline{C}_*(E\mathscr{F})$ and inspection of definitions or by collapse of the universal coefficients spectral sequences. Write $\mathbb{Z}[\underline{\mathscr{F}}]$ for the free Abelian group functor composed with the functor $\underline{\mathscr{F}}$.

Proposition 4.8. Let G be discrete. For a covariant coefficient system N and a contravariant coefficient system M,

$$H_*^G(E\mathscr{F}; N) = \mathrm{Tor}_*^{\mathscr{G}}(\mathbb{Z}[\underline{\mathscr{F}}]; N) \quad \text{and} \quad H_G^*(E\mathscr{F}; M) = \mathrm{Ext}_{\mathscr{G}}^*(\mathbb{Z}[\underline{\mathscr{F}}]; M).$$

CHAPTER VI

The Homotopy Theory of Diagrams

by Robert J. Piacenza

1. Elementary homotopy theory of diagrams

A substantial portion of the homotopy, homology, and cohomology theory of G-spaces X depends only on the underlying diagram of fixed point spaces $\Phi X : \mathscr{G} \longrightarrow \mathscr{U}$. There is a vast and growing literature in which the homotopy theory of spaces is generalized to a homotopy theory of diagrams of spaces that are indexed on arbitrary small indexing categories. The purpose of this chapter is to outline this theory and to demonstrate the connection between diagrams and equivariant theory. A very partial list of sources for further reading is given at the end of this section.

Throughout the chapter, we let $\mathscr{U}$ be the cartesian category of compactly generated weak Hausdorff spaces and let J be a small topological category over $\mathscr{U}$ with discrete object space. Define $\mathscr{U}^J$ to be the category of continuous contravariant $\mathscr{U}$-valued functors on J. Its objects are called either diagrams or J-spaces; its morphisms, which are natural transformations, are called J-maps. Note that $\mathscr{U}^J$ is a topological category: its hom sets are spaces and composition is continuous.

Let I be the unit interval in $\mathscr{U}$. If X and Y are diagrams, then a homotopy from X to Y is a J-map $H : I \times X \longrightarrow Y$, where $I \times X$ is the diagram defined on objects $j \in |J|$ by $(I \times X)(j) = I \times X(j)$ and similarly for morphisms of J. In the usual way homotopy defines an equivalence relation on the J-maps that gives rise to the quotient homotopy category $h\mathscr{U}^J$. We denote the homotopy classes of J-maps from X to Y by $h\mathscr{U}^J(X, Y)$, abbreviated $h(X, Y)$. An isomorphism in

$h\mathscr{U}^J$ will be called a homotopy equivalence.

A J-map is called a J-cofibration if it has the J homotopy extension property, abbreviated $J - HEP$. The basic facts about cofibrations in $\mathscr{U}$ apply readily to J-cofibrations.

The following standard results for spaces are inherited by the category $\mathscr{U}^J$.

THEOREM 1.1 (INVARIANCE OF PUSHOUTS). Suppose given a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] \arrow[drr, "\alpha"] & B \arrow[d] \arrow[drr, "\beta"] & & \\
X \arrow[r] \arrow[drr, "\gamma"'] & Y \arrow[drr, "\delta"] & A' \arrow[r, "f'"] \arrow[d, "i'"] & B' \arrow[d] \\
& & X' \arrow[r] & Y'
\end{tikzcd}$$

in which i and i' are J-cofibrations, f and f' are arbitrary J-maps, α, β, and γ are homotopy equivalences, and the front and back faces are pushouts. Then the induced map δ on pushouts is also a homotopy equivalence.

THEOREM 1.2 (INVARIANCE OF COLIMITS OVER COFIBRATIONS). Suppose given a homotopy commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "i_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "i_1"] \arrow[d, "f^1"] & \cdots \arrow[r] & X^k \arrow[r, "i_k"] \arrow[d, "f^k"] & \cdots \\
Y^0 \arrow[r, "j_0"] & Y^1 \arrow[r, "j_1"] & \cdots \arrow[r] & Y^k \arrow[r, "j_k"] & \cdots
\end{tikzcd}$$

in $\mathscr{U}^J$ where the i_k and j_k are J-cofibrations and the f^k are homotopy equivalences. Then the map $\operatorname{colim}_k f^k : \operatorname{colim}_k X^k \longrightarrow \operatorname{colim}_k Y^k$ is a homotopy equivalence.

The reader will readily accept that other such standard results in the homotopy theory of spaces carry over directly to the homotopy theory of diagrams.

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2. Homotopy Groups

Let I^n be the topological n-cube and ∂I^n its boundary. For an object $j \in |J|$, let $\underline{j} \in \mathscr{U}^J$ denote the associated represented functor; its value on an object k is the space $\mathscr{U}^J(k, j)$.

Definition 2.1. By a pair (X, Y) in $\mathscr{U}^J$, we mean a J-space X together with a sub J-space Y. Morphisms of pairs are defined in the obvious way. Similar definitions apply to triples, n-ads, etc. Let $\phi : \underline{j} \longrightarrow Y$ be a morphism in $\mathscr{U}^J$. By the Yoneda lemma, ϕ is completely determined by the point $\phi(\mathrm{id}_j) = y_0 \in Y(j)$. For each $n \geq 0$, define

$$\pi_n^j(X, Y, \phi) = h((I^n, \partial I^n, \{0\}) \times \underline{j}, (X, Y, Y))$$

where $y_0 = \phi(id_j) \in Y(j)$ serves as a basepoint, and all homotopies are homotopies of triples relative to ϕ. The reader may formulate a similar definition for the absolute case $\pi_n^j(X, \phi)$. For $n = 0$ we adopt the convention that $I^0 = \{0, 1\}$ and $\partial I^0 = \{0\}$ and proceed as above. These constructions extend to covariant functors on $\mathscr{U}^J$. From now on, we shall often drop ϕ from the notation $\pi_n^j(X, Y, \phi)$.

The following proposition follows immediately from the Yoneda lemma.

Proposition 2.2. There are natural isomorphisms $\pi_n^j(X) \cong \pi_n(X(j))$ and $\pi_n^j(X, Y) \cong \pi_n(X(j), Y(j))$ that preserve the group structures when $n \geq 1$ (in the absolute case; the relative case requires $n \geq 2$).

As a direct consequence of Proposition 2.2 we obtain the usual long exact sequences.

PROPOSITION 2.3. For (X, Y) and j as in Definition 2.1, there exist natural boundary maps ∂ and long exact sequences

$$\cdots \longrightarrow \pi^j_{n+1}(X,Y) \xrightarrow{\partial} \pi^j_n(Y) \longrightarrow \pi^j_n(X) \longrightarrow \cdots \longrightarrow \pi^j_0(Y) \longrightarrow \pi^j_0(X)$$

of groups up to $\pi^j_1(Y)$ and pointed sets thereafter.

DEFINITION 2.4. A map $e : (X, Y) \longrightarrow (X', Y')$ of pairs in $\mathscr{U}^J$ is said to be an n-equivalence if $e(j) : (X(j), Y(j)) \longrightarrow (X'(j), Y'(j))$ is an n-equivalence in $\mathscr{U}$ for each $j \in |J|$. A map e is said to be a weak equivalence if it is an n-equivalence for each $n \geq 0$. Observe that e is an n-equivalence if for every $j \in |J|$ and $\phi : j \longrightarrow Y$, $e_* : \pi^j_p(X, Y, \phi) \longrightarrow \pi^j_p(X', Y', e\phi)$ is an isomorphism for $0 \leq p < n$ and an epimorphism for $p = n$. The reader may easily formulate similar definitions for J-maps $e : X \longrightarrow X'$ (the absolute case).

3. Cellular Theory

In this section we adapt May's preferred approach to the classical theory of CW complexes to develop a theory of J-CW complexes.

Let D^{n+1} be the topological $(n + 1)$-disk and S^n the topological n-sphere. Of course, these spaces are homeomorphic to I^{n+1} and ∂I^{n+1} respectively. We shall construct cell complexes over J by the process of attaching cells of the form $D^{n+1} \times \underline{j}$ by attaching morphisms with domain $S^n \times \underline{j}$.

DEFINITION 3.1. A J-complex is an object X of $\mathscr{U}^J$ with a decomposition $X = \operatorname{colim}_{p \geq 0} X^p$ where

$$X^0 = \coprod_{\alpha \in A_0} D^{n_\alpha} \times \underline{j}_\alpha$$

and, inductively,

$$X^p = X^{p-1} \bigcup_f \Big(\coprod_{\alpha \in A_p} D^{n_\alpha} \times \underline{j}_\alpha \Big)$$

for some attaching J-map $f : \coprod_{\alpha \in A_p} S^{n_\alpha - 1} \times \underline{j}_\alpha \longrightarrow X^{p-1}$; here, for each $p \geq 0$, $\{j_\alpha \mid \alpha \in A_p\}$ is a set of objects of J. We call X a J-CW complex if X is a J-complex such that $n_\alpha = p$ for all $p \geq 0$ and $\alpha \in A_p$.

Now J-subcomplexes and relative J-complexes are defined in the obvious way. We adopt the standard terminology for CW-complexes for J-CW-complexes without further comment.

The following technical lemma reduces directly to its space level analog.

LEMMA 3.2. Suppose that $e : Y \longrightarrow Z$ is an n-equivalence. Then we can complete the following diagram in $\mathscr{U}^J$:

$$\begin{tikzcd}
S^{n-1}\times \underline{j} \arrow[rr, "i_0"] \arrow[dd] & & S^{n-1}\times I\times \underline{j} \arrow[dl, "h"'] \arrow[dd] & & S^{n-1}\times \underline{j} \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, crossing over, "e"] & \\
D^n\times \underline{j} \arrow[ur, "f"] \arrow[rr, "i_0"'] & & D^n\times I\times \underline{j} \arrow[ul, dashed, "\tilde{h}"'] & & D^n\times \underline{j}. \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

From here, we proceed exactly as in I§3 to obtain the following results.

THEOREM 3.3 (J-HELP). If (X, A) is a relative J-CW complex of dimension $\leq n$ and $e : Y \longrightarrow Z$ is an n-equivalence, then we can complete the following diagram in $\mathscr{U}^J$:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & A\times I \arrow[dl] \arrow[dd] & & A \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z & & Y \arrow[ll, crossing over, "e"] & \\
X \arrow[ur] \arrow[rr] & & X\times I \arrow[ul, dashed] & & X. \arrow[ll] \arrow[ul, dashed]
\end{tikzcd}$$

THEOREM 3.4 (WHITEHEAD). Let $e : Y \longrightarrow Z$ be an n-equivalence and X be a J-CW complex. Then $e_* : h(X, Y) \longrightarrow h(X, Z)$ is a bijection if X has dimension less than n and a surjection if X has dimension n. If e is a weak equivalence, then $e_* : h(X, Y) \longrightarrow h(X, Z)$ is a bijection for all X.

COROLLARY 3.5. If $e : Y \longrightarrow Z$ is an n-equivalence between J-CW complexes of dimension less than n, then e is a J-homotopy equivalence. If e is a weak equivalence between J-CW complexes, then e is a J-homotopy equivalence.

THEOREM 3.6 (CELLULAR APPROXIMATION). Let (X, A) and (Y, B) be relative J-CW complexes, (X', A') be a subcomplex of (X, A), and $f : (X, A) \longrightarrow (Y, B)$ be a map of pairs in $\mathscr{U}^J$ whose restriction to (X', A') is cellular. Then f is homotopic rel $X' \cup A$ to a cellular map $g : (X, A) \longrightarrow (Y, B)$.

COROLLARY 3.7. Let X and Y be J-CW complexes. Then any J-map $f : X \longrightarrow Y$ is homotopic to a cellular J-map, and any two homotopic cellular J-maps are cellularly homotopic.

Next we discuss the local properties of J-CW complexes. First we develop some preliminary concepts. Let X be a J-space and, for each $j \in |J|$, let $t_j : X(j) \longrightarrow \operatorname{colim}_J X$ be the natural map of $X(j)$ into the colimit. Observe that, for each morphism $s : i \longrightarrow j$ of J, $t_j = t_i \circ X(s)$. For each subspace $A \subseteq \operatorname{colim}_J X$, we define $\check{A}(j) = t_j^{-1}(A)$; for each morphism $s : i \longrightarrow j$ of J, we define $\check{A}(s) : \check{A}(j) \longrightarrow \check{A}(i)$ to be the restriction of $X(s)$. (As usual, we apply the k-ification functor to ensure that all spaces defined above are compactly generated.) One quickly checks that $\check{A}$ is a J-space, that $\operatorname{colim}_J \check{A} = A$, and that there is a natural inclusion $\check{A} \longrightarrow X$. To simplify notation, we write

$$X/J = \operatorname{colim}_J X$$

from now on.

Definition 3.8. A pair $(X, \check{A})$ is a J-neighborhood retract pair (abbreviated J-NR pair) if there exists an open subset U of X/J such that $A \subseteq U$ and a retraction $r : \check{U} \longrightarrow \check{A}$. A pair $(X, \check{A})$ is a J-neighborhood deformation retract pair (abbreviated J-NDR pair) if $(X, \check{A})$ is a J-NR pair and the J-map r is a J-deformation retraction.

Let X be a J-CW complex. The functor colim_J sends the J-space $A \times \underline{j}$ determined by a space A and object j to the space A, and it preserves colimits. Therefore the cellular decomposition of X determines a natural structure of a CW complex on X/J; its attaching maps are the images under the functor colim_J of the attaching J-maps of X. One may also check that if A is a subcomplex of X/J, then $\check{A}$ has a natural structure of a subcomplex of X. In particular, if A^p is the p-skeleton of X/J, then $\check{A}^p = X^p$ is the p-skeleton of X.

Proposition 3.9 (Local contractibility). Let X be a J-CW complex and $A = \{a\}$ be a point of X/J. Then there is an object $j \in |J|$ such that $\check{A} \cong \underline{j}$, and $(X, \check{A})$ is a J-NDR pair.

Proof. Let a be in the p-skeleton but not in the $(p-1)$-skeleton of X/J. Then there is a unique attaching map $f : S^{p-1} \times \underline{j} \longrightarrow X^{p-1}$ such that a is in the interior of D^p. It follows that $\check{A} \cong \underline{j}$. To construct the required neighborhood U, first take an open ball U_1 contained in the interior of B_p and centered at a. Then U_1 is a neighborhood in $(X/J)^p$ that contracts to A. One then extends U_1 inductively cell by cell by the usual space level procedure to construct the required neighborhood U. $\square$

Proposition 3.10. Let X be a J-CW complex and A be a subcomplex of X/J. Then $(X, \check{A})$ is a J-NDR pair.

Proof. It follows from J-HELP that $\check{A} \longrightarrow X$ is a J-cofibration. Just as on the space level, a J-cofibration is the inclusion of a J-NDR pair. $\square$

R. J. Piacenza. Homotopy theory of diagrams and CW-complexes over a category. Canadian J. Math. 43(1991), 814-824.

4. The homology and cohomology theory of diagrams

The ordinary homology and cohomology theories of I§3 are special cases of a construction that applies to the category $\mathscr{U}^J$ for any J. The difference is that the theory in I§3 started with G-CW complexes and then passed to the associated diagrams defined on the orbit category of G, whereas we here exploit the theory of J-CW complexes. There is again a vast literature on the cohomology of diagrams, some relevant references being listed in Section 1.

Define a J-coefficient system to be a continuous contravariant functor $M : J \longrightarrow \mathscr{A}b$. Continuity ensures that M factors through the homotopy category hJ. Let $\mathscr{A}b^{hJ}$ be the category of J-coefficient systems. It is an Abelian category, and we can do homological algebra in it. As in I§4, a covariant homotopy invariant functor $\mathscr{U} \longrightarrow \mathscr{A}b$ induces a functor from J-spaces to J-coefficient systems by composition; we name such functors by underlining the name of the given functor. Of course, we also have the notion of a covariant J-coefficient system.

Let (X, A) be a relative J-CW complex with n skeleton X^n and observe that

$$X^n/X^{n-1} = (\coprod_{j_\alpha} D^n \times \underline{j}_\alpha)/(\coprod_{j_\alpha} S^{n-1} \times \underline{j}_\alpha) \cong S^n \wedge (\underline{j}_\alpha)_+, \tag{4.1}$$

where the $+$ indicates the addition of disjoint basepoints. Define a chain complex $\underline{C}_*(X, A)$ in $\mathscr{A}b^J$, called the J-cellular chains of (X, A), by setting

$$\underline{C}_n(X, A) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z}). \tag{4.2}$$

The connecting homomorphisms of the triples (X^n, X^{n-1}, X^{n-2}) specify the differential

$$d : \underline{C}_n(X, A) \longrightarrow \underline{C}_{n-1}(X, A). \tag{4.3}$$

Clearly (5.1) implies that

$$\underline{C}_n(X, A)(j) = \sum_{j_\alpha} \underline{\tilde{H}}_0(J(j, j_\alpha)_+; \mathbb{Z}). \tag{4.4}$$

The construction is functorial with respect to cellular maps $(X, A) \longrightarrow (Y, B)$.

For a covariant J-coefficient system N, define the cellular chain complex of (X, A) with coefficients N by

$$C_*(X, A; N) = \underline{C}_*(X; A) \otimes_J N, \tag{4.5}$$

where the tensor product on the right is interpreted as the coend over J. Passing to homology, we obtain the cellular homology $H_*(X, A; N)$.

For a contravariant J-coefficient system M, define the cellular cochain complex of (X, A) with coefficients M by

$$C^*(X, A; M) = \mathrm{Hom}_J(\underline{C}_*(X; A), M). \tag{4.6}$$

Passing to cohomology, we obtain the cellular cohomology $H^*(X, A; M)$.

THEOREM 4.7. Cellular homology and cohomology for pairs of J-CW complexes satisfy the standard Eilenberg-Steenrod axioms, suitably reformulated for diagrams.

REMARK 4.8. We may extend the cellular theory to arbitrary pairs of diagrams by means of cellular approximations; see Proposition 4.6. That is, we extend our homology and cohomology theories to theories that carry weak equivalences to isomorphisms. We may also adapt Ilman's construction of equivariant singular theory to construct a singular theory for diagrams. Of course, the singular theory is isomorphic to the cellular theory on the category of J-CW complexes.

S. Illman. Equivariant singular homology and cohomology. Memoirs Amer. Math. Soc. No. 156. 1975.

5. The closed model structure on $\mathscr{U}^J$

Just as the category of spaces has a (closed) model structure in the sense of Quillen, so does the category of G-spaces for any G. This point of view has not been taken earlier since the conclusions are obvious to the experts and perhaps not very helpful to the novice on a first reading. However, since the homotopical properties of categories of diagrams are likely to be less familiar than those of the category of spaces, it is valuable to understand how they inherit model structures from the standard model structure on $\mathscr{U}$, which is the special case of the trivial category J in the definitions here. We use the name q-fibration and q-cofibration for the model structure fibrations and cofibrations to avoid confusion with other kinds of fibrations and cofibrations. The weak equivalences of the model structure will be the weak equivalences that we have already defined; an acyclic q-fibration is one that is a weak equivalence, and similarly for acyclic q-cofibrations. Consider diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

The map g has the left lifting property (LLP) with respect to f if one can always fill in the dotted arrow. The right lifting property (RLP) is defined dually.

Definition 5.1. A J-map $f : X \longrightarrow Y$ is a q-fibration if $f(j) : Y(j) \longrightarrow X(j)$ is a Serre fibration for each object $j \in |J|$. Observe that f is a q-fibration if f has the homotopy lifting property for all objects of the form $I^n \times \underline{j}$. A map $g : A \longrightarrow B$ is a q-cofibration if it has the LLP with respect to all acyclic q-cofibrations.

Theorem 5.2. With the structure just defined, $\mathscr{U}^J$ is a model category.

Proof. Just as as for spaces, one quickly checks Quillen's axioms, using the factorization lemma below to verify the factorization axiom $\underline{M2}$. $\square$

As for spaces, the proof leads directly to the following characterizations of q-cofibrations and of acyclic q-fibrations.

Corollary 5.3. A J-map $g : A \longrightarrow B$ is a q-cofibration if and only if it is a retract of the inclusion $A' \longrightarrow B'$ of a relative J-complex (B', A').

Corollary 5.4. A J-map $f : X \longrightarrow Y$ is an acyclic q-fibration if and only if it has the RLP with respect to each q-cofibration $S^n \times \underline{j} \longrightarrow D^{n+1} \times \underline{j}$.

Lemma 5.5 (Quillen's factorization lemma). Any J-map $f : X \longrightarrow Y$ can be factored as $f = p \circ g$, where g is a q-cofibration and p is an acyclic q-fibration.

PROOF. We construct a diagram

$$\begin{tikzcd} X \arrow[r, "g_0"] \arrow[dr, "f"'] & Z^0 \arrow[r, "g_1"] \arrow[d, "p_0"] & Z^1 \arrow[r] & \cdots \\ & Y & & \end{tikzcd}$$

as follows. Let $Z^{-1} = X$ and $p_{-1} = f$. Having obtained Z^{n-1}, consider the set of all diagrams of the form

$$\begin{tikzcd} S^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "t_\alpha"] \arrow[d] & Z^{n-1} \arrow[d, "p_{n-1}"] \\ D^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "s_\alpha"'] & Y. \end{tikzcd}$$

Forming the coproduct over all of the left vertical arrows, we may define $g_n : Z^{n-1} \longrightarrow Z^n$ by the pushout diagram

$$\begin{tikzcd} \coprod S^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "\coprod t_\alpha"] \arrow[d] & Z^{n-1} \arrow[d, "p_{n-1}"] \\ \coprod D^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "\coprod s_\alpha"'] & Z^n. \end{tikzcd}$$

We have allowed the zero dimensional pair $(D^0, S^{-1}) = (\{pt\}, \emptyset)$ in this construction. Define $p_n : Z^n \longrightarrow Y$ by pushing out along p_{n-1} and the coproduct of the maps s_α. Then let

$$Z = \operatorname{colim} Z^n, \quad p = \operatorname{colim} p_n, \quad \text{and} \quad g = \operatorname{colim} g_n g_{n-1} \cdots g_0.$$

One may check that g has the LLP with respect to each acyclic q-fibration and, by the "small object argument" based on the compactness of the D^n, that p is an acyclic q-fibration. $\square$

Let $\bar{h}\mathscr{U}^J$ be the localization of $h\mathscr{U}^J$ obtained by formally inverting the weak equivalences. The model structure implies that $\bar{h}\mathscr{U}^J$ is equivalent to the homotopy category of J-CW complexes, as we indicate next.

LEMMA 5.6. Let $X = \operatorname{colim} X_n$ taken over a sequence of J-cofibrations such that each X_n has the homotopy type of a J-CW complex. Then X has the homotopy type of a J-CW complex.

PROOF. Up to homotopy, we may approximate the sequence by a sequence of J-CW complexes and cellular inclusions; we then use the homotopy invariance of colimits (Theorem 1.2). $\square$

The following proposition follows easily.

PROPOSITION 5.7. Each J-complex is of the homotopy type of a J-CW complex.

THEOREM 5.8 (APPROXIMATION THEOREM). There is a functor $\Gamma : \mathscr{U}^J \longrightarrow \mathscr{U}^J$ and a natural transformation $\gamma : \Gamma \longrightarrow \mathrm{id}$ such that, for each $X \in \mathscr{U}^J$, ΓX is a J-complex and $\gamma : \Gamma X \longrightarrow X$ is an acyclic q-fibration.

PROOF. Applying Lemma 5.3 to the inclusion of the empty set in X, we obtain an acyclic q-fibration $\gamma : \Gamma X \longrightarrow X$. By the explicit construction, we see that ΓX is a J-complex, Γ is a functor, and γ is a natural transformation. $\square$

The following corollary is immediate from the previous two results.

COROLLARY 5.9. The category $\bar{h}\mathscr{U}^J$ is equivalent to the homotopy category of J-CW complexes.

D. G. Quillen. Homotopical Algebra. Springer Lecture Notes in Mathematics Vol. 43. 1967.
W. G. Dwyer and J. Spalinsky. Homotopy theories and model categories. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 73-126.

6. Another proof of Elmendorf's theorem

The theory of diagrams leads to an alternative proof of Elmendorf's theorem V.3.2, one which gives a precise cellular perspective and illustrates the force of model category techniques. We adopt the notations of V§3.

Observe that the fixed point diagram functor Φ from G-spaces to $\mathscr{G}$-spaces carries $X \times G/H$ to $X \times \underline{G/H}$ for a space X regarded as a G-trivial G-space. Thus it preserves cells. It also preserves the pushouts relevant to cellular theory.

LEMMA 6.1. If

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is a pushout of G-spaces in which i is a closed inclusion, then

$$\begin{tikzcd} \Phi A \arrow[r] \arrow[d, "\Phi i"'] & \Phi X \arrow[d] \\ \Phi B \arrow[r] & Y \end{tikzcd}$$

is a pushout of $\mathscr{G}$-spaces.

Proof. Stripping away the topology we see that this holds on the set level since every G-set is a coproduct of orbits. One may then check that the topologies agree. $\square$

Theorem 6.2. Each $\mathscr{G}$-complex (or $\mathscr{G}$-CW complex) $Y \in \mathscr{G}\mathscr{U}$ is isomorphic to ΦX for some G-complex (or G-CW complex) X. Therefore Φ is an isomorphism between the category of G-complexes (or $\mathscr{G}$-CW complexes) and the category of $\mathscr{G}$-complexes (or $\mathscr{G}$-CW complexes).

Proof. The functor Φ carries G-complexes to $\mathscr{G}$-complexes since it preserves cells, the relevant pushouts, and ascending unions. The assertion follows since Φ is full and faithful: inductively, the attaching maps of Y are in the image of Φ. $\square$

This leads to our alternative version of V.3.2.

Theorem 6.3 (Elmendorf). There is a functor $\Psi : \mathscr{G}\mathscr{U} \longrightarrow G\mathscr{U}$ and a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{id}$ such that ΨX is a G-complex, $\Phi\Psi X$ is a $\mathscr{G}$-complex, and $\varepsilon : \Phi\Psi X \longrightarrow X$ is a weak equivalence of G-spaces for each $\mathscr{G}$-space X. Therefore Φ and Ψ induce an equivalence of categories between $\bar{h}\mathscr{G}\mathscr{U}$ and $\bar{h}G\mathscr{U}$.

Proof. We construct the functor Ψ and transformation ε by using the functor Γ and transformation p given in Theorem 5.7 on the level of diagrams and using Theorem 6.2 to transport from $\mathscr{G}$-complexes to G-complexes. The result follows from the cited results and Corollary 5.8. $\square$

Corollary 6.4. Let Y be a G-space of the homotopy type of a G-CW complex. Then, for any $\mathscr{G}$-space X,

$$hG\mathscr{U}(Y, \Psi X) \cong h\mathscr{G}\mathscr{U}(\Phi Y, X) \cong \bar{h}(\Phi Y, X).$$

Proof. This follows from Theorem 6.3 and generalities about model categories. $\square$

In turn, this implies the following comparison with the original form, V.3.2, of Elmendorf's theorem.

COROLLARY 6.5. Write Ψ' and ε' for the constructions given in V.3.2. For a $\mathscr{G}$-space X, there is a weak equivalence of G-spaces $\xi : \Psi X \longrightarrow \Psi' X$ such that ξ is natural up to homotopy and the following diagram commutes up to homotopy:

$$\begin{tikzcd} \Phi\Psi X \arrow[rr, "\Phi\xi"] \arrow[dr, "\varepsilon"'] & & \Phi\Psi' X \arrow[dl, "\varepsilon'"] \\ & X. & \end{tikzcd}$$

CHAPTER VII

Equivariant Bundle theory and Classifying Spaces

1. The definition of equivariant bundles

Equivariant bundle theory can be developed at various levels of generality. We assume given a subgroup Π of a compact Lie group Γ. We set $G = \Gamma/\Pi$, and we let $q : \Gamma \longrightarrow G$ be the quotient homomorphism. That is, we consider an extension of compact Lie groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Many sources restrict attention to split extensions, but we see little point in that. By far the most interesting case is $\Gamma = G \times \Pi$. When Π is $O(n)$ or $U(n)$, this case will lead to real and complex equivariant K-theory.

Define a principal $(\Pi;\Gamma)$-bundle to be the projection to orbits $p : E \longrightarrow E/\Pi = B$ of a Π-free Γ-space E. Note that G acts on the base space B. Let F be a Γ-space. By a G-bundle with structural group Π, total group Γ, and fiber F, we mean the projection $E \times_\Pi F \longrightarrow B$ induced by a principal $(\Pi;\Gamma)$-bundle E; E is called the associated principal bundle. Although we prefer to think of bundles this way, it is not hard to give an intrinsic characterization of when a G-map $Y \longrightarrow B$ that is a Π-bundle with fiber F is such a $(\Pi;\Gamma)$-bundle.

When $\Gamma = G \times \Pi$, we shall refer to (G,Π)-bundles rather than to $(\Pi; G \times \Pi)$-bundles. Here it is usual to require the fiber F be a Π-space. A principal (G,Π)-bundle E has actions by G and Π that commute with one another; it is usual to write the action of Π on the right and the action of G on the left. Equivariant vector bundles fit into this framework: a $(G, O(n))$-bundle with fiber $\mathbb{R}^n$ is called an n-plane G-bundle, and similarly in the complex case. The tangent and normal bundles of a smooth G-manifold give examples.

EXAMPLE 1.1. A finite G-cover $p : Y \longrightarrow B$ is a G-map that is also a finite cover. Such a map is necessarily a (G, Σ_n)-bundle with fiber the Σ_n-set $F = \{1, \dots , n\}$. Its associated principal (G, Σ_n)-bundle E is the subspace of $\mathrm{Map}(F, Y)$ consisting of the bijections onto fibers of p.

Classical bundle theory readily generalizes to the equivariant context, and we content ourselves with a very brief summary of some of the main points. A principal $(\Pi; \Gamma)$-bundle is said to be trivial if it is equivalent to a bundle of the form

$$q \times \mathrm{id} : \Gamma \times_\Lambda U \longrightarrow G \times_H U,$$

where $H \subset G$, $\Lambda \subset \Gamma$, $\Lambda \cap \Pi = e$, q maps Λ isomorphically onto H, and U is an H-space regarded as a Λ-space by pullback along q. Provided that E and therefore also B are completely regular, a principal $(\Pi; \Gamma)$-bundle $p : E \longrightarrow B$ is locally trivial. If, in addition, B is paracompact, then p is numerable. Numerable $(\Pi; \Gamma)$-bundles satisfy the equivariant bundle covering homotopy property, and a numerable bundle E over $B \times I$ is equivalent to the bundle $(E \times \{0\}) \times I$. Therefore the pullbacks of a numerable $(\Pi; \Gamma)$-bundle along homotopic G-maps are equivalent.

R. K. Lashof. Equivariant bundles. Ill. J. Math. 26(1982).
R. K. Lashof and J. P. May. Generalized equivariant bundles. Bulletin de la Société Mathématique de Belgique 38(1986), 265-271.
L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. (IV§1.) Springer Lecture Notes in Mathematics Vol. 1213. 1986.

2. The classification of equivariant bundles

Let $\mathscr{B}(\Pi; \Gamma)(X)$ be the set of equivalence classes of principal $(\Pi; \Gamma)$-bundles with base G-space X. We assume that X has the homotopy type of a G-CW complex, and we check that this implies that any bundle over X has the homotopy type of a Γ-CW complex. Then Elmendorf's theorem, V.3.2, specializes to give a classification theorem for principal $(\Pi; \Gamma)$-bundles.

DEFINITION 2.1. Define $\mathscr{F}(\Pi; \Gamma)$ to be the family of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and observe that an $\mathscr{F}(\Pi; \Gamma)$-space is the same thing as a Π-free Γ-space. Write

$$E(\Pi; \Gamma) = E\mathscr{F}(\Pi; \Gamma) \quad \text{and} \quad B(\Pi; \Gamma) = E(\Pi; \Gamma)/\Pi,$$

and let

$$\pi : E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$$

be the resulting principal $(\Pi;\Gamma)$-bundle. In the case $\Gamma = G\times\Pi$, write $\mathscr{F}_G(\Pi) = \mathscr{F}(\Pi; G\times\Pi)$,

$$E_G(\Pi) = E(\Pi; G\times\Pi) \text{ and } B_G(\Pi) = B(\Pi; G\times\Pi).$$

Observe that, since $E(\Pi;\Gamma)$ is a contractible space, $B(\Pi;\Gamma)$ is a model for $B\Pi$ that carries a particular action by G.

Theorem 2.2. The bundle $\pi : E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$ is universal . That is, pullback of π along G-maps $X \longrightarrow B(\Pi;\Gamma)$ gives a bijection

$$[X, B(\Pi;\Gamma)]_G \longrightarrow \mathscr{B}(\Pi;\Gamma)(X).$$

It is crucial to the utility of this result to understand the fixed point structure of $B(\Pi;\Gamma)$. For any principal $(\Pi;\Gamma)$-bundle $p : E \longrightarrow B$ and any $H\subset G$, one can check that B^H is the disjoint union of the spaces $p(E^\Lambda)$, where Λ runs over the Π-conjugacy classes of subgroups $\Lambda\subset\Gamma$ such that $\Lambda\cap\Pi = e$ and $q(\Lambda) = H$. Define

$$\Pi^\Lambda \equiv \Pi\cap N_\Gamma\Lambda = \Pi\cap Z_\Gamma\Lambda, \tag{2.3}$$

where $Z_\Gamma\Lambda$ is the centralizer of Λ in Γ; the equality here is an easy observation. Then E^Λ is a principal $(\Pi^\Lambda; W_\Gamma\Lambda)$-bundle and $p(E^\Lambda)$ is its base space. We can go on to analyze the structure of B^H as a W_GH-space. In the case of the universal bundle, we can determine the structure of E^Λ by use of IV.4.7. Putting things together, we arrive at the following conclusion.

Theorem 2.4. For a subgroup H of G,

$$B(\Pi;\Gamma)^H = \coprod B(\Pi^\Lambda),$$

where the union runs over the Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda\cap\Pi = e$ and $q(\Lambda) = H$; as a W_GH-space,

$$B(\Pi;\Gamma)^H = \coprod W_GH \times_{V(\Lambda)} B(\Pi^\Lambda; W_\Gamma\Lambda),$$

where the union runs over the $q^{-1}(N_GH)$-conjugacy classes of such groups Λ and $V(\Lambda) = W_\Gamma\Lambda/\Pi^\Lambda$ is the image of $W_\Gamma\Lambda$ in W_GH.

Here, by use of Lie group theory, $V(\Lambda)$ has finite index in W_GH.

Specializing to $\Gamma = G\times\Pi$, we see that the subgroups Λ of Γ such that $\Lambda\cap\Pi = e$ are exactly the twisted diagonal subgroups

$$\Delta(\rho) = \{(h,\rho(h))|h\in H\}, \tag{2.5}$$

where H is a subgroup of G and $\rho : H \longrightarrow \Pi$ is a homomorphism. Let $N(\rho) = N_{G\times\Pi}\Delta(\rho)$ and observe that

$$N(\rho) = \{(g,\pi)|g\in N_GH \;\text{ and }\; \pi\rho(h)\pi^{-1} = \rho(ghg^{-1}) \text{ for all } h\in H\}.$$

Therefore $\Pi\cap N(\rho)$ coincides with the centralizer

$$\Pi^\rho = \{\pi|\pi\rho(h) = \rho(h)\pi \text{ for all } h\in H\}. \tag{2.6}$$

Let

$$W(\rho) = W_{G\times\Pi}\Delta(\rho) \;\text{ and }\; V(\rho) = W(\rho)/\Pi^\rho \subset W_GH.$$

As usual, let $\mathrm{Rep}(G,\Pi)$ denote the set of Π-conjugacy classes of homomorphisms $G\longrightarrow\Pi$. Define an action of the group W_GH on the set $\mathrm{Rep}(H,\Pi)$ by letting $(gH)\rho$ be the conjugacy class of $g\cdot\rho$, where, for $g\in N_GH$, $g\cdot\rho : H\longrightarrow\Pi$ is the homomorphism specified by $(g\cdot\rho)(h) = \rho(g^{-1}hg)$. Observe that the isotropy group of (ρ) is $V(\rho)$.

Theorem 2.7. For a subgroup H of G,

$$(B_G\Pi)^H = \coprod B(\Pi^\rho),$$

where the union runs over $(\rho)\in\mathrm{Rep}(H,\Pi)$; as a W_GH-space,

$$(B_G\Pi)^H = \coprod W_GH\times_{V(\rho)} B(\Pi^\rho; W(\rho)),$$

where the union runs over the orbit set $\mathrm{Rep}(H,\Pi)/W_GH$.

It is important to observe that the group $W(\rho)$ need not split as a product $V(\rho)\times\Pi^\rho$ in general. Therefore, in order to fully understand the classifying G-spaces for (G,Π)-bundles, one is forced to study the classifying spaces for the more general kind of bundles that we have introduced. These are complicated objects, and their study is in a primitive state. In particular, rather little is known about equivariant characteristic classes. Such classes are understood in Borel cohomology, however. By the universal property of $E(\Pi;\Gamma)$, there is a Γ-map $E\Gamma\longrightarrow E(\Pi;\Gamma)$, which is unique up to homotopy. The induced G-map $E\Gamma/\Pi\longrightarrow B(\Pi;\Gamma)$ is a nonequivariant equivalence and so induces an isomorphism on Borel cohomology. The projection $EG \times E\Gamma \longrightarrow E\Gamma$ is clearly a Γ-homotopy equivalence, and it induces an equivalence

$$EG \times_G (E\Gamma/\Pi) = (EG \times E\Gamma)/\Gamma \longrightarrow E\Gamma/\Gamma = B\Gamma.$$

This already implies the following calculation. We again denote Borel cohomology by H_G^* for the moment.

THEOREM 2.8. With any coefficients, $H_G^*(B(\Pi;\Gamma)) \cong H^*(B\Gamma)$. With field coefficients, $H_G^*(B_G\Pi) \cong H^*(BG) \otimes H^*(B\Pi)$ as an $H^*(BG)$-module.

The interpretation is that the Borel cohomology characteristic classes of a principal (G,Π)-bundle E over X are determined by the $H^*(BG)$-module structure on $H_G^*(X)$ together with the nonequivariant characteristic classes of the Π-bundle $EG \times_G E$ over $EG \times_G X$.

We shall later see that generalized versions of the Atiyah-Segal completion theorem and of the Segal conjecture give calculations of the characteristic classes of (G,Π)-bundles in equivariant K-theory and in equivariant cohomotopy.

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3. Some examples of classifying spaces

It is often valuable to have alternative descriptions of universal bundles. We have Grassmannian models when Π is an orthogonal or unitary group. These lead to good models for the classifying spaces for equivariant K-theory, and, just as nonequivariantly, they are useful for the proof of equivariant versions of the Thom cobordism theorem.

EXAMPLE 3.1. For a real inner product G-space V, let $BO(n,V)$ be the G-space of n-planes in V and let $EO(n,V)$ be the G-space whose points are pairs consisting of an n-plane π in V and a vector $v \in \pi$. The map $EO(n,V) \longrightarrow BO(n,V)$ that sends (π, v) to π is a real n-plane G-bundle. Provided that V is large enough, say the direct sum of infinitely many copies of each irreducible real representation of G, p is a universal real n-plane G-bundle. A similar construction works in the complex case.

Clearly a principal $(\Pi;\Gamma)$-bundle E is universal if and only if E^{Λ} is contractible for $\Lambda \in \mathscr{F}(\Pi;\Gamma)$. Using the fact that the space of G-maps from a free G-CW complex to a nonequivariantly contractible G-space is contractible, one can use this criterion to obtain a simple model that has particularly good naturality properties. Regard EG as a Γ-space via $q : \Gamma \longrightarrow G$ and define

$$\mathrm{Sec}(EG, E\Gamma) \subset \mathrm{Map}(EG, E\Gamma)$$

to be the sub Γ-space consisting of those maps $f : EG \longrightarrow E\Gamma$ such that the composite of $Eq : E\Gamma \longrightarrow EG$ and f is the identity map. Note that

$$\mathrm{Sec}(EG, E(G \times \Pi)) = \mathrm{Map}(EG, E\Pi)$$

since $E(G \times \Pi)$ is homeomorphic to $EG \times E\Pi$.

Theorem 3.2. *The Γ-space $\mathrm{Sec}(EG, E\Gamma)$ is a universal principal $(\Pi;\Gamma)$-bundle and therefore the G-space $\mathrm{Sec}(EG, E\Gamma)/\Pi$ is a model for $B(\Pi;\Gamma)$. In particular, the $G\times\Pi$-space $\mathrm{Map}(EG, E\Pi)$ is a universal principal (G,Π)-bundle and therefore the G-space $\mathrm{Map}(EG, E\Pi)/\Pi$ is a model for $B_G\Pi$.*

Since we are interested in maps from G-CW complexes into classifying spaces, the fact that these models need not have the homotopy types of G-CW complexes need not concern us.

Observe that the map $\pi : E\Gamma \longrightarrow B\Gamma$ induces a natural G-map

$$\alpha : B(\Pi;\Gamma) = \mathrm{Sec}(EG, E\Gamma)/\Pi \longrightarrow \mathrm{Sec}(EG, B\Gamma), \tag{3.3}$$

where $\mathrm{Sec}(EG, B\Gamma)$ is the G-space of maps $f : EG \longrightarrow B\Gamma$ such that the composite of f and $Bq : B\Gamma \longrightarrow BG$ is $\pi : EG \longrightarrow BG$. With $\Gamma = G \times \Pi$, this map is

$$\alpha : B_G\Pi \longrightarrow \mathrm{Map}(EG, B\Pi). \tag{3.4}$$

These maps have bundle theoretic interpretations. Restricting for simplicity to the case $\Gamma = G \times \Pi$, let

$$\mathscr{B}_G(\Pi)(X) \cong [X, B_G\Pi]_G$$

be the set of equivalence classes of (G,Π)-bundles over X and let $\mathscr{B}(\Pi)(X)$ be the set of equivalence classes of nonequivariant Π-bundles over X. By adjunction, a G-map $X \longrightarrow \mathrm{Map}(EG, B\Pi)$ is the same as a map $EG \times_G X \longrightarrow B\Pi$. Thus the represented equivalent of α is the Borel construction on bundles that was relevant to Theorem 2.8; it gives

$$\mathscr{B}_G(\Pi)(X) \longrightarrow \mathscr{B}(\Pi)(EG \times_G X).$$

It is important to know how much information this construction loses, hence it is important to know how near α is to being an equivalence. Elementary covering space theory gives the following result.

PROPOSITION 3.5. If Γ is discrete, then the G-map α of (3.3) is a homeomorphism. If Π, but not necessarily G, is discrete, then the G-map α of (3.4) is a homeomorphism.

An Abelian compact Lie group is the product of a finite Abelian group and a torus. Using ordinary cohomology to study the finite factor and continuous cohomology to handle the torus factor, Lashof, May, and Segal proved another result along these lines.

THEOREM 3.6. If G is a compact Lie group and Π is an Abelian compact Lie group, then the G-map $\alpha : B_G\Pi \longrightarrow \mathrm{Map}(EG, B\Pi)$ is a weak equivalence.

Consequences of the Sullivan conjecture will tell us much more about these maps. To see this, we will need to know the behavior of the maps α on fixed point spaces. We have determined the fixed point spaces $B(\Pi;\Gamma)^H$, and it is clear that

$$\mathrm{Sec}(EG, B\Gamma)^H = \mathrm{Sec}(BH, B\Gamma)$$

is the space of maps $f : BH \longrightarrow B\Gamma$ such that

$$Bq \circ f = Bi : BH \longrightarrow BG,$$

where $i : H \longrightarrow G$ is the inclusion and we take Bi to be the quotient map $EG/H \longrightarrow EG/G$. In particular,

$$\mathrm{Sec}(BH, BG \times B\Pi) = \mathrm{Map}(BH, B\Pi).$$

LEMMA 3.7. Let $\Lambda \subset \Gamma$ satisfy $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$. Define a homomorphism $\mu : H \times \Pi^\Lambda \longrightarrow \Gamma$ by $\mu(q(\lambda), \pi) = \lambda\pi$ and observe that $q \circ \mu = i \circ \pi_1 : H \times \Pi^\Lambda \longrightarrow G$. The restriction of

$$\alpha^H : B(\Pi;\Gamma)^H \longrightarrow \mathrm{Sec}(BH, B\Gamma)$$

to $B(\Pi^\Lambda)$ is the adjoint of the classifying map

$$B\mu : BH \times B(\Pi^\Lambda) = B(H \times \Pi^\Lambda) \longrightarrow B\Gamma.$$

Therefore, if $\Gamma = G \times \Pi$, the restriction of

$$\alpha^H : (B_G\Pi)^H \longrightarrow \mathrm{Map}(BH, B\Pi)$$

to $B(\Pi^\rho)$, $\rho : H \longrightarrow \Pi$, is the adjoint of the map of classifying spaces

$$B\nu : BH \times B(\Pi\rho) = B(H \times \Pi^\rho) \longrightarrow B\Pi,$$

where $\nu : H \times \Pi^\rho \longrightarrow \Pi$ is defined by $\nu(h, \pi) = \rho(h)\pi$.

Consider what happens on components. In nonequivariant homotopy theory, maps between the classifying spaces of compact Lie groups have been studied for many years. One focus has been the question of when passage to classifying maps

$$B : \mathrm{Rep}(G, \Pi) \longrightarrow [BG, B\Pi]$$

is a bijection. We now see that, for $H \subset G$, a map $BH \longrightarrow B\Pi$ not in the image of B corresponds to a principal Π-bundle over BH that does not come from a principal (G, Π)-bundle over an orbit G/H. The equivariant results above imply that there are no such exotic maps if Π is either finite or Abelian. The Sullivan conjecture will give information about general compact Lie groups Π under restrictions on G.

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CHAPTER VIII

The Sullivan Conjecture

1. Statements of versions of the Sullivan conjecture

We defined the homotopy orbit space of a G-space X to be

$$X_{hG} = EG \times_G X,$$

and we defined the homotopy fixed point space of X dually:

$$X^{hG} = \mathrm{Map}(EG, X)^G = \mathrm{Map}_G(EG, X)$$

is the space of G-maps $EG \longrightarrow X$. The projection $EG \longrightarrow *$ induces

$$X^G = \mathrm{Map}(*, X)^G \longrightarrow \mathrm{Map}(EG, X) = X^{hG}.$$

It sends a fixed point to the constant map $EG \longrightarrow X$ at that fixed point. It is very natural to ask how close this map is to being a homotopy equivalence. Thinking equivariantly, it is even more natural to ask how close the G-map

$$\eta : X = \mathrm{Map}(*, X) \longrightarrow \mathrm{Map}(EG, X)$$

is to being a G-homotopy equivalence. Since a G-map $f : X \longrightarrow Y$ that is a nonequivariant equivalence induces a weak equivalence of G-spaces

$$\mathrm{Map}(W, Y) \longrightarrow \mathrm{Map}(W, X)$$

for any free G-CW complex W, such as EG, one cannot expect η to be an equivalence in general. Very little is known about this question for general finite groups. However, for finite p-groups G, to which we restrict ourselves unless we specify otherwise, the Sullivan conjecture gives a beautiful answer. We agree to work in the categories $\bar{h}\mathscr{U}$ and $\bar{h}G\mathscr{U}$, implicitly applying CW approximation. This allows us to ignore the distinction between weak and genuine equivalences.

THEOREM 1.1 (GENERALIZED SULLIVAN CONJECTURE). Let X be a nilpotent finite G-CW complex. Then the natural G-map

$$\hat{X}_p \longrightarrow \mathrm{Map}(EG, \hat{X}_p)$$

is an equivalence.

The hypothesis that X be nilpotent can be removed by applying the Bousfield-Kan simplicial completion on fixed point spaces and then assembling these completed fixed point spaces to a global G-completion by means of Elmendorf's construction. This equivariant interpretation of the Sullivan conjecture was noticed by Haeberly, who also gave some information for finite groups that are not p-groups. Looking at fixed points under $H \subset G$ and noting that EG is a model for EH, we see that the result immediately reduces to the fixed point space level.

THEOREM 1.2 (MILLER, CARLSSON, LANNES). Let X be a nilpotent finite G-CW complex, where G is a finite p-group. Then the natural map

$$(X^G)\hat{_p} \cong (\hat{X}_p)^G \longrightarrow \mathrm{Map}(EG, \hat{X}_p)^G = (\hat{X}_p)^{hG}$$

is an equivalence.

Again, the nilpotence hypothesis is unnecessary provided that one understands $\hat{X}_p$ to mean the Bousfield-Kan completion of X, which generalizes the nilpotent completion that we defined, and takes $(X^G)\hat{_p}$ and not $(\hat{X}_p)^G$ as the source: there is a natural map

$$(X^G)\hat{_p} \longrightarrow (\hat{X}_p)^G,$$

but it is not an equivalence in general. When G acts trivially on X, the result was first proven by Miller, and he deduced the following powerful consequence.

THEOREM 1.3 (MILLER). Let G be a discrete group such that all of its finitely generated subgroups are finite and let X be a connected finite dimensional CW complex. Then $\pi_*F(BG, X) = 0$.

To deduce this from Theorem 1.2, one first observes that any map $BG \longrightarrow X$ induces the trivial map of fundamental groups and so lifts to the universal cover, while a map $\Sigma^n BG \longrightarrow X$ for $n > 0$ trivially lifts to the universal cover. Thus one can assume that X is simply connected. Note that this reduction depends on the fact that we are here working with finite dimensional and not just finite complexes, and one must generalize Theorem 1.2 accordingly; this seems to require trivial action on X. One then applies an inductive argument to reduce to the case $G = \mathbb{Z}/p$. Here the weak equivalence $\hat{X}_p \longrightarrow \mathrm{Map}(BG, \hat{X}_p)$ implies that $\pi_* F(BG, \hat{X}_p) = 0$, and this implies that $\pi_* F(BG, X) = 0$.

The general case of Theorem 1.2 reduces immediately to the case when $G = \mathbb{Z}/p$, by induction on the order of G. To see this, consider an extension

$$1 \longrightarrow C \longrightarrow G \longrightarrow J \longrightarrow 1,$$

where C is cyclic of order p. For any G-space Y, $(Y^{hC})^{hJ}$ is equivalent to Y^{hG}. In fact, by passing to G-fixed points by first passing to C-fixed points and then to J-fixed points, we obtain a homeomorphism

$$\mathrm{Map}(EJ \times EG, Y)^G \cong \mathrm{Map}(EJ, \mathrm{Map}(EG, Y)^C)^J.$$

Since $EJ \times EG$ is a free contractible G-space and EG is a free contractible C-space, this gives the stated equivalence of homotopy fixed point spaces. The equivalence $(X^C)^{\wedge}_p \longrightarrow (\hat{X}_p)^{hC}$ is a J-map, hence it induces an equivalence on passage to J-homotopy fixed point spaces, and the map of Theorem 1.2 coincides with the composite equivalence

$$(X^G)^{\wedge}_p = ((X^C)^J)^{\wedge}_p \longrightarrow ((X^C)^{\wedge}_p)^{hJ} \longrightarrow ((\hat{X}_p)^{hC})^{hJ} \cong (\hat{X}_p)^{hG}.$$

When $G = \mathbb{Z}/p$, Theorem 1.2 was proven independently by Lannes and Miller, using nonequivariant techniques, and by Carlsson, using equivariant techniques. Lannes later gave a variant of his original proof that generalizes the result, uses equivariant ideas, and enjoys a pleasant conceptual relationship to Smith theory. We shall sketch that proof in the following three sections.

There is a basic principle in equivariant topology to the effect that, when working at a prime p, results that hold for p-groups can be generalized to p-toral groups G, which are extensions of the form

$$1 \longrightarrow T \longrightarrow G \longrightarrow \pi \longrightarrow 1.$$

The point is that the circle group can be approximated by the union σ_∞ of its p-subgroups σ_n of $(p^n)th$ roots of unity, and an r-torus T can be approximated by the union τ_∞ of its p-subgroups $\tau_n = (\sigma_n)^r$. It is not hard to see that the map $B\tau_\infty \longrightarrow BT$ induces an isomorphism on mod p homology. Using this basic idea, Notbohm generalized Theorem 1.2 to p-toral groups.

THEOREM 1.4 (NOTBOHM). The generalized Sullivan conjecture, Theorem 1.2, remains true as stated when G is a p-toral group.

Technically, this still works using Bousfield-Kan completion for "p-good" G-spaces X, for which $X \longrightarrow \hat{X}_p$ is a mod p equivalence.

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2. Algebraic preliminaries: Lannes' functors T and Fix

Let V be an elementary Abelian p-group, fixed throughout this section and the next. It would suffice to restrict attention to $V = \mathbb{Z}/p$. The notation V indicates that we think of V ambiguously as both a vector space over $\mathbb{F}_p$ and a group that will act as symmetries of spaces. We refer back to IV.2.3, which gave

$$H^*(X^V) \cong \mathbb{F}_p \otimes_{H^*(BV)} Un(S^{-1}H^*_V(X)) \tag{2.1}$$

for a finite dimensional V-CW complex X.

We begin by describing this in more conceptual algebraic terms. In this section, we let $\mathscr{U}$ be the category of unstable modules over the mod p Steenrod algebra A and let $\mathscr{K}$ be the category of unstable A-algebras. Thus the mod p cohomology of any space is in $\mathscr{K}$. We shall abbreviate notation by setting $H = H^*(BV)$. The celebrated functor $T : \mathscr{U} \longrightarrow \mathscr{U}$ introduced by Lannes is the left adjoint of $H \otimes (\cdot)$: for unstable A-modules M and N,

$$\mathscr{U}(TM, N) \cong \mathscr{U}(M, H \otimes N). \tag{2.2}$$

Observe that the adjoint of the map $M = \mathbb{F}_p \otimes M \longrightarrow H \otimes M$ induced by the unit of H gives a natural A-map $\pi : TM \longrightarrow M$. The key properties of the functor T are as follows.

(2.3) The functor T is exact and commutes with suspension.

(2.4) The functor T commutes with tensor products.

This property implies that if M is an unstable A-algebra, then so is TM. The resulting functor $T : \mathscr{K} \longrightarrow \mathscr{K}$ is also left adjoint to $H \otimes (\cdot)$: for unstable A-algebras M and N,

$$\mathscr{K}(TM, N) \cong \mathscr{K}(M, H \otimes N). \tag{2.5}$$

The Borel cohomology $H^*_V(X)$ is both an unstable A-algebra and an H-module. The action of H is given by a map of A-modules, and the bundle map

$$EV \times_V X \longrightarrow BV$$

induces a map $H \longrightarrow H^*_V(X)$ of unstable A-algebras. We codify these structures in algebraic definitions. Thus let $H\mathscr{U}$ be the category of unstable A-modules M together with an H-module structure given by an A-map $H \otimes M \longrightarrow M$. For such an H-A-module M, define an unstable A-module $\mathrm{Fix}(M)$ by

$$\mathrm{Fix}(M) = \mathbb{F}_p \otimes_{TH} TM \cong \mathbb{F}_p \otimes_H (H \otimes_{TH} TM). \tag{2.6}$$

The notation "Fix" anticipates a connection with (2.1). Here we have used (2.4) to give that TH is an augmented A-algebra and that TM is a TH-module; TH acts on H through the adjoint $TH \longrightarrow H$ of the coproduct $\psi : H \longrightarrow H \otimes H$. We have another adjunction. For unstable H-A-modules M and unstable A-modules N, we have

$$\mathscr{U}(\mathrm{Fix}(M), N) \cong H\mathscr{U}(M, H \otimes N). \tag{2.7}$$

Comparing the adjunctions (2.2) and (2.7), we easily find that, for an unstable A-module M,

$$\mathrm{Fix}(H \otimes M) \cong TM \quad \text{as unstable } A\text{-modules}. \tag{2.8}$$

Less obviously, one can also construct a natural isomorphism

$$H \otimes_{TH} TM \cong H \otimes \mathrm{Fix}(M) \quad \text{as unstable } H\text{-}A\text{-modules}. \tag{2.9}$$

The functor Fix has properties just like those of T.

(2.10) $\quad \mathrm{Fix} : H\mathscr{U} \longrightarrow \mathscr{U}$ is exact and commutes with suspension.

The appropriate tensor product in $H\mathscr{U}$ is $M \otimes_H N$.

(2.11)
There is a natural isomorphism $\mathrm{Fix}(M \otimes_H N) \cong \mathrm{Fix}(M) \otimes \mathrm{Fix}(N)$.

Define $H\backslash\mathscr{K}$ to be the category of unstable A-algebras under H. If M is an unstable A-algebra under H, then its product factors through $M \otimes_H M$ and we deduce from (2.11) that $\mathrm{Fix}(M)$ is an unstable A-algebra. If M is an unstable A-algebra, then (2.8) is an isomorphism of unstable A-algebras. If M is an unstable A-algebra under H, then the isomorphism (2.9) is one of unstable A-algebras under H. We now reach the adjunction that we really want. For an unstable A-algebra M under H and an unstable A-algebra N,

(2.12) $$\mathscr{K}(\mathrm{Fix}(M), N) \cong (H\backslash\mathscr{K})(M, H \otimes N).$$

3. Lannes' generalization of the Sullivan conjecture

Returning to topology, let X be a V-space. Abbreviate

$$\mathrm{Fix}^*_V(X) = \mathrm{Fix}(H^*_V(X)).$$

This is a cohomology theory on V-spaces. The inclusion $i : X^V \longrightarrow X$ induces a natural map

$$j : \mathrm{Fix}^*_V(X) \longrightarrow \mathrm{Fix}^*_V(X^V) \cong TH^*(X^V) \longrightarrow H^*(X^V).$$

Here the middle isomorphism is implied by (2.8) and the last map is an instance of the natural map $\pi : TM \longrightarrow M$. The map j specifies a transformation of cohomology theories on X. By a check on V-spaces of the form $V/W_+ \wedge K$, one finds that, if X is a finite dimensional V-CW complex, then

(3.1) $\quad j : \mathrm{Fix}^*_V(X) \longrightarrow H^*(X^V)$ is an isomorphism.

An alternative proof using the localization theorem is possible. In fact, this must be the case: the only way to reconcile (2.1) and (3.1) is to have an algebraic isomorphism

(3.2) $$\mathrm{Fix}(M) \cong \mathbb{F}_p \otimes_H Un(S^{-1}M)$$

for reasonable M. As a matter of algebra, Dwyer and Wilkerson prove that there is an isomorphism of H-A-algebras

$$H \otimes_{TH} TM \cong Un(S^{-1}M) \tag{3.3}$$

for any unstable H-A-algebra M that is finitely generated as an H-module. Tensoring over H with $\mathbb{F}_p$, this gives (3.2). Combined with (2.9), this gives an entirely algebraic version of the isomorphism

$$H^*(BV) \otimes H^*(X^V) \cong Un(S^{-1}H^*_V(X_{hV}))$$

of IV.2.3. Here, if $M = H^*_V(X)$ is finitely generated over H, the isomorphism (3.2) agrees with that obtained by combining (2.1) and (3.1). Thus we may view (3.1) as another reformulation of Smith theory. This reformulation is at the heart of the Sullivan conjecture, which is a corollary of the following theorem.

THEOREM 3.4 (LANNES). Let X be a V-space whose cohomology is of finite type and let Z be a space (with trivial V-action) whose cohomology is of finite type. Let $\omega : EV \times Z \longrightarrow X$ be a V-map. Then the homomorphism of unstable A-algebras

$$\omega^\# : \mathrm{Fix}^*_V(X) \longrightarrow H^*(Z)$$

induced by ω is an isomorphism if and only if the map

$$\tilde{\omega} : \hat{Z}_p \longrightarrow (\hat{X}_p)^{hV}$$

induced by ω is an equivalence.

The map ω determines and is determined by a map

$$\omega' : BV \times Z \longrightarrow EV \times_V X = X_{hV}$$

of bundles over BV. The map $\omega^\#$ of the theorem is the adjoint via (2.12) of the map under H induced on cohomology by ω'. The map ω induces a map $EV \times \hat{Z}_p \longrightarrow \hat{X}_p$, and the map $\tilde{\omega}$ of the theorem is its adjoint.

To prove the Sullivan conjecture, we take $Z = X^V$ and take $\omega : EG \times X^V \longrightarrow X$ to be the adjoint of the canonical map $X^V \longrightarrow X^{hV}$. Then $\omega^\#$ is the isomorphism j of (3.1), and $\tilde{\omega} : (X^V)^\wedge_p \longrightarrow (\hat{X}_p)^{hV}$ is the map that Theorem 1.2 claims to be an equivalence. Thus we see the Sullivan conjecture as a natural elaboration of Smith theory.

Theorem 3.4 has other applications. In the Sullivan conjecture, we applied it to obtain homotopical information from cohomological information, but its converse implication is also of interest. Taking $Z = X^{hV}$ and letting $\omega : EV \times X^{hV} \longrightarrow X$ be the evaluation map, the theorem specializes to give the following result.

THEOREM 3.5. Let X be a V-space such that the cohomologies of X and of X^{hV} are of finite type. Then the canonical map

$$\mathrm{Fix}_V^*(X) \longrightarrow H^*(X^{hV})$$

is an isomorphism of unstable A-algebras if and only if the canonical map

$$(X^{hV})\hat{_p} \longrightarrow (\hat{X}_p)^{hV}$$

is an equivalence.

When both X and X^{hV} are p-complete, so that $(X^{hV})\hat{_p} \longrightarrow (\hat{X}_p)^{hV}$ is the identity, we conclude that $H^*(X^{hV})$ is calculable as $\mathrm{Fix}_V^*(X)$. This is the starting point for remarkable work of Dwyer and Wilkerson in which they redevelop a great deal of Lie group theory in a homotopical context of p-complete finite loop spaces.

If we specialize to spaces without actions and use (2.8), we get the following nonequivariant version of Theorem 3.4.

THEOREM 3.6. Let Y and Z be spaces with cohomology of finite type and let $\omega : BV \times Z \longrightarrow Y$ be a map. Then the homomorphism of unstable A-algebras $\omega^{\#} : TH^*(Y) \longrightarrow H^*(Z)$ induced by ω is an isomorphism if and only if the map $\tilde{\omega} : \hat{Z}_p \longrightarrow \mathrm{Map}(BV, \hat{Y}_p)$ is an equivalence.

W. G. Dwyer and C. W. Wilkerson. Smith theory and the functor T. Comment. Math. Helv. 66(1991), 1-17.

W. G. Dwyer and C. W. Wilkerson. Homotopy fixed point methods for Lie groups and finite loop spaces. Preprint, 1992.

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4. Sketch proof of Lannes' theorem

We briefly sketch the strategy of the proof of Theorem 3.4. The first step is to reduce it to the nonequivariant version given in Theorem 3.6. It is easy to see that, for a group G and G-space Y, we have an identification

$$(4.1) \qquad Y^{hG} \equiv \mathrm{Map}_G(EG, Y) = \mathrm{Sec}(BG, EG \times_G Y) \equiv \mathrm{Sec}(BG, Y_{hG}),$$

where the right side is the space of sections of the bundle $Y_{hG} \longrightarrow BG$. Let $\mathrm{Map}(BG, BG)_1$ denote the component of the identity map and $\mathrm{Map}(BG, Y_{hG})_1$

denote the space of maps whose projection to BG is homotopic to the identity. We have a fibration

$$\mathrm{Map}(BG, Y_{hG})_1 \longrightarrow \mathrm{Map}(BG, BG)_1$$

with fiber Y^{hG} over the identity map.

Now return to $G = V$. Here easy inspections of homotopy groups show that evaluation at a basepoint gives an equivalence

$$\varepsilon : \mathrm{Map}(BV, BV)_1 \longrightarrow BV$$

and that the composition action of $\mathrm{Map}(BV, BV)_1$ on $\mathrm{Map}(BV, Y_{hV})_1$ induces an equivalence

$$Y^{hV} \times \mathrm{Map}(BV, BV)_1 \longrightarrow \mathrm{Map}(BV, Y_{hV})_1.$$

For a V-space X, the natural map $EV \times \hat{X}_p \longrightarrow (EV \times X)\hat{}_p$ induces a natural map $(\hat{X}_p)_{hV} \longrightarrow (X_{hV})\hat{}_p$, and this map is an equivalence. By (3.7), the map $\tilde{\omega}$ of Theorem 3.4 may be viewed as a map

$$\hat{Z}_p \longrightarrow \mathrm{Sec}(BV, (\hat{X}_p)_{hV}). \tag{4.2}$$

The map ω determines a map $EV \times \hat{Z}_p \longrightarrow \hat{X}_p$, and this in turn determines and is determined by a map

$$BV \times \hat{Z}_p \longrightarrow (\hat{X}_p)_{hV} \tag{4.3}$$

of bundles over BG. The map (3.8) is the composite map of fibers in the following diagram of fibrations

$$\begin{tikzcd}
\hat{Z}_p \arrow[r] \arrow[d] & \mathrm{Map}(BV, \hat{Z}_p) \arrow[r] \arrow[d] & \mathrm{Sec}(BV, (\hat{X}_p)_{hV}) \arrow[d] \\
BV \times \hat{Z}_p \arrow[r] \arrow[d] & \mathrm{Map}(BV, BV \times \hat{Z}_p)_1 \arrow[r] \arrow[d] & \mathrm{Map}(BV, (\hat{X}_p)_{hV})_1 \arrow[d] \\
BV \arrow[r] & \mathrm{Map}(BV, BV)_1 \arrow[r, "="] & \mathrm{Map}(BV, BV)_1.
\end{tikzcd}$$

The left map of fibrations is determined by a chosen homotopy inverse to ε : $\mathrm{Map}(BV, BV)_1 \longrightarrow BV$ and the inclusion of $\hat{Z}_p$ in $\mathrm{Map}(BV, \hat{Z}_p)$ as the subspace of constant functions. Clearly the middle composite is an equivalence if and only if $\tilde{\omega}$ is an equivalence. Applying Theorem 3.6 with Z replaced by $BV \times Z$, Y

taken to be X_{hV} and ω replaced by the adjoint $\nu : BV \times BV \times Z \longrightarrow X_{hV}$ of the composite map

$$BV \times Z \longrightarrow \mathrm{Map}(BV, BV \times Z)_1 \longrightarrow \mathrm{Map}(BV, X_{hV})$$

defined as in the middle row, but before applying completions, we find that the middle composite is an equivalence if and only if the induced map $\nu^{\#} : TH^*(X_{hV}) \longrightarrow H \otimes H^*(Z)$ is an isomorphism. Now (2.9) gives an isomorphism

$$H \otimes_{TH} TH^*(X_{hV}) \cong H \otimes \mathrm{Fix}(H^*(X_{hV}))$$

of unstable H-A-algebras. Its explicit construction parallels the topology in such a way that the map $\omega^{\#} : \mathrm{Fix}^*_V(X) \longrightarrow H^*(Z)$ agrees with $H \otimes_H \nu^{\#}$. This allows us to deduce that $\nu^{\#}$ is an isomorphism if and only if $\omega^{\#}$ is an isomorphism.

It remains to say something about the proof of Theorem 3.6. Since this is nonequivariant topology of the sort that requires us to join with those who use the word "space" to mean "simplicial set", we shall say very little. For a map $\phi : M \longrightarrow N$ of unstable A-algebras, there are certain algebraic functors that one may call $\mathrm{Ext}^{s,t}_{\mathscr{K}}(M, N; \phi)$; for fixed t, they are the left derived functors of a certain functor of derivations $\mathrm{Der}^t_{\mathscr{K}}(\cdot, N; \cdot)$ that is defined on the category of unstable A-algebras over N. The relevance of the functor T comes from the fact that its defining adjunction leads to natural isomorphisms

$$\mathrm{Ext}^{s,t}_{\mathscr{K}}(TM, N; \tilde{\phi}) \cong \mathrm{Ext}^{s,t}_{\mathscr{K}}(M, H \otimes N; \phi)$$

for a map $\phi : M \longrightarrow H \otimes N$ of unstable A-algebras with adjoint $\tilde{\phi}$.

There is an unstable Adams spectral sequence, due originally to Bousfield and Kan. However, the relevant version is a generalization due to Bousfield. For a map $f : X \longrightarrow \hat{Y}_p$, it starts from

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{K}}(H^*(Y), H^*(X); f^*),$$

and it converges (in total degree $t-s$) to $\pi_*(\mathrm{Map}(X, \hat{Y}_p); f)$. Under the hypotheses of Theorem 3.6, the map $\tilde{\omega} : \hat{Z}_p \longrightarrow \mathrm{Map}(BV, \hat{Y}_p)$ induces a map of spectral sequences (for any base point of Z) that is given on the E_2-level by the map

$$\mathrm{Ext}^{s,t}_{\mathscr{K}}(H^*(Z), \mathbb{F}_p) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{K}}(TH^*(Y), \mathbb{F}_p) \cong \mathrm{Ext}^{s,t}_{\mathscr{K}}(H^*(Y), H)$$

induced by $\omega^{\#} : TH^*(Y) \longrightarrow H^*(Z)$. With due care of detail, the deduction that $\tilde{\omega}$ is an equivalence if $\omega^{\#}$ is an isomorphism follows by a comparison of spectral sequences argument. The converse implication is shown by a detailed inductive analysis of the spectral sequence.

An alternative procedure for processing Lannes' algebra to obtain the topological conclusion of Theorem 3.6 has been given by Morel. Using a topological interpretation of the functor T in terms of the continuous cohomology of pro-p-spaces, together with a comparison of Sullivan's p-adic completion functor with that of Bousfield and Kan, he manages to circumvent use of the Bousfield-Kan unstable Adams spectral sequence and thus to avoid use of heavy simplicial machinery.

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D. Sullivan. The genetics of homotopy theory and the Adams conjecture. Annals of Math. 100(1974), 1-79.

5. Maps between classifying spaces

We shall sketch the explanation given by Lannes in a talk at Chicago of how his Theorem 3.6 applies to give a version of results of Dwyer and Zabrodsky that apply the Sullivan conjecture to the study of maps between classifying spaces. Although these authors apparently were not aware of the connection with equivariant bundle theory, what is at issue is precisely the map

$$\alpha^G : \coprod B(\Pi^\rho) = B_G(\Pi)^G \longrightarrow \mathrm{Map}(EG, B\Pi)^G = \mathrm{Map}(BG, B\Pi)$$

that we described in VII.3.7; here the coproduct runs over $(\rho) \in \mathrm{Rep}(G, \Pi)$. The relevant theorem of Lannes is as follows.

Theorem 5.1 (Lannes). If G is an elementary Abelian p-group and Π is a compact Lie group, then the map

$$\coprod B(\Pi^\rho)^\wedge_p \longrightarrow \mathrm{Map}(BG, (B\Pi)^\wedge_p)$$

induced by α^G is an equivalence.

It should be possible to deduce inductively that the result holds in this form for any finite p-group. The original version of Dwyer and Zabrodsky is somewhat different and in some respects a little stronger, although it seems possible to deduce much of one from the other. We say that a map $f : X \longrightarrow Y$ is a "mod p equivalence" if it induces an isomorphism on mod p homology. We say that f is a "strong mod p equivalence" if it satisfies the following conditions.

(i) f induces an isomorphism $\pi_0(X) \longrightarrow \pi_0(Y)$;
(ii) f induces an isomorphism $\pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ for any $x \in X$;
(iii) f induces an isomorphism

$$H_*(\tilde{X}_x, \mathbb{F}_p) \longrightarrow H_*(\tilde{Y}_{f(x)}, \mathbb{F}_p)$$

for any $x \in X$, where $\tilde{X}_x$ and $\tilde{Y}_{f(x)}$ are the universal covers of the components of X and Y that contain x and $f(x)$.

Say that a G-map $f : X \longrightarrow Y$ is a (strong) mod p equivalence if $f^H : X^H \longrightarrow Y^H$ is a (strong) mod p equivalence for each $H \subset G$. In view of VII.3.7, the following statements are equivariant reinterpretations of nonequivariant results of Dwyer and Zabrodsky and Notbohm. In nonequivariant terms, when $\Gamma = G \times \Pi$, their results are statements about the map α^G above.

Theorem 5.2 (Dwyer and Zabrodsky). *If Π is a normal subgroup of a compact Lie group Γ and $G = \Gamma/\Pi$ is a finite p-group, then the G-map $\alpha : B(\Pi; \Gamma) \longrightarrow \mathrm{Sec}(EG, B\Gamma)$ is a strong mod p equivalence.*

Actually, Dwyer and Zabrodsky give the result in this generality for $G = \mathbb{Z}/p$, and they give an inductive scheme to prove the general case when $\Gamma = G \times \Pi$. However, their inductive scheme works just as well to handle the case of general extensions. Their result was generalized to p-toral groups by Notbohm.

Theorem 5.3 (Notbohm). *If Π is a normal subgroup of a compact Lie group Γ and $G = \Gamma/\Pi$ is a p-toral group, then the G-map $\alpha : B(\Pi; \Gamma) \longrightarrow \mathrm{Sec}(EG, B\Gamma)$ is a mod p equivalence.*

However, α need not a strong mod p equivalence in this case: the components of α^H induce injections but not surjections on the fundamental groups of corresponding components.

These results are some of the starting points for beautiful work of Jackowski, McClure, and Oliver, and others, on maps between classifying spaces; these authors have given an excellent survey of the state of the art on this topic.

Lannes' deduces Theorem 4.1 from Theorem 3.6 by taking $Z = \coprod B\Pi^\rho$ and $Y = B\Pi$. The map ω is then the sum of the classifying maps of the homomorphisms $\nu : V \times \Pi^\rho \longrightarrow \Pi$ specified in VII.3.7. The deduction is based on the case $X = *$ of the following calculation.

THEOREM 5.4. Let X be a finite Π-CW complex. Then the natural map

$$TH^*_\Pi(X) \longrightarrow \prod H^*_{\Pi^\rho}(X^{\rho(V)})$$

is an isomorphism, where the product runs over $(\rho) \in \mathrm{Rep}(V, \Pi)$.

PROOF. The proof is an adaptation of methods of Quillen. Embed Π in $U(n)$ for some large n and let F be the G-space $U(n)/S$, where S is a maximal elementary Abelian subgroup of $U(n)$. Quillen shows that the evident diagram of projections

$$X \times F \times F \rightrightarrows X \times F \longrightarrow X$$

induces an equalizer diagram

$$H^*_\Pi(X) \longrightarrow H^*_\Pi(X \times F) \rightrightarrows H^*_\Pi(X \times F \times F).$$

Let

$$j^*(X) = TH^*_\Pi(X)$$

and

$$k^*(X) = \prod_{(\rho) \in \mathrm{Rep}(V,\Pi)} H^*_{\Pi^\rho}(X^{\rho(V)}).$$

These are both Π-cohomology theories in X. Applied to our original diagram of projections, both give equalizers, the first because the functor T is exact and the second by an elaboration of Quillen's argument. We have an induced map from the equalizer diagram for j^* to that for k^*. The isotropy subgroups of the finite Π-CW complexes $X \times F$ and $X \times F \times F$ are elementary Abelian, and it therefore suffices to show that the map

$$TH^*(BW) \cong j^*(\Pi/W) \longrightarrow k^*(\Pi/W) \cong \prod_{(\rho) \in \mathrm{Rep}(V,\Pi)} H^*(E\Pi^\rho \times_{\Pi^\rho} (\Pi/W)^{\rho(V)})$$

is an isomorphism when W is an elementary Abelian subgroup of Π. I learned the details of how to see this from Nick Kuhn. He has shown that T enjoys the property

$$TH^*(BW) \cong \prod_{(\sigma) \in \mathrm{Rep}(V,W)} H^*(BW), \tag{5.5}$$

and the map in cohomology that we wish to show is an isomorphism is in fact induced by a homeomorphism

$$\coprod_{(\rho)\in \mathrm{Rep}(V,\Pi)} E\Pi^{\rho} \times_{\Pi^{\rho}} (\Pi/W)^{\rho(V)} \longrightarrow \coprod_{(\sigma)\in \mathrm{Rep}(V,W)} BW. \tag{5.6}$$

To see the homeomorphism, note that Π acts on the disjoint union over $\rho \in \mathrm{Hom}(V,\Pi)$ of the spaces $(\Pi/W)^{\rho(V)}$; π sends a point $\pi' W$ fixed by $\rho(V)$ to the point $\pi\pi' W$ fixed by the π-conjugate of ρ. It is not hard to check that, as Π-spaces,

$$\coprod_{(\rho)\in \mathrm{Rep}(V,\Pi)} \Pi \times {}_{\Pi^{\rho}} (\Pi/W)^{\rho(V)} \cong \coprod_{\rho\in \mathrm{Hom}(V,\Pi)} (\Pi/W)^{\rho(V)} \cong \coprod_{\sigma\in \mathrm{Hom}(V,W)} \Pi/W.$$

Taking $E\Pi$ as a model for each $E\Pi^{\rho(V)}$, this implies the required homeomorphism. $\square$

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D. G. Quillen. The spectrum of an equivariant cohomology ring: I, II. Annals of Math. 94(1971), 549-572 and 573-602.

CHAPTER IX

An introduction to equivariant stable homotopy

$\mathscr{M}_G(V)$

1. G-spheres in homotopy theory

What is a G-sphereΓ In our work so far, we have only used spheres S^n, which have trivial action by G. Clearly this is contrary to the equivariant spirit of our work. The full richness of equivariant homotopy and homology theory comes from the interplay of homotopy theory and representation theory that arises from the consideration of spheres with non-trivial actions by G. In principle, it might seem reasonable to allow arbitrary G-actions. However, a closer inspection of the role of spheres in nonequivariant topology, both in manifold theory and in homotopy theory, gives the intuition that we should restrict to the linear spheres that arise from representations. Throughout the rest of the book, we shall generally use the term "representation of G" or sometimes "G-module" to mean a finite dimensional real inner product space with a given smooth action of G through linear isometries. We may think of V as a homomorphism of Lie groups $\rho : G \longrightarrow O(V)$. This convention contradicts standard usage, in which representations are defined to be isomorphism classes.

For a representation V, we have the unit sphere $S(V)$, the unit disk $D(V)$, and the one-point compactification S^V; G acts trivially on the point at infinity, which is taken as the basepoint of S^V. The based G-spheres S^V will be central to virtually everything that we do from now on. We agree to think of n as standing for $\mathbb{R}^n$ with trivial G-action, so that S^n is a special case of our definition. For a based G-space X, we write

$$\Sigma^V X = X \wedge S^V \quad \text{and} \quad \Omega^V X = F(S^V, X).$$

Of course, Σ^V is left adjoint to Ω^V.

When do we use trivial spheres and when do we use representation spheresΓ This is a subtle question, and in some of our work the answer may well seem counterintuitive. In defining weak equivalences of G-spaces, we only used homotopy groups defined in terms of trivial spheres, and that is unquestionably the right choice in view of the Whitehead theorem for G-CW complexes. Nevertheless, there are homotopy groups defined in terms of representation spheres, and they often play an important role, although more often implicit than explicit. We may think of a G-representation V as an H-representation for any $H \subseteq G$. For a based G-space X, we define

$$\pi_V^H(X) = [S^V, X]_H \cong [G_+ \wedge_H S^V, X]_G. \tag{1.1}$$

Here the brackets denote based homotopy classes of based maps, with the appropriate equivariance. For a pair (X, A) of based G-spaces, we form the usual homotopy fiber Fi of the inclusion $i : A \longrightarrow X$, and we define

$$\pi_{V+1}^H(X, A) = \pi_V^H(Fi). \tag{1.2}$$

It is natural to separate out the trivial and non-trivial parts of representations. Thus we let $V(H)$ denote the orthogonal complement in V of the fixed point space V^H. We then have the long exact sequence

(1.3)

$$\cdots \longrightarrow \pi_{V(H)+n}^H(X) \longrightarrow \pi_{V(H)+n}^H(X, A) \longrightarrow \cdots \longrightarrow \pi_{V(H)}^H(A) \longrightarrow \pi_{V(H)}^H(X)$$

of groups up to $\pi_{V(H)+1}^H(X)$ and of pointed sets thereafter.

Waner will develop a G-CW theory adapted to a given representation V in the next chapter, and Lewis will use it to study the Freudenthal suspension theorem for these homotopy groups in the chapter that follows. There is a more elementary standard form of the Freudenthal suspension theorem, due first to Hauschild, that suffices for many purposes. Just as nonequivariantly, it is proven by studying the adjoint map $\eta : Y \longrightarrow \Omega^V \Sigma^V Y$. Here one proceeds by reduction to the nonequivariant case and use of obstruction theory. Recall the notion of a ν-equivalence from I§3, where ν is a function from conjugacy classes of subgroups of G to the integers greater than or equal to -1. Define the connectivity function $c^*(Y)$ of a G-space Y by letting $c^H(Y)$ be the connectivity of Y^H for $H \subseteq G$; we set $c^H(Y) = -1$ if Y^H is not path connected.

THEOREM 1.4 (FREUDENTHAL SUSPENSION). The map $\eta : Y \longrightarrow \Omega^V\Sigma^V Y$ is a ν-equivalence if ν satisfies the following two conditions:

(1) $\nu(H) \leq 2c^H(Y) + 1$ for all subgroups H with $V^H \neq 0$, and
(2) $\nu(H) \leq c^K(Y)$ for all pairs of subgroups $K \subseteq H$ with $V^K \neq V^H$.

Therefore the suspension map

$$\Sigma^V : [X, Y]_G \longrightarrow [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if $dim(X^H) \leq \nu(H)$ for all H, and bijective if $dim(X^H) \leq \nu(H) - 1$.

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2. G-Universes and stable G-maps

We next explain how to stabilize homotopy groups and, more generally, sets of homotopy classes of maps between G-spaces. There are several ways to make this precise. The most convenient is that based on the use of universes.

DEFINITION 2.1. A G-universe U is a countable direct sum of representations such that U contains a trivial representation and contains each of its sub-representations infinitely often. Thus U can be written as a direct sum of subspaces $(V_i)^\infty$, where $\{V_i\}$ runs over a set of distinct irreducible representations of G. We say that a universe U is complete if, up to isomorphism, it contains every irreducible representation of G. If G is finite, one example is V^∞, where V is the regular representation of G. We say that a universe is trivial if it contains only the trivial irreducible representation. One example is U^G for a complete universe U. A finite dimensional sub G-space of a universe U is said to be an indexing space in U.

We should emphasize right away that, as soon as we start talking seriously about stable objects, or "spectra", the notion of a universe will become important even in the nonequivariant case.

We can now give a first definition of the set $\{X, Y\}_G$ of stable maps between based G-spaces X and Y.

DEFINITION 2.2. Let U be a complete G-universe. For a finite based G-CW complex X and any based G-space Y, define

$$\{X, Y\}_G = \mathrm{colim}_V [\Sigma^V X, \Sigma^V Y]_G,$$

where V runs through the indexing spaces in U and the colimit is taken over the functions

$$[\Sigma^V X, \Sigma^V Y]_G \longrightarrow [\Sigma^W X, \Sigma^W Y]_G, \quad V \subset W,$$

that are obtained by sending a map $\Sigma^V X \longrightarrow \Sigma^V Y$ to its smash product with the identity map of S^{W-V}.

When G is finite and X is finite dimensional, the Freudenthal suspension theorem implies that if we suspend by a sufficiently large representation, then all subsequent suspensions will be isomorphisms.

COROLLARY 2.3. If G is finite and X is finite dimensional, there is a representation $V_0 = V_0(X)$ such that, for any representation V,

$$\Sigma^V : [\Sigma^{V_0} X, \Sigma^{V_0} Y]_G \xrightarrow{\cong} [\Sigma^{V_0 \oplus V} X, \Sigma^{V_0 \oplus V} Y]_G$$

is an isomorphism.

Let X and Y be finite G-CW complexes. If G is finite, the stable value

$$[\Sigma^{V_0} X, \Sigma^{V_0} Y]_G = \{X, Y\}_G$$

is a finitely generated abelian group. However, if G is a compact Lie group and X has infinite isotropy groups, there is usually no representation V_0 for which all further suspensions Σ^V are isomorphisms, and $\{X, Y\}_G$ is usually not finitely generated.

REMARK 2.4. The groups $\{S^V, X\}_G$ are called equivariant stable homotopy groups of X and are sometimes denoted $\omega_V^G(X)$. However, it is more usual to denote them by $\pi_V^G(X)$, relying on context to resolve the ambiguity between stable and unstable homotopy groups.

The definition of $\{X, Y\}_G$ just given is not the right definition for an infinite complex X. Observe that

$$[\Sigma^V X, \Sigma^V Y]_G \cong [X, \Omega^V \Sigma^V Y]_G.$$

DEFINITION 2.5. Let U be a complete G-universe. For a based G-space X, define

$$QX = \operatorname{colim}_V \Omega^V \Sigma^V X,$$

where V runs over the indexing spaces in U and the colimit is taken over the maps

$$\Omega^V \Sigma^V X \longrightarrow \Omega^W \Sigma^W X, \quad V \subset W,$$

that are obtained by sending a map $S^V \longrightarrow X \wedge S^V$ to its smash product with the identity map of S^{W-V}. Observe that the maps of the colimit system are inclusions.

LEMMA 2.6. Fix an indexing space V in U. For based G-spaces X, there is a natural homeomorphism

$$QX \cong \Omega^V Q \Sigma^V X.$$

PROOF. Clearly QX is homeomorphic to $\operatorname{colim}_{W \supset V} \Omega^W \Sigma^W X$, and similarly for $Q\Sigma^V X$. By the compactness of S^V and the evident isomorphisms of functors $\Sigma^V \Sigma^{W-V} \cong \Sigma^W$ and $\Omega^V \Omega^{W-V} \cong \Omega^W$ for $V \subseteq W$,

$$\operatorname{colim} \Omega^W \Sigma^W X \cong \operatorname{colim} \Omega^V \Omega^{W-V} \Sigma^{W-V} \Sigma^V X \cong \Omega^V \operatorname{colim} \Omega^{W-V} \Sigma^{W-V} \Sigma^V X,$$

where the colimits are taken over $W \supseteq V$. The conclusion follows. $\square$

LEMMA 2.7. If X is a finite G-CW complex, then

$$\{X, Y\}_G \cong [X, QY]_G.$$

PROOF. This is immediate from the compactness of X, which ensures that

$$[X, QY]_G \cong \operatorname{colim}_V [X, \Omega^V \Sigma^V Y]_G. \quad \square$$

For infinite complexes X, it is $[X, QY]_G$ that gives the right notion of the stable maps from X to Y. We shall return to this point in Chapter XII, where we introduce the stable homotopy category of spectra.

3. Euler characteristic and transfer G-maps

We here introduce some fundamentally important examples of stable maps that require the use of representations for their definitions. The Euler characteristic and transfer maps defined here will appear at increasing levels of sophistication and generality as we go on.

Let M be a smooth closed G-manifold. We may embed M in a representation V, say with normal bundle ν. We may then embed a copy of ν as a tubular neighborhood of M in V. Just as for nonequivariant bundles, the Thom complex $T\xi$ of a G-vector bundle ξ is constructed by forming the fiberwise one-point compactification of the bundle, letting G act trivially on the points at infinity, and then identifying all of the points at infinity to a single G-fixed basepoint $*$. We then have the Pontrjagin-Thom map

$$t : S^V \longrightarrow T\nu.$$

It is the based G-map obtained by mapping the tubular neighborhood isomorphically onto ν and mapping all points not in the tubular neighborhood to the basepoint $*$. The inclusion of ν in $\tau_M \oplus \nu$, where τ_M is the tangent bundle of M, induces a based G-map

$$e : T\nu \longrightarrow T(\tau_M \oplus \nu) \cong M_+ \wedge S^V.$$

The composite of these two maps is the "transfer map"

$$(3.1) \qquad \tau(M) = e \circ t : S^V \longrightarrow \Sigma^V M_+$$

associated to the projection $M \longrightarrow \{pt\}$, which we think of as a trivial G-bundle. Of course, this projection induces a map

$$\xi : \Sigma^V M_+ \longrightarrow \Sigma^V S^0 \cong S^V.$$

We define the Euler characteristic of M to be the based G-map

$$(3.2) \qquad \chi(M) = \xi \circ \tau(M) : S^V \longrightarrow S^V.$$

The name comes from the fact that if we ignore the action of G and regard $\chi(M)$ as a nonequivariant map of spheres, then its degree is just the classical Euler characteristic of M. The proof is an interesting exercise in classical algebraic topology, but the fact will become clear from our later more conceptual description of these maps. In fact, from the point of view that we will explain in XV§1, this map *is* the Euler characteristic of M, by definition.

Since V is not well-defined — we just chose some V large enough that we could embed M in it — it is most natural to regard the transfer and Euler characteristics as stable maps

$$(3.3) \qquad \tau(M) \in \{S^0, M_+\}_G \quad \text{and} \quad \chi(M) \in \{S^0, S^0\}_G.$$

Observe that, when $M = G/H$, the map $\tau(G/H)$ of (3.1) can be written as the composite

$$(3.4) \qquad \tau(G/H) : S^V \xrightarrow{t} G_+ \wedge_H S^W \xrightarrow{e} G_+ \wedge_H S^V \cong (G/H)_+ \wedge S^V,$$

where W is the complement of the image in V of the tangent plane $L(H)$ at the identity coset and e is the extension to a G-map of the H-map obtained by smashing the inclusion $S^0 \longrightarrow S^{L(H)}$ with S^W. The unlabelled isomorphism is given by I.2.6.

More generally, for subgroups $K \subset H$ of G, there is a stable transfer G-map $\tau(\pi) : G/K_+ \longrightarrow G/H_+$ associated to the projection $G/H \longrightarrow G/K$. In fact, we may view π as the extension to a G-map

$$G \times_K (K/H) \longrightarrow G/K$$

of the projection $K/H \longrightarrow \{pt\}$, and we may construct the transfer K-map $\tau(K/H)$ starting from an embedding of K/H in a G-representation V regarded as a K-representation by restriction. We then define $\tau(\pi)$ to be the map

(3.5)
$$\tau(\pi) : G/K_+ \wedge S^V \cong G_+ \wedge_K S^V \longrightarrow G_+ \wedge_K (K/H_+ \wedge S^V) \cong G/H_+ \wedge S^V,$$

where the isomorphisms are given by I.2.6 and the arrow is the extension of the K-map $\tau(K/H)$ to a G-map. Note that any G-map $f : G/K_+ \longrightarrow G/H_+$ is the composite of a conjugation isomorphism $c_g : G/K \longrightarrow G/g^{-1}Kg$ and the projection induced by an inclusion $g^{-1}Kg \subset H$. We let $\tau(c_g) = c_{g^{-1}}$. With these definitions, we obtain a contravariantly functorial assignment of stable transfer maps $\tau(f)$ to G-maps f between orbits. Of course, such G-maps may themselves be regarded as stable G-maps between orbits.

4. Mackey functors and coMackey functors

We are headed towards the notions of $RO(G)$-graded homology and cohomology theories, but we start by describing what the coefficients of such theories will look like in the case of "ordinary" $RO(G)$-graded theories.

Recall that the ordinary homology and the ordinary cohomology of G-spaces are defined in terms of covariant and contravariant coefficient systems, which are functors from the homotopy category $h\mathscr{G}$ of orbits to the category $\mathbb{A}b$ of Abelian groups. Let $\mathscr{A}_G$ denote the category that is obtained from $h\mathscr{G}$ by applying the free Abelian group functor to morphisms. Thus $\mathscr{A}_G(G/H, G/K)$ is the free Abelian group generated by $h\mathscr{G}(G/H, G/K)$. Then coefficient systems are the same as additive functors $\mathscr{A}_G \longrightarrow \mathscr{A}b$.

Now imagine what the stable analog might be. It is clear that the sets $\{X, Y\}_G$ are already Abelian groups.

DEFINITION 4.1. Define the Burnside category $\mathscr{B}_G$ to have objects the orbit spaces G/H and to have morphisms

$$\mathscr{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G,$$

with the evident composition. We shall also refer to $\mathscr{B}_G$ as the stable orbit category. Observe that it is an "$\mathscr{A}b$-category": its Hom sets are Abelian groups and composition is bilinear.

We must explain the name "Burnside". The zeroth equivariant stable homotopy group of spheres or equivariant "zero stem" $\{S^0, S^0\}_G$ is a ring under composition. We shall denote this ring by B_G for the moment. It is a fundamental insight of Segal that, if G is finite, then B_G is isomorphic to the Burnside ring $A(G)$. Here $A(G)$ is defined to be the Grothendieck ring of isomorphism classes of finite G-sets with addition and multiplication given by disjoint union and Cartesian product. For a compact Lie group G, tom Dieck generalized this description of B_G by defining the appropriate generalization of the Burnside ring. In this case, $A(G)$ is defined to be the ring of equivalence classes of smooth closed G-manifolds, where two such manifolds are said to be equivalent if they have the same Euler characteristic in B_G; again, addition and multiplication are given by disjoint union and Cartesian product. An exposition will be given in XVII§2.

Definition 4.2. A covariant or contravariant stable coefficient system is a covariant or contravariant additive functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$. A *contravariant* stable coefficient system is called a *Mackey functor*. A *covariant* stable coefficient system is called a *coMackey functor*.

When G is finite, Dress first introduced Mackey functors, using an entirely different but equivalent definition, to study induction theorems in representation theory. We shall explain the equivalence of definitions in XIX§3. The classical examples of Mackey functors are the representation ring and Burnside ring Mackey functors, which send G/H to $R(H)$ or $A(H)$. The generalization to compact Lie groups was first defined and exploited by Lewis, McClure, and myself.

Observe that we obtain an additive functor $\mathscr{A}_G \longrightarrow \mathscr{B}_G$ by sending the homotopy class of a G-map $f : G/H \longrightarrow G/K$ to the corresponding stable map. Therefore a (covariant or contravariant) stable coefficient system has an underlying ordinary coefficient system. Said another way, stable coefficient systems can be viewed as given by additional structure on underlying ordinary coefficient systems.

What is the additional structureΓ Viewed as a stable map, $\tau(G/H)$ is a morphism $G/G \longrightarrow G/H$ in the category $\mathscr{B}_G$, and, more generally, so is $\tau(f)$ for any G-map $f : G/H \longrightarrow G/K$. We shall see in XIX§3 that every morphism of the category $\mathscr{B}_G$ is a composite of stable G-maps of the form f or $\tau(f)$. That is, the extra structure is given by transfer maps. When G is finite, we shall explain algebraically how composites of such maps are computed. In the general compact Lie case, such composites are quite hard to describe. For this reason, it is also quite hard to construct Mackey functors algebraically. However, we have the following concrete example. It may not seem particularly interesting at first sight, but we shall shortly use it to prove an important result called the Conner conjecture.

PROPOSITION 4.3. Let G be any compact Lie group. There is a unique Mackey functor $\underline{\mathbb{Z}} : \mathscr{B}_G \longrightarrow \mathscr{A}b$ such that the underlying coefficient system of $\underline{\mathbb{Z}}$ is constant at $\mathbb{Z}$ and the homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ induced by the stable transfer map $G/K_+ \longrightarrow G/H_+$ associated to an inclusion $H \subset K$ is multiplication by the Euler characteristic $\chi(K/H)$.

PROOF. In XIX§3, we shall give a complete additive calculation of the morphisms of $\mathscr{B}_G$, from which the uniqueness will be clear. The problem is to show that the given specifications are compatible with composition. We do this indirectly. As already noted, we have the Burnside Mackey functor $\underline{A}$. Thought of topologically, its value on G/H is

$$\{G/H_+, S^0\}_G \cong \{S^0, S^0\}_H = B_H,$$

and the contravariant functoriality is clear from this description. Define another Mackey functor $\underline{I}$ by letting $\underline{I}(G/H)$ be the augmentation ideal of $A(H)$. Thought of topologically, its value on G/H is the kernel of the map

$$\{G/H_+, S^0\}_G \longrightarrow \{G_+, S^0\}_G \cong \mathbb{Z}$$

induced by the G-map $G \longrightarrow G/H$ that sends the identity element e to the coset eH. Using XIX.3.2 and the definition of Burnside rings in terms of Euler characteristics, one can check that $\underline{I}$ is a subfunctor of $\underline{A}$. A key point is the identity

$$\chi(Y)\chi(H/K) = \chi(H \times_K Y)$$

of nonequivariant Euler classes for $H \subset K$ and H-spaces Y. One can then define $\underline{\mathbb{Z}}$ to be the quotient Mackey functor $\underline{A}/\underline{I}$; the desired Euler characteristic formula can be deduced from the formula just cited. $\square$

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