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Homotopy Limits and Colimits

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1. Introduction

It is well-known that the canonical projection functor from the category $\mathcal{Top}$ of topological spaces and maps (=continuous functions) to the category $\mathcal{Top}_h$ of topological spaces and homotopy classes of maps does not preserve limits and colimits and that $\mathcal{Top}_h$ has very few limits and colimits. The same holds for the category $\mathcal{Top}^*$ of based spaces and based maps and its homotopy category $\mathcal{Top}_h^*$. Therefore, when dealing with constructions involving homotopies, one often has to substitute limits and colimits by something else, and the homotopy limits and colimits are in many cases the spaces having the universal properties one wants.

Let $\mathscr{C}$ be a small category and $D: \mathscr{C} \to \mathcal{Top}$ [or $D: \mathscr{C} \to \mathcal{Top}^*$] a $\mathscr{C}$-diagram in $\mathcal{Top}$ [respectively $\mathcal{Top}^*$]. Let

$$\mathscr{C}_n(A,B) = \{(f_n, \ldots, f_1) \in (\operatorname{mor} \mathscr{C})^n \mid f_n \circ \cdots \circ f_1 \colon A \to B \text{ is defined in } \mathscr{C}\} \qquad n>0$$

$$\mathscr{C}_0(A,A) = \{(\mathrm{id}_A)\} \qquad \mathscr{C}_0(A,B) = \varnothing \qquad \text{for } A \neq B.$$

(1.1) **Definition.** The *homotopy colimit* of D, h-colim D is

$$\Big(\coprod_{A,B \in \mathscr{C}} \coprod_{n=0}^{\infty} \mathscr{C}_n(A,B) \times I^n \times D(A) \Big) \cup \{*\}/\sim$$

where I is the unit interval and $\{*\}$ an extra point, with the relations

$$(t_n, f_n, \ldots, t_1, f_1; x) = \begin{cases} (t_n, f_n, \ldots, t_2, f_2; x) & f_1 = \mathrm{id} \\ (t_n, f_n, \ldots, f_{i+1}, t_i t_{i-1}, f_{i-1}, \ldots, f_1; x) & f_i = \mathrm{id},\ 1<i \\ (t_n, f_n, \ldots, t_{i+1}, f_{i+1} \circ f_i, t_{i-1}, \ldots, f_1; x) & t_i = 1,\ i<n \\ (t_{n-1}, f_{n-1}, \ldots, f_1; x) & t_n = 1 \\ \big(t_n, f_n, \ldots, f_{i+1}; D(f_i \circ \cdots \circ f_1)(x)\big) & t_i = 0 \\ * & x = \text{base point} \end{cases}$$

with $\{*\}$ as base point for a diagram D in $\mathcal{Top}^*$. The unbased version is obtained by deleting $\{*\}$ and the last relation. The *homotopy limit* of D is defined dually.

To my knowledge, Puppe was the first to use a homotopy colimit as a "substitute" for a colimit. Let C_f be the reduced mapping cone of the based map $f\colon X \to Y$. Then he proved [13] that the sequence

$$X \xrightarrow{f} Y \xrightarrow{Pf} C_f \tag{1.2}$$

induces an exact sequence of based sets

$$[X, Z] \leftarrow [Y, Z] \leftarrow [C_f, Z]$$

where $[X, Z]$ denotes the set of based homotopy classes of maps from X to Z. So C_f, which is the homotopy colimit of the diagram

$$Y \xleftarrow{f} X \to * = \text{one-point space}$$

in $\mathcal{T}op^*$, "substitutes" the cokernel of f. In his second chapter, Puppe showed that the sequence (1.2) is invariant under homotopy equivalences. More precisely, given a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ X' & \xrightarrow{g} & Y' \end{array} \tag{1.3}$$

we can extend it to a diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{Pf} & C_f \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle H} \\ X' & \xrightarrow{g} & Y' & \xrightarrow{Pg} & C_g \end{array}$$

whose right square commutes. Moreover, if h and k are homotopy equivalences, then H is a homotopy equivalence. This result, which requires some work to be proved, is a special case of a more general theorem about homotopy colimits. Note first that (1.3) can be extended to a homotopy commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{f} & X & \longrightarrow & * \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} & & \| \\ Y' & \xleftarrow[g]{} & X' & \longrightarrow & * \end{array}$$

Our general result is (it follows from (4.6) below)

(1.4) **Theorem.** *Let $\mathcal{C}$ be a small category and let D and E be $\mathcal{C}$-diagrams in $\mathcal{T}op$ [or in $\mathcal{T}op^*$ of well-pointed spaces]. If $f\colon D \to E$ is a [based] mor-*

phism of $\mathscr{C}$-diagrams up to compatible [based] homotopies such that for all $A \in \operatorname{ob} \mathscr{C}$ the map $D(A) \to E(A)$ given by f is a homotopy equivalence, then f induces a homotopy equivalence h-colim $D \to$ h-colim E.

Another special case of (1.4) occurring in the literature is due to Milnor [12; Appendix]. Given a topological space X and a sequence of subspaces $X_0 \subset X_1 \subset X_2 \subset \cdots$, he considers the question to what extent is the homotopy type of X determined by the homotopy types of the X_i. For this he considers the "telescope"

$$X_\Sigma = X_0 \times [0,1] \cup X_1 \times [1,2] \cup X_2 \times [2,3] \cup \cdots$$

topologized as a subset of $X \times \mathbb{R}$. It is easy to see that X_Σ is naturally homotopy equivalent to the homotopy colimit of the diagram

$$X_0 \subset X_1 \subset X_2 \subset \cdots.$$

Let Y be another space and $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ a sequence of subspaces; then his main result (which also is a consequence of (1.4)) is

(1.5) **Theorem** (Milnor). *Let f: $X \to Y$ be a map which carries each X_i into Y_i by a homotopy equivalence; then f induces a homotopy equivalence $X_\Sigma \to Y_\Sigma$.*

So the question reduces to the problem of showing that X_Σ is homotopy equivalent to X. This is, for example, the case if $X = \bigcup_{i=0}^{\infty} X_i$ and $X_i \subset X_{i+1}$ is a cofibration.

A more general result along these lines has been proved by tom Dieck [5] using the work of Segal [14].

(1.6) **Definition.** A covering $U = (X_\alpha \mid \alpha \in \mathrm{A})$ of X is called *numerable* if there exists a locally finite partition of unity $(t_\alpha \mid \alpha \in \mathrm{A})$ such that the closure of $t_\alpha^{-1}(0,1]$ is contained in X_α.

(1.7) **Theorem** (tom Dieck). *Let $U = (X_\alpha \mid \alpha \in \mathrm{A})$ and $V = (Y_\alpha \mid \alpha \in \mathrm{A})$ be numerable coverings of X and Y. For any non-empty subset $\sigma \subset \mathrm{A}$ let $X_\sigma = \bigcap_{\alpha \in \sigma} X_\alpha$. Let f: $X \to Y$ be a map which carries each X_σ, $\sigma \subset \mathrm{A}$ finite, into Y_σ by a homotopy equivalence. Then f is a homotopy equivalence.*

This theorem is an immediate consequence of (1.4). By assumption, the map f induces a morphism of the diagram of the X_σ, $\sigma \subset \mathrm{A}$ finite, and their inclusions to the diagram of the Y_σ and their inclusions, and the maps of this morphism are homotopy equivalences. Hence the homotopy colimits are homotopy equivalent. But by a result of Segal [14], the homotopy colimits are homotopy equivalent to X respectively Y.

Milnor's telescope construction can also be used to prove (see [11])

(1.8) **Proposition** (Milnor). *Let $X_0 \subset X_2 \subset \cdots$ be a sequence of cofibrations and $X = \bigcup_{n=0}^{\infty} X_i$. Let k^* be an additive cohomology theory. Then there is an exact sequence*

$$0 \to \lim^1 k^{q-1}(X_i) \to k^q(X) \to \lim k^q(X_i) \to 0$$

where $\lim^p$ *denotes the p-th right derived of the limit.*

One of the standard proofs uses a Puppe sequence argument to show that there is an exact sequence

$$(1.9) \qquad 0 \to \lim^1 k^{q-1}(X_i) \to k^q(X_\Sigma) \to \lim k^q(X_i) \to 0$$

for any sequence of spaces $X_0 \subset X_1 \subset X_2 \subset \cdots$ and that X_Σ has the homotopy type of X under the special assumptions on this sequence stated in the proposition. The sequence (1.9) generalizes to homotopy colimits of arbitrary diagrams: Let $D: \mathscr{C} \to \mathscr{T}\!op$ be a diagram and k^* a generalized cohomology theory. In § 9 we show that there is a spectral sequence

$$E_2^{p,q} \cong \lim^p k^q(D) \Rightarrow k^{p+q}(\text{h-colim}\, D).$$

If D is the diagram of Proposition (1.8), the spectral sequence collapses and induces the short exact sequence (1.9). If D is the diagram

$$X_1 \supset A \subset X_2$$

whose maps are cofibrations, then the spectral sequence collapses and induces the Mayer-Vietoris sequence of $(X_1 \cup_A X_2, X_1, X_2)$.

Given a $\mathscr{C}$-diagram one often wants to substitute some of its spaces by a homotopy equivalent one. One obtains an induced diagram which is only a $\mathscr{C}$-diagram up to compatible or coherent homotopies (a precise definition will be given later). This leads us to consider homotopy $\mathscr{C}$-diagrams which are $\mathscr{C}$-diagrams up to coherent specified homotopies. Such diagrams also occur "in nature": Let ΩX be the space of based loops on a based space X. Define $\lambda_n: (\Omega X)^n \to \Omega X$ by

$$\lambda_n(\omega_1, \ldots, \omega_n)(t) = \omega_i(nt - i + 1), \qquad t \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \subset I.$$

Then the diagram of spaces $(\Omega X)^n$ and of maps $\lambda_{r_1} \times \cdots \times \lambda_{r_n}: (\Omega X)^m \to (\Omega X)^n$ with $m = r_1 + \cdots + r_n$ and of their composites is such a diagram.

Led by Puppe's considerations (see (1.3)) we define as maps between homotopy $\mathscr{C}$-diagrams a morphism between usual diagrams up to coherent homotopy commutativity relations. Introducing a suitable notion of homotopy between such maps we can define the category $\mathscr{H}\mathscr{C}$ of homotopy $\mathscr{C}$-diagrams, provided $\mathscr{C}$ satisfies some weak conditions. There is a canonical functor $J: \mathscr{T}\!op_{\mathscr{k}} \to \mathscr{H}\mathscr{C}$ mapping a space X to the constant homotopy $\mathscr{C}$-diagram on X. Extending the definition of the homotopy limit and colimit to homotopy $\mathscr{C}$-diagrams we show that they are functors from $\mathscr{H}\mathscr{C}$ to $\mathscr{T}\!op_h$ and that h-lim is right adjoint and h-colim left adjoint to J. This justifies the terminology homotopy limit and co-limit, because recall that the functors lim and colim from the category $\mathscr{M}\mathscr{C}$ of $\mathscr{C}$-diagrams to $\mathscr{T}\!op$ are right respectively left adjoint to the canonical inclusion functor $\mathscr{T}\!op \to \mathscr{M}\mathscr{C}$.

The first part of this paper treats the category of homotopy $\mathscr{C}$-diagrams, in Section 5–8 we introduce the homotopy limit and colimit functor and compare them with the definition of Segal [14] who defined the homotopy colimit for commutative diagrams. In the remaining part we introduce the spectral sequences and give some minor applications.

As mentioned before, the notions introduced in this paper are not completely new. Segal has defined homotopy colimits for commutative diagrams. His idea was taken up by Bousfield and Kan who gave a first detailed treatment of homotopy limits and colimits of commutative diagrams in the category of simplicial sets. Our treatment has been developed independently of theirs. Nevertheless there is some overlap in the results. For example, our spectral sequences coincide with theirs if we restrict to commutative diagrams and work semisimplicially. Our treatment is more general because we allow diagrams which commute only up to coherent homotopies so that we for example may change a diagram by homotopies. The connection to the Bousfield-Kan theory is expressed by [4; Chap. IX, § 8] and our results (4.8) and (6.5). They show that our category $\mathscr{H}\mathscr{C}$ is a model category of their category $\mathrm{Ho}(\mathscr{S}^{\mathscr{C}})$. Our method is an outgrowth of the author's joint work with Boardman [3]. The machine developed there enables us to give a more or less satisfactory treatment of the category $\mathscr{H}\mathscr{C}$ and to prove results like Theorem (1.4). I also should mention a paper of Mather [10], in which he defines homotopy limits and colimits for special types of homotopy commutative diagrams, but important homotopy limits and colimits such as the mapping torus are not contained in his concept, nor does he introduce the category of such diagrams.

Some of the results on homotopy colimits have been sketched in [3].

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2. Homotopy Diagrams

Let $\mathscr{T}\!op^{\circ} \subset \mathscr{T}\!op^{*}$ be the full subcategory of well-pointed topological spaces X, i.e. $(X, *)$ is a NDR (=neighbourhood deformation retract), where $* \in X$ is the base point. Here we call a pair of spaces (X, A) a NDR if $A \subset X$ is a closed unbased cofibration. Let $\mathscr{T}\!op_h^{\circ} \subset \mathscr{T}\!op_h^{*}$ denote the associated homotopy categories. There is an inclusion functor

$$(+): \mathcal{T}op \to \mathcal{T}op^{\circ}$$

sending X to $X^+ = X \cup \{*\}$ with the additional point $*$ as base point. Since the theory of homotopy limits and colimits can best be developed in the category $\mathcal{CG}$ of compactly generated spaces [17], called k-spaces, but one often wants results in full generality, we work in $\mathcal{CG}$ and $\mathcal{T}op$ respectively their based versions $\mathcal{CG}^{\circ}, \mathcal{CG}^*, \mathcal{T}op^{\circ}, \mathcal{T}op^*$ simultaneously. Of course, if we deal with k-spaces, products, sums, and other limits and colimits are formed in the category $\mathcal{CG}$.

(2.1) **Definition.** A *pretopological category* is a small category $\mathcal{C}$ whose morphism sets are topologized. If in addition composition is continuous, $\mathcal{C}$ is called *topological*. Call $\mathcal{C}$ *well-pointed* if it is topological and each pair $(\mathcal{C}(A,A), \mathrm{id}_A)$, $A \in \mathrm{ob}\,\mathcal{C}$, is a NDR. A *continuous functor* of pretopological categories is a functor which is continuous as map of the morphism spaces.

A small category in the usual sense is considered as topological category with the discrete topology and hence called *discrete* (not to be mixed up with the category theoretical notion of a discrete category).

Let $\mathcal{TC}$ be the category of small topological categories and continuous functors. We define a functor

$$T: \mathcal{TC} \to \mathcal{TC}$$

as follows: Let $\mathcal{C}_n(A,B) = \{(f_n, f_{n-1}, \ldots, f_1) \in (\mathrm{mor}\,\mathcal{C})^n \mid f_n \circ \cdots \circ f_1 \colon A \to B$ is defined in $\mathcal{C}\}$, $n > 0$, with the subspace topology from $(\mathrm{mor}\,\mathcal{C})^n$, and put

$$\mathcal{C}_0(A,B) = \begin{cases} \{\mathrm{id}_A\} & A = B \\ \varnothing & A \neq B. \end{cases}$$

Define $\mathrm{ob}\,T\mathcal{C} = \mathrm{ob}\,\mathcal{C}$ and

$$T\mathcal{C}(A,B) = \coprod_{n \geq 0} \mathcal{C}_{n+1}(A,B) \times I^n.$$

Composition in $T\mathcal{C}$ is given by

$$(f_n, t_n, \ldots, f_0) \circ (g_m, u_m, \ldots, g_0) = (f_n, t_n, \ldots, f_0, 0, g_m, u_m, \ldots, g_0).$$

If $F: \mathcal{C} \to \mathcal{D}$ is a continuous functor, we define $TF: T\mathcal{C} \to T\mathcal{D}$ by

$$TF(f_n, t_n, \ldots, f_0) = (Ff_n, t_n, \ldots, Ff_0).$$

There is an *augmentation* functor

$$\varepsilon = \varepsilon(\mathcal{C}): T\mathcal{C} \to \mathcal{C}$$

given by $\varepsilon(f_n, t_n, \ldots, f_0) = f_n \circ f_{n-1} \circ \cdots \circ f_0$, which is continuous, and a continuous, non-functorial *standard inclusion*

$$\eta = \eta(\mathscr{C}): \mathscr{C} \to T\mathscr{C}$$

defined by $\eta(f) = (f)$. Both are natural, i.e. if $F: \mathscr{C} \to \mathscr{D}$ is a continuous functor, then

$$\varepsilon(\mathscr{D}) \circ TF = F \circ \varepsilon(\mathscr{C})$$

$$\eta(\mathscr{D}) \circ F = TF \circ \eta(\mathscr{C}).$$

(2.2) **Definition.** Let $\mathscr{C}$ be a pretopological category. A (based) $\mathscr{C}$-*diagram* D consists of a function D_0: ob $\mathscr{C} \to$ ob $\mathscr{T}\!op^*$ and a collection of maps

$$D_{A,B}: \mathscr{C}(A, B) \times D_0 A \to D_0 B$$

one for each pair of objects (A, B) of $\mathscr{C}$ such that

(i) $D_{A,A}(\mathrm{id}_A; x) = x$ for $x \in D_0 A$,

(ii) $D_{A,B}(f; *) = *$, $*$ denotes base points,

(iii) $D_{A,B}(g \circ f; x) = D_{C,B}(g; D_{A,C}(f; x))$ for $f: A \to C, g: C \to B$.

Call a $\mathscr{C}$-diagram D *well-pointed* if each $D_0 A$ is well-pointed. An *unbased* $\mathscr{C}$-*diagram* D consists of a function D_0: ob $\mathscr{C} \to$ ob $\mathscr{T}\!op$ and maps $D_{A,B}: \mathscr{C}(A, B) \times D_0 A \to D_0 B$ satisfying (i) and (iii).

Remark. If we give $\mathscr{C}\mathscr{G}^*(X, Y)$ the k-function space topology respectively $\mathscr{T}\!op^*(X, Y)$ the compact-open topology, then the function D_0 together with the adjoints of the $D_{A,B}$ defines a functor $\mathscr{C} \to \mathscr{C}\mathscr{G}^*$ respectively $\mathscr{C} \to \mathscr{T}\!op^*$ which is continuous on the morphism spaces. The converse always holds in $\mathscr{C}\mathscr{G}^*$ because we have full adjointness there, but not in general in $\mathscr{T}\!op^*$ unless each space $D_0 A$ is locally compact or $\mathscr{C}$ discrete. Hence in $\mathscr{C}\mathscr{G}^*$ or if $\mathscr{C}$ is discrete our definition coincides with the usual one. The same holds for the unbased version. If we consider diagrams in $\mathscr{C}\mathscr{G}$ or $\mathscr{C}\mathscr{G}^*$, we, of course, assume that the spaces $\mathscr{C}(A, B)$ are k-spaces.

(2.3) **Definition.** Let $\mathscr{C}$ be a topological category. A (based) *homotopy* $\mathscr{C}$-*diagram*, or h$\mathscr{C}$-*diagram* for short, is a $T\mathscr{C}$-diagram D such that

$$D_{A,B}(f_n, t_n, \ldots, f_0; x) = \begin{cases} D_{A,B}(f_n, t_n, \ldots, t_2, f_1; x) & f_0 = \mathrm{id} \\ D_{A,B}(f_n, t_n, \ldots, f_{i+1}, t_{i+1} t_i, f_{i-1}, \ldots, f_0; x) & f_i = \mathrm{id},\ 0 < i < n \\ D_{A,B}(f_{n-1}, t_{n-1}, \ldots, f_0; x) & f_n = \mathrm{id} \\ D_{A,B}(f_n, t_n, \ldots, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \ldots, f_0; x) & t_i = 1 \end{cases}$$

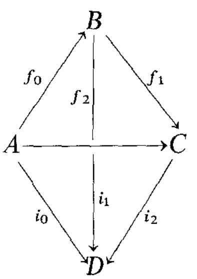
with $x \in D_0 A$, $(f_n, t_n, \dots, f_0) \in T\mathscr{C}(A, B)$. If D is well-pointed or unbased and satisfies the equations, we call it a *well-pointed* respectively *unbased* h$\mathscr{C}$-diagram.

We can define a quotient pretopological category $W\mathscr{C}$ of $T\mathscr{C}$ such that in $\mathscr{C}\mathscr{G}^*$ a h$\mathscr{C}$-diagram is a $W\mathscr{C}$-diagram (note that $W\mathscr{C}$ is topological if we work in $\mathscr{C}\mathscr{G}$). Define ob $W\mathscr{C} = \text{ob}\,\mathscr{C}$ and $W\mathscr{C}(A, B) = T\mathscr{C}(A, B)/\sim$ with the relations

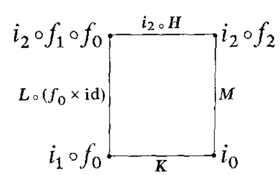
$$(f_n, t_n, \dots, f_0) = \begin{cases} (f_n, t_n, \dots, t_2, f_1) & f_0 = \text{id} \\ (f_n, t_n, \dots, f_{i+1}, t_{i+1} t_i, f_{i-1}, \dots, f_0) & f_i = \text{id},\ 0 < i < n \\ (f_{n-1}, t_{n-1}, \dots, f_0) & f_n = \text{id} \\ (f_n, t_n, \dots, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \dots, f_0) & t_i = 1. \end{cases}$$

Composition is induced by the composition in $T\mathscr{C}$. It is easily checked that a h$\mathscr{C}$-diagram induces a $W\mathscr{C}$-diagram and that the converse holds in $\mathscr{C}\mathscr{G}$, because identifications commute with products.

Example. Let $\mathscr{C}$ be the category given by the commutative diagram



Then a h$\mathscr{C}$-diagram is a homotopy commutative diagram of the same type with specific homotopies H: $f_1 \circ f_0 \simeq f_2$, K: $i_1 \circ f_0 \simeq i_0$, L: $i_2 \circ f_1 \simeq i_1$, M: $i_2 \circ f_2 \simeq i_0$ such that the loop



can be filled in by a specific homotopy.

The example suggests that a h$\mathscr{C}$-diagram is a $\mathscr{C}$-diagram up to coherent homotopies. This is indeed true as the following result shows, which we shall prove later.

(2.5) **Theorem.** *If $\mathscr{C}$ is well-pointed, the augmentation ε: $T\mathscr{C} \to \mathscr{C}$ induces a functor $\bar{\varepsilon}: W\mathscr{C} \to \mathscr{C}$ which is a homotopy equivalence on each morphism space.*

Let $\mathscr{L}_n$ be the category with objects $0, 1, 2, \ldots, n$ and exactly one morphism $i \leqq j$: $i \to j$ if $i \leqq j$.

(2.6) **Definition.** Let $\mathscr{C}$ be a topological category and D and E two $\mathscr{C}$-diagrams. A *homomorphism* f: $D \to E$ is a $(\mathscr{C} \times \mathscr{L}_1)$-diagram whose restriction to $\mathscr{C} \times 0$ is D and to $\mathscr{C} \times 1$ is E, or equivalently, a collection of based maps f_A: $D_0 A \to E_0 A$, one for each $A \in \mathrm{ob}\,\mathscr{C}$, such that

$$\begin{array}{ccc} \mathscr{C}(A,B) \times D_0 A & \xrightarrow{D_{A,B}} & D_0 B \\ \downarrow{\scriptstyle \mathrm{id} \times f_A} & & \downarrow{\scriptstyle f_B} \\ \mathscr{C}(A,B) \times E_0 A & \xrightarrow{E_{A,B}} & E_0 B \end{array}$$

commutes. The f_A are called the *underlying maps* of the homomorphism. A homomorphism of unbased $\mathscr{C}$-diagrams is defined similarly.

Again we find that this definition is equivalent to the usual one if we work in $\mathscr{C}\mathscr{G}^*$ or $\mathscr{C}\mathscr{G}$ or if $\mathscr{C}$ is discrete.

Homomorphisms as maps between h$\mathscr{C}$-diagrams are not good enough for our purposes. Instead we want the diagram of (2.6) commute up to coherent homotopies only.

A continuous functor F: $\mathscr{C} \to \mathscr{D}$ transforms a $\mathscr{D}$-diagram D into a $\mathscr{C}$-diagram $F^*(D)$ by

$$F^*(D)_0 = D_0 \circ (F|\mathrm{ob}\,\mathscr{C}), \qquad F^*(D)_{A,B} = D_{FA,FB} \circ \big(F|\mathscr{C}(A,B) \times \mathrm{id}\big).$$

If $\mathscr{C}$ is topological, F is the augmentation $\varepsilon(\mathscr{C})$: $T\mathscr{C} \to \mathscr{C}$, and D a $\mathscr{C}$-diagram, then $F^*(D)$ is a h$\mathscr{C}$-diagram. Hence we may consider a $\mathscr{C}$-diagram as a h$\mathscr{C}$-diagram. In "geometric" termes, a $\mathscr{C}$-diagram is a h$\mathscr{C}$-diagram with trivial homotopies. Just so, if F: $\mathscr{C} \to \mathscr{D}$ is a continuous functor and D a h$\mathscr{D}$-diagram, then $TF^*(D)$ is a h$\mathscr{C}$-diagram.

Every order-preserving maps α: $\{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ induces a functor $\mathscr{L}_m \to \mathscr{L}_n$ and hence a functor

$$T(\mathrm{Id} \times \alpha)\colon\ T(\mathscr{C} \times \mathscr{L}_m) \to T(\mathscr{C} \times \mathscr{L}_n).$$

Let δ_n^i: $\mathscr{L}_{n-1} \to \mathscr{L}_n$ and σ_n^i: $\mathscr{L}_{n-1} \to \mathscr{L}_n$ be the functors given by the maps

$$\{0, 1, \ldots, n-1\} \ni j \mapsto \begin{cases} j & j < i \\ j+1 & j \geqq i \end{cases} \quad \in \{0, 1, \ldots, n\}$$

$$\{0, 1, \ldots, n+1\} \ni j \mapsto \begin{cases} j & j \leqq i \\ j-1 & j > i \end{cases} \quad \in \{0, 1, \ldots, n\}.$$

Define a simplicial class $\mathscr{S}\mathscr{C}^*$ by taking as n-simplexes all $\mathrm{h}(\mathscr{C}\times\mathscr{L}_n)$-diagrams and defining the face and degeneracy operations d_n^i and s_n^i by $d_n^i(D)=T(\mathrm{Id}\times\delta_n^i)^*(D)$ and $s_n^i(D)=T(\mathrm{Id}\times\sigma_n^i)^*(D)$. Similarly, let $\mathscr{S}\mathscr{C}$ and $\mathscr{S}\mathscr{C}^\circ$ be the simplicial classes whose n-simplexes are all unbased respectively well-pointed $\mathrm{h}(\mathscr{C}\times\mathscr{L}_n)$-diagrams and whose face and degeneracy operations are defined as for $\mathscr{S}\mathscr{C}^*$. The corresponding versions in the category of k-spaces are denoted by $\mathscr{S}\mathscr{C}_{\ell}^*$, $\mathscr{S}\mathscr{C}_{\ell}$, and $\mathscr{S}\mathscr{C}_{\ell}^\circ$.

For convenience, we write A^i for $(A,i)\in\mathrm{ob}(\mathscr{C}\times\mathscr{L}_n)$ and denote the unique morphism $(\mathrm{id}_A, 0\leqq 1)\colon A^0\to A^1$ by j_A.

From now on we state our definitions for (based) $\mathrm{h}\mathscr{C}$-diagrams in $\mathscr{T}\!op^\circ$ only. The corresponding definitions for unbased or well-pointed $\mathrm{h}\mathscr{C}$-diagrams are obtained by substituting "$\mathrm{h}\mathscr{C}$-diagram" by "unbased $\mathrm{h}\mathscr{C}$-diagram" or "well-pointed $\mathrm{h}\mathscr{C}$-diagram".

(2.7) **Definition.** Given two $\mathrm{h}\mathscr{C}$-diagrams D and E. A *homotopy homomorphism*, a *h-morphism* for short, from D to E is a $\mathrm{h}(\mathscr{C}\times\mathscr{L}_1)$-diagram H such that $d^0(H)=E$ and $d^1(H)=D$. We call the collection of maps

$$f_A\colon D_0 A\to E_0 A\colon x\mapsto H_{A^0,A^1}((j_A);\, x)$$

the *underlying maps* of H and say $\{f_A\}$ *carries* a h-morphism. Two h-morphisms H and K from D to E are called *simplicially homotopic* if there is a 2-simplex α in $\mathscr{S}\mathscr{C}^*$ such that $d^0(\alpha)=H$, $d^1(\alpha)=K$, and $d^2(\alpha)=s^0(D)$.

Given h-morphisms $H\colon D\to E$ and $K\colon E\to F$ of $\mathrm{h}\mathscr{C}$-diagrams we run into trouble when we try to define a composite h-morphism $K\circ H\colon D\to F$. We can define a composite by explicit construction but, as usually when homotopies are around, we cannot make composition associative. The way around this difficulty is suggested by the following property of the simplicial classes $\mathscr{S}\mathscr{C}^\circ$ and $\mathscr{S}\mathscr{C}$, which we shall prove later.

(2.8) **Lemma.** *If $\mathscr{C}$ is a well-pointed category, the simplicial classes $\mathscr{S}\mathscr{C}$, $\mathscr{S}\mathscr{C}^\circ$, $\mathscr{S}\mathscr{C}_{\ell}$, and $\mathscr{S}\mathscr{C}_{\ell}^*$ satisfy the restricted Kan extension condition, i.e. given $(n-1)$-simplexes $\alpha_0,\alpha_1,\ldots,\alpha_{r-1},\alpha_{r+1},\ldots,\alpha_n$, where $0<r<n$, such that $d^{j-1}\alpha_i=d^i\alpha_j$ for $0\leqq i<j\leqq n$, $i\neq r\neq j$, then there exists an n-simplex σ such that $d^i\sigma=\alpha_i$, $i\neq r$. In other words, the simplicial classes satisfy the usual Kan extension condition, except that the omitted face in the data is not allowed to be the first or the last.*

(2.9) **Definition.** Let $H\colon D^0\to D^1$ and $K\colon D^1\to D^2$ be h-morphisms of $\mathrm{h}\mathscr{C}$-diagrams. We call $L\colon D^0\to D^2$ a *composite* of H and K if there is a 2-simplex σ in $\mathscr{S}\mathscr{C}^*$ such that $d^0\sigma=K$, $d^1\sigma=L$, and $d^2\sigma=H$.

For a well-pointed $\mathscr{C}$ one deduces easily from (2.8) that composites of h-morphisms of well-pointed or unbased $\mathrm{h}\mathscr{C}$-diagrams or of arbitrary based or unbased $\mathrm{h}\mathscr{C}$-diagrams if we work with k-spaces always exist, that the relation of simplicial homotopy is an equivalence relation on the sets of h-morphism from such a h$\mathscr{C}$-diagram to another one, that the homotopy class of a composite $K \circ H$ depends only on the homotopy classes of K and H, and that up to homotopy composition is associative and has $s^0(D)$ as identity of D. For details see [3; p. 104ff.]. Hence we can form the categories $\mathscr{H}\mathscr{C}^o$, $\mathscr{H}\mathscr{C}$ of well-pointed respectively unbased h$\mathscr{C}$-diagrams and simplicial homotopy classes of the corresponding h-morphisms, and if we work with k-spaces, the categories $\mathscr{H}\mathscr{C}_k^*$ and $\mathscr{H}\mathscr{C}_k$ of arbitrary based respectively unbased h$\mathscr{C}$-diagrams and simplicial homotopy classes of the corresponding h-morphisms.

When we consider h-morphisms from a h$\mathscr{C}$-diagram D to a $\mathscr{C}$-diagram E, which we interprete as h$\mathscr{C}$-diagram, the homotopies inside E are trivial and hence could be deleted. The same holds for h-morphisms from a $\mathscr{C}$-diagram to a h$\mathscr{C}$-diagram. To get rid of unnecessary structure, we define $\mathscr{S}, \mathscr{T} \subset \mathscr{C} \times \mathscr{L}_n$ to be the full subcategories of all objects A^0 respectively A^n, $A \in \operatorname{ob} \mathscr{C}$, and modify the definition of a h-morphism as follows.

(2.10) **Definition.** Call a h$(\mathscr{C} \times \mathscr{L}_n)$-diagram H *source reduced* or $h_{\mathscr{S}}(\mathscr{C} \times \mathscr{L}_n)$-diagram if

$$H_{A,B}(f_p, t_p, \ldots, f_0; x) = H_{A,B}(f_p, t_p, \ldots, t_{i+2}, f_{i+1} \circ f_i \circ \cdots \circ f_0; x)$$

if $f_i \in \mathscr{S}$. Call it *target reduced* or $h_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_n)$-diagram if

$$H_{A,B}(f_p, t_p, \ldots, f_0; x) = H_{A,B}(f_p \circ \cdots \circ f_i \circ f_{i-1}, t_{i-1}, \ldots, t_1, f_0; x)$$

if $f_i \in \mathscr{T}$.

Let D, D' be h$\mathscr{C}$-diagrams, and E a $\mathscr{C}$-diagram. A *source reduced h-morphism* from E to D, a *$\mathscr{S}$-morphism* for short, is a h-morphism $H\colon \varepsilon(\mathscr{C})^*(E) \to D$ such that H is source reduced. Two $\mathscr{S}$-morphisms $H, K\colon E \to D$ are called *simplicially homotopic* if there is a $h_{\mathscr{S}}(\mathscr{C} \times \mathscr{L}_2)$-diagram σ such that $d^0(\sigma) = s^0(D)$, $d^1(\sigma) = K$, $d^2(\sigma) = H$. A $\mathscr{S}$-morphism $L\colon E \to D$ is called a *composite* of the $\mathscr{S}$-morphism $H\colon E \to D'$ and the h-morphism $K\colon D' \to D$ if there is a $h_{\mathscr{S}}(\mathscr{C} \times \mathscr{L}_2)$-diagram τ with $d^0(\tau) = K$, $d^1(\tau) = L$, and $d^2(\tau) = H$.

Analogously, a *target reduced* h-morphism or *$\mathscr{T}$-morphism* from D to E is a h-morphism $H\colon D \to \varepsilon(\mathscr{C})^*(E)$ such that H is target reduced. Two $\mathscr{T}$-morphisms $H, K\colon D \to E$ are called *simplicially homotopic* if there is a h$_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_2)$-diagram σ such that $d^0(\sigma) = H$, $d^1(\sigma) = K$, and $d^2(\sigma) = s^0(D)$. A $\mathscr{T}$-morphism $L\colon D \to E$ is called a *composite* of the h-morphism $H\colon D \to D'$ and the $\mathscr{T}$-morphism $K\colon D' \to E$ if there is a h$_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_2)$-diagram τ with $d^0(\tau) = K$, $d^1(\tau) = L$, and $d^2(\tau) = H$.

There is a modified version of (2.8) which implies that simplicial homotopy is an equivalence relation on the sets of $\mathscr{S}$-morphisms and $\mathscr{T}$-morphisms between appropriate diagrams, that a composite of a $\mathscr{S}$-

morphism with a h-morphism and of a h-morphism with a $\mathscr{T}$-morphism of such diagrams always exists and that its homotopy class depends only on the homotopy class of the h-morphism, $\mathscr{S}$-morphism, or $\mathscr{T}$-morphism.

(2.11) **Lemma.** *Let $\mathscr{C}$ be a well-pointed category. Given well-pointed [or unbased, or if we work with k-spaces arbitrary based or unbased] $\mathrm{h}_{\mathscr{S}}(\mathscr{C}\times\mathscr{L}_{n-1})$-diagrams $\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_n$ where $1 \leqq r < n$, and a well-pointed [respectively unbased, arbitrary based or unbased] $\mathrm{h}(\mathscr{C}\times\mathscr{L}_{n-1})$-diagram α_0 such that $d^{j-1}\alpha_i = d^i\alpha_j$ for $0 \leqq i < j \leqq n$, $i \neq r \neq j$, then there exists such a $\mathrm{h}_{\mathscr{S}}(\mathscr{C}\times\mathscr{L}_n)$-diagram σ with $d^i\sigma = \alpha_i$, $i \neq r$.*

Analogously, given such $\mathrm{h}_{\mathscr{T}}(\mathscr{C}\times\mathscr{L}_{n-1})$-diagrams $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_{n-1}$ where $0 < r \leqq n-1$, and such a $\mathrm{h}(\mathscr{C}\times\mathscr{L}_{n-1})$-diagram α_n with $d^{j-1}\alpha_i = d^i\alpha_j$ for $0 \leqq i < j \leqq n$, $i \neq r \neq j$, then there exists such a $\mathrm{h}_{\mathscr{T}}(\mathscr{C}\times\mathscr{L}_n)$-diagram σ with $d^i\sigma = \alpha_i$, $i \neq r$.

The proofs of (2.8) and (2.11) for k-spaces can be found in [3; chap. IV or VII]. For the topological cases the tools for the proofs are developed in the next section.

3. Extension and Lifting Results

From now on we only consider the based case. The necessary modifications of the proofs for the unbased case are mentioned at the end of each section.

In proving (2.8), for example, we are given a $\mathscr{V}$-diagram E for a subcategory $\mathscr{V}$ of $T\mathscr{C}$, which we have to extend to a $T\mathscr{C}$-diagram. Since $T\mathscr{C}(A,B) = \coprod_{n\geqq 0} \mathscr{C}_{n+1}(A,B)\times I^n$, we do this inductively by constructing the required map for each space $R^n(A,B) = \mathscr{C}_{n+1}(A,B)\times I^n$. Let $Q\mathscr{C}\subset T\mathscr{C}$ be the subcategory of all morphisms $(f_n, t_n, \ldots, f_0)$ with some $t_i = 0$ or 1 or some f_i an identity, let $Q^n(A,B) = R^n(A,B)\cap Q\mathscr{C}(A,B)$, and let $V^n(A,B) = R^n(A,B)\cap\mathscr{V}(A,B)$. Suppose we have constructed the maps $D^k_{A,B}\colon R^k(A,B)\times D_0A \to D_0B$ of a $\mathrm{h}\mathscr{C}$-diagram D for $k<n$. Then $D^n_{A,B}$ is determined by the $D^k_{A,B}$, $k<n$, and the given $\mathscr{V}$-diagram E on

$$(Q^n(A,B)\cup V^n(A,B))\times D_0A \cup R^n(A,B)\times\{*\},$$

because $D^n_{A,B}$ has to satisfy the conditions (2.2) and (2.3). If $V^n(A,B)$ is closed in $R^n(A,B)$ and if $\mathscr{C}$ is well-pointed, the function given by $E_{A,B}$ and the $D^k_{A,B}$ on this subspace is continuous. For the inductive step it remains to extend this map to $R^n(A,B)\times D_0A$. Conditions (2.2) and (2.3) are then automatically satisfied because no morphism in $R^n(A,B) - Q^n(A,B)$ can be decomposed.

The situation is somewhat different for $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$-diagrams, $\mathscr{F} = \mathscr{S}, \mathscr{T}$, because the conditions (2.10) kill the freeness of $T(\mathscr{C}\times\mathscr{L}_n)$.

To give an example, let $(f_n, t_n, \dots, f_0; x)$ be an element of

$$T(\mathscr{C} \times \mathscr{L}_1)(A^0, B^1) \times D_0 A^0 \quad \text{with} \quad f_0\colon A^0 \to C^1.$$

Suppose we inductively have defined the maps $D^k_{A^i, B^j}\colon R^k(A^i, B^j) \times D_0 A^i \to D_0 B^j$ of a $\mathrm{h}_{\mathscr{S}}(\mathscr{C} \times \mathscr{L}_1)$-diagram D for $k<n$. Unless one of the f_i is an identity, or some $t_i = 0, 1$, or $x = *$, the map $D^n_{A^0, B^1}$ is not given on our element. Since f_0 is of the form $f_0 = (f_0', j)$ with $f_0' \in \mathscr{C}$ and $j = (0 \leqq 1)$: $0 \to 1$ in $\mathscr{L}_1$, it can be decomposed $f_0 = (\mathrm{id}_C, j) \circ (f_0', \mathrm{id}_0)$, and condition (2.10) imposes the following condition on D^n to be constructed

$$D^n_{A^0, B^1}(f_n, t_n, \dots, f_0; x) = D^n_{C^0, B^1}\big(f_n, t_n, \dots, t_1, (\mathrm{id}_C, j); D^0_{A^0, C^0}[(f_0', \mathrm{id}_0); x]\big).$$

The example also indicates a way around this difficulty. A morphism $(f_r, t_r, \dots, f_0)$ in $(\mathscr{C} \times \mathscr{L}_n)_{r+1}(A, B) \times I^r = R^r(A, B)$ is called *$\mathscr{F}$-reduced*, $\mathscr{F} = \mathscr{S}, \mathscr{T}$, if

for $\mathscr{F} = \mathscr{S}$: f_0 is of the form $\big(\mathrm{id}_C, (0<i)\big)$ or $A \notin \mathrm{ob}\, \mathscr{S}$, or $r = 0$

for $\mathscr{F} = \mathscr{T}$: f_r is of the form $\big(\mathrm{id}_C, (i<n)\big)$ or $B \notin \mathrm{ob}\, \mathscr{T}$, or $r = 0$.

If $\mathscr{F} = \varnothing$, all elements of $R^r(A, B)$ are called $\mathscr{F}$-reduced. If $P^r_{\mathscr{F}}(A, B) \subset R^r(A, B)$ is the subspace of $\mathscr{F}$-reduced elements, let

$$Q^r_{\mathscr{F}}(A, B) = P^r_{\mathscr{F}}(A, B) \cap Q(\mathscr{C} \times \mathscr{L}_n)(A, B)$$

and let $V^r_{\mathscr{F}}(A, B) = P^r_{\mathscr{F}}(A, B) \cap \mathscr{V}(A, B)$ for a subcategory $\mathscr{V}$ of $T\mathscr{C}$. Now suppose we are given a $\mathscr{V}$-diagram E satisfying (2.3) and (2.10) for $\mathscr{F}$ and, as a partial data of a $h_{\mathscr{F}}(\mathscr{C} \times \mathscr{L}_n)$-diagram D, the maps $D_{A,B}$ for $A, B \in \mathscr{F}$ and $D^k_{A,B} = D_{A,B} | R^k(A, B) \times D_0 A$ for $k<r$, compatible with the given $\mathscr{V}$-diagram. To construct the maps $D^r_{A,B}\colon R^r(A, B) \times D_0 A \to D_0 B$ we need maps $h_{A,B}\colon P^r_{\mathscr{F}}(A, B) \times D_0 A \to D_0 B$ which are compatible with $E_{A,B}$ and the $D^k_{A,B}$ on $\big(Q^r_{\mathscr{F}}(A, B) \cup V^r_{\mathscr{F}}(A, B)\big) \times D_0 A$ and which map $P^r_{\mathscr{F}}(A, B) \times \{*\}$ to $*$. If $\mathscr{F} = \varnothing$, then $h_{A,B}$ is the required extension $D^r_{A,B}$. So let $\mathscr{F} = \mathscr{S}$. It remains to define $D^r_{A,B}$ on the subspace of all elements $(f_r, t_r, \dots, f_0; x)$ such that $f_i = (f_i', j)$ with $j = (0<p)$ for some i and $f_i' \neq \mathrm{id}$ if $i = 0$. Put

$$D^r_{A,B}(f_r, t_r, \dots, f_0; x) = \begin{cases} D^{r-i}_{C,B}\big(f_r, t_r, \dots, t_{i+1}, (\mathrm{id}, j); D^0_{A,C}((f_i', \mathrm{id}_0) \circ (f_{i-1} \circ \cdots \circ f_0; x))\big) & i \neq 0 \\ h_{A,B}\big(f_r, t_r, \dots, t_1, (\mathrm{id}, j); D^0_{A,C}((f_0', \mathrm{id}); x)\big) & i = 0. \end{cases}$$

This is a continuous extension of $h_{A,B}$ to the whole of $R^r(A, B) \times D_0 A$, it is compatible with the $D^k_{A,B}$, $k<r$, and satisfies the conditions (2.2), (2.3), and (2.10), but it is compatible with the $\mathscr{V}$-diagram E only under extra conditions on $\mathscr{V}$ given in the next definition. For $\mathscr{F} = \mathscr{T}$ the reasoning is analogous.

(3.1) **Definition.** Let $\mathscr{F}=\mathscr{S}, \mathscr{T}$, or $\varnothing$. A subcategory $\mathscr{V}\subset T(\mathscr{C}\times\mathscr{L}_n)$ is called *$\mathscr{F}$-admissible* if it satisfies the following conditions

(a) an indecomposable morphism in $\mathscr{V}$ is indecomposable in $T(\mathscr{C}\times\mathscr{L}_n)$,

(b) each $V^k_{\mathscr{F}}(A,B)$ is a closed subspace of $P^k_{\mathscr{F}}(A,B)$,

(c) each pair $\big(P^k_{\mathscr{F}}(A,B), V^k_{\mathscr{F}}(A,B)\cup Q^k_{\mathscr{F}}(A,B)\big)$ is a NDR.

(d) If $\mathscr{F}=\mathscr{S}\,[\mathscr{F}=\mathscr{T}]$ and $(f_k, t_k, \dots, f_0)\in\mathscr{V}$ with $f_0=(f_0', j)$,

$j=(0\leqq p)$ and $f_0'\neq\mathrm{id}\,[f_k=(f_k', j),\ j=(p\leqq n)$ and $f_k'\neq\mathrm{id}]$,

then $(f_k, t_k, \dots, t_1, (\mathrm{id}, j))\in\mathscr{V}$, $[((\mathrm{id}, j), t_k, f_{k-1}, \dots, f_0)\in\mathscr{V}]$ and $\mathscr{V}$ contains the full subcategory of $T(\mathscr{C}\times\mathscr{L}_n)$ of all objects in $\mathscr{S}$ [in $\mathscr{T}$].

(3.2) **Definition.** A sequence of closed subspaces $\varnothing=X_{-1}\subset X_0\subset X_1\subset X_2\subset\cdots$ of a space X is called a *filtration* if $X=\operatorname{colim} X_n$. A *filtered category* is a topological category $\mathscr{C}$ with a filtration $\mathscr{C}^n(A,B)$ of each space $\mathscr{C}(A,B)$.

(3.3) **Definition.** Call a triple $(\mathscr{B}, \mathscr{C}, F)$ consisting of a continuous functor $F\colon \mathscr{B}\to\mathscr{C}$ of filtered categories *acceptable* if

(a) $\operatorname{ob}\mathscr{B}=\operatorname{ob}\mathscr{C}$ and F preserves objects and the filtration,

(b) $\mathscr{B}(A,B)\times X=\operatorname{colim}\big(\mathscr{B}^n(A,B)\times X\big)$ for any space X,

(c) $\mathscr{C}^n(A,B)$ is obtained from $\mathscr{C}^{n-1}(A,B)$ by attaching $\mathscr{B}^n(A,B)$ relative to a subspace $D\mathscr{B}^n(A,B)$ such that $\big(\mathscr{B}^n(A,B), D\mathscr{B}^n(A,B)\big)$ is a NDR, and the induced map $\mathscr{B}^n(A,B)\to\mathscr{C}^n(A,B)$ is F.

(3.4) **Definition.** Let $\mathscr{D}$ be a pretopological category. A family D^t, $t\in I$, of $\mathscr{D}$-diagrams is called a *homotopy* of $\mathscr{D}$-diagrams if D^t_0 is independent of t and the maps

$$I\times\mathscr{D}(A,B)\times D^0_0\,A\to D^0_0\,B$$

$(t,f,x)\mapsto D^t_{A,B}(f;x)$ are continuous.

(3.5) **Proposition.** *Let $\mathscr{F}=\mathscr{S}, \mathscr{T}$, or $\varnothing$. Let $i\colon \mathscr{V}\subset T(\mathscr{C}\times\mathscr{L}_n)$ be a $\mathscr{F}$-admissible subcategory and $\mathscr{C}$ be well-pointed. Suppose given a well-pointed $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$-diagram E and a homotopy of well-pointed $\mathscr{V}$-diagrams D^t satisfying conditions* (2.3) *and* (2.10) *for $\mathscr{F}$, when defined, such that $D^0=i^*(E)$. Then there is a homotopy E^t of well-pointed $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$-diagrams with $E^0=E$ and $D^t=i^*(E^t)$ for all $t\in I$.*

Proof. If $\mathscr{A}$ is the full subcategory of $T(\mathscr{C}\times\mathscr{L}_n)$ of all objects in $\mathscr{F}$, then $\mathscr{A}$ is either in $\mathscr{V}$ in which case $E^t_{A,B}$ for $A,B\in\operatorname{ob}\mathscr{A}$ is given by $D^t_{A,B}$, or $\mathscr{A}\cap\mathscr{V}$ is empty or contains only identities in which case we put $E^t_{A,B}=E^0_{A,B}$ for $A,B\in\operatorname{ob}\mathscr{A}$ and $t\in I$. If not both A and B are in $\operatorname{ob}\mathscr{A}$, we inductively assume that we have constructed maps $f^k_{A,B}\colon I\times R^k(A,B)\times E_0\,A$

$\rightarrow E_0 B$ for $k<r$ such that $f_{A,B}|t\times R^k(A,B)\times E_0 A$ is part of the data of the required $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$-diagram $E^t_{A,B}$. By the considerations above we need a map

$$h_{A,B}\colon I\times P^r_{\mathscr{F}}(A,B)\times E_0 A\rightarrow E_0 B$$

which is already given on the subspace

$$Z=I\times\big(Q^r_{\mathscr{F}}(P,B)\cup V^r_{\mathscr{F}}(A,B)\big)\times E_0 A\cup I\times P^r_{\mathscr{F}}(A,B)\times\{*\}\cup$$
$$0\times P^r_{\mathscr{F}}(A,B)\times E_0 A$$

by the previously defined $f^k_{A,B}$, the condition (2.2 ii), and the requirement $E^0_{A,B}=E_{A,B}$. Since $\big(Q^r_{\mathscr{F}}(A,B)\cup V^r_{\mathscr{F}}(A,B)\big)\times E_0 A\cup P^r_{\mathscr{F}}(A,B)\times\{*\}$ is a NDR of $P^r_{\mathscr{F}}(A,B)\times E_0 A$ by the product theorem for NDRs [15; Thm. 6], Z is a retract of $I\times P^r_{\mathscr{F}}(A,B)\times E_0 A$ so that the required extension exists.

The following result substitutes [2; Thm. 3.17] and translates the methods of [3; IV § 1, 2, 3] to our situation.

(3.6) **Proposition.** *Let $\mathscr{F}=\mathscr{S},\mathscr{T}$, or $\varnothing$, and let $\mathscr{W}\subset T(\mathscr{C}\times\mathscr{L}_n)$ be the full subcategory of objects in $\mathscr{F}$. Given a diagram*

$$\begin{array}{ccc}
\mathscr{V} & \xrightarrow{\;H\;} & \mathscr{A} \\
{\scriptstyle i}\cap & \searrow{\scriptstyle L_t} & \downarrow{\scriptstyle F} \\
T(\mathscr{C}\times\mathscr{L}_n) & & \mathscr{B} \\
\downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle G} \\
\mathscr{C}\times\mathscr{L}_n & \xrightarrow[K]{} & \mathscr{E}
\end{array}\qquad \varepsilon=\varepsilon(\mathscr{C}\times\mathscr{L}_n)$$

of topological categories and continuous functors and a well-pointed $\mathscr{A}$-diagram D such that

(i) *$(\mathscr{A},\mathscr{B},F)$ is acceptable, $\mathscr{V}$ a $\mathscr{F}$-admissible subcategory containing $\mathscr{W}$, and $\mathscr{C}$ well-pointed,*

(ii) *if $f,g\in\mathscr{A}(A,B)$ are such that $F(f)=F(g)$, then $D_{A,B}(f;x)=D_{A,B}(g;x)$ for $x\in D_0 A$,*

(iii) *$H^*(D)$ satisfies the conditions* (2.3) *and* (2.10) *for $\mathscr{F}$ when defined,*

(iv) *$\mathrm{ob}\,\mathscr{B}=\mathrm{ob}\,\mathscr{E}$ and G preserves objects. If $\mathscr{B}'\subset\mathscr{B}$ and $\mathscr{E}'\subset\mathscr{E}$ are the full subcategories of objects in $K\circ\varepsilon(\mathscr{W})$, then $G|\mathscr{B}'\colon\mathscr{B}'\rightarrow\mathscr{E}'$ is an isomorphism with inverse $\bar{G}$ and G is a homotopy equivalence on each morphism space of $\mathscr{B}$,*

(v) *$F\circ(H|\mathscr{W})=\bar{G}\circ K\circ\varepsilon\circ(i|\mathscr{W})$,*

(vi) *L_t is a homotopy through functors from $K\circ\varepsilon\circ i$ to $G\circ F\circ H$ and $L_t|\mathscr{W}=G\circ F\circ(H|\mathscr{W})$ for all $t\in I$.*

Then there exists a $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$-diagram E with the following properties:

(a) *E extends $H^*(D)$*

(b) *If* $r\colon \mathrm{ob}\, T(\mathscr{C}\times\mathscr{L}_n)\to \mathrm{ob}\,\mathscr{A}$ *is given by* $K\circ\varepsilon$, *then* $E_0=D_0\circ r$.

(c) *Let* $Z(A,B)=\mathscr{A}(rA,rB)\times E_0A/\sim$ *with* $(f;x)\sim(g;x)$ *if* $F(f)=F(g)$ *and let* $Z'\subset Z$ *be the subspace of elements of the form* $(f;*)$, *let* $j_{A,B}\colon Z(A,B)\to\mathscr{B}(rA,rB)\times E_0A$ *and* $d_{A,B}\colon Z(A,B)\to E_0B$ *be the maps induced by* $F\times\mathrm{id}_{E_0A}$ *respectively* $D_{A,B}$, *let* $u^r_{A,B}=(K\circ\varepsilon\,|\,P^r_{\mathscr{F}}(A,B))\times\mathrm{id}_{D_0rA}$ *and* $v_{A,B}=(G\,|\,\mathscr{B}(rA,rB))\times\mathrm{id}_{D_0rA}$. *Then* $E^r_{A,B}\,|\,P^r_{\mathscr{F}}(A,B)\times E_0A$ *factors as*

$$P^r_{\mathscr{F}}(A,B)\times E_0A\xrightarrow[f^r_{A,B}]{} Z(A,B)\xrightarrow[d_{A,B}]{} E_0B$$

such that $f^r_{A,B}(P^r_{\mathscr{F}}(A,B)\times *)\subset Z'$, *and there are homotopies*

$$m^r_{A,B}\colon I\times P^r_{\mathscr{F}}(A,B)\times E_0A\to\mathscr{E}(rA,rB)\times E_0A$$

from $u^r_{A,B}$ *to* $v_{A,B}\circ j_{A,B}\circ f^r_{A,B}$ *extending* $(L\,|\,I\times V^r_{\mathscr{F}}(A,B))\times\mathrm{id}_{E_0A}$, *such that* $m^r_{A,B}(I\times P^r_{\mathscr{F}}(A,B)\times *)\subset\mathscr{E}(rA,rB)\times *$.

Moreover, given two $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$*-diagrams* E^0 *and* E^1 *satisfying* (a), (b), (c), *then there is a homotopy* E^t *of* $\mathrm{h}_{\mathscr{F}}(\mathscr{C}\times\mathscr{L}_n)$*-diagrams from* E^0 *to* E^1 *such that each* E^t *satisfies* (a), (b), (c).

For the proof we need the following result (see [3; Appendix 3.5]).

(3.7) *Given a homotopy equivalence* $p\colon Y\to Z$, *a* NDR (X,A), *maps* $f_A\colon A\to Y$ *and* $g\colon X\to Z$ *and a homotopy* $H_A\colon p\circ f_A\simeq g\,|\,A$. *Then there exist extensions* $f\colon X\to Y$ *of* f_A *and* $H\colon p\circ f\simeq g$ *of* H_A.

Proof of (3.6). We construct $E^r_{A,B}$ again by induction on r. For $A,B\in\mathrm{ob}\,\mathscr{W}$ we put $E_{A,B}=H^*(D)_{A,B}$ and $m^r_{A,B}=(L\,|\,I\times V^r_{\mathscr{F}}(P,B))\times\mathrm{id}_{E_0A}$. If not both A and B are in $\mathscr{W}$, we inductively suppose we have constructed $E^k_{A,B}$ and the homotopies $m^k_{A,B}$ for $k<r$.

Induction proceeds if we construct a map $f^r_{A,B}\colon P^r_{\mathscr{F}}(A,B)\times E_0A\to Z(A,B)$ extending the map given by $H^*(D)$ and the $h^k_{A,B}$ for $k<r$ on $(Q^r_{\mathscr{F}}(A,B)\cup V^r_{\mathscr{F}}(A,B))\times E_0A$ such that $f^r_{A,B}(P^r_{\mathscr{F}}(A,B)\times *)\subset Z'$, and a homotopy $m^r_{A,B}\colon u^r_{A,B}\simeq v_{A,B}\circ j_{A,B}\circ f^r_{A,B}$ extending the homotopy given by L and the $m^k_{A,B}$, $k<r$, on $I\times(Q^r_{\mathscr{F}}(A,B)\cup V^r_{\mathscr{F}}(A,B))\times E_0A$ and such that $m^r_{A,B}(I\times P^r_{\mathscr{F}}(A,B)\times *)\subset\mathscr{E}(rA,rB)\times *$. We will prove in an appendix that $j_{A,B}\colon Z(A,B)\to\mathscr{B}(rA,rB)\times E_0A$ and its restriction $j'_{A,B}\colon Z'\to\mathscr{B}(rA,rB)\times *$ are homotopy equivalences. We now apply (3.7) twice: To the diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\ v'_{A,B}\circ j'_{A,B}\ } & \mathscr{E}(rA,rB)\times * \\ \big\uparrow{\scriptstyle (f^r_{A,B})'} & & \big\uparrow{\scriptstyle (u^r_{A,B})'} \\ (Q^r_{\mathscr{F}}(A,B)\cup V^r_{\mathscr{F}}(A,B))\times * & \subset & P^r_{\mathscr{F}}(A,B)\times * \end{array}$$

where ' denotes restrictions. Recall that $f^r_{A,B}$ is already given on $(Q^r_{\mathscr{F}}(A,B)\cup V^r_{\mathscr{F}}(A,B))\times E_0A$. The restriction of $m^r_{A,B}$, which is already defined on $I \times (Q^r_{\mathscr{F}}(A,B) \cup V^r_{\mathscr{F}}(A,B)) \times E_0 A$, makes this diagram commute up to homotopy. Hence we can extend $(f^r_{A,B})'$ to $P^r_{\mathscr{F}}(A,B) \times *$ and $(m^r_{A,B})'$ to a homotopy

$$(m^r_{A,B})'\colon I \times P^r_{\mathscr{F}}(A,B) \times * \to \mathscr{E}(rA, rB) \times *\colon v'_{A,B} \circ j'_{A,B} \circ (f^r_{A,B})' \simeq (u^r_{A,B})'.$$

Now apply (3.7) to

$$\begin{array}{ccc} Z & \xrightarrow{\;v_{A,B} \circ j_{A,B}\;} & \mathscr{E}(rA, rB) \times E_0 A \\ {\scriptstyle (f^r_{A,B})} \uparrow & & \uparrow {\scriptstyle u_{A,B}} \\ (Q^r_{\mathscr{F}}(A,B) \cup V^r_{\mathscr{F}}(A,B)) \times E_0 A \cup P^r_{\mathscr{F}}(A,B) \times * & \subset & P^r_{\mathscr{F}}(A,B) \times E_0 A \end{array}$$

Since $(m^r_{A,B})'$ makes this diagram commute up to homotopy, the required extensions $f^r_{A,B}$ and $m^r_{A,B}$ of $(f^r_{A,B})'$ and $(m^r_{A,B})'$ exist. The map $E^r_{A,B}$: $R^r(A,B) \times E_0 A \to E_0 B$ is then induced by the maps $d_{A,B} \circ f^r_{A,B}$.

The proof of the second part is similar. Let E^0 and E^1 be two $\mathrm{h}_{\mathscr{F}}(\mathscr{C} \times \mathscr{L}_n)$-diagrams satisfying (a), (b), (c) with the maps $f^r_{A,B}$ and $g^r_{A,B}$ and the homotopies $m^r_{A,B}$ and $n^r_{A,B}$. For $A, B \in \operatorname{ob} \mathscr{W}$ define ${}^t E_{A,B} = H^*(D)_{A,B}$. If not both A and B are in $\mathscr{W}$, we inductively suppose we have constructed homotopies

$$F^k_{A,B}\colon I \times P^k_{\mathscr{F}}(A,B) \times E_0 A \to Z(A,B)$$

from $f^k_{A,B}$ to $g^k_{A,B}$, $k < r$, and homotopies of homotopies

$$M^k_{A,B}\colon I \times I \times P^k_{\mathscr{F}}(A,B) \times E_0 A \to \mathscr{E}(rA, rB) \times E_0 A$$

from $m^k_{A,B}$ to $n^k_{A,B}$, $k < r$, satisfying (c) for each $t \in I$. The map $F^r_{A,B}$ and the homotopy $M^r_{A,B}$ to be constructed are already given on

$$I \times (Q^r_{\mathscr{F}}(A,B) \cup V^r_{\mathscr{F}}(A,B)) \times E_0 A \cup \partial I \times P^r_{\mathscr{F}}(A,B) \times E_0 A$$

respectively on

$$I \times I \times (Q^r_{\mathscr{F}}(A,B) \cup V^r_{\mathscr{F}}(A,B)) \times E_0 A \cup \partial I \times I \times P^r_{\mathscr{F}}(A,B) \times E_0 A$$

and have to satisfy extra conditions with respect to the base point of $E_0 A$. We now proceed as in the proof of the first part.

Proof of (2.8) *and* (2.11). We prove the results for $\mathscr{T}$; the other proofs are similar. By assumption we are given $\mathrm{h}_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_{n-1})$-diagrams $D_0, D_1, \ldots, D_{r-1}, D_{r+1}, \ldots, D_{n-1}$, $0 < r \leqq n-1$ and a $\mathrm{h}(\mathscr{C} \times \mathscr{L}_{n-1})$-diagram D_n such that $d^{j-1}(D_i) = d^i(D_j)$ for $0 \leqq i < j \leqq n$. Let $\mathscr{V} \subset T(\mathscr{C} \times \mathscr{L}_n)$ be the subcategory generated by the $d^i T(\mathscr{C} \times \mathscr{L}_n)$, $0 \leqq i \leqq n$, $i \neq r$, where $d^i T(\mathscr{C} \times \mathscr{L}_n) = T(\mathrm{Id}_{\mathscr{C}} \times \delta^i_n)(T(\mathscr{C} \times \mathscr{L}_{n-1}))$. Then the D_i induce a $\mathscr{V}$-diagram D which satisfies the conditions (2.3) and (2.10) for $\mathscr{T}$. Let $\mathscr{B}$ be the quotient category of $\mathscr{V}$ obtained by factoring out the relations (2.3) and (2.10) for $\mathscr{T}$ forgetting the space coordinate. In the proof of [3; Thm. 4.9] one finds that the functor $\mathscr{B} \to \mathscr{C} \times \mathscr{L}_n$ induced by the augmentation is a homotopy equivalence on each morphism space. Now apply (3.6) with $\mathscr{A} = \mathscr{V}$, $\mathscr{E} = \mathscr{C} \times \mathscr{L}_n$, $H = \mathrm{id}$, $K = \mathrm{id}$, G induced by ε, and L_t the constant homotopy. We obtain a $h_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_n)$-diagram E extending the $\mathscr{V}$-diagram D. Hence $d^i E = d^i D = D_i$ for $i \neq r$.

We close this section with a proof of (2.5). It contains an argument which applied to $\mathscr{V}$ in the previous proof shows that $(\mathscr{V}, \mathscr{B}, F)$ is acceptable, where $F: \mathscr{V} \to \mathscr{B}$ is the projection functor, filling a gap we left in the proof of (2.8) and (2.11). Filter $W\mathscr{C}(A, B)$ by the subspaces F_p of morphisms represented by a morphism in some $R^k(A, B)$, $k \leqq p$. Let $N^p(A, B) \subset R^p(A, B)$ be the subspace of all elements $(f_p, t_p, \ldots, f_0)$ such that some $t_i = 1$ or some f_i is an identity. Then F_p is obtained from F_{p-1} by attaching $R^p(A, B)$ relative to $N^p(A, B)$. Since $\mathscr{C}$ is well-pointed, $(R^p(A, B), N^p(A, B))$ and hence (F_p, F_{p-1}) are NDRs. Hence it suffices to show that $N^p(A, B)$ is a SDR (strong deformation retract) of $R^p(A, B)$ because $\bar{\varepsilon} | F_0$ is a homeomorphism. But this follows from [15; Thm. 6].

Evidently, the results of this section also hold for unbased diagrams. In fact, the proofs are easier in this case because the extra considerations for base points are redundant.

4. Properties of *h*-Morphisms

Throughout this section let $\mathscr{C}$ be well-pointed. We list a few properties of the category $\mathscr{H}\mathscr{C}^{\circ}$ of well-pointed h$\mathscr{C}$-diagrams. The proofs can be obtained from the corresponding results of [3; Chap. IV, V] by modifying their proofs in the same manner as we modified the proof of [3; Thm. 4.9] to obtain our statements (2.8) and (2.11) and by substituting [3; Prop. 3.14] by (3.5) where applied.

(4.1) **Proposition.** *Let A, B be well-pointed h$\mathscr{C}$-diagrams and E a well-pointed $\mathscr{C}$-diagram. Two h-morphisms H, K: $A \to B$ [$\mathscr{S}$-morphisms H, K: $E \to A$; $\mathscr{T}$-morphisms H, K: $B \to E$] are simplicially homotopic iff there is a homotopy L through h-morphisms $A \to B$ [$\mathscr{S}$-morphisms $E \to A$; $\mathscr{T}$-morphisms $B \to E$] from H to K.*

(4.2) **Proposition.** *Let H: $D \to E$ be a h-morphism of well-pointed h$\mathscr{C}$-diagrams with underlying maps $\{f_A: D_0 A \to E_0 A\}$. Given a collection of maps $\{g_A: D_0 A \to E_0 A \mid A \in \mathrm{ob}\,\mathscr{C}\}$ such that $f_A \simeq g_A$, there exists a h-morphism K: $D \to E$ homotopic to H having $\{g_A\}$ as underlying maps.*

This result can be generalized; the proof is the same as for (4.2).

(4.3) **Proposition.** *Given a well-pointed h$\mathscr{C}$-diagram D and maps $E^n_{A,B}$: $R^n(A, B) \times D_0 A \to D_0 B$ for $A, B \in \mathrm{ob}\,\mathscr{C}$ and $n \leqq k$, such that the $E^n_{A,B}$*

satisfy the conditions (2.2) *and* (2.3) *if defined and there are homotopies* $F^n_{A,B}(t): R^n(A,B) \times D_0 A \to D_0 B$ *for* $A, B \in \mathrm{ob}\,\mathscr{C}$ *and* $n \leqq k$ *from* $D_{A,B}|R^n(A,B) \times D_0 A$ *to* $E^n_{A,B}$ *such that each* $F^n_{A,B}(t)$ *satisfies* (2.2) *and* (2.3). *Then the* $E^n_{A,B}$ *can be extended to a well-pointed* h$\mathscr{C}$*-diagram E which is homotopic to D in the sense of* (3.4).

(4.4) **Proposition.** *Let* $H: D^0 \to D^1$ *and* $K: D^1 \to D^2$ *be h-morphisms of well-pointed* h$\mathscr{C}$*-diagrams with underlying maps* $\{f_A\}$ *and* $\{g_A\}$. *Then there is a composite of H and K having* $\{g_A \circ f_A\}$ *as underlying maps.*

(4.5) **Proposition.** *Let* $\mathscr{D}$ *be a subcategery of* $\mathscr{C}$ *such that* $\mathrm{ob}\,\mathscr{D} = \mathrm{ob}\,\mathscr{C}$ *and each* $(\mathscr{C}(A,B), \mathscr{D}(A,B))$ *is a* NDR. *Let* $i: T\mathscr{D} \subset T\mathscr{C}$ *and* $j: T(\mathscr{D} \times \mathscr{L}_1) \subset T(\mathscr{C} \times \mathscr{L}_1)$ *be the inclusion functors. Suppose we are given a well-pointed* h$\mathscr{D}$*-diagram D, a well-pointed* h$\mathscr{C}$*-diagram E, and a h-morphism* $H: D \to i^*(E)$ *of* h$\mathscr{D}$*-diagrams whose underlying maps are homotopy equivalences. Then we can extend D to a well-pointed* h$\mathscr{C}$*-diagram D′ and H to a h-morphism* $H': D' \to E$ *of* h$\mathscr{C}$*-diagrams, i.e.* $D = i^*(D')$ *and* $H = j^*(H')$.

(4.6) **Proposition.** *Let* $\mathscr{D}$ *and* $\mathscr{C}$ *be as in* (4.5). *Suppose we are given a h-morphism* $H: D \to E$ *of well-pointed* h$\mathscr{C}$*-diagrams whose underlying maps are homotopy equivalences and a h-morphism* $K': i^*(E) \to i^*(D)$ *of* h$\mathscr{D}$*-diagrams such that K′ is homotopy inverse to* $j^*(H)$, *i.e.* $j^*(H)$ *represents an isomorphism in* $\mathscr{H}\mathscr{D}^o$ *whose inverse is represented by K′. Then there exists an extension* $K: E \to D$ *of K′ such that K is a homotopy inverse of H. In particular, any h-morphism of well-pointed* h$\mathscr{C}$*-diagrams whose underlying maps are homotopy equivalences is a homotopy equivalence, i.e. it represents an isomorphism in* $\mathscr{H}\mathscr{C}^o$.

(4.7) **Remark.** The results (4.1), …, (4.6) also hold in the unbased case. If we work with k-spaces, the assumption that the diagrams are well-pointed can be dropped in (4.1), …, (4.4). For a proof see [3; chap. IV, V].

We now want to give an alternative description of $\mathscr{H}\mathscr{C}^o$, $\mathscr{H}\mathscr{C}$, $\mathscr{H}\mathscr{C}^*_k$, and $\mathscr{H}\mathscr{C}_k$, which to some extend links our theory with the approaches of Bousfield-Kan [4], Quillen, and others. Let $\mathscr{M}\mathscr{C}$ be the category of based $\mathscr{C}$-diagrams and $\mathscr{M}\mathscr{C}^o$ the full subcategory of well-pointed $\mathscr{C}$-diagrams. If we work with k-spaces, the corresponding categories are distinguished by a subscript k, for the unbased versions we drop the superscript. Let Σ be the class of all homomorphisms whose underlying maps are homotopy equivalences, and denote the associated categories of fractions (see [7]) by $\mathscr{M}\mathscr{C}^o[\Sigma^{-1}]$ etc.

(4.8) **Proposition.** *Let* $\mathscr{C}$ *be a well-pointed category such that each* $\mathscr{C}(A,B)$ *is locally compact unless we work with k-spaces.* (X locally compact means that each $x \in X$ has a compact, not necessarily Hausdorff neighbourhood base.) *Then the categories* $\mathscr{H}\mathscr{C}^o$, $\mathscr{H}\mathscr{C}$, $\mathscr{H}\mathscr{C}^o_k$, $\mathscr{H}\mathscr{C}_k$ *are equivalent to* $\mathscr{M}\mathscr{C}^o[\Sigma^{-1}]$, $\mathscr{M}\mathscr{C}[\Sigma^{-1}]$, $\mathscr{M}\mathscr{C}^o_k[\Sigma^{-1}]$, $\mathscr{M}\mathscr{C}_k[\Sigma^{-1}]$ *respectively.*

For k-spaces the proof is in [3; Prop. 4.54]. In the topological case it is a modification of [3; Prop. 4.54] of the same kind as the proof of (5.4) below is a modification of [3; Thm. 4.49].

5. Homotopy Colimits

From now on we always assume that $\mathscr{C}$ is a topological category. If $H: D \to D'$ is a h-morphism of h$\mathscr{C}$-diagrams and if $f: C \to D$ and $g: D' \to E$ are homomorphisms of h$\mathscr{C}$-diagrams, there is a canonical way for defining a composite $g \circ H \circ f$, so that such a composite exists even if $\mathscr{C}$ or the diagrams are not well-pointed.

(5.1) **Definition.** The h-morphism $L: C \to E$ given by

$$L_{A^0, B^1} = g_B \circ H_{A^0, B^1} \circ (\mathrm{id}_{T\mathscr{C}(A^0, B^1)} \times f_A): \ T\mathscr{C}(A^0, B^1) \times C_0 A \to E_0 B$$

is called the *canonical composite* $g \circ H \circ f$ of f, H, and g. (The other data of L is determined by C and E.)

(5.2) **Definition.** (a) If $f: C \to D$ is a homomorphism of h$\mathscr{C}$-diagrams, $H: D \to D'$ a $\mathscr{T}$-morphism and $g: D' \to E$ a homomorphism of $\mathscr{C}$-diagrams, then the formula of (5.1) defines the *canonical composite $\mathscr{T}$-morphism* $g \circ H \circ f$.

(b) If $f: C \to D$ is a homomorphism of $\mathscr{C}$-diagrams, $H: D \to D'$ a $\mathscr{S}$-morphism, and $g: D' \to E$ a homomorphism of h$\mathscr{C}$-diagrams, then the formula of (5.1) defines the *canonical composite $\mathscr{S}$-morphism* $g \circ H \circ f$.

Define a continuous functor $\rho: T(\mathscr{C} \times \mathscr{L}_1) \to T\mathscr{C} \times \mathscr{L}_1$ by taking the identity on $T(\mathscr{C} \times 0)$ and $T(\mathscr{C} \times 1)$ and mapping $f \in T(\mathscr{C} \times \mathscr{L}_1)(A^0, B^1)$ to $(\mathrm{id}_B, j) \circ s^0(f)$, where $j = (0 \leq 1) \in \mathscr{L}_1$ and the image of $s^0: T(\mathscr{C} \times \mathscr{L}_1) \to T\mathscr{C}$ is identified with $T\mathscr{C} \times 0$ in $T\mathscr{C} \times \mathscr{L}_1$. This functor allows us to consider a homomorphism of h$\mathscr{C}$-diagrams as a h-morphism.

Suppose we are given h$\mathscr{C}$-diagrams D, D', E, a h-morphism $H: D \to D'$ and a homomorphism $g: D' \to E$. We construct a h$(\mathscr{C} \times \mathscr{L}_2)$-diagram F by

$$F | d^0 \, T(\mathscr{C} \times \mathscr{L}_2) = \rho^*(g), \qquad F | d^2 \, T(\mathscr{C} \times \mathscr{L}_2) = H,$$

and

$$F_{A^0, B^2} = g_B \circ H_{A^0, B^1} \circ \big(T(\mathrm{Id} \times \sigma^1) \times \mathrm{id}\big)$$

with

$$T(\mathrm{Id} \times \sigma^1) \times \mathrm{id}: \ T(\mathscr{C} \times \mathscr{L}_2)(A^0, B^2) \times D_0 A \to T(\mathscr{C} \times \mathscr{L}_1)(A^0, B^1) \times D_0 A.$$

Then $d^1(F)$ is the canonical composite $g \circ H$, which hence is a composite in the sense of (2.9) of the h-morphism H and the canonical h-morphism $\rho^*(g)$ induced by g. Similarly we can show in the situation of (5.1) and (5.2a) that the canonical composite $H \circ f$ is a composite $H \circ \rho^*(f)$ in the sense of (2.9) respectively (2.10), and in the situation (5.2b) that the canonical composite $g \circ H$ is a composite $\mathscr{S}$-morphism $\rho^*(g) \circ H$ in the sense of (2.10).

(5.3) **Definition.** We call two homomorphisms $f, g: E \to E'$ of $\mathscr{C}$-diagrams *homotopic* if there are homotopies $F_A(t)$: $f_A \simeq g_A$ from the underlying maps $\{f_A\}$ of f to the underlying maps $\{g_A\}$ of g such that each collection $\{F_A(t)$: $E_0 A \to E'_0 A | A \in \operatorname{ob} \mathscr{C}\}$ is the collection of underlying maps of a homomorphism $F(t)$.

Evidently, this definition is equivalent to saying that there is a homotopy of $(\mathscr{C} \times \mathscr{L}_1)$-diagrams from f to g which is constant on $\mathscr{C} \times 0$ and $\mathscr{C} \times 1$.

(5.4) **Proposition.** *Let $\mathscr{C}$ be a topological category such that each $\mathscr{C}(A, B)$ is locally compact unless we work in the category $\mathscr{CG}$ of k-spaces and let D be a h$\mathscr{C}$-diagram. Then there is a $\mathscr{C}$-diagram MD and a $\mathscr{T}$-morphism μ_D: $D \to MD$ such that*

(a) *MD is well-pointed if $\mathscr{C}$ and D are well-pointed,*

(b) *the underlying maps m_A: $D_0 A \to MD_0 A$, $A \in \operatorname{ob} \mathscr{C}$, are inclusions as SDRs,*

(c) *a $\mathscr{T}$-morphism H: $D \to E$ is the canonical composite of μ_D and a unique homomorphism h: $MD \to E$ of $\mathscr{C}$-diagrams,*

(d) *if $\mathscr{C}$ is well-pointed and if H, H': $D \to E$ are simplicially homotopic $\mathscr{T}$-morphisms, then the induced homomorphisms, h, h': $MD \to E$ are homotopic provided D and E are well-pointed or we work with k-spaces.*

Proof. Define MD_0: $\operatorname{ob} \mathscr{C} \to \mathscr{T}\!op^{\circ}$ by

$$MD_0 A = \coprod_{n \geqq 0} \coprod_{B \in \mathscr{C}} \mathscr{C}_{n+1}(B, A) \times I^n \times D_0 B / \sim$$

with the relations

$$(5.5) \qquad (f_n, t_n, \ldots, f_0; x) = \begin{cases} (f_n, t_n, \ldots, t_2, f_1; x) & f_0 = \mathrm{id} \\ (f_n, t_n, \ldots, f_{i+1}, t_{i+1} t_i, f_{i-1}, \ldots, f_0; x) & f_i = \mathrm{id},\ 0 < i < n \\ (f_n, t_n, \ldots, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \ldots, f_0; x) & t_i = 1 \\ (f_n, t_n, \ldots, f_i; D_{B,C}(f_{i-1}, t_{i-1}, \ldots, f_0; x)) & t_i = 0 \\ (\mathrm{id}_A; *) & x = * \end{cases}$$

where $f_{i-1} \circ \cdots \circ f_0: B \to C$. The base point is $(\mathrm{id}_A; *)$. By standard methods one finds that $MD_0 A$ is well-pointed if $\mathscr{C}$ and every $D_0 A$ is well-pointed. If we work with k-spaces or if $\mathscr{C}(A, A')$ is locally compact, the map

$$\mathscr{C}(A, A') \times \Big(\coprod_{n \geqq 0} \coprod_{B \in \mathscr{C}} \mathscr{C}_{n+1}(B, A) \times I^n \times D_0 B\Big) \to \mathscr{C}(A, A') \times MD_0 A$$

is an identification so that the maps $\mathscr{C}(A, A') \times MD_0 A \to MD_0 A'$ given by

$$[f, (f_n, t_n, \ldots, f_0; x)] \mapsto [f \circ f_n, t_n, \ldots, f_0; x]$$

are continuous and define the $\mathscr{C}$-diagram MD.

Let $(f_n, r_n, t_n, f_{n-1}, r_{n-1}, t_{n-1}, \ldots, f_1, r_1, t_1, f_0; x)$ with $(f_n, f_{n-1}, \ldots, f_0) \in \mathscr{C}_{n+1}(A, B)$, $t_i \in I$ and $r_i \in \{0, 1\}$ denote the element $(f'_n, t_n, f'_{n-1}, \ldots, f'_0; x) \in T(\mathscr{C} \times \mathscr{L}_1)(A^0, B^1) \times D_0 A$, where $f'_i = (f_i, (r_i \leqq r_{i+1})) \in \mathscr{C} \times \mathscr{L}_1$, $r_0 = 0$, $r_{n+1} = 1$. Then the $\mathscr{T}$-map μ_D: $D \to MD$ is given by D and MD on $d^i\, T(\mathscr{C} \times \mathscr{L}_1)$, $i = 0, 1$ and the remaining data

$$(\mu_D)_{A^0, B^1}\colon\ T(\mathscr{C} \times \mathscr{L}_1)(A^0, B^1) \times D_0 A \to MD_0 B$$

is defined by

$$(f_n, r_n, t_n, f_{n-1}, r_{n-1}, t_{n-1}, \ldots, f_0; x) \mapsto (f_n \circ \cdots \circ f_i, t_i, f_{i-1}, \ldots, f_1, t_1, f_0; x)$$

if $r_i = 0$ and $r_{i+1} = 1$ (again $r_0 = 0$, $r_{n+1} = 1$). The underlying maps are

$$m_A\colon\ D_0 A \to MD_0 A\colon\ x \mapsto (\mathrm{id}_A; x)$$

and the strong deformation $F\colon I \times MD_0 A \to MD_0 A$ of $MD_0 A$ into $D_0 A$ is given by

$$F(u, (f, t_n, \ldots, f_0; x)) = (\mathrm{id}_A, u, f_n, t_n, \ldots, f_0; x).$$

Given a $\mathscr{T}$-morphism $H\colon D \to E$, then

$$h_A\colon\ MD_0 A \to E_0 A\colon$$
$$(f_n, t_n, \ldots, f_0; x) \mapsto H_{B^0, A^1}(f_n, 0, t_n, f_{n-1}, 0, t_{n-1}, \ldots, f_1, 0, t_1, f_0; x)$$

where $B = \text{source}\, f_0$, defines the unique homomorphism of $\mathscr{C}$-diagrams such that H is the canonical composite $h \circ \mu_D$. If $H, H'\colon D \to E$ are simplicially homotopic $\mathscr{T}$-morphisms, then there is a homotopy H^t through $\mathscr{T}$-maps $D \to E$ from H to H' by (4.1), and H^t induces a homotopy h^t of the induced homomorphisms $MD \to E$.

Let $\mathscr{N}\mathscr{C}^*$ be the category of $\mathscr{C}$-diagrams and homotopy classes of homomorphisms and $\mathscr{N}\mathscr{C}^o$ the full subcategory of well-pointed $\mathscr{C}$-diagrams. A subscript k indicates that we work with k-spaces. Proposition (5.4) enables us to define a functor $M\colon \mathscr{H}\mathscr{C}^o \to \mathscr{N}\mathscr{C}^o$. We send a h$\mathscr{C}$-diagram D to MD and a representing h-morphism $H\colon D \to D'$ to the homotopy class of the homomorphism $MD \to MD'$ induced by some composite $\mathscr{T}$-morphism $\mu_{D'} \circ H$. Since the homotopy class of $\mu_{D'} \circ H$ is independent of the choice of the representative H if $\mathscr{C}$ is well-pointed, the homotopy class of $M(H)$ is independent of the choice of the representative. So if $\mathscr{C}$ is well-pointed and each $\mathscr{C}(A, B)$ locally compact, there is a functor

$$M\colon\ \mathscr{H}\mathscr{C}^o \to \mathscr{N}\mathscr{C}^o.$$

Under the same assumptions there is also a functor

$$J'\colon\ \mathscr{N}\mathscr{C}^o \to \mathscr{H}\mathscr{C}^o$$

sending a $\mathscr{C}$-diagram E to $\varepsilon(\mathscr{C})^*(E)$ and a representing homomorphism $h\colon E \to E'$ to the simplicial homotopy class of $\varepsilon(\mathscr{C} \times \mathscr{L}_1)^*(h)$. We can regard J' as a sort of inclusion functor. Similarly, if we work with k-spaces and if $\mathscr{C}$ is a well-pointed category, there are functors

$$M\colon \mathscr{H}\mathscr{C}_k^* \to \mathscr{N}\mathscr{C}_k^* \quad \text{and} \quad J'\colon \mathscr{N}\mathscr{C}_k^* \to \mathscr{H}\mathscr{C}_k^*.$$

(5.6) **Theorem.** (a) *If $\mathscr{C}$ is a well-pointed category such that each $\mathscr{C}(A, B)$ is locally compact, the functors*

$$M\colon \mathscr{H}\mathscr{C}^\circ \to \mathscr{N}\mathscr{C}^\circ \qquad J'\colon \mathscr{N}\mathscr{C}^\circ \to \mathscr{H}\mathscr{C}^\circ$$
$$D \quad \mapsto MD \qquad\qquad E \quad \mapsto \varepsilon(\mathscr{C})^* E$$

exist. M is fully faithful and left adjoint to J'.

(b) *If we work with k-spaces and $\mathscr{C}$ is well-pointed, the analogous functors $M\colon \mathscr{H}\mathscr{C}_k^* \to \mathscr{N}\mathscr{C}_k^*$ and $J'\colon \mathscr{N}\mathscr{C}_k^* \to \mathscr{H}\mathscr{C}_k^*$ exist. M is left adjoint to J' and its restriction to $\mathscr{H}\mathscr{C}_k^\circ$ is fully faithful.*

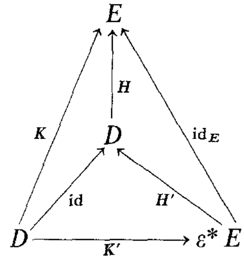
Let D, D' be h$\mathscr{C}$-diagrams and E, E' be $\mathscr{C}$-diagrams which we assume to be well-pointed unless we work with k-spaces. Denote the sets of simplcial homotopy classes of $\mathscr{S}$-morphisms $E \to D$ and $\mathscr{T}$-morphisms $D \to E$ by $\mathscr{S}(E, D)$ respectively $\mathscr{T}(D, E)$. Since any $h_{\mathscr{S}}(\mathscr{C} \times \mathscr{L}_n)$-diagram and any $h_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_n)$-diagram is a $h(\mathscr{C} \times \mathscr{L}_n)$-diagram, there are maps

$$p_{\mathscr{S}}\colon \mathscr{S}(E, D) \to \mathscr{H}\mathscr{C}^\circ(\varepsilon^* E, D) \qquad p_{\mathscr{T}}\colon \mathscr{T}(D, E) \to \mathscr{H}\mathscr{C}^\circ(D, \varepsilon^* E),$$

which are by (2.8) and (2.11) natural with respect to homomorphisms $E \to E'$ and to h-morphisms $D \to D'$. For k-spaces we have to substitute $\mathscr{H}\mathscr{C}^\circ$ by $\mathscr{H}\mathscr{C}_k^*$.

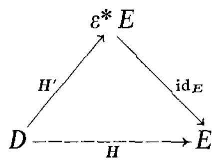
(5.7) **Lemma.** *If $\mathscr{C}$ is well-pointed, the maps $p_{\mathscr{S}}$ and $p_{\mathscr{T}}$ are natural bijections.*

Proof. Let $H, K\colon D \to E$ be two $\mathscr{T}$-morphisms such that $p_{\mathscr{T}}(H) = p_{\mathscr{T}}(K)$. If we consider H and K as h-morphisms, we denote them by H' and K'. Since the identity homomorphism $E \to E$ can be considered as $\mathscr{T}$-morphism $\varepsilon^*(E) \to E$, there is a $h_{\mathscr{T}}(\mathscr{C} \times \mathscr{L}_2)$-diagram τ such that $d^0(\tau) = \mathrm{id}_E$, $d^1(\tau) = H$, and $d^2(\tau) = H'$, and a similar one for K and K'. Now apply (2.11) to the situation



By assumption, the h-morphisms H' and K' are simplicially homotopic so that the face opposite to $\varepsilon^*(E)$ is the only one missing. We fill in and obtain a simplicial homotopy of $\mathscr{T}$-morphisms from H to K.

If $H'\colon D \to \varepsilon^*(E)$ is a h-morphism, we apply (2.11) to the situation



We fill in and obtain a $\mathscr{T}$-morphism $H\colon D \to E$ lifting H' up to simplicial homotopy. This follows from the fact that two h-morphisms $F, G\colon D \to D'$ are simplicially homotopic iff there is a $h(\mathscr{C} \times \mathscr{L}_2)$-diagram σ with $d^0(\sigma) = s^0(D') = \mathrm{id}_{D'}$, $d^1(\sigma) = G$, and $d^2(\sigma) = F$, which can easily be deduced from (2.8).

The proof that $p_{\mathscr{S}}$ is bijective is analogous.

Proof of (5.6). By (5.4) and (5.7) we have natural bijections

$$\mathscr{H}\mathscr{C}^\circ(D, J'E) \cong \mathscr{T}(D, E) \cong \mathscr{N}\mathscr{C}^\circ(MD, E).$$

The front adjunction $D \to J'MD$ is given by μ_D, which is an isomorphism in $\mathscr{H}\mathscr{C}^\circ$ by (4.6). Hence M is fully faithful. The proof is the same for k-spaces.

Let $\mathrm{colim}_{\mathrm{h}}\colon \mathscr{N}\mathscr{C}^* \to \mathscr{T}\!op_h^*$ be the functor mapping each $\mathscr{C}$-diagram to its colimit in the usual based topological sense and a representing homomorphism of a morphism $E \to E'$ in $\mathscr{N}\mathscr{C}^*$ to the based homotopy class of the induced map $\mathrm{colim}\, E \to \mathrm{colim}\, E'$. This homotopy class is independent of the choice of the representing homomorphism. Let

$$J''\colon \mathscr{T}\!op_h^* \to \mathscr{N}\mathscr{C}^*$$

be the functor mapping each space X to the constant $\mathscr{C}$-diagram on X. It is well-known that the homomorphisms from a $\mathscr{C}$-diagram E to the constant diagram on X are in one-to-one correspondence with continuous based maps $\mathrm{colim}\, E \to X$. This bijection is compatible with taking homotopy classes. Hence $\mathrm{colim}_{\mathrm{h}}\colon \mathscr{N}\mathscr{C}^* \to \mathscr{T}\!op_h^*$ is left adjoint to the inclusion functor J''.

Similarly the usual limit functor $\lim\colon \mathscr{M}\mathscr{C}^* \to \mathscr{T}\!op^*$ induces a functor $\lim_{\mathrm{h}}\colon \mathscr{N}\mathscr{C}^* \to \mathscr{T}\!op_h^*$ which is right adjoint to J''. The same holds in both cases for k-spaces.

Although $\mathrm{colim}_{\mathrm{h}}(\mathscr{N}\mathscr{C}^\circ)$ does not lie in $\mathscr{T}\!op_h^\circ$, we shall later on see that $\mathrm{colim}_{\mathrm{h}}(M(\mathscr{H}\mathscr{C}^\circ))$ lies in $\mathscr{T}\!op_h^\circ$. Hence we can define

(5.8) **Definition.** The *homotopy colimit functor* h-colim: $\mathscr{H}\mathscr{C}^\circ \to \mathscr{T}\!op_h^\circ$ [$\mathscr{H}\mathscr{C}_k^* \to \mathscr{C}\mathscr{G}_h^*$] is defined to be the composite $\text{h-colim} = \mathrm{colim}_{\mathrm{h}} \circ M$.

Let $J=J'\circ J''$: $\mathcal{Top}_k^\circ \to \mathcal{HC}^\circ$ $[\mathcal{CG}_k^* \to \mathcal{HC}_k^*]$. Then J sends X to the constant diagram on X, i.e. $(JX)_0 A = X$ for all $A \in \operatorname{ob}\mathcal{C}$ and $(JX)_{A,B}$: $T\mathcal{C}(A,B)\times X \to X$ is the projection. Putting (5.6 b) and the considerations about $\operatorname{colim}_{\mathrm{h}}$ together we obtain

(5.9) **Proposition.** *If we work with k-spaces and $\mathcal{C}$ is a well-pointed category, then* h-colim: $\mathcal{HC}_k^* \to \mathcal{CG}_k^*$ *exists and is left adjoint to* J: $\mathcal{CG}_k^* \to \mathcal{HC}_k^*$.

By (5.6 a) we have the same result for $\mathcal{HC}^\circ \to \mathcal{Top}_k^\circ$ provided each $\mathcal{C}(A,B)$ is locally compact. We shall later on see that the condition of local compactness is unnecessary.

The Unbased Case. Let $\mathcal{C}$ be a well-pointed category. Given an unbased h$\mathcal{C}$-diagram D, we make it into a based one D^+ by substituting each $D_0 A$ by $(D_0 A)^+$. Then the base point of each space of MD^+ is an extra component. By deleting it, we obtain a functor $\mathcal{HC}\to\mathcal{NC}$ or $\mathcal{HC}_k \to \mathcal{NC}_k$. Following it by the functor $\operatorname{colim}_{\mathrm{h}}$: $\mathcal{NC}\to\mathcal{Top}_k$ or $\mathcal{NC}_k \to \mathcal{CG}_k$ we obtain the *unbased homotopy colimit functor*. This functor is left adjoint to the inclusion functor $\mathcal{Top}_k \to \mathcal{HC}$ respectively $\mathcal{CG}_k \to \mathcal{HC}_k$ by the unbased versions of (5.12) below or (5.9).

(5.10) From (5.5) and the construction of the colimit functor one can deduce a direct description of h-colim D of a based h$\mathcal{C}$-diagram D:

$$\text{h-colim}\, D = \Big(\coprod_{A,B\in\mathcal{C}} \coprod_{n\geq 0} \mathcal{C}_n(A,B)\times I^n \times D_0 A\Big)\cup\{*\}/\sim$$

with the relations

$$(t_n, f_n, \ldots, t_1, f_1; x) = \begin{cases} (t_n, f_n, \ldots, t_2, f_2; x) & f_1 = \mathrm{id} \\ (t_n, f_n, \ldots, f_{i+1}, t_i t_{i-1}, f_{i-1}, \ldots, f_1; x) & f_i = \mathrm{id},\ 1<i \\ (t_n, f_n, \ldots, t_{i+1}, f_{i+1}\circ f_i, t_{i-1}, \ldots, f_1; x) & t_i = 1,\ i<n \\ (t_{n-1}, f_{n-1}, \ldots, f_1; x) & t_n = 1 \\ (t_n, f_n, \ldots, f_{i+1}; D_{A,C}(f_i, t_{i-1}, \ldots, f_1; x)) & t_i = 0 \\ * & x = * \end{cases}$$

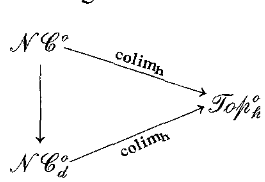
$f_i\circ\cdots\circ f_1$: $A\to C$, $f_n\circ\cdots\circ f_{i+1}$: $C\to B$. In the unbased version we drop $(*)$ and the last relation. Using this description of h-colim D it is an easy exercise to show that h-colim D is well-pointed if $\mathcal{C}$ and D are.

(5.11) *Examples.* (a) If D: $Y \xleftarrow{f} X \xrightarrow{g} Z$, then h-colim D is the reduced double mapping cylinder in the based case and the unreduced one in the unbased case. In particular, if Z is a single point, then h-colim D is the reduced (unreduced) mapping cone C_f.

(b) If D: $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, then h-colim D is the reduced mapping torus in the based case and the unreduced one in the unbased case.

(c) Let G be a topological monoid in $\mathcal{CG}$, i.e. an H-space in $\mathcal{CG}$ with associative multiplication and a strict unit e. Suppose (G, e) is a NDR. Let $\mathcal{C}$ be the category consisting of one object P with $\mathcal{C}(P, P) = G$. Composition is given by the multiplication in G. Let D be the unique h$\mathcal{C}$-diagram on a one-point space. Then the unbased version MD is the total space EG and the unbased h-colim D the base space BG of Milgram's classifying space construction. In particular, EG is contractible by (5.4a). For a proof see [3; VI, § 1].

For the construction (5.10) of h-colim D we did not need that each space $\mathcal{C}(A, B)$ is locally compact. This assumption was only needed in the proof of (5.4) when we showed that MD is a $\mathcal{C}$-diagram. Let $\mathcal{C}_d$ be the category $\mathcal{C}$ with the discrete topology. Then the identity $\mathcal{C}_d \to \mathcal{C}$ is a continuous functor inducing a functor $\mathcal{N}\mathcal{C}^o \to \mathcal{N}\mathcal{C}_d^o$. Now



commutes. Hence it is reasonable to conjecture that (5.9) and its unbased version hold for arbitrary spaces too. We need the assumption that $\mathcal{C}$ is well-pointed, because otherwise $\mathcal{H}\mathcal{C}^o$ might not exist. In fact, we can show

(5.12) **Theorem.** *If $\mathcal{C}$ is a well-pointed category, then* h-colim: $\mathcal{H}\mathcal{C}^o \to \mathcal{T}op_h^o$ *exists and is left adjoint to the inclusion functor J. The same holds for the unbased version.*

Proof. Consider $X \in \mathcal{T}op_h^o$ as a constant $\mathcal{C}$-diagram and let $H: D \to X$ be a $\mathcal{T}$-morphism. Construct MD as in (5.4). Then MD is a $\mathcal{C}_d$-diagram although it might not be a $\mathcal{C}$-diagram and $h: MD \to X$ as defined in (5.4) is a homomorphism of $\mathcal{C}_d$-diagrams inducing a continuous map k: h-colim $D \to X$. Let $\{i_A \colon MD_0\, A \to \operatorname{colim} MD\}$ be the set of universal maps. Then the composite $\{i_A\} \circ \mu_D$ as defined in (5.2) is a $\mathcal{T}$-morphism from D to the constant h$\mathcal{C}$-diagram h-colim D, and k is the unique map such that the canonical composite $k \circ (\{i_A\} \circ \mu_D)$ equals H. A homotopy through $\mathcal{T}$-morphisms $D \to X$ from H to H' induces a homotopy of the induced maps k, k': h-colim $D \to X$. Hence the result follows from (5.7) in the same manner as (5.6) follows from (5.4).

6. Homotopy Limits

For topological spaces X and Y let $F(X, Y)$ denote the space of all maps $X \to Y$ with the compact-open topology or the usual function space topology if we work in $\mathscr{CG}$. If Y has a base-point, the constant map $X \to \{*\} \subset Y$ is a natural base point for $F(X, Y)$ even if X is not based.

(6.1) **Proposition.** *Let $\mathscr{C}$ be a topological category such that each $\mathscr{C}(A, B)$ is locally compact unless we work in $\mathscr{CG}$, and let D be a $\mathrm{h}\mathscr{C}$-diagram. Then there is a $\mathscr{C}$-diagram ND and a $\mathscr{S}$-morphism v_D: $ND \to D$ such that*

(a) *the underlying maps n_A: $ND_0 A \to D_0 A$, $A \in \mathrm{ob}\,\mathscr{C}$, are deformation retractions*

(b) *a $\mathscr{S}$-morphism H: $E \to D$ is the canonical composite of a unique homomorphism h: $E \to ND$ and v_D.*

(c) *If $\mathscr{C}$ is well-pointed and if H, H': $E \to D$ are simplicially homotopic $\mathscr{S}$-morphisms, then the induced homomorphisms h, h': $E \to ND$ are homotopic, provided E and D are well-pointed or we work with k-spaces.*

Proof. Define $ND_0 A$ to be the subspace of all elements

$$\{\alpha_B \mid B \in \mathrm{ob}\,\mathscr{C}\} \in \prod_{B \in \mathscr{C}} F\Big(\coprod_{n \geqq 0} \mathscr{C}_{n+1}(A, B) \times I^n, D_0 B\Big)$$

satisfying

$$\text{(6.2)} \qquad \alpha_B(f_n, t_n, \ldots, f_0) = \begin{cases} \alpha_B(f_n, t_n, \ldots, f_{i+1}, t_{i+1}\, t_i, f_{i-1}, \ldots, f_0) & f_i = \mathrm{id},\ 0 < i < n \\ \alpha_B(f_{n-1}, t_{n-1}, \ldots, f_0) & f_n = \mathrm{id} \\ \alpha_B(f_n, t_n, \ldots, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \ldots, f_0) & t_i = 1 \\ D_{C,B}\big(f_n, t_n, \ldots, f_i;\ \alpha_C(f_{i-1}, t_{i-1}, \ldots, f_0)\big) & t_i = 0 \end{cases}$$

where $f_n \circ \cdots \circ f_i$: $C \to B$ in $\mathscr{C}$. The base point of $ND_0 A$ is the product of the constant maps. If $\mathscr{C}(A, C)$ is locally compact or we work with k-spaces, the correspondence $(f, \{\alpha_B\}) \mapsto \{\bar{\alpha}_B\}$ with

$$\bar{\alpha}_B(f_n, t_n, \ldots, f_0) = \alpha_B(f_n, t_n, \ldots, t_1, f_0 \circ f)$$

defines a continuous map $\mathscr{C}(A, C) \times ND_0 A \to ND_0 C$ extending the function ND_0 to a $\mathscr{C}$-diagram ND. For the definition of v_D we again adopt the notation of the proof of (5.4). The diagrams D and ND and the maps

$$(v_D)_{A^0, B^1}\colon T(\mathscr{C} \times \mathscr{L}_I)(A^0, B^1) \times ND_0 A \to D_0 B$$

given by

$$\big((f_n, r_n, t_n, f_{n-1}, r_{n-1}, t_{n-1}, \ldots, f_0), \{\alpha_C\}\big) \mapsto \alpha_B(f_n, t_n, \ldots, f_i \circ f_{i-1} \circ \cdots \circ f_0)$$

if $r_i = 0$ and $r_{i+1} = 1$, define the $\mathscr{S}$-morphism v_D. The $(v_D)_{A^0, B^1}$ are continuous because $\mathscr{C}(A, B)$ and hence $\mathscr{C}_{n+1}(A, B) \times I^n$ is locally compact or we work in $\mathscr{CG}$. The underlying map n_A: $ND_0 A \to D_0 A$ is given by $n_A(\{\alpha_C\})$

$=\alpha_A(\mathrm{id}_A)$. The correspondence $x \mapsto \{\alpha_C\}$ with

$$\alpha_C(f_n, t_n, \dots, f_0) = D_{A,C}(f_n, t_n, \dots, f_0; x)$$

defines a section $s_A\colon D_0 A \to ND_0 A$ of n_A, and $H_t(\{\alpha_C\}) = \{\alpha_C^t\}$ with

$$\alpha_C^t(f_n, t_n, \dots, f_0) = \alpha_C(f_n, t_n, \dots, f_0, t, \mathrm{id}_A)$$

is a deformation of $ND_0 A$ into the section.

If $H\colon E \to D$ is a $\mathscr{S}$-morphism, then the maps $h_A\colon E_0 A \to ND_0 A$ given by $h_A(x)(f_n, t_n, \dots, f_0) = H_{A^0, B^1}(f_n, 1, t_n, f_{n-1}, 1, t_{n-1}, \dots f_1, 1, t_1, f_0; x)$ define the unique homomorphism $h\colon E \to ND$ such that $H = v_D \circ h$. If $H, H'\colon E \to D$ are simplicially homotopic $\mathscr{S}$-morphism, then there is a homotopy through $\mathscr{S}$-morphisms $E \to D$ from H to H' inducing a homotopy through homomorphisms from h to the homomorphism $h'\colon E \to ND$ induced by H'.

In the same manner as in the previous section we can show

(6.3) **Theorem.** (a) *Let $\mathscr{C}$ be a well-pointed category such that each $\mathscr{C}(A, B)$ is locally compact. Then we can extend the correspondence $D \mapsto ND$ to a functor $N\colon \mathscr{H}\mathscr{C}^o \to \mathscr{N}\mathscr{C}^*$. If $E \in \mathscr{N}\mathscr{C}^o$, there is a natural bijection*

$$\mathscr{H}\mathscr{C}^o(J'E, D) \cong \mathscr{N}\mathscr{C}^*(E, ND).$$

(b) *If we work with k-spaces and if $\mathscr{C}$ is a well-pointed category, we can extend the correspondence $D \mapsto ND$ to a functor $N\colon \mathscr{H}\mathscr{C}_k^* \to \mathscr{N}\mathscr{C}_k^*$ which is right adjoint to $J'\colon \mathscr{N}\mathscr{C}_k^* \to \mathscr{H}\mathscr{C}_k^*$.*

We cannot prove a strong result of the type of (5.6) because ND is in general not well-pointed even if D is.

(6.4) **Definition.** The *homotopy limit functor* h-lim: $\mathscr{H}\mathscr{C}^o \to \mathscr{T}\!op_k^*$ $[\mathscr{H}\mathscr{C}_k^* \to \mathscr{C}\mathscr{G}_k^*]$ is defined to be the composite h-lim $= \lim_h \circ N$.

(6.5) **Theorem.** (a) *If $\mathscr{C}$ is a well-pointed category such that each $\mathscr{C}(A, B)$ is locally compact, then the functor* h-lim: $\mathscr{H}\mathscr{C}^o \to \mathscr{T}\!op_k^*$ *exists. If $X \in \mathscr{T}\!op_k^o$, then there is a natural bijection*

$$\mathscr{H}\mathscr{C}^o(JX, D) \cong \mathscr{T}\!op_k(X, \text{h-colim}\, D).$$

(b) *If we work with k-spaces and if $\mathscr{C}$ is a well-pointed category, the functor* h-lim: $\mathscr{H}\mathscr{C}_k^* \to \mathscr{C}\mathscr{G}_k^*$ *exists. It is right adjoint to the functor $J\colon \mathscr{C}\mathscr{G}_k^* \to \mathscr{H}\mathscr{C}_k^*$.*

The Unbased Case. In the unbased case the functor $N\colon \mathscr{H}\mathscr{C} \to \mathscr{N}\mathscr{C}\,[\mathscr{H}\mathscr{C}_k \to \mathscr{N}\mathscr{C}_k]$ is constructed in exactly the same way as the functor $N\colon \mathscr{H}\mathscr{C}^o \to \mathscr{N}\mathscr{C}^*$, we just do not mention base points. We follow N by the unbased limit functor $\lim_h\colon \mathscr{N}\mathscr{C} \to \mathscr{T}\!op_k[\mathscr{N}\mathscr{C}_k \to \mathscr{C}\mathscr{G}_k]$ and obtain the *unbased homotopy limit functor.* Since we do not have to worry about well-pointedness, we have

(6.6) **Theorem.** (a) *Let $\mathscr{C}$ be a well-pointed category such that each $\mathscr{C}(A,B)$ is locally compact. Then the correspondence $D \mapsto ND$ can be extended to a fully faithful functor $N: \mathscr{H}\mathscr{C} \to \mathscr{N}\mathscr{C}$ which is right adjoint to $J': \mathscr{N}\mathscr{C} \to \mathscr{H}\mathscr{C}$. The homotopy limit functor* h-lim $= \lim_{\mathrm{h}} \circ N$: $\mathscr{H}\mathscr{C} \to \mathscr{T}\!op_{k}$ *exists and is right adjoint to $J: \mathscr{T}\!op_{k} \to \mathscr{H}\mathscr{C}$.*

(b) *If we work with k-spaces and if $\mathscr{C}$ is a well-pointed category, the same holds for the corresponding categories and functors.*

(6.7) From the Eqs. (6.2) and the well-known construction of the limit functor we obtain a direct description of the space h-lim D for a based or unbased h$\mathscr{C}$-diagram D:

$$\text{h-lim}\, D \subset \prod_{A,B\in\mathscr{C}} F\Big(\coprod_{n\geq 0} \mathscr{C}_n(A,B) \times I^n, D_0 B\Big)$$

is the subspace of all elements $\{\alpha_{A,B}: \coprod_{n\geq 0} \mathscr{C}_n(A,B)\times I^n \to D_0 B \mid A, B \in \mathrm{ob}\,\mathscr{C}\}$ satisfying

$$\alpha_{A,B}(f_n, t_n, \ldots, f_1, t_1) = \begin{cases} \alpha_{A,B}(f_n, t_n, \ldots, f_{i+1}, t_{i+1}\, t_i, f_{i-1}, \ldots, t_1) & f_i = \mathrm{id},\ i<n \\ \alpha_{A,B}(f_{n-1}, t_{n-1}, \ldots, t_1) & f_n = \mathrm{id} \\ \alpha_{A,B}(f_n, t_n, \ldots, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \ldots, t_1) & t_i = 1,\ i>1 \\ \alpha_{C,B}(f_n, t_n, \ldots, t_2) & t_1 = 1 \\ D_{E,B}\big((f_n, t_n, \ldots, f_i; \alpha_{A,E}(f_{i-1}, t_{i-1}, \ldots, t_1)\big) & t_i = 0 \end{cases}$$

where

$$f_1: A \to C, \quad f_{i-1} \circ \cdots \circ f_2: C \to E, \quad f_n \circ \cdots \circ f_i: E \to B.$$

If D is a based diagram, then the product of the constant maps is the base point of h-lim D.

(6.8) *Example.* If D: $A \subset X \supset B$, then h-lim D is the space of all paths in X from A to B.

Although we can construct the space h-lim D if the spaces $\mathscr{C}(A,B)$ are not locally compact, we cannot drop this conditions because we needed it for the continuity of the induced homomorphism in the proof of (6.1). The reason actually lies deeper: The functor h-lim can be defined without this assumption if we modify the definition of a $\mathscr{C}$-diagram. Instead of having maps $D_{A,B}$: $\mathscr{C}(A,B) \times D_0 A \to D_0 B$ we have to take maps

$$D_{A,B}: D_0 A \to F\big(\mathscr{C}(A,B), D_0 B\big)$$

satisfying a number of conditions. These two definitions coincide in $\mathscr{C}\mathscr{G}$ but not in $\mathscr{T}\!op$ because we there do not have full adjointness. We do not intend to consider this sort of diagrams because their treatment requires a basic modification of the tools of § 3.

We close this section with a result on base points which occassionally is of interest in applications.

(6.9) **Proposition.** *Let $\mathscr{C}$ be a well-pointed category with finitely many objects such that each $\mathscr{C}(A,B)$ is compact. Assume further that there is a $k<\infty$ such that $g=f_n\circ\cdots\circ f_1$ with $n>k$ implies that some $f_i=\mathrm{id}$. Then the homotopy limit of a well-pointed* h$\mathscr{C}$*-diagram D is well-pointed.*

Proof. Let $E_0A=D_0A\cup I/(*\sim 1)$ with $0\in I$ as base point. Projecting I to $1\in I$, we obtain a based map $p_A\colon E_0A\to D_0A$, which is a based homotopy equivalence. Note that E_0A is well-pointed. Define

$$E_{A,B}\colon T\mathscr{C}(A,B)\times E_0A\to E_0B$$

by $E_{A,B}|T\mathscr{C}(A,B)\times D_0A=D_{A,B}$ and $E_{A,B}(f;t)=t, f\in T\mathscr{C}(A,B),\ t\in I$. We obtain a h$\mathscr{C}$-diagram E such that the p_A form a homomorphism $p\colon E\to D$. By (4.6) and (6.5) the spaces h-lim E and h-lim D have the same based homotopy type. Define a retraction $r_A\colon E_0A\times I\to E_0A\times 0\cup *\times I$ by

$$r_A(x,t)=\begin{cases}(x,0) & x\in D_0A\\ \left(0,\dfrac{t-2x}{1-x}\right) & x\in I,\ t\geqq 2x\\ \left(\dfrac{2x-t}{2-t},0\right) & x\in I,\ t\leqq 2x.\end{cases}$$

Then we obtain a retraction r: (h-lim $E)\times I\to$ (h-lim $E)\times 0\cup *\times I$ as follows. Let $q_B^1\colon E_0B\times 0\cup *\times I\to E_0B$ and $q_B^2\colon E_0B\times 0\cup *\times I\to I$ be the projections. Then $r(\{\alpha_{A,B}\},t)=(\{\beta_{A,B}\},u)$ where

$$\beta_{A,B}\colon \coprod_{n\geqq 0}\mathscr{C}_n(A,B)\times I^n\xrightarrow{\alpha^t_{A,B}}E_0B\times I\xrightarrow{r_B}E_0B\times 0\cup *\times I\xrightarrow{q_B^1}E_0B$$

with $\alpha^t_{A,B}(y)=(\alpha_{A,B}(y),t)$ and where

$$u=\min\{q_B^2\circ r_B\circ\alpha^t_{A,B}(y)\,|\,y\in\mathscr{C}_n(A,B)\times I^n, n\leqq k\}.$$

Since each $\mathscr{C}_n(A,B)$ is compact, r is continuous. Hence h-lim E is well-pointed. By [6; (2.7) and (3.26)] the proposition is proved if there is a map v: h-lim $D\to I$ with $v^{-1}(0)=*$. Since each D_0A is well-pointed, there are maps $u_A\colon D_0A\to I$ with $u_A^{-1}(0)=*$. Put

$$v(\{\alpha_{A,B}\})=\max\{u_B(\alpha_{A,B}(y))\,|\,y\in\mathscr{C}_n(A,B)\times I^n, n\leqq k\}.$$

7. Homotopy Limits and Colimits as Functors from $\mathscr{MC}^*$ to $\mathscr{Top}^*$ or $\mathscr{CG}^*$

For some applications it is desirable to have h-colim and h-lim as functor from the category $\mathscr{MC}^*$ of based $\mathscr{C}$-diagrams and based homomorphisms to $\mathscr{Top}^*$ and $\mathscr{CG}^*$ because passing to $\mathscr{Top}^*_h$ implies a loss of information. Moreover, both functors are then defined for arbitrary topological categories $\mathscr{C}$. Define functors

$$\text{H-lim, H-colim}: \mathscr{M}\mathscr{C}^* \to \mathscr{T}\!op^*$$

as follows: If D is a $\mathscr{C}$-diagram, then H-lim D and H-colim D are the based topological spaces h-lim $\varepsilon^*(D)$ respectively h-colim $\varepsilon^*(D)$ as defined in (6.7) and (5.10). If $g=\{g_A: D_0 A \to E_0 A\}$ is a homomorphism of $\mathscr{C}$-diagrams $g: D \to E$, then

$$\text{H-lim}(g): \text{H-lim}\, D \to \text{H-lim}\, E, \qquad \text{H-colim}(g): \text{H-colim}\, D \to \text{H-colim}\, E$$

are given on representatives by

$$\text{H-lim}(g)(\{\alpha_{A,B}\}) = \{g_B \circ \alpha_{A,B}\}$$
$$\text{H-colim}(g)(t_n, f_n, \dots, f_1; x) = (t_n, f_n, \dots, f_1; g_B(x)) \qquad x \in D_0 B.$$

The unbased version is defined analogously.

Let $P: \mathscr{T}\!op^\circ \to \mathscr{T}\!op_h^\circ$ and $P': \mathscr{M}\mathscr{C}^\circ \to \mathscr{N}\mathscr{C}^\circ$ be the projection functors. Suppose $\mathscr{C}$ is well-pointed so that $\mathscr{H}\mathscr{C}^\circ$ exists. Then put $R = J' \circ P'$: $\mathscr{M}\mathscr{C}^\circ \to \mathscr{N}\mathscr{C}^\circ \to \mathscr{H}\mathscr{C}^\circ$. As an immediate consequence of the definitions we obtain

(7.1) **Proposition.** *Suppose $\mathscr{C}$ is a small well-pointed topological category. Then the diagram*

$$\begin{array}{ccc} \mathscr{M}\mathscr{C}^\circ & \xrightarrow{\text{H-colim}} & \mathscr{T}\!op^\circ \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle P} \\ \mathscr{H}\mathscr{C}^\circ & \xrightarrow{\text{h-colim}} & \mathscr{T}\!op_h^\circ \end{array}$$

commutes. The same holds for the unbased version and for based or unbased k-spaces. If in addition each $\mathscr{C}(A,B)$ is locally compact or we work with k-spaces, the same holds if we replace H-colim *by* H-lim, h-colim *by* h-lim, *and $\mathscr{T}\!op^\circ$, $\mathscr{T}\!op_h^\circ$ by $\mathscr{T}\!op^*$ and $\mathscr{T}\!op_h^*$.*

If we work with k-spaces or if each $\mathscr{C}(A,B)$ is locally compact, the functors H-colim and H-lim factor as

$$\text{H-colim} = \text{colim} \circ \overline{M}, \qquad \text{H-lim} = \lim \circ \overline{N}: \mathscr{M}\mathscr{C}^* \to \mathscr{M}\mathscr{C}^* \to \mathscr{T}\!op^*$$

with $\overline{M}(D) = M(\varepsilon^* D)$, $\overline{N}(D) = N(\varepsilon^* D)$ as defined in (5.4) and (6.1) and

$$\overline{M}(g)(f_n, t_n, \dots, f_0; x) = (f_n, t_n, \dots, f_0; g_B(x)) \qquad x \in D_0 B$$
$$\overline{N}(g)(\{\alpha_B\}) = \{g_B \circ \alpha_B\}.$$

Again we have commutative diagrams

$$\begin{array}{ccc} \mathcal{M}\mathcal{C}^o & \xrightarrow{\bar{M}(\bar{N})} & \mathcal{M}\mathcal{C}^* \\ \downarrow R & & \downarrow P' \\ \mathcal{H}\mathcal{C}^o & \xrightarrow{M(N)} & \mathcal{N}\mathcal{C}^* \end{array}$$

Of course the same holds in the unbased case and for the based and unbased case with k-spaces.

8. Simplicial Parameters

It is the purpose of this section to relate our notions of homotopy limits and colimits with those of Segal [14] and the topological versions of those of Bousfield and Kan [4]. We study the unbased version; the treatment of the based version is analogous. Let $\mathcal{C}$ be a discrete category and D a $\mathcal{C}$-diagram. Then Segal's homotopy colimit $S(D)$ is defined to be the topological realization of the following simplicial space ΓD: The space of n-simplexes is

$$\coprod_{A,B\in\mathcal{C}} \mathcal{C}_n(A,B)\times D_0 A$$

with the following face and degeneracy operations

$$d^i(f_n, f_{n-1}, \dots, f_1; a) = \begin{cases} (f_n, \dots, f_2; D_{A,C}(f_1; a)) & i=0,\ f_1\in\mathcal{C}(A,C) \\ (f_n, \dots, f_{i+1}\circ f_i, \dots, f_1; a) & 0<i<n \\ (f_{n-1}, \dots, f_1; a) & i=n \end{cases}$$

$$s^i(f_n, f_{n-1}, \dots, f_1; a) = (f_n, f_{n-1}, \dots, f_{i+1}, 1, f_i, \dots, f_1; a) \qquad 0\leqq i\leqq n.$$

We give a different description of $S(D)$ which allows us to compare it with our construction. Let Δ^n be the standard n-simplex i.e. the space of all points $(u_1, u_2, \dots, u_n)\in\mathbb{R}^n$ such that $0\leqq u_1\leqq u_2\leqq\cdots\leqq u_n\leqq 1$. Let

$$T(D)_0(B) = \coprod_{\substack{A\in\mathcal{C}\\ n\geqq 0}} \mathcal{C}_{n+1}(A,B)\times\Delta^n\times D_0(A)/\sim$$

with the relations

(8.1) $(f_n, u_n, f_{n-1}, \dots, u_1, f_0; a)$

$$= \begin{cases} (f_n, u_n, \dots, f_{i+1}, u_i, f_{i-1}, u_{i-1}, \dots, f_0; a) & \text{if } f_i=\mathrm{id} \\ (f_n, \dots, f_{i+1}, u_{i+1}, f_i\circ f_{i-1}, u_{i-1}, \dots, f_0; a) & \text{if } u_i=u_{i+1} \\ (f_n\circ f_{n-1}, u_{n-1}, \dots, f_0; a) & \text{if } u_n=1 \\ (f_n, \dots, u_2, f_1; D_{A,C}(f_0, a)) & \text{if } u_1=0 \text{ and } f_0\in\mathcal{C}(A,C). \end{cases}$$

Extend $T(D)_0$ to a $\mathscr{C}$-diagram $T(D)$ by

$$[g,(f_n,u_n,\ldots,f_0)]\mapsto(g\circ f_n,u_n,f_{n-1},\ldots,f_0).$$

The diagram TD can be considered as the topological realization of the $\mathscr{C}$-diagram τD of simplicial spaces defined as follows:

(8.2) The space of n-simplexes of $\tau D_0(B)$ is $\coprod_{A\in\mathscr{C}}\mathscr{C}_{n+1}(A,B)\times D_0(A)$ and the face and degeneracy operations are given by

$$d^i(f_n,f_{n-1},\ldots,f_0;a)=\begin{cases}(f_n,f_{n-1},\ldots,f_i\circ f_{i-1},\ldots,f_0;a) & 0<i\leqq n\\ (f_n,f_{n-1},\ldots,f_1;D_{A,C}(f_0;a)) & i=0 \text{ and } f_0\in\mathscr{C}(A,C)\end{cases}$$

$$s^i(f_n,f_{n-1},\ldots,f_0;a)=(f_n,\ldots,f_i,1,f_{i-1},\ldots,f_0;a)\quad 0\leqq i\leqq n.$$

The simplicial maps $[g,(f_n,f_{n-1},\ldots,f_0;a)]\mapsto(g\circ f_n,f_{n-1},\ldots,f_0;a)$ extend τD_0 to a $\mathscr{C}$-diagram of simplicial spaces. It is easy to check that $\Gamma D=\operatorname{colim}\tau D$ taken in the category of simplicial spaces, and since the topological realization preserves colimits, we obtain

(8.3) **Lemma.** $S(D)=\operatorname{colim} T(D)$.

Given a homomorphism $h\colon D\to E$ of $\mathscr{C}$-diagrams with underlying maps $\{h_A\}$. Then the correspondence $(f_n,u_n,\ldots,f_0;a)\mapsto(f_n,u_n,\ldots,f_0;h_A(a))$ induces a homomorphism $T(h)\colon T(D)\to T(E)$, which makes T into a functor $T\colon\mathscr{M}\mathscr{C}\to\mathscr{M}\mathscr{C}$.

(8.4) **Proposition.** *The functors T and $\overline{M}$ from $\mathscr{M}\mathscr{C}$ to $\mathscr{M}\mathscr{C}$ are naturally isomorphic.*

Proof. The correspondence $(f_n,t_n,\ldots,f_0;a)\mapsto(f_n,u_n,\ldots,f_0;a)$ with $u_i=t_n t_{n-1}\ldots t_1$ determines homeomorphisms $hD_A\colon \overline{M}D_0A\to TD_0(A)$, whose inverses are given by $(f_n,u_n,\ldots,f_0;a)\mapsto(f_n,t_n,\ldots,f_0;a)$ with $t_i=u_i/u_{i+1}$ and the convention that $0/0=1$ and $u_{n+1}=1$. Of course, we use strongly that D is a $\mathscr{C}$-diagram. The hD_A are the underlying maps of a homomorphism $hD\colon \overline{M}D\to TD$, and it follows directly from the definitions that given a homomorphism $f\colon D\to E$ of $\mathscr{C}$-diagrams, then $hE_A\circ\overline{M}f_A=Tf_A\circ hD_A$ for all $A\in\mathscr{C}$, which implies the naturality of $h\colon\overline{M}\to T$.

(8.5) **Corollary.** *The functors* H-colim *and S from $\mathscr{M}\mathscr{C}$ to $\mathscr{T}\!op$ are naturally isomorphic. In particular, if D is a $\mathscr{C}$-diagram, then* H-colim D *is naturally homeomorphic to the topological realization of the simplicial space ΓD.*

Of course, these results can be extended to topological categories $\mathscr{C}$ if we only want the result (8.5). Since T and $\overline{M}$ are only defined if in addition each $\mathscr{C}(A,B)$ is locally compact or if we work with k-spaces, we have to add this extra assumption to obtain (8.4) for more general topological categories. The details are left to the reader.

Segal's construction can be dualized to provide similar results for homotopy limits.

(8.6) **Definition.** Let $X=\{X_n\}$ be a cosimplicial space with face and degeneracy operations $d_n^i\colon X_{n-1}\to X_n$ and $s_n^i\colon X_{n+1}\to X_n$ (for a definition see [7; II, § 2]). Let $\delta_n^i\colon \Delta^{n-1}\to\Delta^n$ and $\sigma_n^i\colon \Delta^{n+1}\to\Delta^n$ be the face and degeneracy maps of the standard n-simplex. The *topological realization* $\|X\|$ of X is defined by

$$\|X\|=\{(y_n|n=0,1,\ldots)\in\prod_{n\geq 0}F(\Delta^n,X^n)|\,y_n\circ\delta_n^i=d^i(y_{n-1})\text{ and }y_n\circ\sigma_n^i=s_i^n(y_{n+1})\}$$

with the subspace topology.

With any $\mathscr{C}$-diagram D we associate a $\mathscr{C}$-diagram $\rho(D)$ of cosimplicial spaces as follows:

(8.7) The space of n-simplexes of $\rho D_0(A)$ is $\prod_{B\in\mathscr{C}}F(\mathscr{C}_{n+1}(A,B),D_0B)$ and the face and degeneracy operations are given by $d^i(\{\alpha_B|B\in\mathscr{C}\})=\{\bar{\alpha}_B|B\in\mathscr{C}\}$ with

$$\bar{\alpha}_B(f_n,\ldots,f_0)=\begin{cases}\alpha_B(f_n,\ldots,f_{i+1}\circ f_i,\ldots,f_0) & 0\leqq i<n\\ D_{C,B}(f_n;\alpha_C(f_{n-1},\ldots,f_0)) & i=n,\ f_n\in\mathscr{C}(C,B)\end{cases}$$

and

$$s^i(\{\alpha_B|B\in\mathscr{C}\})=\{\bar{\alpha}_B|B\in\mathscr{C}\}$$

with

$$\bar{\alpha}_B(f_n,\ldots,f_0)=\alpha_B(f_n,\ldots,f_{i+1},1,f_i,\ldots,f_0)\qquad 0\leqq i\leqq n.$$

We extend ρD_0 to a $\mathscr{C}$-diagram by sending $(g,\{\alpha_B\})$ to $\{\bar{\alpha}_B\}$ where $\bar{\alpha}_B(f_n,\ldots,f_1,f_0)=\alpha_B(f_n,\ldots,f_1,f_0\circ g)$. Let $\Theta D=\lim\rho D$ taken in the category of cosimplicial spaces and cosimplicial maps. The topological realization of $\rho(D)$ is a $\mathscr{C}$-diagram $R(D)$. A homomorphism $h\colon D\to E$ of $\mathscr{C}$-diagrams induces a homomorphism of cosimplicial $\mathscr{C}$-diagrams by $\{\alpha_B\}\to\{h_B\circ\alpha_B\}$ where $\{h_B\}$ are the underlying maps of h. Hence R extends to a functor $\mathscr{M}\mathscr{C}\to\mathscr{M}\mathscr{C}$.

(8.8) **Proposition.** *The functors R and $\bar{N}$ from $\mathscr{M}\mathscr{C}$ to $\mathscr{M}\mathscr{C}$ are naturally isomorphic.*

Proof.

$$R(D)_0A\subset\prod_{n\geq 0}F\big(\Delta^n,\prod_{B\in\mathscr{C}}F(\mathscr{C}_{n+1}(A,B),D_0B)\big)\cong\prod_{B\in\mathscr{C}}F\big(\coprod_{n\geq 0}\mathscr{C}_{n+1}(A,B)\times\Delta^n,D_0B\big)$$

because $\mathscr{C}$ is discrete and Δ^n locally compact. The maps $I^n\to\Delta^n$ given by $(t_n,\ldots,t_1)\mapsto(u_n,\ldots,u_1)$ with $u_i=t_n t_{n-1}\ldots t_i$ determine a map

$$\prod_{n\geq 0}F\big(\Delta^n,\prod_{B\in\mathscr{C}}F(\mathscr{C}_{n+1}(A,B),D_0B)\big)\to\prod_{B\in\mathscr{C}}F\big(\coprod_{n\geq 0}\mathscr{C}_{n+1}(A,B)\times I^n,D_0B\big)$$

sending the subspace $R(D)_0 A$ homeomorphically onto the subspace $\bar{N}(D)_0 A$. These homeomorphisms $h(D)_A$ are the underlying maps of a natural homomorphism $R(D) \to \bar{N}(D)$.

Since the topological realization functor $\|-\|$ preserves limits we have

(8.9) **Corollary.** *The functors* H-lim *and* $\lim \circ R$, *the topological version of the homotopy limit functor of Bousfield and Kan, from* $\mathcal{M}\mathcal{C}$ *to* $\mathcal{T}op$ *are naturally isomorphic. In particular,* H-lim D *is naturally homeomorphic to the topological realization of the cosimplicial space* ΘD.

The results (8.8) and (8.9) can also be proved if $\mathcal{C}$ is a topological category such that each $\mathcal{C}(A, B)$ is locally compact or if $\mathcal{C}$ is an arbitrary topological category and we work with k-spaces.

9. Spectral Sequences for Homotopy Colimits

Throughout this section let $\mathcal{C}$ be a discrete category and D be an unbased h$\mathcal{C}$-diagram. The images of $\coprod_{A, B \in C} \coprod_{n=0}^{p} \mathcal{C}_n(A, B) \times I^n \times D_0 A$ in h-colim D define a filtration $F_p D$ of h-colim D. Let k_* be an arbitrary homology and k^* an arbitrary cohomology theory. For $f \in \mathcal{C}(A, B)$ let $\bar{f}$: $D_0 A \to D_0 B$ denote the map $\bar{f}(x) = D_{A,B}(\eta(f); x)$, where η is the standard inclusion. Since D is a $\mathcal{C}$-diagram up to coherent homotopies, the correspondences $f \mapsto k_q(\bar{f})$ respectively $f \mapsto k^q(\bar{f})$ define a covariant functor $k_q(\eta^* D)$: $\mathcal{C} \to \mathcal{A}b$ and a contravariant functor $k^q(\eta^* D)$: $\mathcal{C} \to \mathcal{A}b$ into the category of abelian groups.

(9.1) **Theorem.** *Let* $\mathcal{C}$ *be a discrete category,* k_* *a homology, and* k^* *a cohomology theory. Assume that* k_* *and* k^* *are additive unless* ob $\mathcal{C}$ *and each space* $\mathcal{C}_n(A, B)$ *is finite. Then*

$$E^2_{q,p} \cong \operatorname{colim}^p k_q(\eta^* D)$$

in the spectral sequence $\{E^r D\}$ *derived from the* k_* *exact couple of the filtration of* h-colim D, *and*

$$E_2^{p,q} \cong \lim{}^p k^q(\eta^* D)$$

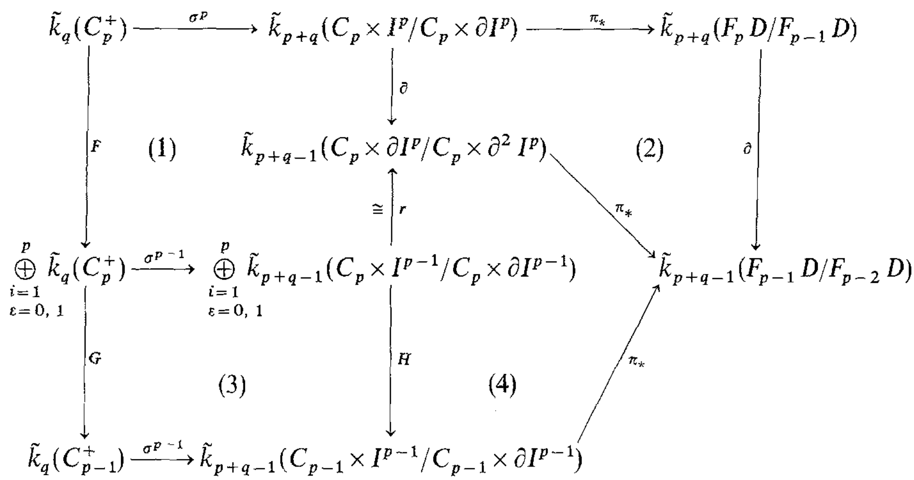
in the spectral sequence $\{E_r D\}$ *derived from the* k^* *exact couple of the filtration of* h-colim D. *Here* colim^p *and* $\lim^p$ *denote the p-th left derived of* colim *and the p-th right derived of* lim.

Proof. $E^1_{p,q} D = k_{p+q}(F_p D, F_{p-1} D)$ and the differential d^1 is the boundary operator of the triple $(F_p D, F_{p-1} D, F_{p-2} D)$. We obtain $F_p D$ from $F_{p-1} D$ by attaching $\coprod_{A,B} \mathcal{C}_p(A, B) \times D_0(A) \times I^p =: C_p \times I^p$ along $R_p \times I^p \cup$

$C_p \times \partial I^p$ where $R_p \subset C_p$ is the space of all elements $(f_p, f_{p-1}, \ldots, f_1; a)$ with some $f_i = \mathrm{id}$. Define maps $d^i\colon C_p \to C_{p-1}$ for $0 \leqq i \leqq p$ by

(9.2)
$$d^i(f_p, f_{p-1}, \ldots, f_1; a) = \begin{cases} (f_p, f_{p-1}, \ldots, f_2; D_{A,B}(\eta f_1; a)) & i=0,\ f_1 \in \mathscr{C}(A,B) \\ (f_p, \ldots, f_{i+1} \circ f_i, \ldots, f_1; a) & 0 < i < p \\ (f_{p-1}, f_{p-2}, \ldots, f_1; a) & i = p. \end{cases}$$

Consider the diagram



Here ∂I^p is the boundary of I^p and $\partial^2 I^p$ the $(p-2)$-skeleton, σ^p is the p-fold suspension isomorphism σ and the sign of σ is determined by the boundary maps

$$\partial I^p \xrightarrow[\text{proj}]{} \partial I^p / \partial^2 I^p \xrightarrow[\mathrm{pr}_1^1]{} I^{p-1} / \partial I^{p-1}$$

where $\mathrm{pr}_j^\varepsilon\colon \partial I^p / \partial^2 I^p \to I^{p-1} / \partial I^{p-1}$ is induced by the projection of I^p to the face $t_j = \varepsilon$, $\varepsilon = 0, 1$. The maps π are induced by the attaching maps. The component (j, ε) of F is multiplication with $(-1)^{j-\varepsilon}$. The maps G and H are induced by the constant map $C_p^+ \to C_{p-1}^+$ on the components $(j, 1)$ for $j > 1$, by d^0 on $(1, 1)$ and by d^j on $(j, 0)$. The map r is the isomorphism of the Mayer-Vietoris sequence of the inclusions

$$\delta_j^\varepsilon\colon (C_p \times I^{p-1} / C_p \times \partial I^{p-1}) \to (C_p \times \partial I^p / C_p \times \partial^2 I^p)$$

sending I^{p-1} to the face $t_j = \varepsilon$. The inverse of r is on its (j, ε)-component given by $\mathrm{pr}_j^\varepsilon$. Then (2) and (3) commute by the naturality of σ and ∂, and (4) commutes by (5.10); here observe that $\pi \circ \delta_j^1 \subset F_{p-2} D$ if $j > 1$. The commutativity of (1) follows from naturality and from

$$\begin{array}{ccccc}
\tilde{k}_{p+q}(I^p/\partial I^p) & \xrightarrow{\partial} & \tilde{k}_{p+q-1}(\partial I^p) & \xrightarrow{\text{proj}_*} & \tilde{k}_{p+q-1}(\partial I^p/\partial^2 I^p) \\
& \searrow^{\sigma^{-1}} & & \swarrow^{(\text{pr}_1^1)_*} & \downarrow{\scriptstyle (\text{pr}_j^\varepsilon)_*} \\
& & \tilde{k}_{p+q-1}(I^{p-1}/\partial I^{p-1}) & = & \tilde{k}_{p+q-1}(I^{p-1}/\partial I^{p-1})
\end{array}$$

in which the square commutes by the choice of the sign of σ and the triangle commutes up to $(-1)^{j-\varepsilon}$.

By the assumptions on k_* we have

$$\tilde{k}_q(C_p^+) = k_q(C_p) = \coprod_{A, B} \mathscr{C}_p(A, B) \times k_q(D_0(A))$$

and the maps $d^i \colon C_p \to C_{p-1}$ induce maps

$$d^{i*} \colon k_q(C_p) \to k_q(C_{p-1})$$

satisfying the identities (9.2) with a replaced by a homology class x in $k_q(D_0(A))$ and $D_{A,B}(\eta f_1; a)$ replaced by $k_q(\eta^* D(f_1))(x)$. The composite map $G \circ F \colon k_q(C_p) \to k_q(C_{p-1})$ is given in terms of the d^{i*} by

$$G \circ F = \sum_{i=0}^{p} (-1)^i d^{i*}.$$

Introduce maps $s^{i*} \colon k_q(C_p) \to k_q(C_{p+1})$ by

$$s^{i*}(f_p, \ldots, f_1; x) = (f_p, \ldots, f_{i+1}, 1, f_i, \ldots, f_1; x)$$

to obtain a simplicial abelian group $k_q(C_*)$. Then $G \circ F$ is the boundary map of the associated chain complex CD_*. By [9, chapter VIII, Thm. 6.1] CD_* is chain equivalent to the normalized chain complex of CD_*, which in turn is given by $(\tilde{k}_q(C_p/R_p),\ p = 0, 1, \ldots)$. Because of the commutativity of

$$\begin{array}{ccccc}
\tilde{k}_q(C_p^+) & & \longrightarrow & & \tilde{k}_q(C_p/R_p) \\
\downarrow{\scriptstyle \sigma^p} & & & & \downarrow{\scriptstyle \sigma^p} \\
\tilde{k}_{p+q}(C_p \times I^p, C_p \times \partial I^p) & \xrightarrow{\pi_*} & \tilde{k}_{p+q}(F_p D/F_{p-1} D) & \xleftarrow[\cong]{\pi_*} & \tilde{k}_{p+q}(C_p \times I^p/(R_p \times I^p \cup C_p \times \partial I^p))
\end{array}$$

$E^2_{p,q} D$ is isomorphic to the homology of the normalized chain complex and hence to the homology of the chain complex CD_*. By [7; appendix II, Prop. 3.3], $H_p(CD_*) = \operatorname{colim}^p(k_q(\eta^* D))$.

The proof for the cohomology spectral sequence is completely dual.

Remark. For $\mathscr{C}$-diagrams this result is an immediate consequence of Segal's spectral sequence of a simplicial space [14] and the result of

Gabriel and Zisman used in the proof (also see [1] for convergence questions and applications). In this case, (9.1) is the topological analogue of the spectral sequence [4; p. 336]. Our argument is strongly influenced by Segal's.

Remark. One obtains similar results for based h$\mathscr{C}$-diagrams and reduced homology and cohomology theories.

10. The Spectral Sequence of a Homotopy Limit

Throughout this section let $\mathscr{C}$ be a discrete category and D a based h$\mathscr{C}$-diagram. The inclusion

$$\coprod_{n=0}^{p} \mathscr{C}_n(A,B)\times I^n \subset \coprod_{n\geq 0} \mathscr{C}_n(A,B)\times I^n$$

induces a projection

$$\text{h-lim}\, D \to (\text{h-lim}\, D) \cap \prod_{A,B\in\mathscr{C}} F\left(\coprod_{n=0}^{p} \mathscr{C}_n(A,B)\times I^n, D_0 B\right).$$

Let G_p be its image. Then we have a cofiltration

(10.1) $$G_0 \xleftarrow{k_1} G_1 \xleftarrow{k_2} G_2 \xleftarrow{k_3} \cdots \leftarrow \text{h-lim}\, D$$

of h-lim D such that h-lim $D \to G_p$ and hence each k_p is surjective.

(10.2) **Lemma.** *The map $k_p\colon G_p \to G_{p-1}$ is a fibration.*

Proof. Let $Q_p(A,B) \subset \mathscr{C}_p(A,B)$ be the subspace of all elements $(f_p, \ldots, f_1)$ with some $f_i = \text{id}$. Then we have a pull back diagram

$$\begin{array}{ccc} G_p & \longrightarrow & \prod_{A,B} F(\mathscr{C}_p(A,B)\times I^p, D_0 B) \\ {\scriptstyle k_p}\downarrow & & \downarrow{\scriptstyle q_p} \\ G_{p-1} & \longrightarrow & \prod_{A,B} F(Q_p(A,B)\times I^n \cup \mathscr{C}_p(A,B)\times \partial I^n, D_0 B) \end{array}$$

in which q_p is the fibration induced by the cofibration

$$Q_p(A,B)\times I^p \cup \mathscr{C}_p(A,B)\times \partial I^p \subset \mathscr{C}_p(A,B)\times I^p.$$

Since the k_p are surjective, the exact sequences of the fibrations $G_{p+1} \to G_p$ with fibre F_p give rise to an exact couple

$$\begin{array}{ccc} D & \xrightarrow{k} & D \\ {\scriptstyle i}\nwarrow & & \swarrow{\scriptstyle \delta} \\ & E & \end{array}$$

with $D^{p,q}=[S^{q-p}X, G_{p+1}]$ and $E^{p,q}=[S^{q-p}X, F_p]$ for $q \geqq p$ and 0 otherwise. The maps i, k, δ have bidegrees $(0,0)$, $(-1,-1)$, $(1,0)$. Here $[X, Y]$ denotes the set of based homotopy classes of based maps from X to Y, and we assume that the functor $[X, -]$ takes values in the category of abelian groups.

(10.3) **Theorem.** *Let $\mathscr{C}$ be a discrete category and D a based* h$\mathscr{C}$*-diagram. Let $X \in \mathscr{T}\!op^*$ be a space such that $[X, -]$ is a functor into abelian groups. Then*

$$E_2^{p,q} D \cong \lim{}^p [S^q X, \eta^* D]$$

in the spectral sequence $\{E_r D\}$ of the exact couple obtained from the cofiltration of h-lim D.

The proof is practically dual to the one of (9.1).

Remark. For $\mathscr{C}$-diagrams this spectral sequence is a topological version of the spectral sequence [4; p. 309]. In [4] the derived $\lim^p$ of lim is studied for not necessarily abelian group valued functors, thus extending the applicability of (10.3), and a convergence proof is included.

11. Weak Limits and Colimits in $\mathscr{T}\!op_h$

As an application of our methods not of our results we prove the following folk theorem.

(11.1) **Theorem.** *The homotopy category $\mathscr{T}\!op_h$ has weak limits and colimits.*

Proof. Let $\mathscr{C}$ be a discrete category and E a $\mathscr{C}$-diagram in $\mathscr{T}\!op_h$. We have to show that there is a space X and a homomorphism $i: E \to X$ from E to the constant $\mathscr{C}$-diagram on X in $\mathscr{T}\!op_h$ such that given a homomorphism $f: E \to Y$ from E to a constant diagram Y in $\mathscr{T}\!op_h$ there exists a morphism $h: X \to Y$ in $\mathscr{T}\!op_h$ such that $f = h \circ i$ as homomorphism. Let D be any lifting of E to $\mathscr{T}\!op$ such that an identity is lifted to an identity (then D is not a $\mathscr{C}$-diagram). Put

$$X = \coprod_{A,B} \mathscr{C}(A,B) \times I \times D_0 B \cup \coprod_B D_0 B$$

with the relations (compare (5.10))

$$(g, t; x) = \begin{cases} x & g = \mathrm{id} \\ D(g; x) & t = 0 \\ x & t = 1. \end{cases}$$

The maps $i_A: D_0 A \to X$ given by $i_A(x) = x$ represent the underlying maps of the homomorphism $i: E \to X$. Given a homomorphism $f: E \to Y$

in $\mathcal{T}op_{h}$. We regard it as a $\mathcal{C}\times\mathcal{L}_1$-diagram and lift it to $\mathcal{T}op$ such that D lifts the $\mathcal{C}\times 0$-part E. Let f_A denote the lift of the morphism $E_0A\to Y$. Then we have maps

$$F_{A,B}\colon \mathcal{C}(A,B)\times I\times D_0A\to Y$$

such that

$$F_{A,B}(g,t;x)=\begin{cases} f_A(x) & g=\mathrm{id}\\ f_B(D(g;x)) & t=0\\ f_A(x) & t=1.\end{cases}$$

Define $h\colon X\to Y$ to be the homotopy class of the map given on D_0B by f_B and on $\mathcal{C}(A,B)\times I\times D_0B$ by $F_{A,B}$.

The construction for weak limits is dual.

12. A Remark on Generalizations

It should be possible to carry out our constructions in any category with a strong notion of homotopy, i.e. in a category containing suitable non-trivial cosimplicial objects substituting cubes or simplices, and having limits and colimits. The category $\mathcal{C}at$ of small categories has these properties. With the correct definition of homotopy one should be able to interpret Boardman's category of finite spectra (see [2] or [16]) as homotopy limit of the diagram

$$\dots\xrightarrow{S}\mathcal{F}\xrightarrow{S}\mathcal{F}\xrightarrow{S}\mathcal{F}\xrightarrow{S}\dots$$

in $\mathcal{C}at$, where $\mathcal{F}$ is the category of finite CW-complexes and S the suspension functor.

Appendix

The following result closes the gap we left in the proof of (3.6).

Proposition. *Let $f\colon X\to Y$ be a filtration preserving map of filtered spaces such that*

(a) *$X\times Z=\mathrm{colim}(X_n\times Z)$ for any space Z.*

(b) *Y_n is obtained from Y_{n-1} by attaching X_n relative to a subspace DX_n such that (X_n,DX_n) is a* NDR *and the induced map $X_n\to Y_n$ is f. Let Z be an arbitrary space and $Y\otimes Z=X\times Z/\sim$ with $(x,z)\sim(x',z)$ if $f(x)=f(x')$. Then the identity $Y\otimes Z\to Y\times Z$ is a homotopy equivalence.*

The proof relies on the following more or less well-known results (see [8; p. 60ff.]).

Lemma 1. *Let (X,A) be a* NDR, *$f\colon A\to Y$ a map and Z an arbitrary space. Then the identity function $(X\times Z)\cup_{A\times Z}(Y\times Z)\to(X\cup_A Y)\times Z$ is a homotopy equivalence.*

Lemma 2. *Let X be a filtered space such that each pair (X_n, X_{n-1}) is a* NDR. *Then the identity* $\operatorname{colim}(X_n \times Z) \to (\operatorname{colim} X_n) \times Z$ *is a homotopy equivalence.*

Proof 1. Let M and M' be the double mapping cylinders of $X \supset A \to Y$ and $X \times Z \supset A \times Z \to Y \times Z$. Since (X, A) and $(X \times Z, A \times Z)$ are NDRs, the natural projections $M \to (X \cup_A Y)$ and $M' \to (X \times Z) \cup_{A \times Z} (Y \times Z)$ are homotopy equivalences. One can show as in [13; Hilfssatz 18] that the identity function $M' \to M \times Z$ is a homotopy equivalence, which implies Lemma 1.

Proof 2. Let T and T' be the telescopes of the X_n and the $X_n \times Z$ (see § 1). Then T and T' are the mapping tori of the diagrams

$$\coprod_{n \geqq 0} X_n \underset{\mathrm{id}}{\overset{f}{\rightrightarrows}} \coprod_{n \geqq 0} X_n, \qquad \coprod_{n \geqq 0} (X_n \times Z) \underset{\mathrm{id}}{\overset{g}{\rightrightarrows}} \coprod_{n \geqq 0} (X_n \times Z)$$

where $f = \coprod f_n$ and $g = \coprod (f_n \times \mathrm{id})$ with $f_n\colon X_n \subset X_{n+1}$. Since the pairs (X_n, X_{n-1}) and $(X_n \times Z,\ X_{n-1} \times Z)$ are NDRs, the natural projections $T \to \operatorname{colim} X_n$ and $T' \to \operatorname{colim}(X_n \times Z)$ are homotopy equivalences. Hence Lemma 2 follows from

Lemma 3. *Let $T(f, g)$ denote the mapping torus of*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

and let Z be any space. Then the identity map

$$T(f \times \mathrm{id}_Z, g \times \mathrm{id}_Z) \to T(f, g) \times Z$$

is a homotopy equivalence.

Proof. Let $h\colon X \times \partial I \to Y$ be given by $h|X \times 0 = f$ and $h|X \times 1 = g$. Then $T(f, g) = X \times I \cup_{X \times \partial I} Y$, and the result follows from Lemma 1.

Proof of the Proposition. Let $p\colon X \times Z \to Y \otimes Z$ be the identification map and let Q_n be the image of $X_n \times Z$. Then the Q_n filter $Y \otimes Z$ and Q_n is obtained from Q_{n-1} by attaching $X_n \times Z$ relative to $DX_n \times Z$. Note that $Q_0 = Y_0 \times Z$ because $X_0 = Y_0$. Inductively assume that the identity $Q_{n-1} \to Y_{n-1} \times Z$ is a homotopy equivalence. By [3; appendix (4.6)] and by Lemma 1 the identity function

$$Q_n = Q_{n-1} \cup_{DX_n \times Z} (X_n \times Z) \to (Y_{n-1} \times Z) \cup_{DX_n \times Z} (X_n \times Z)$$
$$\to (Y_{n-1} \cup_{DX_n} X_n) \times Z = Y_n \times Z$$

is a homotopy equivalence. Since each pair (Q_n, Q_{n-1}) and (Y_n, Y_{n-1}) is a NDR the identity function

$$Y \otimes Z = \operatorname{colim} Q_n \to \operatorname{colim}(Y_n \times Z) \to (\operatorname{colim} Y_n) \times Z = Y \times Z$$

is a homotopy equivalence by [3; appendix (4.4)] and by Lemma 2.

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