The relationship between framed bordism and skew-framed bordism

Pyotr M. Akhmet'ev and Peter J. Eccles*

Abstract

A skew-framing of an immersion is an isomorphism between the normal bundle of the immersion and the Whitney sum of copies of some line bundle. This reduces to a framing if the line bundle is oriented. In this note we investigate the relationship between the bordism groups of framed immersions in Euclidean space and the bordism groups of skew-framed immersions. We show that in certain codimensions (i) framed immersions in Euclidean space are skew-framed boundaries and (ii) framed immersions in Euclidean space are framed bordant to the double cover of a skew-framed immersions. These results are simple consequences of the Kahn–Priddy Theorem and James Periodicity.

1 Framed immersions and skew-framed immersions

Recall that a *framing* of an immersion $M^n \looparrowright \mathbb{R}^{n+k}$ of a smooth n-dimensional manifold M^n into Euclidean $(n+k)$-space is a trivialization of its normal bundle. A manifold which has a framed immersion into some Euclidean space is *stably parallelizable*. A framing may be thought of as an isomorphism between the normal bundle of the immersion and the Whitney sum of k copies of the (unique) oriented line bundle over M. So, by analogy, we may define a *skew-framing* (or *projective framing*) of an immersion to be an isomorphism between the normal bundle of the immersion and the Whitney sum of k copies of some (not necessarily oriented) line bundle over M.

It should be noted that given a self-transverse framed immersion $f\colon M^n \looparrowright \mathbb{R}^{n+k}$ then the double point manifold, a smooth manifold of dimension $n-k$, has an immersion $\Delta_2(f)^{n-k} \looparrowright \mathbb{R}^n$ and the framing of f induces a natural skew-framing on this immersion (see Proposition 5). Hence skew-framed immersions arise naturally in the study of framed immersions.

Clearly, a framing is a particular sort of skew-framing in which the line bundle used is oriented (and so trivial). The main purpose of this note is to observe that given n then, for infinitely many values of k, every framed immersion $M^n \looparrowright \mathbb{R}^{n+k} = \mathbb{R}^{n+k} \times \{0\} \subseteq \mathbb{R}^{n+k} \times [0,1)$ of a manifold without boundary is the boundary of a skew-framed immersion $W^{n+1} \looparrowright \mathbb{R}^{n+k} \times [0,1)$.

More generally, we can use a skew-framed immersion $F\colon W^{n+1} \looparrowright \mathbb{R}^{n+k} \times [0,1]$ of a manifold with boundary $\partial W^{n+1} = M_1^n \sqcup M_2^n$ such that $F^{-1}(\mathbb{R}^{n+k} \times \{i\}) = M_i^n$ to define

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a *skew-framed bordism* between the two skew-framed immersions $f_i = F|M_i^n\colon M_i^n \looparrowright \mathbb{R}^{n+k} \times \{i\}$ in the usual way. The skew-framed bordism group of skew-framed immersions of n-manifolds in codimension k will denoted by $\mathrm{Imm}^{\mathrm{sfr}}(n,k)$ and the framed bordism group of framed immersions of n-manifold in codimension k will denoted by $\mathrm{Imm}^{\mathrm{fr}}(n,k)$. Treating a framing as a skew-framing leads to the *forgetful homomorphism*

$$\sigma\colon \mathrm{Imm}^{\mathrm{fr}}(n,k) \to \mathrm{Imm}^{\mathrm{sfr}}(n,k).$$

To state the theorem precisely it is necessary to recall the definition of some well-known numbers. For a positive integer a, let $\phi(a)$ be the number of positive integers r such that $r < a$ and $r \equiv 0$, 1, 2 or 4 modulo 8. Put $b(a) = 2^{\phi(a)}$.

Theorem 1. *For $n \geq 1$, let M^n be a stably parallelizable compact manifold of dimension n without boundary. Then, for $2m \geq n$, each framed immersion $M^n \looparrowright \mathbb{R}^{n+b(2m+1)-1} = \mathbb{R}^{n+b(2m+1)-1} \times \{0\}$ is the boundary of a non-oriented framed immersion $W^{n+1} \looparrowright \mathbb{R}^{n+b(2m+1)-1} \times [0,1)$ or, equivalently, the forgetful homomorphism*

$$\sigma\colon \mathrm{Imm}^{\mathrm{fr}}(n, b(2m+1)-1) \to \mathrm{Imm}^{\mathrm{sfr}}(n, b(2m+1)-1)$$

is trivial.

On the other hand, it is also possible to associate a framed immersion to each skew-framed immersion in the following manner. Suppose that $(f\colon N^n \looparrowright \mathbb{R}^{n+k}, \phi)$ is a skew-framed immersion with $\phi\colon \nu_f \cong k\kappa$, where ν_f is the normal bundle of the immersion and κ is a line bundle over N^n. Let $p\colon \widetilde{N}^n \to N$ be the double covering of N^n given by the sphere bundle of κ. Then the skew-framing ϕ induces a framing $\widetilde{\phi}\colon \nu_{f\circ p} \cong kp^*(\kappa) \cong \varepsilon^k$ of the immersion $f \circ p\colon \widetilde{N}^n \to N^n \looparrowright \mathbb{R}^{n+k}$. It is routine to check that this construction respects bordisms so that a *transfer homomorphism*

$$\tau\colon \mathrm{Imm}^{\mathrm{sfr}}(n,k) \to \mathrm{Imm}^{\mathrm{fr}}(n,k)$$

may be defined by

$$\tau[f\colon N^n \looparrowright \mathbb{R}^{n+k}, \phi] = [f \circ p\colon \widetilde{N}^n \looparrowright \mathbb{R}^{n+k}, \widetilde{\phi}].$$

Theorem 2. *For $n \geq 1$, let M^n be a stably parallelizable compact manifold of dimension n without boundary. Then, for $2m \geq n$, each framed immersion $M^n \looparrowright \mathbb{R}^{n+b(2m+1)}$ is framed bordant to the double cover of a skew-framed immersion or, equivalently, the transfer homomorphism*

$$\tau\colon \mathrm{Imm}^{\mathrm{sfr}}\big(n, b(2m+1)\big) \to \mathrm{Imm}^{\mathrm{fr}}\big(n, b(2m+1)\big)$$

is an epimorphism.

2 Bordism classes of immersions and the stable homotopy groups of Thom spaces

Theorems 1 and 2 are most readily obtained by interpreting the homomorphisms between the bordism groups as homomorphism between the stable homotopy groups of the appropriate Thom complexes as described below.

The basis of the application of stable homotopy theory to the study of immersions is the observation that the bordism group of immersions of (smooth compact closed) n-manifolds in $\mathbb{R}^{n+k}$ with a $G(k)$-structure (where $G(k)$ is a subgroup of the orthogonal group $O(k)$) is isomorphic to the stable homotopy group $\pi^S_{n+k}MG(k)$. Here $MG(k)$ is the Thom complex of the universal $G(k)$-bundle $i^*\gamma^k$ where $i\colon BG(k) \to BO(k)$ is the inclusion map and γ^k is the universal k-plane bundle over $BO(k)$. This was first observed, in the case $G(k) = O(k)$ by Wells ([7]) using the Smale-Hirsch Theorem, which relates immersions of manifolds to monomorphisms of their tangent bundles, and the Pontrjagin-Thom construction, which gives an isomorphism between bordism groups of embeddings and homotopy groups of Thom complexes. The result for a general subgroup of $O(k)$ follows by the same method.

The simplest case is that of framed immersions for which $G(k)$ is the trivial group and so the universal $G(k)$-bundle can be taken to be the trivial k-dimensional bundle over a point with Thom complex S^k. This gives the well-known result that $\mathrm{Imm}^{\mathrm{fr}}(n,k)$ is isomorphic to the stable stem $\pi^S_{n+k}S^k \cong \pi^S_n$.

For skew-framings the group $G(k)$ is $O(1)$ embedded diagonally in $O(k)$, for a skew-framing of an immersion $M^n \looparrowright \mathbb{R}^{n+k}$ corresponds to a factorization of the classifying map of the normal bundle $M^n \to BO(k)$ through the diagonal map $d\colon BO(1) \to BO(1)^k \to BO(k)$. In this case the universal bundle $d^*\gamma^k$ is the Whitney sum $k\gamma^1$ of k copies of the universal line bundle. If γ^1 is taken to be the canonical line bundle over $\mathbb{R}\mathrm{P}^\infty$ then the Thom complex of $k\gamma^1$ is homeomorphic to the truncated real projective space $P^\infty_k = \mathbb{R}\mathrm{P}^\infty/\mathbb{R}\mathrm{P}^{k-1}$ ([3]). Thus, by the Pontrjagin-Thom-Wells Theorem, the bordism group $\mathrm{Imm}^{\mathrm{sfr}}(n,k)$ is isomorphic to the stable homotopy group $\pi^S_{n+k}P^\infty_k$.

Lemma 3. *The forgetful homomorphism*

$$\sigma\colon \mathrm{Imm}^{\mathrm{fr}}(n,k) \to \mathrm{Imm}^{\mathrm{sfr}}(n,k)$$

corresponds under the Pontrjagin-Thom-Wells isomorphism to the map

$$\pi^S_{n+k}S^k \to \pi^S_{n+k}P^\infty_k$$

induced by the inclusion of the bottom cell.

Proof. The map of universal bundles corresponding to viewing a framing as a skew-framing can be described as $k\gamma^1_0 \to k\gamma^1$ where γ^1_0 is the canonical line bundle over $\mathbb{R}\mathrm{P}^0 \subseteq \mathbb{R}\mathrm{P}^\infty$. This induces a map of Thom complexes $S^k = P^k_k \to P^\infty_k$ which is the inclusion of the bottom cell, as required. □

Lemma 4. *The transfer homomorphism*

$$\tau\colon \mathrm{Imm}^{\mathrm{sfr}}(n,k) \to \mathrm{Imm}^{\mathrm{fr}}(n,k)$$

corresponds under the Pontrjagin-Thom-Wells ismomophism to the map

$$\pi^S_{n+k}P^\infty_k \to \pi^S_{n+k}S^k$$

induced by the co-attaching map of the bottom cell of P^∞_{k-1}, i.e. the map q in the cofibre sequence

$$S^{k-1} \xrightarrow{i} P^\infty_{k-1} \xrightarrow{j} P^\infty_k \xrightarrow{q} S^k. \tag{1}$$

arising from the inclusion of the bottom cell in P^∞_{k-1}.

Proof. Given a skew-framed immersion $(f\colon N^n \looparrowright \mathbb{R}^{n+k}, \phi)$ with $\phi\colon \nu_f \cong k\kappa$, where ν_f is the normal bundle of the immersion and κ is a line bundle over N^n, we can define an embedding, $g\colon \widetilde{N}^n \to E(k\kappa)$, of the double cover of N^n into the total space of the bundle $k\kappa$ by $g(a,x) = (a, 0, \ldots, 0, x)$, for $a \in N^n$ and x a unit vector in the fibre of κ over a. Composing this with the bundle map $E(k\kappa) \to N^n$ gives the double cover map $p\colon \widetilde{N}^n \to N^n$. Let $F\colon E(\nu_f) \looparrowright \mathbb{R}^{n+k}$ be a tubular neighbourhood of the immersion f. Then it is clear that the composition $\widetilde{f}_1\colon \widetilde{N}^n \xrightarrow{g} E(k\kappa) \xrightarrow{\phi^{-1}} E(\nu_f) \xrightarrow{F} \mathbb{R}^{n+k}$ is regularly homotopic to the immersion $\widetilde{f}\colon \widetilde{N}^n \looparrowright \mathbb{R}^{n+k}$.

The embedding g has a tubular neighbourhood $G\colon \widetilde{N}^n \times \mathbb{R}^k \to E(k\kappa)$ given by

$$G\big((a,x),(t_1,t_2,\ldots,t_k)\big) = (a, t_1x, t_2x, \ldots, t_{k-1}x, \exp(t_k)x).$$

Then $\widetilde{N}^n \times \mathbb{R}^k \xrightarrow{G} E(k\kappa) \xrightarrow{\phi^{-1}} E(\nu_f) \xrightarrow{F} \mathbb{R}^{n+k}$ gives a tubular neighbourhood of $\widetilde{f}_1$.

The universal case of the embedding G is the map $S^\infty \times \mathbb{R}^k \to E(k\gamma^1)$ given by

$$(\mathbf{x}, t_1, t_2, \ldots, t_k) \mapsto ([\mathbf{x}], t_1\mathbf{x}, t_2\mathbf{x}, \ldots, t_{k-1}\mathbf{x}, \exp(t_k)\mathbf{x})$$

where γ^1 is the canonical line bundle over the Grassmanian $G_1(\mathbb{R}^\infty) = \mathbb{R}\mathrm{P}^\infty$ which can be taken as the universal line bundle over $BO(1)$.

It follows that the transfer homomorphism corresponds to the homomorphism in stable homotopy induced by the map $T(k\gamma^1) = P_k^\infty \to S^k$ obtained by applying the Pontrjagin-Thom construction to this universal map $S^\infty \times \mathbb{R}^k \to E(k\gamma^1)$.

It is sufficient to consider the embedding $S^N \times \mathbb{R}^k \to E(k\gamma_N^1)$ where γ_N^1 is the canonical line bundle over $\mathbb{R}\mathrm{P}^N$. The total space $E(k\gamma_N^1)$ can be identified with the space $\mathbb{R}\mathrm{P}^{N+k} - \mathbb{R}\mathrm{P}^{k-1}$ via $([\mathbf{x}], t_1\mathbf{x}, t_2\mathbf{x}, \ldots, t_k\mathbf{x}) \mapsto [t_1, t_2, \ldots, t_k, \mathbf{x}]$ for $\mathbf{x} \in S^N$. Under this identification the embedding $S^N \times \mathbb{R}^k \to E(k\gamma_N^1) = \mathbb{R}\mathrm{P}^{N+k} - \mathbb{R}\mathrm{P}^{k-1}$ is given by $(\mathbf{x}, t_1, t_2, \ldots t_k) \mapsto [t_1, t_2, \ldots, t_{k-1}, \exp(t_k), \mathbf{x}]$. Applying the Pontrjagin-Thom construction to this embedding gives a pointed map $T(k\gamma_N^1) = (\mathbb{R}\mathrm{P}^{k+N} - \mathbb{R}\mathrm{P}^{k-1})^* = P_k^{k+N} \to (\mathbb{R}^k)^* \cong S^k$ given by

$$[s_1, s_2, \ldots, s_{k-1}, s_k, \mathbf{x}] \mapsto \begin{cases} (s_1, s_2, \ldots, s_{k-1}, \ln(s_k)) & \text{if } s_k > 0 \\ * & \text{if } s_k = 0, \end{cases}$$

where $\mathbf{x} \in S^N$ and $s_k \geq 0$.

It is now sufficient to recognize this map as the map q in the cofibre sequence

$$S^{k-1} \xrightarrow{i} P_{k-1}^{k+N} \xrightarrow{j} P_k^{k+N} \xrightarrow{q} S^k.$$

To see this observe that the map i is the one point compactification of the map $\mathbb{R}^{k-1} \to P^{k+N} - P^{k-2}$ given by $(t_1, t_2, \ldots, t_{k-1}) \mapsto [t_1, t_2, \ldots, t_{k-1}, 1, \mathbf{0}]$. Hence the map j is the pointed map $(P^{k+N} - P^{k-2})^* \to (P^{k+N} - P^{k-1})^*$ given by $[\mathbf{x}] \mapsto [\mathbf{x}]$ if $[\mathbf{x}] \in P^{k+N} - P^{k-1}$ and $[\mathbf{x}] \mapsto *$ otherwise. It follows that the above formula gives the map q. $\square$

Finally in the section we give the result about the double manifolds of framed immersions referred to earlier.

Proposition 5. *Given a self-transverse framed immersion $f\colon M^n \looparrowright \mathbb{R}^{n+k}$ then the natural immersion of the ($(n-k)$-dimensional) double point manifold $\Delta_2(f) \looparrowright \mathbb{R}^{n+k}$ gives rise to an immersion $\Delta_2(f) \looparrowright \mathbb{R}^n$ and the framing of f induces a natural skew-framing on this immersion.*

Proof. Let

$$\widetilde{\Delta}_2(f) = \{\, (x_1, x_2) \in M \times M \mid f(x_1) = f(x_2),\ x_1 \neq x_2 \,\}.$$

By the self-transversality of f this is a submanifold of $M \times M$ of dimension $n-k$. The group of order 2 acts freely on $\widetilde{\Delta}_2(f)$ by interchanging coordinates and the double point manifold $\Delta_2(f)$ is the quotient space. Clearly f induces an immersion $\delta_2(f)\colon \Delta(f) \looparrowright \mathbb{R}^{n+k}$.

The framing on the immersion f gives a basis for each fibre of the normal bundle of f. Each fibre of the normal space of $\delta_2(f)$ has a natural decomposition as the direct sum of two unordered normal fibres of f each of which has a basis. Thus in this case the group of the bundle is the wreath product $1_k \wr \mathbb{Z}_2 \subseteq O(2k)$ where 1_k is the trivial subgroup of $O(k)$ (see for example [2], 2.1). This subgroup is conjugate to the subgroup $O(1)$ embedded through the diagonal map as $O(1) \xrightarrow{d} O(k) \subseteq O(2k)$. Thus the normal bundle of $\delta_2(f)$ has a normal k-frame field so that by the Smale-Hirsch Theorem there is an immersion of $\Delta_2(f)$ in $\mathbb{R}^n$. The structure group for this immersion is $O(1)$ embedded diagonally in $O(k)$ and so the immersion has a skew-framing as required. □

3 James periodicity and the Kahn-Priddy Theorem

For $0 \leq a \leq b$, the truncated real projective space $\mathbb{R}\mathrm{P}^b/\mathbb{R}\mathrm{P}^{a-1}$ will be denoted by P_a^b. In the case of $a = 0$, $\mathbb{R}\mathrm{P}^{-1}$ is interpreted as the empty set so that P_0^b is $\mathbb{R}\mathrm{P}^b$ with a disjoint base point.

The theorem we need is the following.

Theorem 6 (James Periodicity Theorem). *For each positive integer a and each non-negative integer k there is a homeomorphism*

$$\Sigma^{b(a)} P_k^{k+a-1} = P_k^{k+a-1} \wedge S^{b(a)} \to P_{k+b(a)}^{k+b(a)+a-1}.$$

Proof. This result is implicit in [6]. It may be obtained quite explicitly as follows.

The well-known Hurwitz-Radon-Eckmann theorem in linear algebra (see [5]) tells us that there is a non-singular bilinear map

$$f\colon \mathbb{R}^a \times \mathbb{R}^{b(a)} \to \mathbb{R}^{b(a)}.$$

This induces a homeomorphism

$$(\mathbb{R}\mathrm{P}^{k+a-1} - \mathbb{R}\mathrm{P}^{k-1}) \times \mathbb{R}^{b(a)} \to \mathbb{R}\mathrm{P}^{k+b(a)+a-1} - \mathbb{R}\mathrm{P}^{k+b(a)-1}.$$

by $([x,y],z) \mapsto [x,y,f(x,z)]$ for $x \in \mathbb{R}^a - \{0\}$, $y \in \mathbb{R}^k$, $z \in \mathbb{R}^{b(a)}$.

On taking one point compactifications this gives the homeomorphism in the theorem. □

Adams' version of the Kahn-Priddy Theorem ([1], Formulation 2.3) gives the following result.

Theorem 7 (Kahn-Priddy Theorem). *A stable map $\mathbb{R}\mathrm{P}^{2m} \nrightarrow S^0$ which induces an isomorphism $\pi_1^S \mathbb{R}\mathrm{P}^{2m} \to \pi_1^S (\cong Z/2)$ induces an epimorphism of 2-primary stable homotopy groups ${}_2\pi_n^S \mathbb{R}\mathrm{P}^{2m} \to {}_2\pi_n^S$ for $1 \leq n \leq 2m$.*

These two results now enable us to prove the main results of this paper.

Proof of Theorem 2. For $m \geq 1$, let

$$q\colon P^{\infty}_{b(2m+1)} \to S^{b(2m+1)}$$

be the co-attaching map of the bottom cell of $P^{\infty}_{b(2m+1)-1}$, i.e. the map q in the cofibre sequence (1) of Lemma 4 in the case $k = b(2m+1)$. By Lemma 4, the theorem is equivalent to the statement that q induces an epimorphism of stable homotopy groups $\pi^S_{n+b(2m+1)}(\,)$ for $1 \leq n \leq 2m$.

Firstly, observe that $S^{b(2m+1)} \xrightarrow{i} P^{\infty}_{b(2m+1)} \xrightarrow{q} S^{b(2m+1)}$ is a map of degree 2 since $b(2m+1)$ is even. This means that $q \circ i$ induces isomorphisms of p-primary stable homotopy groups ${}_p\pi^S_{b(2m+1)+n}(\,)$ for $p \neq 2$ and $n \geq 1$ and so q induces epimorphisms of such groups.

Now, by the James Periodicity Theorem, $P^{b(2m+1)+2m}_{b(2m+1)}$ is homeomorphic to $\Sigma^{b(2m+1)} P^{2m}_0$ and so the restriction of q

$$q|\colon P^{b(2m+1)+2m}_{b(2m+1)} \to S^{b(2m+1)}$$

determines a stable map

$$q_0\colon P^{2m}_0 \not\to S^0.$$

Composing this with the inclusion map $\mathbb{R}\mathrm{P}^{2m} \to P^{2m}_0$ gives a stable map

$$q_1\colon \mathbb{R}\mathrm{P}^{2m} \not\to S^0.$$

The cofibre of $q|$ is $\Sigma P^{b(2m+1)+2m}_{b(2m+1)-1}$. Since $b(2m+1)$ is a multiple of 4,

$$\pi^S_{b(2m+1)} P^{b(2m+1)+2m}_{b(2m+1)-1} \cong \pi^S_{b(2m+1)} P^{b(2m+1)+1}_{b(2m+1)-1} \cong \pi^S_4 P^5_3 \quad \text{(by James Periodicity)}.$$

From the cofibre sequence

$$S^3 \xrightarrow{2\iota} S^3 \to P^4_3 \to S^4 \xrightarrow{2\iota} S^4,$$

$\pi^S_4 P^4_3 \cong \mathbb{Z}/2$ generated by the image of the suspension of the Hopf map $\eta_3 \in \pi^S_4 S^3$. This maps to zero under the epimorphism $\pi^S_4 P^4_3 \to \pi^S_4 P^5_3$ (since $Sq^2\colon H^3(P^5_3;\mathbb{Z}/2) \to H^5(P^5_3;\mathbb{Z}/2)$ is non-zero implying that the 5-cell in P^5_3 is attached by the generator of $\pi^S_4 P^4_3$) and so $\pi^S_{b(2m+1)+1} \Sigma P^{b(2m+1)+2m}_{b(2m+1)-1} \cong \pi^S_{b(2m+1)} P^{b(2m+1)+2m}_{b(2m+1)-1} \cong \pi^S_4 P^5_3 = 0$.

This shows that $q|$ induces an epimorphism of $\pi^S_{b(2m+1)+1}(\,)$ or equivalently that q_0 induces an epimorphism of $\pi^S_1(\,)$. It follows that q_1 induces an epimorphism and so an isomorphism of π^S_1, since, as above, the stable map $S^0 \xrightarrow{i} P^{2m}_0 \overset{q_0}{\not\to} S^0$ is a map of degree 2 and so induces the zero homomorphism on $\pi^S_1 \cong Z/2$.

It now follows from the Kahn-Priddy Theorem that the stable map q_1 induces an epimorphism of ${}_2\pi^S_n(\ )$ for $1 \leq n \leq 2m$. The same must therefore be true of the stable map q_0 and so the map q induces an epimorphism of ${}_2\pi^S_{n+b(2m+1)}(\ )$ for $1 \leq n \leq 2m$.

Combining this with the earlier observation about mod p stable homotopy we obtain that the map q induces an epimorphism of $\pi^S_{n+b(2m+1)}(\,)$ for $1 \leq n \leq 2m$ as required. □

Proof of Theorem 1. For $1 \leq n \leq 2m$, continuing the long exact sequence in stable homotopy coming from the cofibre sequence (1) in Lemma 4 with $k = b(2m+1)$, the above result that q_* induces an epimorphism of stable homotopy groups is equivalent to the vanishing of the homomorphisms

$$(\Sigma i)_*\colon \pi^S_{n+b(2m+1)} S^{b(2m+1)} \to \pi^S_{n+b(2m+1)} \Sigma P^{b(2m+1)+2m}_{b(2m+1)-1}.$$

By Lemma 3, this observation is equivalent to Theorem 1. □

Notice that the above arguments show that for an odd prime p there are p-local versions of Theorems 1 and 2 in which $b(2m+1)$ is replaced by any even positive integer.

4 Examples

Proposition 8. *The bordism group* $\mathrm{Imm}^{\mathrm{sfr}}(2,3)$ *is cyclic of order 2.*

Proof. This is clear since, by the Pontrjagin-Thom-Wells Theorem, this bordism group is isomorphic to the stable homotopy group $\pi^S_5 P^\infty_3$. An easy calculation shows that this is cyclic of order 2 generated by an element $\eta^\flat_4$ which maps to the element $\eta_4 \in \pi^S_5 P^\infty_4$ given by composing the double suspension of the Hopf map $\eta \in \pi_3 S^2 \cong \mathbb{Z}$ with the inclusion of the bottom cell. □

A generator for $\mathrm{Imm}^{\mathrm{sfr}}(2,3) \cong \mathbb{Z}/2$ may be described as follows.

Let $f\colon K^2 \looparrowright \mathbb{R}^3$ be the immersion of the Klein bottle obtained by constructing a cylinder on a 'figure eight' and identifying the ends after rotating through an angle π. This represents an element of order 4 in the group $\mathrm{Imm}^{\mathrm{sfr}}(2,1) \cong \mathbb{Z}/8$ ([4]). Let κ be the normal line bundle of this immersion of the Klein bottle. This bundle is the pull back of the canonical line bundle over the circle under the standard bundle map $K^2 \to S^1$ and so the bundle 2κ over K^2 is trivial. We fix an arbitrary bundle isomorphism $2\kappa \cong 2\epsilon$ where ϵ is the trivial line bundle over K^2. Let $g\colon K^2 \looparrowright \mathbb{R}^5$ be the composition of f with the standard inclusion $\mathbb{R}^3 \subset \mathbb{R}^5$ (this is in fact regularly homotopic to an embedding as we are in the stable range). Then there is an isomorphism $\phi\colon \nu_g \cong \kappa \oplus 2\epsilon \cong 3\kappa$ giving a skew-framing on g.

Proposition 9. *This skew-framed immersion* $(g\colon K^2 \looparrowright \mathbb{R}^5, \phi)$ *represents a generator of* $\mathrm{Imm}^{\mathrm{sfr}}(2,3) \cong \mathbb{Z}/2$.

In order to prove this result we describe a geometrical construction corresponding to the homomorphism of stable homotopy induced by the middle map of the cofibre sequence $S^k \to P^\infty_k \to P^\infty_{k+1} \to S^{k+1}$.

Given an immersion $f\colon M^n \looparrowright \mathbb{R}^{n+k}$ with a skew-framing $\nu_f \cong k\kappa$ we can construct a submanifold L^{n-1} of M^n representing the Poincaré dual of $w_1(\kappa) \in H^1(M;\mathbb{Z}/2)$ by taking the transverse self-intersection of the zero section of the bundle κ. The normal bundle of the inclusion $i\colon L \to M$ is the pull-back of the bundle κ and so the immersion $f\circ i\colon L \to M \looparrowright \mathbb{R}^{n+k}$ has a skew-framing $\nu_{f\circ i} \cong (k+1)i^*\kappa$. This construction is clearly bordism invariant and gives a homomorphism $\omega\colon \mathrm{Imm}^{\mathrm{sfr}}(n,k) \to \mathrm{Imm}^{\mathrm{sfr}}(n-1,k+1)$.

Lemma 10. *This homomorphism*

$$\omega\colon \mathrm{Imm}^{\mathrm{sfr}}(n,k) \to \mathrm{Imm}^{\mathrm{sfr}}(n-1,k+1)$$

corresponds under the Ponrtjagin-Thom-Wells isomorphism to the map

$$\pi^S_{n+k} P^\infty_k \to \pi^S_{n+k} P^\infty_{k+1}$$

induced by the collapse of the bottom cell.

Proof. Let $\kappa\colon M \to \mathbb{R}\mathrm{P}^N$ be the map representing the line bundle κ where N is large. Then, if choose κ transverse to $\mathbb{R}\mathrm{P}^{N-1} \subseteq \mathbb{R}^N$, the submanifold L is given by $\kappa^{-1}(\mathbb{R}\mathrm{P}^{N-1})$. Since $k\gamma_N^1$ may be taken to be the normal bundle of $\mathbb{R}\mathrm{P}^N$ in $\mathbb{R}\mathrm{P}^{N+k}$ the skew-structure on M gives the following diagram.

$$\begin{array}{ccccc} L & \xrightarrow{i} & M & \longrightarrow & E(\nu_f) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}\mathrm{P}^{N-1} & \longrightarrow & \mathbb{R}\mathrm{P}^N & \longrightarrow & \mathbb{R}\mathrm{P}^{N+k} \end{array}$$

This induces the following diagram of Thom complexes.

$$\begin{array}{ccc} T(\nu_f) & \longrightarrow & T(\nu_{f\circ i}) \\ \downarrow & & \downarrow \\ P_k^{N+k} & \longrightarrow & P_{k+1}^{N+k} \end{array}$$

This gives the result by applying the Pontrjagin-Thom construction. □

Proof of Proposition 9. In the example, since κ is the normal bundle of the immersion $f\colon K^2 \to \mathbb{R}^3$, the submanifold L is a circle S^1 embedded in K as one 'figure eight' section. The restriction of κ to this circle is trivial and so the induced skew-framing of the immersion $S^1 \to K \looparrowright \mathbb{R}^3$ is a framing, clearly regularly homotopic to the standard Hopf framing representing a generator of $\pi_3 S^2$. It follows that the element of $\pi_5^S P_3^\infty$ represented by $g\colon K \looparrowright \mathbb{R}^5$ maps to $\eta_4 \in \pi_5^S P_4^\infty$ as required. □.

It should be observed from Theorem 2 that the transfer homomorphism $\mathrm{Imm}^{\mathrm{sfr}}(2,3) \to \mathrm{Imm}^{\mathrm{fr}}(2,3)$ is an isomorphism (since $\mathrm{Imm}^{\mathrm{fr}}(2,3) \cong \pi_5 S^3 \cong \mathbb{Z}/2$). Applying the double cover construction to the above immersion $g\colon K^2 \looparrowright \mathbb{R}^5$ of the Klein bottle gives a framed immersion of the torus which represents the generator of $\mathrm{Imm}^{\mathrm{fr}}(2,3)$.

An exact sequence

Finally, we observe that the following result.

Theorem 11. *For all n and $k \geq 0$ the following sequence is exact:*

$$\begin{aligned} \mathrm{Imm}^{\mathrm{fr}}(n,k) \xrightarrow{\sigma} \mathrm{Imm}^{\mathrm{sfr}}(n,k) &\xrightarrow{\omega} \mathrm{Imm}^{\mathrm{sfr}}(n-1,k+1) \\ &\xrightarrow{\tau} \mathrm{Imm}^{\mathrm{fr}}(n-1,k+1) \xrightarrow{\sigma} \mathrm{Imm}^{\mathrm{sfr}}(n-1,k+1). \end{aligned}$$

Proof. This follows from the long exact sequence in stable homotopy arising from the cofibre sequence $S^k \to P_k^\infty \to P_{k+1}^\infty \to S^{k+1}$ using Lemma 3, Lemma 4 and Lemma 10. □

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P.M. Akhmet'ev
Steklov Mathematical Institute RAS
Moscow
Russia
pmakhmet@mi.ras.ru

P.J. Eccles
School of Mathematics
University of Manchester
Manchester
Great Britain
pjeccles@manchester.ac.uk