Geometric approach towards stable homotopy groups of spheres. The Kervaire invariant

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Аннотация

The notion of the geometrical $\mathbb{Z}/2\oplus\mathbb{Z}/2$–control of self-intersection of a skew-framed immersion and the notion of the $\mathbb{Z}/2\oplus\mathbb{Z}/4$-structure (the cyclic structure) on the self-intersection manifold of a $\mathbf{D}_4$-framed immersion are introduced. It is shown that a skew-framed immersion $f : M^{\frac{3n+q}{4}} \looparrowright \mathbb{R}^n$, $0 < q << n$ (in the $\frac{3n}{4}+\varepsilon$-range) admits a geometrical $\mathbb{Z}/2\oplus\mathbb{Z}/2$–control if the characteristic class of the skew-framing of this immersion admits a retraction of the order q, i.e. there exists a mapping $\kappa_0 : M^{\frac{3n+q}{4}} \to \mathbb{RP}^{\frac{3(n-q)}{4}}$, such that this composition $I \circ \kappa_0 : M^{\frac{3n+q}{4}} \to \mathbb{RP}^{\frac{3(n-q)}{4}} \to \mathbb{RP}^{\infty}$ is the characteristic class of the skew-framing of f. Using the notion of $\mathbb{Z}/2\oplus\mathbb{Z}/2$-control we prove that for a sufficiently great n, $n = 2^l-2$, an arbitrary immersed $\mathbf{D}_4$-framed manifold admits in the regular cobordism class (modulo odd torsion) an immersion with a $\mathbb{Z}/2\oplus\mathbb{Z}/4$-structure. In the last section we present an approach toward the Kervaire Invariant One Problem.

1 Self-intersection of immersions and Kervaire Invariant

The Kervaire Invariant One Problem is an open problem in Algebraic topology, for algebraic approach see [B-J-M], [C-J-M]. We will consider a geometrical approach; this approach is based on results by P.J.Eccles, see [E1]. For a geometrical approach see also [C1],[C2].

Let $f : M^{n-1} \looparrowright \mathbb{R}^n$, $n = 2^l-2$, $l > 1$, be a smooth (generic) immersion of codimension 1. Let us denote by $g : N^{n-2} \looparrowright \mathbb{R}^n$ the immersion of self-intersection manifold.

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Definition 1

The Kervaire invariant of f is defined as

$$\Theta(f) =< w_2^{\frac{n-2}{2}}; [N^{n-2}] >,$$

where $w_2 = w_2(N^{n-2})$ is the normal Stiefel-Whitney of N^{n-2}.

The Kervaire invariant is an invariant of the regular cobordism class of the immersion f. Moreover, the Kervaire invariant is a well-defined homomorphism

$$\Theta : Imm^{sf}(n-1,1) \to \mathbb{Z}/2. \qquad (1)$$

The normal bundle $\nu(g)$ of the immersion $g : N^{n-2} \looparrowright \mathbb{R}^n$ is a 2-dimensional bundle over N^{n-2} equipped with a $\mathbf{D}_4$–framing. The classifying mapping $\eta : N^{n-2} \to K(\mathbf{D}_4, 1)$ of this bundle is well-defined. The $\mathbf{D}_4$-structure of the normal bundle or the $\mathbf{D}_4$–framing is the prescribed reduction of the structure group of the normal bundle of the immersion g to the group $\mathbf{D}_4$ corresponding to the mapping η. The pair (g, η) represents an element in the cobordism group $Imm^{\mathbf{D}_4}(n-2, 2)$. The homomorphism

$$\delta : Imm^{sf}(n-1,1) \to Imm^{\mathbf{D}_4}(n-2,2) \qquad (2)$$

is well-defined.

Let us recall that the cobordism group $Imm^{sf}(n-k,k)$ generalizes the group $Imm^{sf}(n-1,1)$. This group is defined as the cobordism group of triples (f, Ξ, κ), where $f : M^{n-k} \looparrowright \mathbb{R}^n$ is an immersion with the prescribed isomorphism $\Xi : \nu(g) \cong k\kappa$, called a skew-framing, $\nu(f)$ is the normal bundle of f, κ is the given line bundle over M^{m-k} with the characteristic class $w_1(\kappa) \in H^1(M^{m-k}; \mathbb{Z}/2)$. The cobordism relation of triples is standard.

The generalization of the group $Imm^{\mathbf{D}_4}(n-2,2)$ is following. Let us define the cobordism groups $Imm^{\mathbf{D}_4}(n-2k, 2k)$. This group $Imm^{\mathbf{D}_4}(n-2k, 2k)$ is represented by triples (g, Ξ, η), where $g : N^{n-2k} \looparrowright \mathbb{R}^n$ is an immersion, Ξ is a dihedral k-framing, i.e. the prescribed isomorphism $\Xi : \nu_g \cong k\eta$, where η is a 2-dimensional bundle over N^{n-2k}. The characteristic mapping of the bundle η is denoted also by $\eta : N^{n-2k} \to K(\mathbf{D}_4, 1)$. The mapping η is the characteristic mapping for the bundle ν_g, because $\nu_g \cong k\eta$.

Obviously, the Kervaire homomorphism (1) is defined as the composition of the homomorphism (2) with a homomorphism

$$\Theta_{\mathbf{D}_4} : Imm^{\mathbf{D}_4}(n-2,2) \to \mathbb{Z}/2. \qquad (3)$$

The homomorphism (3) is called the Kervaire invariant for $\mathbf{D}_4$-framed immersed manifolds.

The Kervaire homomorphisms are defined in a more general situation by a straightforward generalization of the homomorphisms (1) and (3):

$$\Theta^k : Imm^{sf}(n-k,k) \to \mathbb{Z}/2, \qquad (4a)$$

$$\Theta^k_{\mathbf{D}_4} : Imm^{\mathbf{D}_4}(n-2k,2k) \to \mathbb{Z}/2, \qquad (4b)$$

(for $k=1$ the new homomorphism coincides with the homomorphism (3) defined above) and the following diagram

$$\begin{array}{ccccc} Imm^{sf}(n-1,1) & \stackrel{\delta}{\longrightarrow} & Imm^{\mathbf{D}_4}(n-2,2) & \stackrel{\Theta_{\mathbf{D}_4}}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow J^k & & \downarrow J^k_{\mathbf{D}_4} & & || \\ Imm^{sf}(n-k,k) & \stackrel{\delta^k}{\longrightarrow} & Imm^{\mathbf{D}_4}(n-2k,2k) & \stackrel{\Theta^k_{\mathbf{D}_4}}{\longrightarrow} & \mathbb{Z}/2 \end{array} \qquad (5)$$

is commutative. The homomorphism J^k ($J^k_{\mathbf{D}_4}$) is determined by the regular cobordism class of the restriction of the given immersion f (g) to the submanifold in M^{n-1} (N^{n-2}) dual to $w_1(\kappa)^{k-1} \in H^{k-1}(M^{n-1};\mathbb{Z}/2)$ ($w_2(\eta)^{k-1} \in H^{2k-2}(N^{n-2};\mathbb{Z}/2)$).

Let (g,Ξ,η) be a $\mathbf{D}_4$-framed (generic) immersion in the codimension $2k$. Let $h : L^{n-4k} \looparrowright \mathbb{R}^n$ be the immersion of the self-intersection (double points) manifold of g. The normal bundle ν_h of the immersion h is decomposed into a direct sum of k isomorphic copies of a 4-dimensional bundle ζ with the structure group $\mathbb{Z}/2 \int \mathbf{D}_4$. This decomposition is given by the isomorphism $\Psi : \nu_h \cong k\zeta$. The bundle ν_h itself is classified by the mapping $\zeta : L^{n-4k} \to K(\mathbb{Z}/2 \int \mathbf{D}_4, 1)$.

All the triples (h,ζ,Ψ) described above (we do not assume that a triple is realized as the double point manifold for a $\mathbf{D}_4$-framed immersion) up to the standard cobordism relation form the cobordism group $Imm^{\mathbb{Z}/2 \int \mathbf{D}_4}(n-4k,4k)$. The self-intersection of an arbitrary $\mathbf{D}_4$-framed immersion is a $\mathbb{Z}/2 \int \mathbf{D}_4$-framed immersed manifold and the cobordism class of this manifold well-defines the natural homomorphism

$$\delta^k_{\mathbf{D}_4} : Imm^{\mathbf{D}_4}(n-2k,2k) \to Imm^{\mathbb{Z}/2 \int \mathbf{D}_4}(n-4k,4k). \qquad (6)$$

The subgroup $\mathbf{D}_4 \oplus \mathbf{D}_4 \subset \mathbb{Z}/2 \int \mathbf{D}_4$ of index 2 induces the double cover $\bar{L}^{n-4k} \to L^{n-4k}$. This double cover corresponds with the canonical double cover over the double point manifold.

Let $\bar{\zeta} : \bar{L}^{n-4k} \to K(\mathbf{D}_4,1)$ be the classifying mapping induced by the projection homomorphism $\mathbf{D}_4 \oplus \mathbf{D}_4 \to \mathbf{D}_4$ to the first factor. Let $\bar{\zeta} \to L^{n-4k}$ be the 2-dimensional $\mathbf{D}_4$–bundle defined as the pull-back of the universal 2-dimensional bundle with respect to the classifying mapping $\bar{\zeta}$.

Definition 2

The Kervaire invariant $\Theta^k_{\mathbb{Z}/2\int \mathbf{D}_4} : Imm^{\mathbb{Z}/2\int \mathbf{D}_4}(n-4k,4k) \to \mathbb{Z}/2$ for a $\mathbb{Z}/2\int \mathbf{D}_4$-framed immersion (h,Ψ,ζ) is defined by the following formula:

$$\Theta^k_{\mathbb{Z}/2\int \mathbf{D}_4}(h,\Psi,\zeta) =< w_2(\bar{\eta})^{\frac{n-4k}{2}};[L^{n-4k}] > .$$

This new invariant is a homomorphism $\Theta^k_{\mathbb{Z}/2\int \mathbf{D}_4} : Imm^{\mathbb{Z}/2\int \mathbf{D}_4}(n,n-4k) \to \mathbb{Z}/2$ included into the following commutative diagram:

$$\begin{array}{ccc} Imm^{\mathbf{D}_4}(n-2k,2k) & \stackrel{\Theta_{\mathbf{D}_4}}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow \delta^k_{\mathbf{D}_4} & & || \\ Imm^{\mathbb{Z}/2\int \mathbf{D}_4}(n-4k,4k) & \stackrel{\Theta^k_{\mathbb{Z}/2\int \mathbf{D}_4}}{\longrightarrow} & \mathbb{Z}/2. \end{array} \qquad (7)$$

Let us formulate the first main results of the paper. In section 2 the notion of $\mathbb{Z}/2\oplus\mathbb{Z}/2$-control ($\mathbf{I}_b$–control) on self-intersection of a skew-framed immersion is considered. Theorem 1 (for the proof see section 3) shows that under a natural restriction of dimensions the property of $\mathbf{I}_b$-control holds for an immersion in the regular cobordism class modulo odd torsion.

In section 4 we formulate a notion of $\mathbb{Z}/2\oplus\mathbb{Z}/4$–structure (or an $\mathbf{I}_4$–structure, or a cyclic structure) of a $\mathbf{D}_4$-framed immersion. In section 5 we prove Theorem 2. We prove under a natural restriction of dimension that an arbitrary $\mathbf{D}_4$-framed $\mathbf{I}_b$-controlled immersion admits in the regular homotopy class an immersion with a cyclic structure. For such an immersion Kervaire invariant is expressed in terms of $\mathbb{Z}/2\oplus\mathbb{Z}/4$–characteristic numbers of the self-intersection manifold. The proof (based on the two theorems from [A2] (in Russian)) of the Kewrvaire Invariant One Problem is in section 6.

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2 Geometric Control of self-intersection manifolds of skew-framed immersions

In this and the remining sections of the paper by $Imm^{sf}(n-k,k)$, $Imm^{\mathbf{D}_4}(n-2k,2k)$, $Imm^{\mathbb{Z}/2\int \mathbf{D}_4}(n-4k,4k)$, etc., we will denote not the cobordism groups themselves, but the 2-components of these groups. In case the first argument (the dimension of the immersed manifold) is strictly positive, all the groups are finite 2-group.

Let us recall that the dihedral group $\mathbf{D}_4$ is given by the representation (in terms of generators and relations) $\{a, b | a^4 = b^2 = e, [a, b] = a^2\}$. This group is a subgroup of the group $O(2)$ of isometries of the plane with the base $\{f_1, f_2\}$ that keeps the pair of lines generated by the vectors of the base. The element a corresponds to the rotation of the plane through the angle $\frac{\pi}{2}$. The element b corresponds to the reflection of the plane with the axis given by the vector $f_1 + f_2$.

Let $\mathbf{I}_b(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = \mathbf{I}_b \subset \mathbf{D}_4$ be the subgroup generated by the elements $\{a^2, b\}$. This is an elementary 2-group of rank 2 with two generators. These are the transformations of the plane that preserve each line l_1, l_2 generated by the vectors $f_1 + f_2$, $f_1 - f_2$ correspondingly. The cohomology group $H^1(K(\mathbf{I}_b, 1); \mathbb{Z}/2)$ is the elementary 2-group with two generators. The first (second) generator of this group detects the reflection of the line l_2 (of the line l_1) correspondingly. The generators of the cohomology group will be denoted by τ_1, τ_2 correspondingly.

Definition 3

We shall say that a skew-framed immersion (f, Ξ), $f : M^{n-k} \looparrowright \mathbb{R}^n$ has self-intersection of type $\mathbf{I}_b$, if the double-points manifold N^{n-2k} of f is a $\mathbf{D}_4$-framed manifold that admits a reduction of the structure group $\mathbf{D}_4$ of the normal bundle to the subgroup $\mathbf{I}_b \subset \mathbf{D}_4$.

Let us formulate the following conjecture.

Conjecture

For an arbitrary $q > 0$, $q = 2(mod 4)$, there exists a positive integer $l_0 = l_0(q)$, such that for an arbitrary $n = 2^l - 2$, $l > l_0$ an arbitrary element $a \in Imm^{sf}(\frac{3n+q}{4}, \frac{n-q}{4})$ is stably regular cobordant to a stably skew-framed immersion with $\mathbf{I}_b$-type of self-intersection (for the definition of stable framing see [E2], of stable skew-framing see [A1]).

Let us formulate and prove a weaker result toward the Conjecture. We start with the following definition.

Let $\omega : \mathbb{Z}/2 \int \mathbf{D}_4 \to \mathbb{Z}/2$ be the epimorphism defined as the composition $\mathbb{Z}/2 \int \mathbf{D}_4 \subset \mathbb{Z}/2 \int \Sigma_4 \to \Sigma_4 \to \mathbb{Z}/2$, where $\Sigma_4 \to \mathbb{Z}/2$ is the parity of a permutation. Let $\omega^! : Imm^{\mathbb{Z}/2 \int \mathbf{D}_4}(n-4k, 4k) \to Imm^{Ker\omega}(n-4k, 4k)$ be the transfer homomorphism with respect to the kernel of the epimorphism ω.

Let P be a polyhedron with $dim(P) < 2k-1$, $Q \subset P$ be a subpolyhedron with $dim(Q) = dim(P)-1$, and let $P \subset \mathbb{R}^n$ be an embedding. Let us denote by U_P the regular neighborhood of $P \subset \mathbb{R}^n$ of the radius r_P and by U'_Q the regular neighborhood of $Q \subset \mathbb{R}^n$ of the radius r_Q, $r_Q > r_P$. Let us denote $U_Q = U_P \cap U'_Q$.

The boundary ∂U_P of the neighborhood U_P is a codimension one submanifold in $\mathbb{R}^n$. This manifold ∂U_P is a union of the two manifolds with boundaries $V_Q \cup_\partial V_P$, $V_Q = U_Q \cap \partial U_P$, $V_P = \partial U_P \setminus U_Q$ along the common boundary $\partial V_Q = \partial V_P$.

Let us assume that the two cohomology classes $\tau_{Q,1} \in H^1(Q;\mathbb{Z}/2)$, $\tau_{Q,2} \in H^1(Q;\mathbb{Z}/2)$ are given. The projection $U_Q \to Q$ of the neighborhood on the central submanifold determines the cohomology classes $\tau_{U_Q,1}, \tau_{U_Q,2} \in H^1(U_Q;\mathbb{Z}/2)$ as the inverse images of the classes $\tau_{Q,1}, \tau_{Q,2}$ correspondingly.

Let (g, Ξ_N, η), $dim(N) = n-2k$ be a $\mathbf{D}_4$–framed generic immersion, $n-4k > 0$, and $g(N^{n-2k}) \cap \partial U_P$ be an immersed submanifold in $U_Q \subset \partial U_P$. Let us denote $g(N^{n-2k}) \setminus (g(N^{n-2k}) \cap (U_P))$ by N^{n-2k}_{int}, and the complement $N^{n-2k} \setminus N^{n-2k}_{int}$ by N^{n-2k}_{ext}. The manifolds N^{n-2k}_{ext}, N^{n-2k}_{int} are submanifolds in N^{n-2k} of codimension 0 with the common boundary, this boundary is denoted by N^{n-2k-1}_Q. The self-intersection manifold of g is denoted by L^{n-4k}. By the dimensional reason $(n-4k = q << n)$ L^{n-4k} is a submanifold in $\mathbb{R}^n$, parameterized by an embedding h, equipped by the $\mathbb{Z}/2 \int \mathbf{D}_4$-framing of the normal bundle denoted by (Ψ, ζ). The triple (h, Ψ, ζ) determines an element in the cobordism group $Imm^{\mathbb{Z}/2 \int \mathbf{D}_4}(n-4k, 4k)$.

Definition 4

We say that the $\mathbf{D}_4$–framed immersion g is an $\mathbf{I}_b$–controlled immersion if the following conditions hold:

–1. The structure group of the $\mathbf{D}_4$–framing Ξ_N restricted to the submanifold (with boundary) $g(N^{n-2k}_{ext})$ is reduced to the subgroup $\mathbf{I}_b \subset \mathbf{D}_4$ and the cohomology classes $\tau_{U_Q,1}, \tau_{U_Q,2} \in H^1(U_Q;\mathbb{Z}/2)$ are mapped to the generators $\tau_1, \tau_2 \in H^1(N^{n-2k-1}_Q;\mathbb{Z}/2)$ of the cohomology of the structure group of this $\mathbf{I}_b$-framing by the immersion $g|_{N^{n-2k-1}_Q} : N^{n-2k-1}_Q \looparrowright \partial(U_Q) \subset U_Q$.

–2. The restriction of the immersion g to the submanifold $N^{n-2k-1}_Q \subset$

N^{n-2k} is an embedding $g|_{N_Q^{n-2k-1}} : N_Q^{n-2k-1} \subset \partial U_Q$, and the decomposition $L^{n-4k} = L_{int}^{n-4k} \cup L_{ext}^{n-4k} \subset (U_P \cup \mathbb{R}^n \setminus U_P)$ of the self-intersection manifold of g into two (probably, non-connected) $\mathbb{Z}/2 \int \mathbf{D}_4$-framed components is well-defined. The manifold L_{int}^{n-4k} is a submanifold in U_P and the triple $(L_{int}^{n-4k}, \Psi_{int}, \zeta_{int})$ represents an element in $Imm^{Ker\omega}(n-4k, 4k)$ in the image of the homomorphism $\omega^! : Imm^{\mathbb{Z}/2 \int \mathbf{D}_4}(n-4k, 4k) \to Imm^{Ker\omega}(n-4k, 4k)$.

Definition 5

Let $(f, \Xi_M, \kappa) \in Imm^{sf}(n-k, k)$ be an arbitrary element, where $f : M^{n-k} \looparrowright \mathbb{R}^n$ is an immersion of codimension k with the characteristic class $\kappa \in H^1(M^{n-k}; \mathbb{Z}/2)$ of the skew-framing Ξ_M. We say that the pair (M^{n-k}, κ) admits a retraction of order q, if the mapping $\kappa : M^{n-k} \to \mathbb{RP}^\infty$ is represented by the composition $\kappa = I \circ \bar{\kappa} : M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^\infty$. The element $[(f, \Xi_M, \kappa)]$ admits a retraction of order q, if in the cobordism class of this skew-framed immersion there exists a triple $(M'^{n-k}, \Xi_{M'}, \kappa')$ that admits a retraction of order q.

Theorem 1

Let $q = q(l)$ be a positive integer, $q = 2(mod 4)$. Let us assume that an element $\alpha \in Imm^{sf}(\frac{3n+q}{4}, \frac{n-q}{4})$ admits a retraction of the order q and $3n - 12k - 4 > 0$. Then the element $\delta(\alpha) \in Imm^{\mathbf{D}_4}(n-2k, 2k)$, $k = \frac{n-q}{4}$, is represented by a $\mathbf{D}_4$-framed immersion $[(g, \Psi_N, \eta)]$ with $\mathbf{I}_b$-control.

3 Proof of Theorem 1

Let us denote $n - k - q - 1 = 3k - 1$ by s. Let $d : \mathbb{RP}^s \to \mathbb{R}^n$ be a generic mapping. We denote the self-intersection points of d (in the target space) by $\Delta(d)$ and the singular points of d by $\Sigma(d)$.

Let us recall a classification of singular points of generic mappings $\mathbb{RP}^s \to \mathbb{R}^n$ in the case $4s < 3n$, for details see [Sz]. In this range generic mappings have no quadruple points. The singular values (in the target space) are of the following two types:

– a closed manifold $\Sigma^{1,1,0}$;

– a singular manifold $\Sigma^{1,0}$ (with singularities of the type $\Sigma^{1,1,0}$).

The multiple points are of the multiplicities 2 and 3. The set of triple points form a manifold with boundary and with corners on the boundary. These "corner" singular points on the boundary of the triple points manifold coincide with the manifold $\Sigma^{1,1,0}$. The regular part of boundary of triple points is a submanifold in $\Sigma^{1,0}$.

The double self-intersection points form a singular submanifold in $\mathbb{R}^n$ with the boundary $\Sigma^{1,0}$. This submanifold is not generic. After an arbitrary small alteration the double points manifold becomes a submanifold in $\mathbb{R}^n$ with boundary and with corners on the boundary of the type $\Sigma^{1,1,0}$.

Let U_Σ be a small regular neighborhood of the radius ε_1 of the singular submanifold $\Sigma^{1,0}$. Let U_Δ be a small regular neighborhood of the same radius of the submanifold $\Delta(d)$ (this submanifold is immersed with singularities on the boundary). The inclusion $U_\Sigma \subset U_\Delta$ is well-defined.

Let us consider a regular submanifold in Δ obtained by excising a small regular neighborhood of the boundary. This immersed manifold with boundary will be denoted by Δ^{reg}. The (immersed) boundary $\partial\Delta^{reg}$ will be denoted by Σ^{reg}. We will consider the pair of regular neighborhoods $U^{reg}_\Sigma \subset U^{reg}_\Delta$ of the pair $\Sigma^{reg} \subset \Delta^{reg}$ of the radius ε_2, $\varepsilon_2 << \varepsilon_1$. Because $2dim(\Delta^{reg}) < n$, after a small perturbation the manifold Δ^{reg} is a submanifold in U^{reg}_Δ.

Let (f_0, Ξ_0, κ), $f_0 : M^{n-k} \looparrowright \mathbb{R}^n$, $n-k = \frac{3n+q}{4}$ be a skew-framed immersion in the cobordism class α. We will construct an immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$ in the regular homotopy class of f_0 by the following construction.

Let $\kappa_0 : M^{n-k} \to \mathbb{RP}^s$ be a retraction of order q. Let $f : M \looparrowright \mathbb{R}^n$ be an immersion in the regular homotopy class of f_0 under the condition $dist(d \circ \kappa_0, f_0) < \varepsilon_3$. The caliber ε_3 of the approximation is given by the following inequality: $\varepsilon_3 << \varepsilon_2$.

Let $g_1 : N^{n-2k} \looparrowright \mathbb{R}^n$ be the immersion, parameterizing the double points of f. The immersion g_1 is not generic. After a small perturbation of the immersion g_1 with the caliber ε_3 we obtain a generic immersion $g_2 : N^{n-2k} \looparrowright \mathbb{R}^n$.

The immersed submanifold $g_2(N^{n-2k})$ is divided into two submanifolds $g_2(N^{n-2k}_{int})$, $g_2(N^{n-2k}_{ext})$ with the common boundary $g_2(\partial N^{n-2k}_{int}) = g_2(\partial N^{n-2k}_{ext})$ denoted by $g_2(N^{n-2k-1}_Q)$. The manifold $g_2(N^{n-2k}_{int})$ is defined as the intersection of the immersed submanifold $g_2(N^{n-2k})$ with the neighborhood U^{reg}_Δ. The manifold $g_2(N^{n-2k}_{ext})$ is defined as the intersection of the immersed submanifold $g_2(N^{n-2k})$ with the complement $\mathbb{R}^n \setminus (U^{reg}_\Delta)$. We will assume that g_2 is regular along ∂U^{reg}_Δ. Then $g_2(N^{n-2k}_Q)$ is an immersed submanifold in ∂U^{reg}_Δ. By construction the structure group $\mathbf{D}_4$ of the normal bundle of the immersed manifold $g_2(N^{n-2k}_{ext})$ admits a reduction to the subgroup $\mathbf{I}_b \subset \mathbf{D}_4$.

Let us denote by L^{n-4k} the self-intersection manifold of the immersion g_2. This manifold is embedded into $\mathbb{R}^n$ by $h : L^{n-4k} \subset \mathbb{R}^n$. The normal bundle of this embedding h is equipped with a $\mathbb{Z}/2 \int \mathbf{D}_4$-framing denoted by Ψ_L and the characteristic class of this framing is denoted by ζ_L. By the analogous construction the manifold L^{n-4k} is decomposed as the union of the two manifolds over a common boundary, denoted by Λ: $L^{n-4k} = L^{n-4k}_{ext} \cup_{\Lambda} L^{n-4k}_{int}$. The manifold (with boundary) L^{n-4k}_{int} is embedded by h into U^{reg}_{Δ}, the manifold L^{n-4k}_{ext} (with the same boundary) is embedded in the complement $\mathbb{R}^n \setminus U^{reg}_{\Delta}$. The common boundary Λ is embedded into $\partial U^{reg}_{\Delta}$.

The manifold L^{n-4k} is a $\mathbb{Z}/2 \int \mathbf{D}_4$-framed submanifold in $\mathbb{R}^n$. Let us describe the reduction of the structure group of this manifold to a corresponding subgroup in $\mathbb{Z}/2 \int \mathbf{D}_4$. We will describe the subgroups $\mathbf{I}_{2,j}(\mathbb{Z}/2 \oplus \mathbf{D}_4) \subset \mathbb{Z}/2 \int \mathbf{D}_4$, $j = x, y, z$. We will describe the transformations of $\mathbb{R}^4$ in the standard base (f_1, f_2, f_3, f_4) determined by generators of the groups.

Let us consider the subgroup $\mathbf{I}_{2,x}$. The generator c_x (a generator will be equipped with the index corresponding to the subgroup) defines the transformation of the space by the following formula: $c_x(f_1) = f_3$, $c_x(f_3) = f_1$, $c_x(f_2) = f_4$, $c_x(f_4) = f_2$.

For the generator a_x (of the order 4) the transformation is the following: $a_x(f_1) = f_2$, $a_x(f_2) = -f_1$, $a_x(f_3) = f_4$, $a_x(f_4) = -f_3$. The generator b_x (of order 2) defines the transformation of the space by the following formula: $b_x(f_1) = f_2$, $b_x(f_2) = f_1$, $b_x(f_3) = f_4$, $b_x(f_4) = f_3$. From this formula the subgroup $\mathbf{D}_4 \subset \mathbf{D}_4 \oplus \mathbb{Z}/2$ is represented by transformations that preserve the subspaces (f_1, f_2), (f_3, f_4). The generator of the cyclic subgroup $\mathbb{Z}/2 \subset \mathbf{D}_4 \oplus \mathbb{Z}/2$ permutes these planes.

The subgroups $\mathbf{I}_{2,y}$ and $\mathbf{I}_{2,x}$ are conjugated by the automorphism OP : $\mathbb{Z}/2 \int \mathbf{D}_4 \to \mathbb{Z}/2 \int \mathbf{D}_4$ given in the standard base by the following formula: $f_1 \mapsto f_1$, $f_2 \mapsto f_3$, $f_3 \mapsto f_2$, $f_4 \mapsto f_4$. Therefore the generator $c_y \in \mathbf{I}_{2,y}$ is determined by the following transformation: $c_y(f_1) = f_2$, $c_y(f_2) = f_1$, $c_y(f_3) = f_4$, $c_y(f_4) = f_3$. The generator a_y (of the order 4) is given by $a_y(f_1) = f_3$, $a_y(f_3) = -f_1$, $a_y(f_2) = f_4$, $a_y(f_4) = -f_2$. The generator b_y (of the order 2) is given by $b_y(f_1) = f_3$, $b_y(f_3) = f_1$, $b_y(f_2) = f_4$, $b_y(f_4) = f_2$.

Let us describe the subgroup $\mathbf{I}_{2,z}$. In this case the generator c_z defines the transformation of the space by the following formula: $c_z(f_i) = -f_i$, $i = 1, 2, 3, 4$.

For the generator a_z (of order 4) the transformation is the following: $a_z(f_1) = f_2$, $a_z(f_2) = f_3$, $a_z(f_3) = f_4$, $a_z(f_4) = f_1$. The generator b_x (of the order 2) defines the transformation of the space by the following formula: $b_z(f_1) = f_2$, $b_z(f_2) = f_1$, $b_z(f_3) = f_4$, $b_z(f_4) = f_3$.

Obviously, the restriction of the epimorphism $\omega : \mathbb{Z}/2 \int \mathbf{D}_4 \to \mathbb{Z}/2$ to the subgroups $\mathbf{I}_{2,x}, \mathbf{I}_{2,y} \subset \mathbb{Z}/2 \int \mathbf{D}_4$ is trivial and the restriction of this homomorphism to the subgroup $\mathbf{I}_{2,z}$ is non-trivial.

The subgroup $\mathbf{I}_3 \subset \mathbf{I}_{2,x}$ is defined as the subgroup with the generators c_x, b_x, a_x^2. This is an index 2 subgroup isomorphic to the group $\mathbb{Z}/2^3$. The image of this subgroup in $\mathbb{Z}/2 \int \mathbf{D}_4$ coincides with the intersection of arbitrary pair of subgroups $\mathbf{I}_{2,x}$, $\mathbf{I}_{2,y}$, $\mathbf{I}_{2,z}$. The subgroup $\mathbf{I}_3 \subset \mathbf{I}_{2,y}$ is generated by c_y, b_y, a_y^2. Moreover, one has $c_y = b_x$, $b_y = c_x$, $a_y^2 = a_x^2$. It is easy to check that the following relations hold: $c_z = a_x^2$, $a_z^2 = c_x = b_y$, $b_z = b_x = c_y$. Therefore $Ker(\omega|_{\mathbf{I}_{2,z}})$ coincides with the subgroup $\mathbf{I}_3 \subset \mathbf{I}_{2,z}$.

The subgroups $\mathbf{I}_{2,x}, \mathbf{I}_{2,y}, \mathbf{I}_{2,z}, \mathbf{I}_3$ in $\mathbb{Z}/2 \int \mathbf{D}_4$ are well-defined. There is a natural projection $\pi_b : \mathbf{I}_3 \to \mathbf{I}_b$.

We will also consider the subgroup $\mathbf{I}_{2,x\downarrow} \subset \mathbb{Z}/2 \int \mathbf{D}_4$ from geometrical considerations. This subgroup is a quadratic extension of the subgroup $\mathbf{I}_{2,x}$ such that $\mathbf{I}_{2,x} = Ker\omega|_{\mathbf{I}_{2,x\downarrow}} \subset \mathbf{I}_{2,x\downarrow}$. An algebraic definition of this group will not be required.

In the following lemma we will describe the structure group of the framing of the triad $(L_{int}^{n-4k} \cup_{\Lambda} L_{ext}^{n-4k})$. The framings of the spaces of the triad will be denoted by $(\Psi_{\int} \cup_{\Psi_{\Lambda}} \cup \Psi_{ext}, \zeta_{int} \cup_{\zeta_{\Lambda}} \cup \zeta_{ext})$.

Lemma 1

There exists a generic regular deformation $g_1 \to g_2$ of the caliber $3\varepsilon_3$ such that the immersed manifold $g_2(N_{ext}^{n-2k})$ admits a reduction of the structure group of the $\mathbf{D}_4$-framing to the subgroup $\mathbf{I}_b \subset \mathbf{D}_4$. The manifold L_{int}^{n-4k} is divided into the disjoint union of the two manifolds (with boundaries) denoted by $(L_{int,x\downarrow}^{n-4k}, \Lambda_{x\downarrow})$, $(L_{int,y}^{n-4k}, \Lambda_y)$.

1. The structure group of the framing $(\Psi_{int,x\downarrow}, \Psi_{\Lambda_{x\downarrow}})$ for the submanifold (with boundary) $(L_{int,x\downarrow}^{n-4k}, \Lambda_{x\downarrow})$ is reduced to the subgroups $(\mathbf{I}_{2,x\downarrow}, \mathbf{I}_{2,z})$. (In particular, the 2-sheeted cover over $L_{int,x\downarrow}^{n-4k}$, classified by ω (denoted by $\tilde{L}_{int,x}^{n-4k} \to L_{int,x\downarrow}^{n-4k}$) is, generally speaking, a non-trivial cover.)

2. The structure group of the framing $(\Psi_{int,y}, \Psi_{\Lambda})$ for the submanifold (with boundary) $(L_{int,y}^{n-4k}, \Lambda_y)$ is reduced to the subgroup $(\mathbf{I}_{2,y}, \mathbf{I}_3)$. (In particular, the 2-sheeted cover $\tilde{L}_{int,y}^{n-4k} \to L_{int,y}^{n-4k}$ classified by ω, is the trivial cover.) Moreover, the double covering $\tilde{L}_x^{n-4k}$ over the component $L_{x\downarrow}^{n-4k}$ is naturally diffeomorphic to $\tilde{L}_y^{n-4k}$ and this diffeomorphism agrees with the restriction of the automorphism $OP : \mathbb{Z}/2 \int \mathbf{D}_4 \to \mathbb{Z}/2 \int \mathbf{D}_4$ on the subgroup $\mathbf{I}_{2,x}$, $OP(\mathbf{I}_{2,x}) = \mathbf{I}_{2,y}$.

3. The structure group of the framing $(\Psi_{ext}, \zeta_{ext})$ for the submanifold (with boundary) $h(L_{ext}^{n-4k}, \Lambda^{n-4k}) \subset (\mathbb{R}^n \setminus U_{\Delta}^{reg}, \partial(U_{\Delta}^{reg}))$ is reduced to the

subgroup $\mathbf{I}_{2,z}$. (In particular, the 2-sheeted cover $\tilde{L}^{n-4k}_{ext} \to L^{n-4k}_{ext}$ classified by ω, is, generally speaking, a nontrivial cover.)

Proof of Lemma 1

Components of the self-intersection manifold $g_1(N^{n-2k}) \setminus (g_1(N^{n-2k}) \cap U_\Sigma)$ (this manifold is formed by double points $x \in g_1(N^{n-2k})$, $x \notin U_\Sigma$ with inverse images $\bar{x}_1, \bar{x}_2 \in M^{n-k}$) are classified by the following two types.

Type 1. The points $\kappa(\bar{x}_1)$, $\kappa(\bar{x}_2)$ in $\mathbb{RP}^s$ are ε_2-close.

Type 2. The distances between the points $\kappa(\bar{x}_1)$, $\kappa(\bar{x}_2)$ in $\mathbb{RP}^s$ are greater then the caliber ε_2 of the regular approximation. Points of this type belong to the regular neighborhood U_Δ (of the radius ε_1).

Let us classify components of the triple self-intersection manifold $\Delta_3(f)$ of the immersion f. The a priori classification of components is the following.

A point $x \in \Delta_3(f)$ has inverse images $\bar{x}_1, \bar{x}_2, \bar{x}_3$ in M^{n-k}.

Type 1. The images $\kappa(\bar{x}_1), \kappa(\bar{x}_2), \kappa(\bar{x}_3)$ are ε_2-close in $\mathbb{RP}^s$.

Type 2. The images $\kappa(\bar{x}_1), \kappa(\bar{x}_2)$ are ε_2-close in $\mathbb{RP}^s$ and the distance between the images $\kappa(\bar{x}_3)$ and $\kappa(\bar{x}_1)$ (or $\kappa(\bar{x}_2)$) are greater than the caliber ε_2 of the approximation.

Type 3. The pairwise distances between the points $\kappa(\bar{x}_1), \kappa(\bar{x}_2), \kappa(\bar{x}_3)$ greater than the caliber ε_2 of the approximation.

By a general position argument the component of the type 3 does not intersect $d(\mathbb{RP}^s)$. Therefore the immersion f can be deformed by a small ε_2-small regular homotopy inside the ε_3-regular neighborhood of the regular part of $d(\mathbb{RP}^s)$ such that after this regular homotopy $\Delta_3(f)$ is contained in the complement of U^{reg}_Δ. The codimension of the submanifold $\bar{\Delta}_2(d) \subset \mathbb{RP}^s$ is equal to $n-3k+1 = q+k+1$ and greater then $dim(\Delta_3(f)) = n-3k$. By analogical arguments the component of triple points of the type 1 is outside U^{reg}_Δ.

Let us classify components of the quadruple self-intersection manifold $\Delta_4(f)$ of the immersion f. A point $x \in \Delta_4(f)$ has inverse images $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$ in M^{n-k}. The a priori classification is the following.

Type 1. The images $\kappa(\bar{x}_1), \kappa(\bar{x}_2)$ are ε_2-close in $\mathbb{RP}^s$ and the pairwise distances between the images $\kappa(\bar{x}_1)$ (or $\kappa(\bar{x}_2)$), $\kappa(\bar{x}_3)$ and $\kappa(\bar{x}_4)$) are greater than the caliber ε_2 of the approximation.

Type 2. The two pairs $(\kappa(\bar{x}_1), \kappa(\bar{x}_2))$ and $(\kappa(\bar{x}_3), \kappa(\bar{x}_4))$ of the images are ε_2-close in $\mathbb{RP}^s$ and the distance between the images $\kappa(\bar{x}_1)$ (or $\kappa(\bar{x}_2)$) and $\kappa(\bar{x}_3)$ (or $\kappa(\bar{x}_4)$) are greater than the calibre ε_2 of the approximation. (The described component is the complement of the regular ε_2 neighborhood of the triple points manifold of $d(\mathbb{RP}^s)$.)

Type 3. Images $\kappa(\bar{x}_1), \kappa(\bar{x}_2)$ and $\kappa(\bar{x}_3)$ on $\mathbb{RP}^s$ are pairwise ε_2-close in $\mathbb{RP}^s$ and the distance between the images $\kappa(\bar{x}_1)$ (or $\kappa(\bar{x}_2)$, or $\kappa(\bar{x}_3)$) and $\kappa(\bar{x}_4)$ is greater than the caliber ε_2 of the approximation.

Type 4. All the images $\kappa(\bar{x}_1), \kappa(\bar{x}_2)$, $\kappa(\bar{x}_3)$ and $\kappa(\bar{x}_4)$ are pairwise ε_2-close in $\mathbb{RP}^s$.

Let us prove that there exists a generic f such that the components of the type 1 and the type 3 are empty. For the component of the type 3 the proof is analogous to the proof for the component of the type 1.

Let us prove that there exists a generic deformation $g_1 \to g_2$ with the caliber $3\varepsilon_3$ such that after this deformation in the neighborhood U_{Δ}^{reg} there are no self-intersection points of g_2 obtained by a generic resolution of triple points of f of the types 1 and 2. Let us start with the proof for triple points of the type 1.

For a generic small alteration of the immersion g_2 inside U_{Δ}^{reg} the points of the type 1 of the triple points manifold $\Delta_3(f)$ are perturbed into a component of the self-intersection points on L^{n-4k}. This component is classified by the following two subtypes:

– Subtype **a**. Preimages of a point are $(\bar{x}_2, \bar{x}_1), (\bar{x}_2, \bar{x}'_1)$.

—Subtype **b**. Preimages of a point are $(\bar{x}_1, \bar{x}'_1), (\bar{x}_1, \bar{x}_2)$.

In the formula above the points with the common indeces have ε_3-close projections on the corresponding sheet of $d(\mathbb{RP}^s)$. The two points in a pair form a point on N^{n-2k} and a couple of pairs forms a point on the component of L^{n-4k}.

Let us prove that there exists a $2\varepsilon_3$-small regular deformation $g_1 \to g_2$, such that the component of $h(L^{n-4k}) \cap U_{\Delta}^{reg}$ of the subtype **a** is empty. Let K^{s-k} be the intersection manifold of $f(M^{n-k})$ with $d(\mathbb{RP}^s)$ (this manifold is immersed into the regular part in $\mathbb{RP}^s$). By a general position argument, because $2s < n - 2k$, a generic perturbation $r \to r'$ of the immersion $r : K^{s-k} \looparrowright \mathbb{RP}^s \to \mathbb{R}^n$ is an embedding. Therefore there exists a $2\varepsilon_2$-small deformation of immersed manifold $r(K^{s-k}) \to r'(K^{s-k})$ in $\mathbb{R}^n$, such that the regular ε_2-neighborhood of the submanifold $r'(K^{s-k})$ has no self-intersection. The deformation of the immersed manifolds $r(K^{s-k}) \to r'(K^{s-k})$ is extended to the deformation of $g_1(N^{n-2k})$ in the regular neighborhoods of the constructed one-parameter family of immersed manifolds. After the described regular deformation the immersed manifold $g_2(N^{n-2k})$ has no self-intersection components of the subtype **a**. The case of the self-intersection of the subtype **b** is analogous.

Let us describe a generic deformation $g_1 \to g_2$ with the support in U_{Δ}^{reg} that resolves self-intersection corresponding to quadruple points of f of the type 2. This deformation could be arbitrarily small. After this deformation the component $\Delta_4(f)$ of the type 2 is resolved into two components of L^{n-4k} of different subtypes. These two components will be denoted by L^{n-4k}_x, L^{n-4k}_y.

The immersed submanifold $g_2(N^{n-2k}) \cap U^{reg}_{\Delta}$ is divided into two components. The first component is formed by pairs of points $(\bar{x}, \bar{x}')$ with the $3\varepsilon_3$-close images $(\kappa(\bar{x}), \kappa(\bar{x}')$ on $\mathbb{RP}^s$. This component is denoted by $g_2(N^{n-2k}_x)$. The last component of $g_2(N^{n-2k}) \cap U^{reg}_{\Delta}$ is denoted by $g_2(N^{n-2k}_y)$. This component is formed by pairs of points $(\bar{x}, \bar{x}')$ with the projections $(\kappa(\bar{x}), \kappa(\bar{x}'))$ on different sheets of $\mathbb{RP}^s$.

The component $L^{n-4k}_{x\downarrow}$ is defined by pairs $(\bar{x}_1, \bar{x}'_1), (\bar{x}_2, \bar{x}'_2)$. The component L^{n-4k}_y is defined by pairs $(\bar{x}_1, \bar{x}_2), (\bar{x}'_1, \bar{x}'_2)$. A common index of points in the pair means that the images of the points are ε_3-close on $\mathbb{RP}^s$. Each pair determines a point on N^{n-2k} with the same image of g_2. It is easy to see that the component $L^{n-4k}_{x\downarrow}$ is the self-intersection of $g_2(N^{n-2k}_x)$ and the component L^{n-4k}_y is the self-intersection of $g_2(N^{n-2k}_y)$.

It is easy to see that the structure groups of the components agree with the corresponding subgroup described in the lemma. The component $L^{n-4k}_{x\downarrow}$ admits a reduction of the structure group to the subgroup $\mathbf{I}_{2,x\downarrow} \subset \mathbb{Z}/2 \int \mathbf{D}_4$. The component L^{n-4k}_y admits a reduction of the structure group to the subgroup $\mathbf{I}_{2,y}$. Moreover, it is easy to see that the covering $\tilde{L}^{n-4k}_{x\downarrow}$ over L^{n-4k}_x induced by the epimorphism $\omega : \mathbb{Z}/2 \int \mathbf{D}_4 \to \mathbb{Z}/2$ with the kernel $\mathbf{I}_{2,x} \subset \mathbb{Z}/2 \int \mathbf{D}_4$ is naturally diffeomorphic to L^{n-4k}_y. Also it is easy to see that this diffeomorphism agrees with the transformation OP of the structure groups of the framing over the components.

The last component of L^{n-4k} is immersed in the ε_2-neighborhood of $d(\mathbb{RP}^s)$ outside of U^{reg}_{Δ} and will be denoted by L^{n-4k}_z. The structure group of the framing of this component is $\mathbf{I}_{2,z}$. Lemma 1 is proved.

The last part of the proof of the Theorem 1

Let us construct a pair of polyhedra $(P', Q') \subset \mathbb{R}^n$, $dim(P') = 2s - n = n - 2k - q - 2$, $dim(Q') = dim(P') - 1$. Obviously, $dim(P') < 2k - 1$. Take a generic mapping $d' : \mathbb{RP}^s \to \mathbb{R}^n$. Let us consider the submanifold with boundary $(\Delta'^{reg}, \partial\Delta'^{reg}) \subset \mathbb{R}^n$ (see the denotation in Lemma 1). Let $\eta_{\Delta'^{reg}} : (\Delta'^{reg}, \partial\Delta'^{reg}) \to (K(\mathbf{D}_4, 1), K(\mathbf{I}_b, 1))$ be the classifying mapping for the double point self-intersection manifold of d.

By a standard argument we may modify the mapping d into d' such that the mapping $\eta_{\Delta^{reg}}$ is a homotopy equivalence of pairs up to the dimension $q+1$. After this modification $d' \to d$ we define $(P, Q) = (\Delta^{reg}, \partial\Delta^{reg}) \subset \mathbb{R}^n$ and the mapping $\eta_{\Delta^{reg}}$ is a $(q+1)$-homotopy equivalence.

The subpolyhedron Q is equipped with two cohomology classes $\kappa_{Q,1}, \kappa_{Q,2} \in H^1(Q;\mathbb{Z}/2)$. Because Σ is a submanifold in $\mathbb{RP}^s$, the restriction of the characteristic class $\kappa \in H^1(\mathbb{RP}^s;\mathbb{Z}/2)$ to $H^1(\Sigma;\mathbb{Z}/2)$ is well-defined. The inclusion $i_Q : Q \subset U_\Sigma$ determines the cohomology class $(i_Q)^*(\kappa) \in H^1(Q;\mathbb{Z}/2)$. The cohomology class $\kappa_{Q,1}$ is defined as the characteristic class of the canonical double points covering over Σ. The class $\kappa_{Q,2}$ is defined by the formula $\kappa_{Q,2} = (i_Q)^*(\kappa) + \kappa_{Q,1}$.

The immersed manifold (with boundary) $(N^{n-2k} \cap U_\Sigma) \looparrowright U_\Sigma$ is equipped with an $\mathbf{I}_b$-framing. Obviously the classes $\kappa_{Q,1}, \kappa_{Q,2} \in H^1(U_\Sigma;\mathbb{Z}/2) = H^1(Q;\mathbb{Z}/2)$ restricted to $H^1(g_2(N^{n-2k}_{ext});\mathbb{Z}/2)$ (recall that $g_2(N^{n-2k}_{ext}) = g_2(N^{n-2k}) \cap (\mathbb{R}^n \setminus U_\Delta)$) agree with the two generated cohomology classes ρ_1, ρ_2 of the $\mathbf{I}_b$-framing correspondingly.

Let us define the immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ with $\mathbf{I}_b$-control over (P,Q). Let us start with the immersion $g_2 : N^{n-2k} \looparrowright \mathbb{R}^n$ constructed in the lemma. By a $2\varepsilon_2$–small generic regular deformation we may deform the immersion g_2 into g_3, such that this deformation pushes the component $g_2(N^{n-2k}_x)$ out of U^{reg}_Δ. Therefore the component $L^{n-4k}_{x\downarrow} \subset L^{n-4k}$ of the self-intersection of g_2 is also deformed out of U^{reg}_Δ.

The immersed manifold (with boundary) $g_3(N^{n-2k}) \cap (\mathbb{R}^n \setminus U^{reg}_\Delta)$ is equipped with an $\mathbf{I}_b$-framing of the normal bundle. Obviously, the classes $\kappa_{Q,1}, \kappa_{Q,2} \in H^1(U_\Sigma;\mathbb{Z}/2) = H^1(Q;\mathbb{Z}/2)$, restricted to $H^1(g_2(N^{n-2k}) \cap U_\Delta;\mathbb{Z}/2)$, agree with the two generated cohomological classes of the $\mathbf{I}_b$-framing. The immersed manifold $g_3(N^{n-2k}) \cap U^{reg}_\Delta$ coincides with $g_2(N^{n-2k}_y)$ and has the general structure group of the framing. This immersed manifold has the self-intersection manifold (with boundary) $h(L^{n-4k}) \cap U^{reg}_\Delta$ with the reduction of the structure group to the pair of the subgroups $(\mathbf{I}_{2,y}, \mathbf{I}_3)$.

Let us prove that the immersed manifold (with boundary) $h(L^{n-4k}) \cap U^{reg}_\Delta$ is $\mathbb{Z}/2 \int \mathbf{D}_4$-framed cobordant (relative to the boundary) to a $\mathbb{Z}/2 \int \mathbf{D}_4$-framed manifold decomposed into the disjoint union of a closed $\mathbb{Z}/2 \int \mathbf{D}_4$-framed manifold that is the image of the transfer homomorphism $\omega^!$ and a relative $\mathbf{I}_3$-framed manifold.

Take a $\mathbb{Z}/2 \int \mathbf{D}_4$-framed manifold $(\tilde{L}^{n-4k}, \tilde{\Psi}, \tilde{\zeta})$ that is defined as the image of $\mathbb{Z}/2 \int \mathbf{D}_4$-framed manifold (L^{n-4k}, Ψ, ζ) by the transfer homomorphism (a double covering) with respect to the cohomology class $\omega \in H^1(\mathbb{Z}/2 \int \mathbf{D}_4;\mathbb{Z}/2)$. Recall that the manifold $\tilde{L}^{n-4k}$ is obtained by gluing the manifold $\tilde{L}^{n-4k}_x \cup \tilde{L}^{n-4k}_y$ with the manifold $\tilde{L}^{n-4k}_z$ along the common boundary $\tilde{\Lambda}^{n-4k-1}$. Note that the group of the framing of the last manifold $\tilde{\Lambda}^{n-4k-1}_z$ is the subgroup $\mathbf{I}_3 \subset \mathbb{Z}/2 \int \mathbf{D}_4$.

Let $OP\alpha$ be the $\mathbb{Z}/2 \int \mathbf{D}_4$–framed immersion obtained from an arbitrary $\mathbb{Z}/2 \int \mathbf{D}_4$-framed immersion α by changing the structure group of the framing by the transformation OP. The $\mathbb{Z}/2\int \mathbf{D}_4$-framed manifold (with boundary) $(\tilde{L}_y^{n-4k}, \tilde{\Psi}_y, \tilde{\zeta}_y)$ coincides with the two disjoint copies of $\mathbb{Z}/2\int \mathbf{D}_4$-framed manifold (with boundary) $OP(\tilde{L}_y^{n-4k}, \tilde{\Psi}_y, \tilde{\zeta}_y)$.

Let us put $\alpha_1 = -OP(\tilde{L}^{n-4k}, \tilde{\Psi}, \tilde{\zeta})$. Let us define the sequence of $\mathbb{Z}/2\int \mathbf{D}_4$-framed immersions $\alpha_2 = -2OP\alpha_1$, $\alpha_3 = -2OP\alpha_2$, $\dots$, $\alpha_j = -2OP\alpha_{j-1}$.

Obviously, the $\mathbf{D}/4\int \mathbb{Z}/2$-framed immersion $\alpha_1 + \alpha_2 = \alpha_1 + 2OP\alpha_1^{-1}$ is represented by 3 copies of the manifold $\tilde{L}^{n-4k}$. The second and the third copies are obtained from the first copy by the mirror image and the changing of structure group of the framing. The manifold $-OP[\tilde{L}^{n-4k}] \cup 2[\tilde{L}^{n-4k}]$ contains, in particular, a copy of $-OP[\tilde{L}_x^{n-4k}]$ inside the first component and the union $[\tilde{L}_y^{n-4k} \cup L_y^{n-4k}]$ of the mirror two copies of $-OP[\tilde{L}_x^{n-4k}]$ in the second and the third component. Therefore the manifold $-OP[\tilde{L}^{n-4k}] \cup 2[\tilde{L}^{n-4k}]$ is $\mathbb{Z}/2\int \mathbf{D}_4$-framed cobordant to a $\mathbb{Z}/2\int \mathbf{D}_4$-framed manifold, obtained by gluing the union of a copy of $-OP[\tilde{L}_x^{n-4k}]$ and 4 copies of $\tilde{L}_y^{n-4k}$ by a $\mathbf{I}_3$-framing manifold along the boundary. This cobordism is relative with respect to the submanifold $-OP[\tilde{L}_z^{n-4k}] \cup 2[L_z^{n-4k}] \subset -OP[L^{n-4k}] \cup 2L^{n-4k}$.

By an analogous argument it is easy to prove that the element $\aleph = \sum_{j=1}^{j_0} \alpha_j$ is $\mathbb{Z}/2\int \mathbf{D}_4$-framed cobordant to the manifold obtained by gluing the union $-OP[\tilde{L}_x^{n-4k}] \cup 2^j(-OP)^{j-1}[\tilde{L}_y^{n-4k}]$ by an $\mathbf{I}_3$-manifold along the boundary. Moreover, this cobordism is relative with respect to all copies of $\tilde{L}_z^{n-4k}$ (with various orientations). If j_0 is great enough, the manifold (with $\mathbf{I}_3$-framed boundary) $2^j(-OP)^{j_0-1}[\tilde{L}_y^{n-4k}]$ is cobordant relative to the boundary to an $\mathbf{I}_3$-framed manifold.

Therefore the manifold L_y^{n-4k} is $\mathbb{Z}/2\int \mathbf{D}_4$-framed cobordant relative to the boundary to the union of an $\mathbf{I}_3$-framed manifold with the same boundary and a closed manifold that is the double cover with respect to ω over a $\mathbb{Z}/2\int \mathbf{D}_4$-framed manifold. This cobordism is realized as a cobordism of the self-intersection of a $\mathbf{D}_4$-framed immersion with support inside U_Δ^{reg}. This cobordism joins the immersion g_3 with a $\mathbf{D}_4$–framed immersion g_4. After an additional deformation of g_4 inside a larger neighborhood of Δ^{reg} the relative $\mathbf{I}_b$-submanifold of the self-intersection manifold of g_4 is deformed outside of U_Δ^{reg}. The $\mathbf{D}_4$-framed immersion obtained as the result of this cobordism admits an $\mathbf{I}_b$-control. The Theorem 1 is proved.

4 An $\mathbf{I}_4$-structure (a cyclic structure) of a $\mathbf{D}_4$-framed immersion

Let us describe the subgroup $\mathbf{I}_4 \subset \mathbb{Z}/2 \int \mathbf{D}_4$. This subgroup is isomorphic to the group $\mathbb{Z}/2 \oplus \mathbb{Z}/4$. Let us recall that the group $\mathbb{Z}/2 \int \mathbf{D}_4$ is the transformation group of $\mathbb{R}^4$ that permutes the 4-tuple of the coordinate lines and two planes (f_1, f_2), (f_3, f_4) spanned by the vectors of the standard base (f_1, f_2, f_3, f_4) (the planes can remin fixed or be permuted by a transformation).

Let us denote the generators of $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ by l, r correspondingly. Let us describe the transformations of $\mathbb{R}^4$ given by each generator. Consider a new base (e_1, e_2, e_3, e_4), given by $e_1 = f_1 + f_2$, $e_2 = f_1 - f_2$, $e_3 = f_3 + f_4$, $e_4 = f_3 - f_4$. The generator r of order 4 is represented by the rotation in the plane (e_2, e_4) through the angle $\frac{\pi}{2}$ and the reflection in the plane (e_1, e_3) with respect to the line $e_1 + e_3$. The generator l of order 2 is represented by the central symmetry in the plane (e_1, e_3).

Obviously, the described representation of $\mathbf{I}_4$ admits invariant (1,1,2)-dimensional subspaces. We will denote subspaces by $\lambda_1, \lambda_2, \tau$.

The lines λ_1, λ_2 are generated by the vectors $e_1 + e_3$, $e_1 - e_3$ correspondingly. The subspace τ is generated by the vectors e_2, e_4. The generator r acts by the reflection in λ_2 and by the rotation in τ throught the angle $\frac{\pi}{2}$. The generator l acts by reflections in the subspaces λ_1, λ_2.

In particular, if the structure group $\mathbb{Z}/2 \int \mathbf{D}_4$ of a 4-dimensional bundle $\zeta : E(\zeta) \to L$ admits a reduction to the subgroup $\mathbf{I}_4$, then the bundle is decomposed into the direct sum $\zeta = \lambda_1 \oplus \lambda_2 \oplus \tau$ of $1, 1, 2$–dimensional subbundles.

Definition 6

Let $(g : N^{n-2k} \looparrowright \mathbb{R}^n, \Xi_N, \eta)$ be an arbitrary $\mathbf{D}_4$-framed immersion. We shall say that this immersion is an $\mathbf{I}_b$–immersion (or a cyclic immersion), if the structure group $\mathbb{Z}/2 \int \mathbf{D}_4$ of the normal bundle over the double points manifold L^{n-4k} of this immersion admits a reduction to the subgroup $\mathbf{I}_4 \subset \mathbb{Z}/2 \int \mathbf{D}_4$. In this definition we assume that the pairs (f_1, f_2), (f_3, f_4) are the vectors of the framing for the two sheets of the self-intersection manifold at a point in the double point manifold L^{n-4k}.

In particular, for a cyclic $\mathbb{Z}/2 \int \mathbf{D}_4$-framed immersion there exists the mappings $\kappa_a : L^{n-4k} \to K(\mathbb{Z}/2, 1)$, $\mu_a : L^{n-4k} \to K(\mathbb{Z}/4, 1)$ such that the characteristic mapping $\zeta : L^{n-4k} \to K(\mathbb{Z}/2 \int \mathbf{D}/4, 1)$ of the $\mathbb{Z}/2 \int \mathbf{D}_4$-framing of the normal bundle over L^{n-4k} is reduced to a mapping with the target $K(\mathbf{I}_b, 1)$ such that the following equation holds:

$$\zeta = i(\kappa_a \oplus \mu_a),$$

where $i : \mathbb{Z}/2 \oplus \mathbb{Z}/4 \to \mathbf{I}_4$ is the prescribed isomorphism.

The following Proposition is proved by a straightforward calculation.

Proposition 2

Let (g, Ψ_N, η) be a $\mathbf{D}_4$–framed immersion, that is a cyclic immersion. Then the Kervaire invariant, appearing as the top line of the diagram (7), can be calculated by following formula:

$$\Theta_a =< \kappa_a^{\frac{n-4k}{2}} \mu_a^*(\tau)^{\frac{n-4k-2}{4}} \mu_a^*(\rho); [L] >, \qquad (8)$$

where $\tau \in H^2(\mathbb{Z}/4; \mathbb{Z}/2)$, $\rho \in H^1(\mathbb{Z}/4; \mathbb{Z}/2)$ are the generators.

Proof of Proposition 2

Let us consider the subgroup of index 2, $\mathbf{I}_b \subset \mathbf{I}_4$. This subgroup is the kernel of the epimorphism $\chi' : \mathbf{I}_4 \to \mathbb{Z}/2$, that is the restriction of the characteristic class $\chi : \mathbb{Z}/2 \int \mathbf{D}_4 \to \mathbb{Z}/2$ of the canonical double cover $\bar{L} \to L$ to the subgroup $\mathbf{I}_4 \subset \mathbb{Z}/2 \int \mathbf{D}_4$. Obviously, the characteristic number (8) is calculated by the formula

$$\Theta_a =< \hat{\kappa}_a^{\frac{n-4k}{2}} \hat{\rho}_a^{\frac{n-4k}{2}}; \bar{L} >, \qquad (9)$$

where the characteristic class $\hat{\kappa}_a \in H^1(\bar{L}; \mathbb{Z}/2)$ is induced from the class $\kappa_a \in H^1(L; \mathbb{Z}/2)$ by the canonical cover $\bar{L} \to L$, and the class $\hat{\rho}_a \in H^1(\bar{L}; \mathbb{Z}/2)$ is obtained by the transfer of the class $\rho \in H^1(L; \mathbb{Z}/4)$.

Note that $\hat{\kappa}_a = \tau_1$, $\hat{\rho}_a = \tau_2$, where τ_1, τ_2 are the two generating $\mathbf{I}_b$–characteristic classes. Therefore $\hat{\kappa}_a \hat{\rho}_a = \tau_1 \tau_2 = w_2(\eta)$, where η is the two-dimensional bundle that determines the $\mathbf{D}_4$–framing (over the submanifold $\bar{L}^{n-4k} \subset N^{n-2k}$ this framing admits a reduction to an $\mathbf{I}_b$-framing) of the normal bundle for the immersion g of N^{n-2k} into $\mathbb{R}^n$.

Therefore the characteristic number, given by the formula (8) in the case when the $\mathbb{Z}/2 \int \mathbf{D}_4$ framing over L^{n-4k} is reduced to an $\mathbf{I}_4$-framing, coincides with the characteristic number, given by the formula (9). Proposition 2 is proved.

Definition 7

We shall say that a $\mathbf{D}_4$-framed immersion (g, Ξ_N, η) admits a $\mathbf{I}_4$–structure (a cyclic structure), if for the double points manifold L^{n-4k} of g there exist mappings $\kappa_a : L^{n-4k} \to K(\mathbb{Z}/2,1)$, $\mu_a : L^{n-4k} \to K(\mathbb{Z}/4,1)$ such that the characteristic number (8) coincides with Kervaire invariant, see Definition 2.

Theorem 2

Let (g, Ψ, η) be a $\mathbf{D}_4$-framed immersion, $g : N^{n-2k} \looparrowright \mathbb{R}^n$, that represents a regular cobordism class in the image of the homomorphism $\delta : Imm^{sf}(n-k,k) \to Imm^{\mathbf{D}_4}(n-2k,2k)$, $n-4k=62$, $n=2^l-2$, $l \ge 13$, and assume the conditions of the Theorem 1 hold, i.e. the residue class $\delta^{-1}(Imm^{sf}(n-k,k)$ (this class is defined modulo odd torsion) contains a skew-framed immersion that admits a retraction of order 62.

Then in the $\mathbf{D}_4$-framed cobordism class $[(g,\Psi,\eta)] = \delta[(f,\Xi,\kappa)] \in Imm^{\mathbf{D}_4}(n-2k,2k)$ there exists a $\mathbf{D}_4$-framed immersion that admits an $\mathbf{I}_4$–structure (a cyclic structure).

5 Proof of Theorem 2

Let us formulate the Geometrical Control Principle for $\mathbf{I}_b$–controlled immersions.

Let us take an $\mathbf{I}_b$–controlled immersion (see Definition 4) $(g, \Xi_N, \eta; (P,Q), \kappa_{Q,1}, \kappa_{Q,2})$, where $g : N \looparrowright \mathbb{R}^n$ is a $\mathbf{D}_4$-framed immersion, equipped with a control mapping over a polyhedron $i_P : P \subset \mathbb{R}^n$, $dim(P) = 2k-1$; $Q \subset P$ $dim(Q) = dim(P)-1$. The characteristic classes $\kappa_{Q,i} \in H^1(Q;\mathbb{Z}/2)$, $i=1,2$ coincide with characteristic classes $\kappa_{i,N_Q} \in N_Q^{n-2k-1}$ by means of the mapping $\partial N_{int}^{n-2k} = N_Q^{n-2k} \to Q$, where $N_{int}^{n-2k} \subset N^{n-2k}$, $N_{int}^{n-2k} = g^{-1}(U_P)$, $U_P \subset \mathbb{R}^n$.

Proposition 3. Geometrical Control Principle for $\mathbf{I}_b$–controlled immersions

Let $j_P : P \subset \mathbb{R}^n$ be an arbitrary embedding; such an embedding is unique up to isotopy by a dimensional reason, because $2dim(P)+1 = 4k-1 < n$. Let $g_1 : N^{n-2k} \to \mathbb{R}^n$ be an arbitrary mapping, such that the restriction $g_1|_{N_{int}} : (N_{int}^{n-2k}, N_Q^{n-2k-1}) \looparrowright (U_P, \partial U_P)$ is an immersion (the restriction

$g|_{N_Q^{n-2k-1}}$ is an embedding) that corresponds to the immersion $g|_{N_{int}^{n-2k}}$: $(N_{int}^{n-2k}, N_Q^{n-2k-1}) \looparrowright (U_P, \partial U_P)$ by means of the standard diffeomorphism of the regular neighborhoods $U_{i_P} = U_{j_P}$ of subpolyhedra $i(P)$ and $j(P)$. (For a dimension reason there is a standard diffeomorphism of U_{i_P} and U_{j_P} up to an isotopy.)

Then for an arbitrary $\varepsilon > 0$ there exists an immersion $g_\varepsilon : N^{n-2k} \looparrowright \mathbb{R}^n$ such that $dist_{C^0}(g_1, g_\varepsilon) < \varepsilon$ and such that g_ε is regular homotopy to an immersion g and the restrictions $g_\varepsilon|_{N_{int}^{n-2k}}$ and $g_1|_{N_{int}^{n-2k}}$ coincide.

We start the proof of Theorem 2 with the following construction. Let us consider the manifold $Z = S^{\frac{n}{2}+64}/i \times \mathbb{RP}^{\frac{n}{2}+64}$. This manifold is the direct product of the standard lens space $(mod 4)$ and the projective space. The cover $p_Z : \hat{Z} \to Z$ over this manifold with the covering space $\hat{Z} = \mathbb{RP}^{\frac{n}{2}+64} \times \mathbb{RP}^{\frac{n}{2}+64}$ is well-defined.

Let us consider in the manifold Z a family of submanifolds X_i, $i = 0, \dots, \frac{n+2}{64}$ of the codimension $\frac{n+2}{2}$, defined by the formulas $X_0 = S^{\frac{n}{2}+64}/i \times \mathbb{RP}^{63}$, $X_1 = S^{\frac{n}{2}+32}/i \times \mathbb{RP}^{95}, \dots$, $X_j = S^{\frac{n}{2}-32(j-2)-1}/i \times \mathbb{RP}^{32(j+2)-1}, \dots$, $X_{\frac{n+2}{64}} = S^{63}/i \times \mathbb{RP}^{\frac{n}{2}+64}$. The embedding of the corresponding manifold in Z is defined by the Cartesian product of the two standard embeddings.

The union of the submanifolds $\{X_i\}$ is a stratified submanifold (with singularities) $X \subset Z$ of the dimension $\frac{n}{2} + 127$, the codimension of maximal singular strata in X is equal to 64. The covering $p_X : \hat{X} \to X$, induced from the covering $p_Z : \hat{Z} \to Z$ by the inclusion $X \subset Z$, is well-defined. The covering space $\hat{X}$ is a stratified manifold (with singularities) and decomposes into the union of the submanifolds $\hat{X}_0 = \mathbb{RP}^{\frac{n}{2}+64} \times \mathbb{RP}^{63}, \dots, \hat{X}_j = \mathbb{RP}^{\frac{n}{2}-32(j-2)} \times \mathbb{RP}^{32(j+2)-1}, \dots, \hat{X}_{\frac{n+2}{64}} = \mathbb{RP}^{63} \times \mathbb{RP}^{\frac{n}{2}+64}$. Each manifold $\hat{X}_i$ of the family is the 2-sheeted covering space over the manifold X_i over the first coordinate. Let us define $d_1(j) = \frac{n}{2} - 32(j-2)$, $d_2(j) = 32(j+2) - 1$. Then the formula for X_i is the following: $X_j = \mathbb{RP}^{d_1(j)} \times \mathbb{RP}^{d_2(j)}$.

The cohomology classes $\rho_{X,1} \in H^1(X;\mathbb{Z}/4)$, $\kappa_{X,2} \in H^1(X;\mathbb{Z}/2)$ are well-defined. These classes are induced from the generators of the groups $H^1(Z;\mathbb{Z}/4)$, $H^1(Z;\mathbb{Z}/2)$. Analogously, the cohomology classes $\kappa_{\hat{X},i} \in H^1(\hat{X};\mathbb{Z}/4)$, $i = 1, 2$ are well-defined. The cohomology class $\kappa_{\hat{X},1}$ is induced from the class $\rho_{X,1} \in H^1(X;\mathbb{Z}/4)$ my means of the transfer homomorphim, and $\kappa_{\hat{X},2} = (p_X)^*(\kappa_{X,2})$.

Let us define for an arbitrary $j = 0, \dots, (\frac{n+2}{64})$ the space J_j and the mapping $\varphi_j : X_j \to J_j$. We denote by $Y_1(k)$ the space $S^{31}/i * \dots * S^{31}/i$ of the join of k copies, $k = 1, \dots, (\frac{n+2}{64} + 1)$, of the standard lens space S^{31}/i.

Let us denote by $Y_2(k)$, $k = 2, \dots, (\frac{n+2}{64} + 2)$, $Y_2(k) = \mathbb{RP}^{31} * \dots * \mathbb{RP}^{31}$ the joins of the k copies of the standard projective space $\mathbb{RP}^{31}$. Let us define $J_j = Y_1(\frac{n+2}{64} - j + 2)) \times Y_2(j+2)$ $Q = Y_1(\frac{n+2}{64} + 2) \times Y_2(\frac{n+2}{64} + 2)$. For a given j the natural inclusions $J_j \subset Q$ are well-defined. Let us denote the union of the considered inclusions by J.

The mapping $\varphi_j : X_j \to J_j$ is well-defined as the Cartesian product of the two following mappings. On the first coordinate the mapping is defined as the composition of the standard 2-sheeted covering $\mathbb{RP}^{d_1(j)} \to S^{\frac{n}{2}-64(j-1)}/i$ and the natural projection $S^{d_1(j)}/i \to Y_1(d_1(j))$. On the second coordinate the mapping is defined by the natural projection $\mathbb{RP}^{d_2(j)} \to Y_2(j+1)$.

The family of mappings φ_j determines the mapping $\varphi : \hat{X} \to J$, because the restrictions of any two mappings to the common subspace in the origin coincide.

For $n+2 \ge 2^{13}$ the space J embeddable into the Euclidean n-space by an embedding $i_J : J \subset \mathbb{R}^n$. Each space $Y_1(k)$, $Y_2(k)$ in the family is embeddable into the Euclidean (2^6k-1-k)–space. Therefore for an arbitrary j the space J_j is embaddable into the Euclidean space of dimension $n + 126 - \frac{n+2}{64}$. In particular, if $n + 2 \ge 2^{13}$ the space J_j is embeddable into $\mathbb{R}^n$. The image of an arbitrary intersection of the two embeddings in the family belongs to the standard coordinate subspace. Therefore the required embedding i_J is defined by the gluing of embeddings in the family.

Let us describe the mapping $\hat{h} : \hat{X} \to \mathbb{R}^n$. By ε we denote the radius of a (stratified) regular neighborhood of the subpolyhedron $i_J(J) \subset \mathbb{R}^n$. Let us consider a small positive ε_1, $\varepsilon_1 << \varepsilon$, (this constant will be defined below in the proof of Lemma 4) and let us consider a generic PL ε_1–deformation of the mapping $i_J \circ \varphi : \hat{X} \to J \subset \mathbb{R}^n$. The result of the deformation is denoted by $\hat{h} : \hat{X} \to \mathbb{R}^n$.

Let us define the positive integer k from the equation $n - 4k = 62$. In the prescribed regular homotopy class of an $\mathbf{I}_b$-controlled immersion $f : N^{n-2k} \looparrowright \mathbb{R}^n$ we will construct another $\mathbf{I}_b$–controlled immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ that admits a $\mathbf{I}_b$–structure.

Let the immersion f be controlled over the embedded subpolyhedron $\psi_P : P \subset \mathbb{R}^n$. Let $\psi_Q : Q \to \hat{X}$ be a generic mapping such that $\kappa_{Q,i} = \psi_Q \circ \kappa_{\hat{X},i}$, $i = 1, 2$. By the previous definition the manifolds N^{n-2k}_{int}, N^{n-2k}_{ext} with the common boundary N_Q^{n-2k-1}, $N^{n-2k} = N^{n-2k}_{int} \cup_{N_Q^{n-2k-1}} N^{n-2k}_{ext}$ are well-defined.

Let $\eta : N^{n-2k}_{ext} \to K(\mathbf{I}_b, 1) \subset K(\mathbf{D}_4, 1)$ be the characteristic mapping of the framing Ξ_N, restricted to $N^{n-2k}_{ext} \subset N^{n-2k}$. The restriction of this mapping to the boundary $\partial N^{n-2k}_{ext} = N_Q^{n-2k-1}$ is given by the composition $\partial N_Q^{n-2k-1} \to Q \to K(\mathbf{I}_b, 1) \subset K(\mathbf{D}_4, 1)$. The target space for the mapping

η is the subspace $K(\mathbf{I}_b,1) \subset K(\mathbf{D}_4,1)$. This mapping is determined by the cohomology classes $\kappa_{N^{n-2k}_{ext},s} \in H^1(N^{n-2k}_{ext},Q;\mathbb{Z}/2)$, $s=1,2$.

Let us define the mapping $\lambda : N^{n-2k}_{ext} \to \hat{X}$ by the following conditions. This mapping transforms the cohomology classes $\kappa_{\hat{X},i}$ into the classes $\kappa_i \in H^1(N^{n-2k}_{ext};\mathbb{Z}/2)$ and also the restriction $\lambda|_{N_Q^{n-2k-1}}$ coincides with the composition of the projection $N_Q^{n-2k-1} \to Q$ and the mapping $\psi_Q : Q \to \hat{X}$. The boundary conditions for the mapping ψ_Q are $\kappa_{Q,i} = \psi_Q \circ \kappa_{\hat{X},i}$, $i=1,2$. The submanifold with singularities $\hat{X} \subset \hat{Z}$ contains the skeleton of the space $\hat{Z}$ of the dimension $\frac{n}{2}+62$. Because $n-2k = \frac{n}{2}+31$, the mapping λ is well-defined.

Let us denote the composition $\hat{h} \circ \lambda : N^{n-2k}_{ext} \to \hat{X} \to \mathbb{R}^n$ by g_1. Let us denote the mapping $\hat{h} \circ \psi_Q : Q \to \hat{X} \to \mathbb{R}^n$ by φ_Q. One can assume that the mapping φ_Q is an embedding. Moreover, without loss of generality one may assume that this embedding is extended to a generic embedding $\varphi_P : P \subset \mathbb{R}^n$ such that the embedded polyhedron $\varphi_P : P \subset \mathbb{R}^n$ does not intersect $g_1(N^{n-2k}_{ext})$.

Let us denote by $U_\varphi(P)$ a regular neighborhood of the subpolyhedron $\varphi_P(P) \subset \mathbb{R}^n$ (we may assume that the radius of this neighborhood is equal to ε). Up to an isotopy a regular neighborhood $U_\varphi(P)$ is well-defined, in particular, this neighborhood does not depend on the choice of a regular embedding of P, moreover $U_\varphi(P)$ and $U(P)$ are diffeomorphic.

Without loss of generality after an additional small deformation we may assume that the restriction $g_1|_{N^{n-2k}_{int}}$ is a regular immersion $g_1 : N^{n-2k}_{int} \subset \mathbb{R}^n$ with the image inside $U_\varphi(P)$. In particular, the restriction of g_1 to the boundary $N_Q^{n-2k-1} = \partial(N^{n-2k}_{int})$ is a regular embedding $N_Q^{n-2k-1} \subset \partial U(P)$. The immersion $g_1|_{N_{int}}$ is conjugated to the immersion $f|_{N_{int}}$ by means of a diffeomorphism of $U_\varphi(P)$ with $U(P)$.

By Proposition 3, for an arbitrary $\varepsilon_2 > 0$, $\varepsilon_2 << \varepsilon_1 << \varepsilon$, there exists an immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ in the regular homotopy class of f, such that g coincides with g' (and with g_1) on N^{n-2k}_{int} and, moreover, $dist(g,g_1) < \varepsilon_2$.

Let us consider the self-intersection manifold L^{n-4k} of the immersion g. This manifold is a submanifold in $\mathbb{R}^n$. Let us construct the mappings $\kappa_a : L^{n-4k} \to K(\mathbb{Z}/2,1)$, $\mu_a : L^{n-4k} \to K(\mathbb{Z}/4,1)$. Then we check the conditions (8) and (9).

The manifold L^{n-4k} is naturally divided into two components. The first component L^{n-4k}_{int} is inside $U_{\varphi_P}(P)$. The last component (we will denote this component again by L^{n-4k}) consists of the last self-intersection points. This component is outside the ε–neighborhood of the submanifold with singularities $h(X)$. The mappings κ_a, μ_a over L^{n-4k}_{int} are defined as the trivial mappings. Let us define the mappings κ_a, μ_a on L^{n-4k}.

Let us consider the mapping $\varphi : \hat{X} \to J$ and the singular set (polyhedron) Σ of this mapping. This is the subpolyhedron $\Sigma \subset \{\hat{X}^{(2)} = \hat{X} \times \hat{X} \setminus \Delta_{\hat{X}}/T'\}$, where $T' : \hat{X}^(2) \to \hat{X}^{(2)}$– is the involution of coordinates in the delated product $\hat{X}^{(2)}$ of the space $\hat{X}$. The subpolyhedron (it is convenient to view this polyhedron as a manifold with singularities) Σ is naturally decomposed into the union of the subpolyhedra $\Sigma(j)$, $j = 0, \dots, \frac{n+2}{128}$. The subpolyhedron $\Sigma(j)$ is the singular set of the mapping $\varphi(j) : \mathbb{RP}^{d_1(j)} \times \mathbb{RP}^{d_2(j)} \to S^{d_1(j)}/i \times \mathbb{RP}^{d_2(j)} \to J_j$. This subpolyhedron consists of the singular points of the mapping φ in the inverse image $(\varphi)^{-1}(J_j) = \mathbb{RP}^{d_1(j)} \times \mathbb{RP}^{d_2(j)}$ of the subspace $J_j \subset J$.

Let us consider the subspace $\Sigma^{reg} \subset \Sigma$, consisting of points on strata of length 0 (regular strata) and of length 1 (singular strata of the codimension 32) after the regular ε_2 –neighborhoods ($\varepsilon_2 << \varepsilon_1$) of the diagonal Δ^{diag} and the antidiagonal $\Delta^{antidiag}$ of Σ^{reg} are cut out.

The manifold with singularities Σ^{reg} admits a natural compactification (closure) in the neighborhood of Δ^{diag} and $\Delta^{antidiag}$; the result of the compactification will be denoted by K_{reg}.

The space RK, called the space of resolution of singularities, equipped with the natural projection $RK \to K_{reg}$ is defined by the analogous construction; see the short English translation of [A1], Lemma 7. The cohomology classes $\rho_{RK,1} \in H^1(RK;\mathbb{Z}/4)$, $\kappa_{RK,2} \in H^1(RK;\mathbb{Z}/2)$ are well-defined. The cohomology classes $\kappa_{K_{reg},1} \in H^1(RK;\mathbb{Z}/2)$, $\kappa_{RK,1} \in H^1(RK;\mathbb{Z}/2)$ are the images of the class $\kappa_{\Sigma,1} \in H^1(\Sigma;\mathbb{Z}/2)$ with respect to the inclusion $K_{reg} \subset \Sigma$ and the projection $RK \to K_{reg}$. The class classifies the transposition of the two non-ordered preimages of a point in the singular set.

Let us consider the restrictions of the classes $\kappa_{K_{reg},1}, \kappa_{RK,1}, \kappa_{\Sigma,1}$ to neighborhoods of the diagonal and the antidiagonal. The natural projection $\Delta^{diag} \to \hat{X}$ is well-defined. The restrictions of the classes ρ_1 and κ_2 to neighborhoods of the diagonal coincide with the restrictions of the classes $\rho_{\hat{X},1} \in H^1(\hat{X};\mathbb{Z}/4)$, $\kappa_{\hat{X},2} \in H^1(\hat{X};\mathbb{Z}/2)$. (These classes $\rho_{\hat{X},1}, \kappa_{\hat{X},2}$ are extended to neighborhoods of the diagonal).

Let us recall that the mapping $\hat{h} : \hat{X} \to \mathbb{R}^n$ is defined as the result of an ε_1–small regular deformation of the mapping $\hat{X} \to X \xrightarrow{h} \mathbb{R}^n$. The singular set of the mapping $\hat{h}$ will be denoted by $\Sigma_{\hat{h}}$. This is a 128–dimensional polyhedron, or a manifold with singularities in the codimensions $32, 64, 96, 128$. Moreover, the inclusion $\Sigma_{\hat{h}} \subset \hat{X}^{(2)}$ is well-defined. The image of this inclusion is in the regular ε_1–small neighborhood of the singular polyhedron $\Sigma \subset X^{(2)}$.

Let us denote by $\Sigma^{reg}_{\hat{h}}$ the part of the singular set after cutting out the regular ε_1–neighborhood of the points in singular strata of length at least 2 (of the codimension 64) and self-intersection points of all singular strata (these strata are also of the codimension 64). The boundary $\partial\Sigma_{\hat{h}}$ is a submanifold with singularities in $\hat{X}$ and therefore. by a general position argument, we may also assume that the boundary $\partial\Sigma^{reg}_{\hat{h}}$ is a regular submanifold with singularities in $\hat{X}$.

Additionally, by general position arguments, the intersection of the image $Im(\lambda(N^{n-2k}_{ext}))$ inside the singular set $\Sigma_{\hat{h}}$ (this is a polyhedron of the dimension 62) on X are outside (with respect to the caliber ε) of the projection of the singular submanifold with singularities (this singular part is of the codimension 64) in the complement of the regular submanifold with singularities $\Sigma^{reg}_{\hat{h}} \subset \Sigma_{\hat{h}}$. Therefore the image $Im(\lambda(N^{n-2k}_{ext}))$ is inside the regular part $\Sigma^{reg}_{\hat{h}} \subset \Sigma_{\hat{h}}$.

Let us denote by $L^{62}_{cycl} \subset L^{62}$ the submanifold (with boundary) given by the formula $L^{62}_{cycl} = L^{62} \cap U_{\Sigma^{reg}}$. The mappings κ_a, ρ_a are extendable from $U_{\Sigma^{reg}}$ to $L^{62}_{cycl} \subset L^{62}$. Let us prove that these mappings are extendable to mappings $\kappa_a : L^{62} \to K(\mathbb{Z}/2,1)$, $\rho_a : L^{62} \to K(\mathbb{Z}/4,1)$.

The complement of the submanifold $L^{62}_{cycl} \subset L^{62}$ is denoted by $L^{62}_{\mathbf{I}_3} = L^{62} \setminus L^{62}_{cycl}$. The submanifold $L^{62}_{\mathbf{I}_3}$ is a submanifold in the regular ε–neighborhood of $h(X) \subset \mathbb{R}^n$. Obviously, the structure group of the $\mathbb{Z}/2 \int \mathbf{D}_4$–framing of the normal bundle of the manifold (with boundary) $L^{62}_{\mathbf{I}_3}$ is reduced to the subgroup $\mathbf{I}_3 \subset \mathbb{Z}/2 \int \mathbf{D}_4$.

Let us consider the mapping of pairs $\mu_a \times \kappa_a : (L^{62}_{cycl}, \partial L^{62}_{cycl}) \to (K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1), K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,1))$. Let us consider the natural projection $\pi_b : \mathbf{I}_3 \to \mathbf{I}_b$. The extension of the mapping $\mu_a \times \kappa_a$ to the required mapping $L^{62} \to K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1)$ is given by the composition $L^{62}_{\mathbf{I}_3} \to K(\mathbf{I}_3,1) \stackrel{\pi_{b,*}}{\longrightarrow} K(\mathbf{I}_b,1) \subset K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1)$, where $\kappa_1 \in K(\mathbf{I}_b;\mathbb{Z}/2)$ determines the inclusion $K(\mathbf{I}_b,1) \subset K(\mathbb{Z}/2,1) \subset K(\mathbb{Z}/4,1)$.

Let us formulate the results in the following lemma.

Lemma 4

–1. Let $n \geq 2^{13} - 2$ and k, $n - 4k = 62$ satisfy the conditions of Theorem 1 (in particular, an arbitrary element in the group $Imm^{sf}(n-k,k)$ admits a retraction of the order 62. Then for arbitrarily small positive numbers ε_1, ε_2, $\varepsilon_1 >> \varepsilon_2$ (the numbers ε_1, ε_2 are the calibers of the regular deformations in the construction of the PL–mapping $\hat{h} : \hat{X} \to \mathbb{R}^n$ and of the immersion $g : N^{n-2k} \looparrowright \mathbb{R}^n$ correspondingly) there exists the mapping $m_a = (\kappa_a \times \mu_a) : \Sigma^{reg}_h \to K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1)$ under the following condition. The restriction

$m_a|_{\partial\Sigma_h^{reg}}$ (by $\partial\Sigma_h^{reg}$ is denoted the part of the singular polyhedron consisting of points on the diagonal) has the target $K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,1) \subset K(\mathbb{Z}/4,1)\times K(\mathbb{Z}/2,1)$ and is determined by the cohomological classes $\kappa_{\hat{X},1}, \kappa_{\hat{X},2}$.

–2. The mappings κ_a, μ_a induces a mapping $(\mu_a \times \kappa_a) : L^{62} \to K(\mathbb{Z}/4,1)\times K(\mathbb{Z}/2,1)$ on the self-intersection manifold of the immersion g.

Let us prove that the mapping $(\mu_a \times \kappa_a)$ constructed in Lemma 4 determines a $\mathbb{Z}/2 \oplus \mathbb{Z}/4$–structure for the $\mathbf{D}_4$–framed immersion g. We have to prove the equation (9).

Let us recall that the component L^{62}_{int} of the self-intersection manifold of the immersion g is a $\mathbb{Z}/2 \int \mathbf{D}_4$–framed manifold with trivial Kervaire invariant: the corresponding element in the group $Imm^{\mathbb{Z}/2\int \mathbf{D}_4}(62, n-62)$ is in the image of the transfer homomorphism. Therefore it is sufficient to prove the equation

$$< m_a^*(\rho\tau^{15}t^{31}); [L^{62}] >= \Theta,$$

or, equivalently, the equation

$$< (\hat{\rho}_a^{31}\hat{\kappa}_a^{31}); [\hat{L}^{62}] >= \Theta, \qquad (10)$$

where $\hat{L} \to L$ is the canonical cover over the self-intersection manifold, $\hat{L} \subset N^{n-2k}_{ext}$ is the canonical inclusion.

By Herbert's theorem (see [A1] for the analogous construction) we may calculate the right side of the equation by the formula

$$< \eta^*(w_2(\mathbf{I}_b))^{\frac{n-2k}{2}}; [N^{n-2k}_{ext}/\sim] > . \qquad (11)$$

In this formula by $N^{n-2k}_{ext}/\sim$ is denoted the quotient of the boundary $\partial N^{n-2k}_{ext} = N_Q^{n-2k-1}$ that is contracted onto the polyhedron Q with the loss of the dimension. Note that the mapping $m_a|_{N_Q^{n-2k-1}}$ is obtained by the composition of the mapping $p_Q : N^{n-2k-1} \to Q$ with a loss of dimension with the mapping $Q \to K(\mathbf{I}_b,1)$, the last mapping is determined by the cohomology classes $\kappa_{i,Q} \in H^1(Q;\mathbb{Z}/2)$, $i=1,2$. Therefore, $m_{a*}([N^{n-2k}_{ext}/\sim]) \in H_{n-2k}(\mathbf{I}_b;\mathbb{Z}/2)$ is a permanent cycle and the integration over the cycle $[N^{n-2k}_{ext}/\sim]$ of the inverse image of the universal cohomology class in (11) is well-defined.

It is convenient to consider the characteristic number Θ_a as the value of a homomorphism $H_{n-2k}(X;\mathbb{Z}/2) \to \mathbb{Z}/2$ on the cycle $\lambda_*[N^{n-2k}_{ext}/\sim] \in H_{n-2k}(X;\mathbb{Z}/2)$. This homomorphism is the result of the calculation of the characteristic class $w_2(\mathbf{I}_b) \in H^2(K(\mathbf{I}_b,1);\mathbb{Z}/2)$ on the prescribed cycle, i.e. on the image of the fundamental cycle $[N^{n-2k}_{ext}/\sim]$ with respect to the mapping $N^{n-2k}_{ext}/\sim \longrightarrow \hat{X} \to K(\mathbf{I}_b,1)$. The cycle $\lambda_*[N^{n-2k}_{ext}/\sim] \in H_{n-2k}(X;\mathbb{Z}/2)$ is the modulo 2 reduction of an integral homology class. Therefore this cycle is given by a sum of fundamental classes of the product of the two odd-dimensional projective spaces, the sum of the dimensions of this spaces being equal to $n-2k$.

Let us consider an arbitrary submanifold $S^{k_1}/i \times \mathbb{RP}^{k_2} \subset X$, $k_1+k_2 = \frac{n}{2}+31$, k_1, k_2 being odd. Let us consider the cover $\mathbb{RP}^{k_1} \times \mathbb{RP}^{k_2} \to S^{k_1}/i \times \mathbb{RP}^{k_2}$ and the composition $\mathbb{RP}^{k_1} \times \mathbb{RP}^{k_2} \subset \hat{X} \overset{\hat{h}}{\looparrowright} \mathbb{R}^n$ after an ε_1–small generic perturbation. Let us denote this mapping by s_{k_1,k_2}.

The self-intersection manifold of the generic mapping $s_{k_1,k_2} : \mathbb{RP}^{k_1} \times \mathbb{RP}^{k_2} \to \mathbb{R}^n$ is a manifold with boundary denoted by $\Lambda^{62}_{k_1,k_2}$. The mapping

$$\mu_a \times \kappa_a : (\Lambda^{62}_{k_1,k_2}, \partial N^{n-2k}_{k_1,k_2}) \to (K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1), K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,1))$$

is well-defined. The 61-dimensional homology fundamental class $[\partial\Lambda]$ is integral, therefore the image of this fundamental class $(\mu_a \times \kappa_a)_*([\partial\Lambda^{62}_{k_1,k_2}]) \in H_{61}(K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1);\mathbb{Z}/2)$ is trivial for a dimensional reason.

Therefore the homology class

$$(\mu_a \times \kappa_a)_*([\Lambda^{62}_{k_1,k_2}, \partial\Lambda^{62}_{k_1,k_2}]) \in$$

$$H_{62}(K(\mathbb{Z}/4,1) \times K(\mathbb{Z}/2,1), K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,1);\mathbb{Z}/2)$$

is well-defined. Let us consider the (permanent) homology class

$$(\mu_a \times \kappa_a)^!_*([\bar{\Lambda}^{62}_{k_1,k_2}]) \in H_{62}(K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,1);\mathbb{Z}/2), \qquad (12)$$

defined from the relative class above by the transfer homomorphism.

To prove (10) it is sufficient to prove that the class (12) coincides with the characteristic class

$$p_{*,b} \circ \hat{\eta}_*([\hat{\Lambda}]) \in H_{62}(K(\mathbf{I}_b,1);\mathbb{Z}/2)$$

under the following isomorphism of the target group $\mathbf{I}_b = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. By this isomorphism the prescribed generators in $H^1(\mathbb{Z}/2 \oplus \mathbb{Z}/2;\mathbb{Z}/2)$ are identified with the cohomology classes $\tau_1, \tau_2 \in H^1(K(\mathbf{I}_b,1);\mathbb{Z}/2)$ (compare with Lemma 8 in [A1]). Theorem 2 is proved.

6 Kervaire Invariant One Problem

In this section we will prove the following theorem.

Main Theorem

There exists an integer l_0 such that for an arbitrary integer $l \ge l_0$, $n = 2^l - 2$ the Kervaire invariant given by the formula (1) is trivial.

Proof of Main Theorem

Take the integer k from the equation $n - 4k = 62$. Consider the diagram (5). By the Retraction Theorem [A2], Section 8 there exists an integer l_0 such that for an arbitrary integer $l \ge l_0$ an arbitrary element $[(f, \Xi, \kappa)]$ in the 2-component of the cobordism group $Imm^{sf}(\frac{3n+q}{4}, \frac{n-q}{4})$ admits a retraction of order 62. By Theorem 2 in the cobordism class $\delta[(f, \Xi, \kappa)]$ there exists a $\mathbf{D}_4$-framed immersion (g, Ψ, η) with an $\mathbf{I}_4$-structure.

Take the self-intersection manifold L^{62} of g and let $L_0^{10} \subset L^{62}$ be the submanifold dual to the cohomology class $\kappa_a^{28}\mu_a^*(\tau)^{12} \in H^{52}(L^{62}; \mathbb{Z}/2)$. By a straightforward calculation the restriction of the normal bundle of L^{62} to the submanifold $L_0^{10} \subset L^{62}$ is trivial and the normal bundle of L_0^{10} is the Whitney sum $12\kappa_a \oplus 12\mu_a$, where κ_a is the line $\mathbb{Z}/2$-bundle, μ_a is the plane $\mathbb{Z}/4$-bundle with the characteristic classes κ_a, $\mu_a^{ast}(\tau)$ described in the formula (8). By Lemma 6.1 (in the proof of this lemma we have to assume that the normal bundle of the manifold L_0^{10} is as above) and by Lemma 7.1 [A2] the characteristic class (8) is trivial. The Main Theorem is proved.

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