Geometric approach to stable homotopy groups of spheres II. The Kervaire invariant

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Abstract

A solution to the Kervaire invariant problem is presented. We introduce the concepts of abelian structure on skew-framed immersions, bicyclic structure on $\mathbb{Z}/2^{[3]}$–framed immersions, and quaternionic-cyclic structure on $\mathbb{Z}/2^{[4]}$–framed immersions. Using these concepts, we prove that for sufficiently large n, $n = 2^{\ell} - 2$, an arbitrary skew-framed immersion in Euclidean n-space $\mathbb{R}^n$ has zero Kervaire invariant. Additionally, for $\ell \geq 12$ (i.e., for $n \geq 4094$) an arbitrary skew-framed immersion in Euclidean n-space $\mathbb{R}^n$ has zero Kervaire invariant if this skew-framed immersion admits a compression of order 16.

1 Self-intersections of immersions and the Kervaire invariant

The Kervaire invariant one problem was for many years an open problem in algebraic topology. For algebraic approaches to the problem see Snaith [S], Barratt-Jones-Mahowald [B-J-M] and Cohen-Jones-Mahowald [C-J-M]. Recently, Hill, Hopkins, and Ravenel obtained a solution of this problem for all dimensions, except $n = 126$ (see [H-H-R]). We consider an alternative geometric approach. For a different geometric approach see Carter [C1], [C2].

The proof is based on a paper by P.J. Eccles [E1], identifying the Kervaire invariant with the number of multiple points of an immersion using the Kahn-Priddy map $MO(1) = P^{\infty} \to S^0$. Since all Steenrod squares are non-zero in the mapping cone of this map, an element $\alpha \in \pi^S_{2k-2}MO(1)$ is detected by Sq^k on a_{k-1} if and only if its image is detected by the secondary operation coming from Sq^kSq^k (k a power of 2 and $n+2 = 2k$). By W. Browder's result

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[B] this gives the geometric interpretation of the Kervaire invariant. Namely, Peter Eccles showed that the Kervaire invariant can be interpreted as the parity of the number of $(2k-2)$-fold points of a corresponding immersion. But with our approach we do not directly make use of this.

Using the equivalence of the James-Hopf invariant and the Steenrod-Hopf invariant this is also equivalent to the stable Hopf invariant $j_2(\alpha) \in \pi^S_{2k-2} D_2 MO(1)$ having Hurewicz image a^2_{k-1}. This element represents the double point manifold of an immersion corresponding to α. Forgetting the additional structure on the double point manifold maps $j_2(\alpha)$ to an element $\beta \in \pi^S_{2k-2} MO(2)$ with Hurewicz image a^2_{k-1}.

If α corresponds to an immersion f then β corresponds to an immersion g, as in the following paragraph. By dualizing to cohomology and using the Thom isomorphism one can check that $h(\beta) = a^2_{k-1}$ if and only if the characteristic number $\langle w_2^{k-1}, [N] \rangle = 1$ which is our interpretation of the Kervaire invariant. More detailed information can be found in [A-E]. As far as I know, there is no explicitly geometric proof of Eccles' Theorem on the Kervaire invariant.

Consider a smooth immersion $f : M^{n-1} \looparrowright \mathbb{R}^n$, $n = 2^\ell - 2$, $\ell > 1$ in general position and having codimension 1. We denote by $g : N^{n-2} \looparrowright \mathbb{R}^n$ the immersion of the manifold of self-intersections.

Definition 1. The *Kervaire invariant* of the immersion f is defined by the formula

$$\Theta^{sf}(f) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle, \tag{1}$$

where $\eta_N = w_2(N^{n-2})$ denotes the second normal Stiefel-Whitney class of the manifold N^{n-2}.

The Kervaire invariant is an invariant of the regular cobordism class of the immersion f. Moreover, the Kervaire invariant determines a homomorphism

$$\Theta^{sf} : Imm^{sf}(n-1, 1) \to \mathbb{Z}/2, \tag{2}$$

the cobordism group (and cobordism groups mentioned below) of immersions is defined in [A1]. The normal bundle ν_g of the immersion $g : N^{n-2} \looparrowright \mathbb{R}^n$ is a 2-dimensional bundle over N^{n-2}, which is naturally equipped with a $\mathbf{D}$-framing, where $\mathbf{D}$ denotes the dihedral group of order 8. The classifying map of this bundle (and its corresponding characteristic class) are denoted by $\eta_N : N^{n-2} \to K(\mathbf{D}, 1)$. The pair (g, η_N) represents an element of the cobordism group $Imm^{\mathbf{D}}(n-2,2)$. The passage from f to (g,η_N) gives rise to a well defined homomorphism

$$\delta^{\mathbf{D}} : Imm^{sf}(n-1,1) \to Imm^{\mathbf{D}}(n-2,2). \qquad (3)$$

The cobordism group $Imm^{sf}(n-k,k)$ generalizes the cobordism group $Imm^{sf}(n-1,1)$. The new group is defined as the cobordism group of triples (f,Ξ,κ_M), where $f : M^{n-k} \looparrowright \mathbb{R}^n$ is an immersion of a compact closed manifold; moreover there is given a morphism of bundles (a bundle map) $\Xi : \nu_f \cong k\kappa_M$, which is invertible, i.e., which is a fiberwise isomorphism, and which is called a skew-framing, where ν_f denotes the normal bundle of the immersion f and κ_M is a given line bundle over M^{n-k}, whose characteristic class is also denoted by $\kappa_M \in H^1(M^{n-k};\mathbb{Z}/2)$. The relation of cobordism on the set of triples is the standard one (see section 1 of [A1] for more details for the definition of the cobordism relation).

The group $Imm^{\mathbf{D}}(n-2,2)$ is generalized in the following way. We shall define cobordism groups $Imm^{\mathbf{D}}(n-2k,2k)$. Each element of the group $Imm^{\mathbf{D}}(n-2k,2k)$ is represented by a triple (g,Ψ,η_N), where $g : N^{n-2k} \looparrowright \mathbb{R}^n$ is an immersion, Ψ is a dihedral framing of codimension $2k$, i.e., a fixed isomorphism $\Xi : \nu_g \cong k\eta_N$, and where η_N is a 2-dimensional bundle over N^{n-2k} with structure group $\mathbf{D}$, τ is the universal 2-bundle over $K(\mathbf{D},1)$. The characteristic mapping of this bundle, and also the corresponding characteristic Euler class (respectively, the universal characteristic Euler class) will be denoted also by $\eta_N : N^{n-2k} \to K(\mathbf{D},1)$, $\eta_N \in H^2(N^{n-2k};\mathbb{Z}/2)$ (respectively, $\tau \in H^2(K(\mathbf{D},1);\mathbb{Z}/2)$).

The mapping η_N is also called characteristic for the bundle ν_g, since $\nu_g \cong k\eta_N$.

We define the Kervaire homomorphism (2) (see (1)) as the composition of the homomorphism (3) and a homomorphism

$$\Theta^{\mathbf{D}} : Imm^{\mathbf{D}}(n-2,2) \to \mathbb{Z}/2, \quad \Theta^{\mathbf{D}}(g,\Psi,\eta_N) = \langle \eta_N^{\frac{n-2}{2}}; [N^{n-2}] \rangle. \qquad (4)$$

The homomorphism (4) is called the Kervaire invariant of a **D**–framed immersion.

The Kervaire homomorphism can be defined in more general situations by means of a direct generalization of the homomorphisms (2) and (4):

$$\Theta^{sf}_k : Imm^{sf}(n-k,k) \to \mathbb{Z}/2, \qquad \Theta^{sf}_k := \Theta^{\mathbf{D}}_k \circ \delta^{\mathbf{D}}_k. \qquad (5)$$

$$\Theta^{\mathbf{D}}_k : Imm^{\mathbf{D}}(n-2k,2k) \to \mathbb{Z}/2, \qquad \Theta^{\mathbf{D}}_k[(g,\Psi,\eta_N)] = \langle \eta_N^{\frac{n-2k}{2}}; [N^{n-2k}] \rangle. \quad (6)$$

For $k = 1$ the new homomorphism (5) coincides with the homomorphism (4) already defined; moreover the following diagram, in which the homomorphisms J^{sf} and $J^{\mathbf{D}}$ were defined in the first part of the paper [A1] (Proposition 2), is commutative:

$$\begin{array}{ccccc} Imm^{sf}(n-1,1) & \stackrel{\delta^{\mathbf{D}}}{\longrightarrow} & Imm^{\mathbf{D}}(n-2,2) & \stackrel{\Theta^{\mathbf{D}}}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow J^{sf}_k & & \downarrow J^{\mathbf{D}}_k & & || \\ Imm^{sf}(n-k,k) & \stackrel{\delta^{\mathbf{D}}_k}{\longrightarrow} & Imm^{\mathbf{D}}(n-2k,2k) & \stackrel{\Theta^{\mathbf{D}}_k}{\longrightarrow} & \mathbb{Z}/2. \end{array} \qquad (7)$$

We shall need to generalize formula (6) for immersions with framing of a more general form. Denote by $\mathbb{Z}/2^{[s]}$ the wreath product of 2^{s-1} copies of the cyclic group $Z/2$. This group is a subgroup of the orthogonal group $O(2^{s-1})$, and can be defined in the following way:

Transformations in $Z/2[s]$ leave invariant the collection of $(s-1)$ sets Υ_s, Υ_{s-1}, $\dots$, Υ_2 of coordinate subspaces. The set of subspaces Υ_i, $2 \le i \le s$ consists of the 2^{i-1} coordinate subspaces $(Lin(\mathbf{e}_1, \dots, \mathbf{e}_{2^{s-i}}), \dots, Lin(\mathbf{e}_{2^{s-1}-2^{s-i}+1}, \dots, \mathbf{e}_{2^{s-1}}))$, spanned by the orthonormal basis vectors. The blocks of basis vectors are disjoint and all of the same size.

In particular, in this new notation the dihedral group $\mathbf{D}$ will be denoted by $\mathbb{Z}/2^{[2]}$. This group is defined as the subgroup of orthogonal transformations of the plane, carrying the set $\Upsilon_2 = \{Lin(e_1), Lin(e_2)\}$ of lines into itself. In this paper we shall make use of the groups $\mathbb{Z}/2^{[s]}$ for $2 \le s \le 5$. By definition, there is an inclusion $\mathbb{Z}/2^{[s]} \subset \mathbb{Z}/2 \wr \Sigma(2^{s-1})$, which coincides with the inclusion of a 2-Sylow subgroup of the symmetric group $\Sigma(2^s)$. E.g., the dihedral group $\mathbb{Z}/2^{[2]}$ is a 2-Sylow subgroup of $\Sigma(4)$.

Consider an immersion $g : N^{n-k2^{s-1}} \looparrowright \mathbb{R}^n$ in general position and of codimension $k2^{s-1}$. We say that the immersion g is $\mathbb{Z}/2^{[s]}$-framed (with multiplicity k), if an isomorphism $\Psi : \nu_g \cong k\eta_N$ is given between the normal bundle ν_g of the immersion g and the Whitney sum of k copies of a 2^{s-1}-dimensional bundle η_N with structure group $\mathbb{Z}/2^{[s]}$.

The bundle η_N is classified by a mapping $\eta_N : N^{n-k2^{s-1}} \to K(\mathbb{Z}/2^{[s]}, 1)$. (The corresponding characteristic class is also denoted by η_N.) The characteristic class of the universal 2^{s-1}-dimensional $\mathbb{Z}/2^{[s]}$-bundle over $K(\mathbb{Z}/2^{[s]}, 1)$ is denoted by $\tau_{[s]}$. Therefore, $\eta_N^*(\tau_{[s]}) = \eta_N$. The mapping η_N is also called a characteristic map for the bundle ν_g, since $\nu_g \cong k\eta_N$.

The set of all possible triples (g, Ψ, η_N), as described above, generate the cobordism group $Imm^{\mathbb{Z}/2^{[s]}}(n - k2^{s-1}, k2^{s-1})$. In some considerations we use an additional index in the notation, connected with the structure group. For example, a representative of the group $Imm^{\mathbb{Z}/2^{[2]}}(n-2k, 2k)$ will sometimes be denoted by $(g_{[2]}, \Psi_{[2]}, \eta_{N_{[2]}})$ and so on.

The manifold of self-intersections of an arbitrary $\mathbb{Z}/2^{[s]}$-framed immersion admits a natural $\mathbb{Z}/2^{[s+1]}$-framed immersion. Thus, the manifold of self-intersections yields a triple (h, Λ, ζ_L), where $h : L^{n-k2^s} \looparrowright \mathbb{R}^n$ is an immersion, $\Lambda : \nu_h \cong k\zeta_L$ and $\zeta_L : L^{n-k2^s} \to K(\mathbb{Z}/2^{[s+1]}, 1)$ is the classifying map of the 2^s-dimensional bundle ζ_L. We therefore obtain a homomorphism

$$\delta_k^{\mathbb{Z}/2^{[s+1]}} : Imm^{\mathbb{Z}/2^{[s]}}(n-k2^{s-1}, k2^{s-1}) \to Imm^{\mathbb{Z}/2^{[s+1]}}(n-k2^s, k2^s), \qquad (8)$$

$s \geq 1$, assigning to the normal cobordism class $[(g, \Psi, \eta_N)]$ the normal cobordism class $[(h, \Lambda, \zeta_L)]$.

In this formula the positive integer k indicates the multiplicity of the framing and, for $s = 1$, k is equal to the codimension of the immersion. In this case the index $\mathbb{Z}/2^{[s]}$ is replaced by the index sf.

A subgroup $i_{[s+1]} : \mathbb{Z}/2^{[s]} \subset \mathbb{Z}/2^{[s+1]}$ is defined as the subgroup of transformations of the subspace $Lin(\mathbf{e}_1, \dots, \mathbf{e}_{2^{s-1}}) = \mathbb{R}^{2^{s-1}} \subset \mathbb{R}^{2^s}$, generated by the first 2^{s-1} basis vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_{2^{s-1}}\}$, and acting as the identity on the remaining basis vectors.

A subgroup of index 2

$$\bar{i}_{[s+1]} : \mathbb{Z}/2^{[s]} \times \mathbb{Z}/2^{[s]} \subset \mathbb{Z}/2^{[s+1]} \qquad (9)$$

is defined as the subgroup of transformations leaving invariant each subspace in the set Υ_2.

The subgroup (9) induces a double covering $\pi_{[s+1]} : K(\mathbb{Z}/2^{[s]} \times \mathbb{Z}/2^{[s]}, 1) \to K(\mathbb{Z}/2^{[s+1]}, 1)$. The characteristic mapping $\zeta_L : L^{n-k2^s} \to K(\mathbb{Z}/2^{[s+1]}, 1)$ induces a double covering $\pi_{[s+1],L} : \bar{L}^{n-k2^s} \to L^{n-k2^s}$ from the covering $\pi_{[s+1]}$ over the classifying space. The double covering $\pi_{[s+1],L}$ can be defined geometrically, namely it coincides with the canonical double covering of the manifold L^{n-k2^s} of points of self-intersection of the $\mathbb{Z}/2^{[s]}$-framed immersion (g, Ψ, η_N) (see [A1], section 1, formula (3)).

The projection $p_{[s]} : \mathbb{Z}/2^{[s]} \times \mathbb{Z}/2^{[s]} \to \mathbb{Z}/2^{[s]}$ onto the first factor induces a mapping $p_{[s]} : K(\mathbb{Z}/2^{[s]} \times \mathbb{Z}/2^{[s]}, 1) \to K(\mathbb{Z}/2^{[s]}, 1)$.

For the manifold of self-intersections (h, Λ, ζ_L) of an arbitrary $\mathbb{Z}/2^{[s]}$–framed immersion (g, Ψ, η_N), we consider the double covering $\bar{\zeta}_L : \bar{L}_{[s]}^{n-k2^s} \to K(\mathbb{Z}/2^{[s]} \times \mathbb{Z}/2^{[s]}, 1)$ over the classifying mapping ζ_L, which is induced from the covering $\pi_{[s+1],L}$. This covering coincides with the canonical double covering over the classifying mapping $\zeta_L : L^{n-k2^s} \to K(\mathbb{Z}/2^{[s+1]}, 1)$, which is defined by geometric considerations. The characteristic class

$(p_{[s]} \circ \bar{\zeta}_L)^*(\tau_{[s]} \times \tau_{[s]}) \in H^{2^s}(\bar{L}^{n-k2^s}_{[s]};\mathbb{Z}/2)$, $\tau_{[s]} \in H^{2^{s-1}}(K(\mathbb{Z}/2^{[s]},1);\mathbb{Z}/2)$ coincides with the characteristic class $\pi_{[s+1],L} \circ \bar{\zeta}_L(\tau_{[s+1]})$.

We define the mapping $i_{tot} = i_{[s]} \circ \dots \circ i_{[3]}$ from the tower:

$$K(\mathbf{D},1) \stackrel{i_{[3]}}{\longrightarrow} K(\mathbb{Z}/2^{[3]},1) \stackrel{i_{[4]}}{\longrightarrow} \dots \stackrel{i_{[s]}}{\longrightarrow} K(\mathbb{Z}/2^{[s]},1) \stackrel{i_{[s+1]}}{\longrightarrow} K(\mathbb{Z}/2^{[s+1]},1). \quad (10)$$

There is defined a tower of canonical double coverings

$$\bar{L}^{n-k2^s}_{[2]} \stackrel{\pi_{[3]}}{\longrightarrow} \bar{L}^{n-k2^s}_{[3]} \stackrel{\pi_{[4]}}{\longrightarrow} \dots \stackrel{\pi_{[s]}}{\longrightarrow} \bar{L}^{n-k2^s}_{[s]} \stackrel{\pi_{[s+1]}}{\longrightarrow} L^{n-k2^s}. \quad (11)$$

This tower of coverings is endowed with characteristic mappings to the diagram (10). There is defined a sequence of characteristic classes

$$\bar{\zeta}^*_{[2],L}(\tau_{[2]}) \in H^2(\bar{L}^{n-k2^s}_{[2]};\mathbb{Z}/2), \dots, \zeta^*_{[s+1],L}(\tau_{[2]}) \in H^{2^s}(L^{n-k2^s};\mathbb{Z}/2). \quad (12)$$

Each element in this sequence is induced from the characteristic class of the corresponding universal space in (10). We denote by

$$\pi_{tot} = \pi_{[s]} \circ \dots \circ \pi_{[3]} : \bar{L}^{n-k2^s}_{[2]} \to L^{n-k2^s} \quad (13)$$

the covering defined as the composition of the coverings in the diagram (11). The tower of coverings (11) and the sequence of characteristic classes (12) are defined not only for a $\mathbb{Z}/2^{[s+1]}$– framed immersion which occur as the parametrization of a manifold of self-intersection of a suitable $\mathbb{Z}/2^{[s]}$–framed immersions, but also for an arbitrary $\mathbb{Z}/2^{[s+1]}$–framed immersion.

Definition 2. The Kervaire invariant $\Theta^k_{\mathbb{Z}/2^{[s+1]}}$ of an arbitrary $\mathbb{Z}/2^{[s+1]}$–framed immersion (h,Λ,ζ_L) is defined by the following formula:

$$\Theta^{\mathbb{Z}/2^{[s+1]}}_k(h,\Lambda,\zeta_L) = \langle (\bar{\zeta}^*_{[2],L}(\tau_{[2]}))^{\frac{n-k2^s}{2}}; [\bar{L}_{[2]}] \rangle, \quad (14)$$

where $[\bar{L}_{[2]}]$ denotes the fundamental class of the covering manifold in the sequence (11).

The invariant just constructed defines a homomorphism $\Theta^{\mathbb{Z}/2^{[s]}}_k$: $Imm^{\mathbb{Z}/2^{[s]}}(n, n-k2^{d-1}) \to \mathbb{Z}/2$, which is included in the following commutative diagram:

$$\begin{array}{ccc} Imm^{\mathbb{Z}/2^{[s]}}(n-k2^{s-1},k2^{s-1}) & \stackrel{\Theta^{\mathbb{Z}/2^{[s]}}_k}{\longrightarrow} & \mathbb{Z}/2 \\ \downarrow \delta^{\mathbb{Z}/2^{[d+1]}}_k & & \| \\ Imm^{\mathbb{Z}/2^{[s+1]}}(n-k2^s,k2^s) & \stackrel{\Theta^{\mathbb{Z}/2^{[s+1]}}_k}{\longrightarrow} & \mathbb{Z}/2. \end{array} \quad (15)$$

For an arbitrary $s \geq 2$ the diagram (15) is commutative. For $s = 2$ this is proved in [A1], Lemma 11, for $s > 2$ the proof is analogous.

In section 2 we define the concept of an $\mathbf{I}_{b\times\dot{b}}$–structure (abelian structure) on a skew-framed immersion representing an element of the cobordism group $Imm^{sf}(n-k,k)$. It is proved in Theorem 8 that under an appropriate dimensional restriction and modulo elements of odd order, an arbitrary cobordism class of skew-framed immersions admits an $\mathbf{I}_{b\times\dot{b}}$-structure. The hypothesis of this theorem presupposes the existence of a compression of characteristic classes of skew-framed immersions, see Definition 7. The compression theorem 27 will be proved somewhere else.

Let us define a positive integer σ (c.f. (1)[A1]) by the following formula:

$$\sigma = \left[\frac{\ell}{2}\right] - 1. \qquad (16)$$

In particular, for $\ell = 12$, $\sigma = 5$. In section 3 we formulate the concept of an $\mathbf{I}_a \times \dot{\mathbf{I}}_a$–structure (bicyclic structure) on a $\mathbb{Z}/2^{[3]}$–framed immersion. In Corollary 21 it is proved that, under the conditions of Theorem 8 (in this theorem the natural number n can be taken to be greater or equal to 254 or greater, if the Compression Theorem for $q = 16$ is satisfied; this condition corresponds to the following: the adjoint element in the stable homotopy group of spheres belongs to the image of the suspension of the order 17, i.e. is borne on the sphere of the dimension $S^{2^\ell-18}$) an arbitrary element of the group

$$\mathrm{Im}(\delta_k^{\mathbb{Z}/2^{[3]}} \circ \delta_k^{\mathbb{Z}/2^{[2]}} : Imm^{sf}(n-k,k) \to Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k)), \qquad (17)$$

$$k = \frac{n - m_\sigma}{16}, \quad m_\sigma = 2^\sigma - 2, \sigma \geq 5, n \geq 254 \qquad (18)$$

is represented by a $\mathbb{Z}/2^{[3]}$–framed immersion with bicyclic structure. For such an immersion the Kervaire invariant can be evaluated in terms of an $\mathbf{I}_a \times \dot{\mathbf{I}}_a$–characteristic class of the manifold of self-intersections.

In section 4 we formulate the concept of a $\mathbf{Q} \times \mathbb{Z}/4$–structure (quaternionic-cyclic structure or briefly quaternionic structure) on a $\mathbb{Z}/2^{[4]}$–framed immersion. In Corollary 26 it is proved that, under the conditions of Theorem 6, an arbitrary element of the group

$$\mathrm{Im}(\delta_k^{\mathbb{Z}/2^{[4]}} \circ \delta_k^{\mathbb{Z}/2^{[3]}} \circ \delta_k^{\mathbb{Z}/2^{[2]}} : Imm^{sf}(n-k,k) \to$$

$$Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k)) \qquad (19)$$

is represented by a $\mathbb{Z}/2^{[4]}$–framed immersion with quaternionic-cyclic structure. For such an immersion the Kervaire invariant can be evaluated in terms of a $\mathbf{Q} \times \mathbb{Z}/4$–characteristic class of the manifold of self-intersections.

The following two diagrams explain the plan of the proof. In the first diagram the structure groups $\mathbb{Z}/2^{[s]}$, $2 \le s \le 5$ of parameterizations of self-intersections of immersions with structure group $\mathbb{Z}/2^{[s-1]}$ are given, as well as the names of the structures on immersions corresponding to each subgroup:

$$\begin{array}{cccc}
\mathbf{I}_{b\times\dot{b}} & \subset & \mathbb{Z}/2^{[2]} & \frac{\textit{Abelian}}{\textit{structure}} \\
\downarrow & & i_{[3]} \downarrow & \\
\mathbf{E}_{b\times\dot{b}} & \subset & \mathbb{Z}/2^{[3]} & \frac{\textit{cyclic}-\textit{Abelian}}{\textit{structure}} \\
\downarrow & & i_{[4]} \downarrow & \\
\mathbf{J}_a \times \dot{\mathbf{J}}_a & \subset & \mathbb{Z}/2^{[4]} & \frac{\textit{bicyclic}}{\textit{structure}} \\
\downarrow & & i_{[5]} \downarrow & \\
\mathbf{Q} \times \mathbb{Z}/4 & \subset & \mathbb{Z}/2^{[5]} & \frac{\textit{quaternionic}-\textit{cyclic}}{\textit{structure}}
\end{array} \qquad (20)$$

In the following diagram the natural homomorphisms of cobordism groups of immersions that will be used are shown, and the Kervaire invariants on each of these groups are indicated:

$$\begin{array}{ccccc}
Imm^{\mathbb{Z}/2^{[2]}}(n-2,2) & \stackrel{J_k^{\mathbb{Z}/2^{[2]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k) & \stackrel{\Theta_k^{\mathbb{Z}/2^{[2]}}}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow \delta^{\mathbb{Z}/2^{[3]}} & & \downarrow \delta_k^{\mathbb{Z}/2^{[3]}} & & \| \\
Imm^{\mathbb{Z}/2^{[3]}}(n-4,4) & \stackrel{J_k^{\mathbb{Z}/2^{[3]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k) & \stackrel{\Theta_k^{\mathbb{Z}/2^{[3]}}}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow \delta^{\mathbb{Z}/2^{[4]}} & & \downarrow \delta_k^{\mathbb{Z}/2^{[4]}} & & \| \\
Imm^{\mathbb{Z}/2^{[4]}}(n-8,8) & \stackrel{J_k^{\mathbb{Z}/2^{[4]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k) & \stackrel{\Theta^k_{\mathbb{Z}/2^{[4]}}}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow \delta^{\mathbb{Z}/2^{[5]}} & & \downarrow \delta_k^{\mathbb{Z}/2^{[5]}} & & \| \\
Imm^{\mathbb{Z}/2^{[5]}}(n-16,16) & \stackrel{J_k^{\mathbb{Z}/2^{[5]}}}{\longrightarrow} & Imm^{\mathbb{Z}/2^{[5]}}(n-16k,16k) & \stackrel{\Theta_k^{\mathbb{Z}/2^{[5]}}}{\longrightarrow} & \mathbb{Z}/2
\end{array} \qquad (21)$$

In view of the commutativity of this diagram, it suffices to show that the Kervaire invariant defined in the last row of the diagram is zero. This is proved with the help of the concept of biquaternionic structure.

The structure group of a $\mathbb{Z}/2^{[5]}$–framed immersion that parameterizes the manifold of self-intersection points of a $\mathbb{Z}/2^{[4]}$–framed immersion contains the subgroup $\mathbf{Q} \times \mathbb{Z}/4$, the direct product of the quaternion group $\mathbf{Q}$ of order 8 and the cyclic group $\mathbb{Z}/4$ of order 4.

The cohomology group $H^{4i}(\mathbf{Q};\mathbb{Z})$, for $i \ge 1$, (see [At], section 13) contains a characteristic class of order 8. In the part 1 (see [A1]) the Hopf invariant of a corresponded skew-framed immersion is estimated using this characteristic class. The Main Theorem (see section 6) is proved analogously.

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2 Geometric control of the manifold of self-intersections of a skew-framed immersion

In this and the following sections, we shall make use of the cobordism groups $Imm^{sf}(n-k,k)$, $Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k)$. It is well known that if the first argument in parentheses, denoting the dimension of the immersed manifold, is strictly positive, then the indicated group is a finite group.

The dihedral group $\mathbb{Z}/2^{[2]}$ is defined by its presentation:

$$\{a,b \,|\, a^4=b^2=e, [a,b]=a^2\}.$$

This group is a subgroup of the orthogonal group $O(2)$, namely, the group of orthogonal transformations of the standard Euclidean plane $Lin(\mathbf{e}_1,\mathbf{e}_2)$, preserving the unordered pair of lines generated by a basis of orthogonal unit vectors $\{\mathbf{e}_1,\mathbf{e}_2\}$. The element a is represented by a rotation of the plane through an angle $\frac{\pi}{2}$. The element b is represented by a reflection of the plane with respect to the line $l_1 = Lin(\mathbf{e}_1+\mathbf{e}_2)$, generated by the vector $\mathbf{e}_1+\mathbf{e}_2$.

Consider the subgroup $\mathbf{I}_{b\times\dot{b}} = \mathbf{I}_b \times \dot{\mathbf{I}}_b \subset \mathbb{Z}/2^{[2]}$ of the dihedral group, generated by the elements $\{b, ba^2\}$. Notice that this is an elementary 2-group of rank 2. This is the subgroup of $O(2)$ consisting of transformations preserving individually each of the lines l_1, l_2 in the directions of the vectors $\mathbf{f}_1=\mathbf{e}_1+\mathbf{e}_2$, $\mathbf{f}_2=\mathbf{e}_1-\mathbf{e}_2$ respectively. The cohomology group $H^1(K(\mathbf{I}_b\times\dot{\mathbf{I}}_b,1);\mathbb{Z}/2)$ is also an elementary 2-group with two generators. We now describe these generators.

Let us define the cohomology classes

$$\kappa_b \in H^1(K(\mathbf{I}_b\times\dot{\mathbf{I}}_b,1);\mathbb{Z}/2)), \quad \kappa_{\dot{b}} \in H^1(K(\mathbf{I}_b\times\dot{\mathbf{I}}_b,1);\mathbb{Z}/2)). \qquad (22)$$

We denote by $p_b : \mathbf{I}_{b\times\dot{b}} \to \mathbf{I}_b$ the homomorphism, whose kernel consists of the reflection with respect to the bisector of the second coordinate angle and the identity. Define $\kappa_b = p_b^*(t_b)$, where $e \ne t_b \in H^1(K(\mathbf{I}_b,1);\mathbb{Z}/2) \cong \mathbb{Z}/2$. Denote by $p_{\dot{b}} : \mathbf{I}_b \times \dot{\mathbf{I}}_b \to \dot{\mathbf{I}}_b$ the homomorphism, whose kernel consists of the reflection with respect to the bisector of the first coordinate angle and the identity, or equivalently whose kernel consists of the composition of the central symmetry and the symmetry with respect to the second coordinate angle and the identity. Define $\kappa_{\dot{b}} = p_{\dot{b}}^*(t_{\dot{b}})$, where $e \ne t_{\dot{b}} \in H^1(K(\dot{\mathbf{I}}_b,1);\mathbb{Z}/2) \cong \mathbb{Z}/2$.

We next define a group $\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}$ and an epimorphism $\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}/2^{[2]}$. Consider the automorphism

$$\chi^{[2]} : \mathbf{I}_{b\times\dot{b}} \to \mathbf{I}_{b\times\dot{b}} \qquad (23)$$

determined by the external conjugation of the subgroup $\mathbf{I}_{b\times\dot{b}} \subset \mathbb{Z}/2^{[2]}$ by the element $ab \in \mathbb{Z}/2^{[2]}$, this element correspond to the reflection with respect to the line $Lin(\mathbf{e}_1)$..

Define the automorphism (the notation is similar)

$$\chi^{[2]} : \mathbb{Z}/2^{[2]} \to \mathbb{Z}/2^{[2]}, \qquad (24)$$

by the reflection in the line $Lin(\mathbf{e}_1)$. It is easy to see that the inclusion $\mathbf{I}_{b\times\dot{b}} \subset \mathbb{Z}/2^{[2]}$ commutes with (23) and (24) such that the corresponding diagram is commutative.

Define the group

$$\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z} \qquad (25)$$

as the factorgroup of the group $\mathbf{I}_{b\times\dot{b}} * \mathbb{Z}$ (the free product of the groups $\mathbf{I}_{b\times\dot{b}}$ and $\mathbb{Z}$) by the relation $zxz^{-1} = \chi^{[2]}(x)$, where $z \in \mathbb{Z}$ is the standard generator, $x \in \mathbf{I}_{b\times\dot{b}}$ is an arbitrary element. The group (25) is a particular example of a semi-direct product of groups $A \rtimes_\phi B$, $A = \mathbf{I}_{b\times\dot{b}}$, $B = \mathbb{Z}$, where $\phi : B \to \mathrm{Aut}(A)$ is a homomorphism and the set $A \times B$ is equipped with the operation $(a_1,b_1) * (a_2,b_2) \mapsto (a_1\phi_{b_1}(b_2), b_1b_2)$.

The classifying space $K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1)$ is a skew-product of the standard circle S^1 and the space $K(\mathbf{I}_{b\times\dot{b}},1)$, the map $K(\mathbf{I}_{b\times\dot{b}},1) \to K(\mathbf{I}_{b\times\dot{b}},1)$, of the shift in the cyclic covering, associated with the cyclic covering over $K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1)$ is induced by the automorphism $\chi^{[2]}$. Denote the standard fibration by

$$p_{b\times\dot{b}} : K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1) \to S^1. \qquad (26)$$

Take a marked point $pt_{S^1} \in S^1$. and define the subspace

$$K(\mathbf{I}_{b\times\dot{b}}, 1) \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \tag{27}$$

as the inverse image of the marked point pt_{S^1} by the mapping (26).

Let us consider the homology groups $H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2)$, $H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z})$ (below the coefficients $\mathbb{Z}/2$ are omitted).

The standard basis of the group $H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z})$ is sufficiently complicated and we do not need its description. The basis of the group $H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z})$ is described by the Künneth formula:

$$0 \to \bigoplus_{i_1+i_2=i} H_{i_1}(K(\mathbf{I}_b, 1); \mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{I}}_b, 1); \mathbb{Z}) \longrightarrow H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}) \tag{28}$$

$$\longrightarrow \bigoplus_{i_1+i_2=i-1} Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{I}_b, 1); \mathbb{Z}), H_{i_2}(K(\dot{\mathbf{I}}_b, 1); \mathbb{Z}) \to 0.$$

The standard basis of the group $H_i(K(\mathbf{I}_{b\times\dot{b}}, 1))$ is following:

$$x \otimes y/(x \otimes y) + (y \otimes x),$$

where $x \in H_j(K(\mathbf{I}_b, 1))$, $y \in H_{i-j}(K(\dot{\mathbf{I}}_b, 1))$.

In particular, in the case of odd i the group $H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z})$ contains the fundamental classes of the following submanifolds $\mathbb{RP}^i \times pt \subset \mathbb{RP}^i \times \mathbb{RP}^i \subset K(\mathbf{I}_b, 1) \times K(\dot{\mathbf{I}}_b, 1)$, $pt \times \mathbb{RP}^i \subset \mathbb{RP}^i \times \mathbb{RP}^i \subset K(\mathbf{I}_b, 1) \times K(\dot{\mathbf{I}}_b, 1)$. Denote the corresponding elements by

$$t_{b,i} \in H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}), t_{\dot{b},i} \in H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}). \tag{29}$$

Let us define the analogous homology groups with local coefficients system:

$$H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]), \tag{30}$$

$$H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]). \tag{31}$$

The following epimorphism

$$p_{b\times\dot{b}} : \mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z}, \tag{32}$$

is well defined by the formula $x * y \mapsto y$, $x \in \mathbf{I}_{b\times\dot{b}}$, $y \in \mathbb{Z}$.. The following epimorphism

$$\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \qquad (33)$$

is defined by the formula $p_{b\times\dot{b}}$ (mod 2).

Let us define the group (31). Consider the group ring $\mathbb{Z}[\mathbb{Z}/2] = \{a + bt\}$, $a, b \in \mathbb{Z}$, $t \in \mathbb{Z}/2$. The generator $t \in \mathbb{Z}[\mathbb{Z}/2]$ is represented by the involution $\chi^{[2]} : K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$, the restriction of this involution on the subspace $K(\mathbf{I}_{b\times\dot{b}}, 1) \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$ is the reflection, which is induced by the permutation automorphism $\mathbf{I}_b \times \dot{\mathbf{I}}_b \to \dot{\mathbf{I}}_{\dot{b}} \times \mathbf{I}_b$. Consider the following local system of the coefficients $\rho_t : \mathbb{Z}/2[\mathbb{Z}/2] \to Aut(K(\mathbf{I}_{b\times\dot{b}}, 1) \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$, which identifies the chain $(a + bt)\sigma$, with a support on a simplex $\sigma \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$ with the chain $(at + b)\chi^{[2]}(\sigma)$. The group (31) is well defined. The group (30) is defined analogously.

The description of the groups (30), (31) are sufficiently complicated and we will not use this description. Let us define a subgroup

$$D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2]) \subset H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]) \qquad (34)$$

by the formula: $D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2]) = \mathrm{Im}(H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]))$, where the homomorphism $H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2])$ is induced by the inclusion of the subgroup (73). The natural epimorphism

$$H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2]) \qquad (35)$$

is well defined.

The following natural homomorphism $H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z})) \otimes \mathbb{Z}[\mathbb{Z}/2] \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2])$, is an isomorphism by the universal coefficients formula. This calculation do not use the structure of $\mathbb{Z}[\mathbb{Z}/2]$–module, and use the additive isomorphism $\mathbb{Z}[\mathbb{Z}/2] \equiv \mathbb{Z} \oplus \mathbb{Z}$ and additivity of the functor $Tor^{\mathbb{Z}}$. Analogously, $H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}/2)) \otimes \mathbb{Z}/2[\mathbb{Z}/2] \equiv H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}/2[\mathbb{Z}/2])$.

The subgroup (34) is generated by the following elements: $X + Yt$, $X, Y \in H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z})$. The following equivalence relation determines the equivalence of the two representatives: $X \equiv \chi^{[2]}_*(X)t$, where the automorphism

$$\chi^{[2]}_* : H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}) \qquad (36)$$

is induced by the automorphism (23). The automorphism (23) induces also the automorphism

$$\chi_*^{[2]} : H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]). \qquad (37)$$

The subgroup

$$D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]) \subset H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \qquad (38)$$

is defined analogously to (34). The description of this subgroup is more simple, because this subgroup is generated by the elements $X + tY$, $X = x \otimes y, Y = x' \otimes y'$, where $\chi_*^{[2]}(x \otimes y) = y \otimes x$.

Define the homomorphism

$$\Delta^{[2]} : H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}) \qquad (39)$$

by the formula $\Delta^{[2]}(X+Yt) = X + \chi_*^{[2]}(Y)$. Let us prove that the homomorphism (39) is naturally factorized to the homomorphism

$$\Delta^{[2]} : D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}), \qquad (40)$$

which we denote the same. This follows from the following observation: the kernel of the homomorphism (35) is generated by elements $X - \chi_*^{[2]}(X)t$.

Define the analogously homomorphisms with $\mathbb{Z}/2$–coefficients:

$$\Delta^{[2]} : D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1)). \qquad (41)$$

Let us consider the following composition of the homomorphisms:

$$H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to \qquad (42)$$

$$D_i(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}}, 1); \mathbb{Z}),$$

where the left homomorphism is the natural inclusion, the middle is the homomorphism (35), the right homomorphism in this composition is (40). It is easy to check that the composition coincides with the identity homomorphism.

Let us define the forgetful homomorphism

$$forg_* : H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}). \qquad (43)$$

This homomorphism is induced by the forgetful mapping of the local coefficient system, analogously to the homomorphism $\Delta^{[2]}$.

Let us consider the homomorphism (32) and let us consider the mapping (26), which is associated with this homomorphism. Assume $\theta \in H^1(S^1;\mathbb{Z}[\mathbb{Z}/2])$ is the generator. Take the element $p^*_{b\times\dot b}(\theta) \in H^1(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$ and define the obstruction (44) by the formula $o(x) = x \cap p^*_{b\times\dot b}(\theta)$. It is easy to check, that $\chi^{[2]}_*(o(x)) = o(x)$, where the automorphism $\chi^{[2]}_*$ is defined by the formula (37).

Consider the subgroup (34) and the epimorphism (35). Take an element $x \in H_i(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$. Let us describe the total obstruction

$$o(x) \in D_{i-1}(\mathbf{I}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2]) \qquad (44)$$

for the following inclusion: $x \in D_i(\mathbf{I}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2]) \subset H_i(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$. Let us consider the generator $\theta \in H^1(S^1;\mathbb{Z}[\mathbb{Z}/2])$. Take the element $p^*_{b\times\dot b}(\theta) \in H^1(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$ and define the obstruction (44) by the formula $o(x) = x \cap p^*_{b\times\dot b}(\theta)$.

From the definition of the obstruction (44) is obvious that the value of this obstruction one can calculate as following: apply to a singular cycle with local coefficient, which represents a prescribed homology class, the homomorphism (43), and then intersects the singular cycle of the space $K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1)$ with the subspace (27).

Evidently, $\chi^{[2]}_*(o(x)) = o(x)$, where the automorphism $\chi^{[2]}_*$ is given by the formula (37).

In the case $i = 2s$ basis elements $y \in D_{i-1}(\mathbf{I}_{b\times\dot b};\mathbb{Z}[\mathbb{Z}/2])$, which satisfies the equation $\chi^{[2]}_*(y) = y$, are the following:

1. $y = r$, $r = t_{b,2s-1} + t_{\dot b,2s-1}$, where $t_{d,2s-1}, t_{\dot b,2s-1} \in H_{2s-1}(K(\mathbf{I}_{b\times\dot b},1))$ are given by the formula (29).

2. $y = z(i_1,i_2)$, where $z(i_1,i_2) = tor(r_{b,i_1}, r_{\dot b,i_2}) + tor(r_{b,i_2}, r_{\dot b,i_1})$, $i_1, i_2 \equiv 1 \pmod 2$, $i_1 + i_2 = 2s-2$, $tor(r_{b,i_1}, r_{\dot b,i_2}) \in Tor^{\mathbb{Z}}(H_{i_1}(K(\mathbf{I}_b,1);\mathbb{Z}), H_{i_2}(K(\dot{\mathbf{I}}_b,1);\mathbb{Z}))$. The elements $\{z(i_1,i_2)\}$ are in the kernel of the homomorphism

$$D_{i-1}(\mathbf{I}_{b\times\dot b};\mathbb{Z}[\mathbb{Z}/2]) \to D_{i-1}(\mathbf{I}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (45)$$

given by the homomorphism $\mathbb{Z} \to \mathbb{Z}/2$ of coefficients.

Elements R, $Z(i_1,i_2) \in H_{2s}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$, which satisfies the relation $o(R) = r$, $o(R) = r$, $o(Z(i_1,i_2)) = z(i_1,i_2)$ are defined the following way. Each element are given by cycle, which is represented by the the product of the corresponding $2s-1$–cycle $f : C_{2s-1} \to K(\mathbf{I}_{b\times\dot b},1)$ with the circle. The mapping of the cycle into $K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1)$ is determined by the composition of the mapping $f \times id : C_{2s-1} \times S^1 \to K(\mathbf{I}_{b\times\dot b},1) \times S^1$ with the standard 2-sheeted covering $K(\mathbf{I}_{b\times\dot b},1) \times S^1 \to K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1)$.

By means of the obstruction (44) we shall prove the following lemma.

Lemma 3. *The group* $H_{2s}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$ *is isomorphic to the direct sum of the subgroup* $D_{2s}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}[\mathbb{Z}/2])$ *and the subgroup generated by the elements* R, $\{Z(i_1,i_2)\}$. *The elements in the subgroup* $\bigoplus_{i_1+i_2=2s}H_{i_1}(K(\mathbf{I}_b,1);\mathbb{Z})\otimes H_{i_2}(K(\dot{\mathbf{I}}_b,1);\mathbb{Z})\subset D_{2s}(\mathbf{I}_b\times\dot{\mathbf{I}}_b;\mathbb{Z}[\mathbb{Z}/2])$, *and the elements* R *generate the image* $\mathrm{Im}(A)$ *of the homomorphism*

$$A: H_{2s}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])\to \tag{46}$$

$$H_{2s}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]),$$

which is induced by the reduction of the local coefficients system modulo 2.

Proof of Lemma 3

Let $x \in H_{2s}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$ is not in the subgroup $D_{2s}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}[\mathbb{Z}/2])$. Then $o(x)\neq 0$ (see (44) for $i=2s$. Therefore x as an element in a residue class with respect to the considered subgroup is expressed by means of the elements R, $Z(i_1,i_2)$. The subgroup of all values of the obstruction (44) is a direct factor in $H_{2s}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$, because this obstructions are realized by a linear combination elements R, $\{Z(i_1,i_2)\}$. The elements $Z(i_1,i_2)$ belong to $\mathrm{Ker}(A)$ and, therefore these elements are not a generators of $\mathrm{Im}(A)$. Lemma 3 is proved.

Define the inclusion $i_{\mathbf{I}_d,\mathbf{I}_{b\times\dot{b}}}:\mathbf{I}_d\subset\mathbf{I}_b\times\dot{\mathbf{I}}_b=\mathbf{I}_{b\times\dot{b}}$, as the diagonal inclusion. The subgroup coincides with the kernel of the homomorphism

$$\omega^{[2]}:\mathbf{I}_b\times\dot{\mathbf{I}}_b\to\mathbb{Z}/2, \tag{47}$$

this homomorphism is given by the formula $(x\times y)\mapsto xy$.

Define the epimorphism

$$\Phi^{[2]}:\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}\to\mathbb{Z}/2^{[2]} \tag{48}$$

by the formula: $\Phi^{[2]}(z)=ab$, $z\in\mathbb{Z}$ is the generator (the element ab corresponds to the reflection of the first vector of the standard base with respect to the standard representation $\mathbb{Z}^{[2]}\subset O(2)$), the restriction $\Phi^{[2]}|_{\mathbf{I}_b\times\dot{\mathbf{I}}_b\times\{0\}}$: $\mathbf{I}_b\times\dot{\mathbf{I}}_b\subset\mathbb{Z}/2^{[2]}$ is the standard inclusion. Therefore $\Phi^{[2]}|_{\mathbf{I}_{b\times\dot{b}}\times\{1\}}:\mathbf{I}_b\times\dot{\mathbf{I}}_b=$

$\mathbf{I}_{b\times\dot{b}} \subset \mathbb{Z}/2^{[2]}$ is the conjugated inclusion by the exterior automorphism in the subgroup $\mathbf{I}_b \times \dot{\mathbf{I}}_b = \mathbf{I}_{b\times\dot{b}} \subset \mathbb{Z}/2^{[2]}$.

Define

$$(\Phi^{[2]})^*(\tau_{[2]}) = \tau_{b\times\dot{b}},$$

where $\tau_{b\times\dot{b}} \in H^2(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b \int_{\chi^{[2]}} \mathbb{Z}), 1))$, $\tau_{[2]} \in H^2(K(\mathbb{Z}^{[2]}, 1)$.

Definition 4. Let an element $y \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k, 2k)$ be represented by a $\mathbb{Z}/2^{[2]}$–framed immersion (g, Ψ, η_N), $g : N^{n-2k}_{b\times\dot{b}} \looparrowright \mathbb{R}^n$. We say that the $\mathbb{Z}/2^{[2]}$-framed immersion (g, Ψ, η_N) is an $\mathbf{I}_{b\times\dot{b}}$–immersion (abelian immersion) if the following two conditions are satisfied.

1. The structure mapping $\eta_N : N^{n-2k}_{b\times\dot{b}} \to K(\mathbb{Z}/2^{[2]}, 1)$ is represented as the composition of a mapping

$$\eta_{b\times\dot{b},N} : N^{n-2k}_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \qquad (49)$$

and the mapping $\Phi^{[2]} : K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \to K(\mathbb{Z}/2^{[2]}, 1)$.

2. Consider the submanifold

$$N^{n-16k}_{\eta^{7k}} \subset N^{n-2k}_{b\times\dot{b}}, \qquad (50)$$

which represents the Euler class $[\eta_N^{7k}]^{op} \in H_{n-16k}(N^{n-2k}_{b\times\dot{b}}; \mathbb{Z}/2)$ of the bundle $7k\eta_N$ (the considered class is Poincaré dual to the cohomology class $\eta_N^{7k} \in H^{7k}(N^{n-2k}_{b\times\dot{b}}; \mathbb{Z}/2))$. It is required that the restriction of the mapping (49) on the submanifold (50) is given by the composition of the mapping

$$\eta_{Ab} : N^{n-16k}_{\eta^{7k}} \to K(\mathbf{I}_{b\times\dot{b}}, 1) \qquad (51)$$

with the standard inclusion $K(\mathbf{I}_{b\times\dot{b}}, 1) \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$.

Definition 5. Let a skew-framed immersion (f, Ξ, κ), $f : M^{n-k} \looparrowright \mathbb{R}^n$ represents an element $x \in Imm^{sf}(n-k, k)$, where $n > 16k$. Let the $\mathbb{Z}/2^{[2]}$–framed immersion (g, Ψ, η_N), $g : N^{n-2k} \looparrowright \mathbb{R}^n$ be the immersion of the manifold of self-intersections of the immersion f, so g represents the element $y = \delta^k_{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k, 2k)$. We say that the skew-framed immersion (f, Ξ, κ_M) admits an abelian structure ($\mathbf{I}_{b\times\dot{b}}$–structure) if the self-intersection manifold N^{n-2k} of the immersion f is decomposed into two components (possibly, non-connected):

$$N^{n-2k} = N^{n-2k}_{b\times\dot{b}} \cup N^{n-2k}_{[2]}. \qquad (52)$$

Each component is equipped by a $\mathbb{Z}/2^{[2]}$–framed immersion into $\mathbb{R}^n$. The following two conditions are satisfies:

1. For the characteristic mapping $\eta_N|_{N^{n-2k}_{b\times\dot{b}}}$ of the $\mathbb{Z}/2^{[2]}$–framing over first component in the formula (52) is equipped by a reduction into the subspace $K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1) \subset K(\mathbb{Z}/2^{[2]},1)$, defined by the mapping (49). For the mapping (49) the property 2 of Definition 4 is satisfied. Namely, the restriction of the considered mapping on a submanifold $N^{n-16k}_{\eta^{7k}} \subset N^{n-2k}_{b\times\dot{b}}$, which is determined by the formula (50), allows an additional reduction into the subspace $K(\mathbf{I}_{b\times\dot{b}},1) \subset K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1)$, given by the formula (51).

2. The $\mathbb{Z}/2^{[2]}$–framed immersion, which is defined by the restriction of the immersion g on the second component in the formula (52), has the trivial Kervaire invariant: the characteristic class (6) for the considered component is trivial.

Example

Assume a skew-framed immersion (f,Ξ,κ_M), $f: M^{n-k} \looparrowright \mathbb{R}^n$, represents an element $x \in Imm^{sf}(n-k,k)$, $n > 16k$. Assume that a $\mathbb{Z}/2^{[2]}$–framed immersion (g,Ψ,η_N), $g: N^{n-2k} \looparrowright \mathbb{R}^n$, which is the immersion of self-intersection manifold of f, is a $\mathbf{I}_{b\times\dot{b}}$–immersion in the sense of Definition 4. Then the skew-framed immersion (f,Ξ,κ_M) admits an abelian structure, for which the second component in the formula (52) is empty.

The justification of the example

Let us define the map, determined by an abelian structure by the formula (49). The Conditions 1 and 2 in Definition 4 implies the conditions 1 and 2 in Definition 5. The abelian structure is well defined.

The fundamental class with local coefficients of a framed immersion

Consider a $\mathbb{Z}/2^{[2]}$–framed immersion (g,Ψ,η_N), $g: N^{n-2k}_{b\times\dot{b}} \looparrowright \mathbb{R}^n$, and assume that a mapping (49), which determines a reduction of the characteristic mapping, is well-defined. Assume that the manifold $N^{n-2k}_{b\times\dot{b}}$ is connected. Assume that a marked point $pt \in N^{n-2k-1}_{pt}$ is fixed. Assume that the image of the marked point by the mapping (49) is a point in the subspace (27). Let us prove that the image of the fundamental class $[N^{n-2k}_{b\times\dot b};pt]$ by the mapping (49) determines an element

$$\eta_{b\times\dot b,*}([N^{n-2k}_{b\times\dot b};pt]) \in H_{n-2k}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2]). \qquad (53)$$

Denote by

$$N^{n-2k-1}_{pt} \subset N^{n-2k}_{b\times\dot b} \qquad (54)$$

the submanifold, which is defined as the regular preimage of the subspace (73). The restriction of the reduction mapping on the submanifold (54) determines a $\mathbf{I}_{b\times\dot b}$–reduction of the restriction of the characteristic mapping η_N to this manifold.

Consider the skeleton of the space $K(\mathbf{I}_{b\times\dot b},1)$, which is realized as a $O(2)/\mathbf{I}_{b\times\dot b}$-bundle over the Grassman manifold $Gr_O(2,n)$ of 2-planes in n–dimensional space. Denote this skeleton by

$$KK(\mathbf{I}_{b\times\dot b},1) \subset K(\mathbf{I}_{b\times\dot b},1). \qquad (55)$$

The following free involution

$$\chi^{[2]}: KK(\mathbf{I}_{b\times\dot b},1) \to KK(\mathbf{I}_{b\times\dot b},1), \qquad (56)$$

acts on the space (55), this involution corresponds to the automorphism (23) (and denote the same). Let us consider the skeleton of the pair of spaces (73) as the cylinder of the involution (55), and denote this cylinder by

$$KK(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1) \subset K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1). \qquad (57)$$

The involution (56) induces the involution

$$\chi^{[2]}: KK(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1) \to KK(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1), \qquad (58)$$

which is extended to the involution on the hole space $K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1)$.

The universal $\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z}$–bundle over the skeleton (57) is well-defined, denote this bundle by $\tau_{b\times\dot b,\int}$. Denote the restriction of the bundle $\tau_{b\times\dot b,\int}$ on the subspace (55) by $\tau_{b\times\dot b}$.

The mapping (49) determines a $\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z}$–reduction of the characteristic mapping of the framing Ψ and the framing $\Psi_{b\times\dot b}: \nu_g \equiv k\eta^*_{b\times\dot b}(\tau_{b\times\dot b,\int})$ is well-defined.

Let us visualize this reduction by the following way. The normal bundle ν_g of the immersion g is decomposed into the Whitney sum of the k copies of a 2-dimensional bundle. Let us denote the first term of this decomposition by $\nu_{b\times\dot{b},\int} \subset \nu_g$. The bundle $\nu_{b\times\dot{b},\int}$ is a twisted Whitney sum of the two line bundles $\kappa_b \tilde{\oplus} \kappa_{\dot{b}}$. These two line bundles κ_b, $\kappa_{\dot{b}}$ are permuted by the parallel transformation over a path $l \subset N^{n-2k}_{b\times\dot{b}}$, if the projection of the path by the mapping (26) represents a generator in $H_1(S^1)$.

Let us consider an arbitrary cell α of a regular cell decomposition of the manifold $N^{n-2k}_{b\times\dot{b}}$. Assume that a path ϕ_α, which is joins the center of the cell α with the marked point $pt \in N^{n-2k}_{b\times\dot{b}}$ is given. The restriction of the bundle $\nu_{b\times\dot{b},\int}$ to the cell α is classified by the mapping $\eta_{b\times\dot{b}}(\alpha,\phi_\alpha) : \alpha \to KK(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$. A change of ϕ_α in a residue class of paths with the fixed ends in the group $H_1(S^1)$ determines another mapping $\eta_{b\times\dot{b}}(\alpha,\phi_\alpha)$, which is defined by means of the composition of the classified mapping mapping with the involution (58). In the cell complex of the space (57) with $\mathbb{Z}[\mathbb{Z}/2]$–local coefficients a change of the path, which is attached to the cell α, corresponds to the change of the base of the corresponding generator by the multiplication on the element t and the change of the parametrization mapping for the cell by the involution (58). Therefore, the element (53) is well-defined.

Let us prove that the image of the fundamental class of the target manifold by the mapping (51) determines the element

$$\eta^{loc}_{Ab,*}([\eta^{7k}_N]^{op}) \in D_{m_\sigma}(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]), \tag{59}$$

which is mapped into the element

$$\eta_{Ab,*}([\eta^{7k}_N]^{op}) \in H_{m_\sigma}(K(\mathbf{I}_{b\times\dot{b}}, 1)). \tag{60}$$

by means of the homomorphism (41).

Consider the submanifold (50) and consider the decomposition of this manifold into connected components:

$$N^{n-16k}_{\eta^{7k}} \equiv \cup_i N^{n-16k}_{i,\eta^{7k}} \subset N^{n-2k}_{b\times\dot{b}}. \tag{61}$$

On each connected component of the manifold (61) let us take a marked point $pt_i \in N^{n-16k}_{i,\eta^{7k}}$. Take a path ρ_i on the manifold $N^{n-2k}_{b\times\dot{b}}$ from the point pt_i to the point pt. For an arbitrary i the isomorphism of the fiber $\kappa_b \oplus \kappa_{\dot{b}}$ over the point pt and the fiber over the point pt_i is well-defined.

Therefore the following mapping

$$\eta_{N_{i,\eta^{7k}}}(\rho_i) : N^{n-16k}_{i,\eta^{7k}} \to K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1). \tag{62}$$

is well-defined. The immersed manifold $N^{n-16k}_{\eta^{7k}}$ is equipped with a framing Ψ_{Ab}. This framing over each component $N^{n-16k}_{i,\eta^{7k}}$ of $N^{n-16k}_{\eta^{7k}}$ is totally determined by the choice of the prescribed coordinate system in the fiber over the marked point $pt_i \in N^{n-16k}_{i,\eta^{7k}}$.

At the opposite site, the choice of the coordinate system in the fiber over pt_i is changed by the transformation on the element $s(i) \in \mathbb{Z}/2$ from a residue class of the subgroup $\mathbf{I}_{b\times\dot{b}} \subset \mathbb{Z}/2^{[2]}$. For the non-trivial residue class the transformation is given by the element $ab \in \mathbf{D} \setminus \mathbf{I}_b$. Denote the element $\eta_{Ab,i,*}(\rho_i)(N^{n-16k}_{i,\eta^{7k}}) \in H_{m_\sigma}(\mathbf{I}_{b\times\dot{b}})$, by $x_i(\rho_i)$. Let us define the element

$$X_i \in D_{m_\sigma}(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]) \tag{63}$$

is equal to $x_i(\rho_i) + 0t$, if the framing Ψ_{Ab} over the manifold $N^{n-16k}_{i,\eta^{7k}}$ is agree with the framing, which is obtained by means of the parallel translation of the framing $\Psi_{b\times\dot{b}}$ along the path ρ_i, and define $Xi_i = 0 + \chi^{[2]}_*(x_i)t$, if the considered framings are not agree. The element (59) is well-defined. By the construction, the element (59) does not depended on a choice of a path ρ_i and a reduction Ψ_{Ab}.

The following lemma is proved by a straightforward calculation.

Lemma 6. *Let us assume that the mapping* (49) *is well defined as in Definition* 5*. Then the following two properties are satisfied.*

–1. The element (59)*, which is constructed by means of the mapping* (51)*, has the image with respect to the homomorphism* (41)*, such that the decomposition of this image over the standard base of the group* $H_{m_\sigma}(K(\mathbf{I}_b \times \dot{\mathbf{I}}_b, 1))$ *contains not more then one non-trivial element, which is determined by the coefficient of the monomial* $t_{b,i} \otimes t_{\dot{b},i}$*, see.* (29)*,* $i = \frac{m_\sigma}{2} = \frac{n-16k}{2}$*. This coefficient coincides with the characteristic number* (6) *for the* $\mathbb{Z}/2^{[2]}$*–framed immersion* (g, Ψ, η_N)*.*

–2. The element (53) *belongs to the subgroup* (38)*,* $i = n - 16k$*.*

Proof of Lemma 6

Let us prove the statement 1 of the lemma in the case $m_\sigma = 14$. The general case is analogous. Consider the manifold $N^{14}_{\eta^{7k}}$, in the proof we denote this manifold by N^{14} for short. The manifold N^{14} is equipped with the mapping (51), in the proof we denote this mapping by $\eta : N^{14} \to K(\mathbf{I}_{b\times\dot{b}}, 1)$ for short. Consider all characteristic numbers with $\mathbb{Z}/2$–coefficients for the mapping η, which are induced from the universal characteristic classes (22) by the mapping (51). We will define the induced classes the same.

Because N^{14} is oriented, non-trivial numbers, possibly, are following: $\kappa_b\kappa_{\dot{b}}^{13}$, $\kappa_b^3\kappa_{\dot{b}}^{11}$, $\kappa_b^5\kappa_{\dot{b}}^9$, $\kappa_b^7\kappa_{\dot{b}}^7$, $\kappa_b^9\kappa_{\dot{b}}^5$, $\kappa_b^3\kappa_{\dot{b}}^{11}$, $\kappa_b\kappa_{\dot{b}}^{13}$.

Let us prove that the characteristic number $\kappa_b \kappa_{\dot{b}}^{13}$ is trivial. Consider a submanifold $K^3 \subset N^{14}$, which is dual to the characteristic class $\kappa_b \kappa_{\dot{b}}^{10}$. The normal bundle ν_N of the manifold N^{14} is isomorphic to the bundle $k\kappa_b \oplus k\kappa_{\dot{b}}$, where $k \equiv 0 \pmod 8$. The restriction of the bundle ν_N on the submanifold K^3 is trivial. Therefore, the normal bundle ν_K of the manifold K^3 is stably equivalent to the bundle $\kappa_b \oplus 2\kappa_{\dot{b}}$. The characteristic class $w_2(K^3)$ is trivial, in particular, $\langle \kappa_{\dot{b}}^3; [K^3] \rangle = 0$. We have $\langle \kappa_{\dot{b}}^3; [K^3] \rangle = \langle \kappa_b \kappa_{\dot{b}}^{13}; [N^{14}] \rangle$. This proves that the characteristic number $\kappa_b \kappa_{\dot{d}}^{13}$ is trivial. Analogically, the following characteristic numbers $\kappa_b^5 \kappa_{\dot{b}}^9$, $\kappa_b^9 \kappa_{\dot{b}}^5$, $\kappa_b \kappa_{\dot{b}}^{13}$ are trivial.

Let us prove that the characteristic number $\kappa_b^3 \kappa_{\dot{b}}^{11}$ is trivial. Take a submanifold $K^7 \subset N^{14}$, which is dual to the characteristic class $\kappa_b^2 \kappa_{\dot{b}}^5$. the normal bundle ν_K of the manifold K^6 is stably equivalent to the bundle $2\kappa_b \oplus 5\kappa_{\dot{b}}$. Because $w_5(K^7) = 0$, the characteristic class $\kappa_{\dot{b}}^5$ is trivial. In particular, $\langle \kappa_b \kappa_{\dot{b}}^6; [K^7] \rangle = 0$. The following equation is satisfied: $\langle \kappa_b \kappa_{\dot{b}}^6; [K^7] \rangle = \langle \kappa_b^3 \kappa_{\dot{b}}^{11}; [N^{14}] \rangle$. This proves that the characteristic number $\kappa_b^3 \kappa_{\dot{b}}^{11}$ is trivial. Analogously, the characteristic number $\kappa_b^{11} \kappa_{\dot{b}}^3$ is trivial.

Obviously, the characteristic number $\langle \kappa_b^7 \kappa_{\dot{b}}^7; [N^{14}] \rangle$ coincides with the characteristic number (6). The statement 1 of the lemma is proved.

Let us prove the statement 2. Because the manifold $N^{n-16k}_{b \times \dot{b}}$ in the case $\sigma \geq 5$ is oriented. Consider the decomposition of the element (53) over the base of the group $H_{n-16k}(K(\mathbf{I}_{b \times \dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])$, using Lemma 3. By Lemma 34, which is proved in Section 6, the element R is not involved to the considered expansion. Another case, the mapping $\eta_{b \times \dot{b}}$ is not satisfied Condition 2 in Definition 5. The statement 2 is proved. Lemma 6 is proved.

Definition 7. Let $[(f, \Xi, \kappa_M)] \in Imm^{sf}(n-k, k)$, $f : M^{n-k} \looparrowright \mathbb{R}^n$, $\kappa_M \in H^1(M^{n-k}; \mathbb{Z}/2)$ be skew-framed by Ξ. We say that the pair (M^{n-k}, κ_M) admits a compression of order q if the mapping $\kappa_M : M^{n-k} \to \mathbb{RP}^\infty$ can be represented as a composition $\kappa = I \circ \kappa'_M : M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^\infty$, where I denotes the inclusion. We say that the element $[(f, \Xi, \kappa_M)]$ admits a compression of order q if this cobordism class contains a triple $(f', \Xi', \kappa_{M'})$, so that the pair $(M'^{n-k}, \kappa_{M'})$ admits a compression of order q.

Theorem 8. *Let $m_\sigma = 2^\sigma - 2$, $\sigma \geq 5$, $n \geq 4m_\sigma + 6$. Assume that the element $\alpha \in Imm^{sf}(n - \frac{n-n_\sigma}{16}, \frac{n-m_\sigma}{16})$ admits a compression of order $q = \frac{m_\sigma}{2} - 1$ (in particular, in the case $\sigma = 5$, $q = 16$). Then the element α admits an $\mathbf{I}_{b \times \dot{b}}$–structure.*

Let

$$d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^n \times \mathbb{R}^n \tag{64}$$

be an arbitrary $T_{\mathbb{RP}^{n-k}}$, $T_{\mathbb{R}^n}$–equivariant mapping, which is transversal along the diagonal $\mathbb{R}^n_{diag} \subset \mathbb{R}^n \times \mathbb{R}^n$ like in the formula (41)[A1]. (We will use only the case $k' = k + q + 1$, as in denotations of Theorem 8.)

Denote $(d^{(2)})^{-1}(\mathbb{R}^n_{diag})/T_{\mathbb{RP}^{n-k}}$ by $\mathbf{N} = \mathbf{N}(d^{(2)})$. Let us call this polyhedron the polyhedron of (formal) intersection of the equivariant mapping $d^{(2)}$.

The polyhedron $\mathbf{N}$, generally speaking, contains a nonempty boundary $\partial\mathbf{N}$ this boundary is represented by critical points of the mapping $d^{(2)}$. Denote by $\mathbf{N}_\circ$ the open polyhedron $\mathbf{N} \setminus \partial\mathbf{N}$, denote by $U(\partial\mathbf{N})_\circ$ the deleted regular neighborhood of the boundary $\partial\mathbf{N}$.

Let us assume the following equivariant mapping (64) and the following generic PL–mapping

$$d : \mathbb{RP}^{n-k} \to \mathbb{R}^n \tag{65}$$

are given.

Definition 9. Let us say that a formal (equivariant) mapping $d^{(2)}$, given by (64), is holonomic if this mapping is the extension of a mapping

$$d : \mathbb{RP}^{n-k} \to \mathbb{R}^n. \tag{66}$$

Definition 10. Let a formal (equivariant) mapping (64) be given. Let us say that $d^{(2)}$ admits an abelian structure, if the following condition is satisfied.

– On the open polyhedron $\mathbf{N}_\circ$ of formal self-intersection points of $d^{(2)}$ the following mapping

$$\eta_{b\times\dot{b}\circ} : (\mathbf{N}_\circ, U(\partial\mathbf{N})_\circ) \to (K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1), K(\mathbf{I}_{b\times\dot{b}}, 1)), \tag{67}$$

is well-defined. This mapping determines a reduction of the structure mapping

$$\eta_\circ : (\mathbf{N}_\circ, U(\partial\mathbf{N})_\circ) \to (K(\mathbb{Z}/2^{[2]}, 1), K(\mathbf{I}_{b\times\dot{b}}, 1)),$$

which satisfies the boundary conditions (about the notion "‘structure mapping"’ see [A1, formula (48)]).

The following lemma is proved in [A3].

Lemma 11. *Assume that the following dimensional restriction*

$$n - k' \equiv -1 \pmod 4, \quad k' \geq 7, \quad n \equiv 0 \pmod 2 \tag{68}$$

is satisfied. Then there exists a formal (equivariant) mapping $d^{(2)}$, which admits an abelian structure in the sense of Definition 10.

Proof of Theorem 8

Put $k = \frac{n-n_\sigma}{16}$. Let the element α be represented by a skew-framed immersion (f, Ξ, κ_M), $f : M^{n-k} \looparrowright \mathbb{R}^n$. By assumption there exists a compression $\kappa'_M : M^{n-k} \to \mathbb{RP}^{n-k-q-1}$ such that the composition $M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset K(\mathbf{I}_d, 1)$ coincides with the mapping $\kappa_M : M^{n-k} \to K(\mathbf{I}_d, 1)$.

Let us define k' as the maximal integer, which is less or equal to $k+q+1$, and such that $n - k' \equiv -1 \pmod 4$. In the case $\sigma = 5$ we get $k' = k+q+1$, in the case $\sigma \geq 6$ we get $k' = k+q-1$. Because $q = \frac{m_\sigma}{2} \geq 14$, for $k' \geq 7$ the both dimensional restrictions (68) are satisfied.

By Lemma 11 there a formal (equivariant) mapping $d^{(2)}$, which admits an abelian structure in the sense of Definition 10.

Let us construct a skew-framed immersion (f_1, Ξ_1, κ_1), for which the immersed $\mathbb{Z}/2^{[2]}$–framed self-intersection manifold contains a closed component $N^{n-2k}_{b\times\dot{b}}$, as it is required in the formula (52).

Define the immersion $f_1 : M^{n-k} \looparrowright \mathbb{R}^n$ using [Corollary 31, A1] as a result of a C^0–small regular deformation of the composition $d \circ \kappa : M^{n-k} \to \mathbb{RP}^{n-k-q-1} \subset \mathbb{RP}^{n-k} \to \mathbb{R}^n$ in the prescribed regular homotopy class of the immersion $f : M^{n-k} \looparrowright \mathbb{R}^n$. Define a caliber of the deformation $d \circ \kappa \mapsto f_a$ much less then the radius of the regular neighborhood of the polyhedron of self-intersection points of the mapping d.

Denote by $N^{n-2k}_{b\times\dot{b}}$ the self-intersection manifold of the immersion f_1. The following decomposition of the manifold into the union of two manifolds with boundaries along the common boundary is well defined:

$$N^{n-2k}_{b\times\dot{b}} = N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})} \cup_\partial N^{n-2k}_{reg}. \tag{69}$$

In this formula $N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})}$ is a manifold with boundary, which is immersed into a regular (immersed) neighborhood $U_{\mathbf{N}(d)}$ of the polyhedron with the boundary $\mathbf{N}(d)$ of self-intersection points of the mapping d. The manifold N^{n-2k}_{reg} with boundary is immersed into a regular immersed neighborhood (denote this immersed neighborhood by U_{reg}) of self-intersection points of the mapping d outside of critical points. The common boundary of the manifolds $N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})}$, N^{n-2k}_{reg} is a closed manifold of the dimension $n-2k-1$, this manifold is immersed into the boundary $\partial(U_{reg})$ of the immersed neighborhood U_{reg}.

Denote the $\mathbb{Z}/2^{[2]}$–framed immersion of the self-intersection (69) by $g_{b\times\dot{b}}$. Let us prove that the $\mathbb{Z}/2^{[2]}$–framing over (69) is reduced to a framing $(\eta_{b\times\dot{b}}, \Psi_{b\times\dot{b}})$ with the structure group $\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}$.

Define a mapping $N^{n-2k}_{b\times\dot{b}} \stackrel{\eta_{b\times\dot{b}}}{\longrightarrow} K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$, which is determined the required reduction of the characteristic mapping of the framing. Define on the submanifold $N^{n-2k}_{reg} \subset N^{n-2k}_{b\times\dot{b}}$ the mapping $\kappa_{b,N_{reg}} : N^{n-2k}_{reg} \to K(\mathbf{I}_b, 1)$ by the composition of the projection $N^{n-2k}_{reg} \to \mathbb{RP}^{n-k'}$ and the inclusion $\mathbb{RP}^{n-k'} \subset \mathbb{RP}^{\infty} = K(\mathbf{I}_b, 1)$.

The cohomology class $\kappa_{\dot{b},N_{reg}} \in H^1(N^{n-2k}_{reg}; \mathbb{Z}/2)$ is defined as the orientation class of the line bundle $p \otimes \kappa_{b,N_{reg}}$, this bundle is the tensor product of the line bundle, associated with the canonical double covering $p : \bar{N}^{n-2k}_{reg} \to N^{n-2k}_{reg}$ and the line bundle with the characteristic class $\kappa_{d,N_{reg}}$, namely

$$\kappa_{\dot{b},N_{reg}} = p \otimes \kappa_{b,N_{reg}}.$$

The pair of the cohomology classes $\kappa_{\dot{b},N_{reg}}$, $\kappa_{b,N_{reg}}$ determines the required mapping

$$\eta_{b\times\dot{b},N_{reg}} : N^{n-2k}_{reg} \to K(\mathbf{I}_{b\times\dot{b}}, 1) \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1). \tag{70}$$

The denotations are agree with the definition of universal characteristic classes (22).

Let us define on the first component $N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})}$ in the formula (69) the mapping

$$\eta_{b\times\dot{b},\mathbf{N}(d^{(2)})} : N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}), 1) \tag{71}$$

as the composition of the projection $N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})} \to \mathbf{N}(d)$ and the mapping $\eta_{b\times\dot{b}} : \mathbf{N}(d^{(2)}) \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}), 1)$, which is constructed in Lemma 11.

Restrictions of the mappings $\eta_{b\times\dot{b},\mathbf{N}(d^{(2)})}$, $\eta_{b\times\dot{b},N_{reg}}$ to the common boundaries $\partial N^{n-2k}_{b\times\dot{b},\mathbf{N}(d^{(2)})}$, ∂N^{n-2k}_{reg} are homotopic, because the mapping $\eta_{\mathbf{N}(d^{(2)})} : \mathbf{N}(d^{(2)}) \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}), 1)$ satisfies the boundary conditions on $\partial\mathbf{N}(d^{(2)})$. Therefore the mapping

$$\eta_{b\times\dot{b}} : N^{n-2k}_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}), 1) \tag{72}$$

is well defined as the result of the gluing of the two mappings $\eta_{b\times\dot{b},\mathbf{N}(d^{(2)})}$ и $\eta_{b\times\dot{b},\mathbf{N}_{reg}}$. The mapping (72) determines a reduction of the characteristic mapping of the $\mathbb{Z}/2^{[2]}$–framing of the immersion $g_{b\times\dot{b}}$.

Let us comes back to the immersion f_a of the manifold (**??**). Consider the submanifold $M^{n-8k}_{2,a} \subset M^{n-2k}$, which is defined as the inverse image $M^{n-8k}_{2,a} = (\kappa'_M)^{-1}(\mathbb{RP}^{n-8k-q-1})$ of the projective subspace $\mathbb{RP}^{n-8k-q-1} \subset \mathbb{RP}^{n-k-q-1}$ of the codimension $7k$. Define the immersion $f_{2,a} : M^{n-8k}_{2,a} \looparrowright \mathbb{R}^n$ by the restriction of the immersion f_a on the submanifold $M^{n-8k}_{2,a}$.

By construction of f_a the image of $f_{2,a}$ is inside a regular neighborhood of the image of the submanifold $\mathbb{RP}^{n-8k-q-1} \subset \mathbb{RP}^{n-7k-k'}$ by the mapping d. Because $n-8k-q-1 = n - \frac{n}{2} + \frac{m_\sigma}{2} - \frac{m_\sigma}{2} - 1 = \frac{n}{2} - 1$, we get $d(\mathbb{RP}^{n-8k-q-1})$ is an embedded submanifold.

Denote by $N_{2,a}^{n-16k}$ the self-intersection manifold of the immersion $f_{2,a}$. We have $\dim(N_{2,a}) = n-16k = m_\sigma$. $n-16k = m_\sigma$. The following inclusion is well-defined:

$$N_{2,a}^{n-16k} \subset N_{a,reg}^{n-2k}, \tag{73}$$

where the manifold $N_{a,reg}^{n-2k}$ is determined by the formula (**??**).

In particular, a reduction of the classifying mapping $\eta_{2,a} : N_{2,a}^{n-16k} \to K(\mathbb{Z}/2^{[2]},1)$ into the subspace $K(\mathbf{I}_{b\times\dot{b}},1) \subset K(\mathbb{Z}/2^{[2]},1)$ is well-defined:

$$\eta_{Ab,a} : N_{2,a}^{n-16k} \to K(\mathbf{I}_{b\times\dot{b}},1). \tag{74}$$

Therefore in the case $\sigma \geq 5$ without loss of a generality we may assume that

$$N_2^{n-16k} \cap N_{b\times\dot{b},\mathbf{N}(d)}^{n-2k} = \emptyset. \tag{75}$$

The triple (f_1, Ξ_1, κ_1) defines the required skew-framed immersion in the cobordism class x. Let us define the second component in the formula (52) as the empty component. The reduction mapping (74) (recall that $N_{2,reg}^{n-16k}$ is re-denoted by $N_{\eta^{7k}}^{n-16k}$) coincides with the required reduction, which is given by the formula (51)/ This proves Property 1 from Definition 5. Property 2 is analogous.

Арф-инвариант, определеный при помощи $\mathbb{Z}/2^{[2]}$–оснащенного многообразия (69) совпадает с Арф-инвариантом исходного элемента x. The Arf-invariant, which is defined by the $\mathbb{Z}/2^{[2]}$–framed immersion of the manifold (69) coincides with the Arf-invariant of the element x.

Let us present a sketch of the proof without the assumption that the mapping $d^{(2)}$ is holonomic. Analogously to [Theorem 23, A1], let us generalize the construction above and let us define a $\mathbb{Z}/2^{[2]}$–framed immersion $(g_{b\times\dot{b}}, \Psi, \eta)$ of the manifold (69), which satisfies Conditions 1 and 2, but, probably, which is not an immersion of a self-intersection of a framed immersion Evidently, there exists a skew-framed immersion (f, Ξ, κ), which represents the element x, for which an arbitrary $\mathbb{Z}/2^{[2]}$–framed immersion, in particular, the $\mathbb{Z}/2^{[2]}$–framed immersion $(g_{b\times\dot{b}}, \Psi, \eta)$, is a closed (probabely, non-connected) component of the self-intersection manifold of this skew-framed immersion (f, Ξ, κ). The second component in the formula (52) is defined as the last component of the self-intersection manifold of the skew-framed immersion (f, Ξ, κ). Properties 1 and 2 are evident. Theorem 8 is proved.

3 $\mathbf{E}_{b\times\dot{b}}$–structure of a $\mathbb{Z}/2^{[2]}$–framed immersion and $\mathbf{J}_a \times \dot{\mathbf{J}}_a$-structure (bicyclic structure) of a $\mathbb{Z}/2^{[3]}$–framed immersion

We define the group $\mathbf{I}_a$ as the cyclic subgroup of order 4 in the dihedral group $\mathbf{I}_a \subset \mathbb{Z}/2^{[2]}$, see section 2 of [A1]. We shall now define an analogous subgroup

$$i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \subset \mathbb{Z}/2^{[4]}, \tag{76}$$

which is isomorphic to the direct product of the two cyclic groups of the order 4.

Recall that the group $\mathbb{Z}/2^{[4]}$ is defined in terms of a basis $(\mathbf{e}_1,\dots,\mathbf{e}_8)$ of the Euclidean space $\mathbb{R}^8$. We denote the generators of the factors of the group $\mathbf{J}_a \times \dot{\mathbf{J}}_a$ by a and $\dot{a}$ respectively. We shall describe transformations in $\mathbb{Z}/2^{[4]}$, corresponding to each generator. We introduce a new basis $\{\mathbf{f}_1,\dots,\mathbf{f}_8\}$, by the formulas $\mathbf{f}_{2i-1} = \frac{\mathbf{e}_{2i-1}+\mathbf{e}_{2i}}{\sqrt{2}}$, $\mathbf{f}_{2i} = \frac{\mathbf{e}_{2i-1}-\mathbf{e}_{2i}}{\sqrt{2}}$, $i = 1,\dots,4$.

We shall show that the group of transformations $\mathbf{J}_a \times \dot{\mathbf{J}}_a$ has invariant orthogonal $(2,2,2,2)$–dimensional subspaces, which we denote by $\mathbb{R}^2_{a,+}$, $\mathbb{R}^2_{a,-}$, $\mathbb{R}^2_{\dot{a},+}$, $\mathbb{R}^2_{\dot{a},-}$.

The subspace $\mathbb{R}^2_{a,+} = Lin(\mathbf{f}_1+\mathbf{f}_5, \mathbf{f}_3+\mathbf{f}_7)$ is generated by the pair of vectors $(\mathbf{f}_1+\mathbf{f}_5, \mathbf{f}_3+\mathbf{f}_7)$. The subspace $\mathbb{R}^2_{a,-} = Lin(\mathbf{f}_1-\mathbf{f}_5, \mathbf{f}_3-\mathbf{f}_7)$ is generated by the pair of vectors $(\mathbf{f}_1-\mathbf{f}_5, \mathbf{f}_3-\mathbf{f}_7)$. The subspace $\mathbb{R}^2_{\dot{a},+} = Lin(\mathbf{f}_2+\mathbf{f}_4, \mathbf{f}_6+\mathbf{f}_8)$ is generated by the pair of vectors $(\mathbf{f}_2+\mathbf{f}_4, \mathbf{f}_6+\mathbf{f}_8)$. The subspace $\mathbb{R}^2_{\dot{a},-} = Lin(\mathbf{f}_2-\mathbf{f}_4, \mathbf{f}_6-\mathbf{f}_8)$ is generated by the pair of vectors $(\mathbf{f}_2-\mathbf{f}_4, \mathbf{f}_6-\mathbf{f}_8)$.

It is convenient to pass to a new basis

$$\frac{\mathbf{f}_1+\mathbf{f}_5}{\sqrt{2}} = \mathbf{h}_{1,+}, \frac{\mathbf{f}_1-\mathbf{f}_5}{\sqrt{2}} = \mathbf{h}_{1,-}, \frac{\mathbf{f}_3+\mathbf{f}_7}{\sqrt{2}} = \mathbf{h}_{2,+}, \frac{\mathbf{f}_3-\mathbf{f}_7}{\sqrt{2}} = \mathbf{h}_{2,-}, \tag{77}$$

$$\frac{\mathbf{f}_2+\mathbf{f}_4}{\sqrt{2}} = \dot{\mathbf{h}}_{1,+}, \frac{\mathbf{f}_2-\mathbf{f}_4}{\sqrt{2}} = \dot{\mathbf{h}}_{1,-}, \frac{\mathbf{f}_6+\mathbf{f}_8}{\sqrt{2}} = \dot{\mathbf{h}}_{2,+}, \frac{\mathbf{f}_6-\mathbf{f}_8}{\sqrt{2}} = \dot{\mathbf{h}}_{2,-}. \tag{78}$$

The pairs of vectors $(\mathbf{h}_{1,+}, \mathbf{h}_{2,+})$, $(\mathbf{h}_{1,-}, \mathbf{h}_{2,-})$ are bases for the subspaces $\mathbb{R}^2_{a,+} = Lin(\mathbf{h}_{1,+}, \mathbf{h}_{2,+})$, $\mathbb{R}^2_{a,-} = Lin(\mathbf{h}_{1,-}, \mathbf{h}_{2,-})$ respectively. In addition, the pairs of vectors $(\dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+})$, $(\dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$ are bases for the subspaces $\mathbb{R}^2_{\dot{a},+} = Lin(\dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+})$, $\mathbb{R}^2_{\dot{a},-} = Lin(\dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$ respectively.

The generator a of order 4 is represented by a rotation through angle $\frac{\pi}{2}$ in each of the planes $\mathbb{R}^2_{a,+}$, $\mathbb{R}^2_{a,-}$ and by the central symmetry in the plane $\mathbb{R}^2_{\dot{a},-}$, (Evidently, the image of the generator a commutes with the presentation of the generator $\dot{a}$ see below). The generator $\dot{a}$ is represented by a rotation through angle $\frac{\pi}{2}$ in the planes $\mathbb{R}^2_{\dot{a},+}$ $\mathbb{R}^2_{\dot{a},-}$ and by the central symmetry in the plane $\mathbb{R}^2_{a,-}$. So the subgroup (76) is well defined.

Define the subgroup $i_{\mathbf{E}_{b\times\dot{b}},\mathbf{J}_a\times\dot{\mathbf{J}}_a} : \mathbf{E}_{b\times\dot{b}} \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a$, as the direct product of the diagonal subgroup $\mathbf{I}_a \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a$ with the elementary subgroup $\dot{\mathbf{J}}_d \subset \dot{\mathbf{J}}_a$ of the second factor. The subgroup $\mathbf{E}_{b\times\dot{b}} \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a$ coincides with the preimage of the subgroup $\mathbb{Z}/2 \subset \mathbb{Z}/4$ with respect to the homomorphism

$$\omega^{[4]} : \mathbf{I}_a \times \dot{\mathbf{I}}_a \to \mathbb{Z}/4, \qquad (79)$$

defined by the formula $(x \times y) \mapsto xy$.

Remark

The group $\mathbf{E}_{b\times\dot{b}}$ is isomorphic to the group $\mathbf{H}_{b\times\dot{b}}$, which is defined in [A1, Section 2]. The corresponding subgroups in $\mathbb{Z}/2^{[3]}$ are distinguished.

Define the subgroup $i_{\mathbf{I}_{b\times\dot{b}},\mathbf{E}_{b\times\dot{b}}} : \mathbf{I}_{b\times\dot{b}} \subset \mathbf{E}_{b\times\dot{b}}$, as the kernel of the homomorphism

$$\omega^{[3]} : \mathbf{E}_{b\times\dot{b}} \to \mathbb{Z}/2, \qquad (80)$$

which is determined by the formula $(x \times y) \mapsto x$ in terms of the generators of the ambiance group.

Consider the diagonal subgroup $\mathbb{Z}/2^{[3]} \subset \mathbb{Z}/2^{[4]}$, this subgroup is generated by transformations in the direct sum of the subspaces $diag(\mathbb{R}^2_{a,+}, \mathbb{R}^2_{\dot{a},+})$, $diag(\mathbb{R}^2_{a,-}, \mathbb{R}^2_{\dot{a},-})$. This subgroup is the transformation group of the unit vectors, which are collinear to the vectors $\mathbf{h}_{1,+} + \dot{\mathbf{h}}_{1,+}, \mathbf{h}_{2,+} + \dot{\mathbf{h}}_{2,+}, \mathbf{h}_{1,-} + \dot{\mathbf{h}}_{2,-}, \mathbf{h}_{2,-} + \dot{\mathbf{h}}_{2,-}$. This collection of vectors gives the standard base in the subspace $diag(\mathbb{R}^2_{a,+}, \mathbb{R}^2_{\dot{a},+}) \oplus diag(\mathbb{R}^2_{a,-}, \mathbb{R}^2_{\dot{a},-})$. The complement to this subspace is given by the formula $antidiag(\mathbb{R}^2_{a,+}, \mathbb{R}^2_{\dot{a},+}) \oplus antidiag(\mathbb{R}^2_{a,-}, \mathbb{R}^2_{\dot{a},-})$. In this complement the standard base is given analogously.

In the notations above the diagonal subgroup $\mathbb{Z}/2^{[2]} \subset \mathbb{Z}/2^{[3]}$ is the transformation group of the unit vectors, which are collinear to the vectors $\mathbf{h}_{1,+} + \dot{\mathbf{h}}_{1,+} + \mathbf{h}_{2,+} + \dot{\mathbf{h}}_{2,+}$, $\mathbf{h}_{1,-} + \dot{\mathbf{h}}_{2,-} + \mathbf{h}_{2,-} + \dot{\mathbf{h}}_{2,-}$. It is easy to check that the first vector is collinear to the vector $\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_5 + \mathbf{e}_7$ and the second vector is collinear to the vector $\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6$.

There is an inclusion $i_{\mathbf{E}_{b\times\dot{b}}} : \mathbf{E}_{b\times\dot{b}} \subset \mathbb{Z}/2^{[3]}$, which is compatible with the inclusion (76). Moreover, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbf{I}_{b\times\dot b} & \stackrel{i_{b\times\dot b}}{\longrightarrow} & \mathbb{Z}/2^{[2]} \\
i_{b\times\dot b,\mathbf{E}_{b\times\dot b}} \downarrow & & i^{[3]} \downarrow \\
\mathbf{E}_{b\times\dot b} & \stackrel{i_{\mathbf{E}_{b\times\dot b}}}{\longrightarrow} & \mathbb{Z}/2^{[3]} \\
i_{\mathbf{E}_{b\times\dot b},\mathbf{J}_a\times\dot{\mathbf{J}}_a} \downarrow & & i^{[4]} \downarrow \\
\mathbf{J}_a \times \dot{\mathbf{J}}_a & \stackrel{i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array}
\qquad (81)
$$

Let us define automorphisms

$$\chi^{[3]} : \mathbf{E}_{b\times\dot b} \to \mathbf{E}_{b\times\dot b}, \qquad (82)$$

$$\chi^{[4]} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \to \mathbf{J}_a \times \dot{\mathbf{J}}_a. \qquad (83)$$

of order 2.

Let us also define automorphisms

$$\chi^{[3]} : \mathbb{Z}/2^{[3]} \to \mathbb{Z}/2^{[3]}, \qquad (84)$$

$$\chi^{[4]} : \mathbb{Z}/2^{[4]} \to \mathbb{Z}/2^{[4]}, \qquad (85)$$

of order 2, which are denoted the same. The automorphism (84) is defined by the permutation of the corresponding basis vectors of the subspace $diag(\mathbb{R}^2_{a,+},\mathbb{R}^2_{\dot a,+}) \oplus diag(\mathbb{R}^2_{a,-},\mathbb{R}^2_{\dot a,-})$ with the indexes a and $\dot a$.

Define the automorphism (83) such that its restriction to the diagonal subgroup $diag(\mathbf{J}_a,\dot{\mathbf{J}}_a) = \mathbf{I}_a \subset \mathbf{E}_{b\times\dot b}$ coincides with the identity, and the restriction of this automorphism to the subgroup $\mathbf{I}_{b\times\dot b} \subset \mathbf{E}_{b\times\dot b}$ coincides with the automorphism $\chi^{[2]}$. Evidently, the automorphism (83) is uniquely well defined.

The automorphism (85) is defined by means of the standard bases of the subspaces $diag(\mathbb{R}^2_{a,+},\mathbb{R}^2_{\dot a,+}) \oplus diag(\mathbb{R}^2_{a,-},\mathbb{R}^2_{\dot a,-})$, $antidiag(\mathbb{R}^2_{a,+},\mathbb{R}^2_{\dot a,+}) \oplus antidiag(\mathbb{R}^2_{a,-},\mathbb{R}^2_{\dot a,-})$ by the permutation of the basis vectors with the indexes a and $\dot a$.

Recall that the group $\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z}$ is defined by the formula (25). Define the analogous subgroups

$$\mathbf{E}_{b\times\dot b}\int_{\chi^{[3]}}\mathbb{Z}, \qquad (86)$$

$$(\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}, \qquad (87)$$

as the semi-direct product of the corresponding groups with involutions with the group $\mathbb{Z}$.

Namely, define the group (86) as the factorgroup of the group $\mathbf{E}_{b\times\dot{b}} * \mathbb{Z}$ by the relations $zxz^{-1} = \chi^{[3]}(x)$, where $z \in \mathbb{Z}$ is the generator, $x \in \mathbf{E}_{b\times\dot{b}}$ is an arbitrary element. The classifying space $K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)$ is the semi-direct product of the standard circle S^1 and the space $K(\mathbf{E}_{b\times\dot{b}}, 1)$. The shift map $K(\mathbf{E}_{b\times\dot{b}}, 1) \to K(\mathbf{E}_{b\times\dot{b}}, 1)$ of the cyclic cover over $K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)$ is induced by the automorphism $\chi^{[3]}$. The definition of the group (87) is analogous.

Let us consider homology groups $H_i(K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1))$, $H_i(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1))$. In particular, for an odd $* = i$, the second group contains the elements, which is represented by the fundamental classes of the submanifolds $S^i/\mathbf{i} \times pt \subset S^i/\mathbf{i} \times S^i/\mathbf{i} \subset K(\mathbf{J}_a, 1) \times K(\dot{\mathbf{J}}_a, 1)$, $pt \times S^i/\mathbf{i} \subset S^i/\mathbf{i} \times S^i/\mathbf{i} \subset K(\mathbf{J}_a, 1) \times K(\dot{\mathbf{J}}_a, 1)$, denote the elements by

$$t_{a,i}, \in H_i(K(\mathbf{J}_a, 1)); \quad t_{\dot{a},i} \in H_i(K(\dot{\mathbf{J}}_a, 1)). \tag{88}$$

The homology groups with local coefficients $\mathbb{Z}[\mathbb{Z}/2]$ are defined analogously to (30), (31), (34). For example:

$$D_i(\mathbf{J}_a \times \dot{\mathbf{J}}_a; \mathbb{Z}[\mathbb{Z}/2]) \subset H_i(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]), \tag{89}$$

$$D_i(\mathbf{E}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2]) \subset H_i(K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]). \tag{90}$$

Analogous groups are defined with local $\mathbb{Z}/2[\mathbb{Z}/2]$–coefficients:

$$D_i(\mathbf{J}_a \times \dot{\mathbf{J}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) \subset H_i(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]), \tag{91}$$

$$D_i(\mathbf{E}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]) \subset H_i(K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]). \tag{92}$$

Analogously to (40), (41) the following homomorphism (isomorphisms) are well-defined:

$$\Delta^{[4]} : D_i(\mathbf{J}_a \times \dot{\mathbf{J}}_a; \mathbb{Z}/2[\mathbb{Z}/2]) \to H_i(K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1)), \tag{93}$$

$$\Delta^{[3]} : D_i(\mathbf{E}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2]) \to H_i(K(\mathbf{E}_{b\times\dot{b}}, 1)). \tag{94}$$

The following lemma is analogous to Lemma 3.

Lemma 12. *–1. The group* $H_{2s}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2])$ *(correspondingly,* $H_{2s}(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2])$*) contains the direct factor* $D_{2s}(\mathbf{J}_a \times \dot{\mathbf{J}}_a; \mathbb{Z}[\mathbb{Z}/2])$ *(correspondingly, contains the direct factor* $D_{2s}(\mathbf{E}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2])$*)*

–2. The base of the subgroup $\bigoplus_{i_1+i_2=2s} H_{i_1}(K(\mathbf{J}_a, 1); \mathbb{Z}) \otimes H_{i_2}(K(\dot{\mathbf{J}}_a, 1); \mathbb{Z}) \subset D_{2s}(\mathbf{J}_a \times \dot{\mathbf{J}}_a; \mathbb{Z}[\mathbb{Z}/2])$ *determines the base of the subgroup* $\mathrm{Im}(B) \cap D_{2s}(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[Z/2])$*, where*

$$B : H_{2s}(K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_{2s}(K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])$$

is the reduction homomorphism of modulo 2. (Correspondingly, the base of the subgroup $\bigoplus_{i_1+i_2=2s} H_{i_1}(K(\mathbf{I}_a, 1); \mathbb{Z}) \otimes H_{i_2}(K(\mathbb{Z}/2, 1); \mathbb{Z}) \subset D_{2s}(\mathbf{E}_{b\times\dot{b}}; \mathbb{Z}[\mathbb{Z}/2])$ *(see the homomorphism* (80)*) determines the base of the subgroup* $\mathrm{Im}(B) \cap D_{2s}(\mathbf{I}_{b\times\dot{b}}; \mathbb{Z}/2[\mathbb{Z}/2])$*,*

$$B : H_{2s}(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2]) \to H_{2s}(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]).$$

Define the epimorphism

$$\omega^{[4]} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z} \to \mathbb{Z}/4. \qquad (95)$$

This epimorphism is defined by the extension of the homomorphism (79) from the subgroup $\mathbf{J}_a \times \dot{\mathbf{J}}_a$ to the hole group, the generator of the factor $\mathbb{Z}$ is in the kernel of the epimorphism (95).

Analogous epimorphism

$$\omega^{[3]} : \mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}} \mathbb{Z} \to \mathbb{Z}/4. \qquad (96)$$

is well defined.

The representation $\Phi^{[2]}$ by the formula (48) is generalized as follows:

$$\Phi^{[3]} : \mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}} \mathbb{Z} \to \mathbb{Z}/2^{[3]}, \qquad (97)$$

$$\Phi^{[4]} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z} \to \mathbb{Z}/2^{[4]}, \qquad (98)$$

where the image of the generator of the cyclic group $\mathbb{Z}$ in $\mathbb{Z}/2^{[3]}$ (correspondingly, in $\mathbb{Z}/2^{[4]}$) is represented by the automorphism (84) ((85)).

The automorphisms $\chi^{[i]}$, $i = 2, 3, 4$, in the images and in the preimages of the diagram (81) correspond with respect to horizontal homomorphisms. Therefore the following diagram of groups is well defined:

$$
\begin{array}{ccc}
\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z} & \stackrel{\Phi^{[2]}}{\longrightarrow} & \mathbb{Z}/2^{[2]} \\
i_{b\times\ bb,\mathbf{E}_{b\times\dot{b}}} \downarrow & & i^{[3]} \downarrow \\
\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z} & \stackrel{\Phi^{[3]}}{\longrightarrow} & \mathbb{Z}/2^{[3]} \\
i_{\mathbf{E}_{b\times\dot{b}},\mathbf{J}_a\times\dot{\mathbf{J}}_a} \downarrow & & i^{[4]} \downarrow \\
(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]}.
\end{array} \qquad (99)
$$

Definition 13. Let a $\mathbb{Z}/2^{[3]}$–framed ($\mathbb{Z}/2^{[4]}$–framed) immersion (h, Λ, ζ_L), $h : L^{n-4k} \looparrowright \mathbb{R}^n$ ($h : L^{n-8k} \looparrowright \mathbb{R}^n$) represent an element $z \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$ ($z \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$). We say that this $\mathbb{Z}/2^{[3]}$–framed ($\mathbb{Z}/2^{[4]}$–framed) immersion is an $\mathbf{E}_{b\times\dot{b}}$–framed ($\mathbf{J}_a \times \dot{\mathbf{J}}_a$–framed) immersion if the following two conditions are satisfied:

1. The structure mapping $\zeta_L : L^{n-4k} \to K(\mathbb{Z}/2^{[3]}, 1)$ (correspondingly, $\zeta_L : L^{n-8k} \to K(\mathbb{Z}/2^{[4]}, 1)$) admits a reduction: this mapping is the composition of the mapping

$$\zeta_{\mathbf{E}_{b\times\dot{b}}} : L^{n-4k} \to K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1) \qquad (100)$$

(correspondingly,

$$\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : L^{n-8k} \to K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)) \qquad (101)$$

and the mapping $\Phi^{[3]} : K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1) \to K(\mathbb{Z}/2^{[3]}, 1)$ (correspondingly, the mapping $\Phi^{[4]} : K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \to K(\mathbb{Z}/2^{[4]}, 1)$).

–2. The mapping $\bar{\zeta}_{\mathbf{E}_{b\times\dot{b}}} : \bar{L}^{n-4k} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$ (correspondingly, the mapping $\bar{\zeta}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : \bar{L}^{n-8k} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$), which is defined by means of the 2-sheeted (correspondingly, by means of the 4-sheeted) covering over the mapping $\zeta_{\mathbf{E}_{b\times\dot{b}}} : L^{n-4k} \to K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)$ (correspondingly, over the mapping $\zeta_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} : L^{n-8k} \to K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)$)satisfies Condition 2 of Definition 4.

Let us clarifies Condition 2 in the definition above. Consider a submanifold

$$\bar{L}^{n-16k}_{\eta^{6k}} \subset \bar{L}^{n-4k}, \tag{102}$$

which represents the homology Euler class $[\eta_{\bar{L}_{\eta^{6k}}}]^{op} \in H_{n-16k}(\bar{L}^{n-4k}; \mathbb{Z}/2)$ of the bundle $3k\eta$ over $\bar{L}^{n-4k}$, where $2\eta = p^*(\zeta_L)$, $p : \bar{L}^{n-4k} \to L^{n-4k}$ is the canonical 2-sheeted covering.

(Correspondingly, a submanifold

$$\bar{L}^{n-16k}_{\eta^{4k}} \subset \bar{L}^{n-8k}, \tag{103}$$

which represents the Euler class $[\eta_{\bar{L}_{\eta^{4k}}}]^{op} \in H_{n-16k}(\bar{L}^{n-8k}; \mathbb{Z}/2)$ of the bundle $4k\eta$ over $\bar{L}^{n-8k}$,where $4\eta_{\bar{L}} = p^*(\zeta_L)$, $p : \bar{L}^{n-8k} \to L^{n-8k}$ is the canonical 4-sheeted covering.)

The restriction of the mapping (100) to the submanifold (102) in the regular $\mathbb{Z}/2^{[2]}$–framed cobordism class has to be represented by the composition of the mapping

$$\eta_{\bar{L}_{\eta^{6k}}} : \bar{L}^{n-16k}_{\eta^{6k}} \to K(\mathbf{I}_{b\times\dot{b}}, 1) \tag{104}$$

and the standard inclusion $K(\mathbf{I}_{b\times\dot{b}}, 1) \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$.

(Correspondingly, the restriction of the mapping (101) on the submanifold (103) in the regular $\mathbb{Z}/2^{[3]}$-framed cobordism class has to be presented by the composition of the mapping

$$\eta_{\bar{L}_{\eta^{4k}}} : \bar{L}^{n-16k}_{\eta^{4k}} \to K(\mathbf{E}_{b\times\dot{b}}, 1) \tag{105}$$

and the standard inclusion $K(\mathbf{E}_{b\times\dot{b}}, 1) \subset K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)$.)

Let us investigate the characteristic class 14 for the cobordism group of $\mathbf{E}_{b\times\dot{b}}$–framed (correspondingly, $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–framed) immersions.

The cohomology group $H^4(K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}}, 1); \mathbb{Z}/2)$ (correspondingly, $H^8(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}}, 1); \mathbb{Z}/2)$ contains an element $\tau_{\mathbf{E}_{b\times\dot{b}}}$ (correspondingly,

$\tau_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$), which is defined in by the equation (106) (correspondingly, (107)) below.

Consider the mapping $\Phi^{[3]} : K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}},1) \to K(\mathbb{Z}/2^{[3]},1)$ (correspondingly, $\Phi^{[4]} : K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}},1) \to K(\mathbb{Z}/2^{[4]},1)$) and consider the pull-back $(\Phi^{[3]})^*(\tau_{[3]})$ (correspondingly, $(\Phi^{[4]})^*(\tau_{[4]})$) of the characteristic Euler class $\tau_{[3]} \in H^4(K(\mathbb{Z}/2^{[3]},1);\mathbb{Z}/2)$ (correspondingly, $\tau_{[4]} \in H^8(K(\mathbb{Z}/2^{[4]},1);\mathbb{Z}/2)$) of the universal bundle.

Define

$$(\Phi^{[3]})^*(\tau_{[3]}) = \tau_{\mathbf{E}_{b\times\dot{b}}} \in H^4(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1);\mathbb{Z}/2), \tag{106}$$

(correspondingly,

$$\Phi^{[4]})^*(\tau_{[4]}) = \tau_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \in H^8(K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z},1);\mathbb{Z}/2). \tag{107}$$

In section 1, for a $\mathbb{Z}/2^{[s+1]}$—framed immersion (h,Λ,ζ_L), together with a 2^s–dimensional characteristic class $\zeta_L \in H^{2^s}(L^{n-k2^s};\mathbb{Z}/2)$, we also considered a 2-dimensional characteristic class $\bar{\zeta}_{[2],L} \in H^2(\bar{L}_{[2]}^{n-k2^sk};\mathbb{Z}/2)$.

For a mapping $\zeta_{\mathbf{E}_{b\times\dot{b}}} : L^{n-4k} \to K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1)$ (correspondingly, for a mapping $\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : L^{n-8k} \to K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z},1)$) as an analog of the characteristic class $\bar{\zeta}^*_{[2],L}(\tau_{[2]})$ there serves the characteristic class $\bar{\zeta}^*_{b\times\dot{b}}(\tau_{b\times\dot{b}}) \in H^2(\bar{L}^{n-4k};\mathbb{Z}/2)$, for $s=3$ (correspondingly, $\bar{\zeta}^*_{a\times\dot{a}}(\tau_{a\times\dot{a}}) \in H^2(\bar{L}^{n-8k};\mathbb{Z}/2)$, for $s=4$, in this formula the covering $\bar{L}^{n-8k} \to L^{n-8k}$ is a 4-sheeted covering).

Define the mapping $\bar{\zeta}_{\mathbf{E}_{b\times\dot{b}}}$ as 2-sheeted covering over the mapping $\zeta_{\mathbf{E}_{b\times\dot{b}}}$ with respect to the subgroup $i_{b\times\dot{b},\mathbf{E}_{b\times\dot{b}}} : \mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z} \subset \mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z}$. Define the mapping $\bar{\zeta}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ as 2-sheeted covering over the mapping $\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ with respect to the subgroup $i_{b\times\dot{b},a\times\dot{a}} : \mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z} \subset (\mathbf{J}_a\times J_{\dot{a}})\int_{\chi^{[4]}}\mathbb{Z}$.

The characteristic class $\bar{\zeta}^*_{\mathbf{E}_{b\times\dot{b}}}(\tau_{b\times\dot{b}})$ (correspondingly, the class $\bar{\zeta}^*_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}(\tau_{b\times\dot{b}})$) is induced from the universal class $\tau_{b\times\dot{b}} \in H^2(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}),1);\mathbb{Z}/2)$ by the mapping $\bar{\zeta}_{\mathbf{E}_{b\times\dot{b}}} : \bar{L}^{n-4k} \to K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}),1)$ (correspondingly, by the mapping $\bar{\zeta}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : \bar{L}^{n-8k} \to K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1)$, in this formula the mapping $\bar{\zeta}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ is the 4-sheeted covering over the mapping $\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$). We need to define dual homology classes and the analogous formulas to (59).

Let us consider the immersion $h : L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}} \looparrowright \mathbb{R}^n$ (correspondingly, the immersion $h : L^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \looparrowright \mathbb{R}^n$) as in Definition 13. Let the mapping

$$\zeta_{\mathbf{E}_{b\times\dot{b}}} : L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1) \tag{108}$$

be given (correspondingly, the mapping

$$\zeta_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} : L^{n-8k}_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \to K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)) \tag{109}$$

be given).

Over the target space of the mapping (108) the universal 4-dimensional $\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}$-bundle is well defined. (Correspondingly, Over the target space of the mapping (109) the universal 8-dimensional $(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[3]}} \mathbb{Z}$-bundle is well defined.) Assume that the manifold $L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}}$ (correspondingly, the manifold $L^{n-8k}_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$) is connected, and the mapping (108) (correspondingly, the mapping (109)) is punctured.

Let us define the element

$$\zeta^{loc}_{\mathbf{E}_{b\times\dot{b}},*}([L_{\mathbf{E}_{b\times\dot{b}}}]) \in H_{n-4k}(K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2]) \tag{110}$$

(correspondingly, the element

$$\zeta^{loc}_{\mathbf{J}_a \times \dot{\mathbf{J}}_a,*}([L_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}]) \in H_{n-8k}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])) \tag{111}$$

analogously to the formula (63) for the element (59). Below this element is defined even for a non-punctured reduction mapping, assuming that the target manifold is the self-intersection manifold of a connected $\mathbb{Z}/2^{[2]}$-framed (correspondingly, $\mathbb{Z}/2^{[3]}$-framed) immersed manifold.

Let us assume that a $\mathbb{Z}/2^{[3]}$–framed immersion $h : L^{n-4k} \looparrowright \mathbb{R}^n$ (correspondingly, a $\mathbb{Z}/2^{[4]}$–framed immersion $h : L^{n-8k} \looparrowright \mathbb{R}^n$) is the immersion of self-intersection manifold of a $\mathbb{Z}/2^{[2]}$–framed immersion (g, η_N, Ψ), $g : N^{n-2k} \looparrowright \mathbb{R}^n$ (correspondingly, of a $\mathbb{Z}/2^{[3]}$–framed immersion (g, η_N, Ψ), $g : N^{n-4k} \looparrowright \mathbb{R}^n$).

Assume that in the manifold N^{n-2k} a closed connected component $N^{n-2k}_{b\times\dot{b}} \subset N^{n-2k}$ is marked, comp. with the formula (52) (correspondingly, in the manifold N^{n-4k} a closed connected component $N^{n-4k}_{b\times\dot{b}} \subset N^{n-4k}$ is marked). Moreover, a punctured mapping

$$\eta_{b\times\dot{b}} : N^{n-2k}_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \tag{112}$$

(correspondingly, a punctured mapping

$$\eta_{b\times\dot{b}} : N^{n-4k}_{b\times\dot{b}} \to K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)), \tag{113}$$

which determines a reduction of the characteristic mapping of the $\mathbb{Z}/2^{[2]}$–framing (correspondingly, of the $\mathbb{Z}/2^{[3]}$–framing) over the marked component is given.

Denote by $L^{n-4k}_{b\times\dot{b}} \subset L^{n-4k}$ (correspondingly, by $L^{n-8k}_{b\times\dot{b}} \subset L^{n-8k}$) the component of the self-intersection manifold of the immersion g, restricted to the marked component $N^{n-2k}_{b\times\dot{b}}$ (correspondingly, to the marked component $N^{n-4k}_{b\times\dot{b}}$). Assume that the manifold $L^{n-4k}_{b\times\dot{b}} \subset L^{n-4k}$ is decomposed into 2 subcomponents as in the following formula:

$$L^{n-4k}_{b\times\dot{b}} = L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}} \cup L^{n-4k}_{b\times\dot{b},[3]} \tag{114}$$

(correspondingly, the manifold $L^{n-8k}_{b\times\dot{b}} \subset L^{n-8k}$ is decomposed into 2 subcomponents as in the following formula:

$$L^{n-8k}_{b\times\dot{b}} = L^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \cup L^{n-8k}_{b\times\dot{b},[4]}). \tag{115}$$

Let us assume that the first component $L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}}$ in the formula (114), (correspondingly, to the first component $L^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ in the formula (115)) which generally speaking, is non-connected and a reduction mapping (108) (correspondingly, a reduction mapping (109)) of the characteristic mapping is given, we do not assume that the reduction mapping is punctured.

Let us consider the immersion of the canonical 2-sheeted covering

$$\bar{L}^{n-4k}_{\mathbf{E}_{b\times\dot{b}}} \looparrowright N^{n-2k}_{b\times\dot{b}} \tag{116}$$

(correspondingly, the immersion of the canonical 2-sheeted covering

$$\bar{L}^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \looparrowright N^{n-4k}_{b\times\dot{b}} \tag{117}$$

over the self-intersection manifold $L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}}$ (correspondingly, over the self-intersection manifold $L^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$).

Denote by

$$\eta_{\mathbf{E}_{b\times\dot{b}}} : \bar{L}^{n-4k}_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \tag{118}$$

(correspondingly, by

$$\eta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : \bar{L}^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \to K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1)) \tag{119}$$

the restriction of the mapping (112) (correspondingly, the restriction of the mapping (113)) on the 2-sheeted covering.

Assume, that the mapping (118) is homotopic to the corresponding 2-sheeted covering over the mapping(108). (Correspondingly, assume, that the mapping (119) is homotopic to the corresponding 2-sheeted covering over the mapping (109).)

The characteristic mapping (112) (correspondingly, the mapping (119)) determines the following homology class

$$\eta^{loc}_{\mathbf{E}_{b\times\dot{b}},*}([\bar{L}_{\mathbf{E}_{b\times\dot{b}}}]) \in H_{n-4k}(K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (120)$$

which coincides with the transfer to the corresponding 2-sheeted covering of the element (110). (Correspondingly the homology class

$$\eta^{loc}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,*}([\bar{L}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}]) \in H_{n-8k}(K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]) \qquad (121)$$

is well-defined. This homology class coincides with the transfer to the corresponding 2-sheeted covering of the element (111).)

Definition 14. Let (g,Ψ,η_N) be a $\mathbb{Z}/2^{[2]}$–framed immersion, $g: N^{n-2k} \looparrowright \mathbb{R}^n$, which represents an element $y \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k,2k)$, assuming $n > 16k$. Let (h,Λ,ζ_L) be a $\mathbb{Z}/2^{[3]}$–framed immersion, $h: L^{n-4k} \looparrowright \mathbb{R}^n$, is an immersion of self-intersection points of the immersion g, which represents an element $z = \delta_k^{\mathbb{Z}/2^{[3]}}(y) \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k,4k)$.

Assume that a closed component $N^{n-2k}_{b\times\dot{b}}$ of the manifold N^{n-2k} is punctured (comp. with the formula (52)). Assume that the self-intersection manifold $L^{n-4k}_{b\times\dot{b}}$ of the immersion $g|_{N^{n-2k}_{b\times\dot{b}}}$ is decomposed into two components like in the formula (114).

We say that $\mathbb{Z}/2^{[2]}$–framed immersion (g,Ψ,η_N) admits an $\mathbf{E}_{b\times\dot{b}}$–structure, if on the component $N^{n-2k}_{b\times\dot{b}}$ a punctured mapping (112) is given, and this mapping determines a reduction of the restriction of the characteristic mapping η_N; on the component $L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}}$ a mapping (108) is given (we do not assume that this mapping is punctured), which determines a reduction of the restriction of the classifying mapping ζ_L and the canonical double covering over this reduction mapping is homotopic to the mapping (112). Additionally, the following 3 conditions are satisfy.

–1. The $\mathbb{Z}/2^{[2]}$–framed immersion (g,Ψ,η_N) satisfies Condition 2 from Definition 5 (recall, that this condition means that the Arf-invariant is calculated using the component $N^{n-2k}_{b\times\dot{b}}$).

–2. The punctured mapping (112) satisfies Property 2 of Lemma 6 (recall, that this condition means that the element $\eta_{b\times\dot{b},*}([N^{n-2k}_{b\times\dot{b}},pt]) \in$

$H_{n-2k}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2])$, which is constructed by the formula (53) for the mapping (112), belongs to the subgroup $D_{n-2k}(\mathbf{I}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2])$.

–3. The element

$$\eta^{loc}_{b\times\dot b,*}([\bar L_{b\times\dot b}]) \in H_{n-4k}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (122)$$

which is defined analogously to the element (120), but using the both components from the formula (114),and the element (120), which is constructed only for the first component in the formula (114) are related in the group $H_{n-4k}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$ by the following equation:

$$\eta^{loc}_{b\times\dot b,*}([\bar L_{b\times\dot b}]) = r^{!}(\zeta^{loc}_{\mathbf{E}_{b\times\dot b},*}([L_{\mathbf{E}_{b\times\dot b}}])), \qquad (123)$$

where

$$r^{!}: H_{n-4k}(K(\mathbf{E}_{b\times\dot b}\int_{\chi^{[3]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2]) \to \qquad (124)$$

$$H_{n-4k}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])$$

is the transfer homomorphism, which is associated with the right upper inclusion of the subgroup 2 in the diagram (99), this inclusion is re-denoted by

$$r: \mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z} \subset \mathbf{E}_{b\times\dot b}\int_{\chi^{[3]}}\mathbb{Z} \qquad (125)$$

for short.

Let us express in the framework of Definition 14 the homology class (59) from the homology class (120) and the normal Euler class of the immersion. Define the element

$$\eta^{loc}_{b\times\dot b,*}([\bar\zeta_L^{3k}]^{op}) \in H_{m_\sigma}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (126)$$

where the homology class $([\bar\zeta_L^{3k}]^{op}) \in H_{m_\sigma}(\bar L^{n-4k}_{\mathbf{E}_{b\times\dot b}};\mathbb{Z}/2[\mathbb{Z}/2])$ is defined as the result of the intersection of the fundamental class $[\bar L_{\mathbf{E}_{b\times\dot b}}] \in H_{n-4k}(\bar L^{n-4k}_{\mathbf{E}_{b\times\dot b}};\mathbb{Z}/2[\mathbb{Z}/2])$ with the Euler class $\bar\zeta_L^{3k} \in H^{12k}(\bar L^{n-4k}_{\mathbf{E}_{b\times\dot b}};\mathbb{Z}/2[\mathbb{Z}/2])$ of the bundle $3k\bar\zeta_L$.

Lemma 15. *Assume that a $\mathbb{Z}/2^{[2]}$–framed immersion (g,Ψ,η_N) is given, and this immersion allows $\mathbf{E}_{b\times\dot b}$–structure in the sense of Definition* 14. *Then the following properties are satisfied:*

–1. The element (126) *belongs to the subgroup* (38), *$i = m_\sigma$ and this element is lifted to the subgroup* (34). *The image of this element in the group $H_{m_\sigma}(K(\mathbf{I}_{b\times\dot b},1))$ by means of the homomorphism* (41) *satisfies Property 1 fro Lemma* 6, *analogously to the element* (59).

–2. The element $\zeta^{loc}_{L_{\mathbf{E}_{b\times\dot b}},*}([L_{\mathbf{E}_{b\times\dot b}}])$, which is defined by the formula (110), *belongs to the subgroup $D_{n-4k}(\mathbf{E}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2])$, which is defined by the formula* (92) *for $i = n-4k$, and this elements is lifted to the subgroup* (90).

–3. The Arf-invariant of the $\mathbb{Z}/2^{[2]}$–framed immersion (g,Ψ,η_N) coincides with the Arf-invariant which is calculated by the formula (14) *only for $\mathbb{Z}/2^{[3]}$–framed immersed component $L^{n-4k}_{\mathbf{E}_{b\times\dot b}}$.*

Proof of Lemma 15

Prove Statement 1. Consider the immersion $g_{b\times\dot b} : N^{n-2k}_{b\times\dot b} \looparrowright \mathbb{R}^n$ of the marked component and apply to this immersion the Herbert Theorem with the local coefficient system $\mathbb{Z}/2[\mathbb{Z}/2]$. (The statement and a proof of this version of the Herbert Theorem is analogous to the statement and the proof of the Herbert Theorem with $\mathbb{Z}/2$-coefficients, see [A1], Proposition 8, and the reference there.) We get that homology class

$$[\bar L_{b\times\dot b}] \in H_{n-4k}(N^{n-2k}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2])) \qquad (127)$$

coincides with the homology class $[\eta^k_{N_{b\times\dot b}}]^{op} \in H_{n-4k}(N^{n-2k}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2])$. By assumption, the equation (123) is satisfied, therefore we have:

$$\eta^{loc}_{b\times\dot b,*}([\eta^k_{N_{b\times\dot b}}]^{op}) = \eta^{loc}_{b\times\dot b,*}([\bar L_{\mathbf{E}_{b\times\dot b}}]). \qquad (128)$$

In this formula we re-express the right side using the transfer homomorphism:

$$\eta^{loc}_{b\times\dot b,*}([\eta^k_{N_{b\times\dot b}}]^{op}) = r^!(\zeta^{loc}_{\mathbf{E}_{b\times\dot b},*}([L_{\mathbf{E}_{b\times\dot b}}])). \qquad (129)$$

Consider the product of the homology classes in the both sides of the equation with the cohomology class $\eta^*_{b\times\dot b}\circ r^*(\zeta^{3k}) \in H^{12k}(N^{n-2k}_{b\times\dot b};\mathbb{Z}/2[\mathbb{Z}/2])$. In the right side of the equation we have the homology class $r^!(\zeta^{loc}_{\mathbf{E}_{b\times\dot b},*}([L_{\mathbf{E}_{b\times\dot b}}]))\cap r^*((\zeta^{[3]})^{3k})$, which coincides with the homology class $r^!(\zeta^{loc}_{\mathbf{E}_{b\times\dot b},*}([L_{\mathbf{E}_{b\times\dot b}}])\cap (\zeta^{[3]})^{3k})$, namely, with the homology class (126). In the right side of the equation we have the homology class $\eta^{loc}_{b\times\dot b,*}([\eta^{7k}_{N_{b\times\dot b}}]^{op})$. Let us apply the homomorphism (41) to this homology classes, we get that the class coincides with (59). For this class the required property is proved in Lemma 6. Statement 1 is proved.

Prove Statement 2. In the case $\sigma \geq 5$ the codimension of $\mathbb{Z}/2^{[3]}$–framed immersed manifold $L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}}$ is even, therefore the considered manifold is oriented and its fundamental class belongs to the homology group with integer coefficients.

The transfer homomorphism

$$r^! : H_{n-4k}(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}; \mathbb{Z}/2[\mathbb{Z}/2]) \to H_{n-4k}(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}; \mathbb{Z}/2[\mathbb{Z}/2])$$

with $\mathbb{Z}/2[\mathbb{Z}/2]$–coefficients is a monomorphism. This follows from the following fact: the element R, which is described in Lemma 3, belongs to the image of the transfer homomorphism. Therefore it is sufficient to prove that the element (120) belongs to the subgroup (38), $i = n - 4k$. Using the equation (123) it is sufficient to prove that the element (122) belongs to the considered group. The homology class (122) is expressed from the homology class (53). By Property 1 from Definition (14) the homology class (53) belongs to the subgroup (38), $i = n - 2k$. Statement 2 is proved.

Statement 3 follows from Statement 2 above, and the property of the homology class (53),which is formulated in Statement 1 of Lemma 6. Lemma 15 is proved.

Definition 16. Let (g, Ψ, η_N) be a $\mathbb{Z}/2^{[3]}$–framed immersion, $g : N^{n-4k} \looparrowright \mathbb{R}^n$, which represents an element $y \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$, assuming $n > 16k$. Let (h, Λ, ζ_L) be a $\mathbb{Z}/2^{[4]}$–framed immersion, $h : L^{n-8k} \looparrowright \mathbb{R}^n$, is an immersion of self-intersection points of the immersion g, which represents an element $z = \delta_k^{\mathbb{Z}/2^{[4]}}(y) \in Imm^{\mathbb{Z}/2^{[4]}}(n - 8k, 8k)$.

Assume that a component $N^{n-4k}_{b\times\dot{b}}$ of the manifold N^{n-4k} is marked and punctured. Assume that the self-intersection manifold $L^{n-8k}_{b\times\dot{b}}$ of the immersion $g|_{N^{n-4k}_{b\times\dot{b}}}$ is decomposed into two components, like in the formula (115).

We will say that the $\mathbb{Z}/2^{[3]}$–framed immersion (g, Ψ, η_N) admits an $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure, if on the component $N^{n-4k}_{b\times\dot{b}}$ a punctured mapping (113) is well-defined, and this mapping determines a reduction of the restriction of the characteristic mapping η_N; on the component $L^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ a mapping (109) is given (we do not assume that this mapping is punctured), which determines a reduction of the restriction of the classifying mapping ζ_L and the canonical double covering over this reduction mapping is homotopic to the mapping (113). Additionally, the following 3 conditions are satisfy.

–1. The mapping (118) satisfies Condition 3 from Lemma 15 (recall, that this condition means that the Arf-invariant is calculated using the component $N^{n-4k}_{b\times\dot{b}}$).

–2. The mapping (118) satisfies Condition 2 from Lemma 15.

–3. The element

$$\eta^{loc}_{b\times\dot{b},*}([\bar{L}_{b\times\dot{b}}]) \in H_{n-8k}(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (130)$$

which is defined analogously to the element (121), but using the both components from the formula (115),and the element (121), which is constructed only for the first component in the formula (115) are related in the group $H_{n-8k}(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$ by the following relation:

$$\eta^{loc}_{b\times\dot{b},*}([L^{-}_{b\times}b]) = r^{!}(\zeta^{loc}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,*}([L_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}])), \qquad (131)$$

where

$$r^{!}: H_{n-8k}(K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2]) \to \qquad (132)$$

$$H_{n-8k}(K(\mathbf{E}_{b\times\dot{\mathbf{I}}_b}\int_{\chi^{[3]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])$$

is the transfer homomorphism, which is associated with the right middle inclusion of the subgroup 2 in the Diagram (99), which is re-denoted by

$$r: \mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z} \subset (\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z} \qquad (133)$$

for short.

Lemma 17. *Assume a $\mathbb{Z}/2^{[3]}$–framed immersion (g,Ψ,η_N) is given, which admits an $\mathbf{J}_a\times\dot{\mathbf{J}}_a$–structure in the sense of Definition* 16. *The following two properties are satisfied:*

–1. In the group $H_{m_\sigma}(K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$ the element (121) *is equal to the element $\eta^{loc}_{b\times\dot{b},*}([\eta^{3k}_{N_{b\times\dot{b}}}]^{op})$. In particular, the images of these elements by the homomorphism* (94) *for $i = m_\sigma$ are equal in the group $H_{m_\sigma}(K(\mathbf{E}_{b\times\dot{b}},1))$.*

–2. The Arf-invariant of the $\mathbb{Z}/2^{[3]}$–framed immersion (g,Ψ,η_N) coincides with the Arf-invariant, which is calculated by the formula (14), *using only the component $L^{n-8k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$.*

Proof of Lemma 17

The proof is analogous to the proof of Lemma 15.

Example 18. Let the $\mathbb{Z}/2^{[2]}$–framed (correspondingly, $\mathbb{Z}/2^{[3]}$–framed) immersion $(g, \eta_N, \Psi,)$, $g : N^{n-2k} \looparrowright \mathbb{R}^n$ (correspondingly, $g : N^{n-4k} \looparrowright \mathbb{R}^n$) be represented an element $y \in Imm^{\mathbb{Z}/2^{[2]}}(n-2k, 2k)$ (correspondingly, an element $y \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$), $n > 16k$. Assume that the considered immersion is a $\mathbf{I}_{b\times\dot{b}}$–framed immersion (correspondingly, is a $\mathbf{E}_{b\times\dot{b}}$–framed immersion) in the sense of Definition 4 (correspondingly, in the sense of Definition 13. Let a $\mathbb{Z}/2^{[3]}$–framed (correspondingly, a $\mathbb{Z}/2^{[4]}$–framed) immersion (h, ζ_L, Λ), $h : L^{n-4k} \looparrowright \mathbb{R}^n$ (correspondingly, $h : L^{n-8k} \looparrowright \mathbb{R}^n$) be represented the element $z = \delta_k^{\mathbb{Z}/2^{[3]}}(y) \in Imm^{\mathbb{Z}/2^{[3]}}(n-4k, 4k)$ (correspondingly, $z = \delta_k^{\mathbb{Z}/2^{[4]}}(y) \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$) and be the immersion of the self-intersection manifold of the immersion (g, η_N, Ψ). Assume that the immersion (h, ζ_L, Λ) is a $\mathbf{E}_{b\times\dot{b}}$–framed immersion (correspondingly, is a $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–framed immersion) in the sense of Definition 13. Then the $\mathbb{Z}/2^{[2]}$–framed (correspondingly, the $\mathbb{Z}/2^{[3]}$–framed) immersion (g, Ψ, η_N) admits a $\mathbf{E}_{b\times\dot{b}}$–structure (correspondingly, a $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure), which is defined by the reduction mapping $\zeta_{\mathbf{E}_{b\times\dot{b}}}$ (correspondingly, by $\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$) of the characteristic mapping ζ_L (comp. with Example 19 in [A1]).

Justification of the example 18

The example is obvious.

The following theorems are analogs of Theorem 8.

Lemma 19. *Assume that the* $\mathbb{Z}/2^{[2]}$*–framed immersion* (g, Ψ, η_N) *represents an element* $y \in Imm^{\mathbb{Z}/2^{[2]}}(n - \frac{n-n_\sigma}{8}, \frac{n-n_\sigma}{8})$, $n \geq 254$, *see* (18), *and a reduction of the characteristic mapping* η_N *by the mapping* (112) *is given, such that Conditions 1,2, from Definition* 14 *are satisfied. Then the element* $\delta_k^{\mathbb{Z}/2^{[3]}}(y)$ *in the group* $Imm^{\mathbb{Z}/2^{[3]}}(n - \frac{n-m_\sigma}{4}, \frac{n-m_\sigma}{4})$ *is represented by a* $\mathbb{Z}/2^{[3]}$*–framed immersion* (h, Λ, ζ_L), *which admits a* $\mathbf{E}_{b\times\dot{b}}$*–structure of the* $\mathbb{Z}/2^{[2]}$*–framed immersion* (g, Ψ, η_N) *in the sense of Definition* 14.

Lemma 20. *Assume that the* $\mathbb{Z}/2^{[3]}$*–framed immersion* (g, Ψ, η_N) *represents an element* $z \in Imm^{\mathbb{Z}/2^{[3]}}(n - \frac{n-n_\sigma}{4}, \frac{n-n_\sigma}{4})$, $n \geq 254$, *see* (18), *and a reduction* *of the characteristic mapping η_N by the mapping* (113) *is given, such that Conditions 1,2, from Definition* 16 *are satisfied.*

Then the element $\delta_k^{\mathbb{Z}/2^{[4]}}(z)$ in the group $Imm^{\mathbb{Z}/2^{[4]}}(n - \frac{n-m_\sigma}{2}, \frac{n-m_\sigma}{2})$ is represented by a $\mathbb{Z}/2^{[4]}$–framed immersion (h, Λ, ζ_L), which admits a $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure of the $\mathbb{Z}/2^{[3]}$–framed immersion (g, Ψ, η_N) in the sense of Definition 16*.*

Corollary 21. *Assume that the assumptions and the dimensional restriction of Theorem* 8 *is satisfied. Then the element $\delta_k^{\mathbb{Z}/2^{[4]}} \circ \delta_k^{\mathbb{Z}/2^{[3]}} \circ \delta_k^{\mathbb{Z}/2^{[2]}}(\alpha)$, which is defined by means of the composition of the homomorphisms* (8)*, $k = \frac{n-m_\sigma}{16}$, is represented by a $\mathbb{Z}/2^{[4]}$–framed immersion (h, Λ, ζ_L), which admits a bicyclic structure ($\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure) in the sense of Definition* 16*.*

Moreover, the projection of the element

$$\zeta^{loc}_{\mathbf{J}_a \times \dot{\mathbf{J}}_a,*}([\zeta^k_{L_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}}]^{op}) \in H_{m_\sigma}(K(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])) \qquad (134)$$

into the direct factor (91)*, $i = m_\sigma$, after the expansion over the standard base, involves not more then the only basic element $t_{a,i} \otimes t_{\dot{a},i}$, see.* (88)*, $i = \frac{m_\sigma}{2} = \frac{n-16k}{2}$. The coefficient at this basic element coincides with the Arf-invariant* (6)*, which is calculated for the $\mathbb{Z}/2^{[2]}$–framed immersion (g, η_N, Ψ).*

Proof of Corollary 21

By Theorem 8 without loss of a generality, we may assume that the element $y = \delta_k^{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[2]}}(n - \frac{n-m_\sigma}{8}, \frac{n-m_\sigma}{8})$ is represented by a $\mathbb{Z}/2^{[2]}$–framed immersion (g, η_N, Ψ), such that the self-intersection manifold of this immersion contains a closed marked component $N^{n-8k}_{b \times \dot{b}}$, like in the formula (52), and the mapping (49) on this marked component is well-defined. Then the both conditions in Lemma 6 are satisfied, therefore Conditions 1 and 2 from Definition 14 are satisfied.

By Lemma 19 we may assume that the element y is represented by a $\mathbb{Z}/2^{[2]}$–framed immersion, which admits a $\mathbf{E}_{b \times \dot{b}}$–structure. An immersion, which represents the element $z = \delta_k^{\mathbb{Z}/2^{[3]}}(y)$ contains a marked component $N^{n-4k}_{b \times \dot{b}}$. By Lemma 20 a $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure of an immersion, which is represent the element z is well-defined. Properties 1 and 2 in Definition 16 follow from Lemma 15.

Let us consider the image of the element (134) from the group $H_{m_\sigma}(K(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])$ by the composition of the following two transfers:

$$H_{m_\sigma}(K(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}/2[\mathbb{Z}/2])) \to$$

$$H_{m_\sigma}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])),$$

which are induced by the inclusions (133), (125). Properties 1 from Lemmas 15, 17 imply that the image of the element (134) coincides to the element (59) from the subgroup

$$D_{m_\sigma}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}/2[\mathbb{Z}/2]) \subset H_{m_\sigma}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])).$$

The image by the composition of the two transfers of the projection of the element (134) onto the subgroup (91), $i = m_\sigma$, is also coincides with (59).

Properties of the expansion of the element (59) are given by Lemma 6, Statement 1. We get the required properties for the element (134). Corollary 21 is proved.

4 $\mathbf{Q}\times\mathbb{Z}/4$–structure (quaternionic-cyclic structure) on $\mathbb{Z}/2^{[4]}$–framed immersion

Let us recall the definition of the quaternion subgroup $\mathbf{Q} \subset \mathbb{Z}/2^{[3]}$, which contains the subgroup $\mathbf{I}_a \subset \mathbf{Q}$, see [A1], section 2, formulas (22),(23),(24).

Let us define subgroups:

$$i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \subset \mathbf{Q}\times\mathbb{Z}/4, \qquad (135)$$

$$i_{\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{Q}\times\mathbb{Z}/4 \subset \mathbb{Z}/2^{[5]}, \qquad (136)$$

$$i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2 \subset \mathbb{Z}/2^{[5]}. \qquad (137)$$

Define the subgroup (135). Define the epimorphism on the subgroup $\mathbf{J}_a \times \dot{\mathbf{J}}_a \to \mathbb{Z}/4 \subset \mathbf{Q}$ by the formula $(x\times y) \mapsto xy$. The kernel of this epimorphism coincides with the antidiagonal subgroup $\dot{\mathbf{I}}_a = antidiag(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a$, this subgroup a the direct factor. This factor is mapped onto the subgroup $\mathbb{Z}/4$ by the formula $(x\times x^{-1}) \mapsto x$. The complement of this factor is the subgroup $\mathbf{J}_a \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a$. The subgroup (135) is well defined.

To define the subgroups (136), (137) consider the basis $(\mathbf{h}_{1,+}, \mathbf{h}_{2,+}, \mathbf{h}_{1,-}, \mathbf{h}_{2,-}, \dot{\mathbf{h}}_{1,+}, \dot{\mathbf{h}}_{2,+}, \dot{\mathbf{h}}_{1,-}, \dot{\mathbf{h}}_{2,-})$ in the space $\mathbb{R}^8$, determined by (77), (78).

We define an analogous basis of $\mathbb{R}^{16}$. This basis consists of 16 vectors, this set of the basis vectors is divided two subsets:

$$\mathbf{h}_{1,*,**}, \mathbf{h}_{2,*,**}; \qquad (138)$$

$$\dot{\mathbf{h}}_{1,*,**}, \dot{\mathbf{h}}_{2,*,**}. \qquad (139)$$

where the symbols $*$, $**$ independently takes the values $+, -$.

Let us define the subgroup (136). The representation $i_{\mathbf{Q}\times\mathbb{Z}/4}$ is given such that the generator $\mathbf{j}$ of the factor $\mathbf{Q} \subset \mathbf{Q} \times \mathbb{Z}/4$ acts in each 4-dimensional subspace

$$diag(Lin(\mathbf{h}_{1,*,**}, \mathbf{h}_{2,*,**}, \mathbf{h}_{1,*,-**}, \mathbf{h}_{2,*,-**}), \qquad (140)$$
$$Lin(\dot{\mathbf{h}}_{1,*,**}, \dot{\mathbf{h}}_{2,*,**}, \dot{\mathbf{h}}_{1,*,-**}, \dot{\mathbf{h}}_{2,*,-**})),$$

$$diag(Lin(\mathbf{h}_{1,-*,**}, \mathbf{h}_{2,-*,**}, \mathbf{h}_{1,-*,-**}, \mathbf{h}_{2,-*,-**}), \qquad (141)$$
$$Lin(\dot{\mathbf{h}}_{1,-*,**}, \dot{\mathbf{h}}_{2,-*,**}, \dot{\mathbf{h}}_{1,-*,-**}, \dot{\mathbf{h}}_{2,-*,-**})),$$

$$antidiag(Lin(\mathbf{h}_{1,*,**}, \mathbf{h}_{2,*,**}, \mathbf{h}_{1,*,-**}, \mathbf{h}_{2,*,-**}), \qquad (142)$$
$$Lin(\dot{\mathbf{h}}_{1,*,**}, \dot{\mathbf{h}}_{2,*,**}, \dot{\mathbf{h}}_{1,*,-**}, \dot{\mathbf{h}}_{2,*,-**})),$$

$$antidiag(Lin(\mathbf{h}_{1,-*,**}, \mathbf{h}_{2,-*,**}, \mathbf{h}_{1,-*,-**}, \mathbf{h}_{2,-*,-**}), \qquad (143)$$
$$Lin(\dot{\mathbf{h}}_{1,-*,**}, \dot{\mathbf{h}}_{2,-*,**}, \dot{\mathbf{h}}_{1,-*,-**}, \dot{\mathbf{h}}_{2,-*,-**}))$$

by the standard transformations, given by the matrix (23), [A1].

Let us note that each 4-dimensional space, described above, corresponds to one of the two subspaces $\mathbb{R}^2_{a,*}$, or to one of the two subspaces $\mathbb{R}^2_{\dot{a},*}$, the definition of this subspaces is given below the formulas (77), (78). The generator $\mathbf{i} \in \mathbf{Q}$ is represented in the direct sum of the two copies of the corresponding spaces, according to the representation of the generator of the group $\mathbf{J}_a$ given by the matrix (23) [A1]. The generator of the second factor $\mathbb{Z}/4 \subset \mathbf{Q} \times \mathbb{Z}/4$

$\mathbf{i} \in \mathbf{Q}$ is represented in the direct sum of the two copies of the corresponding spaces, according to the representation of the generator of the subgroup $\dot{\mathbf{I}}_a \equiv antidiag(\mathbf{I}_a \times \dot{\mathbf{I}}_a) \subset \mathbf{I}_a \times \dot{\mathbf{I}}_a$. The representation (136) is well defined.

Let us define the representation (137) as follows. The factor $\mathbf{J}_a \times \dot{\mathbf{J}}_a \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2$ is represented in each 4-dimensional subspace (140)-(143) by the formula (76), this formula is applied in each subspace with the prescribed basis. The factor $\mathbb{Z}/2 \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2$ is represented:

–in 8-dimensional subspace, the direct sum of the subspaces (140), (142) by the identity;

–in 8-dimensional subspace, the direct sum of the subspaces (141), (143) by the central symmetry;

the representation (137) is well-defined.

We define an order 4 automorphism $\chi^{[5]}$ of the subgroup $\mathbf{Q} \times \mathbb{Z}/4$. The restriction of this automorphism to the subgroup (135) coincides with the automorphism $\chi^{[4]}$. The extension of the automorphism $\chi^{[4]}$ on the subgroup to the automorphism $\chi^{[5]}$ on the group is defined by the identity on the generator $\mathbf{j}$. It is easy to verify that the automorphism described above is uniquely well defined.

Consider the homomorphism

$$p_{\mathbf{Q}} : \mathbf{Q} \times \mathbb{Z}/4 \to \mathbf{Q}, \tag{144}$$

which is the projection on the first factor. The kernel of the homomorphism $p_{\mathbf{Q}}$ coincides with the image of the antidiagonal subgroup $\dot{\mathbf{I}}_a \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a$ by the inclusion (135). Evidently, the following equation is satisfied:

$$\chi^{[5]} \circ p_{\mathbf{Q}} = p_{\mathbf{Q}}. \tag{145}$$

Analogously, define the automorphism (involution) $\chi^{[5]}$ of the group $\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2$ (we denote this new automorphism the same). Define the homomorphism

$$p_{\mathbb{Z}/4\times\mathbb{Z}/2} : \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2 \tag{146}$$

with the kernel $\dot{\mathbf{I}}_a \subset \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2$. Evidently, the following equation is satisfied: $\chi^{[5]} \circ p_{\mathbb{Z}/4\times\mathbb{Z}/2} = p_{\mathbb{Z}/4\times\mathbb{Z}/2}$.

Define the following groups

$$(\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, \tag{147}$$

$$(\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}, \tag{148}$$

as semi-direct products of the corresponding groups with automorphisms and the cyclic group $\mathbb{Z}$. (see the analogous definitions (25), (86), (87)).

Let us define the epimorphism

$$\omega^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} \to \mathbf{Q}, \tag{149}$$

the restriction of this epimorphism on the subgroup (135) coincides with the epimorphism (144). It is sufficient to use the formula (145), define $z \in Ker(p_{\mathbf{Q}})$, where $z \in \mathbb{Z}$ is the generator.

Evidently, the following epimorphism

$$\omega^{[5]} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/4 \times \mathbb{Z}/2, \tag{150}$$

is well-defined, we denote this epimorphism the same way like the epimorphism (149).

Define the automorphism (involution) of the group $\mathbb{Z}/2^{[5]}$, which is also denoted by $\chi^{[5]}$. In the standard basis of the subspaces (140)-(143) the automorphism $\chi^{[5]}$ is given by the same formulas as the automorphism $\chi^{[4]}$, each of this space is an invariant space of $\chi^{[5]}$. From the definition it is easy to verify that $\chi^{[5]}$ commutes with (136), (137).

Moreover, the following homomorphisms are well defined:

$$\Phi^{[5]} : (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}, \tag{151}$$

$$\Phi^{[5]} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} \to \mathbb{Z}/2^{[5]}. \tag{152}$$

These homomorphisms are analogous to the homomorphism (98) and are included into the following commutative diagrams (153), (154), (see the analogous diagram (81)):

$$\begin{array}{ccc} (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]} \times \Phi^{[4]})}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{Q} \times \mathbb{Z}/4} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]}, \end{array} \tag{153}$$

In this diagram the left vertical homomorphism

$$i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{Q} \times \mathbb{Z}/4} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z} \to (\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}$$

is induced by the homomorphism (135), the right vertical homomorphism

$$i_{[5]} : \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \subset \mathbb{Z}/2^{[5]}$$

is defined by the formula (9).

$$\begin{array}{ccc} (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z} & \stackrel{\Phi^{[4]} \times \Phi^{[4]}}{\longrightarrow} & \mathbb{Z}/2^{[4]} \times \mathbb{Z}/2^{[4]} \\ i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2} \downarrow & & i_{[5]} \downarrow \\ (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z} & \stackrel{\Phi^{[5]}}{\longrightarrow} & \mathbb{Z}/2^{[5]}, \end{array} \qquad (154)$$

In this diagram the left vertical homomorphism

$$i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2} : (\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z} \to (\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}$$

is induced by the inclusion of the factor.

The following definition is analogous to Definition 13 (cf Definition 15 of [A1]).

Definition 22. Let a $\mathbb{Z}/2^{[5]}$–framed immersion (h, Λ, ζ_L), $h : L^{n-16k} \looparrowright \mathbb{R}^n$ represent an element $z \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$. We say that this $\mathbb{Z}/2^{[5]}$–framed immersion is an $\mathbf{Q} \times \mathbb{Z}/4$–framed immersion if the following two conditions are satisfied.

–1. The structure mapping $\zeta_L : L^{n-16k} \to K(\mathbb{Z}/2^{[5]}, 1)$ is represented as a composition of a mapping $\zeta_{\mathbf{Q} \times \mathbb{Z}/4} : L^{n-16k} \to K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}), 1)$ and the mapping $\Phi^{[5]} : K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}), 1) \to K(\mathbb{Z}/2^{[5]}, 1)$, this mapping is induced by the homomorphism (151).

–2. The mapping $\bar{\zeta}_{\mathbf{Q} \times \mathbb{Z}/4} : \bar{L}^{n-16k} \to K(\mathbf{I}_{b \times \dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1)$, which is defined as the 8-sheeted covering over the mapping $\zeta_L : L^{n-4k} \to K(\mathbb{Z}/2^{[5]}, 1)$, which (by Condition 1) satisfies Condition 1 of Definition 4, satisfies also Condition 2 of Definition 4.

The cohomology group $H^{16}(K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1); \mathbb{Z}/2)$ contains an element $\tau_{\mathbf{Q} \times \mathbb{Z}/4}$, this element is defined by the following equation (155). Consider the mapping $\Phi^{[5]} : K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}), 1) \to K(\mathbb{Z}/2^{[5]}, 1)$ and take the pull-back $(\Phi^{[5]})^*(\tau_{[5]})$ of the Euler class $\tau_{[5]} \in H^{16}(K(\mathbb{Z}/2^{[5]}), 1); \mathbb{Z}/2)$ of the universal bundle. Define

$$(\Phi^{[5]})^*(\tau_{[5]}) = \tau_{\mathbf{Q} \times \mathbb{Z}/4} \in H^{16}(K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}), 1); \mathbb{Z}/2). \qquad (155)$$

Let us assume that the manifold L^{n-16k} is the self-intersection manifold of a $\mathbb{Z}/2^{[4]}$–framed immersion (h,Λ,ζ_L), and the immersion of this manifold into $\mathbb{R}^n$ is a $\mathbb{Z}/2^{[5]}$–framed immersion which is a $\mathbf{Q}\times\mathbb{Z}/4$–framed immersion.

The mapping $\bar{\zeta}_{b\times\dot{b}}$ is defined as the 8-sheeted covering over the mapping ζ_L with respect to the subgroup $i_{\mathbf{I}_{b\times\dot{b}},\mathbf{Q}\times\dot{\mathbf{J}}_a} : \mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z} \subset (\mathbf{Q}\times\dot{\mathbf{J}}_a)\int_{\chi^{[5]}}\mathbb{Z}$. Over the preimage of this mapping, i.e. over the manifold L^{n-16k}, the corresponding 8-sheeted covering $p_{b\times\dot{b},\mathbf{Q}\times\dot{\mathbf{J}}_a}$ is well-defined. Let us re-denote the characteristic class $\bar{\zeta}_{[2],L}$ by $\bar{\zeta}_{b\times\dot{b}} \in H^2(\bar{L}^{n-16k}_{b\times\dot{b}};\mathbb{Z}/2)$.

Quaternionic-cyclic structure

Let a $\mathbb{Z}/2^{[4]}$–framed immersion (g,Ψ,η_N), $g : N^{n-8k} \looparrowright \mathbb{R}^n$ represent an element $y \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k,8k)$, assuming $n > 16k$. Additionally, let us assume that there exists a mapping

$$\eta_{a\times\dot{a}} : N^{n-8k}_{a\times\dot{a}} \to K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}),1). \qquad (156)$$

Let a $\mathbb{Z}/2^{[5]}$–framed immersion (h,Λ,ζ_L), $h : L^{n-16k} \looparrowright \mathbb{R}^n$, is the self-intersection manifold of the restriction of the immersion g on the component $N^{n-8k}_{a\times\dot{a}}$ (denote this restriction by $g_{a\times\dot{a}}$. Note that the following equation $n-16k = m_\sigma$, where m_σ is defined by the equation (18), is satisfied.

Let us assume that the manifold $L^{n-16k}_{a\times\dot{a}}$ is represented by the following disjoint union of two components:

$$L^{n-16k}_{a\times\dot{a}} = L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup L^{n-16k}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}. \qquad (157)$$

Let as assume that the following mapping

$$\lambda = \zeta_{\mathbf{Q}\times\mathbb{Z}/4} \cup \zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} : L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup L^{n-16k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} \to \qquad (158)$$

$$K((\mathbf{Q}\times\mathbb{Z}/4)\int_{\chi^{[5]}}\mathbb{Z}),1) \cup K((\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2)\int_{\chi^{[5]}}\mathbb{Z}),1).$$

is well-defined.

The following definition is analogous to Definition 20 from [A1].

Definition 23. Let a $\mathbb{Z}/2^{[4]}$–framed immersion (g,Ψ,η_N), equipped with a punctured mapping (156) of a marked component $N^{n-8k}_{a\times\dot{a}} \subset N^{n-8k}$, which is defined the reduction of the characteristic mapping $\eta_{[4],N}$ be given. Assume that the Arf-invariant for (g,Ψ,η_N) is totally determined by means of the marked component.

Let us say that a $\mathbb{Z}/2^{[4]}$–framed immersion (g, Ψ, η_N), admits a $\mathbf{Q} \times \mathbb{Z}/4$–structure if, respectively to the formula (157), the mapping (158) is well-defined, moreover, the pair of mappings $(\eta_{a\times\dot{a}}, \lambda)$ satisfies the following conditions:

–1. The pair of mappings $(\eta_{a\times\dot{a},N}, \zeta_{\mathbf{Q}\times\mathbb{Z}/4})$ are related by the following commutative diagram:

$$\begin{array}{ccccc} \bar{L}^{m_\sigma}_{a\times\dot{a},\mathbf{Q}\times\mathbb{Z}/4} & \looparrowright & N^{n-8k} & \stackrel{\eta_{a\times\dot{a}}}{\longrightarrow} & K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}), 1) \\ \pi_{a\times\dot{a},\mathbf{Q}\times\mathbb{Z}/4} \downarrow & & & & i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,\mathbf{Q}\times\mathbb{Z}/4} \downarrow \\ L^{m_\sigma}_{\mathbf{Q}\times\mathbb{Z}/4} & & & \stackrel{\zeta_{\mathbf{Q}\times\mathbb{Z}/4}}{\longrightarrow} & K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}), 1), \end{array} \qquad (159)$$

where the right vertical mapping $\pi_{a\times\dot{a},\mathbf{Q}\times\mathbb{Z}/4} : \bar{L}^{m_\sigma}_{a\times\dot{a},\mathbf{Q}\times\mathbb{Z}/4} \to L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4}$ is the canonical 2-sheeted covering over the component of self-intersection manifold of the immersion $g_{a\times\dot{a}}$ in the formula (157), the left vertical mapping is induced by the corresponding subgroup.

–2. The pair of mappings $(\eta_{a\times\dot{a}}, \zeta_{L_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}})$ are related by the following commutative diagram:

$$\begin{array}{ccccc} \bar{L}^{m_\sigma}_{a\times\dot{a},\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} & \looparrowright & N^{n-8k} & \stackrel{\eta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,N}}{\longrightarrow} & K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}), 1) \\ \pi_{a\times\dot{a},\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} \downarrow & & & & i_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} \downarrow \\ L^{m_\sigma}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} & & & \stackrel{\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}}{\longrightarrow} & K((\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}), 1), \end{array} \qquad (160)$$

where the left vertical mapping $\pi_{a\times\dot{a},\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} : \bar{L}^{n-16k}_{a\times\dot{a},\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} \to L^{n-16k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$ is the canonical 2-sheeted covering over the component of the self-intersection manifold of the immersion $g_{a\times\dot{a}}$ in the formula (157), the left vertical mapping is induced by the corresponding subgroup.

Example 24. Let a $\mathbb{Z}/2^{[4]}$–framed immersion (g, Ψ, η_N), $g : N^{n-8k} \looparrowright \mathbb{R}^n$ be represented an element $y \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$ and be an $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–framed immersion, (see Definition 13), where $n > 16k$.

Let a $\mathbb{Z}/2^{[5]}$–framed immersion (h, Λ, ζ_L), $h : L^{n-16k} \looparrowright \mathbb{R}^n$, which is the immersion of the self-intersection manifold of (g, Ψ, η_N) be represented the element $z = \delta^{\mathbb{Z}/2^{[5]},k}(y) \in Imm^{\mathbb{Z}/2^{[5]}}(n-16k, 16k)$ and be an $\mathbf{Q} \times \mathbb{Z}/4$–framed immersion (see Definition 22). Assume that the restriction of the reduction mapping of ζ_L and the the restriction of the mapping $\eta_{a\times\dot{a}}$ to the canonical 2-sheeted covering $\bar{L}^{n-16k}$ of L^{n-16k} are homotopic.

Then the $\mathbb{Z}/2^{[4]}$–framed immersion (g,η_N,Ψ), which is equipped with the mapping $\eta_{a\times\dot{a}}$, admits an $\mathbf{Q}\times\mathbb{Z}/4$–structure, given by the reduction $\zeta_{\mathbf{Q}\times\mathbb{Z}/4}$ of the classifying mapping $\zeta_{[5],L}$. The manifold in the decomposition (157) contains the empty second component (cf Example 21 of [A1].

Justification of the example 24

Consider the manifold L^{m_σ} of self-intersection points of the immersion g, given by the formula (157), and define the decomposition of this manifold such that L^{m_σ} coincides with the first component $L^{m_\sigma}_{\mathbf{Q}\times\mathbb{Z}/4}$, i.e. the second component $L^{m_\sigma}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$ is empty. The commutativity of the diagram (159) follows from the diagram (153) and from the definition 22, the diagram (160) is represented by the empty manifold.

The following theorem is analogous to Lemmas 19, 20.

Theorem 25. *Assume that the dimensional restriction is* $n=2^\ell-2$, $\ell\geq 11$, $\sigma\geq 5$ *(in Theorem* 8 *is required* $\ell\geq 8$*). Assume that a* $\mathbb{Z}/2^{[4]}$*–framed immersion* (g,Ψ,η_N) *represents an element* $y\in Imm^{\mathbb{Z}/2^{[4]}}(n-\frac{n-n_\sigma}{2},\frac{n-n_\sigma}{2})$, *and a mapping* (109)*, which determines a* $(\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}$*–reduction of the characteristic mapping* η_N *is given.*

Then the element $\delta_k^{\mathbb{Z}/2^{[5]}}(y)$ *in the group* $Imm^{\mathbb{Z}/2^{[5]}}(n-\frac{n-m_\sigma}{2},\frac{n-m_\sigma}{2})$ *is represented by a* $\mathbb{Z}/2^{[5]}$*–framed immersion* (h,ζ_L,Λ) *and using this immersion a* $\mathbf{Q}\times\mathbb{Z}/4$*–structure of the* $\mathbb{Z}/2^{[4]}$*–framed immersion* (g,η_N,Ψ) *is defined.*

The following corollary is analogous to the Corollary 21.

Corollary 26. *Assume that the hypotheses of Theorem* 8 *hold under the stronger dimensional restrictions from Theorem* 25*. Then for an arbitrary* x *the element*

$$\delta_k^{\mathbb{Z}/2^{[5]}}\circ\delta_k^{\mathbb{Z}/2^{[4]}}\circ\delta_k^{\mathbb{Z}/2^{[3]}}\circ\delta_k^{\mathbb{Z}/2^{[2]}}(x), \tag{161}$$

defined by the composition of homomorphisms (8)*,* $k=\frac{n-n_\sigma}{16}$*, is represented by a* $\mathbb{Z}/2^{[5]}$*–framed immersion* (h,Λ,ζ_L) *and using this immersion a* $\mathbf{Q}\times\mathbb{Z}/4$*–structure of the element* $\delta_k^{\mathbb{Z}/2^{[4]}}\circ\delta_k^{\mathbb{Z}/2^{[3]}}\circ\delta_k^{\mathbb{Z}/2^{[2]}}(x)$ *is defined.*

With respect to the decomposition (157) *the following homology classes are well-defined:*

$$\bar{\zeta}^{loc}_*([\bar{L}_{a\times\dot{a},\mathbf{Q}\times\mathbb{Z}/4}]) \in H_{m_\sigma}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])), \qquad (162)$$

$$\bar{\zeta}^{loc}_*([\bar{L}_{a\times\dot{a},\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}]) \in H_{m_\sigma}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])). \qquad (163)$$

The projection of the element (162) *to the factor* (89) *by the homomorphism* (93) *is expended over the standard basis. This expansion contains not more then one nontrivial element, which is defined by the coefficient of the monomial* $t_{a,i} \otimes t_{\dot{a},i}$*, see.* (88)*,* $i = \frac{m_\sigma}{2} = \frac{n-16k}{2}$*. This coefficient coincides with the characteristic number* (6)*, which is calculated for the* $\mathbb{Z}/2^{[2]}$*–framed immersion* (g,Ψ,η_N)*. The element* (163) *is trivial.*

Proof of Corollary 26

Let us consider the immersion $g_{a\times\dot{a}}$, which is defined as the restriction of the immersion g on the marked component $N^{n-8k}_{a\times\dot{a}} \subset N^{n-8k}$ (Recall, that for the marked component a reduction of the characteristic mapping is given, and the Arf-invariant of the cobordism class x is determined by the cobordism class of the marked component.) Let us apply to the immersion $g_{a\times\dot{a}}$ the Herbert's theorem with $\mathbb{Z}/2[\mathbb{Z}/2]$–local coefficients system, see the analogous construction in the proof of Lemma 15.

From this theorem we get that the sum of the homology classes (162), (163) is equal to the homology class (134). In Corollary 21 the required property for the class (134) is proved. It is sufficiently to prove that the homology class (163) is trivial. Consider the projection $\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2 \to \mathbf{J}_a \times \dot{\mathbf{J}}_a$, which is induced the projection $(\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2)\int_{\chi^{[5]}}\mathbb{Z} \to (\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}$. Consider the corresponding mapping

$$p : K((\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2)\int_{\chi^{[5]}}\mathbb{Z},1) \to K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z},1).$$

The element (163) is obtained from the element

$$p_* \circ \zeta^{loc}_*([L_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}]) \in H_{m_\sigma}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])). \qquad (164)$$

by means of the composition with the canonical 2-sheeted covering. Therefore the element (163) is trivial. Corollary 26 is proved.

5 Solution of the Kervaire invariant problem

In this section we prove the following result.

Main Theorem

There exists a natural number l_0 such that for an arbitrary integer $\ell \ge l_0$ the Kervaire invariant defined by formula (2) is trivial. (Recall, $n = 2^{\ell} - 2$, $\ell \ge 12$, $m_{\sigma} = 2^{\sigma} - 2$, where σ is defined by the equation (16). We may put $m_{\sigma} = 30$, $\sigma = 5$) Assume that an element $x \in Imm^{sf}(n-1,1)$ admits a compression of the order $m_{\sigma}+2$, then $\Theta^{sf}(x) = 0$.

Remark

The Main Theorem could be proved under weaker dimensional assumption then $n \ge 4094$ by the considered approach. See a remark in the Introduction of the part III [A3] of the paper.

Theorem 27. ***Compression Theorem*** *For an arbitrary positive integer d there exists a positive integer $\ell = \ell(d)$ such that for an arbitrary element in the 2-component of the cobordism group $Imm^{sf}(2^{l'}-3,1)$, assuming $l' \ge \ell$, admits a compression of the order $(d-1)$ (see Definition 7).*

Remark

The proof of Theorem 27 is presented in Section 7. By the Pontryagin-Thom construction (in the form by Welles) the cobordism group $Imm^{sf}(2^l - 3,1)$ is isomorphic to the stable homotopy group $\Pi_{2^l-2}(K(\mathbb{Z}/2,1))$. The space $Q(K(\mathbb{Z}/2,1))$ is 2-primary. This implies that the cobordism group $Imm^{sf}(2^l-3,1)$ has no odd torsions. From Theorem 27 an explicit sub-exponential estimation of the dimension $2^{\ell}-2$, $\ell \ge l_0 = l_0(d)$, for which an arbitrary element in $Imm^{sf}(2^l-3,1)$ admits a $d-1$-compression, could be possible. Prof. D. Ravenel in [R] gave an explicit formula for $l_0(d)$ for small d.

To prove the Main Theorem is sufficiently to prove that the residue class in the cobordism group $Imm^{sf}(2^{l'}-3,1)$ which is determined by the non-trivial Arf-invariant, is the empty, or, contains an element x, which admits a compression of the order 16. The condition of a compression of the order q, $3q < 2^{l'} - 2$ of an element x is equivalent to the following condition: the adjoined element $y \in \Pi_{2^{l'}-2}$ to the element x admits a $q+1$–desuspension in the unstable domain, i.e. the stable homotopy class y is represented with sphere-of-origin $2^{l'} - 2 - q$. Accordingly to the result [R-S], if the group Π_{126} contains an element with Kervaire-invariant 1, it will have a representative with sphere-of-origin 116.

Proof of Main Theorem from Corollary 26

The proof is analogous to the proof of Proposition 29 [A1]. Compute a positive integer k from the equation $n - 16k = m_\sigma$, $k \ge 7$, $k \equiv 0 \pmod 2$, this is possible if the condition $\ell \ge 9$ is satisfied. By Theorem (25) we have $\ell \ge 12$. Consider a triple $(f : M^{n-k} \looparrowright \mathbb{R}^n, \kappa, \Xi)$, representing the given element $x \in Imm^{sf}(n-k, k)$.

Consider the element $\delta_k^{\mathbb{Z}/2^{[4]}} \circ \delta_k^{\mathbb{Z}/2^{[3]}} \circ \delta_k^{\mathbb{Z}/2^{[2]}}(x) \in Imm^{\mathbb{Z}/2^{[4]}}(n-8k, 8k)$, see the formula (161), represented by a $\mathbb{Z}/2^{[4]}$–framed immersion (g, η_N, Ψ).

Denote by L^{n-16k} the self-intersection manifold of the immersion g. By Corollary 26 we may assume that the triple (g, η_N, Ψ) admits a $\mathbf{Q} \times \mathbb{Z}/4$–structure (see Definition 23).

Let us assume that the classifying mapping η_N satisfies the condition of Example 24. This means that the following equalities are satisfied:

$$\eta_N = i_{\mathbf{J}_a \times \dot{\mathbf{J}}_a, \mathbb{Z}/2^{[4]}} \circ \eta_{a \times \dot{a}},$$

$$\zeta_L = i_{\mathbf{Q} \times \mathbb{Z}/4, \mathbb{Z}/2^{[5]}} \circ \zeta_{\mathbf{Q} \times \mathbb{Z}/4}.$$

Let us prove the Theorem under this assumption.

Let us denote by $\tilde{N}^{n-8k-2} \subset N^{n-8k}$ the submanifold, representing the Euler class of the vector bundle $\eta^*_{a \times \dot{a}}(\psi_+)$, where by ψ_+ denotes a 2-dimensional vector bundle over the classifying space $K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)$, given by the formula

$$\psi_+ = (\omega^{[4]})^*(\psi_{\mathbb{Z}/4}), \qquad (165)$$

where $\psi_{\mathbb{Z}/4}$ is the universal 2-bundle over $K(\mathbb{Z}/4, 1)$, the mapping $\omega^{[4]}$ is induced by the homomorphism (95). Because the classifying map η admits a $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–reduction, the submanifold $\tilde{N}^{n-2k-2} \subset N^{n-2k}$ is co-oriented (we do not use that k is even).

Let us denote by $\tilde{g} : \tilde{N}^{n-8k-2} \looparrowright \mathbb{R}^n$ the restriction of the immersion g to the submanifold $\tilde{N}^{n-8k-2} \subset N^{n-8k}$, assuming that the immersion $\tilde{g}$ is generic. The normal bundle of the immersion $\tilde{g}$ is the Whitney sum $\nu_g \oplus \tilde{\nu}_{\mathbf{J}_a}$, where ν_g denotes the normal bundle of the immersion g, restricted to $\tilde{N}^{n-8k-2} \subset N^{n-8k}$ (this bundle has the structure group $(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}$, this group determines a reduction the structure group of $\mathbb{Z}/2^{[4]}$–framing Ψ), by $\tilde{\nu}_{\mathbf{J}_a}$ is denoted the normal bundle of the submanifold $\tilde{N}^{n-8k-2} \subset N^{n-8k}$ (this bundle is an $(\omega^{[4]})^*(\mathbb{Z}/4)$-bundle).

Let us denote by $\tilde{L}^{n-16k-4}$ the self-intersection manifold of the immersion $\tilde{g}$. The manifold $\tilde{L}^{n-16k-4}$ is a submanifold of the manifold L^{n-16k}, $\tilde{L}^{n-16k-4} \subset L^{n-16k}$. The immersion $\tilde{h}h|_{\tilde{L}} : \tilde{L}^{n-16k-4} \looparrowright \mathbb{R}^n$ is well-defined.

The normal bundle of this immersion $\tilde{h}$ is isomorphic to the Whitney sum $\nu_h \oplus \tilde{\nu}_{\mathbf{Q}}$, where by ν_h is denoted the restriction of the normal bundle of the immersion h over the submanifold $\tilde{L}^{n-16k-4} \subset L^{n-16k}$, by $\tilde{\nu}_{\mathbf{Q}}$ denotes the normal bundle of the submanifold $\tilde{L}^{n-16k-4} \subset L^{n-16k}$.

Let us repeat the proof of Lemma 7 [A1] (the commutativity of the left square of the diagram (8) of [A1]). This arguments proves that the submanifold $\tilde{L}^{n-16k-4} \subset L^{n-16k}$ represents the Euler class of the bundle $(\zeta^*_{\mathbf{Q}\times\mathbb{Z}/4})^*(\psi_{\mathbf{Q}\times\mathbb{Z}/4})$, where $\psi_{\mathbf{Q}\times\mathbb{Z}/4}$ is the $SO(4)$-bundle over $K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1)$, given by the representation (23) – (25) of [A1]) as follows:

$$\psi_{\mathbf{Q}\times\mathbb{Z}/4} = (\omega^{[5]})^*(\psi_{\mathbf{Q}}), \tag{166}$$

the mapping

$$\omega^{[5]} : K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1) \to K(\mathbf{Q}, 1)$$

corresponds to the epimorphism (149).

Consider 2-sheeted covering $p_{a\times\dot{a},\mathbf{Q}\times\dot{\mathbf{J}}_a} : K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1) \to K((\mathbf{Q} \times \dot{\mathbf{J}}_a) \int_{\chi^{[5]}} \mathbb{Z}, 1)$ over the universal space and denote the bundle $p^*_{a\times\dot{a},\mathbf{Q}\times\mathbb{Z}/4}(\psi_{\mathbf{Q}\times\mathbb{Z}/4})$ by $\psi^!_{\mathbf{Q}\times\mathbb{Z}/4}$, where the bundle $\psi_{\mathbf{Q}\times\mathbb{Z}/4}$ is defined by the formula (166).

For the universal bundle $\psi^!_{\mathbf{Q}\times\mathbb{Z}/4}$ the following formula is satisfied:

$$\psi^!_{\mathbf{Q}\times\mathbb{Z}/4} = \psi_+ \oplus \psi_-,$$

where the bundle ψ_+ (this bundle is given by the formula (165)) admits a lift $\psi_{+,U}$ to a complex $U(1)$–bundle, the bundle ψ_- is an $SO(2)$–bundle, obtained from $\psi_{+,U}$ by means of the complex conjugation and forgetting the complex structure. The bundles ψ_+, ψ_- satisfy the equation: $e(\psi_+) = -e(\psi_-)$.

Let us denote by $m \in H^{16k}(N^{n-8k}; \mathbb{Z}[\mathbb{Z}/2])$ the cohomology class, with local coefficient system (this cohomology class is defined analogously to the homology class (53)), which is dual to the fundamental class of the oriented submanifold $\bar{L}^{n-16k} \subset N^{n-8k}$ in the oriented manifold N^{n-8k}. Let us denote by $e_g \in H^{16k}(N^{n-8k};\mathbb{Z}[\mathbb{Z}/2])$ the Euler class of the normal bundle ν_g of the immersion g. By the analog of the Herbert theorem for the immersion $g : N^{n-8k} \looparrowright \mathbb{R}^n$ with the self-intersection manifold L^{n-16k} (an analogous theorem was formulated in Theorem 1.1 of [E-G], the case $r = 1$, but only with integer coefficients) the following formula is satisfied:

$$e_g + m = 0, \qquad (167)$$

where the both cohomology classes in this formula are defined by means of the $\mathbb{Z}[\mathbb{Z}/2]$-local coefficients system.

Let us denote by $\tilde{m} \in H^{16k-4}(N^{n-8k};\mathbb{Z}[\mathbb{Z}/2])$ the cohomology class, dual to the fundamental class of the oriented submanifold $\bar{\tilde{L}}^{n-16k-4} \subset \tilde{N}^{n-8k-2} \subset N^{n-8k}$ in the oriented manifold N^{n-8k}. Let us denote by $e_{\tilde{g}} \in H^{16k-4}(N^{n-8k};\mathbb{Z}[\mathbb{Z}/2])$ the cohomology class, dual to the Euler class of the normal bundle $\nu_{\tilde{g}}$ of the immersion $\tilde{g}$ on the submanifold $\tilde{N}^{n-8k} \subset N^{n-8k}$.

By the analog of the Herbert theorem for the immersion $\tilde{g} : \tilde{N}^{n-8k} \looparrowright \mathbb{R}^n$ with the self-intersection manifold $\tilde{L}^{n-16k-4}$ the following formula is satisfied:

$$e_{\tilde{g}} + \tilde{m} = 0, \qquad (168)$$

where the both the cohomology classes in this formula are also defined by means of the $\mathbb{Z}[\mathbb{Z}/2]$-local coefficients system.

Because $\bar{\lambda} = \eta_{a \times \dot{a}}$, we may use the equation: $\bar{\lambda}^*(\psi^!_{\mathbf{Q}\times\mathbb{Z}/4}) = \eta^*_{a\times\dot{a}}(\psi_+) \oplus \eta^*_{a\times\dot{a}}(\psi_-)$. The following equation is satisfied:

$$\tilde{m} = m \cdot e(\eta^*_{a\times\dot{a}}(\psi_+)) \cdot e(\eta^*_{a\times\dot{a}}(\psi_-)),$$

where the right side is the product of the three cohomology classes: m and the two Euler classes of the corresponding bundles. The following equation is satisfied: $e_{\tilde{g}} = e_g \cdot e^2(\eta^*_{a\times\dot{a}}(\psi_+))$.

The equation (168) can be rewritten in the following form:

$$e_g \cdot e^2(\eta^*_{a\times\dot{a}}(\psi_+)) + e_g \cdot e(\eta^*_{a\times\dot{a}}(\psi_+)) \cdot e(\eta^*_{a\times\dot{a}}(\psi_-))) = 0. \qquad (169)$$

Then we may take into account (167) and the equation $e(\eta^*_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}(\psi_-)) = -e(\eta^*_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}(\psi_+))$. Let us rewrite the previous formula as follows:

$$2e_g \cdot e^2(\eta^*_{a\times\dot{a}}(\psi_+)) = 0. \qquad (170)$$

Let us prove that the equation (170) implies that the expansion of the expression (162) with respect to the standard base does not involves the generator $t_{a,i} \otimes t_{\dot{a},i}$, see (88), $i = \frac{m_\sigma}{2} = \frac{n-16k}{2}$. (29). The homology class

$[N] \cap e_g$ is equal to the homology class $[\tilde{L}]$, therefore from the expression (170) follows that the homology class $2\eta_{a\times\dot{a},*}([\tilde{L}])$ in the group $H_{m_\sigma-4}(K(\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1); \mathbb{Z}[\mathbb{Z}/2])$ is trivial.

The homology class $\eta_{a\times\dot{a},*}([\tilde{L}])$ is calculated by the formula

$$\eta_{a\times\dot{a},*}([\tilde{L}]) = \eta_{a\times\dot{a},*}([L]) \cap (\omega^{[4],*}[\tau])^2, \qquad (171)$$

where $\omega^{[4]}$ is defined by the formula (95), and where $\tau \in H^2(K(\mathbb{Z}/4, 1); \mathbb{Z})$ is the generator.

Let us consider the expansion of the homology class (171) over the standard basis of the group $H_{m_\sigma}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}); \mathbb{Z}[\mathbb{Z}/2])$, using Lemma 12. All basis elements of the subgroup $D(\mathbf{J}_a \times \dot{\mathbf{J}}_a; \mathbb{Z}[\mathbb{Z}/2]) \subset H_{m_\sigma}(K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}); \mathbb{Z}[\mathbb{Z}/2])$ (elements of this subgroups are detected by means of the monomomorphism (93)) except, probably, the elements $t_{a,\frac{m_\sigma}{2}+4} \otimes t_{\dot{a},\frac{m_\sigma}{2}-4}$ and $t_{a,\frac{m_\sigma}{2}-4} \otimes t_{\dot{a},\frac{m_\sigma}{2}+4}$ are involved with even coefficients.

In the expansion of the element (171) over the standard basis the elements $t_{a,\frac{m_\sigma}{2}} \otimes t_{\dot{a},\frac{m_\sigma}{2}-4}$, $t_{a,\frac{m_\sigma}{2}-4} \otimes t_{\dot{a},\frac{m_\sigma}{2}}$ are involved with coefficients of the same parity as the parity of the corresponding coefficients in the expansion of the homology class (162). By the formula (170) we get that all this two coefficients are even.

By Corollary 26 the characteristic number (6) is trivial. The theorem in the particular case is proved.

Let us prove the theorem in a general case. Let us consider the pair of mappings $(\eta_{a\times\dot{a}}, \lambda)$, where $\eta_{a\times\dot{a}} : N^{n-8k}_{a\times\dot{a}} \to K((\mathbf{J}_a \times \dot{\mathbf{J}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)$, $\lambda = \zeta_{\mathbf{Q}\times\mathbb{Z}/4} \cup \zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} : L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup L^{n-16k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \to K((\mathbf{Q} \times \mathbb{Z}/4) \int_{\chi^{[5]}} \mathbb{Z}, 1) \cup K((\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2) \int_{\chi^{[5]}} \mathbb{Z}, 1)$ and $L^{n-16k} = L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup L^{n-16k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$, where these two mappings are determined by the quaternionic structure of the $\mathbb{Z}/4^{[4]}$–framed immersion (g, η_N, Ψ).

Let us consider the manifold $L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup L^{n-16k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$, defined by the formula (157), and the manifolds $\bar{L}^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4} \cup \bar{L}^{n-4k}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$, which are the canonical 2-sheeted covering manifolds over the components of $L^{n-16k}_{a\times\dot{a}}$.

The formula (167) is satisfied, where the cohomology class m (this class is dual to the fundamental class $[\bar{L}_{a\times\dot{a}}]$ of the submanifold $\bar{L}^{n-16k}_{a\times\dot{a}} \subset N^{n-8k}_{a\times\dot{a}}$) decomposes into the following sum:

$$m = m_{\mathbf{Q}\times\mathbb{Z}/4} + m_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}, \qquad (172)$$

correspondingly to the type of the components of the self-intersection manifold.

Let us consider the submanifold $\tilde{N}^{n-8k-2} \subset N^{n-8k}$, representing the Euler class of the bundle $\eta^*_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}(\psi^+)$. The following immersion $\tilde{g} : \tilde{N}^{n-8k-2} \looparrowright \mathbb{R}^n$ is well defined by the restriction of the immersion $g_{a\times\dot{a}}$ to the submanifold $\tilde{N}^{n-8k-2} \subset N^{n-8k}_{a\times\dot{a}}$. Let us denote by $\tilde{L}^{n-16k-4}$ the self-intersection manifold of the immersion $\tilde{g}$.

The following inclusion $\tilde{L}^{n-16k-4} \subset L^{n-16k}_{a\times\dot{a}}$ is well-defined. In particular, the manifold $\tilde{L}^{n-16k-4}$ is representing by the union of the following two components: $\tilde{L}^{n-16k-4} = \tilde{L}^{n-16k-4}_{\mathbf{Q}\times\mathbb{Z}/4} \cup \tilde{L}^{n-16k-4}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}$. From Lemma 34 of [A1] we will prove the following. The submanifold $\tilde{L}^{n-16k-4}_{\mathbf{Q}\times\mathbb{Z}/4} \subset L^{n-16k}_{\mathbf{Q}\times\mathbb{Z}/4}$ represents the Euler class of the bundle $\zeta^*_{\mathbf{Q}\times\mathbb{Z}/4}(\psi_{\mathbf{Q}\times\mathbb{Z}/4})$.

The submanifold $\tilde{L}^{n-2k-4}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times Z/2} \subset L^{n-2k}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}$ represents the Euler class of the bundle $\zeta^*_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}(\psi_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times i\mathbb{Z}/2})$, where $\psi_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}$ is the universal 4-bundle over the space $K((\mathbf{I}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2)\int_{\chi^{[5]}} \mathbb{Z}, 1)$. (comp. with an analogous definition in the proof of Theorem 12 [A1]).

The cohomology class $\tilde{m}$ is well-defined as in the formula (168), moreover the following formula is satisfied:

$$\tilde{m} = \tilde{m}_{\mathbf{Q}\times\mathbb{Z}/4} + \tilde{m}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}, \tag{173}$$

where the terms in the right side of the formula are defined as the homology classes, dual to the fundamental classes $[\bar{\tilde{L}}_{\mathbf{Q}\times\mathbb{Z}/4}]$, $[\bar{\tilde{L}}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}]$ of the canonical coverings over the corresponding component.

The following formula determines a relation between the cohomology classes $m_{\mathbf{Q}\times\mathbb{Z}/4}$ and $\tilde{m}_{\mathbf{Q}\times\mathbb{Z}/4}$:

$$\tilde{m}_{\mathbf{Q}\times\mathbb{Z}/4} = m_{\mathbf{Q}\times\mathbb{Z}/4} \cdot e(\eta^*_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}(\psi_+)) \cdot e(\eta^*_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}(\psi_-)).$$

The following formula determines a relation between the cohomology classes $m_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$ and $\tilde{m}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$:

$$\tilde{m}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} = m_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} \cdot e^2(\eta^*_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}(\psi_+)).$$

To prove the last formula we use the following formula:

$$\bar{\zeta}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}(\psi^!_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}) = \eta^*_{a\times\dot{a}}(\psi_+ \oplus \psi_+).$$

In this formula the mapping $\eta_{a\times\dot{a}}$ is the restriction of the corresponding mapping to the immersed submanifold $\tilde{\bar{L}}^{n-16k-4} \subset \tilde{N}^{n-8k-2} \subset N^{n-8k}_{a\times\dot{a}}$, $\psi^!_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$ is the pull-back of the universal $SO(4)$–bundle over $K((\mathbf{J}_a \times \dot{\mathbf{J}}_a \times \mathbb{Z}/2)\int_{\chi^{[5]}} \mathbb{Z}, 1)$ to the 2-sheeted covering $K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z}, 1)$. Obviously, the bundle $\psi^!_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$ is the Whitney sum of two copies of the pull-back of the universal $SO(2)$-bundles over $K(\mathbb{Z}/4,1)$ by the mapping $K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z},1) \to K(\mathbb{Z}/4 \times \mathbb{Z}/2,1) \to K(\mathbb{Z}/4,1)$.

The analog of the formula (169) is the following:

$$e_g \cdot e^2(\eta^*_{a\times\dot{a}}(\psi_+)) - m_{\mathbf{Q}\times\dot{\mathbf{J}}_a} \cdot e^2(\eta^*_{a\times\dot{a}}(\psi_+)) + \tag{174}$$

$$m_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2} \cdot e^2(\eta^*_{a\times\dot{a}}(\psi_+)) = 0.$$

Let us multiply both sides of the formula (173) by the cohomology class $e^2(\eta^*_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}(\psi_+))$ and take the sum with the opposite sign with (174), we get:

$$2m_{\mathbf{Q}\times\mathbb{Z}/4} \cdot e^2(\eta^*_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}(\psi_+)) = 0. \tag{175}$$

This is an analog of the formula (170).

Let us prove that the Kervaire invariant of the $\mathbb{Z}/2^{[4]}$–framed immersion (g,Ψ,η) is trivial. The expansion of the element (162) over the standard basis does not involved the monomial $t_{a,i} \otimes t_{\dot{a},i}$, см. (88), $i = \frac{m_\sigma}{2} = \frac{n-16k}{2}$. The proof is analogous to the previous case using Corollary 26 and the formula (175) instead of the formula (170).

The Main Theorem is proved.

6 Proof of Lemmas 19, 20 and Theorem 25

We shall prove first Lemmas 19 and 20. Let us define the positive integer m_σ by the formula (18) (in this section to simplify the calculation of dimensions we assume that $\sigma \ge 6$ All the constructions are well-defined in the case $\sigma = 5$).

Let us denote by $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ the direct product of the two standard lens spaces (mod 4), namely

$$ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} = S^{n-\frac{n-m_\sigma}{8}+9}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{8}+9}/\mathbf{i}. \tag{176}$$

Obviously,

$$2n > \dim(ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}) = \frac{7n+m_\sigma}{4} + 18 > n.$$

On the space $ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ the standard involution $\chi^{[4]} : ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \to ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$ is given by the formula $\chi^{[4]}(x \times y) = (y \times x)$.

Let us define a polyhedron (a submanifold with singularities)

$$X_{a\times\dot{a}} \subset ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}. \tag{177}$$

Let us consider the following family: $\{X_j, \quad j = 0, \dots, j_{max}\}$ of submaniflods $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$:

$$X_0 = S^{n - \frac{n-m_\sigma}{8} + 9}/\mathbf{i} \times S^7/\mathbf{i},$$

$$X_1 = S^{n - \frac{n-m_\sigma}{8} + 1}/\mathbf{i} \times S^{15}/\mathbf{i}, \quad \dots$$

$$X_j = S^{n - \frac{n-m_\sigma}{8} + 9 - 8j}/\mathbf{i} \times S^{8j+7}/\mathbf{i},$$

$$X_{j_{max}} = S^7/\mathbf{i} \times S^{n - \frac{n-m_\sigma}{8} + 9}/\mathbf{i},$$

where

$$j_{max} = \frac{7n + m_\sigma + 16}{64}. \tag{178}$$

The dimension of each submanifold in the family is equal to $n - \frac{n-m_\sigma}{8} + 16$ and the codimension in $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ is equal to $n - \frac{n-m_\sigma}{8} + 2$. Let us define an embedding

$$X_j \subset ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$$

as the product of the two standard inclusions. Obviously, we get $\chi^{[4]}(X_{2j}) = X_{j_{max} - j}$.

The subpolyhedron $X_{a \times \dot{a}} = \bigcup_{j=0}^{j_{max}} X_j \subset ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}$ is well defined. This subpolyhedron is invariant with respect to the involution $\chi^{[4]}$. The restriction of the involution $\chi^{[4]}$ on the subpolyhedron $X_{a \times \dot{a}}$ denote the same.

Let us consider the following sequence of the index 2 subgroups, as in the Diagram (81):

$$\mathbf{I}_{b \times \dot{b}} \stackrel{\mathrm{i}_{b \times \dot{b}, \mathbf{E}_{b \times \dot{b}}}}{\longrightarrow} \mathbf{E}_{b \times \dot{b}} \stackrel{\mathrm{i}_{b \times \dot{b}, \mathbf{J}_a \times \dot{\mathbf{J}}_a}}{\longrightarrow} \mathbf{J}_a \times \dot{\mathbf{J}}_a. \tag{179}$$

Let us define the following tower of 2-sheeted covers, associated with the sequence (179):

$$ZZ_{\mathbf{I}_{b \times \dot{b}}} \longrightarrow ZZ_{\mathbf{E}_{b \times \dot{b}}} \longrightarrow ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a}. \tag{180}$$

The bottom space of the tower (180) is the standard skeleton of the Eilenberg–Mac Lane space: $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \subset K(\mathbf{J}_a, 1) \times K(\dot{\mathbf{J}}_a, 1)$. The tower of 2-sheeted coverings

$$K(\mathbf{I}_{b \times \dot{b}}, 1) \to K(\mathbf{E}_{b \times \dot{b}}, 1) \to K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1),$$

associated with the sequence (179) is well-defined. This tower determines the tower (180) by means of the inclusion $ZZ_{\mathbf{J}_a \times \dot{\mathbf{J}}_a} \subset K(\mathbf{J}_a, 1) \times K(\dot{\mathbf{J}}_a, 1) = K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1)$.

Let us define the following tower of 2-sheeted covers:

$$X_{b\times\dot{b}} \longrightarrow X'_{b\times\dot{b}} \longrightarrow X_{a\times\dot{a}}. \qquad (181)$$

The bottom space of the tower (181) is the subspace of the bottom space of the tower (180), the inclusion is given by the inclusion $X_{a\times\dot{a}} \subset ZZ_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}$, see (177). The tower (181) is defined as the restriction of the tower (180) over this subspace of the bottom.

Let us describe the top polyhedron of the tower (181), which is a subpolyhedron $X_{b\times\dot{b}} \subset ZZ_{\mathbf{I}_{b\times\dot{b}}}$ explicitly. Let us define the family $\{X_0, X_2, \dots, X_{j_{max}}\}$ of the standard submanifolds in $ZZ_{\mathbf{I}_{b\times\dot{b}}} = \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9}$ by the following formulas:

$$X_0 = \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \times \mathbb{R}\mathrm{P}^{7} \subset \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9}, \dots \qquad (182)$$

$$X_j = \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9-8j} \times \mathbb{R}\mathrm{P}^{8j+7} \subset \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9}, \dots$$

$$X_{j_{max}} = \mathbb{R}\mathrm{P}^{7} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \subset \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9}.$$

In this formula j_{max} is defined by the formula (178). The subpolyhedron $X_{b\times\dot{b}} \subset ZZ_{\mathbf{I}_{b\times\dot{b}}}$ is defined as the union of the standard submanifolds in this family. The description of the subpolyhedron $X'_{b\times\dot{b}} \subset ZZ_{\mathbf{E}_{b\times\dot{b}}}$ is obvious and omitted.

On the space $ZZ_{\mathbf{E}_{b\times\dot{b}}}$ (correspondingly, $ZZ_{\mathbf{I}_{b\times\dot{b}}}$) the standard involution is well-defined

$$\chi^{[3]} : ZZ_{\mathbf{E}_{b\times\dot{b}}} \to ZZ_{\mathbf{E}_{b\times\dot{b}}} \qquad (183)$$

(correspondingly,

$$\chi^{[2]} : ZZ_{\mathbf{I}_{b\times\dot{b}}} \to ZZ_{\mathbf{I}_{b\times\dot{b}}}) \qquad (184)$$

by the formula $\chi^{[3]}(x \times y) = (y \times x)$ (correspondingly, $\chi^{[2]}(x \times y) = (y \times x)$). The subpolyhedron $X_{b\times\dot{b}} \subset ZZ_{\mathbf{I}_{b\times\dot{b}}}$ is invariant with respect to the involution $\chi^{[2]}$. The restriction of the involution $\chi^{[2]}$ on $X_{b\times\dot{b}}$) is denoted the same.

Let us define the submanifold $YY_{\mathbf{E}_{b\times\dot{b}}} \subset ZZ_{\mathbf{E}_{b\times\dot{b}}}$ by the following formula:

$$YY_{\mathbf{E}_{b\times\dot{b}}} = (\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9})/\mathbf{i} \subset$$

$$\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}+9})/\mathbf{i} \subset ZZ_{\mathbf{E}_{b\times\dot{b}}}.$$

Let us define the family of the standard submanifolds in $YY_{\mathbf{E}_{b\times\dot{b}}} = (\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9} \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9})/\mathbf{i}$.

$$Y_0 = (\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9} \times \mathbb{R}\mathrm{P}^{7})/\mathbf{i} \subset (\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9}) \times \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{4}+9})/\mathbf{i}, \dots \qquad (185)$$

$$Y_j = (\mathbb{RP}^{n-\frac{n-m_\sigma}{4}+9-8j} \times \mathbb{RP}^{8j+7})/\mathbf{i} \subset (\mathbb{RP}^{n-\frac{n-m_\sigma}{4}+9} \times \mathbb{RP}^{n-\frac{n-m_\sigma}{4}+9})/\mathbf{i}, \dots$$
$$Y_{j'_{max}} = (\mathbb{RP}^7 \times \mathbb{RP}^{n-\frac{n-m_\sigma}{4}+9})/\mathbf{i} \subset (\mathbb{RP}^{n-\frac{n-m_\sigma}{4}+9} \times \mathbb{RP}^{n-\frac{n-m_\sigma}{4}+9})\mathbf{i},$$

where

$$j'_{max} = \frac{3n + m_\sigma + 8}{32}. \qquad (186)$$

In the formula (185) the action of the generator $\mathbf{i} \in \mathbb{Z}/4$ is defined as the diagonal action, which is the standard on the factors.

Define the polyhedron

$$Y_{b\times\dot{b}} \subset YY_{\mathbf{E}_{b\times\dot{b}}} \qquad (187)$$

as the union of the submanifold of the family $\{Y_j\}$.

The polyhedron $Y_{b\times\dot{b}}$ is equipped with the involutions $\chi^{[3]}$. The definition of the involutions is standard.

The maps (188), (189) correspond to the inclusions of the subgroups in (179) and are commuted with the corresponded restrictions of maps in the tower of cover (181).

Let us denote the cylinder of the involution $\chi^{[2]}$ on $X_{b\times\dot{b}}$ by $X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1$.

The following natural inclusion is well defined:

$$\eta_X : X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1 \subset K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}), 1). \qquad (188)$$

The space $Y_{b\times\dot{b}} \int_{\chi^{[3]}} S^1$ and the standard inclusion

$$\eta_Y : Y_{b\times\dot{b}} \int_{\chi^{[3]}} S^1 \subset K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1). \qquad (189)$$

are defined analogously.

Let us define a polyhedron J_X. For an arbitrary $j = 0, 2, \dots, j_{max}$, where j_{max} is given by the formula (178), let us define the polyhedron $J_j = S^{n-\frac{n-m_\sigma}{8}-8j+7} \times S^{8j+7}$ The spheres (the factors of this Cartesian product) $S^{n-\frac{n-m_\sigma}{8}-8j+9}$, S^{8j+7} is re-denoted by $J_{j,1}$, $J_{j,2}$ correspondingly. Using this notation we have:

$$J_j = J_{j,1} \times J_{j,2}.$$

The standard inclusion $i_{J_j} : J_{j,1} \times J_{j,2} \subset S^{\frac{n-m_\sigma}{8}+9} \times S^{\frac{n-m_\sigma}{8}+9}$ is well-defined, each factor embeds in the sphere-image as the standard subsphere. The union $\bigcup_{j=0}^{j_{max}} Im(i_{J_j})$ of the images of these inclusions will be denoted by

$$J_X \subset S^{\frac{n-m_\sigma}{8}+9} \times S^{\frac{n-m_\sigma}{8}+9}. \qquad (190)$$

The polyhedron J_X is well-defined.

Let us define a polyhedron (a manifold with singularities) J_Y, this polyhedron is a subpolyhedron on the polyhedron J_X:

$$\mathrm{i}_{J_Y,J_X} : J_Y \subset J_X. \tag{191}$$

For an arbitrary $j = 0, \dots, j'_{max}$, where j'_{max} is given by the formula (186), let us define the polyhedron $J_j = S^{n-\frac{n-m_\sigma}{4}-8j+9} \times S^{8j+7}$ The spheres (the factors of this Cartesian product) $S^{n-\frac{n-m_\sigma}{4}-8j+9}$, S^{8j+7} are re-denoted by $J_{2j,1}$, $J_{2j,2}$ correspondingly. Using this notation we have again the formula (**??**).

The standard inclusion $\mathrm{i}_{J_j} : J_{j,1} \times J_{j,2} \subset S^{\frac{n-m_\sigma}{4}+9} \times S^{\frac{n-m_\sigma}{4}+9}$ is well-defined, each factor embeds in the sphere-image as the standard subsphere. The union $\bigcup_{j=0}^{j'_{max}} \mathrm{Im}(i_{J_j})$ of the images of these inclusions will be denoted by $J_Y \subset J_X \subset S^{\frac{n-m_\sigma}{8}+9} \times S^{\frac{n-m_\sigma}{8}+9}$. The polyhedron J_Y is well-defined.

Define the standard 4-sheeted covering with ramification

$$\varphi_{X_{d\times\dot{d}}} : X_{b\times\dot{b}} \to J_X \tag{192}$$

the standard 2-sheeted covering with ramification

$$\varphi_{Y_{b\times\dot{b}}} : Y_{b\times\dot{b}} \to J_Y. \tag{193}$$

Consider the $\mathbf{I}_{b\times\dot{b}}$–covering $\bar{X}_{b\times\dot{b}} \to X_{b\times\dot{b}}$. The total space $\bar{X}_{b\times\dot{b}}$ is the union of products of pairs of spheres, each product is the 4-sheeted covering over the corresponding pair of the projective spaces in the formula (182). Define the standard 16-sheeted covering with ramification $\bar{X}_{b\times\dot{b}} \to J_X$. This covering over each product of a pair of spheres is defined as the Cartesian product of joins of the corresponding cyclic $\mathbf{J}_a$ и $\dot{\mathbf{J}}_a$–coverings over S^1. The $\mathbf{I}_{b\times\dot{b}}$–action on $\bar{X}_{b\times\dot{b}}$ is commuted with the covering with ramification, constructed above. Therefore the covering (192) is well defined by factorization of the considered caverning with ramification with respect to considered action. The definition (193) is analogous.

The polyhedra J_X and J_Y are equipped by involutions χ_{J_X}, χ_{J_Y}, which are defined analogously to the involution $\chi^{[2]}$, $\chi^{[3]}$. The cylinders of this involutions are well defined, denote those cylinders by $J_Y \int_\chi S^1$, $J_X \int_\chi S^1$. The covering with ramification (192) commutes with the action of the involutions χ_{J_X}, $\chi^{[2]}$. Therefore the following covering with ramification

$$c_X : X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1 \to J_X \int_\chi S^1, \tag{194}$$

is well defined, this covering admits a 4-sheeted factor, which is denoted by $\hat{c}_X$.

The covering with ramification

$$c_Y : Y_{b\times \dot{b}} \int_{\chi^{[3]}} S^1 \to J_Y \int_{\chi} S^1, \qquad (195)$$

is defined analogously to (194). This covering admits a natural 2-sheeted factor, which is denoted by $\hat{c}_Y$.

Lemma 28. *There exists an embedding*

$$\mathrm{i}_{J_X \int_\chi S^1} : (J_X \int_{\chi} S^1) \times D^{10} \subset D^{n-11} \times S^1 \times D^{10} \subset D^{n-10} \times D^{10} \subset \mathbb{R}^n, (196)$$

where D^i is the standard i-dimensional disk (of a small radius) $D^{n-11} \times S^1 \subset D^{n-10}$ is the standard embedded solid torus.

Proof of Lemma 28

Let us define the collection of $j_{max}+1$ standard coordinate subspaces of dimension $\frac{7n+m_\sigma}{8}+6$ in the Euclidean space $\mathbb{R}^{n-13}$, each coordinate space in the collection contains the origin. Let us consider the Euclidean space $\mathbb{R}^{2n-\frac{n-m_\sigma}{4}+20}$, and let us fix the Cartesian product structure $\mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \times \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10}$. Inside the first factor of this Cartesian product let us take the collection of $\frac{j_{max}}{+}1$ coordinate subspaces V_{2j}, $j=0,\dots,j_{max}$, $\dim(V_j) = 8j+8$. The space V_j is the standard coordinate codimension 2 subspace in the space V_{j+2}, the space $V_{j_{max}}$ coincides with the first factor of the Cartesian product. Inside the second factor of this Cartesian product let us take the collection of $\frac{j_{max}}{+}1$ coordinate subspaces

$$W_{2j}, \quad j=0,2,\dots,j_{max},$$

$\dim(W_j) = n - \frac{n-m_\sigma}{8} + 10 - 8j$. The space W_{j+1} is the standard coordinate subspace in the space W_j of the codimension 8, the space W_0 coincides with the second factor of the Cartesian product.

Let us consider the following collection of subspaces:

$$\{V_j \times W_j \subset \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \times \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10}\}, \qquad (197)$$

$\dim(V_j \times W_j) = \frac{7n+m_\sigma}{8}+18$, which contain the origin. Let us consider the subspace $U = \bigcup_{j=0}^{j_{max}} V_j \times W_j$, $U \subset \mathbb{R}^{\frac{7n+m_\sigma}{4}+20}$, in this formula the union is taken over the spaces of the collection (197).

Take a one-parametric family of orthogonal projections $\pi(t) : \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \times \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \to \mathbb{R}^{n-11}$, $t \in S^1$, which satisfies the following condition.

–1. The following equation $\pi(t+180^o) = I_{antidiag} \circ \pi(t)$ is satisfied, where $t \in S^1$, $I_{antidiag} : \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \times \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \to \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \times \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10}$ is an orthogonal mapping, which is the identity on the diagonal and is antipodal on the antidiagonal.

–2. The kernel $Ker(\pi)$ of the projections $\pi(t)$ for each t is a linear subspace, which is denoted by $L(t) \subset \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10} \times \mathbb{R}^{n-\frac{n-m_\sigma}{8}+10}$, $\dim(L(t)) = \frac{3n+m_\sigma}{4} + 26$. In the family of projections with the boundary condition $\pi(t)$ the space $L(t)$ for an arbitrary t intersects each subspace $V_j \times W_j$ of the collection (197) only at the origin.

Evidently, the family of projections $\pi(t)$, with the required properties exists. For example, we may take first $\pi(0)$ as an embedding on the antidiagonal and is the identity on the diagonal. Because the dimension of the diagonal is even, this condition determines a family $\pi(t)$, which satisfies Condition 1. Then take a small alteration of the family, keeping the boundary condition.

By the general position argument in the case $\sigma \geq 5$ the following equality is satisfied: $\dim(V_j \times W_j) + \dim(L(t)) = \frac{7n+m_\sigma}{8} + 18 + \frac{3n+m_\sigma}{4} + 26 \leq 2(\frac{7n+m_\sigma}{8} + 10) - 1$. Therefore there exists a family $\pi(t)$, which is satisfies Condition 2 for each t.

Let us denote the constructed family of embeddings by $i_U(t) : U \subset \mathbb{R}^{n-11}$. Let us denote the family of embeddings of the standard unite disk, which is associated with the family $i_U(t)$ by $i_{\bar{U}}(t) : \bar{U}(t) \subset D^{n-13}$. In this formula by $\bar{U}(t)$ is defined the union of the image of the standard unite disk with the center at the origin, which is associated with the union $U(t)$ of the vector spaces of the collection (197) for the prescribed value of the parameter t. By the involution $I_{antidiag}$ the induced involution $\bar{I}_{antidiag} : \bar{U}(0) \to \bar{U}(0)$ is well-defined.

Consider the embedding $\mathrm{i} : \mathrm{S}^1 \times \mathrm{D}^{\mathrm{n}-11} \subset \mathbb{R}^{\mathrm{n}-10}$. Take the composition of the one-parameter family of embeddings $i_{\bar{U}}(t)$, $t \in [0, 180^0]$, with the one-parameter family i, this composition determines an embedding, denoted by $i_{\bar{U}}(t)$, $t \in [0, 180^0]$.

The required embedding (196) is defined by the composition

$$\mathrm{i}_J = (\mathrm{i}_U \times \mathrm{id}_{\mathbb{R}^{10}}) \circ (\mathrm{i}_{J,U} \times \mathrm{id}_{D^{10}}) : (J_X \int_\chi S^1) \times D^{10} \subset$$

$$(\bar{U} \int_{\bar{I}_{antidiag}} S^1) \times D^{10} \subset \mathbb{R}^{n-10} \times \mathbb{R}^{10} = \mathbb{R}^n.$$

In this formula $\mathrm{i}_{J_X,U} : J_X \int_\chi S^1 \subset \bar{U} \int_{\bar{I}_{antidiag}} S^1$ is the embedding, which is constructed by means of the collection of the standard embeddings: $\{Z_{j,1} \times Z_{j,2} \subset V_j \times W_j\}$, $\mathrm{id}_{D^{10}} : D^{10} \subset \mathbb{R}^{10}$ is the standard embedding, $\mathrm{id}_{\mathbb{R}^{10}} : \mathbb{R}^{10} \subset \mathbb{R}^{10}$ is the standard diffeomorphism. Lemma 28 is proved.

An $\mathbf{E}_{b\times\dot{b}}$–structure for a formal mapping $d^{(2)}_X$

Let us consider an arbitrary equivariant generic PL–mapping

$$d^{(2)}_X : (X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1)^2 \to \mathbb{R}^n \times \mathbb{R}^n. \qquad (198)$$

Such a (equivariant) mapping let us called a *formal* mapping.

Let us define an (open) polyhedron of a (formal) self-intersection of the mapping $d^{(2)}_X$ as the quotient of the inverse image of the diagonal $diag(\mathbb{R}^n \times \mathbb{R}^n) \subset \mathbb{R}^n \times \mathbb{R}^n$ without points on the diagonal of the origin space. Denote this polyhedron by $\mathbf{N}^{(2)} = \mathbf{N}^{(2)}(d^{(2)}_X)_\circ$. The closure $\mathbf{N}^{(2)}$ of the polyhedron $\mathbf{N}^{(2)}_\circ$ contains boundary, this boundary will be denoted by $\partial \mathbf{N}^{(2)}_X$) (an analogous construction is in [A1], the formula (44)).

In the case the mapping (198) is defined by the extension of a PL-mapping

$$d_X : X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1 \to \mathbb{R}^n, \qquad (199)$$

the polyhedron $\mathbf{N}^{(2)}_\circ$ coincides with the polyhedron $\mathbf{N}(d_X)$ of self-intersection of the mapping d_X. In this case the boundary $\partial(Cl(\mathbf{N}^{(2)}))$ coincides with the polyhedron of critical points of the mapping d_X.

Consider the subpolyhedron

$$X_{b\times\dot{b}} \subset X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1, \qquad (200)$$

which is defined as the fiber over the marked point of the fibration $X_{b\times\dot{b}} \int_{\chi} S^1 \to S^1$. The restriction of the equivariant mapping (198) to the subpolyhedron (200) denote by

$$d^{[2]}_X : X^2_{b\times\dot{b}} \to \mathbb{R}^n \times \mathbb{R}^n. \qquad (201)$$

The self-intersection polyhedron of the formal mapping $d^{[2]}_X$ denote by $\mathbf{N}^{[2]} = \mathbf{N}^{[2]}(d^{[2]}_X)_\circ \subset \mathbf{N}^{(2)}(d^{(2)}_X)_\circ$.

Suppose the polyhedron $\mathbf{N}^{(2)}_\circ$ contains a marked closed component, which is denoted by

$$\mathbf{N}^{(2)}_{\mathbf{E}_{b\times\dot{b}}} \subset \mathbf{N}^{(2)}(d^{(2)}_X)_\circ. \qquad (202)$$

Then the polyhedron $\mathbf{N}^{[2]}_\circ$ also contains a marked closed, which is denoted by

$$\mathbf{N}^{[2]}_{\mathbf{E}_{b\times\dot{b}}} \subset \mathbf{N}^{[2]}(d^{[2]}_X)_\circ. \qquad (203)$$

The following inclusion

$$\mathbf{N}^{[2]}_{\mathbf{E}_{b\times\dot{b}}} \subset \mathbf{N}^{(2)}_{\mathbf{E}_{b\times\dot{b}}}$$

is well-defined.

The structure mapping

$$\zeta_\circ : \mathbf{N}^{(2)}_\circ \to K(\mathbb{Z}/2^{[3]}, 1),$$

which is analogous to the structure mapping ([A1], formula (43) is well-defined. Let us consider the restriction of the structure mapping $\zeta_\circ$ on the marked component of the self-intersection polyhedron:

$$\zeta : \mathbf{N}^{(2)}_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbb{Z}/2^{[3]}, 1).$$

Assume that this mapping admits a reduction, given by a mapping

$$\zeta_{\mathbf{E}_{b\times\dot{b}}\int} : \mathbf{N}^{(2)}_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbf{E}_{b\times\dot{b}} \int_{\chi^{[3]}} \mathbb{Z}, 1). \qquad (204)$$

In this case the restriction of the structure mapping

$$\zeta : \mathbf{N}^{[2]}_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbb{Z}/2^{[3]}, 1)$$

on the marked component (203) admits a reduction by the mapping

$$\zeta_{\mathbf{E}_{b\times\dot{b}}} : \mathbf{N}^{[2]}_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbf{E}_{b\times\dot{b}}, 1). \qquad (205)$$

Generic homology classes

Let us formulate a homological condition. Let (s_1, s_2) be an an arbitrary pair of integers satisfying the following conditions: $s_1 = 1 \pmod 2$, $s_2 = 1 \pmod 2$, $s_1 + s_2 = n - \frac{n-m_\sigma}{8}$.

For an arbitrary pair (s_1, s_2), described below, we shall construct a manifold $X(s_1, s_2) = \mathbb{RP}^{s_1} \times \mathbb{RP}^{s_2}$ and consider the embedding $i_{X(s_1,s_2)} : X(s_1, s_2) \subset X_{b\times\dot{b}} \subset X_{b\times\dot{b}} \int_\chi S^1$, which is defined as the extension of the embedding $X(s_1, s_2) \subset X_{b\times\dot{b}}$, which is the Cartesian product of the two standard coordinate inclusions

By the construction for each pair (s_1, s_2) the following inclusion

$$\mathrm{i}_{X(s_1,s_2)} : X(s_1, s_2) \subset X_{b\times\dot{b}} \stackrel{\eta_{X_{b\times\dot{b}}}}{\longrightarrow} K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1) \qquad (206)$$

is well-defined.

Additionally, define the manifold $\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}-1} \times S^1$, which we denote by X_∞ and define the embeddings

$$i_{X_\infty} : X_\infty \subset X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1, \qquad (207)$$

$$\mathrm{i}_{X_\infty} : X_\infty \subset X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1 \stackrel{\eta_{X_{b\times\dot{b}}}}{\longrightarrow} K(\mathbf{I}_{b\times\dot{b}} \int_{\chi^{[2]}} \mathbb{Z}, 1). \qquad (208)$$

Consider the standard 2-sheeted covering $p_{X_{b\times\dot{b}}\times S^1} : X_{b\times\dot{b}} \times S^1 \to X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1$, for which the embedded circle $pt \times S^1 \subset X_{b\times\dot{b}} \times S^1$ is mapped to the circle $p_{X_{b\times\dot{b}}\times S^1}(pt\times S^1)$, the image of this circle by the standard projection $X_{b\times\dot{b}} \int_{\chi} S^1 \to S^1$ is projected as the standard 2–sheeted covering $S^1 \to S^1$. Define the embedding $\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}-1} \times S^1 \subset X_{b\times\dot{b}} \times S^1$ as the product of the standard embedding $\mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}-1} \subset X_{b\times\dot{b}}$ on the first factor with the identity mapping $S^1 \to S^1$. Define the mapping $i_{X_\infty} : \mathbb{R}\mathrm{P}^{n-\frac{n-m_\sigma}{8}-1} \times S^1 \to X_{d\times\dot{d}} \int_{\chi} S^1$ as the composition of this embedding with the projection $p_{X_{d\times\dot{d}}\times S^1}$. The embedding (207) is well-defined.

Analogously to the case of framing immersions (see Definition of the homology class (53)) let us prove that the images of the fundamental classes by means of the mappings (206), (207) determine the following homology classes

$$\{[X(s_1,s_2)] = \mathrm{i}_{X(s_1,s_2);*}[X(s_1,s_2)], \quad [X_\infty] = i_{X_\infty,*} : [X_\infty] \in \qquad (209)$$

$$H_{n-\frac{n-m_\sigma}{8}}(X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1; \mathbb{Z}/2[\mathbb{Z}/2]).$$

Consider the following compositions:

$$\{\mathrm{i}_{J_X \int_\chi S^1} \circ \mathrm{i}_{X(s_1,s_2)} : X(s_1,s_2) \to X_{b\times\dot{b}} \subset X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1 \to S^1 \times D^{n-1} \subset \mathbb{R}^n\},$$

$$\{\mathrm{i}_{J_X \int_\chi S^1} \circ \mathrm{i}_{X_\infty} : X_\infty \to X_{b\times\dot{b}} \to X_{b\times\dot{b}} \int_{\chi^{[2]}} S^1 \to S^1 \times D^{n-1} \subset \mathbb{R}^n\}.$$

Assume that the formal mapping (201) is the formal extension of a mapping $d_{X_{b\times\dot{b}} \int_\chi S^1}$. In this case the image of the polyhedron $J_X \int_\chi S^1$ by this mapping is a immersed framed manifold with singularities. This observation allow to define the elements (209) analogously to the regular case. For a formal mapping (201) this construction is analogous.

The homology class

$$[X_\infty] \in H_{n-\frac{n-m_\sigma}{8}}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])$$

is not in the subgroup (38), the last classes of the collection (209) belong to this subgroup. The collection (209) corresponds to the standard basis of the subgroup

$$\mathrm{Im}(H_i(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}[\mathbb{Z}/2]) \to H_i(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2])),$$

which is described in Lemma 3.

Assuming that the formal mapping (201) is holonomic. In this case each manifold $X(s_1,s_2)$ determines the polyhedron of the self-intersection of the mapping $d_{X_{b\times\dot{b}}\int_\chi S^1}\circ \mathrm{i}_{X(s_1,s_2)}$. In a general case the analogous polyhedron of the is well-defined, denote this polyhedron by $NX(s_1,s_2)_\circ = NX(s_1,s_2)(d^{[2]})_\circ$. By the construction the following inclusion is well-defined:

$$NX(s_1,s_2)_\circ \subset N(d^{[2]}_X)_\circ. \tag{210}$$

Define the closed subpolyhedron

$$NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}} \subset N^{[2]}_{\mathbf{E}_{b\times\dot{b}}}; \quad NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}} \subset NX(s_1,s_2)_\circ \tag{211}$$

by the intersection of the polyhedron (210) with the component (203).

The restriction $\zeta_{b\times\dot{b}}|_{NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}}}$ of the structure mapping (205) to the subpolyhedron (211) denote by

$$\zeta_{\mathbf{E}_{b\times\dot{b}},NX(s_1,s_2)} : NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}} \to K(\mathbf{E}_{b\times\dot{b}},1). \tag{212}$$

Consider the fundamental class

$$[NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}}] \in H_{\dim(NX)}(NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}}),$$

where $\dim(NX(s_1,s_2)) = n - \frac{n-m_\sigma}{4}$.

The manifold $X(s_1,s_2)$ is oriented and the codimension of the formal mapping $d^{[2]}_{X(s_1,s_2)}$ is even. Therefore the following collection of the homology classes is well-defined:

$$\{\zeta_{*,\mathbf{E}_{b\times\dot{b}}}([NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}}]) \in \tag{213}$$

$$\mathrm{Im}D_{n-\frac{n-m_\sigma}{4}}(\mathbf{E}_{b\times\dot{b}};\mathbb{Z}[\mathbb{Z}/2]) \to D_{n-\frac{n-m_\sigma}{4}}(\mathbf{E}_{b\times\dot{b}};\mathbb{Z}/2[\mathbb{Z}/2])\}.$$

The transfer homomorphism

$$! : D_{n-\frac{n-m_\sigma}{4}}(\mathbf{E}_{b\times\dot{b}};\mathbb{Z}[\mathbb{Z}/2]) \to D_{n-\frac{n-m_\sigma}{4}}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}[\mathbb{Z}/2]), \qquad (214)$$

which is associated with the double covering $K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1) \to K(\mathbf{E}_{b\times\dot{b}}\int_{\chi^{[3]}}\mathbb{Z},1)$ is well-defined. The collection of permanent homology classes (modulo 2)

$$\zeta^!_{\mathbf{E}_{b\times\dot{b}},*}([NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}}]) \in D_{n-\frac{n-m_\sigma}{4}}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}/2[\mathbb{Z}/2])\} \qquad (215)$$

is well-defined.

Define an extra collection of homology classes of the group $D_{n-\frac{n-m_\sigma}{4}}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}/2[\mathbb{Z}/2])$. Let us consider the polyhedron (210), generally speaking, with a boundary. The standard compactification $\bar{N}X(s_1,s_2)$ of the canonical 2-sheeted covering $\overline{NX}(s_1,s_2)$ over $\overline{NX}(s_1,s_2)$ is well-defined. The polyhedron $\overline{NX}(s_1,s_2)$ contains the following closed subpolyhedron $\overline{NX}(s_1,s_2)_{b\times\dot{b}}$ as a marked component. The polyhedron $\overline{NX}(s_1,s_2)$ itself, is not closed, is equipped with the natural embedding into the polyhedron $X(s_1,s_2)\subset X_{b\times\dot{b}}$.

Consider the mapping

$$\eta_{X_{b\times\dot{b}}} : X_{b\times\dot{b}} \to K(\mathbf{I}_{b\times\dot{b}},1), \qquad (216)$$

which is defined by means of the mapping (188). The induced mapping $\eta_{X_{b\times\dot{b}}}|_{C\overline{NX}(s_1,s_2)}$ determines the permanent homology class

$$\eta_{b\times\dot{b},*}([C\overline{NX}(s_1,s_2)]) \in D_{\dim(NX))}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}/2[\mathbb{Z}/2]). \qquad (217)$$

It is easy to calculate $\dim(NX)) = n - \frac{n-m_\sigma}{4}$. Obviously, the homology class (217) is defined as the reduction modulo 2 of the corresponding integer homology class.

The following equation for each pair (s_1,s_2) is well defined:

$$\eta_{b\times\dot{b},*}([C\overline{NX}(s_1,s_2)]) = \zeta^!_{\mathbf{E}_{b\times\dot{b}},*}([NX(s_1,s_2)_{\mathbf{E}_{b\times\dot{b}}}]). \qquad (218)$$

This equation determines the identity of the two homology classes (217) and (215) in the group $D_{\dim(NX)}(\mathbf{I}_{b\times\dot{b}}\mathbb{Z}/2[\mathbb{Z}/2])$.

Definition 29. Let us say that the formal mapping $d^{(2)}_X$ (198) admits an $\mathbf{E}_{b\times\dot{b}}$–structure, if there exists a mapping (204), which determines a reduction of the restriction of the structure mapping on the marked component. Moreover, the restriction (205) of the mapping (204) is elated with the mapping (188) by the family of equations (218).

Proposition 30. *There exists a formal mapping* (198)*, for which the polyhedron of the formal self-intersection contains a closed component* (202 *along which the self-intersection is holonomic. An* $\mathbf{E}_{b\times\dot{b}}$*–structure in the sense of Definition* 29 *is well-defined, which is given by the mapping* (205)*.*

The sketch of the proof of Proposition 30 is in [A3,Section 7].

An $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure for a formal mapping $d^{(2)}_Y$

Definition of a $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure is analogous to Definition (29) of an $\mathbf{E}_{b\times\dot{b}}$–structure. In this definition polyhedra are replaced by and their factors, and structure mappings are replaced corresponding quadratic extension.

Definition 31. Let us say that a formal mapping

$$d^{(2)}_Y : (Y_{b\times\dot{b}} \int_{\chi^{[3]}} S^1)^2 \to \mathbb{R}^n \times \mathbb{R}^n \qquad (219)$$

with holonomic self-intersection along a closed marked component

$$\mathbf{N}^{(2)}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \subset \mathbf{N}^{(2)}(d^{(2)}_Y)_\circ \qquad (220)$$

admits an $\mathbf{J}_a \times \dot{\mathbf{J}}_a$–structure, if there exists a mapping

$$\zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} : \mathbf{N}^{[2]}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a} \to K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1), \qquad (221)$$

which determine a reduction of the restriction of the structure mapping on the marked component, and which is related with the mapping (189) by the following family of equations:

$$\eta_{b\times\dot{b},*}([C\overline{NY}(s_1,s_2)]) = \zeta^{!}_{\mathbf{J}_a\times\dot{\mathbf{J}}_a,*}([NY(s_1,s_2)_{\mathbf{J}_a\times\dot{\mathbf{J}}_a}]).$$

Proposition 32. *There exists a formal mapping* (219)*, for which the polyhedron of formal self-intersection contains a closed marked component* (220)*.Additionally, a* $\mathbf{J}_a \times \dot{\mathbf{J}}_a$*–structure in the sense of Definition* 31 *is well-defined by the mapping* (221)*.*

Construction of $\mathbb{Z}/2^{[2]}$–framed immersion, which admits an $\mathbf{E}_{b\times\dot{b}}$–structure, proof of Lemmas 19 and 20

Let us prove Lemma 19. Let $y = [(g,\eta_N,\Psi)]$ be given, where g is a $\mathbb{Z}/2^{[2]}$–framed immersion, $g : N^{n-\frac{n-m_\sigma}{8}} \looparrowright \mathbb{R}^n$, $\eta_N : N^{n-\frac{n-m_\sigma}{8}} \to K(\mathbb{Z}^{[2]},1)$ is the characteristic class of the $\mathbb{Z}/2^{[2]}$–framing Ψ.

By the condition of the lemma there exists a mapping

$$\eta_{b\times\dot{b}} : N^{n-\frac{n-m_\sigma}{8}} \to K((\mathbf{I}_{b\times\dot{b}})\int_{\chi^{[2]}}\mathbb{Z},1), \tag{222}$$

which determines a reduction of the characteristic mapping η_N. This mapping satisfies the Conditions 1 and 2 of Definition 14. The mapping $\eta_{b\times\dot{b}}$ determines (up to a homotopy) the mapping

$$\eta_{b\times\dot{b},X} : N^{n-\frac{n-m_\sigma}{8}} \to X_{b\times\dot{b}}\int_{\chi^{[2]}} S^1, \tag{223}$$

because the polyhedron $X_{b\times\dot{b}}\int_{\chi^{[2]}} S^1$ is a subspace of the Eilenberg-Mac Lane space $K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z},1)$, and this subspace contains the standard skeleton of the dimension $n-\frac{n-m_\sigma}{8}+1 = \dim(N)+1$.

Analogously to [Theorem 23,A1] let us define a $\mathbb{Z}/2^{[2]}$-framed immersion (g,Ψ,η_N), which is determined in the group $Imm^{\mathbb{Z}/2^{[2]}}(n-\frac{n-m_\sigma}{8},\frac{n-m_\sigma}{8})$ the given element y. Additionally, the manifold N^{n-2k} contains a closed marked component $N^{n-2k}_{b\times\dot{b}} \subset N^{n-2k}$, and the self-intersection manifold $L^{n-\frac{n-m_\sigma}{4}}_{b\times\dot{b}}$ of the restriction immersion $g_{b\times\dot{b}} = g|_{N^{n-2k}_{b\times\dot{b}}}$, contains a closed component $L^{n-\frac{n-m_\sigma}{4}}_{\mathbf{E}_{b\times\dot{b}}}$, which is the first component in the formula (114).

By geometrical arguments a natural projection

$$L^{n-4k}_{\mathbf{E}_{b\times\dot{b}}} \to N^{(2)}_{\mathbf{E}_{b\times\dot{b}}}, \tag{224}$$

$n-4k = n-\frac{n-m_\sigma}{4}$. is well-defined.

Define the mapping (108), which is required in Definition 14. Consider the mapping (204), which is constructed in Proposition 30 and define the mapping (108) by the composition of the mapping (224) with (204).

The following Lemma is analogous to Lemma 33 in [A1].

Lemma 33. *Assume $x \in Imm^{[2]}(n-2k,2k)$ be an arbitrary element, which is represented by a triple $(g',\eta_{N'},\Xi')$. (We will use the assumption that the characteristic mapping $\eta_{N'}$ admits an $\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}$–reduction, given a mapping*

$\eta_{b\times\dot b}$.) Let $y \in Imm^{[3]}(n-4k,4k)$ be an arbitrary element, which is represented by a triple (h,ζ_L,Λ). Then there exists another triple (g,η_N,Ξ), $g : N^{n-2k} \looparrowright \mathbb{R}^n$, which represents an element x, for which the immersion g is self-transversal. The self-intersection manifold of g, is an $\mathbb{Z}/2^{[3]}$–immersed manifold, represented the element $\delta^{[3]}(x)$, contains a closed component L^{n-4k}. Moreover, the characteristic mapping ζ_L admits an $\mathbf{E}_{b\times\dot b}\int_{\chi^{[3]}}\mathbb{Z}$–reduction by a mapping $\zeta_{\mathbf{E}_{b\times\dot b}}$, for which the canonical 2-sheeted covering mapping $\bar\zeta_{\mathbf{E}_{b\times\dot b}} : \bar L^{n-4k} \to K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1)$ is induced from the mapping $\eta_{b\times\dot b}$ by the immersion $\bar L^{n-4k} \looparrowright N^{n-2k}$.

By construction the immersion $g|_{VN^{n-2k}}$, and therefore, the immersion (**??**) is a $\mathbb{Z}/2^{[2]}$–framed immersion, and its characteristic mapping admits a $\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z}$–reduction. Let us apply Lemma 33 and construct a $\mathbb{Z}/2^{[2]}$–framed immersion (g,η_N,Ψ), for which the characteristic mapping admits an $\mathbf{I}_{b\times\dot b}\int_{\chi^{[3]}}\mathbb{Z}$ – reduction, and for which the $\mathbb{Z}^{[3]}$–framed immersion of the self-intersects manifold contains a closed component (224), for which the mapping (108) determines an $\mathbf{E}_{b\times\dot b}$–reduction of the characteristic mapping.

Let us check that the immersion (g,η_N,Ψ) admits an $\mathbf{E}_{b\times\dot b}$–structure. Let us check Condition 3 in Definition П 14. The proof of this condition is analogous to the calculation of the degree of the mapping κ_0 in the proof of [Proposition 28, A1].

Let us decompose the image of the fundamental class of the manifold $N^{n-2k}_{b\times\dot b}$ by the mapping (223) over the basic homology classes (209). The homology class $[X_\infty]$ is not required, because of Property 2 of Definition 14.

Each homology class (209) satisfies the equation (218). Therefore the image of the fundamental class by the mapping (223) also satisfies by the same relation (123). The Condition 3 from Definition 14 is proved.

Condition 1 follows by construction, because the manifold N^{n-2k} contains the only component $N^{n-2k}_{b\times\dot b}$. Let us proof Condition 2 from Definition 14.

Consider the restriction of the mapping (188) on the subpolyhedron X_∞, which is defined by the formula (207). Consider the mapping

$$\mathrm{i}_{X(s_1,s_2)} : X_\infty \to K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1),$$

which is defined by the formula (206). Consider the image of the homology class

$$\mathrm{i}_{X(s_1,s_2),*}([((\bar w_{\frac{n-m_\sigma}{8}})^7)^{op}]) \in H_{m_\sigma}(K(\mathbf{I}_{b\times\dot b}\int_{\chi^{[2]}}\mathbb{Z},1);\mathbb{Z}/2[\mathbb{Z}/2]), \qquad (225)$$

where by $\bar w_{\frac{n-m_\sigma}{8}}$ is denoted the normal Stiefel-Whitney class of the manifold X_∞ of the dimension $\frac{n-m_\sigma}{8}$.

Lemma 34. *The homology class* (225) *is not in the subgroup* $D_{m_\sigma}(\mathbf{I}_{b\times\dot{b}};\mathbb{Z}/2[\mathbb{Z}/2]) \subset H_{m_\sigma}(K(\mathbf{I}_{b\times\dot{b}}\int_{\chi^{[2]}}\mathbb{Z}),1);\mathbb{Z}/2[\mathbb{Z}/2])$*, in particular, this homology class in non-trivial.*

Proof of Lemma 34

Let us prove that the total obstruction (44) for the homology class (225) is non-trivial. Apply to the considered homology class the forgetful homomorphism (43) and consider this homology class with local coefficients as an integer homology class.

Let us calculate the homology class (225). At the firs step let us calculate the homology class

$$((\bar{w}_{\frac{n-m_\sigma}{8}})^7)^{op} \in H_{m_\sigma}(\mathbb{RP}^{n-\frac{n-m_\sigma}{8}-1}\times S^1;\mathbb{Z}/2[\mathbb{Z}/2]). \qquad (226)$$

A direct calculation shows (in this calculation the inequality $\sigma \ge 5$ is used: the integer $\frac{n-m_\sigma}{8}$ is divided by 4), that the cohomology class $\bar{w}_{\frac{n-m_\sigma}{8}} \in H^{\frac{n-m_\sigma}{8}}(\mathbb{RP}^{n-\frac{n-m_\sigma}{8}-1}\times S^1;\mathbb{Z}/2)$ is the generic class of the first factor. Therefore, the cohomology class $(\bar{w}_{\frac{n-m_\sigma}{8}})^7$ is also the generic class of the first factor. Consider the homology class (226) and let us divide this class by the 1-dimensional cohomology class, which is induced from the generic cohomology class $H^1(S^1;\mathbb{Z})$ by the standard projection $X_\infty \to K((\mathbf{I}_b\times\dot{\mathbf{I}}_b)\int_{\chi^{[2]}}\mathbb{Z}) \to S^1$. Consider the image of this class by the mapping $\mathrm{i}_{X(s_1,s_2)}$. This image coincides with the class (29) for $i = m_\sigma - 1$ and is non-trivial. Lemma 34 is proved.

The last step of the proof of Lemma 19**. The proof of Property 2 in Lemma** 15

Let us assume that the homology class (110) is not in the subgroup (92), $i = n-4k$. This means that the image of the fundamental class of the manifold $N^{n-2k}_{b\times\dot{b}}$ by the mapping (223) is decomposed over the basis of this homology group, such that the basic class $[X_\infty]$ is not involved. Then by Lemma 34, Property 2 of Definition 14 is not satisfied. Statement 2 of Lemma 15 is proved. Lemma 19 is proved.

The proof of Lemma 20 is analogous.

Proof of Theorem 25

Let us start to prove Theorem 25. Let us define the space

$$ZZ_{\mathbf{I}_a\times\dot{\mathbf{I}}_a} = S^{\frac{n+m_\sigma}{2}+1}/\mathbf{i}\times S^{\frac{n+m_\sigma}{2}+1}/\mathbf{i}. \qquad (227)$$

(The space (176) is not used below). Obviously, $\dim(ZZ_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}) = n+m_\sigma+2 > n$.

Let us define a family $\{Z_1,\dots,Z_{j_{max}}\}$, of standard submanifolds in the manifold $ZZ_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}$, where

$$j_{max} = \frac{n+m_\sigma+4}{m_\sigma+2}, \qquad (228)$$

by the following formula:

$$Z_1 = S^{n-\frac{n-m_\sigma}{2}+1} \times S^{\frac{m_\sigma}{2}}/\mathbf{i} \subset S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i}, \dots \qquad (229)$$

$$Z_j = S^{n-\frac{n-m_\sigma}{2}+1-(\frac{m_\sigma}{2}+1)j}/\mathbf{i} \times S^{(\frac{m_\sigma}{2}+1)j-1}/\mathbf{i} \subset S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i}, \dots$$

$$Z_{j_{max}} = S^{\frac{m_\sigma}{2}}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i} \subset S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i} \times S^{n-\frac{n-m_\sigma}{2}+1}/\mathbf{i}.$$

In the formulas j_{max} is defined by the formula (228). The subpolyhedron

$$Z_{a\times\dot{a}} \subset ZZ_{\mathbf{I}_a\times\dot{\mathbf{I}}_a} \qquad (230)$$

is defined as the union of the standard submanifolds of the family (229). Obviously, $\dim(Z_{a\times\dot{a}}) = \frac{n+2m_\sigma+2}{2}$.

The standard involution

$$\chi^{[4]} : ZZ_{\mathbf{I}_a\times\dot{\mathbf{I}}_a} \to ZZ_{\mathbf{I}_a\times\dot{\mathbf{I}}_a}, \qquad (231)$$

id defined analogously to the involution (184), (183). The subpolyhedron (230) is invariant with respect to the involution (231).

Let us consider the standard cell decomposition of the space $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1) = K(\mathbf{I}_a, 1) \times K(\dot{\mathbf{I}}_a, 1)$, this decomposition is defined as the Cartesian product of the standard cell decompositions of the factors.

The following standard inclusion is well defined

$$Z_{a\times\dot{a}} \subset K(\mathbf{J}_a \times \dot{\mathbf{J}}_a, 1). \qquad (232)$$

The skeleton (232) is invariant with respect to the involution (231). Therefore the inclusion

$$Z_{a\times\dot{a}} \int_{\chi^{[4]}} S^1 \subset Z_{a\times\dot{a}} \int_{\chi^{[4]}} S^1 \subset K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1), \qquad (233)$$

which is an analog of the inclusions (188), (189) is well defined.

Let us define a polyhedron J_Z, this polyhedron is the analog of the polyhedron J_X, see (190). Let us denote by $JJ_{\mathbf{I}_a}$ the space of join of $j_{max}+1$ copies of the standard lens spaces $S^{\frac{m_\sigma}{2}}/\mathbf{i}$. Let us denote by $JJ_{\mathbf{I}_a}$ the space of join of $j_{max}+1$ copies of the standard lens spaces $S^{\frac{m_\sigma}{2}}/\mathbf{i}$. In this formulas j_{max} is defined by (228).

Define a subpolyhedron $J_{\bar{Z},j} \subset JJ_{\mathbf{I}_a} \times JJ_{\dot{\mathbf{I}}_a}$ by the formula

$$J_{Z,j} = JJ_{\mathbf{Q},j} \times JJ_{\mathbf{I}_a,j}, 1 \le i \le j_{max},$$

where $JJ_{\mathbf{I}_a,j} \subset JJ_{\mathbf{I}_a}$ is the subjoin with the coordinates $0 \le i \le j_{max} - j$, $JJ_{\mathbf{I}_a,i} \subset JJ_{\mathbf{I}_a}$ is the subjoin with the coordinates $1 \le i \le j$. Define a subpolyhedron $J_{Z,j} \subset JJ_{\mathbf{Q}} \times JJ_{\dot{\mathbf{Q}}}$ by the formula

$$JJ_{Z,j} = JJ_{\mathbf{Q},j} \times JJ_{\dot{\mathbf{Q}},j}, 1 \le i \le j_{max},$$

where $JJ_{\mathbf{Q},j} \subset JJ_{\mathbf{Q}}$ is the subjoin with the coordinates $0 \le i \le j_{max} - j$, $JJ_{\dot{\mathbf{Q}},j} \subset JJ_{\dot{\mathbf{Q}}}$ is the subjoin with the coordinates $1 \le i \le j$.

Let us define J_Z by the formula:

$$J_Z = \bigcup_{j=1}^{j_{max}} J_{Z,j} \subset JJ_{\mathbf{Q}} \times JJ_{\mathbf{I}_a}. \qquad (234)$$

Let us define $J_{\bar{Z}}$ by the formula:

$$J_{\bar{Z}} = \cup_{j=1}^{j_{max}} J_{\bar{Z},j} \subset JJ_{\mathbf{I}_a} \times JJ_{\dot{\mathbf{I}}_a}. \qquad (235)$$

On the polyhedra $Z_{\mathbf{I}_a \times \dot{\mathbf{I}}_a}$, J_Z a free involutions are well defined. The both involutions are denoted by $T_{\mathbf{Q}}$, the involutions corresponds to the quadratic extension (135). The polyhedron (234) is invariant with respect to the involution $T_{\mathbf{Q}}$, the inclusion of the factorpolyhedra denote by

$$(J_Z)/T_{\mathbf{Q}} = \cup_{j=1}^{j_{max}} J_{Z,j}/T_{\mathbf{Q}} \subset JJ_{\mathbf{I}_a}/T_{\mathbf{Q}} \times JJ_{\dot{\mathbf{I}}_a}/T_{\mathbf{Q}}. \qquad (236)$$

The following standard 4-sheeted covering with ramification is well defined:

$$c_Z : Z_{a \times \dot{a}} \to J_Z, \qquad (237)$$

This covering is factorized to 1-sheeted covering with ramification $\hat{c}_Z$. The covering (237) is defined by the composition of the 2-sheeted covering with ramification

$$Z_{a \times \dot{a}}/T_{\mathbf{Q}} \to (J_Z)/T_{\mathbf{Q}} \qquad (238)$$

and the standard 2-sheeted covering $Z_{a \times \dot{a}} \to Z_{a \times \dot{a}}/T_{\mathbf{Q}}$. The covering (237) is invariant with respect to the standard involution, which corresponds to the involution (231), this involution is not re-denoted.

The following embeddings $S^{\frac{m_\sigma}{2}}/\mathbf{Q} \subset \mathbb{R}^{m_\sigma - 1}$, $S^{\frac{m_\sigma}{2}}/\mathbf{i} \subset \mathbb{R}^{m_\sigma - 1}$ are well defined. This embeddings in the case $\sigma = 5$ was constructed by Hirsch [Hi] (see also [Ro]). The join of the several copies of the considered embeddings determines the following embedding:

$$J_{Z,j} \subset \mathbb{R}^{m_\sigma(j_{max}+2)-2}. \tag{239}$$

The following lemma is the analog of Lemma 28.

Lemma 35. *There exists an embedding*

$$\mathrm{i}_{\mathrm{J_Z}} : \mathrm{J_Z} \int_{\chi} \mathrm{S}^1 \subset \mathbf{D}^{\mathrm{n}-2} \times \mathrm{S}^1 \subset \mathbb{R}^{\mathrm{n}-1} \subset \mathbb{R}^{\mathrm{n}}. \tag{240}$$

Denote by $\tilde{Z}_{a\times\dot{a}} \subset Z_{a\times\dot{a}}$ the inclusion of the standard skeleton of dimension $n - \frac{n-m_\sigma}{2}$. Let us consider a generic PL–mapping $d_{\tilde{Z}} : \tilde{Z}_{a\times\dot{a}} \to D^{n-1} \times S^1 \subset \mathbb{R}^n$. Let us assume that the mapping $d_{\tilde{Z}}$ is generic. Let us consider a polyhedron of self-intersection of the map $d_{\tilde{Z}}$ and let us denote this polyhedron by $\mathbf{N}(d_{\tilde{Z}})$. Obviously, $\dim(\mathbf{N}(d_{\tilde{Z}})) = m_\sigma$. In particular, in the case $n = 2^{11}$ we get $\dim(\mathbf{N}(d_{\tilde{Z}})) = 30$. The polyhedron $\mathbf{N}(d_{\tilde{Z}})$ contains the subpolyhedron of critical points of the mapping $d_{\tilde{Z}}$. This polyhedron is called the boundary of the polyhedron $\mathbf{N}(d_{\tilde{Z}})$ and is denoted by $\partial\mathbf{N}(d_{\tilde{Z}}) \subset \mathbf{N}(d_{\tilde{Z}})$. Hence, the standard inclusion $\partial\mathbf{N}(d_{\tilde{Z}}) \subset \tilde{Z}_{a\times\dot{a}}$ is well-defined. Because of the upper row of the diagram (153), the inclusion of the subgroup $a \times \dot{a} \subset \mathbb{Z}/2^{[4]}$ is well-defined.

The following mapping called the structure mapping is well defined:

$$\zeta_{\mathbf{N}(d_{\tilde{Z}})} : (\mathbf{N}(d_{\tilde{Z}}), \partial\mathbf{N}(d_{\tilde{Z}})) \to (K(\mathbb{Z}/2^{[5]}, 1), K((\mathbf{I}_a \times \dot{\mathbf{I}}_a) \int_{\chi^{[4]}} \mathbb{Z}, 1)), \tag{241}$$

the restriction of this mapping to the boundary of the polyhedron of self-intersection of the mapping $d_{\tilde{Z}}$ is the composition of the standard inclusion $\partial\mathbf{N}(d_{\tilde{Z}}) \to K(\mathbf{I}_a \times \dot{\mathbf{I}}_a)$ and the mapping (232). The space $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1)$ is equipped with a mapping (this mapping is an inclusion in the homotopy category) into $K(\mathbb{Z}/2^{[4]}, 1)$, this mapping is defined by the composition of the mapping $K(\mathbf{I}_a \times \dot{\mathbf{I}}_a, 1) \subset K(\mathbb{Z}/2^{[4]}, 1)$ and the diagonal embedding $K(\mathbb{Z}/2^{[4]}, 1) \times K(\mathbb{Z}/2^{[4]}, 1) \subset K(\mathbb{Z}/2^{[5]}, 1)$.

Definition 36. Let us say that the mapping $d_{\tilde{Z}}$ admits $\mathbf{Q} \times \mathbb{Z}/4$–structure, if for the mapping (241) the following conditions are satisfied.

–1. The polyhedron $\mathbf{N}(d_{\tilde{Z}})$ is decomposed into two components:

$$\mathbf{N}(d_{\tilde{Z}}) = \mathbf{N}_{\mathbf{Q}\times\mathbb{Z}/4} \cup \mathbf{N}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}, \tag{242}$$

where the boundary $\partial\mathbf{N}(d_{\tilde{Z}})$ is contained in the component $\mathbf{NI}_a \times \dot{\mathbf{I}}_a \times \mathbb{Z}/2$.

–2. The restriction of the mapping (241) to the component $\mathbf{N}_{\mathbf{Q}\times\mathbb{Z}/4}$ admits a reduction to a mapping into the subspace $K((\mathbf{Q}\times\mathbb{Z}/4)\int_{\chi^{[5]}}\mathbb{Z},1)$ i.e. this restriction is given by the following composition:

$$\zeta_{\mathbf{Q}\times\mathbb{Z}/4} : \mathbf{N}_{\mathbf{Q}\times\mathbb{Z}/4} \to K((\mathbf{Q}\times\mathbb{Z}/4)\int_{\chi^{[5]}}\mathbb{Z},1) \to K(\mathbb{Z}/2^{[5]},1),$$

where the mapping of the classifying spaces $K((\mathbf{Q}\times\mathbb{Z}/4)\int_{\chi^{[5]}}\mathbb{Z},1) \to K(\mathbb{Z}/2^{[5]},1)$ is induced by the homomorphism (136).

–3. The restriction of the mapping (241) to the component $\mathbf{N}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}$ is given by the following composition:

$$\zeta_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2} : (\mathbf{N}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}, \partial\mathbf{N}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2})$$
$$\longrightarrow (K((\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2)\int_{\chi^{[5]}}\mathbb{Z},1), K((\mathbf{I}_a\times\dot{\mathbf{I}}_a)\int_{\chi^{[4]}}\mathbb{Z},1),$$

where the mapping of the classifying spaces $K(\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2,1) \to K(\mathbb{Z}/2^{[5]},1)$ is induced by the homomorphism (137).

The following lemma is analogous to Lemmas 30, 32. The proof of this lemma is analogous to the proof of Lemma 33 of [A1], see part III of the paper.

Lemma 37. *There exists a mapping $d_{\tilde{Z}} : \tilde{Z}_{a\times\dot{a}} \to D^{n-1}\times S^1 \subset \mathbb{R}^n$, admitting (a relative) $\mathbf{Q}\times\mathbb{Z}/4$–structure (see Definition (36)).*

The mapping $d_{\bar{Y}}$ is defined as a generic alteration of the following composition:

$$\tilde{Z}_{a\times\dot{a}}\int_{\chi^{[4]}} S^1 \subset Z_{a\times\dot{a}}\int_{\chi^{[4]}} S^1 \stackrel{\varphi}{\longrightarrow} J_Z\int_{\chi} S^1 \stackrel{\mathrm{i}_{J_Z}}{\subset} D^{n-2}\times S^1 \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n, \quad (243)$$

where $c_Z : Z_{a\times\dot{a}}\int_{\chi^{[4]}} S^1 \to J_Z\int_{\chi} S^1$ is the 4–sheeted covering with ramification, defined by means of the covering (237)*, $\tilde{Z}_{a\times\dot{a}}\int_{\chi^{[4]}} S^1 \subset Z_{a\times\dot{a}}\int_{\chi^{[4]}} S^1$ $J_{\tilde{Z}} \subset J_Z$ is the standard embedding, determined by means of the inclusion* (232)*, $\mathrm{i}_J : J_Z \subset D^{n-2}\times S^1 \subset \mathbb{R}^n$ is the inclusion, constructed in Lemma* 35*.*

Proof of Theorem 25

Consider the mapping (156), which is determined a reduction of the restriction of the characteristic mapping η_N on the marked component. Without loss of the generality, we may assume that the image of the map $\eta_{a\times\dot{a}}$ is contained in the subspace $\tilde{Z}_{a\times\dot{a}}\int_{\chi^{[4]}} S^1 \subset K((\mathbf{J}_a\times\dot{\mathbf{J}}_a)\int_{\chi^{[4]}}\mathbb{Z},1)$. Let us define an $\mathbb{Z}/2^{[4]}$–immersion $g_{a\times\dot{a}} : N_{a\times\dot{a}}^{\frac{n+m_\sigma}{2}} \looparrowright \mathbb{R}^n$, which is in the regular homotopy class of the immersion $g|_{N_{a\times\dot{a}}^{\frac{n+m_\sigma}{2}}}$, and the self-intersection of this immersion satisfies the required conditions from Definition 23.

Let us consider the composition $N_{a\times\dot{a}}^{\frac{n+m_\sigma}{2}} \stackrel{\eta_{a\times\dot{a}}}{\longrightarrow} \tilde{Z}_{a\times\dot{a}} \int_{\chi^{[4]}} S^1 \stackrel{d_{\tilde{Z}}}{\longrightarrow} \mathbb{R}^n$. Let us consider a C^0–small alteration of this composition into an immersion g_1 in the regular homotopy class of the immersion g. Let us denote the self-intersection manifold of the immersion g_1 by $L_{a\times\dot{a}}^{m_\sigma}$. The caliber of the deformation above is chosen so small that the manifold $L_{a\times\dot{a}}^{m_\sigma}$ is decomposed into two disjoint components as in the formula (158).

The component $L_{\mathbf{Q}\times\mathbb{Z}/4}^{m_\sigma}$ is contained in a small regular neighborhood of the image of the first component $\mathbf{N}_{\mathbf{Q}\times\mathbb{Z}/4}$ from the decomposition (242). The component $L_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}^{m_\sigma}$ is represented as the union of the two submanifolds with boundaries along the common boundary. The first of the manifolds with boundary is immersed into a small regular neighborhood of the image of the second components $\mathbf{N}_{\mathbf{I}_a\times\dot{\mathbf{I}}_a\times\mathbb{Z}/2}$ from the decomposition (242). The second manifold with boundary in immersed into a small neighborhood of regular values of the mapping $d_{\tilde{Z}}$. The common boundary is immersed into a small regular neighborhood of critical values of the map $d_{\tilde{Z}}$. Therefore, for the manifold $L_{a\times\dot{a}}^{m_\sigma}$ the formula (157) is satisfied and the induced mapping (158) is well-defined.

The immersion $g_{a\times\dot{a}}$ is a $\mathbb{Z}/2^{[4]}$–framed immersion in the prescribed cobordism class of the restriction of the triple (g, η_N, Ψ) on the marked component. By this condition the pair of mappings (156), (158) is well-defined, the mapping $\eta_{a\times\dot{a}} : N_{a\times\dot{a}}^{\frac{n+m_\sigma}{2}} \to K((\mathbf{J}_a \times \dot{\mathbf{J}}_a)\int_{\chi^{[4]}} \mathbb{Z}, 1)$, corresponds to the mapping $(\lambda = \zeta_{\mathbf{Q}\times\mathbb{Z}/4} \cup \zeta_{\mathbf{J}_a\times\dot{\mathbf{J}}_a\times\mathbb{Z}/2}$, and the Conditions 1 and 2 from Definition 23 are satisfied. Theorem 25 is proved.

7 Compression Theorem

Let us prove the Compression Theorem 27.

Remark

A sketch of an alternative proof of the Compression Theorem 27 was proposed by D. Ravenel, see [R].

Let M^{n-k} be a closed manifold of dimension $(n-k)$, $\varphi : M^{n-k} \looparrowright \mathbb{R}^m$ be an immersion of this manifold into $\mathbb{R}^n$ in the codimension k, Ξ_M is a skew-framing of the immersion φ with the characteristic class κ_M of this skew-framing. Additionally, let us assume that the manifold M^{n-k} is equipped with the family of 1-dimensional cohomology classes modulo 2:

$$A_j = \{\kappa_i\}, \quad \kappa_i \in H^1(M;\mathbb{Z}/2), \quad 0 \le i \le j, \quad \kappa_0 = \kappa_M. \tag{244}$$

This collection of cohomology classes is represented by the collection of classifying maps:

$$A_j = \{\kappa_i : M^{n-k} \to \mathbb{RP}^\infty\}, \quad i = 0,1,\dots,j, \quad \kappa_0 = \kappa_M. \tag{245}$$

Definition 38. The cobordism group of immersions $Imm^{sf;A_j}(n-k;k)$ is represented by triples (φ, Ξ_M, A_j), where:

—$\varphi : M^{n-k} \looparrowright \mathbb{R}^n$–is an immersion of a closed $(n-k)$-dimensional manifold into Euclidean space,

— Ξ_M is skew-framing of the immersion φ,

—A_j is a collection of cohomology classes, described in (245).

The cobordism relation of triples is the standard.

Remark

In the case $j = 0$ the cobordism group $Imm^{sf;A_j}(n-k;k)$ coincides with the cobordism group $Imm^{sf}(n-k;k)$ of skew-framed immersions.

A natural homomorphism

$$J^k_{sf;A_j} : Imm^{sf;A_j}(n-1,1) \to Imm^{sf;A_j}(n-k,k) \tag{246}$$

is defined as follows. Let us assume that the triple $(\varphi_0, \Xi_{M_0}, A_j(M_0))$ represents an element in the cobordism group $Imm^{sf;A_j}(n-1,1)$. Let us consider the following triple $(\varphi, \Xi_M, A_j(M))$, where the immersion $\varphi : M^{n-k} \looparrowright \mathbb{R}^n$ define as follows. The manifold M^{n-k} is a submanifold in the manifold M_0^{n-1}, the fundamental class of this submanifold represents the homological Euler class of the bundle $(k-1)\kappa_{M_0}$, the immersion φ is defined as the restriction of the immersion φ_0 on M^{n-k}. The skew-framing Ξ_M of the immersion φ is defined by the standard construction like in the case $j = 0$, the collection $A_j(M)$ of cohomology classes is the restriction of the collection $A_j(M_0)$ on the submanifold $M^{n-k} \subset M_0^{n-1}$.

Let us generalize Definition 7 for the cobordism group $Imm^{sf;A_j}(n-k,k)$.

Definition 39. Let $[(\varphi, \Xi_M, A_j)] \in Imm^{sf;A_j}(n-k,k)$. We shall say that the element $[(\varphi, \Xi_M, A_j)]$ admits a compression of the order q, if in its cobordism class there exists a triple $(\varphi', \Xi'_M, A'_j)]$, such that the pair (M^{n-k}, κ_0) admits a compression of the order q is the sense of the Definition 7.

Let us define the transfer homomorphism

$$r^!_j : Imm^{sf;A_j}(n-k,k) \to Imm^{sf;\hat{A}_{j-1}}(n-k,k) \qquad (247)$$

with respect to the cohomology class κ_j.

Let $x \in Imm^{sf;A_j}(n-k,k)$ be an element represented by a triple (φ, Ξ_M, A_j). Let us define the 2-sheeted cover

$$p_j : \hat{M}^{n-k} \to M^{n-k}$$

as the regular cover with the characteristic class $w_1(p_j) = \kappa_j \otimes \kappa_M \in H^1(M^{n-k}; \mathbb{Z}/2)$. (We will denote below by p_j the linear bundle over M^{n-k} with the characteristic class $w_1(p_j)$ and also the characteristic class $w_1(p_j)$ itself.)

Let us define a skew-framing $\Xi_{\hat{M}}$. Let us consider the immersion $\varphi : M^{n-k} \looparrowright \mathbb{R}^n$. Let us denote the immersion $\varphi \circ p_j$ by $\hat{\varphi}$. Let us denote the normal bundle of the immersion φ by ν_M. Let us denote the normal bundle of the immersion $\varphi \circ p_j$ by $\nu_{\hat{M}}$. Let us define the skew-framing $\Xi_{\hat{M}}$ of the immersion $\varphi \circ p_j$ by the formula $\Xi_{\hat{M}} = p^*_j(\Xi_M)$.

Let us define the collection of the cohomology classes $\hat{A}_{j-1}$ by the following formula:

$$\hat{A}_{j-1} = \{\hat{\kappa}_j = p_j \circ \kappa_j, \quad j = 0, \dots, j-1\}.$$

We will define

$$r^!_j(x) = [(\hat{\varphi}, \Xi_{\hat{M}}, \hat{A}_{j-1})].$$

Example 40. In the case $j = 0$ the transfer homomorphism (247) is given by the following formula:

$$r_0 : Imm^{sf}(n-k,k) \to Imm^{fr}(n-k,k)$$

The properties of this homomorphism was considered in [A-E].

Let us consider the homomorphism, given by the composition of the $(j_2 - j_1)$ transfer homomorphisms:

$$r^!_{j_1+1,\dots,j_2} : Imm^{sf;A_{j_2}}(n-k,k) \to Imm^{sf;\hat{A}_{j_1}}(n-k,k), \quad j_2 \geq j_1. \qquad (248)$$

In the case $j_1 = 0$, the homomorphism (248) will be denoted by

$$r^!_{tot} : Imm^{sf;A_{j_2}}(n-k,k) \to Imm^{sf}(n-k,k). \qquad (249)$$

Let us describe the homomorphism (248) explicitly. Let us assume that an element $x \in Imm^{sf;A_{j_2}}(n-k,k)$ is given by a triple $(\varphi, \Xi_M, A_{j_2})$. Let us consider the the following subcollection $\{\kappa_{j_1+1}, \dots, \kappa_{j_2}\}$ of the last $(j_2 - j_1)$ cohomology classes of the collection A_{j_2}. Let us define the $2^{j_2-j_1}$–sheeted cover $p : \hat{M}^{n-k} \to M^{n-k}$, given by the following collection of the cohomology classes:

$$\{\kappa_{j_1+1} \otimes \kappa_0, \dots, \kappa_{j_2} \otimes \kappa_0\}.$$

The immersion $\hat{\varphi} : \hat{M}^{n-k} \looparrowright \mathbb{R}^n$ is given by the following composition:

$$\hat{\varphi} = \varphi \circ p.$$

The normal bundle of the immersion $\hat{\varphi}$ is equipped with the skew-framing $p^*(\Xi_M) = \Xi_{\hat{M}}$. The collection of cohomology classes

$$\hat{A}_{j_1} = \{\hat{\kappa}_0 = \kappa_0 \circ p, \dots \hat{\kappa}_{j_1} = \kappa_{j_1} \circ p\}$$

is well-defined by the classifying maps.

The cobordism class of the triple $(\hat{\varphi}, \Xi_{\hat{M}}, \hat{A}_{j_1})$ determines the element $r^!_{j_1+1,\dots,j_2}(x)$.

Proposition 41. *For an arbitrary even positive integer k, $k = 0(mod 2)$, $2^l - 2 = n > k > 0$, there exists a positive integer $\psi = \psi(k)$, such that the total transfer homomorphism* (249) *for $j_2 = j$:*

$$r_{tot} : Imm^{sf;A_\psi}(n-k,k) \to Imm^{sf}(n-k,k) \qquad (250)$$

is trivial.

Proof of the Proposition 41

Let us start with the following lemma.

Lemma 42. *The cobordism group $Imm^{sf;A_j}(n-k,k)$ is a finite 2-group if $n-k > 0$.*

Proof of Lemma 42

The cobordism group $Imm^{sf}(n-k,k)$ by the Pontrjagin-Thom construction is the stable homotopy group $\Pi_{n-k}(P_{k-1})$, where $P_{k-1} = \mathbb{R}\mathrm{P}^{\infty}/\mathbb{R}\mathrm{P}^{k-1}$, see f.ex. [A-E]. By the standard arguments the cobordism group $Imm^{sf;A_j}(n-k,k)$ is a stable homotopy group $\Pi_{n-k}(P_{k-1} \times \prod_{i=1}^{j} \mathbb{R}\mathrm{P}^{\infty}_j))$. The space $P_{k-1} \times \prod_{i=1}^{j} \mathbb{R}\mathrm{P}^{\infty}_j)$ is a space of the 2-homotopy type. The Lemma 42 is proved.

Let us define the following sequence of positive integers $s_{n-k}, \dots, s_1$, where the indexes decrease from $(n-k)$ to 1:

$$s_{n-k} = ord(\Pi_{n-k}), \quad s_{n-k-1} = ord(\Pi_{n-k-1}), \quad \dots, \quad s_1 = ord(\Pi_1).$$

In this formula we denote by $ord(\Pi_i)$ the logarithm of the maximal order of an element in the 2-component of the i-th stable homotopy group of spheres. Then let us define $\psi(i) = \sum_{j=i}^{n-k} s_j$ and let us define the required integer by the following formula:

$$\psi = \psi(1) + \sigma, \tag{251}$$

where $\sigma = ord(Imm^{sf;A_{\psi(1)}}(n-k,k))$ (see Lemma 42). Therefore we have $\psi \ge \psi(1) \ge \psi(2) \ge \dots \ge \psi(n-k)$.

Let $x \in Imm^{sf;A_{\psi}}(n-k,k)$ be an element represented by a triple $(\varphi, \Xi_M, A_{\psi})$. Let us consider the element

$$r^!_{\psi(1)+1,\dots,\psi}(x), \tag{252}$$

represented by a triple $(\hat{\varphi}, \Xi_{\hat{M}}, \hat{A}_{\psi-\psi(1)})$, given by the transfer homomorphism with respect to the subcollection $\{\kappa_{\psi(1)+1}, \dots, \kappa_{\psi}\}$ of the cohomology classes in the collection $A_{\psi}(M)$ (the last σ classes in the collection A_{ψ}). Let us prove that the element (252) admits a compression of the order 0, see the Definition 39.

Let us denote the product $\mathbb{R}\mathrm{P}^{\infty}_0 \times \prod_{j=1}^{j=j_0} \mathbb{R}\mathrm{P}^{\infty}(j)$ by $X(j_0)$ and let us consider the map

$$\lambda_M : M^{n-k} \to X(s), \tag{253}$$

were $s = \psi_1$, defined as the direct product of the classifying maps of the collection $A_{\psi-\psi(1)}$ of cohomology classes (the classes $\kappa_{\psi(1)} + 1, \dots, \kappa_{\psi}$ are omitted). Let us denote again the space $X(\psi(1))$ by X for short.

Let us consider a natural filtration

$$\dots \subset X^{(n-k+1)} \subset X^{(n-k)} \subset \dots \subset X^{(1)} \subset X. \tag{254}$$

This filtration is the direct product of the two standard coordinate filtrations. Each stratum $X^{(i)} \setminus X^{(i+1)}$, $i = 1, \dots n-k$ is an union of open cells of the codimension i. Each cell is determined by the corresponded multi-index $\mu = (m_1, \dots, m_{\psi(n-k)})$, $m_1 + \dots + m_{\psi(n-k)} = i$, $m_i \ge 0$, each coordinate of the multi-index shows the codimension of the skeleton of the corresponded coordinate projective space that contains the given cell.

Let us assume that the map λ is in a general position with respect to the filtration (254). Let us denote by $L^0 \subset M^{n-k}$ a 0–dimensional submanifold in M^{n-k}, defined as the inverse image of the stratum X^{n-k} of this filtration.

Let us consider the triple $(\hat{\varphi}, \Xi_{\hat{M}}, \hat{A}'_{\psi-\psi(1)})$. Let us prove that this triple is cobordant to a triple $(\hat{\varphi}', \Xi_{\hat{M}'}, \hat{A}'_{\psi-\psi(1)})$ such that the mapping $\hat{\lambda}' : \hat{M}'^{n-k} \to X$, constructed by means of the collection of the cohomology classes $A_{\hat{M}'}$, satisfies the following property:

$$\hat{\lambda}_1^{-1}(X^{(n-k)}) = \emptyset. \qquad (255)$$

By the assumption k is even. Therefore, because n is even, the manifold M^{n-k} is oriented. Let us consider an arbitrary choose the orientation of each cell in $X^{(n-k)} \setminus X^{(n-k+1)}$. Let us denote by $lk(\mu)$ the integer coefficient of self-intersection of the image $\lambda(M^{n-k})$ with the oriented cell of a multi-index μ. Let us denote analogically by $lk(\hat{\mu})$ the integer coefficient of self-intersection of the image $\hat{\lambda}(\hat{M}^{n-k})$ with the oriented cell of a multi-index μ. Obviously, the collection of the integer $\{lk(\hat{\mu})\}$ is obtain from the collection of the corresponded integers $\{lk(\mu)\}$ by the multiplication on $2^{\psi-\psi(1)}$.

Let $A'_{\psi(1)}$ be the subcollection of A_ψ, consists of the first $\psi(1)$ cohomology classes. By the construction, the exponent of the group $Imm^{sf;A_{\psi(1)}}(n-k,k)$ is equal to $2^{\psi-\psi(1)}$. Therefore the disjoint union of the $2^{\psi-\psi(1)}$ copies of the triple $(\varphi, \Xi_M, A'_{\psi(1)})$ determines the trivial element in the cobordism group $Imm^{sf;A_{\psi(1)}}(n-k,k)$.

Let us consider the triple $(2^\sigma)(-\varphi, -\Xi_M, A'_{\psi(1)})$. This triple is define as the disjoin union of $2^\sigma = \psi - \psi(1)$ copies of the triple $(-\varphi, -\Xi_M, A'_{\psi(1)})$ (the orientation of M^{n-k} is changed and the immersion φ and the skew-framing Ξ_M are changed to the opposite). The collection of the coefficients for the triple $(2^\sigma)(-\varphi, -\Xi_M, A'_{\psi(1)})$ will be denoted by $\{lk(2^\sigma(-\mu))\}$. Obviously, $\{lk(2^\sigma(-\mu))\} = -2^\sigma\{lk(\mu)\}$.

Let us consider the triple $(\hat{\varphi}', \Xi_{\hat{M}'}, \hat{A}'_{\psi(1)})$, defined as the disjoint union of the triple $(\hat{M}^{n-k}, \Xi_{\hat{M}}, \hat{A}_{\psi(1)})$ with the triple $(2^\sigma)(-\varphi, -\Xi_M, A'_{\psi(1)})$. The new triple $(\hat{\varphi}', \Xi_{\hat{M}'}, \hat{A}'_{\psi(1)})$ and the triple $(\hat{M}^{n-k}, \Xi_{\hat{M}}, \hat{A}_{\psi(1)})$ represent the common element in the cobordism group $Imm^{sf;\hat{A}_{\psi(1)}}(k, n-k)$. The mapping $\hat{\lambda}' : \hat{M}'^k \to X$, constructed by means of the collection $\hat{A}'_{\psi(1)}$ of the cohomology classes is well-defined. The collection of the intersection coefficients, defined for the mapping $\hat{\lambda}'$ will be denoted by $\hat{l}k'(\mu)$. Obviously for an arbitrary multi-index μ we have $\hat{l}k'(\mu) = 0$.

A normal surgery of the triple $(\hat{\varphi}', \Xi_{\hat{M}'}, \hat{A}'_{\psi(1)})$ to a triple $(\hat{\varphi}'', \Xi_{\hat{M}''}, \hat{A}''_{\psi(1)})$ by 1-handles is defined such that the the map λ'' defined by means of the collection $\hat{A}''_{\psi(1)})$ of cohomology classes satisfy the condition (255):

$$\hat{\lambda}''^{-1}(X^{(n-k)}) = \emptyset.$$

This gives the first step of the proof.

Let us describe the next steps of the proof. Let us denote the triple $(\hat{\varphi}'', \Xi_{M''}, A''_{\psi(1)})$ by $(\varphi, \Xi_M, A_{\psi(1)})$ again. The map $\lambda : \hat{M}^{n-k} \to X$, constructed by means of the collection $A_{\psi(1)}$ of cohomology classes satisfy the condition (255). Let us consider the triple $(\hat{\varphi}, \Xi_{\hat{M}}, \hat{A}_{\psi(1)})$, given by the following element

$$r^!_{\kappa_{\psi(2)+1},\dots,\kappa_{\psi(1)}}(\varphi, \Xi_M, A_{\psi(1)})$$

in the cobordism group $Imm^{sf;\hat{A}_{\psi(2)}}(n-k,k)$.

Let us define the space $X(\psi(2))$ by the Cartesian product of infinite–dimensional projective spaces with indexes $(0,1,\dots,\psi(2))$:

$$X(\psi(2)) = \mathbb{RP}^\infty_0 \times \prod_{j=1}^{j=\psi(2)} \mathbb{RP}^\infty(j).$$

The standard coordinate inclusion

$$i_{\psi(2)} : X(\psi(2)) \subset X(\psi(1)) \tag{256}$$

is well-defined. The space $X(\psi(2))$ is equipped with the following standard stratification:

$$\dots \subset X^i(\psi(2)) \subset X^{i-1}(\psi(2)) \subset \dots \subset X(\psi(2)). \tag{257}$$

The inclusion (256) is agree with the stratifications (254), (257) of the origin and the target spaces.

The collection $A_{\psi(2)}$ of cohomology classes determines the map $\hat{\lambda} : \hat{M}^{n-k} \to X(\psi(2))$. The condition (255) implies the following analogical condition:

$$\hat{\lambda}^{-1}(X^{(n-k)}(\psi(2))) = \emptyset. \tag{258}$$

Let us denote by $\hat{L}^1 \subset \hat{M}^{n-k}$ a 1-dimensional submanifold in $\hat{M}^{n-k}$ given by the following formula:

$$\hat{L}^1 = \hat{\lambda}^{-1}(X^{(n-k-1)}(\psi(2))).$$

The restriction of the cohomology classes of the collection $\hat{A}_{\psi(2)}$ on the submanifold $\hat{L}^1$ is trivial. In particular, the submanifold $\hat{L}^1$ is framed.

The components of the manifold $\hat{L}^1$ are equipped with the collection of the multi-indexes corresponded to the top cells of the space $X^{n-k-1}(\psi(2))$. A fixed multi-index determines a disjoint collection of 2^{s_1} copies of 1-dimensional framed manifold (probably, non-connected) and the copies are pairwise diffeomorphic as a framed manifolds.

A framed 2-dimensional manifold $\tilde{K}^2$ with a framed boundary $\partial(\tilde{K}^2) = (\tilde{L}^1)$ is well-defined. This framed manifold determines the body of a handle for the normal surgery of the triple $(\hat{\varphi}, \Xi_{\hat{M}}, \hat{A}_{\psi(2)})$ to a triple $(\hat{\varphi}', \Xi_{\hat{M}'}, \hat{A}'_{\psi(2)})$ such that the collection $\hat{A}'_{\psi(2)}$ of cohomology classes determines the map

$$\hat{\lambda}' : \hat{M}'^{n-k} \to X(\psi(2)),$$

satisfied the condition (258).

The next steps of the proof are analogical to the step 1. The parameter i denoted the dimension of the obstruction is changed from 2 up to $n-k$. In each step the analogical condition to the conditions (255), (258) is considered. At the last step of the proof we have a framed manifold (M^{n-k}, Ξ_M), equipped with a collection $\hat{A}_{\psi(n-k)}$ of the trivial cohomological classes. The framed manifold represented by $\psi(n-k)$ disjoint copies of the framed manifold (M^{n-k}, Ξ_M) is a framed boundary (and therefore a skew-framed boundary). The Proposition 41 is proved.

Let us describe an algebraic obstruction for the compression of the given order.

Lemma 43. *An arbitrary element $x \in Imm^{sf;A_j}(n-k,k)$ admits a compression of the order i, $i \le n-k$, if and only if the element*

$$J^{k'}_{sf}(x) \in Imm^{sf;A_j}(n-k',k')$$

($i \le n-k' \le n-k$) admits a compression of the same order i.

Theorem 44. *For an arbitrary element $x \in Imm^{sf;A_j}(n-k,k)$ the total obstruction for a compression of an order q ($0 \le q \le n-k$) is given by the element $J^q_{sf;\mathbf{A}_j}(x) \in Imm^{sf;A_j}(q,n-q)$.*

To prove Lemma 43 and Theorem 44, let us formulate an auxiliary lemma. Let as assume that a triple (φ, Ξ_M, A_j) represents an element $x \in Imm^{sf;A_j}(n-k,k)$. Let us additionally assume that this element admits a compression of the order $(i-1)$. This means that in the triple

(φ, Ξ_M, A_j) can be taken in its cobordism class such that the characteristic class $\kappa_M \in H^1(M^{n-k};\mathbb{Z}/2)$ of the skew-framing Ξ_M is given by the following composition:

$$\kappa_M : M^{n-k} \to \mathbb{RP}^{n-k-i} \subset \mathbb{RP}^{\infty},$$

$i < n-k$. We shall denote the map $M^{n-k} \to \mathbb{RP}^{n-k-i}$ described above again by κ_M.

Let us consider the manifold $Q^i \subset M^{n-k}$, given by the formula:

$$Q^i = \kappa_M^{-1}(pt), \quad pt \in \mathbb{RP}^{n-k-i}. \tag{259}$$

The manifold is equipped with the natural framing Ψ_Q, because the restriction of the skew-framing Ξ_M over the submanifold $Q^i \subset M^{n-k}$ is a framing.

Moreover, the restriction of cohomology classes of the collection A_j over the submanifold $Q^i \subset M^{n-k}$ determines the collection $A_j(Q)$ of cohomology classes on Q^i. Note that the class κ_0 in the collection A_Q is the trivial class. The immersion $\varphi_Q : Q^i \looparrowright \mathbb{R}^n$ is defined as the restriction of the immersion φ over the submanifold $Q^i \subset M^{n-k}$. A triple $(\varphi_Q, \Psi_Q, A_j(Q))$ determines an element $J^i_{sf;\mathbf{A}_j}(x) = y \in Imm^{sf;A_j}(i, n-i)$.

Lemma 45. *The element $x = [(\varphi, \Xi_M, A_M)] \in Imm^{sf,A_j}(n-k,k)$, which admits a compression of the order $(i-1)$, admits a compression of the order i if and only if the element $J^i_{sf;\mathbf{A}_j}(x) = y = [(\varphi_Q, \Xi_Q, A_j(Q))] \in Imm^{sf,A_j}(i, n-i)$ is trivial.*

Proof of Lemma 45

At the first step let us prove that if $y = 0$ then x admits a compression of the order i. Let us consider a skew-framed in the codimension $(n-i)$ $(i+1)$–dimensional manifold (P^{i+1}, Ξ_P) with boundary $\partial P^{i+1} = Q^i$, equipped with the collection $A_j(P)$ of cohomology classes, such that the restriction of the skew-framing Ξ_P over the boundary Q^i is a framing coincided with the framing Ψ_Q and the restriction of the collection $A_j(P)$ over the boundary Q^i coincides with the collection $A_j(Q)$.

Let us describe a normal surgery of the skew-framed manifold (M^{n-k}, Ξ_M) into a skew-framed manifold (T^{n-k}, Ξ_T). Let us construct a manifold with boundary called the body of a handle. Let us consider the manifold P^{i+1} and let us denote the $(n-i)$–dimensional normal bundle over P^{i+1} by ν_P. The normal bundle ν_P is equipped with the skew-framing Ξ_P, i.e. the bundle map (an isomorphism on each fiber)

$$\nu_P \to (n-i)\kappa_P \tag{260}$$

is well-defined. Let us denote by U_P the disk bundle over P spanned by the first $(k-i)$ factor in the Whitney sum (260). The manifold U_P is a skew-framed manifold in the codimension $(n-k)$ manifold with boundary will be called the body of a handle.

The boundary ∂U_P of the body of the handle contains a submanifold $Q^i \times D^{n-k-i}$, the total space of the disk bundle over the manifold Q^i. Let us consider the Cartesian product $M^{n-k} \times I$ of the manifold M^{n-k} and the unite segment $I = [0,1]$. A second copy of the manifold $Q^i \times D^{n-k-i}$ is embedded into the submanifold $M^{n-k} \times \{1\} \subset \partial(M^{n-k} \times I)$, this is a regular neighborhood of the submanifold $Q^i \times \{1\} \subset M^{n-k} \times \{1\}$. Let us define the manifold T^{n-k} by the following formula:

$$T^{n-k} = \partial^+((M^{n-k} \times I) \cup_{Q^i \times D^{n-k-i}} U_P), \tag{261}$$

where by ∂^+ is denoted the "upper" component of the cobordism i.e. the component that contains the last part $\partial U_P \setminus (Q^i \times D^{n-k-i})$ of the boundary of the body U_P.

After the standard operation called "smoothing the corners" the PL–manifold T^{n-k} becomes a smooth closed smooth manifold. The immersion $\varphi_T : T^{n-k} \looparrowright \mathbb{R}^n$ (this immersion is well-defined up to a regular homotopy), the skew framing Ξ_T with the characteristic class κ_T (i.e. the bundle fiberwise isomorphism $\nu_T \to k\kappa_T$) are well-defined. The manifold T^{n-k} is equipped with the collection $A_j(T)$ of the collection of characteristic classes, each class in the collection is determined by the gluing of the corresponded classes of the two components in the decomposition (261). The class $\kappa_0(T)$ of the collection $A_j(T)$ coincides with the characteristic class κ_T of the skew-framing Ξ_T. The triple $(\varphi_T, \Xi_T, A_j(T))$ determines an element in the cobordism group $Imm^{sf;A_j}(n-k,k)$ and by the construction $[(\varphi, \Xi_M, A_j)] = [(\varphi_T, \Xi_T, A_j(T))]$.

Let us prove that the element $[(\varphi_T, \Xi_T, A_j(T))]$ admits a compression of the order i. We will prove that the characteristic class κ_T is represented by a classifying map $\kappa_T : T^{n-k} \to \mathbb{R}\mathrm{P}^{n-k-i-1} \subset \mathbb{R}\mathrm{P}^{\infty}$. Take a positive integer b big enough and let us consider $\kappa_M : M^{n-k} \to \mathbb{R}\mathrm{P}^b$, such that

$$\kappa_M^{-1}(\mathbb{R}\mathrm{P}^{b-n+k+i}) = Q^i,$$

$Q^i \subset M^{n-k}$, $\mathbb{R}\mathrm{P}^{b-n+k+i} \subset \mathbb{R}\mathrm{P}^b$.

Let us consider the mapping $g : P^{i+1} \to \mathbb{R}\mathrm{P}^{b-n+k+i}$, the restriction of this mapping to the component of the boundary $\partial P^{i+1} = Q^i$ coincides with the map $\kappa_M|_{Q^i}$. Let us consider the "thickening" $h : U_P \to \mathbb{R}\mathrm{P}^b$ of the map g, this map h is defined by the standard extension of the map g to the body of the handle U_P.

The map $g' : M^{n-k} \cup_{Q^i \times D^{n-k-i}} U_P \to \mathbb{RP}^b$, $g'|_{U_P} = g$ is well-defined and the restriction $g'|_{T^{n-k} \subset M^{n-k} \cup U_P}$ does not meet the submanifold $\mathbb{RP}^{b-n-k+i} \subset \mathbb{RP}^b$. The space $\mathbb{RP}^b \setminus \mathbb{RP}^{b-n-k+i}$ is retracted to the its subspace $\mathbb{RP}^{n-k-i-1}$ by a deformation, the required compression of the map κ_T of the order i is constructed. We have proved that the element x admits a compression of the order i.

Let us prove the inverse statement: assume that the element $x = [(\varphi, \Xi_M, A_j)]$ admits a compression of the order i, then the triple $(\varphi_Q, \Psi_Q, A_j(Q)))$, Q^i is given by the equation (259) determines the trivial element in the cobordism group $Imm^{sf;A_j}(i, n-i)$.

Let $(\varphi_W, \Xi_W, A_j(W))$ be a triple, where W^{n-k+1} is a manifold with boundary, $\partial W^{n-k+1} = M^{n-k} \cup M_1^{n-k}$; (φ_W, Ξ_W) is a skew-framed immersion of the manifold W into $\mathbb{R}^n \times I$; $A_j(W)$ is a collection of characteristic classes. Moreover, the triple $(\varphi_W, \Xi_W, A_j(W))$ determines a cobordism between the triples $(\varphi_M, \Xi_M, A_j(M))$ and $(\varphi_{M'}, \Xi_{M'}, A_j(M'))$, where the pair $(M'^{n-k}, \kappa_{M'})$ (the cohomology class $\kappa_{M'}$ is the characteristic class of the skew-framing $\Xi_{M'}$ and this class is included into the collection $A_j(M')$) admits a compression of the order i, i.e. the classifying map $\kappa_{M'} = \kappa_W|_{M'}$ is given by the following composition:

$$\kappa_{M'} : M'^{n-k} \to \mathbb{RP}^{n-k-i} \subset \mathbb{RP}^{\infty}.$$

Let us consider the standard submanifold $\mathbb{RP}^{b-n-k+i} \subset \mathbb{RP}^b$, this submanifold intersects the submanifold $\mathbb{RP}^{n-k-i} \subset \mathbb{RP}^b$ at a point $pt \in \mathbb{RP}^{n-k-i} \setminus \mathbb{RP}^{n-k-i-1}$ and does not intersect the standard submanifold $\mathbb{RP}^{n-k-i-1} \subset \mathbb{RP}^{n-k-i}$. The image $Im(\kappa_M(M^{n-k}))$ is in the submanifold $\mathbb{RP}^{n-k-i} \subset \mathbb{RP}^b$, the image $Im(\kappa_{M'}(M'^{n-k}))$ is in the submanifold $\mathbb{RP}^{n-k-i-1} \subset \mathbb{RP}^{n-k-i}$.

Let us denote by P^{i+1} the submanifold $F^{-1}(\mathbb{RP}^{b-n+k+i})$ (we assume that F is transversal along the submanifold $\mathbb{RP}^{b-n+k+i} \subset \mathbb{RP}^b$). By the construction $\partial P^{i+1} = Q^i$. Let us define a skew-framing Ξ_P in the codimension $(n-i)$ as the direct sum of a skew-framing of the submanifold $P^{i+1} \subset W^{n-k+1}$ and the skew-framing Ξ_W, restricted to the submanifold $P^{i+1} \subset W^{n-k+1}$.

The restriction of the skew-framing Ξ_W on $\partial W^{n-k+1} = Q^{n-k}$ coincides with the skew-framing Ψ_Q with the trivial characteristic class κ_Q (i.e. the skew-framing Ψ_Q is the framing). The restriction of the collection $A_j(P)$ of cohomology classes on $\partial W^{n-k+1} = Q^{n-k}$ coincides with the collection $A_j(Q)$. This proves that the triple (φ_Q, Ψ_Q, A_Q), is a boundary. Lemma 45 is proved.

Proof of Theorem 44

Let us assume that a compression of the order $(i-1)$, $i < q$ for an element $x \in Imm^{sf;A_j}(n-k, k)$, $x = [(\varphi, \Xi_M, A_M)]$ is well-defined. By

Lemma 45, the obstruction to a compression of the order i of the element $x \in Imm^{sf;A_j}(n-k,k)$ represented by the same triple (φ, Ξ_M, A_M), coincides with the obstruction of a compression of the same order i for the element $J^{sf}_{k'}(x) \in Imm^{sf;A_j}(n-k',k')$. Therefore, by induction over i, the total obstruction for a compression of the order q for the element x is trivial if and only if the total obstruction for a compression of the order q for the element $J^{sf}_{k'}(x)$ is trivial. Theorem 44 is proved.

To prove the Compression Theorem 27 the following construction by U.Koshorke of the total obstruction for a homotopy of a bundle map into a bundle monomorphism on each fiber of the bundles (see [K]) is required.

Let $\alpha \to Q^q$, $\beta \to Q^q$ be a pair of the vector bundles over the smooth manifold Q^q (we do not assume that the manifold Q^q is closed) $dim(\alpha) = a$, $dim(\beta) = b$, $dim(Q^q) = q$, $2(b-a+1) < q$. Let $u : \alpha \to \beta$ be a generic vector bundle morphism. let us denote by $\Sigma \subset Q^q$ a submanifold, given by the formula:

$$\Sigma = \{x \in Q^q | Ker(u_x : \alpha_x \to \beta_x) \neq 0\}. \qquad (262)$$

This manifold Σ is the singular manifold of the bundle morphism u. Note that under the presented dimensional restrictions, for a generic vector bundle morphism u we have $rk(u) \geq a-1$.The codimension of the submanifold $\Sigma \subset Q^q$ is equal to $b-a+1$.

Let us describe the normal bundle of the submanifold (262), this bundle will be denoted by ν_Σ. Let us denote by $\lambda : E(\lambda) \to \Sigma$ the linear subbundle, determined as the subbundle of kernels of the morphism u over the singular submanifold $\Sigma \subset Q^q$. Therefore, the inclusion of the bundles over Σ $\varepsilon : \lambda \subset \alpha$ is well-defined. Let us denote by Λ_α the bundle over Σ, this bundle is the orthogonal complement to the subbundle $\varepsilon(\lambda) \subset \alpha$. A natural vector-bundle morphism over Σ (isomorphism of fibers) $v : \Lambda_\alpha \subset \beta$ is well-defined. Let us define the bundle Λ_β over Σ as the orthogonal complement to the subbundle $v(\Lambda_\alpha)$ in the bundle $\alpha|_\Sigma$. The normal bundle $\nu(\Sigma)$ is determined by the following formula:

$$\nu(\Sigma) = \lambda \otimes \Lambda_\beta. \qquad (263)$$

If the manifold Q^q has a boundary ∂Q and the vector bundles morphism u is the morphism of the bundles over the manifold with boundary, then the singular submanifold $\partial\Sigma \subset \partial Q$ of the restriction $u|_{\partial Q}$ is a boundary of the submanifold $\Sigma \subset Q^q$ with the normal bundle, given by the same formula (263).

In the paper [K] (in this paper there is a reference to the previous papers by the same author) a cobordism group of embeddings of of manifolds in Q^q (in this construction the manifold Q^q is closed) of codimension $b-a+1$ with an additional structure of the normal bundle, given by the equation (263) is defined. For an arbitrary generic vector bundle morphism $u : \alpha \to \beta$ an element in this cobordism group is well-defined. This element is the total obstruction of a homotopy of the vector bundle morphism u to a fiberwise monomorphism.

Let $\hat{\nu} : E(\hat{\nu}) \to \mathbb{RP}^{2^k-1}$ be a vector bundle, $\dim(\hat{\nu}) = n+1-2^k$, $2^k < n+2$ over the standard projective space, isomorphic to the following Whitney sum: $\hat{\nu} \equiv (n+1-2^k)\kappa_{\mathbb{RP}}$, where $\kappa_{\mathbb{RP}}$ is the canonical line bundle over $\mathbb{RP}^{2^k-1}$. Let us denote the Whitney sum $\hat{\nu} \oplus \kappa$ by ν, $\dim(\nu) = (n-2^k+2)$. The standard projection $\pi : \nu \to \hat{\nu}$ with the kernel κ is well-defined. The bundle $\nu_{\mathbb{RP}}$ in the case

$$b(2^k) \leq n+2 \qquad (264)$$

(the positive integer $b(r)$, $r = 2^k$ is equal to the corresponded power of 2, see [A-E]) is isomorphic to the normal bundle of the projective space $\mathbb{RP}^{2^k-1}$.

Let us define an admissible s–family of sections (singularities in the family of sections are possible) of the bundle $\hat{\nu}_{\mathbb{RP}}$.

Definition of an admissible family of sections of the bundle $\hat{\nu}_{\mathbb{RP}}$

We shall say that a generic s–family of sections

$$\hat{\psi} = \{\hat{\psi}_1, \dots, \hat{\psi}_s\}, \quad \hat{\psi} : s\varepsilon \to \hat{\nu}_{\mathbb{RP}}$$

of the bundle $\hat{\nu}_{\mathbb{RP}}$ is admissible, if there exists a regular s–family

$$\psi = \{\psi_1, \dots, \psi_s\}, \quad \psi : s\varepsilon \to \nu_{\mathbb{RP}}$$

of sections of the bundle $\nu_{\mathbb{RP}}$ satisfied the condition: $\pi \circ \psi = \hat{\psi}$.

Lemma 46. *Let us assume that $s \leq n+2-2^{k+1}-k-1$ and $n \equiv -2 \pmod{2^{2^k}}$, $n > 0$. Then the bundle $\hat{\nu}_{\mathbb{RP}}$ has an admissible s-family of sections.*

Proof of Lemma 46

By the Davis table the projective space $\mathbb{RP}^{2^k-1}$ is immersable into the Euclidean space $\mathbb{R}^{2^{k+1}-k-1}$ (this is not the lowest possible dimension of the target Euclidean space of immersions). By the equation (264), and because $b(k) \le 2^k$ for $n = 2^{2^k}$, the bundle $\nu_{\mathbb{RP}}$ is the normal bundle over the projective space $\mathbb{RP}^{2^k-1}$. Therefore the bundle $\nu_{\mathbb{RP}}$ admits a generic regular s–family of section, denoted by ψ. The projection $\hat{\psi} = \pi \circ \psi$ of this regular family is the admissible s–family of sections of the bundle $\hat{\nu}_{\mathbb{RP}}$. Lemma 46 is proved.

Let us consider an admissible generic s–family of sections $\hat{\psi}$ of the bundle $\hat{\nu}_{\mathbb{RP}}$. Let us denote by $\Sigma \subset \mathbb{RP}^{2^k-1}$ the singular manifold of the family $\hat{\psi}$. This denotation corresponds to (262), if we take $\alpha \equiv s\varepsilon$, $\beta \equiv \hat{\nu}$, $u = \psi$. In the following lemma we will describe the normal bundle ν_Σ of the submanifold $\Sigma \subset \mathbb{RP}^{2^k-1}$.

Lemma 47. *Let us assume that*

$$s = n + 2 - 2^{k+1} - k - 1, \quad k \ge 2, \tag{265}$$

where $n = 2^{2^k} - 2$. Then the singular submanifold $\Sigma \subset \mathbb{RP}^{2^k-1}$ of an admissible s–family of sections of the vector-bundle $\hat{\nu}_{\mathbb{RP}}$ is a smooth submanifold of dimension k, the normal bundle ν_Σ is equipped with a skew-framing $\Xi_\Sigma : (2^k - 1 - k)\kappa_{\mathbb{RP}} \equiv \nu_\Sigma$, the characteristic class κ_Ξ of this skew-framing coincides with the restriction $\kappa_{\mathbb{RP}}|_{\Sigma \subset \mathbb{RP}^{2^k-1}}$.

Proof of Lemma 47

Let us describe the normal bundle ν_Σ by means of Koschorke's Theorem. Moreover, let us define a skew-framing of this bundle. Let us denote by $\lambda \subset s\varepsilon$ the subbundle of the kernels of the family $\hat{\psi}$ over the submanifold Σ. By the assumption the the family $\hat{\psi}$ is admissible, therefore:

$$\lambda = \kappa_\Sigma, \tag{266}$$

where by κ_Σ the vector bundle $\kappa_{\mathbb{RP}}|_\Sigma$ is denoted. Let us denote the orthogonal complement to $\kappa_\Sigma \subset s\varepsilon$ over Σ by $s\varepsilon - \kappa_\Sigma$. Let us denote the orthogonal complement to $\hat{\psi}(s\varepsilon - \kappa_\Sigma)$ in the bundle $\hat{\nu}_{\mathbb{RP}}|_\Sigma$ by Λ. By the construction:

$$(s\varepsilon - \kappa_\Sigma) \oplus \kappa_\Sigma = s\varepsilon. \tag{267}$$

Let us prove that the bundle ν_Σ satisfies the equation:

$$\nu_\Sigma \equiv (n + 2 - 2^k - k - 1)\kappa_\Sigma. \tag{268}$$

By the Koschorke Theorem the following isomorphism of vector bundles is well-defined:

$$\Lambda \otimes \kappa_\Sigma \equiv \nu_\Sigma.$$

This equation is equivalent to the equation:

$$(s\varepsilon) \otimes \kappa_\Sigma \equiv \nu_\Sigma \oplus \kappa_\Sigma.$$

This proves the equation (268). The isomorphism (268) defines a skew-framing Ξ_Σ of the bundle ν_Σ with the characteristic class κ_Σ. The Lemma 47 is proved.

Remark

Because the restriction of the normal bundle $\nu_{\mathbb{RP}}$ over Σ is isomorph to the trivial bundle by the canonical isomorphism, the skew-framing Ξ_Σ determines the skew-framing of the normal bundle (in the Euclidean space) of the manifold Σ.

Let us consider an element

$$x \in Imm^{sf,A_j}(2^k-2, n-2^k+2), \quad n = -2(mod 2^k),$$

given by the cobordism class of a triple $(\varphi, \Xi_M, \mathbf{A}_j(M))$. Put $m = 2^k - 2$. Let us consider the vector bundle $\nu \to \mathbb{RP}^{2^k-1}$, $\dim(\nu) = n - m$. The normal bundle ν_M of the manifold M^m is given by the formula:

$$\nu_M = \kappa_M^*(\nu_{\mathbb{RP}}).$$

Let us consider the map

$$\lambda_M : M^m \to \mathbb{RP}^{2^k-1} \times \prod_{i=1}^{j} \mathbb{RP}^\infty, \tag{269}$$

see (253), constructed by means of the collection A_j. Let us consider the standard projection $\pi_0 : \mathbb{RP}^{2^k-1} \times \prod_{i=1}^{j} \mathbb{RP}^\infty \to \mathbb{RP}^{2^k-1}$ on the factor $\mathbb{RP}^{2^k-1}$. The composition $\pi_0 \circ \lambda_M : M^m \to \mathbb{RP}^{2^k-1}$ coincides with the map κ_M.

Let us define the subbundle $\hat{\nu}_M \subset \nu_M$ of the codimension 1 (i.e. of the dimension $\dim(\hat{\nu}_M) = n + 1 - 2^k$) by the formula:

$$\hat{\nu}_M = \kappa_M^*(\hat{\nu}_{\mathbb{RP}}).$$

Let us define a family of sections $\hat{\xi}_M = \{\hat{\xi}_1, \dots, \hat{\xi}_s\}$, $s = n + 3 - 2^{k+1} + k$ of the bundle $\hat{\nu}_M$. This collection is the pull-back image of an admissible collection of sections $\hat{\psi} = \{\hat{\psi}_1, \dots \hat{\psi}_s\}$ of the bundle $\hat{\nu}_M$ by the map κ_M. Let us denote by

$$N^{k-1} \subset M^m \tag{270}$$

the submanifold of singular sections. It is not hard to prove that $\dim(N^{k-1}) = (k-1)$. The manifold N^{k-1} is equipped with the collection $A_j(N)$ of cohomology classes, a class of $A_j(N)$ is defined as the restriction of the corresponded class of the collection A_M over the submanifold $N^{k-1} \subset M^m$. The class $\kappa_M|_N$ in the collection $A_j(N)$ is denoted by κ_N. Let us denote the immersion $\varphi|_N$ by φ_N.

Let us denote by ν_N the normal bundle of the immersed (embedded by the general position arguments) manifold $\varphi_N(N^{k-1})$ in the Euclidean space $\mathbb{R}^n$. The normal bundle ν_N is isomorph to the Whitney sum $\nu_N = \nu_M|_N \oplus \nu_{N \subset M}$, where by $\nu_{N \subset M}$ is denoted the normal bundle of the submanifold $N^{k-1} \subset M^m$ inside the manifold M^m.

By Lemma 47 and by the transversality of the map κ_M along the submanifold $\Sigma \subset \mathbb{RP}^{2^k-1}$, the bundle $\nu_{N \subset M}$ is equipped with the skew-framing $\Xi_{N \subset M}$ with the characteristic class κ_N. The bundle $\nu_M|_N$ is also equipped with a skew-framing with the same characteristic class (see an analogical Remark after Lemma 47). This gives a skew-framing Ξ_N of the immersion φ_N of codimension $(n-k+1)$. Let us denote by $A_j(N)$ the collection of cohomology classes from the group $H^1(N^{k-1};\mathbb{Z}/2)$, this collection is the restriction of the collection A_j over the submanifold $N^{k-1} \subset M^m$.

Lemma 48. *The triple $(\varphi_N, \Xi_N, A_j(N))$ determines an element $x_{k-1} \in Imm^{sf,A_j}(k-1, n-k+1)$, this element is the total obstruction of a compression of the order $(k-1)$ for the element $x \in Imm^{sf,A_j}(2^k-2, n-2^k+2)$.*

Proof of Lemma 48

Let us consider the submanifold $\Sigma^k \subset \mathbb{RP}^{2^k-1}$ of singularities of the family of sections $\hat{\psi}$, let us re-denote this manifold by Σ_0^k. This manifold is equipped by the following natural stratification (a filtration):

$$\emptyset \subset \Sigma_k^0 \subset \cdots \subset \Sigma_1^{k-1} \subset \Sigma_0^k \subset \mathbb{RP}^{2^k-1}. \qquad (271)$$

The submanifold Σ_i, $dim(\Sigma_i) = k - i$ in (271) is defined as the singular submanifold of the subfamily of the first $(s-i)$ sections in $\hat{\psi}$. By the straightforward calculations follows that the fundamental class $[\Sigma_i]$ of the corresponded submanifold in the filtration in the group $H_{k-i}(\mathbb{RP}^{2^k-1};\mathbb{Z}/2)$ represents the only generator of this group: this homology class is dual to the characteristic class $\bar{w}_{n+1-2^k-k+i}(n+1-2^k)\kappa_{\mathbb{RP}}$ of the bundle $\hat{\nu}_{\mathbb{RP}}$.

Without loss of the generality we assume that the map $\kappa_M : M^m \to \mathbb{RP}^{2^k-1}$ is transversal along the stratification (271). Let us denote the inverse image of the stratification (271) by

$$N_{k-1}^0 \subset N_{k-2}^1 \subset \cdots \subset N_0^{k-1} \subset M^m. \qquad (272)$$

The top manifold N_0^{k-1} of the filtration (272) coincides with manifold N^{k-1}, defined above.

Let us prove the lemma by the induction over the parameter i, $i = 0, \dots, k-1$. Let us assume that the image of the map $\kappa_M : M^m \to \mathbb{R}\mathrm{P}^{2^k-1}$ is in the standard projective subspace $\mathbb{R}\mathrm{P}^{2^k-2-i} \subset \mathbb{R}\mathrm{P}^{2^k-1}$. In this case $N_{k-i}^{i-1} = \emptyset$. By the standard argument we may assume that the stratum Σ_{k-i}^i intersects in the general position the standard submanifold $\mathbb{R}\mathrm{P}^{2^k-1-i}$ of the complementary dimension at the only point. (The index of self-intersection of this two submanifolds in the manifold $\mathbb{R}\mathrm{P}^{2^k-1}$ is odd and well-defined modulo 2.)

The framed manifold N_{k-i-1}^i is the regular preimage of the marked point by the map κ_M (the image of this map is in the submanifold $\mathbb{R}\mathrm{P}^{2^k-2-i}$). Let us denote an element represented by the cobordism class of the triple $(\varphi_{N_{k-i-1}}, \Xi_{N_{k-i-1}}, A_j(N_{k-i-1}))$ in the cobordism group $Imm^{sf;A_j}(i, n-i)$ by x_i. By Lemma 47 the condition $x_i = 0$, $i \le k-2$ is satisfied if and only if there exists a normal cobordism of the map κ_M to the map $\kappa_{M'} : M'^m \to \mathbb{R}\mathrm{P}^{2^k-2-i} \subset \mathbb{R}\mathrm{P}^{2^k}$. Therefore a compression of the order $i+1$, $i+1 \le k-1$ of an element x is well defined. If we put $i+1 = k-1$ we have a compression of the order $(k-1)$. Lemma 48 is proved.

The following Proposition is the main in the proof of Theorem 27.

Proposition 49. *Let $n = -2(mod(2^k))$, $n > 2^k$, $x \in Imm^{sf;A_j}(2^k-2, n-2^k+2)$ be an arbitrary element in the kernel of the homomorphism* (279) *(in this formula $b(k) = 2^k$). Let $x_{k-1} \in Imm^{sf;A_j}(k-1, n-k+1)$ be the total obstruction for a compression of the order $(k-1)$ of the element x. Then the element x_{k-1} is in the image of the transfer homomorphism, i.e. there exists an element $y_{k-1} \in Imm^{sf,A_{j+1}}(k-1, n-k+1)$, for which $r_{j+1}(y_{k-1}) = x_{k-1}$. Assuming $k-1$ is even, the element y_{k-1} is in the kernel of the homomorphism* (279) *(where we assume that $b(k) = k+1$)*

Proof of Proposition 49

Let us assume that the cobordism class of the element x is given by a triple $(\varphi_M, \Xi_M, A_j(M))$, where $\varphi_M : M^m \looparrowright \mathbb{R}^n$ is an immersion, $\dim(M^m) = m = 2^k - 2$. Because $codim(\varphi_M)$ is odd, M^m is oriented. Assume that $[(\varphi_M, \Xi_M, A_j(M))]$ is in the kernel of (279) (in this formula $b(k) = 2^k$). (This assumption is satisfied for $j = 0$, because characteristic numbers for M^m are trivial.)

Let us consider the normal bundle $\nu_M \cong (n-2^k+2)\kappa_M$ over the manifold M^m and the subbundle $\hat{\nu}_M \subset \nu_M$ of the codimension 1, $\hat{\nu}_M \equiv (n-2^k+1)\kappa_M$. Let us prove that there exists a regular s–family of sections $\hat{\psi}$ of the bundle $\hat{\nu}_M$, $s = n+2+k-2^{k+1}$.

Let us denote by $D(\kappa_M)$ a manifold with boundary, the total space of the disk bundle, associated with line bundle κ_M. The vector bundle $\hat{\nu}_M$ is lifted to the vector bundle over $D(\kappa_M)$ (we will denote this lift again by $\hat{\nu}_M$). By the R.Cohen theorem [C] there exists an immersion $D(\kappa_M) \looparrowright \mathbb{R}^{2^{k+1}-2-k}$, because $\dim(D(\kappa_M)) = 2^k-1$ and $\alpha(2^k-1) = k$. The proof of the statement is in Theorem 50.

Equivalently, the bundle $\hat{\nu}_M$ admits an s–family of regular sections. The tautological lift, denoted by ψ, of the regular s–family $\hat{\xi}$ of the bundle $\hat{\nu}_M$ to a regular s–family of sections of the bundle $\hat{\nu}_M$ is defined.

Let us consider the admissible s–family of sections $\hat{\psi}$ of the bundle $\hat{\nu}_M$. This family of sections is defined as the pull-back of admissible s–family of sections of the bundle $\hat{\nu}_{\mathbb{RP}}$, see Lemma 46. A regular s–family of sections ψ of the bundle ν_M is defined as the lift of the admissible s–family $\hat{\psi}$.

Let us consider the manifold $M^m \times I$ and let us define the bundle $\nu_{M\times I}$ by the formula $\nu_{M\times I} = p_M^*(\nu_M)$, where $p_M : M^m \times I \to M$ is the standard projection on the second factor. Let us define the bundle $\hat{\nu}_{M\times I}$ by the formula $\hat{\nu}_{M\times I} = p_M^*(\hat{\nu}_M)$. Let us consider a generic s–family of sections $X = \{\chi_1, \dots \chi_s\}$ of the bundle $\nu_{M\times I}$ with the following boundaries conditions:

$$X = \psi \quad over \quad M^m \times \{1\}, \tag{273}$$

$$X = \xi \quad over \quad M^m \times \{0\}. \tag{274}$$

Let us denote by $V^{k-1} \subset M^m \times I$ the singular subset of the family X. By the general position argument this subset is a closed submanifold in $M^m \times I$, because over the boundary $\partial(M^m \times I)$ the family X is regular. Let us denote by

$$\hat{X}\{\hat{\chi}_1, \dots, \hat{\chi}_s\}$$

the projection of the s–family X into the s–family of sections of the vector bundle $\hat{\nu}_{M\times I}$. The s–family $\hat{X}$ satisfies the following boundary conditions:

$$\hat{X} = \hat{\psi} \quad over \quad M^m \times \{1\}, \tag{275}$$

$$\hat{X} = \hat{\xi} \quad over \quad M^m \times \{0\}. \tag{276}$$

Let us denote by $\hat{K}^k \subset M^m \times I$ the subset of singular sections of the s–family $\hat{X}$. This subset $\hat{K}^k$ is a k–dimensional manifold with boundary. The only component of the boundary $\partial\hat{K}^k$ is a submanifold of $M^m \times \{1\})$, this component coincides with the submanifold $N^{k-1} \subset M^m \times \{1\}$ of the singularities of the admissible family ψ, see (270).

By Lemma 47, the triple (φ_N, Ξ_N, A_N) is well-defined. Here $\varphi_N = \varphi|_N$, Ξ_N is the skew-framing of this immersion, $A_j(N)$ is the restriction of the collection A_j on $N^{k-1} \subset M^m \times \{1\}$. The triple (φ_N, Ξ_N, A_N) represents the total obstruction $x_{k-1} \in Imm^{sf;A_j}(k-1, n-k+1)$ for a compression of the order $(k-1)$ of the element $x = [(\varphi, \Xi_M, \kappa_M)]$.

Let us use the formula (263) to calculate the normal bundle of the submanifold $\hat{K}^k \subset M^m \times I$ and of the line normal bundle of the submanifold $V^{k-1} \subset \hat{K}^k$.

Let us denote by λ the line bundle over V^{k-1} of kernels of the s–family of sections X. Let us prove that the normal bundle $\nu_{V \subset M \times I}$ of the submanifold $V^{k-1} \subset M^m \times I$ is given by the formula:

$$\nu_{V \subset M \times I} \equiv \varepsilon \oplus (m-k+1)\lambda, \quad m = 2^k - 2. \tag{277}$$

The restriction of the normal bundle $\nu_{M \times I}|_V$ over the submanifold V^{k-1} is isomorph to the bundle $(n+2-2^k)\kappa_{M \times I}$, and therefore, because $b(k-1) \le 2^k$, is isomorph to the trivial bundle:

$$(n+2-2^k)\kappa_{M \times I} \equiv (n+2-2^k)\varepsilon. \tag{278}$$

This isomorphism is canonical, i.e. does not depends of M^m and of V^{k-1}. Let us define the vector bundle Λ as the orthogonal complement to the line subbundle λ in the trivial bundle of linear combinations of the base sections. The bundle Λ represents the stable vector bundle $-\bar{\lambda}$. Therefore the orthogonal complement of the subbundle $\Xi(\Lambda)$ in the vector bundle $\nu_{M \times I}$ is the subbundle in the vector bundle $\nu_M|_V$ isomorph to the vector bundle $\lambda \oplus (2^k - k - 1)\varepsilon$. By the Koschorke Theorem, the normal bundle of the submanifold $V^{k-1} \subset M^{2^k-2} \times I$ is isomorph to the vector bundle:

$$\nu_{V \subset M \times I} \equiv \lambda \otimes (\lambda \oplus (2^k - k - 1)\varepsilon) \equiv \varepsilon \oplus (2^k - k - 1)\lambda.$$

This vector bundle $\nu_{V \subset M \times I}$ also represents the stable normal bundle of the manifold V^{k-1} because of the equation (278). The formula (277) is proved.

The restriction of the immersion $\varphi \times Id|_V$ is regular homotopic to an immersion $\varphi_V : V^{k-1} \looparrowright \mathbb{R}^n \times \{1\}$. By the computation the normal bundle of the immersion φ_V is equipped with a skew-framing, denoted by Ξ_V. The collection of cohomology classes $A_j(V)$ is defined by the formula $A_j(V) =$

$A_j(M \times I)|_V$, where $A_j(M \times I)$ is induced from the given collection $A_j(M)$ of cohomology classes on M^m by the projection p_M. Let us define the collection of cohomology classes $A_{j+1}(V)$ by the addition to the collection $A_j(V)$ the last cohomology class $\kappa_{j+1} = \lambda \otimes \kappa_{M\times I}$. The triple $y_{k-1} = (\varphi_V, \Xi_V, A_{j+1}(V))$ determines an element in the cobordism group $Imm^{sf;A_{j+1}}(k-1, n-k+1)$.

Let us denote by $U_V \subset \hat{K}^k$ a small closed regular neighborhood of the submanifold $V^{k-1} \subset \hat{K}^k$. The line bundle of kernels of the family $\hat{X}$ of sections over the submanifold (with boundary) $U_V \subset M^m \times I$ is isomorph to the line bundle $p^*_{U_V,V}(\lambda)$, where $p_{U_V,V} : U_V \to V$ is the projection of the neighborhood on the central submanifold. Let us denote the line bundle $p^*_{U_V,V}(\lambda)$ by $\hat{\lambda}$. The orthogonal complement to the subbundle $\hat{\lambda}$ in the vector bundle of the linear combinations of the base sections over U_V will be denote again by $\hat{\Lambda}$.

Let us consider the subbundle $\hat{X}(\hat{\Lambda})$ in the vector bundle $\hat{\nu}_{M\times I}$. By the analogical calculations, using the canonical isomorphisms of the vector bundles over U_V: $(2^{k-1})\lambda \equiv (2^{k-1})\kappa_M \equiv (2^{k-1})\varepsilon$, the orthogonal complement in $\hat{\nu}_{M\times I}$ of the subbundle $\hat{X}(\hat{\Lambda})$, denoted by $-\hat{X}(\hat{\Lambda})$, is isomorph to the bundle $\lambda \oplus (2^{k-1} - k - 1)\varepsilon \oplus (2^{k-1} - 1)\kappa_M$. By the Koschorke Theorem, the stable isomorphism class of the normal bundle $\nu_{U_V \subset M\times I}$ of the submanifold $U_V \subset M^m \times I$ is given by the formula:

$$\nu_{U_V \subset M\times I} \equiv (-k-1)\hat{\lambda} \oplus (-\kappa_M \otimes \hat{\lambda}).$$

In particular, from this calculation follows that the line normal bundle of the submanifold $V^{k-1} \subset K^k$ is isomorph to the line bundle $\lambda \otimes \kappa_M$.

Let us denote ∂U_V by Q^{k-1}. The space Q^{k-1} is a closed manifold, $\dim(Q^{k-1}) = k - 1$. The normal bundle $\nu_{Q\subset M\times I}$ of the submanifold $Q^{k-1} \subset M^m \times I$ is given by the formula:

$$\nu_{Q\subset M\times I} \equiv \varepsilon \oplus (m - k + 1)\kappa_M.$$

The restriction of the immersion $\varphi \times id : M^m \times I \looparrowright \mathbb{R}^n \times I$ to the submanifold $Q^{k-1} \subset M^m \times I$ is regular homotopic to a skew-framed immersion $\hat{\varphi}_Q : Q^{k-1} \looparrowright \mathbb{R}^n \times \{1\}$ of the codimension $(n - k + 1)$ with the sklew-framing, denoted by $\hat{\Xi}_Q$, and with the characteristic class of this skew-framing $\hat{\kappa}_Q = \kappa_{M\times I}|_Q$. The manifold Q^{k-1} is equipped by the collection $\hat{A}_j(Q)$ of cohomology classes, $\hat{A}_j(Q) = A_j(M \times I)|_{Q\subset M\times I}$. The triple $(\varphi_Q, \Xi_Q, A_j(Q))$ determines an element in the cobordism group $Imm^{sf;A_j}(k-1, n-k+1)$.

The manifold $K^k \setminus U_V$ has the boundary consists of the two components: $\partial(K^k \setminus U_V) = Q^{k-1} \cup K^{k-1}$. The restriction of the immersion $\varphi \times Id|_{(K^k\setminus U_V)}$ is regular homotopic to an immersion $\varphi_K : K^k \looparrowright \mathbb{R}^n \times I$ with the following the boundary conditions: $\varphi_K|Q = \varphi_Q : Q^{k-1} \looparrowright \mathbb{R}^n \times \{1\}$, $\varphi_K|N = \varphi_N :$

$N^{n-1} \looparrowright \mathbb{R}^n \times \{0\}$. The immersion φ_K is a skew-framed immersion with a skew-framing Ξ_K and with the characteristic class $\kappa_K = \kappa_{M\times I}|_K$ of this skew-framing. The manifold K^k is equipped by the collection of cohomology classes $A_j(K) = A_j(M \times I)|_{K \subset M\times I}$.

The triple $x_{k-1} = (\varphi_N, \Xi_N, A_j(N))$ determines an element in the cobordism group $Imm^{sf;A_j}(k-1, n-k+1)$. The element x_{k-1} is the total obstruction to a compression of the element $x = [(\varphi, \Xi_M, A_j)]$ of the order $(k-1)$. The element $[(\varphi_Q, \hat{\Xi}_Q, \hat{A}_j(Q))]$ is the image by the transfer homomorphism $r^!_{j+1}$ of the element $y_{k-1} = [(\varphi_V, \Xi_V, A_{j+1}(V))]$. The elements $x_{k-1} = [(\varphi_N, \Xi_N, A_j(N))]$ and $r^!_{j+1}(y_{k-1}) = [(\varphi_Q, \hat{\Xi}_Q, \hat{A}_j(Q))]$ are equal. The cobordism between the elements x_{k-1} and $r^!_{j+1}(y_{k+1})$ is given by the triple $(\varphi_K, \Xi_K, \mathbf{A}_j(K))$.

Assume that $k-1$ is even. Let us prove that y_{k-1} is in the kernel of (279) (in this formula we assume that $b(k) = k+1$). The construction of y_{k-1} from x is generalized for the image of x by (279) (in this formula $b(k) = 2^k$). Because x is in the kernel of (279), y_{k-1} is also in the kernel of (279). Proposition 49 is proved.

Proof of the Compression Theorem 27

Let us define a positive integer $\psi = \psi(q)$, by the formula (251) for $n - k = q - 1$. By Proposition 41 the total transfer homomorphism (250), which is defined on the group $Imm^{sf;A_\psi}(d-1, n-d+1)$ is the trivial homomorphism. Let us define a positive integer $l(d) = \exp_2(\exp_2 \dots \exp_2(d) \dots + 1)$, where the number of the iterations of the function $\exp_2(x+1) = 2^{x+1}$ is equal to ψ and the initial value is $x = d-1$.

Let l' be an arbitrary power of 2, $l' \geq l(d)$. Let us define $n = l' - 2$. Let us prove that an arbitrary element in $Imm^{sf}(n-1, 1)$ admits a compression of the order $d-1$.

Let us define $n_0 = l(d) - 2$, by the assumption $n_0 \leq n$. Let us define the following sequence of ψ integers: $2n_1 = log_2(n_0+2) - 2$, $2n_2 = log_2(n_1+2) - 2$, $\dots$, $2n_\psi = log_2(n_{\psi-1}+2) - 2$. All this integers are positive and $n_\psi = d-1$.

Let $x_0 \in Imm^{sf}(n_0, n-n_0)$ be the image an arbitrary element $x \in Imm^{sf}(n-1, 1)$ by $J^{sf} : Imm^{sf}(n-1, 1) \to Imm^{sf}(n_0, n-n_0)$. Denote by x_{d-1} the total obstruction of a compression of the order $d-1$ for the element x. The element x_{d-1} coincides with the total obstruction of a compression of the order $(d-1)$ for the element x_{n_0}.

Let us consider the total obstruction $x_{n_1} \in Imm^{sf}(n_1, n-n_1)$ of the retraction of the order n_1 for the element x_{n_0}. The element x_{n_1} is in the kernel of the homomorphism (279) (in this formula we put $b(k) - 2 = n_1$ is even), because characteristic classes of M^{n_1} are trivial). By Proposition 49, there exists an element $y_{n_1} \in Imm^{sf;\{\kappa_1\}}(n_1, n-n_1)$, such that the image of the element y_{n_1} by the transfer homomorphism is equal to x_{n_1}.

Let us consider the total obstruction $y_{n_2} \in Imm^{sf;\kappa_1}(n_2, n-n_2)$ of a compression of the order n_2 for an element y_{n_1}. By the Proposition 49 there exists an element $z_{n_2} \in Imm^{sf;\kappa_1,\kappa_2}(n_2, n-n_2)$ such that the element $r_2(z_{n_2})$ is the total obstruction of a compression of the order n_2 for the element y_{n_1}. The element $r_{tot}(z_{n_2}) = r_1 \circ r_2(z_{n_2}) = J^{sf}_{n_2}(x_{n_0}) = x_{n_2}$ is the total obstruction for a compression of the order n_2 of the element x_{n_1}. The same element x_{n_2} is the total obstruction for a compression of the order (n_2) for the initial element x.

By the induction we prove that the total obstruction of a compression of the order $(d-1)$ for the element x is in the image of the total transfer homomorphism of the multiplicity ψ, i.e. this total obstruction is equal to $r_{tot}(z)$, $z \in Imm^{sf;A_\psi}(d-1, n-d+1)$. By Proposition 41 we have $r_{tot}(z) = 0$. Therefore, $x_{d-1} = 0$. The Compression Theorem 27 is proved.

In the proof of Proposition 49 we used the following fact. Denote $b = b(k) = 2^{2k}$. Assume that $n = 2^l - 2$, $l \ge b(k)$.

Theorem 50. *Let a cobordism class $x \in Imm^{sf,A_r}(b(k)-2, n-b(k)+2)$ is represented by a skew-framed immersion $(f : M^{b(k)-2} \looparrowright \mathbb{R}^n, \Xi_M)$, $\dim(M^{b(k)-2}) = b(k)-2$, where the manifold $M^{b(k)-2}$ is equipped with a mapping* (269). *Assume that the element x belongs to the kernel of the forgetful homomorphism*

$$Imm^{sf,A_r}(b(k)-2, n-b(k)+2) \to \Omega_{b(k)-2}(\prod_{i=0}^{r} \mathbb{RP}^\infty_i). \qquad (279)$$

Then in the regular cobordism class $[x]$ there exists an element for which the manifold $M^{b(k)}$ admits an immersion into the Euclidean space $\mathbb{R}^{2b(k)-3-k}$ with a non-degenerate skew cross section given by a linear bundle κ_M.

Remark

Theorem 50 is a corollary of the R.Cohen's Immersion Theorem [C].

The main step of the proof of Theorem 50 is the following lemma.

Lemma 51. *Let* (N^j, Ξ_N), $j = 0, \dots, k-1$, *be a framed (in particular, oriented) manifold, equipped with a mapping*

$$\lambda_N : N^j \to \prod_{i=1}^{r} \mathbb{RP}^\infty_i.$$

Let us assume that the Hurewicz image

$$(\lambda_N)_*([N]) \in H_j(\prod_{i=1}^{r} \mathbb{RP}^\infty_i; \mathbb{Z}) \tag{280}$$

with integer coefficients is trivial. Then there exists a skew-boundary (W^k, Ψ_W, λ_W) *in codimension* $b(k)-1$, $\partial(W^k_W) = N^{k-1}$, $\Psi_W|_{\partial W^k} = \Xi_N$, $\lambda_W|\partial W = N^k = \lambda_N$, *where*

$$\lambda_W : W^k \to \mathbb{RP}^\infty_0 \times \prod_{i=1}^{r} \mathbb{RP}^\infty_i,$$

and $\kappa_W = w_1(\Psi_W)$ *coincides with the projection of the mapping* λ_W *on the factor* $\mathbb{RP}^\infty_0$.

A sketch of the proof of Lemma 51

To prove Lemma 51 let us consider the Atiyah–Hirzebruch spectral sequence for the cobordism group of framed immersions of the dimensions $0, \dots, k-1$ (the stable homotopy group) of the space $\prod_{i=1}^{r} \mathbb{RP}^\infty_i$. Let us consider the Atiyah–Hirzebruch spectral sequence for cobordism groups of skew-framed immersions of dimensions $0, \dots, k-1$ in the codimension $b(k)-1$ of the space $\mathbb{RP}^\infty_0 \times \prod_{i=1}^{r} \mathbb{RP}^\infty_i$. There is a natural mapping of the first spectral sequence to the second spectral sequence. By the main result of [A-E] all higher coefficients in E_2-terms of the kernel are trivial. Therefore the kernel of E^∞–therm is totally described by the Hurewicz image (280) of a cobordism class of a corresponding mapping of a framed manifold. This Hurewicz image is trivial by the assumption. Therefore an arbitrary framed manifold (N^k, Ξ_N) is a skew-framed boundary in codimension $b(k)-1$. Lemma Lemma 51 is proved.

A sketch of the proof of Theorem 50

By the assumption the normal bundle ν_M is isomorphic to the Whitney sum $Cb(k)\kappa_M$, where κ_M is the given line bundle over $M^{b(k)-2}$, $b(k) = 2^{2k}$, C in a positive integer. Let us calculate the Koschorke construction to prove that the normal bundle $Cb(k)\kappa_M$ of $M^{b(k)-2}$ admits a regular skew-section $\kappa_M \oplus k\varepsilon \oplus (C-1)b(k)\varepsilon$. The construction is given by the induction over the index $j = 0, \dots, k$.

The bundle ν_M admits a regular family of $2k+(C-1)b(k)$ sections. Denote by $I \subset \nu_M$ the orthogonal complement to the subbundle $(C-1)b(k)\varepsilon \subset \nu_M$, which consists of the first $(C-1)b(k)$ sections of the given family. At a j-th step of the induction $j = 0, \dots, k$, let us assume that there exists a morphism (this morphism could be tot a fiberwise isomorphism) ρ_j of the bundle $(2k-j+1)\varepsilon \oplus \kappa_M$ into the subbundle $I \subset \nu_M$, such that the following conditions are satisfied:

–1. Denote by $\hat{\rho}_j$ the restriction of the morphism ρ_j on the subbundle $(j+1)\varepsilon \oplus \kappa_M \subset (2k-j+1)\varepsilon \oplus \kappa_M$. It is required that $\hat{\rho}_j$ is regular (a fiberwise isomorphism).

–2. Denote by $\tilde{\rho}_j$ the restriction of the morphism ρ_j at the subbundle $(2k-j+1)\varepsilon \subset (2k-j+1)\varepsilon \oplus \kappa_M$. It is required that $\tilde{\rho}_j$ is regular.

Let us prove the base of the induction $j = 0$. Consider the manifold $M^{b(k)-2}$ as a manifold with the prescribed orientation, equipped with the collection A_r of cohomology classes. This manifold determines an element in the cobordism group $\Omega_{b(k)-2}(\prod_{i=0}^{r} \mathbb{RP}_i^{\infty})$, which is given by the image of the element x by the homomorphism Ω. The obstruction of the existence of a regular morphism $\hat{\rho}_0$ for such a manifold is well-defined and is trivial, because x belongs to the kernel of (279). An extension of the morphism $\hat{\rho}_0$ to a morphism ρ_0, which satisfies the condition –2, is well defined.

Let us prove the step $j \mapsto j+1$ of the induction. Consider the morphism ρ_j and denote the restriction of this morphism on the subbundle $(j+2)\varepsilon \oplus \kappa_M \subset (2k-j+1)\varepsilon \oplus \kappa_M$ by $\hat{\rho}_j^{\circ}$ (one more section then in the family $\hat{\rho}_j^{\circ}$). The morphism $\hat{\rho}_j^{\circ}$, generally speaking, is not regular. Denote by $N^{j+1} \subset M^{b(k)-2}$ the singular submanifold. The conditions –1 and –2 imply that the restriction of the cohomology class κ_M on N^{j+1} is trivial and that N_{j+1} is a framed submanifold in $M^{b(k)-2}$, and, therefore, a framed manifold. Let us prove that this framed manifold satisfies conditions of Lemma 51.

Let us prove that the Hurewicz image (280) is trivial. There are the two cases:

–a. j is even;

–b. j is odd.

Let us consider the case a. Take the collection of section ρ_{j+2}, which is restricted to the subbundle $(j+3)\varepsilon \oplus \kappa_M \subset (2k-j+1)\varepsilon \oplus \kappa_M$, denote this restriction by $\hat{\rho}_j^{\circ\circ}$ (two extra sections with respect to $\hat{\rho}_j^{\circ}$). Denote the singular manifold of the morphism $\hat{\rho}_j^{\circ\circ}$ by K^{j+2}. Evidently, $w_1(K^{j+2}) = \kappa_M|_K$. The local fundamental class of the manifold manifold K^{j+2}, equipped by the restriction of the collection A_r, determines an element y in $H_{j+2}(\mathbb{RP}_0^\infty \times \prod_{i=1}^r \mathbb{RP}_i^\infty; \mathbb{Z}^{tw})$, where the coefficients are integers and twisted with respect to the cohomology class κ_M. Because the element x belongs to the kernel of (279), $y = 0$. Therefore the homology class $[K^{j+2}] \cap \kappa_M = [L^{j+1}]$, which is considered as an element in the group $H_{j+1}(\mathbb{RP}_0^\infty \times \prod_{i=1}^r \mathbb{RP}_i^\infty; \mathbb{Z})$ is trivial. This proves that the statement in the case –a.

Let us consider the case b. Take the collection of section ρ_j°, which contains a regular subfamily ρ_j. Because the restriction κ_M on the manifold N^{j+1} is trivial, the tensor product of the morphism ρ_j° and the line bundle κ_M is well defined as the morphism with the same singular manifold. In particular, $\rho_j^{\circ} \otimes \kappa_M$ determines a morphism of the bundle $(C-1)b(k)\kappa_M \oplus (j+1)\kappa_M \oplus \varepsilon$ into the bundle $I \otimes \kappa_M$, which is the orthogonal complement of the subbundle $(C-1)b(k)\kappa_M \subset Cb(k)\varepsilon$. Because $j+1$ is even, and because x is in the kernel of (279), N^{j+1} determines the element is the kernel of (280). This proves that the statement in the case –b.

The obstruction of the existence of a regular morphism $\hat{\rho}_{j+1}$ with the trivial kernel over the singular manifold is given by a cobordism class of a mapping $\lambda_N : N^j \to \prod_{i=1}^r \mathbb{RP}_i^\infty$ of a framed j-dimensional manifold, which is considered as a skew-framed manifold in codimension $b(k)-1$ with local twisted coefficients system, associated with κ_M (note that $\kappa_M|_{N^{j+1}}$ is trivial, but in the regular cobordism class this property is not assumed).

Therefore, there exists ρ_{j+1}, which satisfies the condition –1. Let us prove that there exists ρ_{j+1}, which satisfies the both conditions –1 and –2. By the construction a homotopy of the morphism $\hat{\rho}_j^{\circ}$ into the morphism $\hat{\rho}_{j+1}$ has singularities with the trivial kernels. Evidently, there exists a homotopy $\tilde{\rho}_j$ into $\tilde{\rho}'_j$ of the bundle $(2k-j+1)\varepsilon$, which has singularities with the trivial kernels, and the restriction of $\tilde{\rho}'_j$ on the subbundle $(j+1)\varepsilon$ coincides with the restriction $\hat{\rho}_{j+1}$ on the considered subbundle. Therefore there exists a morphism ρ'_{j+1} of the bundle $(2k-j+1)\varepsilon \oplus \kappa_M$ (one more sections then in ρ_{j+1}), for which the condition –1 is satisfies, and instead of the condition –2 the following condition is satisfied: the restriction of ρ'_{j+1} on the subbundle $(2k-j+1)\varepsilon \subset (2k-j+1)\varepsilon \oplus \kappa_M$ has the trivial kernel. By general position arguments the restriction ρ'_{j+1} on the trivial line bundle is regular. Let us restrict ρ'_{j+1} on the subbundle $(2k-j)\varepsilon \subset (2k-j+1)\varepsilon \oplus \kappa_M$. There exists a small generic deformation $\rho'_{j+1}|_{(2k-j)\varepsilon} \mapsto \tilde{\rho}_{j+1}$, for which the family $\tilde{\rho}_{j+1}$ is regular. This proves the condition 2. The induction is well-defined.

By the last step of the induction, there exists a regular section ρ_{k-1}.

Theorem 50 is proved.

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