Convenient Categories of Topological Spaces for Homotopy Theory

By

RAINER M. VOGT

For many questions in homotopy theory, the category $\mathscr{T}$ of topological spaces is not a very good one to work in. For example, if $q\colon X \to Y$ is an identification map then $1 \times q\colon Z \times X \to Z \times Y$ need not be one. Or take the free topological monoid over a space, then one only knows that its multiplication is continuous on compact subsets. So many attempts have been made to find a suitable category, closely related to the category $\mathscr{T}$, in which a variety of constructions can be made without further assumptions on the spaces involved. In recent years, the following three categories have enjoyed increasing popularity:

1) The category $\mathscr{W}$ of spaces having the homotopy type of a CW-complex [1]. It allows a semi-efficient theory of homotopy type.

2) The category $\mathscr{C}\mathscr{G}$ of compactly generated Hausdorff spaces [3]. A space X is in $\mathscr{C}\mathscr{G}$ if it is Hausdorff and $A \subset X$ is closed provided its intersection with each compact subset of X is closed.

3) The category $\mathscr{Q}\mathscr{T}$ of quasi-topological spaces and quasi-continuous maps [2]. A quasi-topological space is a set X together with a collection of sets $Q(C, X)$ of functions $C \to X$, one for each compact Hausdorff space C, such that

(a) the constant functions $C \to X$ are in $Q(C, X)$;
(b) if $f\colon C \to C'$ is a continuous map and $r \in Q(C', X)$, then $r \circ f \in Q(C, X)$;
(c) if $f\colon C \to C'$ is a continuous surjection, then $r \in Q(C', X)$ iff $r \circ f \in Q(C, X)$;
(d) if C is the disjoint union of C_1 and C_2, then $r \in Q(C, X)$ iff

$$r \mid C_i \in Q(C_i, X),\ i = 1, 2.$$

A function $f\colon X \to Y$ is called quasi-continuous if $r \in Q(C, X)$ implies that

$$f \circ r \in Q(C, Y).$$

Both categories $\mathscr{C}\mathscr{G}$ and $\mathscr{Q}\mathscr{T}$ are suited for the study of H-spaces, classifying spaces, infinite symmetric products etc. Unfortunately both have some disadvantages: Many topologists dislike working with things that are not topological spaces. This may be the reason why the category $\mathscr{C}\mathscr{G}$ is more popular than $\mathscr{Q}\mathscr{T}$. But $\mathscr{C}\mathscr{G}$ has the disadvantage that its colimits are not what they are supposed to be. More precisely, the forgetful functor $\mathscr{C}\mathscr{G} \to \mathscr{S}ets$ does not preserve colimits. For example, a quotient space of a space in $\mathscr{C}\mathscr{G}$ need not be in $\mathscr{C}\mathscr{G}$.

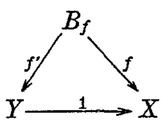
The aim of this paper is to construct full subcategories of $\mathscr{T}$ which enjoy all the nice properties of $\mathscr{C}\mathscr{G}$ but do not have this disadvantage. Among our examples, we have a category which contains $\mathscr{C}\mathscr{G}$ and is closely related to $\mathscr{Q}\mathscr{T}$. In fact, it is isomorphic to the image of the functor $\mathscr{T} \to \mathscr{Q}\mathscr{T}$ which maps each topological space to its associated quasi-topological space.

Some of the results of this paper have been announced in [4]. I want to thank Professor A. Kock, Professor D. Puppe, and Dr. Th. Bröcker for helpful comments.

1. The Construction. Let $\mathscr{S}$ be a non-empty full subcategory of $\mathscr{T}$. For any topological space X, let $\mathscr{S}/X$ be the category whose objects are all maps $f\colon B_f \to X$ in $\mathscr{T}$, where $B_f \in \operatorname{ob} \mathscr{S}$, and whose morphisms from f to g are all maps $h\colon B_f \to B_g$ in $\mathscr{S}$ such that $f = g \circ h$. The spaces B_f, $f \in \operatorname{ob} \mathscr{S}/X$, and the maps $h\colon B_f \to B_g$ form a (may be big) diagram $D(X)$ in $\mathscr{T}$. Define $k(X) = \varinjlim D(X)$.

Lemma 1.1. *For any $X \in \operatorname{ob} \mathscr{T}$, there is a canonical choice of $k(X)$ such that X and $k(X)$ have the same underlying sets.*

Proof. Let Y be the topological space given by $|Y| = |X|$, where $|Z|$ denotes the underlying set of the space Z, and $U \subset Y$ open iff $f^{-1}(U)$ is open for all $f \in \operatorname{ob} \mathscr{S}/X$. Then the identity function $1\colon Y \to X$ is continuous, and each $f \in \operatorname{ob} \mathscr{S}/X$ factors as



in $\mathscr{T}$. Given maps $h_f\colon B_f \to Z$, one for each vertex B_f of $D(X)$, such that $h_g \circ u = h_f$ for any morphism $u\colon B_f \to B_g$ of $D(X)$, then there exists a unique map $h\colon Y \to Z$ such that $h \circ f' = h_f$. The map h is defined as follows: For each $y \in Y$, there exists a B_f and an $x \in B_f$ such that $f'(x) = y$. Put $h(y) = h_f(x)$. Note that this definition is forced upon us. Suppose there exists a $z \in B_g$, some B_g, such that $g'(z) = y$. Then we can find a B_r and morphisms $u\colon B_r \to B_f$ and $v\colon B_r \to B_g$ in $D(X)$ such that $u(B_r) = x$ and $v(B_r) = z$. Hence

$$h_f(x) = h_f \circ u(B_r) = h_r(B_r) = h_g \circ v(B_r) = h_g(z),$$

so that h is well-defined. To show the continuity of h, let $U \subset Z$ be open. Then

$$f'^{-1}(h^{-1}(U)) = h_f^{-1}(U)$$

is open for all f. Therefore $h^{-1}(U)$ is open in Y. The space Y is the canonical choice for $k(X)$.

Proposition 1.2. (a) *The identity function $k(X) \to X$ is continuous.*

(b) *$k(X)$ has the finest topology such that any map from $B \in \operatorname{ob} \mathscr{S}$ to X factors through the identity function $k(X) \to X$.*

(c) *If $B \in \operatorname{ob} \mathscr{S}$, then there exists a one-one correspondence between maps $B \to X$ and $B \to k(X)$.*

(d) *$k(B) = B$ for $B \in \operatorname{ob} \mathscr{S}$.*

(e) *$k(k(X)) = k(X)$ for all X in $\mathscr{T}$.*

(f) *If the composites* $h \circ f\colon B \to k(X) \to Y$ *are continuous for all maps* $f\colon B \to k(X)$ *with* $B \in \operatorname{ob} \mathscr{S}$, *then* h *is continuous.*

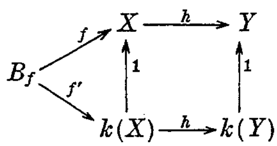
(g) *If the standard simplexes are in* $\mathscr{S}$, *then the identity function* $k(X) \to X$ *induces isomorphisms of singular homology and cohomology groups.*

(h) *If the standard spheres* Σ^n *and the cylinders* $\Sigma^n \times I$, $n = 0, 1, 2, \ldots$ *are in* $\mathscr{S}$, *then the identity function* $k(X) \to X$ *induces isomorphisms of homotopy groups.*

Proof. (a) and (b) follow from the canonical choice of $k(X)$. Property (c) is a consequence of (b). If $B \in \operatorname{ob} \mathscr{S}$, then it is a terminal object of $D(B)$, which implies (d). Property (e) follows from the definition of $k(X)$, and (f) from the definition of a colimit. The properties (g) and (h) are immediate consequences of (c).

Lemma 1.3. *For any map* $h\colon X \to Y$ *in* $\mathscr{T}$, *the function* $k(h) =: h\colon k(X) \to k(Y)$ *is continuous.*

Proof. In the following commutative diagram



the composite $h \circ f$ is continuous. Hence, by (c), the composite $h \circ f'$ and therefore, by (f), the function $k(h) = h$ are continuous.

Let $\mathscr{K}$ be the full subcategory of $\mathscr{T}$ consisting of all objects $k(X)$, $X \in \operatorname{ob} \mathscr{T}$. Then k is a functor from $\mathscr{T}$ to $\mathscr{K}$. In abuse of notation, we often consider k as a functor from $\mathscr{T}$ to $\mathscr{T}$ by composing it with the inclusion $\mathscr{K} \subset \mathscr{T}$.

Corollary 1.4. *The inclusion functor* $i\colon \mathscr{K} \to \mathscr{T}$ *is left adjoint to the functor* $k\colon \mathscr{T} \to \mathscr{K}$. *In fact, we have an equality*

$$\mathscr{K}(X, k(Y)) = \mathscr{T}(i(X), Y)$$

$X \in \operatorname{ob} \mathscr{K}$, $Y \in \operatorname{ob} \mathscr{T}$. (Here we consider the maps as functions on their underlying sets.)

Proof. Apply 1.2 (a) and (e).

Proposition 1.5. *Given full subcategories* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *of* $\mathscr{T}$ *which give rise to functors* $k_i\colon \mathscr{T} \to \mathscr{K}_i$, $i = 1, 2$.

(a) *If* $\mathscr{S}_1 \subset \mathscr{S}_2$, *then* $\mathscr{K}_1 \subset \mathscr{K}_2$.

(b) *If* $\mathscr{S}_1 \subset \mathscr{S}_2 \subset \mathscr{K}_1$, *then* $\mathscr{K}_1 = \mathscr{K}_2$ *and* $k_1 = k_2$.

Proof. (a) Let $X \in \operatorname{ob} \mathscr{K}_1$, and let $U \subset X$ be a subset such that $f^{-1}(U)$ is open for all maps $f\colon B \to X$ with $B \in \operatorname{ob} \mathscr{S}_2$. Then this holds in particular if $B \in \operatorname{ob} \mathscr{S}_1$. Hence U is open in X and therefore $X \in \operatorname{ob} \mathscr{K}_2$.

(b) Let $X \in \operatorname{ob} \mathscr{T}$. Then $k_i(X)$ has the finest topology such that $f\colon B \to X$ factors through $k_i(X)$ if $B \in \operatorname{ob} \mathscr{S}_i$, $i = 1, 2$. Hence the topology of $k_1(X)$ is finer than the one of $k_2(X)$. On the other hand, let $f\colon B \to X$ be a map and $B \in \operatorname{ob} \mathscr{S}_2$. Then, by 1.2 (e), the function $f\colon B \to k_1(X)$ is continuous. Hence the topology of $k_2(X)$ is finer than the one of $k_1(X)$.

Remark 1.6. The construction of the functor k from $\mathscr{S}$ is known to category theorists as the Kan extension of the inclusion functor $\mathscr{S} \subset \mathscr{T}$.

Remark 1.7. Some topologists may prefer to consider the following category $\mathscr{K}'$: Its objects are the topological spaces, and its morphisms from X to Y are all functions $h: X \to Y$ such that the composites $h \circ f: B \to X \to Y$ are continuous for all $f \in \mathscr{S}/X$. It follows easily from Proposition 1.2 that $\mathscr{K}$ and $\mathscr{K}'$ are equivalent categories. We prefer to stick to the version $\mathscr{K}$.

2. Properties of $\mathscr{K}$. Limits and colimits.

Theorem 2.1. *Let D be any diagram in $\mathscr{K}$ (it may be big).*
(a) *If $\varinjlim D$ exists in $\mathscr{T}$, then it exists in $\mathscr{K}$.*
(b) *If $\varprojlim D$ exists in $\mathscr{T}$, then it exists in $\mathscr{K}$.*
(c) *The functor $k: \mathscr{T} \to \mathscr{K}$ preserves limits and the functor $i: \mathscr{K} \subset \mathscr{T}$ colimits.*
(d) *The forgetful functor $\mathscr{K} \to \mathscr{S}ets$ preserves limits and colimits.*
In particular, $\mathscr{K}$ is complete and cocomplete.

Proof. Statement (c) holds because i is left adjoint to k, and (d) is an immediate consequence of (c). Let $C = \varinjlim D$ and $L = \varprojlim D$, both in $\mathscr{T}$. Then $k(L) = \varprojlim k(D)$ by (c). But $k(D) = D$ by 1.2 (e). Let $\{i_B: B \to C,\ B \in \operatorname{ob} D\}$ be the collection of universal maps. Since $k(B) = B$, the function $i_B: B \to k(C)$ is continuous. Hence $1: C \to k(C)$ is continuous. On the other hand, $1: k(C) \to C$ is continuous, whence $k(C) = C$. So $C \in \operatorname{ob} \mathscr{K}$.

Corollary 2.2. *A quotient space of a space in $\mathscr{K}$ is in $\mathscr{K}$.*

Proof. A quotient space is a colimit.

Subspaces. One cannot expect that any subspace of a space in $\mathscr{K}$ is again in $\mathscr{K}$. In fact, counter examples can be found [3; 2.3].

Let $X \in \operatorname{ob} \mathscr{T}$. We denote the space given by a subset A of X with the relative topology by A_r, and define $A_k = k(A_r)$. A function $Z \to A_r$, $Z \in \operatorname{ob} \mathscr{T}$, is continuous iff its composite with the inclusion $A_r \subset X$ is continuous. The space A_k has the same property for spaces in $\mathscr{K}$.

Proposition 2.3. *Let $X \in \operatorname{ob} \mathscr{K}$ and $A \subset X$. A function $f: Z \to A_k$, where $Z \in \operatorname{ob} \mathscr{K}$, is continuous iff the composite*

$$g: Z \xrightarrow{f} A_k \subset X$$

is continuous.

Proof. Suppose g is continuous. We have to show that the composites

$$f \circ r: B \to Z \to A_k$$

are continuous for all maps r with $B \in \operatorname{ob} \mathscr{S}$. Since the composite $Z \to A_k \to A_r$ is continuous, the maps $f \circ r$ are continuous by 1.2 (b).

We next show that under certain conditions on $\mathscr{S}$ and A the topologies of A_r and A_k coincide.

Axiom 1. *If A is a closed subset of an object in $\mathcal{S}$, then A_r is in $\mathcal{K}$.*

Axiom 1*. *If A is an open subset of an object in $\mathcal{S}$, then A_r is in $\mathcal{K}$.*

Proposition 2.4. *If $\mathcal{S}$ satisfies Axiom 1 [Axiom 1*] and A is a closed [open] subset of a space in $\mathcal{K}$, then $A_r = A_k$.*

Proof. Let A be a closed subset of a space X in $\mathcal{K}$. For any map $f\colon B_f \to X$, let $A_f = f^{-1}(A)$. Substituting the vertices B_f and the morphisms $h\colon B_f \to B_g$ in $D(X)$ by A_f and $h|A_f$, we obtain a diagram D, which by assumption lies in $\mathcal{K}$. Let U be a subset of A_r such that $(f|A_f)^{-1}U$ is closed for all maps $(f|A_f)$. Then $f^{-1}(U)$ is closed in B_f, hence U closed in X and therefore in A. Using the same arguments as in the proof of Lemma 1.1, one sees that A_r is the colimit of D. Hence $A_r = A_k$.

The second part of the proposition follows similarly.

3. Products and Function Spaces. Throughout the sections 3 and 4 we require that $\mathcal{S}$ satisfies the following axiom.

Axiom 2. (a) *The cartesian product of two spaces in $\mathcal{S}$ is again in $\mathcal{S}$.*
(b) *If $X \in \operatorname{ob} \mathcal{S}$ and $Y \in \operatorname{ob} \mathcal{T}$, then the evaluation map*

$$e_{X,Y}\colon \mathcal{T}_t(X, Y) \times X \to Y$$

is continuous. Here $\mathcal{T}_t(X, Y)$ is $\mathcal{T}(X, Y)$ with the compact-open topology and $e_{X,Y}$ is defined by $e_{X,Y}(f, x) = f(x)$.

To avoid confusion, we denote the cartesian product of two spaces X and Y in $\mathcal{K}$ by $X \times Y$ and their category theoretical product in $\mathcal{K}$ by $X \otimes Y$.

It is well-known that the evaluation map has the following universal property: Given a map $f\colon X \times Y \to Z$, there exists a unique map $\hat{f}\colon X \to \mathcal{T}_t(Y, Z)$, called the adjoint of f, such that

(3.1)
$$\begin{array}{ccc} \mathcal{T}_t(Y,Z) \times Y & \xrightarrow{e_{Y,Z}} & Z \\ & {\scriptstyle \hat{f}\times 1}\nwarrow \quad \nearrow{\scriptstyle f} & \\ & X \times Y & \end{array}$$

commutes. This holds even if $e_{Y,X}$ is not continuous. Necessarily, $\hat{f}(x)(y) = f(x, y)$, which implies that $\hat{f}$ is unique even as a function between the underlying sets.

If $e_{Y,Z}$ is continuous, diagram (3.1) induces a function

$$l\colon \mathcal{T}_t(X, \mathcal{T}_t(Y, Z)) \to \mathcal{T}_t(X \times Y, Z)\,.$$

Proposition 3.2. *If X and Y are in $\mathcal{S}$, then l is a natural homeomorphism.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{T}_t(X, \mathcal{T}_t(Y,Z)) \times X \times Y & \xrightarrow{e_1 \times 1} & \mathcal{T}_t(Y,Z) \times Y \\ {\scriptstyle l\times 1\times 1}\downarrow & \nearrow{\scriptstyle \hat{e}_3 \times 1} & \downarrow{\scriptstyle e_2} \\ \mathcal{T}_t(X \times Y, Z) \times X \times Y & \xrightarrow{e_3} & Z \end{array}$$

with $e_1 = e_{X,\mathcal{T}_t(Y,Z)}$, $e_2 = e_{Y,Z}$, $e_3 = e_{X\times Y,Z}$. Since l makes the square commute, it is continuous by the universal property of e_3. The lower triangle commutes by definition of $\hat{e}_3$. Hence $\hat{e}_3 \circ (l \times 1) = e_1$ because of the universal property of e_2. By the universal property of e_1, there exists a unique map

$$h: \mathscr{T}_t(X \times Y, Z) \to \mathscr{T}_t(X, \mathscr{T}_t(Y, Z))$$

such that $e_1 \circ (h \times 1) = \hat{e}_3$. Now

$$e_3 \circ ((l \circ h) \times 1 \times 1) = e_2 \circ (e_1 \times 1) \circ (h \times 1 \times 1) = e_2 \circ (\hat{e}_3 \times 1) = e_3\,,$$
$$e_1 \circ ((h \circ l) \times 1) = \hat{e}_3 \circ (l \times 1) = e_1\,.$$

Hence $l \circ h = 1$ and $h \circ l = 1$ by the universal properties of e_3 and e_1.

Corollary 3.3. (a) *If $X \in \operatorname{ob} \mathscr{S}$, then the functor $- \times X: \mathscr{T} \to \mathscr{T}$ preserves colimits.*
(b) *If $X \in \operatorname{ob} \mathscr{S}$ and $Y \in \operatorname{ob} \mathscr{K}$, then $X \times Y = X \otimes Y$.*

Proof. (a) holds since $\mathscr{T}_t(X, -): \mathscr{T} \to \mathscr{T}$ is a right adjoint of $- \times X$. By definition, $X \otimes Y = \varinjlim D(X \times Y)$. Since $X \times D(Y)$ is a cofinal subdiagram of $D(X \times Y)$, part (a) implies

$$X \otimes Y = \varinjlim (X \times D(Y)) = X \times \varinjlim D(Y) = X \times Y\,.$$

We next want to show a version of Proposition 3.2 for the case that X and Y are in $\mathscr{K}$. Since $\mathscr{T}_t(X, Y)$ need not be in $\mathscr{K}$ even if X and Y are, we define

$$\mathscr{K}_t(X, Y) = k(\mathscr{T}_t(X, Y))\,.$$

This definition makes sense for arbitrary topological spaces. If we know

(3.4) Given a map $f: X \otimes Y \to Z$, where X and Y are in $\mathscr{K}$, then the adjoint $\hat{f}$, defined as in (3.1), is a continuous map from X to $\mathscr{K}_t(Y, Z)$.

and

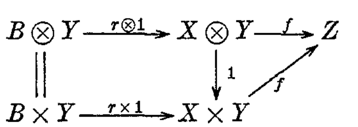
(3.5) The evaluation maps $e_{Y,Z}$ of (3.1) are continuous as maps from $\mathscr{K}_t(Y, Z) \otimes Y$ to Z, provided that Y is in $\mathscr{K}$.

then we can obtain the following result in the same manner as Proposition 3.2.

Theorem 3.6. *Let X and Y be spaces in $\mathscr{K}$. Then the correspondence $\hat{f} \to f$ is a natural homeomorphism*

$$\mathscr{K}_t(X, \mathscr{K}_t(Y, Z)) \cong \mathscr{K}_t(X \otimes Y, Z)\,.$$

Proof of (3.4). Let $B \in \operatorname{ob} \mathscr{S}$ and $r: B \to X$ be a map. The commutativity of



shows that $f \circ (r \times 1)$ is continuous and hence has an adjoint. Since each $x \in X$ is in the image of some r, there is a factorization

$$\begin{array}{ccc} B & \xrightarrow{\widehat{f\circ(r\times 1)}} & \mathscr{T}_t(Y,Z) \\ \downarrow r & & \uparrow 1 \\ X & \xrightarrow{\hat{f}} & \mathscr{K}_t(Y,Z) \end{array}$$

The continuity of $\hat{f}$ follows now from 1.2.

Proof of (3.5). Let $B \in \mathrm{ob}\, \mathscr{S}$ and $r = (r_1, r_2)\colon B \to \mathscr{K}_t(Y, Z) \otimes Y$ be a map. The statement follows from the commutativity of

$$\begin{array}{ccccccc} B & \xrightarrow{\quad\quad r \quad\quad} & & \mathscr{K}_j(Y,Z)\otimes Y & & \xrightarrow{\quad e_{Y,Z} \quad} & Z \\ \downarrow \text{diagonal} & & \nearrow 1\otimes r_2 & & & & \uparrow e_{B,Z} \\ B\otimes B & \xrightarrow{r_1\otimes 1} & \mathscr{K}_t(Y,Z)\otimes B & \xrightarrow{r_2^*\otimes 1} & \mathscr{K}_t(B,Z)\otimes B & \xrightarrow{1} & \mathscr{T}_t(B,Z)\times B \end{array}$$

Theorem 3.6 has a number of interesting consequences.

Theorem 3.7. *Let X be a space in $\mathscr{K}$.*

(a) *The functor $\mathscr{K}_t(X, -)\colon \mathscr{K} \to \mathscr{K}$ preserves limits. In particular*

$$\mathscr{K}_t(X, Y \otimes Z) \cong \mathscr{K}_t(X, Y) \otimes \mathscr{K}_t(X, Z)$$

for Y and Z in $\mathscr{K}$.

(b) *The functor $-\otimes X\colon \mathscr{K} \to \mathscr{K}$ preserves colimits.*

(c) *The functor $\mathscr{K}_t(-, X)\colon \mathscr{K} \to \mathscr{K}$ transfers colimits to limits.*

Proof. $\mathscr{K}_t(X, -)$ is a right adjoint of $-\otimes X$, which implies (a) and (b). To prove (c), we have to show that $\mathscr{K}_t(-, X)$ as a functor from the dual category $\mathscr{K}^{op}$ of $\mathscr{K}$ to $\mathscr{K}$ preserves limits. You can also consider $\mathscr{K}_t(-, X)$ as a functor from $\mathscr{K}$ to $\mathscr{K}^{op}$. Now

$$\mathscr{K}(Y, \mathscr{K}_t(Z, X)) \cong \mathscr{K}(Y \otimes Z, X) \cong \mathscr{K}(Z, \mathscr{K}_t(Y, X)) = \mathscr{K}^{op}(\mathscr{K}_t(Y, X), Z).$$

Hence $\mathscr{K}_t(-, X)\colon \mathscr{K}^{op} \to \mathscr{K}$ has a left adjoint.

Corollary 3.8. *Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be identification maps between spaces in $\mathscr{K}$. Then $f \otimes g\colon X \otimes Y \to X' \otimes Y'$ is an identification map.*

Proof. Since $f \otimes g = (f \otimes 1) \circ (1 \otimes g)$ and since composites of identification maps are identification maps, it suffices to prove the result for $g = 1_Y$. But X' is a colimit, which is preserved by $-\otimes Y$.

A similar result can be shown for inclusions.

Definition. Let X and Y be spaces in $\mathscr{K}$. A map $f\colon X \to Y$ is called an *inclusion in $\mathscr{K}$* if a function $h\colon Z \to X$ with $Z \in \mathrm{ob}\, \mathscr{K}$ is continuous whenever $f \circ h$ is.

Using just the definition we can show

Proposition 3.9. *If $f\colon X \to X'$ and $g\colon Y \to Y'$ are inclusions in $\mathscr{K}$, then so is*

$$f \otimes g\colon X \otimes Y \to X' \otimes Y'.$$

Another consequence of Theorem 3.6 is

Theorem 3.10. *If X and Y are spaces in $\mathscr{K}$, then the composition of maps induces a continuous map*

$$c\colon \mathscr{K}_t(Y,Z)\otimes\mathscr{K}_t(X,Y)\to\mathscr{K}_t(X,Z).$$

Proof. The map c is the adjoint of the composite

$$e_{Y,Z}\circ(1\otimes e_{X,Y})\colon \mathscr{K}_t(Y,Z)\otimes\mathscr{K}_t(X,Y)\otimes X\to\mathscr{K}_t(Y,Z)\otimes Y\to Z.$$

It is well-known that the function $l\colon \mathscr{T}_t(X,\mathscr{T}_t(Y,Z))\to\mathscr{T}_t(X\times Y,Z)$ of (3.1) is defined and is a bijection if Y is locally compact.

Definition. A space Y is called *locally compact*, if each neighbourhood of any point $y\in Y$ contains a compact (not necessarily Hausdorff) neighbourhood of y.

Proposition 3.11. *Let X and Y be spaces in $\mathscr{K}$ and Y locally compact. Then*

$$X\otimes Y=X\times Y.$$

Proof. By definition, $X=\varinjlim D(X)$ because X is in $\mathscr{K}$. Since l is a bijection, the functor $-\times Y\colon\mathscr{T}\to\mathscr{T}$ preserves colimits. Since the colimits in $\mathscr{T}$ and in $\mathscr{K}$ coincide we obtain from 3.3

$$X\otimes Y=\varinjlim(D(X)\otimes Y)=\varinjlim(D(X)\times Y)=(\varinjlim D(X))\times Y=X\times Y.$$

4. The Based Category. In this section we sketch that the category $\mathscr{K}_*$ of based spaces in $\mathscr{K}$ enjoys the same nice properties as $\mathscr{K}$. Since $\mathscr{K}_*$ can be considered as the category $\mathscr{K}$ under a one-point space P, the following result follows from formal arguments.

Proposition 4.1. *The category $\mathscr{K}_*$ is complete and cocomplete.*

This result can also be obtained in the manner of 2.1 by deriving $\mathscr{K}_*$ from the category $\mathscr{T}_*$ of based topological spaces. The colimits of $\mathscr{K}_*$ are the same as the ones of $\mathscr{T}_*$. The limits of $\mathscr{K}_*$ are the ones of $\mathscr{K}$ but with a distinguished base point. More precisely, the forgetful functor $\mathscr{K}_*\to\mathscr{K}$ preserves limits.

One of the advantages of $\mathscr{K}_*$ over $\mathscr{T}_*$ is that it has a wellbehaved smash product functor. Let $(X_\alpha,\alpha\in\mathsf{A})$ be any set of spaces in $\mathscr{K}_*$. Let $W_{\alpha\in\mathsf{A}}X_\alpha$ be the subset of those points of the product $\prod_{\alpha\in\mathsf{A}}X_\alpha$ in $\mathscr{K}_*$ which have at least one coordinate at the base point.

Definition. The *smash product* $\bigwedge_{\alpha\in\mathsf{A}}X_\alpha$ is the quotient $\left(\prod_{\alpha\in\mathsf{A}}X_\alpha\right)\Big/\left(W_{\alpha\in\mathsf{A}}X_\alpha\right)$.

Proposition 4.2. *Let Γ be the disjoint union of the sets A and B. Then there is a natural homeomorphism*

$$\Big(\bigwedge_{\alpha\in\mathsf{A}}X_\alpha\Big)\wedge\Big(\bigwedge_{\beta\in\mathsf{B}}X_\beta\Big)\cong\bigwedge_{\gamma\in\Gamma}X_\gamma.$$

Proof. In the following diagram, let s, r, p, q be the obvious identification maps and h the bijection making the diagram commute.

$$\begin{array}{ccccc}
\prod_{\gamma\in\Gamma} X_\gamma & \xrightarrow{\cong} \left(\prod_{\alpha\in A} X_\alpha\right) \otimes \left(\prod_{\beta\in B} X_\beta\right) \xrightarrow{p\otimes q} & \left(\bigwedge_{\alpha\in A} X_\alpha\right) \otimes \left(\bigwedge_{\beta\in B} X_\beta\right) \\
\downarrow s & & \downarrow r \\
\bigwedge_{\gamma\in\Gamma} X_\gamma & \xrightarrow{\qquad\qquad h \qquad\qquad} & \left(\bigwedge_{\alpha\in A} X_\alpha\right) \wedge \left(\bigwedge_{\beta\in B} X_\beta\right)
\end{array}$$

Since both s and $r \circ (p \otimes q)$ are identifications, the function h is a homeomorphism.

Corollary 4.3. *The functor* $- \wedge -: \mathscr{K}_* \times \mathscr{K}_* \to \mathscr{K}_*$ *is associative.*

We next want to prove an exponential law for the smash product. We consider $\mathscr{K}_*(X, Y)$ as a subset of $\mathscr{K}_t(X, Y)$ forgetting the base points, and we define

$$\mathscr{K}_{*t}(X, Y) = \mathscr{K}_*(X, Y)_k \subset \mathscr{K}_t(X, Y)$$

(see section 2). The base point of $\mathscr{K}_{*t}(X, Y)$ is the constant map.

Theorem 4.4. *The evaluation map induces a based natural homeomorphism*

$$\mathscr{K}_{*t}(X, \mathscr{K}_{*t}(Y, Z)) \cong \mathscr{K}_{*t}(X \wedge Y, Z).$$

Proof. Define $e'_{X,Y}: \mathscr{K}^*_t(X, Y) \wedge X \to Y$ to be the function given on representatives by the evaluation map $e_{X,Y}$. It is continuous because of the commutativity of

$$\begin{array}{ccc}
\mathscr{K}_{*t}(X, Y) \otimes X & \xrightarrow{i\otimes 1} & \mathscr{K}_t(X, Y) \otimes X \\
\downarrow p & & \downarrow e_{X,Y} \\
\mathscr{K}_{*t}(X, Y) \wedge X & \xrightarrow{e'_{X,Y}} & Y
\end{array}$$

where p is the identification map and i the inclusion.

Let $f: Z \wedge X \to Y$ be any map in $\mathscr{K}_*$ and $q: Z \otimes X \to Z \wedge X$ the identification. The composite $f \circ q$ has an adjoint $r: Z \to \mathscr{K}_t(X, Y)$, which factors as

$$\begin{array}{ccc}
Z & \xrightarrow{\quad r \quad} & \mathscr{K}_t(X, Y) \\
 & {}_{g}\searrow \quad \nearrow_{i} & \\
 & \mathscr{K}_{*t}(X, Y) &
\end{array}$$

By 2.3, the function g is continuous. Since it can be considered as a based map, we define g to be the adjoint of f in $\mathscr{K}_*$. By definition, $f = e'_{X,Y} \circ (g \wedge 1_X)$ and g is the unique map satisfying this equation.

Theorem 4.4 now follows in the same manner as 3.2.

We can again draw a number of consequences like in section 3. Let us mention just one.

Theorem 4.5. *The functor* $X \wedge -: \mathscr{K}_* \to \mathscr{K}_*$ *preserves colimits. In particular, there is a natural based homeomorphism*

$$X \wedge \left(\bigvee_{\alpha\in A} Y_\alpha\right) \cong \bigvee_{\alpha\in A} (X \wedge Y_\alpha)$$

where $\bigvee_{\alpha\in A} Y_\alpha$ *is the wedge (one-point union) of the family* $(Y_\alpha, \alpha \in A)$.

5. Examples. (i) Let $\mathscr{S}$ be the category consisting of a one-point space only. Denote the resulting category $\mathscr{K}$ by $\mathscr{DG}$. Since the functor $k\colon \mathscr{T} \to \mathscr{DG}$ maps each topological space to the discrete space on its underlying set, the category $\mathscr{DG}$ is not particularly interesting.

(ii) Let $\mathscr{S}$ be the category of all compact Hausdorff spaces. Let $\mathscr{HG}$ denote its corresponding category $\mathscr{K}$.

Theorem 5.1. (a) *$\mathscr{S}$ satisfies each of our axioms so that all of our previous results hold in $\mathscr{HG}$.*

(b) *The category $\mathscr{CG}$ of compactly generated Hausdorff spaces* [3] *is contained in $\mathscr{HG}$.*

(c) *If X is a locally compact Hausdorff space and $Y \in \operatorname{ob} \mathscr{HG}$, then $X \times Y = X \otimes Y$.*

(d) *The identity map $k(X) \to X$, $X \in \operatorname{ob} \mathscr{T}$, induces isomorphisms of homotopy and singular homology and cohomology groups.*

Proof. Let X be a Hausdorff space such that $A \subset X$ is closed iff its intersection with each compact subset of X is closed. Then X is in $\mathscr{HG}$ because the compact subsets of X together with the inclusions form a cofinal diagram in $D(X)$. This implies (b). Examples of such spaces X are the locally compact Hausdorff spaces. So (c) follows from 3.11. Statement (d) holds by 1.2. It is well-known that $\mathscr{S}$ satisfies Axiom 1 and Axiom 2. Since any open subset of a compact Hausdorff space is locally compact, it is in $\mathscr{HG}$. Hence Axiom 1* holds too.

The category $\mathscr{HG}$ is closely related to the category $\mathscr{QT}$ of quasitopological spaces [2].

Define functors

$$\mathscr{T} \xrightarrow{Q} \mathscr{QT} \xrightarrow{P} \mathscr{T}$$

as follows: $Q(X) = (|X|, \{Q(C, |X|) = \mathscr{T}(C, X)\})$, and $Q(f) = f$. The space $Z = P(Y, \{Q(C, Y)\})$ has Y as underlying set and $U \subset Z$ is open iff $r^{-1}(U)$ is open in C for all $r \in Q(C, Y)$ and all C. On morphisms, we define $P(f) = f$.

Let $\mathscr{QH}$ be the image of $\mathscr{Q}$ in $\mathscr{QT}$. Let

$$\mathscr{HG} \xrightarrow{q} \mathscr{QH} \xrightarrow{p} \mathscr{HG}$$

be the functors given by $q = Q|\mathscr{HG}$ and $p = k \circ (P|\mathscr{QH})$. One verifies easily

Proposition 5.2. *The functor $q\colon \mathscr{HG} \to \mathscr{QH}$ is an isomorphism of categories with inverse p.*

(iii) Let $\mathscr{S}$ be the category of locally compact Hausdorff spaces. It is easy to verify that $\mathscr{S}$ satisfies the axioms. We have seen that this $\mathscr{S}$ is contained in $\mathscr{HG}$. Since all compact Hausdorff spaces are in $\mathscr{S}$, the corresponding category $\mathscr{K}$ is again $\mathscr{HG}$, by 1.5 (b).

(iv) Let $\mathscr{S}$ be the category of locally compact spaces. Let $\mathscr{LG}$ denote its corresponding category $\mathscr{K}$.

Theorem 5.3. (a) *$\mathscr{S}$ satisfies all axioms so that all our results hold for $\mathscr{LG}$.*

(b) $\mathscr{DG} \subset \mathscr{CG} \subset \mathscr{HG} \subset \mathscr{LG}$.

(c) *If X is locally compact and $Y \in \mathscr{LG}$, then $X \otimes Y = X \times Y$.*

(d) *The identity map $k(X) \to X$, $X \in \operatorname{ob} \mathscr{T}$, induces isomorphisms of homotopy and singular homology and cohomology groups.*

Proof. It is well-known that Axiom 2 holds for $\mathscr{S}$, and it is easy to check that open or closed subspaces of objects in $\mathscr{S}$ are again in $\mathscr{S}$. Hence all axioms hold. The statements (b), (c), and (d) follow from 1.5 (a), 3.3 (b), and 1.2 respectively.

Remark. We do not know whether $\mathscr{H}\mathscr{G} \subset \mathscr{L}\mathscr{G}$ is a proper inclusion or not.

The general problem is to find a full subcategory $\mathscr{F}$ of $\mathscr{T}$, as big as possible, such that all our results hold for $\mathscr{F}$. The category $\mathscr{L}\mathscr{G}$ is the biggest one we found. It contains all the spaces one usually deals with in homotopy theory such as the CW-complexes.

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Anschrift des Autors:
Rainer M. Vogt
Mathematisches Institut der Universität
66 Saarbrücken 11