

ON THE UNICITY OF THE HOMOTOPY THEORY OF HIGHER CATEGORIES

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ABSTRACT. We propose four axioms that a quasicategory should satisfy to be considered a reasonable homotopy theory of (∞, n) -categories. This axiomatization requires that a homotopy theory of (∞, n) -categories, when equipped with a small amount of extra structure, satisfies a simple, yet surprising, universal property. We further prove that the space of such quasicategories is homotopy equivalent to $(\mathbf{RP}^\infty)^n$. In particular, any two such quasicategories are equivalent. This generalizes a theorem of Toën when $n = 1$, and it verifies two conjectures of Simpson. We also provide a large class of examples of models satisfying our axioms, including those of Joyal, Kan, Lurie, Rezk, and Simpson.

1. INTRODUCTION

The chaotic prehistory of higher category theory, whose origins can be traced to an assortment of disciplines, has borne a diverse subject with a range of apparatuses intended to formalize the imagined common themes in the notions of operad, monoidal category, n -category, homotopy algebra (of various flavors), higher stack, and even aspects of manifold theory. The rush to put the ideas of higher category theory on a firm foundational footing has led to a rich landscape — but also to a glut of definitions, each constructed upon different principles. Until now, there has been very little machinery available to compare the various notions of higher category that have been proposed.

The search for a useful comparison of the various notions of higher category has been an elusive and long-standing goal [28]. In their proposal for the 2004 IMA program on n -categories, Baez and May underlined the difficulties of this *Comparison Problem*:

It is not a question as to whether or not a good definition exists. Not one, but many, good definitions already do exist [. . .]. There is growing general agreement on the basic desiderata of a good definition of n -category, but there does not yet exist an axiomatization, and there are grounds for believing that only a partial axiomatization may be in the cards.

In what follows we provide just such an axiomatization for the theory of (∞, n) -categories. We propose simple but powerful axioms for a homotopy theory to be considered a reasonable *homotopy theory of (∞, n) -categories* [Df. 6.8]. We are then able to address the Comparison Problem very precisely:

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- We show that these axioms uniquely characterize the homotopy theory of (∞, n) -categories up to equivalence. [Rk. 6.9]
- We show that the topological group of self-equivalences of this homotopy theory is homotopy equivalent to the discrete group $(\mathbb{Z}/2)^n$ [Th. 8.12].
- We provide a recipe with which one may generate models of (∞, n) -categories satisfying these axioms [Th. 9.2], and we apply it to show that two of the best-known models of (∞, n) -categories satisfy our axioms [Th. 11.15 and Th. 12.6].

For the theory of (∞, n) -categories, we regard this as a complete solution to the Comparison Problem. Our result verifies two conjectures proposed by Simpson [37, Conjectures 2 and 3] and also permits us to generalize beautiful and difficult theorems of Bergner [10] and Toën [39].

The Comparison Problem when $n = 1$. The Comparison Problem for $(\infty, 1)$ -categories has already received a great deal of attention, and by now there is a solution that is both elegant and informative. The theory of $(\infty, 1)$ -categories, otherwise known as the *homotopy theory of homotopy theories*, admits a number of different descriptions. The most popular of these are the Quillen model categories

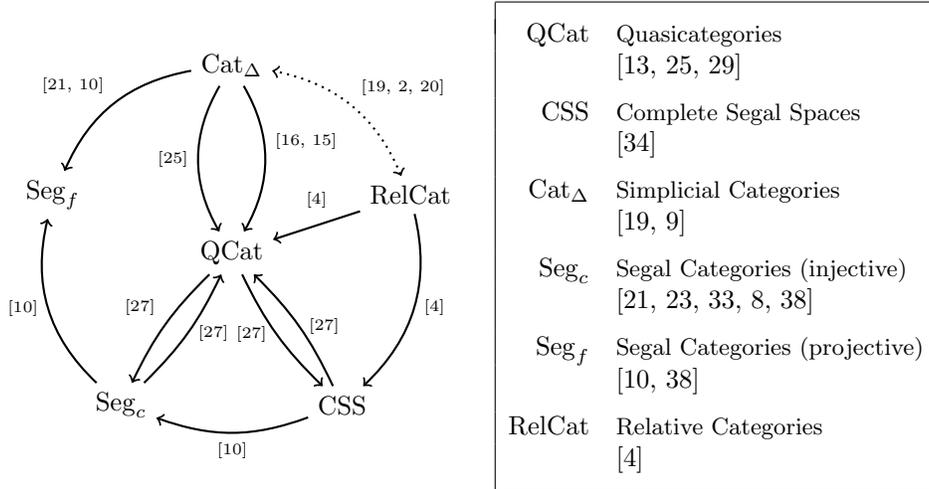


FIGURE 1. Some right Quillen equivalences among models for the homotopy theory of homotopy theories.

depicted in Fig. 1. These model categories are connected by the intricate web of Quillen equivalences depicted,¹ but this diagram fails to commute, even up to natural isomorphism; so although these model categories are all Quillen equivalent, there is an overabundance of ostensibly incompatible Quillen equivalences available.

This unpleasant state of affairs was resolved by Toën’s groundbreaking work [39], inspired by conjectures of Simpson [37]. Toën sets forth seven axioms necessary for a model category to be regarded as a *model category of $(\infty, 1)$ -categories*, and he shows that any model category satisfying these axioms is Quillen equivalent to

¹The dotted arrow connecting the simplicial categories with the relative categories is not a Quillen equivalence, but it is an equivalence of relative categories. See [19, 2, 20]

Rezk’s theory of complete Segal spaces (CSS). More importantly, Toën shows that these Quillen equivalences are *unique up to coherent homotopy* in a sense we now describe.

Any model category \mathcal{A} gives rise to a simplicial category $\mathbb{L}^H\mathcal{A}$ via the Dwyer–Kan *hammock localization* [17]. This is a functorial construction under which Quillen equivalences are carried to equivalences of simplicial categories [18]. Consequently, Fig. 1 can be regarded as a diagram of equivalences in the model category Cat_Δ . As such, its commutativity (up to coherent homotopy) is governed by the derived space of automorphisms of any one of the objects involved. Toën computes [39, Th. 6.3] that the derived automorphism space of $\mathbb{L}^H(\text{CSS})$ in Cat_Δ is equivalent to the discrete group $\mathbb{Z}/2$.

Thus Toën’s theorem immediately implies that, modulo a potential $\mathbb{Z}/2$ ambiguity, Fig. 1 commutes up to higher coherent homotopy. A more careful consideration shows that this $\mathbb{Z}/2$ -ambiguity is not a problem; Toën’s theorem also shows that the action of $\mathbb{Z}/2$ arises from the formation of the *opposite homotopy theory*, and so the two possibilities are distinguished by considering the restriction to the subcategory consisting of the terminal object and the 1-cell.

Our results. One lesson drawn from the Comparison Problem for $n = 1$ is that the essential structure of a homotopy theory is best captured by working *within* one of the six models from Fig. 1. As a matter of convenience we employ the language of *quasicategories* as our chosen model of the homotopy theory of homotopy theories. We do this primarily because Joyal and Lurie have thoroughly developed the necessary technology of limits, colimits, and presentability in this setting, and this material has appeared in print [29]. We do not use the quasicategory model in any essential way; in fact, if this technology were transported to any of the other models in Fig. 1, our results could be proven there with no more difficulty.

Within a quasicategory is a distinguished class of objects: the *0-truncated objects*; these are objects X such that for any object A the mapping space $\text{Map}(A, X)$ is homotopy equivalent to a discrete space. The 0-truncated objects in the quasicategory \mathcal{S} of spaces are precisely the spaces that are homotopy equivalent to sets. Put differently, the 0-truncated spaces are precisely the spaces whose homotopy theory is trivial. Furthermore, the homotopy theory of spaces is in a strong sense controlled by a single 0-truncated object — the one-point space. To be more precise, \mathcal{S} is freely generated under homotopy colimits by the one-point space, so that any homotopy-colimit-preserving functor $\mathcal{S} \rightarrow \mathcal{C}$ is essentially uniquely specified by its value on the one-point space. As we presently explain, the homotopy theory of (∞, n) -categories is similarly controlled by a small collection of its 0-truncated objects.

Historically, the passage from the strict theory of n -categories to a weak theory thereof is concerned with allowing what were once *equations* to be replaced with coherent *isomorphisms*. Thus the n -categories for which all k -isomorphisms are identities must by necessity have a trivial weak theory, regardless of the form that this weak theory will ultimately take. In §3 we define the *gaunt n -categories* to be precisely those n -categories in which all k -isomorphisms are identities.² We observe [Rk. 4.7] that the monoidal category of auto-equivalences of the category Gaunt_n is equivalent to the discrete category $(\mathbb{Z}/2)^n$.

²Gaunt n -categories were called *rigid* in [35], but alas this conflicts with standard terminology.

The *homotopy hypothesis* is a basic desideratum for any good theory of higher categories. It asks that the homotopy theory of n -groupoids (those n -categories in which all morphisms are equivalences) inside the homotopy theory of weak n -categories be equivalent to the homotopy theory of n -types. Or, more generally, that inside the homotopy theory of (∞, n) -categories the theory of ∞ -groupoids is equivalent to the homotopy theory of spaces. In light of this principle one may think of the gaunt n -categories as what is left of higher category theory after one strips away all traces of topology.

Any purported homotopy theory \mathcal{C} of weak n -categories must necessarily include the gaunt n -categories among the 0-truncated objects. In fact, the subcategory $\tau_{\leq 0}\mathcal{C}$ of 0-truncated objects must be equivalent to the category Gaunt_n of gaunt n -categories; however, perhaps unsurprisingly, this property alone is not enough to determine the homotopy theory of (∞, n) -categories. We are then presented with the problem of finding further properties that characterize this homotopy theory.

The existence of internal mapping objects in the quasicategory of (∞, n) -categories is an obviously desirable property. One of the fundamental results pertaining to Rezk's theory of complete Segal Θ_n -spaces is that, in this model, these internal homs exist at the model category level [35] (i.e., it is a cartesian model category). So it is reasonable to require the existence of internal homs at the quasicategory level. In fact, we will go further and demand the existence of internal homs for the quasicategory of *correspondences*.

Correspondences serve several roles both in classical category theory and in the theory of $(\infty, 1)$ -categories. Also called a *profunctor*, *distributor*, or *module*, a correspondence between categories \mathcal{A} and \mathcal{B} is defined to be a functor

$$\mathcal{A} \times \mathcal{B}^{\text{op}} \rightarrow \text{Set}.$$

The existence of internal homs for Cat readily implies the existence of internal homs for the categories of correspondences. While the above is the more common definition, a correspondence may equivalently be understood as a single functor $\mathcal{M} \rightarrow C_1$ to the 1-cell (a.k.a. free walking arrow). The categories \mathcal{A} and \mathcal{B} may then be recovered as the fibers of \mathcal{M} over the objects 0 and 1. Both this description and the existence of internal homs holds for the theory of $(\infty, 1)$ -categories [26, Th. 7.9]. Moreover there is an analogous higher theory of correspondences for gaunt n -categories, with the role of the 1-cell played by any higher cell C_k ($0 \leq k \leq n$) [§ 5].

As the 0-truncated objects in any quasicategory are closed under fiber products, any purported homotopy theory \mathcal{C} of weak n -categories must also necessarily contain this theory of higher correspondences, at least for $\tau_{\leq 0}\mathcal{C}$ the 0-truncated objects. The category Υ_n — which is defined [Df. 5.6] as the smallest subcategory containing the cells that is closed under retracts and fiber products over cells — is the minimal subcategory of gaunt n -categories for which there is a good theory of higher correspondences, and as such should have an important role in any potential weak theory of n -categories. Moreover, the building blocks of higher category theory — various kinds of cells and pasting diagrams — are themselves objects of Υ_n .

These considerations go some distance toward motivating our axiomatization, which consists of four axioms and is given in Df. 6.8. Our first axiom, the axiom of **strong generation**, requires that a homotopy theory \mathcal{C} of (∞, n) -categories be generated under canonical homotopy colimits by an embedded copy of Υ_n . This

generalizes the case in which $n = 0$, where $\mathcal{C} = \mathcal{S}$ and Υ_0 consists of the single one-point space.

Now, as Υ_n contains the cells, one may define the k -correspondences of \mathcal{C} as the overcategory $\mathcal{C}/_{C_k}$ over the k -cell. The next axiom is that **correspondences have internal homs**. As we have seen, this extends the theory of correspondences to the (∞, n) -categorical context.

These two axioms can be understood to provide \mathcal{C} with a set of *generators* and *relations* in the ∞ -categorical sense. Our third axiom, the axiom of **fundamental pushouts**, imposes further relations by demanding that a *finite* list of very simple homotopy pushouts of objects in Υ_n do not create unexpected homotopy theory. These pushouts are just enough to express the fiber products of cells $(C_i \times_{C_j} C_k)$ as iterated colimits of cells (Axiom (C.3.a-c)) and to ensure the proper notion of equivalence (Axiom (C.3.d)).

We then define, somewhat optimistically, a *homotopy theory of (∞, n) -categories* as any homotopy theory that is freely generated by Υ_n , subject only to the requirement that these two classes of relations hold. In other words the homotopy theory of (∞, n) -categories is **universal** with respect to the first three properties. Employing techniques discussed by Dugger [14] in the context of model categories, we are able to show that such a universal homotopy theory exists [Th. 7.6].

Of course a homotopy theory \mathcal{C} is unique up to a contractible choice once the generators have been identified; it turns out that such a choice amounts to an identification of its subcategory $\tau_{\leq 0}\mathcal{C}$ of 0-truncated objects with the category Gaunt_n [Cor. 8.6]. This identification is not unique, but it is unique up to the action of $(\mathbb{Z}/2)^n$ on Gaunt_n via the formation of the opposite in each degree [Lm. 4.6], whence we deduce that the space of quasicategories that satisfy our axioms is $B(\mathbb{Z}/2)^n$ [Th. 8.12]. We may informally summarize the situation by saying that our axioms uniquely characterize the homotopy theory of (∞, n) -categories, up to orientation of the k -morphisms for each $1 \leq k \leq n$.

To lend credence to our axioms, we must show that some widely accepted models of (∞, n) -categories satisfy them. In § 9 we begin this task by studying models of the homotopy theory of (∞, n) -categories that arise as localizations of the homotopy theory of simplicial presheaves on a suitable category. We present a very general recognition principle [Th. 9.2] for these sorts of models, and use it to show that the complete Segal Θ_n -spaces of [35] and the n -fold complete Segal spaces of [1] are each homotopy theories of (∞, n) -categories [Th. 11.15 and Th. 12.6]. These proofs are rather technical, but as a consequence, we find that these homotopy theories are equivalent [Cor. 12.7], reproducing a key result of Bergner and Rezk [12].

In the final section § 13, we show that our uniqueness result also guarantees that the *combinatorial model category* of (∞, n) -categories is unique up to Quillen equivalence. Using a range of results of Bergner, Lurie, and Simpson (some of which remain unpublished), we describe how many of the remaining purported homotopy theories of (∞, n) -categories may also be shown to satisfy our axioms.

The $(\infty, n + 1)$ -category of (∞, n) -categories. Since this work first appeared we have often been asked what these results tell us about the $(\infty, n + 1)$ -category of (∞, n) -categories. We would like to take the opportunity to address this point now.

As a consequence of our axiomatization it follows that the quasicategory of (∞, n) -categories is cartesian closed. This means that it admits products and that

for each object $X \in \text{Cat}_{(\infty, n)}$ the functor $X \times (-)$ admits a right adjoint, the internal hom. Suppose, for the sake of argument, that there exists a process which produces a quasicategory $\mathcal{V}\text{Cat}$ of ‘categories weakly enriched in \mathcal{V} ’ for any ‘monoidal quasicategory’ \mathcal{V} . Suppose further that this process satisfies the following requirements:

- (a) A cartesian closed quasicategory is in particular a monoidal quasicategory. Moreover for each cartesian closed quasicategory there exists a distinguished object $\underline{\mathcal{V}} \in \mathcal{V}\text{Cat}$ (\mathcal{V} is enriched in itself);
- (b) The construction of $\mathcal{V}\text{Cat}$ and, when it makes sense, $\underline{\mathcal{V}}$ are functorial in \mathcal{V} ; and
- (c) If \mathcal{V} is a homotopy theory of (∞, n) -categories (with the categorical product), then $\mathcal{V}\text{Cat}$ is a homotopy theory of $(\infty, n + 1)$ -categories.

Such a process, paired with the results of this paper, would immediately specify the $(\infty, n + 1)$ -category of (∞, n) -categories as well. It seems immensely plausible that such a process should exist, though we do not address that in this paper. At the time of this writing the authors know of no general theory of enriched higher categories which satisfies all of these requirements, though the work of Haugseng-Gepner appears promising [22]. However there are such constructions which apply to a limited class of quasicategories \mathcal{V} , such as those which arise from sufficiently nice monoidal model categories. Some of these are touched upon in the final section § 13.

Acknowledgments. We hope that the preceding paragraphs have highlighted the tremendous influence of the ideas and results of Carlos Simpson and Bertrand Toën on our work here. Without the monumental foundational work of André Joyal and Jacob Lurie on quasicategories, it would not be possible even to state our axioms, let alone prove anything about them. We thank Charles Rezk for his remarkably influential work on complete Θ_n -spaces, which lead to the discovery of the importance of the class of gaunt n -categories, and, moreover, for spotting critical mistakes in earlier versions of this paper. We would also like to thank Ricardo Andrade, Dimitri Ara, and David Carchedi for pointing out places which needed minor corrections.

2. STRICT n -CATEGORIES

Definition 2.1. A small *strict 0-category* is a set. Proceeding recursively, for any positive integer n , a small *strict n -category* is a small category enriched in small $(n - 1)$ -categories. A *functor* between strict n -categories will mean a functor in the enriched sense. We denote by Cat_n the category of small strict n -categories and functors.

For the rest of this paper we will hold the convention that, unless otherwise stated, all strict n -categories are small (see also Rk. 5.5). A set can be regarded as a 1-category with only identity morphisms, and this gives an inclusion functor $\text{Set} \hookrightarrow \text{Cat} = \text{Cat}_1$. By induction this yields an inclusion $\text{Cat}_{(n-1)} \hookrightarrow \text{Cat}_n$ for any positive integer n . Strict n -categories in the image of $\text{Cat}_0 = \text{Set}$ will be called *discrete*.

If idempotents split in a given ambient category \mathcal{C} with pullbacks, then *internal category objects* admit the following equivalent single sorted description: a category internal to \mathcal{C} consists of an object C together with a *category structure* $(*, s, t)$, consisting of self maps $s, t: C \rightarrow C$ satisfying equations

$$s^2 = ts = s \quad t^2 = st = t$$

together with an associative partial multiplication (the composition)

$$* : C \times_C^{s,t} C \rightarrow C$$

over $C \times C$ admitting units. This last means that $* \circ (s \times \text{id}) = \text{pr}_2 : C \times_C^{s,t} C \rightarrow C$ and $* \circ (\text{id} \times s) = \text{pr}_1 : C \times_C^{s,t} C \rightarrow C$. If \mathcal{C} is concrete — so that we may speak of elements of C — these equations would be written $x * f = f$ and $g * x = g$ where x is in the image of s (or equivalently in the image of t).

A strict n -category is a category that is not *internal to* but *enriched in* strict $(n-1)$ -categories. This means that a strict n -category may be viewed as an internal category which satisfies an additional *globularity condition* ensuring that the image of the source (or target) map consists of a discrete $(n-1)$ -category. Unraveling this we see that a strict n -category consists of a single set C of *cells* together with a family of category structures $(*_i, s_i, t_i)$ for $1 \leq i \leq n$. Whenever $i < j$, the maps $(*_i, s_i, t_i)$ preserve the category structure $(*_j, s_j, t_j)$; they are functors. Moreover the images of s_i and t_i consist of identity cells with respect to the j^{th} composition (this last requirement is the globularity condition).

Example 2.2. The following are some important examples of strict n -categories:

- The 0-cell C_0 is the singleton set, viewed as a strict n -category. This is also the terminal strict n -category.
- The the empty n -category \emptyset . Later it will be convenient to denote this category by $\partial C_0 = \emptyset$.
- The 1-category E is the “walking isomorphism”, that is the groupoid, equivalent to the terminal category, with two objects.
- By induction the k -cell C_k is a strict k -category (and hence a strict n -category for any $n \geq k$) defined as follows: The k -cell C_k has two objects \top and \perp . These objects only have identity endomorphisms. The remaining hom $(k-1)$ -categories are $\text{hom}_{C_k}(\perp, \top) = \emptyset$, and $\text{hom}_{C_k}(\top, \perp) = C_{k-1}$. There is a unique composition making this a strict k -category.
- The maximal sub- $(k-1)$ -category of the k -cell, denoted ∂C_k , is the $(k-1)$ -category of “walking parallel $(k-1)$ -morphisms”.
- A finite ordinal S gives rise to a 1-category Δ^S , whose object consist of the elements of S and where there is a unique morphism $s \rightarrow s'$ if and only if $s \leq s'$. The simplex category of finite ordinals will be denoted Δ , as usual.

Generalizing the fourth example, let C be a strict n -category. We obtain a strict $(n+1)$ -category σC , the *suspension* of C , as follows. As a set we have $\sigma C = C \cup \{\top, \perp\}$. The first category structure on σC has constant source and target maps, $s_1 = \text{const}_\top$ and $t_1 = \text{const}_\perp$. There is a unique composition making this a category structure $(*_1, s_1, t_1)$. For $j > 1$, the j^{th} category structure on σC is the unique one which agrees with the $(j-1)^{\text{th}}$ category structure on C and makes σC into a strict n -category. The k -fold suspension of the zero cell C_0 is the k -cell $\sigma^k(C_0) = C_k$.

The suspension functor preserves both pullback and pushout squares in Cat . We have $\sigma(\emptyset) \cong C_0 \sqcup C_0 \cong \partial C_1$, and hence by induction the k -fold suspension of \emptyset is $\sigma^k(\emptyset) \cong \sigma^{k-1}(C_0 \cup^\emptyset C_0) \cong C_{k-1} \cup^{\partial C_{k-1}} C_{k-1} \cong \partial C_k$. From this description the canonical functor $\partial C_k \rightarrow C_{k-1}$ arises as the suspension of the unique functor $C_0 \sqcup C_0 \rightarrow C_0$.

There are many other functors relating the categories Cat_k for various k . We describe a few. The fully-faithful inclusion i of strict k -categories into strict n -categories gives rise to a sequence of adjoint functors: $\pi \dashv i \dashv j$. The right adjoint $j_k : \text{Cat}_n \rightarrow \text{Cat}_k$ associates to a strict n -category X the maximal sub- k -category $j_k X$.

Proposition 2.3. *The following two properties are equivalent and characterize the maximal sub- k -category $j_k X$:*

- For all k -categories Y , every functor $Y \rightarrow X$ factors uniquely through $j_k X$.
- For all cells C_i with $0 \leq i \leq k$, every functor $C_i \rightarrow X$ factors uniquely through $j_k X$. \square

The following proposition is standard.

Proposition 2.4. *The cells $(C_i, 0 \leq i \leq n)$ generate Cat_n under colimits, i.e. the smallest full subcategory of Cat_n containing the cells and closed under colimits is all of Cat_n . \square*

3. GAUNT n -CATEGORIES

Definition 3.1. A strict n -category X is *gaunt* if for all $1 \leq k \leq n$ X is local with respect to the functor $\sigma^{k-1} E \rightarrow \sigma^{k-1}(\ast) = C_{k-1}$, that is, the following natural map is a bijection,

$$\text{Cat}_k(C_{k-1}, X) \rightarrow \text{Cat}_k(\sigma^{k-1} E, X).$$

Equivalently, for any $k \leq n$, the only invertible k -morphisms of X are the identities.

Lemma 3.2 ([35, Pr. 11.23]). *In a gaunt n -category, any equivalence is an identity. \square*

Remark 3.3. The suspension σX of a gaunt n -category is gaunt. Since the truncation $\pi E = C_0$, if X is gaunt, then so is $j_k X$ for all k .

Rezk observed [35, § 10] that E may be formed in Cat as a pushout of more elementary n -categories, namely as the pushout

$$K := \Delta^3 \sqcup_{(\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}})} (\Delta^0 \sqcup \Delta^0).$$

Thus a strict n -category is gaunt if and only if for each $k \geq 0$ the following natural map is a bijection:

$$\text{Fun}(C_k, X) \rightarrow \text{Fun}(\sigma^k(\Delta^3), X) \times_{\text{Fun}(\sigma^k(\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}}), X)} \text{Fun}(\sigma^k(\Delta^0 \sqcup \Delta^0), X).$$

Corollary 3.4 (of the definition). *The inclusion of the gaunt n -categories into all strict n -categories admits a left adjoint L^G that realizes the full subcategory Gaunt_n of Cat_n spanned by the gaunt n -categories as a localization. \square*

It follows from this corollary that gaunt n -categories form a presentable category Gaunt_n . In fact, we have the following result.

Lemma 3.5. *The category Gaunt_n is finitely presentable.*

Proof. It is enough to show that the inclusion $\text{Gaunt}_n \hookrightarrow \text{Cat}_n$ commutes with filtered colimits. To this end, suppose Λ a filtered category, and suppose $D: \Lambda \rightarrow \text{Cat}_n$ a diagram such that for any object $\alpha \in \Lambda$, the n -category D_α is gaunt. We claim that the colimit $D = \text{colim}_{\alpha \in \Lambda} D_\alpha$ (formed in Cat_n) is gaunt as well. This claim now follows from the fact that both C_k and $\sigma^k(E)$ are compact objects in Cat_n . \square

Remark 3.6. As a consequence of the previous result, the category Gaunt_n can be identified with the category of Ind-objects of the full subcategory $\text{Gaunt}_n^\omega \subset \text{Gaunt}_n$ spanned by the compact objects of Gaunt_n . That is [32, Cor. 2.1.9'], for any category \mathcal{D} that admits all filtered colimits, if $\text{Fun}^\omega(\text{Gaunt}_n, \mathcal{D})$ denotes the full subcategory of $\text{Fun}(\text{Gaunt}_n, \mathcal{D})$ spanned by those functors that preserve filtered colimits, then the restriction functor

$$\text{Fun}^\omega(\text{Gaunt}_n, \mathcal{D}) \rightarrow \text{Fun}(\text{Gaunt}_n^\omega, \mathcal{D})$$

is an equivalence.

Corollary 3.7. *The smallest full subcategory of Gaunt_n closed under colimits and containing the cells of dimension $\leq k$ is Gaunt_k .*

Proof. The inclusion of gaunt k -categories commutes with colimits (as it admits a right adjoint, see Rk. 3.3) and so it is enough to consider just the case $k = n$. This now follows readily from Cor. 3.4 and Pr. 2.4. \square

4. AUTOMORPHISMS OF THE CATEGORY OF GAUNT n -CATEGORIES.

We now demonstrate that the category of auto-equivalences of the category of gaunt n -categories is the discrete group $(\mathbb{Z}/2)^n$.

Lemma 4.1. *There is a unique natural endo-transformation of the identity functor on the category of gaunt n -categories.*

Proof. Such a natural transformation consists of component maps (i.e. functors) $\eta_X: X \rightarrow X$ for each gaunt n -category X . We will show that $\eta_X = \text{id}_X$ for all X . The functor η_X induces, for all $0 \leq k \leq n$ a map on sets of k -cells, $(\eta_X)_k: X_k \rightarrow X_k$, and since a functor is completely determined by the map on cells, it is enough to show that each $(\eta_X)_k$ is the identity. By naturality of η it is enough to show that the single functor $\eta_{C_n} = \text{id}_{C_n}$. We will prove that $\eta_{C_k} = \text{id}_{C_k}$ by inducting on k . When $k = 0$ there is a unique such functor, so $\eta_{C_0} = \text{id}_{C_0}$. By induction η_{C_k} is a functor which restricts to the identity functor on ∂C_k . There is only one functor with this property, namely $\eta_{C_k} = \text{id}_{C_k}$. \square

Definition 4.2. The *globular category* \mathbb{G}_n consists of the full subcategory of Gaunt_n consisting of the i -cells C_i for $i \leq n$. An *n -globular set* is a presheaf of sets on \mathbb{G}_n . The *k -cells* X_k of a globular set is the set obtained by evaluating the presheaf X on C_k .

Remark 4.3. The globular sets considered here are sometimes called *reflexive globular sets*, the difference being that our globular category \mathbb{G}_n includes *degeneracies* $C_k \rightarrow C_{k-1}$. We will have no use for non-reflexive globular sets in this paper, and so drop the distinction.

Let C be a set with a category structure $(*, s, t)$. By reversing the order of composition, we may obtain a new category structure on C , the *opposite* $(*^{\text{op}}, t, s)$. This operation leaves the identity cells unchanged, and so is compatible with the globularity condition. This implies that for each element $I \in (\mathbb{Z}/2)^n$ we obtain an auto-equivalence r_I of the category of strict n -categories. The strict n -category $r_I C$ is that whose underlying set is the same and whose i^{th} -category structure either agrees with that of C or is the opposite of that of C , according to whether the i^{th} -coordinate of I is trivial or not. Composition of the r_I corresponds to the group structure in $(\mathbb{Z}/2)^n$, and each functor restricts to an equivalence $r_I: \mathbb{G}_n \rightarrow \mathbb{G}_n$.

Proposition 4.4. *Every auto-equivalence of \mathbb{G}_n is isomorphic to r_I for some element $I \in (\mathbb{Z}/2)^n$.*

Proof. First we observe that the k -cell $C_k \in \mathbb{G}_n$ is the unique object such that, up to isomorphism, there exists precisely k other objects of \mathbb{G}_n which occur as proper retracts (namely all the cells C_i with $0 \leq i < k$). Thus every auto-equivalence F of \mathbb{G}_n must fix the objects identically, $F(C_k) = C_k$.

Next we observe that for $i < j$, there exists precisely one epimorphism $C_j \twoheadrightarrow C_i$. Since epimorphisms are preserved by any equivalence of categories, this unique epimorphism is also preserved by F . Similarly for each $0 \leq i < n$, there exist precisely two monomorphisms $C_i \hookrightarrow C_n$; these are either preserved identically by F or permuted. Thus every auto-equivalence determines a symbol $I \in (\mathbb{Z}/2)^n$ which records whether the i^{th} pair of monomorphisms $C_i \hookrightarrow C_n$ is permuted or not. The symbol corresponding to r_I is, of course, I .

Finally, we observe that the symbol I in fact determines the functor F . Indeed, every morphisms $C_i \rightarrow C_j$ in \mathbb{G}_n admits a factorization

$$C_i \twoheadrightarrow C_k \hookrightarrow C_n \twoheadrightarrow C_j$$

for some $C_k \in \mathbb{G}_n$. Thus $F = r_I$. \square

Lemma 4.5. *Let F be an auto-equivalence of the category of gaunt n -categories. Then F restricts to an equivalence between \mathbb{G}_n and its essential image in the gaunt n -categories, i.e. $F(C_k) \cong C_k$ for all $0 \leq k \leq n$.*

Proof. The proper retracts of C_n are precisely the cells C_k for $0 \leq k < n$ and, as before, each C_k is distinguished as the unique such retract such that, up to isomorphism, there exists precisely k other objects which occur as further proper retracts (the cells of even lower dimension). Thus it is enough to show that $F(C_n) \cong C_n$ for the n -cell alone, as this implies the analogous statement $F(C_k) \cong C_k$ for all $0 \leq k \leq n$.

Recall that a *generator* of a category \mathcal{C} is an object X such that $\mathcal{C}(X, -) : \mathcal{C} \rightarrow \text{Set}$ is faithful [31, pg. 127]. The collection of generators is preserved under any auto-equivalence. The n -cell C_n is a generator for Cat_n and hence also Gaunt_n , however the k -cells C_k for $k < n$ are not generators. Thus no proper retract of C_n is a generator. This property characterizes the n -cell (up to isomorphism) among all generators: it is the unique generator such that every proper retract *fails* to be a generator. If this characterization holds, then any auto-equivalence necessarily preserves the n -cell up to automorphism and the lemma follows.

In fact we will prove a stronger statement, that the n -cell itself is a retract of every generator. For this consider the gaunt n -category ∂C_{n+1} . This is the free-walking-pair-of-parallel- n -morphisms, and may be written as

$$\partial C_{n+1} = C_n \cup^{\partial C_n} C_n.$$

There are exactly two non-identity n -morphisms in ∂C_{n+1} and this gives rise to two morphism from C_n to ∂C_{n+1} (which send the non-trivial n -morphism of C_n to one or the other of these n -morphisms). These two functors differ only on the non-identity n -morphism of C_n , the induce the identical functor on the maximal $(n-1)$ -subcategory ∂C_n .

Suppose that X is a generator. Then there must exist a functor $X \rightarrow C_n$ whose image includes the non-trivial cell of C_n , for otherwise $\mathcal{C}(X, -)$ would not be able to distinguish the above pair of maps $C_n \rightarrow \partial C_{n+1}$, contradicting the fact that X

is a generator. Thus there exists a functor $X \rightarrow C_n$, and an n -cell of X which maps via this functor to the non-trivial n -cell of C_n . Since C_n is the free n -category on an n -cell, there is a section $C_n \rightarrow X$, which hits this n -cell of X . In other words we have exhibited C_n as a retract of X . \square

Lemma 4.6. *Any auto-equivalence of the category Gaunt_n is isomorphic to a functor of the form r_I for some $I \in (\mathbb{Z}/2)^n$.*

Proof. By the previous lemma, every auto-equivalence F restricts to an auto-equivalence of \mathbb{G}_n , which by Pr. 4.4 is necessarily of the form r_I for some $I \in (\mathbb{Z}/2)^n$. It suffices to prove that $F \circ r_I \simeq r_I \circ F \simeq \text{id}$, and so without loss of generality we may assume $r_I = \text{id}$, that is, that F restricts to the identity functor on \mathbb{G}_n . Now in this case, the isomorphisms

$$\text{Gaunt}_n(C_k, X) \cong \text{Gaunt}_n(F(C_k), F(X)) = \text{Gaunt}_n(C_k, F(X))$$

are natural in both C_k and X , whence one obtains a natural isomorphism $\gamma: U \cong U \circ F$, where $U: \text{Gaunt}_n \rightarrow \text{Fun}(\mathbb{G}_n^{\text{op}}, \text{Set})$ is the forgetful functor from gaunt n -categories to globular sets. It thus remains to show that this natural isomorphism is compatible with the compositions $*_j$.

For this, write

$$w_j: C_j \rightarrow C_j \cup^{C_{j-1}} C_j$$

for the morphism that corepresents the composition law $*_j$. Since F preserves pushouts, one obtains a commutative diagram

$$\begin{array}{ccc} C_j & \xrightarrow{w_j} & C_j \cup^{C_{j-1}} C_j \\ \parallel & & \parallel \\ F(C_j) & \xrightarrow{F(w_j)} & F(C_j) \cup^{F(C_{j-1})} F(C_j) \\ & & \downarrow \cong \\ & & F(C_j \cup^{C_{j-1}} C_j). \end{array}$$

Hence for any gaunt n -category X , we obtain a commutative diagram

$$\begin{array}{ccc} \text{Gaunt}_n(C_j \cup^{C_{j-1}} C_j, X) & \xrightarrow{*_j} & \text{Gaunt}_n(C_j, X) \\ \cong \downarrow & & \downarrow \cong \\ \text{Gaunt}_n(F(C_j \cup^{C_{j-1}} C_j), F(X)) & & \text{Gaunt}_n(F(C_j), F(X)) \\ \cong \downarrow & & \downarrow \\ \text{Gaunt}_n(F(C_j) \cup^{F(C_{j-1})} F(C_j), F(X)) & & \text{Gaunt}_n(F(C_j), F(X)) \\ \parallel & & \parallel \\ \text{Gaunt}_n(C_j \cup^{C_{j-1}} C_j, F(X)) & \xrightarrow{*_j} & \text{Gaunt}_n(C_j, F(X)) \end{array}$$

in which the top and bottom morphisms are exactly the composition functors. Hence the natural isomorphism γ is compatible with compositions, whence it lifts to a natural isomorphism $\text{id} \cong F$, as desired. \square

Remark 4.7. It follows from the results of this section that the category of auto-equivalences of Gaunt_n is equivalent to the discrete group $(\mathbb{Z}/2)^n$.

Combining these results with Rm. 3.6 we obtain:

Corollary 4.8. *For each regular cardinal κ , the automorphisms of the category Gaunt_n^κ of κ -compact gaunt n -categories is the discrete group $(\mathbb{Z}/2)^n$.*

5. CORRESPONDENCES AND THE CATEGORY Υ_n

The category of of gaunt n -categories is cartesian closed. Consequently, Gaunt_n is a category enriched in strict n -categories, hence a (large) strict $(n+1)$ -category. In fact any skeleton of Gaunt_n is a (large) *gaunt* $(n+1)$ -category.

A *correspondence* between ordinary categories A and B has several equivalent definitions. The two of most relevance here are:

- (a) a functor $m: A^{\text{op}} \times B \rightarrow \text{Set}$;
- (b) a category M with a functor $M \rightarrow C_1$, together with equivalences $A \simeq M_0$ and $B \simeq M_1$, where M_i denotes the fiber over the object $i \in C_1$.

Consequently we regard the category Cat/C_1 as the *category of correspondences*. The notion of correspondence generalizes to gaunt n -categories in a manner we now describe.

Definition 5.1. A *correspondence* (or *k-correspondence*, for clarity) of gaunt n -categories is an object of Gaunt_n/C_k , that is a gaunt n -category M equipped with a functor $M \rightarrow C_k$.

Example 5.2. If X is a gaunt $(n-k)$ -category, then $\sigma^k(!): \sigma^k(X) \rightarrow \sigma^k(\text{pt}) = C_k$ is naturally a k -correspondence.

Proposition 5.3. *There is a canonical equivalence of categories between the category of k -correspondences Gaunt_n/C_k and the category of triples (M_s, M_t, m) where $M_s, M_t \in \text{Gaunt}_n/C_{k-1}$ are $(k-1)$ -correspondences and*

$$m: M_s^{\text{op}} \times_{C_{k-1}} M_t \rightarrow \sigma^{k-1}(\text{Gaunt}_{n-k})$$

is a functor of $(k-1)$ -correspondences, where Gaunt_{n-k} is regarded as a (large) gaunt $(n-k+1)$ -category via the internal hom.

The equivalence is implemented by sending $M \rightarrow C_k$ to the fibers M_s and M_t over the inclusions $s, t: C_{k-1} \rightarrow C_k$. The functor m may be obtained by considering certain associated lifting diagrams. We leave the details as an amusing exercise for the interested reader, as they will not be needed here.

Corollary 5.4. *The categories of correspondences, Gaunt_n/C_k , are cartesian closed.*

Proof. When $k=0$ this is merely the statement that Gaunt_n is cartesian closed. For general k , the result follows from induction, using the equivalence from Pr. 5.3. \square

Remark 5.5. If desired it is possible to avoid the use of *large* n -categories in the above. Instead of considering all correspondences, one considers only those correspondences in which the size of M is bounded by a fixed cardinality κ . A similar version of the above proposition holds with Gaunt_{n-k} replaced by a *small* gaunt $(n-k+1)$ -category of gaunt $(n-k)$ -categories whose size is similarly bounded by κ . Allowing κ to vary, one still recovers the above corollary.

We regard the existence of internal homs for correspondences as one of the key defining properties of the theory of n -categories, and an analog of this forms one

of our main axioms for the theory of (∞, n) -categories. A weak theory of n -categories should contain the cells and a similar theory of correspondences. This suggests that the following category Υ_n should have an important role in any potential weak theory of n -categories.

Definition 5.6. The category Υ_n is the smallest full subcategory of Gaunt_n containing the globular category \mathbb{G}_n of cells that is closed under retracts and fiber products over cells, $(X, Y) \mapsto X \times_{C_k} Y$.

Example 5.7. The simplex category Δ is a subcategory of Υ_1 . To see this first observe that $[0] = C_0$, $[1] = C_1$, and that $[2]$ is a retract of $C_1 \times C_1$. Hence $[0], [1], [2] \in \Upsilon_1$. From this we see that $[n] \in \Upsilon_1$ from either the formula

$$[n] = \underbrace{[2] \times_{[1]} [2] \times_{[1]} \cdots \times_{[1]} [2]}_{n-1 \text{ times}},$$

or by observing that $[n]$ is a retract of $[1]^{\times n}$. More generally, one may show that Joyal's category Θ_n of “ n -disks” (Df. 10.2) is a subcategory of Υ_n .

We will now examine the fiber products of cells in detail. We aim to express these fiber products as simple iterated colimits of cells. Let $\varphi : C_i \rightarrow C_j$ and $\psi : C_k \rightarrow C_j$ be a pair of maps ($i, j, k \geq 0$). A map of cells $\varphi : C_i \rightarrow C_j$ either factors as a composite $C_i \rightarrow C_0 \rightarrow C_j$ or is a suspension $\varphi = \sigma(\phi)$ for some map $\phi : C_{i-1} \rightarrow C_{j-1}$. This leads to several cases. We will first consider the case in which φ fails to be the suspension of a map of lower dimensional cells. In this case we have a diagram of pullback squares

$$\begin{array}{ccccc} C_i \times F & \longrightarrow & F & \longrightarrow & C_k \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \psi \\ C_i & \longrightarrow & C_0 & \longrightarrow & C_j \end{array}$$

Here F is the fiber of $\psi : C_k \rightarrow C_j$ over the unique object in the image of φ . There are four possibilities:

- The image of ψ may be disjoint from the image of φ , in which case $F = \partial C_0 = \emptyset$. Hence F and also $C_i \times F$ are the empty colimit of cells.
- The fiber may be a zero cell, $F = C_0$, in which case $C_i \times F \cong C_i$ is trivially a colimit of cells.
- The fiber may be an m -cell $F \cong C_m$ for some $1 \leq m \leq k$, but we have $i = 0$. In this case $C_i \times F \cong F \cong C_m$ is again trivially a colimit of cells.
- The fiber may be an m -cell $F \cong C_m$ for some $1 \leq m \leq k$, and we have $i \geq 1$. In this case we have (cf. [35, Pr. 4.9])

$$C_i \times C_m \cong (C_i \cup^{C_0} C_m) \cup^{\sigma(C_{i-1} \times C_{m-1})} (C_m \cup^{C_0} C_i)$$

where for each pushout $C_x \cup^{C_0} C_y$, the object C_0 is included into the final object of C_x and the initial object of C_y .

As the suspension functor σ commutes with pullback squares, a general pullback of cells is the suspension of one of the three kinds just considered. Moreover, as the suspension functor also commutes with pushout squares, the above considerations give a recipe for writing any fiber product of cells as an iterated pushout of cells. This will be made precise in Lm. 5.10 below by considering presheaves on Υ_n .

The inclusion $\Upsilon_n \rightarrow \text{Gaunt}_n$ induces a fully-faithful nerve functor

$$\nu : \text{Gaunt}_n \rightarrow \text{Fun}(\Upsilon_n^{\text{op}}, \text{Set}).$$

In particular we may regard gaunt n -categories as particular presheaves of sets on the category Υ_n (precisely which presheaves is determined in Cor. 8.6). The nerve functor commutes with all limits, hence in particular fiber products.

Notation 5.8. Let S_{00} consist of the union of the following four finite sets of maps of presheaves on Υ_n :

$$\left\{ \nu C_{i-1} \cup^{\nu(\partial C_{i-1})} \nu C_{i-1} \rightarrow \nu(\partial C_i) \mid 0 \leq i \leq n-1 \right\}$$

(when $i = 0$, we interpret this as the empty presheaf mapping to the nerve of the empty n -category),

$$\left\{ \nu C_j \cup^{\nu C_i} \nu C_j \rightarrow \nu(C_j \cup^{C_i} C_j) \mid 0 \leq i < j \leq n \right\},$$

$$\left\{ \nu(C_{i+j} \cup^{C_i} C_{i+k}) \cup^{\nu\sigma^{i+1}(C_{j-1} \times_{C_{k-1}})} \nu(C_{i+k} \cup^{C_i} C_{i+j}) \rightarrow \nu(C_{i+j} \times_{C_i} C_{i+k}) \right. \\ \left. \mid 0 \leq i \leq n, 0 < j, k \leq n-i \right\},$$

$$\left\{ \nu\sigma^k(\Delta^3) \cup^{\nu\sigma^k(\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}})} \nu\sigma^k(\Delta^0 \sqcup \Delta^0) \rightarrow \nu C_k \mid 0 \leq k \leq n \right\}.$$

Let S_0 be the smallest class of morphisms $U \rightarrow V$ in $\text{Fun}(\Upsilon_n^{\text{op}}, \text{Set})$, closed under isomorphism, containing S_{00} , and closed under the operation $H \times_{C_i} (-)$ for all $H \in \Upsilon_n$, $H \rightarrow C_i$, and $V \rightarrow C_i$.

Lemma 5.9. *Let $X \in \text{Gaunt}_n$. Then the presheaf of sets νX is local with respect to the morphisms of S_0 .*

Proof. Forming each of the pushouts of S_{00} in Gaunt_n yields an equivalence, so X is local with respect to S_{00} . Let $S'_0 \subseteq S_0$ consist of those morphisms $(f : U \rightarrow V) \in S_0$ such that νX is local with respect to f for all gaunt X . We have observed that S'_0 contains S_{00} . It is also closed under isomorphism. We claim that it is closed under the operation $H \times_{C_i} (-)$ for all $H \in \Upsilon_n$.

Let $(U \rightarrow V) \in S'_0$. For each $G \in \Upsilon_n$, we have

$$\begin{aligned} \text{Hom}(H \times_{C_i} G, X) &\cong \text{Hom}_{C_i}(H \times_{C_i} G, X \times C_i) \\ &\cong \text{Hom}_{C_i}(G, \underline{\text{Hom}}_{C_i}(H, X \times C_i)) \end{aligned}$$

where Hom_{C_i} denotes functors of i -correspondences and $\underline{\text{Hom}}_{C_i}(A, B)$ denotes the internal hom of Gaunt_n / C_i . As $\underline{\text{Hom}}_{C_i}(H, X \times C_i)$ is gaunt, it is local with respect to $U \rightarrow V$. The result now follows easily by comparing the morphism sets in Gaunt_n and Gaunt_n / C_i . \square

Lemma 5.10. *Let $S_{00}^{-1} \text{Fun}(\Upsilon_n^{\text{op}}, \text{Set})$ denote the full subcategory of presheaves of sets which are local with respect to the morphisms of S_{00} . Let $\varphi : C_i \rightarrow C_j$ and $\psi : C_k \rightarrow C_j$ be an arbitrary pair of maps ($i, j, k \geq 0$). Then $\nu(C_i \times_{C_j} C_k)$ is contained in the smallest full subcategory of $S_{00}^{-1} \text{Fun}(\Upsilon_n^{\text{op}}, \text{Set})$ containing the nerves of cells and closed under isomorphism and the formation of colimits.*

Proof. Recall that ν commutes with limits. Let m ($\leq i, j, k$) be the largest integer such that $\varphi = \sigma^m(g)$ and $\psi = \sigma^m(f)$ are both m -fold suspensions of maps, $g : C_{i-m} \rightarrow C_{j-m}$ and $f : C_{k-m} \rightarrow C_{j-m}$. Suppose, without loss of generality, that φ

is not an $(m+1)$ -fold suspension of a map. Our previous considerations have shown that there exists a diagram of pullback squares

$$\begin{array}{ccccc}
 \sigma^m(C_{i-m} \times_{C_0} F) & \longrightarrow & \sigma^m(F) & \longrightarrow & C_k \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \psi = \sigma^m(f) \\
 C_i = \sigma^m(C_{i-m}) & \xrightarrow{\sigma^m(!)} & C_m = \sigma^m(C_0) & \xleftarrow{\sigma^m(g)} & C_j
 \end{array}$$

where as before F denotes the fiber of $f : C_{k-m} \rightarrow C_{j-m}$ over the image of g . We have already seen that there are four cases to consider. First if $F = \emptyset$, then

$$C_i \times_{C_j} C_k \cong \sigma^m(C_{i-m} \times_{C_0} F) \cong \sigma^m(\emptyset) \cong \partial C_m.$$

In this case the collection of morphisms (a) in S_{00} provide an iterative construction of $\nu\partial C_m$ as a colimit in $S_{00}^{-1} \text{Fun}(\Upsilon_n^{\text{op}}, \text{Set})$ of cells.

Next, if $F \cong C_0$, then

$$C_i \times_{C_j} C_k \cong \sigma^m(C_{i-m} \times_{C_0} F) \cong \sigma^m(C_{i-m}) \cong C_i$$

is already a cell. Similarly suppose that $F \cong C_\ell$ for $1 \leq \ell \leq k-m$, but that $i = m$. In this case

$$C_i \times_{C_j} C_k \cong \sigma^m(C_0 \times_{C_0} F) \cong \sigma^m(C_\ell) \cong C_{m+\ell}$$

is again already a cell. Finally let us suppose that $F \cong C_\ell$ with $\ell > 0$ and $i = m+p$ for $p > 0$. In this case we have,

$$C_i \times_{C_j} C_k \cong C_{p+m} \times_{C_m} C_{\ell+m}$$

is precisely the fiber product considered in the set of morphisms (c) among S_{00} . One readily observes that morphisms (b) and (c) in S_{00} provide an inductive construction of this fiber product as an iterated colimit of cells in $S_{00}^{-1} \text{Fun}(\Upsilon_n^{\text{op}}, \text{Set})$. \square

Remark 5.11. In the above considerations we have not made use of the collection of morphisms (d) in S_{00} . These morphisms are unnecessary for the above arguments and we only include them in S_{00} (and hence also in S_0) for notational consistency in later sections. (See Not. 7.5 and Df. 6.8 (C.3)).

6. AXIOMS OF THE HOMOTOPY THEORY OF (∞, n) -CATEGORIES

Definition 6.1 ([30, 4.4.2]). A functor $f : \mathcal{C} \rightarrow \mathcal{D}$ between quasicategories is said to *strongly generate* the quasicategory \mathcal{D} if the identity transformation $\text{id} : f \rightarrow f$ exhibits the identity functor $\text{id}_{\mathcal{D}}$ as a left Kan extension of f along f .

Remark 6.2 ([30, 4.4.4]). Suppose \mathcal{C} a small quasicategory, and suppose \mathcal{D} a locally small quasicategory that admits small colimits. Then a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ factors as $\mathcal{C} \rightarrow \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$, where $\mathcal{P}(\mathcal{C})$ denotes the quasicategory of functors \mathcal{C}^{op} to the quasicategory \mathcal{S} of spaces [29, Df. 5.1.0.1]. Now f strongly generates \mathcal{D} if and only if the functor $\mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$ exhibits \mathcal{D} as a localization of $\mathcal{P}(\mathcal{C})$.

Moreover, a right adjoint to $\mathcal{P}(\mathcal{C}) \rightarrow \mathcal{D}$ is given by the composite

$$\mathcal{D} \xrightarrow{j} \mathcal{P}(\mathcal{D}) \xrightarrow{f^*} \mathcal{P}(\mathcal{C}),$$

where j denotes the Yoneda embedding.

Lemma 6.3. *Suppose \mathcal{C} a small quasicategory, and suppose \mathcal{D} a locally small quasicategory that admits small colimits. Suppose $f: \mathcal{C} \rightarrow \mathcal{D}$ a functor that strongly generates \mathcal{D} . For any quasicategory \mathcal{E} admitting all small colimits, let $\text{Fun}^{\text{L}}(\mathcal{D}, \mathcal{E}) \subset \text{Fun}(\mathcal{D}, \mathcal{E})$ denote the full sub-quasicategory consisting of those functors that preserve small colimits. Then the natural functor $\text{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{E})$ restricts to a fully faithful functor*

$$\text{Fun}^{\text{L}}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{E}).$$

Proof. By Remark 6.2, \mathcal{D} is a localization of $\mathcal{P}(\mathcal{C})$, whence we obtain a fully faithful embedding $\text{Fun}^{\text{L}}(\mathcal{D}, \mathcal{E}) \rightarrow \text{Fun}^{\text{L}}(\mathcal{P}(\mathcal{C}), \mathcal{E})$. Now by [29, 5.1.5.6], left Kan extension induces an equivalence $\text{Fun}(\mathcal{C}, \mathcal{E}) \simeq \text{Fun}^{\text{L}}(\mathcal{P}(\mathcal{C}), \mathcal{E})$. See [29, 5.5.4.20]. \square

Notation 6.4. For any quasicategory \mathcal{C} , denote by $\tau_{\leq 0}\mathcal{C}$ the full subcategory of \mathcal{C} spanned by the 0-truncated [29, 5.5.6.1] objects. This quasicategory is in fact the nerve of an ordinary category.

Remark 6.5. Suppose \mathcal{C} a presentable quasicategory. Then the inclusion $\tau_{\leq 0}\mathcal{C} \hookrightarrow \mathcal{C}$ admits a left adjoint (also denoted $\tau_{\leq 0}$), which exhibits $\tau_{\leq 0}\mathcal{C}$ as an accessible localization of \mathcal{C} [29, 5.5.6.18]. Consequently, for any finite category K , any colimit functor $f: K^{\triangleright} \rightarrow \tau_{\leq 0}\mathcal{C}$, any colimit diagram $F: NK^{\triangleright} \rightarrow \mathcal{C}$, and any homotopy $\eta: f|_K \simeq F|_K$, the canonical map $r: \tau_{\leq 0}F(+\infty) \rightarrow f(+\infty)$ is an equivalence in \mathcal{C} .

Definition 6.6. Let κ be a regular cardinal. We shall say that a presentable quasicategory \mathcal{C} is κ -reduced if \mathcal{C} is κ -compactly generated [29, Df. 5.5.7.1] and if the full subcategory $(\tau_{\leq 0}\mathcal{C})^{\kappa} \subseteq \mathcal{C}$ of κ -compact 0-truncated objects strongly generates \mathcal{C} . We will say \mathcal{C} is reduced if it is κ -reduced for some κ .

Recall [29, Rk. 5.5.6.21] that the 0-truncation $\tau_{\leq 0}\mathcal{C}$ of a presentable ∞ -category \mathcal{C} is an accessible localization; consequently, $\tau_{\leq 0}\mathcal{C}$ is itself presentable. It therefore follows from [29, Rk. 5.4.2.13] that for any regular cardinal κ , the ∞ -category $(\tau_{\leq 0}\mathcal{C})^{\kappa}$ is essentially small.

Example 6.7. The quasicategory \mathcal{S} of spaces is reduced (even ω -reduced); more generally, any 1-localic ∞ -topos [29, Df. 6.4.5.8] is reduced. The quasicategory of quasicategories is shown to be reduced in [30, Ex. 4.4.9].

Definition 6.8. We shall say that a quasicategory \mathcal{C} is a *homotopy theory of (∞, n) -categories* if there is a fully faithful functor $f: \Upsilon_n \hookrightarrow \tau_{\leq 0}\mathcal{C}$ such that the following axioms are satisfied.

(C.1) **Strong generation.** The composite

$$\Upsilon_n \hookrightarrow \tau_{\leq 0}\mathcal{C} \hookrightarrow \mathcal{C}$$

strongly generates \mathcal{C} . In particular \mathcal{C} is presentable.

(C.2) **Correspondences have internal homs.** For any morphism $\eta: X \rightarrow f(C_i)$ of \mathcal{C} , the fiber product functor

$$\eta^*: \mathcal{C}_{/f(C_i)} \rightarrow \mathcal{C}_{/X}$$

preserves colimits. Since \mathcal{C} is presentable this is equivalent to the existence of internal homs for the categories of correspondences $\mathcal{C}_{/f(C_i)}$. (hence admits a right adjoint, the *internal hom*).

(C.3) **Fundamental pushouts.** Each of the finite number of maps comprising S_{00} (see Not. 5.8) is an equivalence, that is:

- (a) For each integer $1 \leq i \leq n-1$ the natural morphism

$$f(C_{i-1}) \cup^{f(\partial C_{i-1})} f(C_{i-1}) \rightarrow f(\partial C_i)$$

is an equivalence, as is the natural map from the empty colimit to $f(\emptyset)$.

- (b) For each pair of integers $0 \leq i < j \leq n$, the natural morphism

$$f(C_j) \cup^{f(C_i)} f(C_j) \rightarrow f(C_j \cup^{C_i} C_j)$$

is an equivalence.

- (c) For each $0 \leq i \leq n$, each $0 < j, k \leq n-i$, and every nondegenerate morphism $C_{i+j} \rightarrow C_i$ and $C_{i+k} \rightarrow C_i$, the natural morphism

$$(f(C_{i+j} \cup^{C_i} C_{i+k})) \cup^{f(\sigma^{i+1}(C_{j-1} \times C_{k-1}))} (f(C_{i+k} \cup^{C_i} C_{i+j})) \rightarrow f(C_{i+j} \times_{C_i} C_{i+k}),$$

is an equivalence.

- (d) For any $0 \leq k \leq n$, the natural morphism

$$(f(C_k) \sqcup f(C_k)) \cup^{(f(\sigma^k C_1) \sqcup f(\sigma^k C_1))} f(\sigma^k[3]) \rightarrow f(C_k)$$

in \mathcal{C} is an equivalence.

- (C.4) **Universality.** For any quasicategory \mathcal{D} and any fully faithful functor $g: \Upsilon_n \hookrightarrow \tau_{\leq 0}\mathcal{D}$ satisfying the conditions above, there exist a localization $L: \mathcal{C} \rightarrow \mathcal{D}$ and an equivalence $L \circ f \simeq g$.

In this situation, we say that the functor f exhibits \mathcal{C} as a homotopy theory of (∞, n) -categories.

We denote by $\text{Thy}_{(\infty, n)}$ the maximal Kan simplicial set contained in the full subcategory of the quasicategory of quasicategories [29, 3.0.0.1] spanned by the homotopy theories of (∞, n) -categories.

Remark 6.9. Clearly any homotopy theory of (∞, n) -categories is reduced. Axioms (C.1) ensures that the localization in axiom (C.4) is essentially unique, from which it follows that any two homotopy theories of (∞, n) -categories are equivalent. Consequently, the space $\text{Thy}_{(\infty, n)}$ is either empty or connected. In the following sections, we show that not only is the space $\text{Thy}_{(\infty, n)}$ nonempty, but it contains a class of familiar quasicategories.

7. A CONSTRUCTION OF THE HOMOTOPY THEORY OF (∞, n) -CATEGORIES

By Rk. 6.2, it follows from Axiom (C.1) that a homotopy theory of (∞, n) -categories \mathcal{C} may be exhibited as an accessible localization of the quasicategory of presheaves of spaces on $\Upsilon_n \subseteq \text{Gaunt}_n$. Many well-known examples of homotopy theories of (∞, n) -categories are exhibited as an accessible localizations of presheaves of spaces on smaller categories. Thus, although this section is primarily concerned with the construction of a universal quasicategory satisfying axioms (C.1-4), we also consider this more general situation.

Notation 7.1. Let $i: R \rightarrow \Upsilon_n$ be a functor, and let T be a strongly saturated class of morphisms in $\mathcal{P}(\mathcal{R})$ of small generation. Denote by $i^*: \mathcal{P}(\Upsilon_n) \rightarrow \mathcal{P}(\mathcal{R})$ the precomposition with the functor i ; denote by $i_!: \mathcal{P}(\mathcal{R}) \rightarrow \mathcal{P}(\Upsilon_n)$ the left adjoint, given by Kan extension along i ; and denote by $i_*: \mathcal{P}(\mathcal{R}) \rightarrow \mathcal{P}(\Upsilon_n)$ the right adjoint, given by right Kan extension along i .

We now wish to formulate conditions on (\mathcal{R}, T, i) that will guarantee Axioms (C.1-4) for the quasicategory $\mathcal{C} = T^{-1}\mathcal{P}(\mathcal{R})$. We begin with the following trivial observation.

Lemma 7.2. *The quasicategory $\mathcal{C} = T^{-1}\mathcal{P}(\mathcal{R})$ is strongly generated by the composite $\Upsilon_n \rightarrow \tau_{\leq 0}\mathcal{C} \rightarrow \mathcal{C}$.*

It is now convenient to have a formulation of the troublesome Axiom (C.2) that can be verified exclusively in the presheaf quasicategory $\mathcal{P}(\mathcal{R})$. To this end, we introduce the following.

Definition 7.3. Suppose $Z \in \mathcal{P}(\mathcal{R})$. For some integer $0 \leq k \leq n$, we say that a morphism $Z \rightarrow C_k$ is *degenerate* if it factors through an inclusion $C_j \hookrightarrow C_k$ for $j < k$; otherwise, we say that it is *nondegenerate*.

Proposition 7.4. *For $\mathcal{C} = T^{-1}\mathcal{P}(\mathcal{R})$ as above, Axiom (C.2) is equivalent to the following condition.*

(C.2-bis) There is a subset $T_0 \subset T$ that generates T as a strongly saturated class for which the following condition holds. For any object W of \mathcal{R} , any functor $i(W) \rightarrow C_k$ of Υ_n , any morphism $U \rightarrow V$ of T_0 , and any nondegenerate morphism $V \rightarrow C_k$ of $\mathcal{P}(\mathcal{R})$, the induced morphism

$$U \times_{C_k} i(W) \rightarrow V \times_{C_k} i(W)$$

lies in T .

Proof. For any T -local object X of $\mathcal{P}(\mathcal{R})$, a morphism $Y \rightarrow X$ represents an object of $(T^{-1}\mathcal{P}(\mathcal{R}))_{/X}$ if and only if, for any morphism $U \rightarrow V$ of T_0 , the square

$$\begin{array}{ccc} \text{Map}(V, Y) & \longrightarrow & \text{Map}(U, Y) \\ \downarrow & & \downarrow \\ \text{Map}(V, X) & \longrightarrow & \text{Map}(U, X). \end{array}$$

is homotopy cartesian, since the horizontal map at the bottom is an equivalence. For this, it suffices to show that the induced map on homotopy fibers over any vertex of $\text{Map}(V, X)$ is an equivalence. Unpacking this, we obtain the condition that for any morphism $V \rightarrow X$, the map

$$\text{Map}_{/X}(V, Y) \rightarrow \text{Map}_{/X}(U, Y)$$

is a weak equivalence. We therefore deduce that $(T^{-1}\mathcal{P}(\mathcal{R}))_{/X}$ may be exhibited as a localization $T_X^{-1}(\mathcal{P}(\mathcal{R}))_{/X}$, where T_X is the strongly saturated class generated by the set of diagrams of the form

$$\begin{array}{ccc} & \phi & \\ U & \longrightarrow & V \\ & \searrow & \swarrow \\ & X & \end{array}$$

in which $\phi \in T_0$.

Now suppose $\eta: Z \rightarrow C_k$ a morphism of $T^{-1}\mathcal{P}(\mathcal{R})$. Since colimits are universal in $\mathcal{P}(\mathcal{R})$ [29, § 6.1.1], the functor

$$\mathcal{P}(\mathcal{R})_{/C_k} \rightarrow \mathcal{P}(\mathcal{R})_{/Z}$$

given by pullback along η preserves all colimits, and the universal property of localizations guarantees that the composite

$$\mathcal{P}(\mathcal{R})_{/C_k} \rightarrow \mathcal{P}(\mathcal{R})_{/Z} \rightarrow T_Z^{-1}(\mathcal{P}(\mathcal{R})_{/Z}) \simeq (T^{-1}\mathcal{P}(\mathcal{R}))_{/Z}$$

descends to a colimit-preserving functor

$$(T^{-1}\mathcal{P}(\mathcal{R}))_{/C_k} \simeq T_{C_k}^{-1}(\mathcal{P}(\mathcal{R})_{/C_k}) \rightarrow T_Z^{-1}(\mathcal{P}(\mathcal{R})_{/Z}) \simeq (T^{-1}\mathcal{P}(\mathcal{R}))_{/Z}$$

(which then must also be given by the pullback along η) if and only if, for any diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ & \searrow & \swarrow \\ & C_k & \end{array}$$

in which $0 \leq k \leq n$ and $\phi \in T_0$, the induced morphism $U \times_{C_k} Z \rightarrow V \times_{C_k} Z$ lies in T .

It is clear that it suffices to check this only for nondegenerate morphisms $V \rightarrow C_k$. It now remains only to show that it suffices to check this for objects Z among the essential image of \mathcal{R} . This follows from the fact that the class T is strongly saturated and the fact that \mathcal{R} generates $\mathcal{P}(\mathcal{R})$ under colimits. \square

Pr. 7.4 now suggests an approach to defining a strongly saturated class S of morphisms of $\mathcal{P}(\Upsilon_n)$ such that the localization $S^{-1}\mathcal{P}(\Upsilon_n)$ will satisfy axioms (C.1-4); this will yield our first example of a homotopy theory of (∞, n) -categories.

Notation 7.5. We consider the quasicategory $\mathcal{P}(\Upsilon_n)$ and the Yoneda embedding

$$f: \Upsilon_n \hookrightarrow \tau_{\leq 0} \mathcal{P}(\Upsilon_n) \hookrightarrow \mathcal{P}(\Upsilon_n).$$

Let S_{00} denote the image of the finite set of morphisms of the same name as defined in Not. 5.8, which also represent the morphisms that appeared in (C.3). Let S_0 be the smallest class of morphisms of $\mathcal{P}(\Upsilon_n)$, closed under isomorphism, containing S_{00} , and closed under the operation $X \times_{C_i} (-)$ for all $X \in \Upsilon_n$. One may check that S_0 has countably many isomorphism classes of maps and agrees with the essential image of the class, also called S_0 , introduced in Not. 5.8. Let S be the strongly saturated class of morphisms of $\mathcal{P}(\Upsilon_n)$ generated by the class S_0 .

Theorem 7.6. *The quasicategory $\text{Cat}_{(\infty, n)} := S^{-1}\mathcal{P}(\Upsilon_n)$ satisfies axioms (C.1-4); that is, it is a homotopy theory of (∞, n) -categories.*

Proof. By construction, and Pr. 7.4, $\text{Cat}_{(\infty, n)}$ satisfies axioms (C.1-3). Any quasicategory satisfying axiom (C.1) must be a localization of $\mathcal{P}(\Upsilon_n)$. By the above proposition, any strongly saturated class of morphisms in $\mathcal{P}(\Upsilon_n)$ resulting in a localization satisfying axioms (C.2) and (C.3) must contain S , hence must be a further localization of $\text{Cat}_{(\infty, n)}$. Therefore $\text{Cat}_{(\infty, n)}$ also satisfies (C.4). \square

8. THE SPACE OF THEORIES OF (∞, n) -CATEGORIES

Th. 7.6 implies in particular that the space $\text{Thy}_{(\infty, n)}$ is nonempty (and thus connected). We now compute the homotopy type of this space precisely. We begin by computing the subcategory $\tau_{\leq 0} \text{Cat}_{(\infty, n)}$ of 0-truncated objects of $\text{Cat}_{(\infty, n)}$.

Lemma 8.1. *The Yoneda embedding factors through a fully-faithful inclusion*

$$\Upsilon_n \hookrightarrow \tau_{\leq 0} \text{Cat}_{(\infty, n)}.$$

This induces a fully-faithful nerve functor

$$\nu : \text{Gaunt}_n \hookrightarrow \tau_{\leq 0} \text{Cat}_{(\infty, n)}.$$

Proof. The 0-truncated objects of $\mathcal{P}(\Upsilon_n)$ consist precisely of those presheaves of spaces taking values in the 0-truncated spaces, i.e. those presheaves which are equivalent to ordinary set-valued presheaves on Υ_n . The 0-truncated objects of $\text{Cat}_{(\infty, n)} = S^{-1} \mathcal{P}(\Upsilon_n)$ consist of precisely those 0-truncated objects of $\mathcal{P}(\Upsilon_n)$ which are S -local. By Lm. 5.9, the nerve of every gaunt n -category is S -local, and so the result follows. \square

Notation 8.2. Now let $\Upsilon_n^{(0)} = \mathbb{G}_n$ be the globular category of cells and inductively define $\Upsilon_n^{(k)}$ to be the full subcategory of Υ_n spanned by the set

$$\left\{ X \in \Upsilon_n \left| \begin{array}{l} \text{there exists a colimit diagram } f : K^{\triangleright} \rightarrow \text{Cat}_{(\infty, n)} \\ \text{such that } f(+\infty) \simeq X \text{ and } f(K) \subset \Upsilon_n^{(k-1)} \end{array} \right. \right\}.$$

That is, $\Upsilon_n^{(k)} \subset \Upsilon_n$ consists of colimits, formed in $\text{Cat}_{(\infty, n)}$, of diagrams of objects of $\Upsilon_n^{(k-1)}$.

Proposition 8.3. *The collection $\{\Upsilon_n^{(k)}\}$ forms an exhaustive filtration of Υ_n .*

Proof. In any quasicategory, retractions of 0-truncated objects are the same as retractions inside the ordinary category of 0-truncated objects. This follows as a retract (in the sense of a *weak retraction diagram* [29, Df. 4.4.5.4]) of a 0-truncated object is automatically 0-truncated, and moreover any *strong retraction diagram* [29, Df. 4.4.5.4] of such objects factors essentially uniquely through the 0-truncated subcategory. By [29, Pr. 4.4.5.13] it then follows that a retract of an object of $\Upsilon_n^{(k)}$ is an object of $\Upsilon_n^{(k+1)}$. Thus the union $\cup_k \Upsilon_n^{(k)} \subseteq \Upsilon_n$ contains the cells and is closed under retractions.

It now suffices to show that this union is closed under fiber products over cells. To prove this, we induct on the statement:

- If $X, Y \in \Upsilon_n^{(k)}$ then $X \times_{C_i} Y \in \Upsilon_n^{(k+n+1)}$ for all $X \rightarrow C_i$ and $Y \rightarrow C_i$.

Since $\text{Cat}_{(\infty, n)}$ satisfies Axiom (C.2), the above statement for $k = i$ implies the statement for $k = i + 1$. The base case ($k = 0$) follows from the proof of Lm. 5.10, which is essentially an induction argument using Axiom (C.3). \square

Corollary 8.4. *The quasicategory $\text{Cat}_{(\infty, n)}$ is generated under colimits by the cells $C_k \in \Upsilon_n$; that is, the smallest subcategory of $\text{Cat}_{(\infty, n)}$ containing the cells and closed under colimits is $\text{Cat}_{(\infty, n)}$ itself.*

Proof. As $\text{Cat}_{(\infty, n)}$ is strongly generated by Υ_n , it suffices to show that the representables are generated by the cells *under colimits in $\text{Cat}_{(\infty, n)}$* . This follows again by induction on the filtration of Pr. 8.3: by definition $\Upsilon_n^{(k)}$ is generated by the cells if $\Upsilon_n^{(k-1)}$ is generated by them, and $\Upsilon_n^{(0)} = \mathbb{G}_n$ consisting of precisely the cells, is trivially generated by them. \square

Corollary 8.5. *The cells detect equivalences in $\text{Cat}_{(\infty, n)}$, that is $f : X \rightarrow Y$ is an equivalence in $\text{Cat}_{(\infty, n)}$ if and only if it induces equivalences $\text{Map}(C_k, X) \rightarrow \text{Map}(C_k, Y)$ for all $0 \leq k \leq n$.*

Proof. Since $\text{Cat}_{(\infty, n)}$ is a localization of $\mathcal{P}(\Upsilon_n)$, a map $f : X \rightarrow Y$ is an equivalence in $\text{Cat}_{(\infty, n)}$ if and only if it induces an equivalence of mapping spaces

$$\text{Map}(H, X) \rightarrow \text{Map}(H, Y)$$

for every $H \in \Upsilon_n$. In particular, an equivalence induces equivalences for $H = C_k$ for all $0 \leq k \leq n$.

Conversely, suppose that $f : X \rightarrow Y$ induces equivalences of cellular mapping spaces. Then we claim that it induces equivalences of mapping spaces for all H . This follows from induction on the following statement using the filtration $\{\Upsilon_n^{(k)}\}$:

- If $H \in \Upsilon_n^{(k)}$, then $\text{Map}(H, X) \rightarrow \text{Map}(H, Y)$ is an equivalence.

The inductive statement for $k = i$ follows the statement for $k = i - 1$, as H is a colimit of objects of $\Upsilon_n^{(k-1)}$. The base case $k = 0$ is precisely the assumption that $f : X \rightarrow Y$ induces equivalences of cellular mapping spaces. \square

Corollary 8.6. $\tau_{\leq 0} \text{Cat}_{(\infty, n)} \simeq \text{Gaunt}_n$. *In particular a presheaf of sets on Υ_n is isomorphic to the nerve of a gaunt n -category if and only if it is S -local.*

Proof. The nerve of a gaunt n -category is S -local (cf. Lm. 5.9). Conversely, for any $X \in \tau_{\leq 0} \text{Cat}_{(\infty, n)} \subseteq \text{Fun}(\Upsilon_n^{\text{op}}, \text{Set})$, we may restrict to \mathbb{G}_n to obtain a globular set H_X . For $0 \leq j < i \leq n$, apply X to the unique nondegenerate i -cell $\mu : C_i \rightarrow C_i \cup^{C_j} C_i$ connecting the initial and terminal vertices; this gives rise to the various compositions

$$X(C_i) \times_{X(C_j)} X(C_i) \cong X(C_i \cup^{C_j} C_i) \rightarrow X(C_i).$$

By examining the maps

$$X(C_i) \times_{X(C_j)} X(C_i) \times_{X(C_j)} X(C_i) \cong X(C_i \cup^{C_j} C_i \cup^{C_j} C_i) \rightarrow X(C_i)$$

corresponding to the unique nondegenerate i -cell

$$C_i \rightarrow C_i \cup^{C_j} C_i \cup^{C_j} C_i$$

connecting the initial and terminal vertices, we find that these compositions are associative, and by examining the maps $X(C_j) \rightarrow X(C_i)$ induced by the nondegenerate cell $C_i \rightarrow C_j$, we find that these compositions are unital. From this we deduce that H_X forms a strict n -category. Finally, since X is local with respect to $K_k \rightarrow C_k$, it follows that H_X is gaunt. Now map $A \rightarrow X$, with $A \in \Upsilon_n$ induces a map $A \rightarrow \nu H_X$, and hence we have a map $X \rightarrow \nu H_X$ in $\tau_{\leq 0} \text{Cat}_{(\infty, n)}$. By construction this is a cellular equivalence, whence $X \simeq \nu H_X$. \square

Definition 8.7. For any nonnegative integer n , an n -categorical pair (\mathcal{C}, F) consists of a reduced quasicategory \mathcal{C} (Df. 6.6), along with an equivalence

$$F : \tau_{\leq 0} \mathcal{C} \simeq \text{Gaunt}_n.$$

When such an equivalence F exists, we shall say that the reduced quasicategory \mathcal{C} admits an n -categorical pair structure.

Remark 8.8. One can easily employ Lm. 4.6 to show that the space of n -categorical pairs whose underlying quasicategory is fixed is a torsor under $(\mathbb{Z}/2)^n$.

Now write $\text{Aut}(\mathcal{C})$ for the full sub-quasicategory of $\text{Fun}(\mathcal{C}, \mathcal{C})$ spanned by the auto-equivalences.

Proposition 8.9. *For any quasicategory \mathcal{C} that admits an n -categorical pair structure, the quasicategory $\text{Aut}(\mathcal{C})$ is equivalent to the (discrete) group $(\mathbb{Z}/2)^n$.*

Proof. Let κ be a regular cardinal such that \mathcal{C} is strongly generated by the κ -compact 0-truncated objects $(\tau_{\leq 0}\mathcal{C})^\kappa$. We observe that the existence of an equivalence $\tau_{\leq 0}\mathcal{C} \simeq \text{Gaunt}_n$, Lm. 4.1, Lm. 4.6, and Cor. 4.8 guarantee that $\text{Aut}((\tau_{\leq 0}\mathcal{C})^\kappa)$ in $\text{Fun}((\tau_{\leq 0}\mathcal{C})^\kappa, (\tau_{\leq 0}\mathcal{C})^\kappa)$ is equivalent to the discrete group $(\mathbb{Z}/2)^{\times n}$. It therefore suffices to exhibit an equivalence of quasicategories $\text{Aut}(\mathcal{C}) \simeq \text{Aut}((\tau_{\leq 0}\mathcal{C})^\kappa)$.

Clearly $\text{Aut}(\mathcal{C})$ is a full sub-quasicategory of the the full sub-quasicategory $\text{Fun}^{\text{L}}(\mathcal{C}, \mathcal{C})$ of $\text{Fun}(\mathcal{C}, \mathcal{C})$ spanned by those functors that preserve small colimits. Since $(\tau_{\leq 0}\mathcal{C})^\kappa$ strongly generates \mathcal{C} , it follows from Lemma 6.3 that the inclusion $(\tau_{\leq 0}\mathcal{C})^\kappa \hookrightarrow \mathcal{C}$ induces a fully faithful functor

$$\text{Fun}^{\text{L}}(\mathcal{C}, \mathcal{C}) \hookrightarrow \text{Fun}((\tau_{\leq 0}\mathcal{C})^\kappa, \mathcal{C}).$$

Moreover, any auto-equivalence of \mathcal{C} restricts to an auto-equivalence of $\tau_{\leq 0}\mathcal{C}$ and hence an autoequivalence of $(\tau_{\leq 0}\mathcal{C})^\kappa$. Thus restriction furnishes us with a fully faithful functor from $\text{Aut}(\mathcal{C})$ to $\text{Aut}((\tau_{\leq 0}\mathcal{C})^\kappa) \simeq (\mathbb{Z}/2)^n$.

It remains to show that the restriction functor is essentially surjective. For this, suppose $(\tau_{\leq 0}\mathcal{C})^\kappa \rightarrow (\tau_{\leq 0}\mathcal{C})^\kappa$ an auto-equivalence. One may form the left Kan extension $\Phi: \mathcal{C} \rightarrow \mathcal{C}$ of the composite

$$\phi: (\tau_{\leq 0}\mathcal{C})^\kappa \rightarrow (\tau_{\leq 0}\mathcal{C})^\kappa \hookrightarrow \mathcal{C}$$

along the inclusion $(\tau_{\leq 0}\mathcal{C})^\kappa \hookrightarrow \mathcal{C}$. One sees immediately that Φ is an equivalence, and moreover its restriction to $(\tau_{\leq 0}\mathcal{C})^\kappa$ coincides with ϕ . \square

Remark 8.10. If \mathcal{C} is a quasicategory that admits an n -categorical pair structure, the choice of an equivalence $F: \tau_{\leq 0}\mathcal{C} \rightarrow \text{Gaunt}_n$ kills the auto-equivalences. That is, the fiber of the induced functor

$$F^*: \text{Aut}(\mathcal{C}) \rightarrow \text{Fun}^{\text{L}}(\tau_{\leq 0}\mathcal{C}, \text{Gaunt}_n)$$

over F is contractible.

Notation 8.11. For n -categorical pairs (\mathcal{C}, F) and (\mathcal{D}, G) , write $\text{Fun}^{\text{L}}((\mathcal{C}, F), (\mathcal{D}, G))$ for the fiber of the functor

$$\text{Fun}^{\text{L}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}^{\text{L}}(\mathcal{C}, \text{Gaunt}_n)$$

over the object $F \circ \tau_{\leq 0} \in \text{Fun}^{\text{L}}(\mathcal{C}, \text{Gaunt}_n)$.

Theorem 8.12. *The Kan complex $\text{Thy}_{(\infty, n)}$ of homotopy theories of (∞, n) -categories is $B(\mathbb{Z}/2)^n$.*

Proof. Cor. 8.6 provides an identification $F: \tau_{\leq 0}\text{Cat}_{(\infty, n)} \simeq \text{Gaunt}_n$. Axiom (C.1) ensures that $\text{Cat}_{(\infty, n)}$ is strongly generated by $\Upsilon_n \subseteq \text{Gaunt}_n$, since Υ_n is essentially small, it follows that there exists a regular cardinal κ such that $\Upsilon_n \subseteq \text{Gaunt}_n^\kappa$. (In fact one may take $\kappa = \omega$, but we will not need this precise bound.) It follows that $\text{Cat}_{(\infty, n)}$ is also strongly generated by Gaunt_n^κ (cf. [29, Pr. 4.3.2.8]). Thus $\text{Cat}_{(\infty, n)}$ admits an n -categorical pair structure. The result now follows from the results of this section. \square

9. PRESHEAVES OF SPACES AS MODELS FOR (∞, n) -CATEGORIES

Many well-known examples of homotopy theories of (∞, n) -categories are exhibited as an accessible localizations of presheaves of spaces on smaller categories. Consequently, in this section, we identify conditions on an ordinary category \mathcal{R} and a class T of morphisms of $\mathcal{P}(\mathcal{R})$ that will guarantee that the localization $T^{-1}\mathcal{P}(\mathcal{R})$ is a homotopy theory of (∞, n) -categories.

Notation 9.1. Suppose \mathcal{R} an ordinary category, and suppose $i: \mathcal{R} \rightarrow \Upsilon_n$ a functor. Suppose T_0 a set of morphisms of $\mathcal{P}(\mathcal{R})$, and write T for the strongly saturated class of morphisms of $\mathcal{P}(\mathcal{R})$ it generates.

Theorem 9.2. *Suppose that the following conditions are satisfied.*

- (R.1) $i^*(S_0) \subseteq T$.
- (R.2) $i_!(T_0) \subseteq S$.
- (R.3) Any counit $R \rightarrow i^*i_!(R) = i^*(i(R))$ is in T for any $R \in \mathcal{R}$.
- (R.4) For each $0 \leq k \leq n$, there exists an object $R_k \in \mathcal{R}$ such that $i(R_k) \cong C_k \in \Upsilon_n$.

Then $i^*: \text{Cat}_{(\infty, n)} \rightarrow T^{-1}\mathcal{P}(\mathcal{R})$ is an equivalence, and $T^{-1}\mathcal{P}(\mathcal{R})$ is a homotopy theory of (∞, n) -categories.

Proof. Condition (R.1) implies both that i_* carries T -local objects to S -local objects and that we obtain an adjunction:

$$L^T \circ i^*: S^{-1}\mathcal{P}(\Upsilon_n) \rightleftarrows T^{-1}\mathcal{P}(\mathcal{R}): i_*$$

Similarly, condition (R.2) implies that i^* carries S -local objects to T -local objects and that we obtain a second adjunction:

$$L^S \circ i_!: T^{-1}\mathcal{P}(\mathcal{R}) \rightleftarrows S^{-1}\mathcal{P}(\Upsilon_n): i^*$$

Since i^* sends S -local objects to T -local objects, $i^* \simeq L^T \circ i^*$ when restricted to the S -local objects of $\mathcal{P}(\Upsilon_n)$. Thus $i^*: \mathcal{P}(\Upsilon_n) \rightarrow \mathcal{P}(\mathcal{R})$ restricts to a functor

$$i^*: \text{Cat}_{(\infty, n)} = S^{-1}\mathcal{P}(\Upsilon_n) \rightarrow T^{-1}\mathcal{P}(\mathcal{R}),$$

that admits a left adjoint $L^S \circ i_!$ and a right adjoint i_* .

Notice that $i_!(R) \cong i(R)$ in $\mathcal{P}(\Upsilon_n)$, where we have identified \mathcal{R} and Υ_n with their images under the Yoneda embedding in, respectively, $\mathcal{P}(\mathcal{R})$ and $\mathcal{P}(\Upsilon_n)$. Thus by (R.3) the counit map applied to $r \in \mathcal{R}$,

$$R \rightarrow i^* \circ L^S \circ i_!(R) \cong i^* \circ L^S i(R) \cong i^* i(R)$$

becomes an equivalence in $T^{-1}\mathcal{P}(\mathcal{R})$ (the last equality follows from Lm. 8.1, as the image of i consists of S -local objects). The endofunctor $i^* \circ L^S \circ i_!: T^{-1}\mathcal{P}(\mathcal{R}) \rightarrow T^{-1}\mathcal{P}(\mathcal{R})$ is a composite of left adjoints, hence commutes with colimits. Therefore, as $T^{-1}\mathcal{P}(\mathcal{R})$ is strongly generated by \mathcal{R} , the functor $i^* \circ L^S \circ i_!$ is determined by its restriction to \mathcal{R} . It is equivalent the left Kan extension of its restriction to \mathcal{R} . Consequently $i^* \circ L^S \circ i_!$ is equivalent to the identity functor.

On the other hand, for each $X \in \text{Cat}_{(\infty, n)}$, consider the counit map $X \rightarrow i_* i^* X$. For each k , we have natural equivalences,

$$\begin{aligned} \text{Map}(C_k, i_* i^* X) &\simeq \text{Map}(i(R_k), i_* i^* X) \\ &\simeq \text{Map}(i^* i(R_k), i^* X) \\ &\simeq \text{Map}(R_k, i^* X) \\ &\simeq \text{Map}(i(R_k), X) \simeq \text{Map}(C_k, X), \end{aligned}$$

which follow from (R.3), (R.4), the identity $i_*(R_k) \cong i(R_k)$, and the fact that $i^* X$ is T -local. By Cor. 8.5 this implies that the counit $X \rightarrow i_* i^* X$ is an equivalence. Thus i^* is a functor with both a left and right inverse, hence is itself an equivalence of quasicategories. \square

Remark 9.3. Note that if the functor $i: \mathcal{R} \rightarrow \Upsilon_n$ is fully-faithful, then condition (R.3) is automatic. Note also that (R.3) and (R.4) together imply that the presheaves $i^*(C_k)$ on \mathcal{R} are each T -equivalent to representables R_k .

Condition (R.1) appears to be the most difficult to verify in practice. Heuristically, it states that T contains *enough morphisms*. To verify it, it will be convenient to subdivide it into two conditions.

Lemma 9.4. *Condition (R.1) is implied by the conjunction of the following.*

(R.1-bis(a)) $i^*(S_{00}) \subset T$.

(R.1-bis(b)) *For any morphism $U' \rightarrow V'$ of T_0 , and for any morphisms $V' \rightarrow i^*(C_i)$ and $H \rightarrow C_i$ with $H \in \Upsilon_n$, the pullback*

$$U' \times_{i^*(C_i)} i^* H \rightarrow V' \times_{i^* C_i} i^* H$$

lies in T .

Proof. First, consider the subclass $T' \subset T$ containing those morphisms $U' \rightarrow V'$ of T such that for any nondegenerate morphisms $V' \rightarrow C_k$ and $H \rightarrow C_k$, the pullback

$$U' \times_{C_k} H \rightarrow V' \times_{C_k} H$$

lies in T . Since colimits in $\mathcal{P}(\mathcal{R})$ are universal, one deduces immediately that the class T' is strongly saturated. Hence (R.1-bis(b)) implies that $T' = T$. Thus T is closed under pullbacks along morphisms $H \rightarrow C_k$ and contains $i^* S_{00}$ (by (R.1-bis(a))), hence contains all of $i^* S_0$. \square

There are two main examples to which we shall apply Th. 9.2: Rezk's model of complete Segal Θ_n -spaces [§ 11] and the model of n -fold complete Segal spaces [§ 12].

10. STRICT n -CATEGORIES AS PRESHEAVES OF SETS

A category internal to an ordinary category \mathcal{D} may be described as a simplicial object in \mathcal{D} , that is a \mathcal{D} -valued presheaf $C: \Delta^{\text{op}} \rightarrow \mathcal{D}$, which satisfies the following *strong Segal conditions*. For any nonnegative integer m and any integer $1 \leq k \leq m - 1$, the following square is a pullback square:

$$\begin{array}{ccc} C([m]) & \longrightarrow & C(\{k, k+1, \dots, m\}) \\ \downarrow \lrcorner & & \downarrow \\ C(\{0, 1, \dots, k\}) & \longrightarrow & C(\{k\}). \end{array}$$

Thus a strict n -category consists of a presheaf of sets on the category $\Delta^{\times n}$ which satisfies the Segal condition in each factor and further satisfies a globularity condition. Equivalently a strict n -category is a presheaf of sets on $\Delta^{\times n}$ which is local with respect to the classes of maps $\text{Segal}_{\Delta^{\times n}}$ and $\text{Glob}_{\Delta^{\times n}}$ defined below.

Notation 10.1. Objects of $\Delta^{\times n}$ will be denoted $\mathbf{m} = ([m_k])_{k=1, \dots, n}$. Let

$$j : \Delta^{\times n} \rightarrow \text{Fun}((\Delta^{\times n})^{\text{op}}, \text{Set})$$

denote the Yoneda embedding. Let

$$\boxtimes : \text{Fun}(\Delta^{\text{op}}, \text{Set}) \times \text{Fun}((\Delta^{\times n-1})^{\text{op}}, \text{Set}) \rightarrow \text{Fun}((\Delta^{\times n})^{\text{op}}, \text{Set})$$

be the essentially unique functor that preserves colimits separately in each variable and sends $(j[k], j(\mathbf{m}))$ to $j([k], \mathbf{m})$. Let Segal_{Δ} denote the collection of maps that corepresent the Segal squares:

$$\text{Segal}_{\Delta} = \{j\{0, 1, \dots, k\} \cup^{j\{k\}} j\{k, k+1, \dots, m\} \rightarrow j[m] \mid 1 \leq k \leq m-1\}$$

and inductively define

$$\text{Segal}_{\Delta^{\times n}} = \{\text{Segal}_{\Delta} \boxtimes j(\mathbf{m}) \mid \mathbf{m} \in \Delta^{\times n-1}\} \cup \{j[k] \boxtimes \text{Segal}_{\Delta^{\times n-1}} \mid [k] \in \Delta\}.$$

Moreover for each $\mathbf{m} \in \Delta^{\times n}$, let $\widehat{\mathbf{m}} = ([\widehat{m}_j])_{1 \leq j \leq n}$ be defined by the formula

$$[\widehat{m}_j] = \begin{cases} [0] & \text{if there exists } i \leq j \text{ with } [m_i] = [0], \text{ and} \\ [m_j] & \text{else} \end{cases},$$

and let

$$\text{Glob}_{\Delta^{\times n}} = \{j(\mathbf{m}) \rightarrow j(\widehat{\mathbf{m}}) \mid \mathbf{m} \in \Delta^{\times n}\}.$$

The presheaf underlying a strict n -category C will be called its *nerve* νC .

Strict n -categories may also be described as certain presheaves on the category Θ_n , the opposite of Joyal's category of n -disks [24].

Definition 10.2 ([7, Df. 3.1]). Let C be a small category. The *wreath product* $\Delta \wr C$ is the category

- whose objects consist of tuples $([n]; c_1, \dots, c_n)$ where $[n] \in \Delta$ and $c_i \in C$, and
- whose morphisms from $([m]; a_1, \dots, a_m)$ to $([n]; b_1, \dots, b_n)$ consist of tuples $(\phi; \phi_{ij})$, where $\phi : [m] \rightarrow [n]$, and $\phi_{ij} : a_i \rightarrow b_j$ where $0 < i \leq m$, and $\phi(i-1) < j \leq \phi(i)$.

The category Θ_n is defined inductively as a wreath product: $\Theta_1 = \Delta$, and $\Theta_n = \Delta \wr \Theta_{n-1}$. In particular this gives rise to embeddings $\sigma : \Theta_{n-1} \rightarrow \Theta_n$, given by $\sigma(o) = ([1]; o)$, and $\iota : \Delta \rightarrow \Theta_n$ given by $\iota([n]) = ([n]; ([0]), \dots, ([0]))$.

There is a fully-faithful embedding $i : \Theta_n \hookrightarrow \text{Cat}_n$ as a dense subcategory [7, Th. 3.7]. The image under i of $([m]; a_1, \dots, a_m)$ may be described inductively as the following colimit:

$$i([m]; o_1, \dots, o_m) = \sigma(i(o_1)) \cup^{C_0} \sigma(i(o_2)) \cup^{C_0} \dots \cup^{C_0} \sigma(i(o_m)).$$

This colimit, taken in Cat_n , is a series of pushouts in which C_0 is embedded into $\sigma(i(o_k))$ via \top and into $\sigma(i(o_{k+1}))$ via \perp as described after Ex. 2.2. There is no possible confusion by the meaning of σ , as $i(\sigma(o)) = \sigma(i(o))$ for all $o \in \Theta_{n-1}$.

As i is dense, the corresponding nerve functor $\nu : \text{Cat}_n \rightarrow \text{Fun}(\Theta_n^{\text{op}}, \text{Set})$ is fully-faithful. The essential image consists of precisely those presheaves which are local

with respect to the class of maps Segal_{Θ_n} defined inductively to be the union of $\sigma\text{Segal}_{\Theta_{n-1}}$ and the following:

$$\left\{ \begin{array}{l} j(\{0, \dots, k\}; o_1, \dots, o_k) \cup^{j(\{k\})} j(\{k, \dots, m\}; o_{k+1}, \dots, o_m) \rightarrow j(\{m\}; o_1, \dots, o_m) \\ 0 \leq k \leq m, \quad o_i \in \Theta_{n-1} \end{array} \right\}.$$

We will call this latter class Se_{Θ_n} for later reference.

The category Θ_n consists of gaunt n -categories and without much difficulty one may check that it is a full subcategory of Υ_n . Thus Υ_n may equivalently be described as the smallest subcategory of Gaunt_n containing Θ_n which is closed under retracts and fiber products over cells.

Notation 10.3. Let K denote the simplicial set

$$\Delta^3 \cup^{(\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}})} (\Delta^0 \sqcup \Delta^0)$$

obtained by contracting two edges in the three simplex.

Rezk observed [35, § 10] that K detects equivalences in nerves of categories, and consequently it may be used to formulate his completeness criterion. We shall use it to identify the gaunt n -categories. To this end set

$$\begin{aligned} \text{Comp}_{\Delta} &= \{K \rightarrow j[0]\} \\ \text{Comp}_{\Delta^{\times n}} &= \{\text{Comp}_{\Delta} \boxtimes j(\mathbf{0})\} \cup \{j[k] \boxtimes \text{Comp}_{\Delta^{\times n-1}}\} \\ \text{Comp}_{\Theta_n} &= \iota_! \text{Comp}_{\Delta} \cup \sigma_! \text{Comp}_{\Theta_{n-1}}. \end{aligned}$$

where

$$\iota_! : \text{Fun}(\Delta^{\text{op}}, \text{Set}) \rightarrow \text{Fun}(\Theta_n^{\text{op}}, \text{Set}) \quad \text{and} \quad \sigma_! : \text{Fun}(\Theta_{n-1}^{\text{op}}, \text{Set}) \rightarrow \text{Fun}(\Theta_n^{\text{op}}, \text{Set})$$

are given by left Kan extension along ι and σ , respectively.

Corollary 10.4. *A presheaf of sets on $\Delta^{\times n}$ is isomorphic to the nerve of a gaunt n -category if and only if it is local with respect to the classes $\text{Segal}_{\Delta^{\times n}}$, $\text{Glob}_{\Delta^{\times n}}$, and $\text{Comp}_{\Delta^{\times n}}$. A presheaf of sets on Θ_n is isomorphic to the nerve of a gaunt n -category if and only if it is local with respect to the classes Segal_{Θ_n} and Comp_{Θ_n} .*

Proof. Being local with respect $\text{Segal}_{\Delta^{\times n}}$ and $\text{Glob}_{\Delta^{\times n}}$ (or to Segal_{Θ_n} for Θ_n -presheaves) implies that the presheaf is the nerve of a strict n -category. Such an n -category is gaunt if and only if it is local with respect to the morphisms $\sigma^k(E) \rightarrow \sigma^k(C_0)$. This last follows from locality with respect to $\text{Comp}_{\Delta^{\times n}}$ (or, respectively, with respect to Comp_{Θ_n}) because the square

$$\begin{array}{ccc} [1] \sqcup [1] & \xleftarrow{i_{0,2} \sqcup i_{1,3}} & [3] \\ \downarrow & & \downarrow \\ [0] \sqcup [0] & \xrightarrow{\quad \quad \quad} & \lrcorner E \end{array}$$

is a pushout square of strict n -categories. \square

The description of Θ_n as an iterated wreath product gives rise to a canonical functor $\delta_n : \Delta^{\times n} \rightarrow \Theta_n$, described in [7, Df. 3.8], which sends $[k_1] \times [k_2] \times \dots \times [k_n]$

to the object

$$([k_n]; \underbrace{i_{\Delta \times (n-1)}([k_1] \times \cdots \times [k_{n-1}]), \dots, i_{\Delta \times (n-1)}([k_1] \times \cdots \times [k_{n-1}])}_{k_n \text{ times}}).$$

This object may be thought of as generated by a $k_1 \times k_2 \times \cdots \times k_n$ grid of cells. If X is a strict n -category then its nerve $\nu_{\Delta \times n} X$ in $\text{Fun}((\Delta^{\times n})^{\text{op}}, \text{Set})$ is obtained by the formula $\delta_n^* \nu_{\Theta_n} X$, where ν_{Θ_n} is the nerve induced from the inclusion $\Theta_n \rightarrow \text{Cat}_n$ [7, Pr. 3.9, Rk. 3.12].

Proposition 10.5. *Joyal's category Θ_n is the smallest full subcategory of Gaunt_n containing the grids (the full subcategory of Θ_n spanned by the image of δ_n) and closed under retracts. Furthermore, the morphisms in the set Segal_{Θ_n} may be obtained as retracts of the set $(\delta_n)_!(\text{Segal}_{\Delta^{\times n}})$.*

Proof. Both statements follow by induction. First note that Θ_n itself is closed under retracts [7, Pr. 3.14]. In the base case, the category $\Theta_1 = \Delta$ consists of precisely the grids, and the sets of morphisms agree $\text{Segal}_{\Theta_1} = \text{Segal}_{\Delta}$. Now assume, by induction, that every object of $o \in \Theta_{n-1}$ is the retract of a grid $\delta_n(\mathbf{m}^o)$, for some object $\mathbf{m}^o = [m_1^o] \times \cdots \times [m_{n-1}^o] \in \Delta^{\times n}$. In fact, given any finite collection of objects $\{o_i \in \Theta_{n-1}\}$ they may be obtained as the retract of a single grid. This grid may be obtained as the image of $\mathbf{k} = [k_1] \times \cdots \times [k_{n-1}]$, where k_j is the maximum of the collection $\{m_j^{o_i}\}$. It now follows easily that the object $([n]; o_1, \dots, o_i) \in \Theta_n$ is a retract of the grid coming from the object $[n] \times \mathbf{k}$.

To prove the second statement we note that there are two types of maps in Segal_{Θ_n} , those in $\sigma_!(\text{Segal}_{\Theta_{n-1}})$ and the maps

$$j(\{0, \dots, k\}; o_1, \dots, o_k) \cup^{j(\{k\})} j(\{k, \dots, m\}; o_{k+1}, \dots, o_m) \rightarrow j([m]; o_1, \dots, o_m)$$

for $0 \leq k \leq m$ and $o_i \in \Theta_{n-1}$. This later map is a retract of the image under $(\delta_n)_!$ of the map

$$(j\{0, 1, \dots, k\} \cup^{j\{k\}} j\{k, \dots, m\} \rightarrow j[m]) \boxtimes j(\mathbf{m})$$

which is a map in $\text{Segal}_{\Delta^{\times n}}$. Here \mathbf{m} is such that $(\{0, \dots, k\}; o_1, \dots, o_k)$ is the retract of the grid corresponding to $[k] \times \mathbf{m}$.

The former class of morphisms in Segal_{Θ_n} , those in $\sigma_!(\text{Segal}_{\Theta_{n-1}})$, are also retracts on elements in $\text{Segal}_{\Delta^{\times n}}$. Specifically, if $\sigma_!(f) \in \sigma_!(\text{Segal}_{\Theta_{n-1}})$, then by induction f is the retract of $(\delta_{n-1})_!(g)$ for some $g \in \text{Segal}_{\Delta^{\times n-1}}$. One may then readily check that $\sigma(f)$ is the retract of $j[1] \boxtimes g \in \text{Segal}_{\Delta^{\times n}}$. \square

11. REZK'S COMPLETE SEGAL Θ_n -SPACES FORM A HOMOTOPY THEORY OF (∞, n) -CATEGORIES

Here we consider Joyal's full subcategory Θ_n of Cat_n [24, 6, 7]; write $i: \Theta_n \rightarrow \Upsilon_n$ for the inclusion functor. Rezk [35, 11.4] identifies the set of morphisms $\mathcal{T}_{n, \infty}$ of $\mathcal{P}(\Theta_n)$ consisting of the union of Segal_{Θ_n} and Comp_{Θ_n} ³. Let us write T_{Θ_n} for the saturated class generated by $\mathcal{T}_{n, \infty}$, and let us write $\text{CSS}(\Theta_n)$ for the localization $T_{\Theta_n}^{-1} \mathcal{P}(\Theta_n)$. We now show that $\text{CSS}(\Theta_n)$ is a homotopy theory of (∞, n) -categories.

³Rezk use a slightly different generating set based on the full decomposition of $[n]$ as the union

$$[1] \cup^{[0]} [1] \cup^{[0]} \dots \cup^{[0]} [1].$$

Both Rezk's set of generators and the union $\text{Segal}_{\Theta_n} \cup \text{Comp}_{\Theta_n}$ are readily seen to produce the same saturated class T_{Θ_n} . We find it slightly more convenient to use the later class of generators.

Remark 11.1. It follows from [29, A.3.7.3] that $\text{CSS}(\Theta_n)$ is canonically equivalent to the simplicial nerve of the cofibrant-fibrant objects in the simplicial model category $\Theta_n \text{Sp}_\infty$ considered by Rezk — i.e., the left Bousfield localization of the injective model category of simplicial presheaves on Θ_n with respect to the set $\mathcal{J}_{n,\infty}$.

Lemma 11.2. *The saturated class T_{Θ_n} contains the set $i^*(S_{00})$.*

Proof. The set S_{00} consists of the union of four subsets of maps, corresponding to the four families of fundamental pushouts of types (a), (b), (c), and (d) in Axiom (C.3). The second and last subsets corresponding to the (b) and (d) families pullback to morphisms which are contained in the generating set of T_{Θ_n} . Thus it remains to prove that the same holds for the remaining families (a) and (c). In particular, we wish to show that for each $0 \leq i \leq n$, each $0 \leq j, k \leq n - i$, and every nondegenerate morphism $C_{i+j} \rightarrow C_i$ and $C_{i+k} \rightarrow C_i$, the natural morphism

$$(11.2.1) \quad f(C_{i+j} \cup^{C_i} C_{i+k}) \cup^{f(\sigma^{i+1}(C_{j-1} \times C_{k-1}))} (f(C_{i+k} \cup^{C_i} C_{i+j}) \rightarrow f(C_{i+j} \times_{C_i} C_{i+k}),$$

is contained in T_{Θ_n} where the pushout is formed as in Not. 5.8.

In fact a stronger statement holds (cf. [35, Pr. 4.9]). For each object $o \in \Theta_n$ we have a natural bijection of sets

$$\text{hom}(o, C_{i+j} \times_{C_i} C_{i+k}) \cong \text{hom}(o, C_{i+j} \cup^{C_i} C_{i+k}) \cup^{\text{hom}(o, C_{i+m})} \text{hom}(o, C_{i+k} \cup^{C_i} C_{i+j}).$$

Thus Eq. 11.2.1 is in fact an equivalence in the presheaf category $\mathcal{P}(\Theta_n)$. In particular the family (c) pulls back to a family of equivalences, which are hence contained in T_{Θ_n} . An virtually identical argument applies to the family (a), which also consists of morphisms pulling back to equivalences of presheaves. \square

The functor $\sigma : \Theta_{n-1} \rightarrow \Theta_n$ gives rise to a functor $\sigma_! : \mathcal{P}(\Theta_{n-1}) \rightarrow \mathcal{P}(\Theta_n)$, left adjoint to σ^* . The classes Segal_{Θ_n} and Comp_{Θ_n} are defined inductively using the 1-categorical analog of $\sigma_!$, but may also be defined using $\sigma_!$. We therefore collect some relevant properties of this functor in the next two lemmas.

Lemma 11.3. *The functor $\sigma_!$ preserves both pushouts and pullbacks, sends $T_{\Theta_{n-1}}$ -local objects to T_{Θ_n} -local objects, and satisfies $\sigma_!(T_{\Theta_{n-1}}) \subseteq T_{\Theta_n}$.*

Proof. The functor $\sigma_!$ is a left adjoint, hence it preserves all colimits in $\mathcal{P}(\Theta_n)$, in particular pushouts. Moreover, $\sigma_!$ sends the generators of $T_{\Theta_{n-1}}$ to generators of T_{Θ_n} . Together these imply the containment $\sigma_!(T_{\Theta_{n-1}}) \subseteq T_{\Theta_n}$. Direct computations, which we leave to the reader, show that $\sigma_!$ sends $T_{\Theta_{n-1}}$ -local objects to T_{Θ_n} -local objects and that the following formula holds,

$$\text{hom}([n]; o_1, \dots, o_n), \sigma(X \times_Y Z) = \coprod_{\substack{i_k : [n] \rightarrow [1] \\ 0 \leq k \leq n+1}} \text{Map}(o_k, X \times_Y Z).$$

where $i_k : [n] \rightarrow [1]$ maps $i \in [n]$ to $0 \in [1]$ if $i < k$ and to $1 \in [1]$ otherwise. In the above formula when $k = 0$ or $n + 1$ the space $\text{Map}(o_k, X \times_Y Z)$ is interpreted as a singleton space. From this it follows that σ preserves fiber products. \square

The following observation is clear from the definition of σ and the morphisms in Θ_n .

Lemma 11.4. *Suppose that $V \in \Theta_n$ is of the form $V \cong \sigma(W)$ for some $W \in \Theta_{n-1}$. Suppose that $f : V \rightarrow C_i$ is nondegenerate and $i > 0$. Then $f = \sigma(g)$ for a unique nondegenerate $g : W \rightarrow C_{i-1}$.*

More generally, the Θ -construction gives rise, for each $[m] \in \Delta$, to functors

$$\begin{aligned} \sigma^{[m]} : \Theta_{n-1}^{\times m} &\rightarrow \Theta_n \\ (o_1, \dots, o_m) &\mapsto ([m]; o_1, \dots, o_m). \end{aligned}$$

The case $\sigma^{[1]} = \sigma$ has just been considered, and as in that case we obtain functors

$$\sigma_1^{[m]} : \mathcal{P}(\Theta_{n-1})^{\times m} \rightarrow \mathcal{P}(\Theta_n)$$

by left Kan extension in each variable. These functors were also considered by Rezk [35, § 4.4], and we adopt a similar notation: $\sigma_1^{[m]}(X_1, \dots, X_m) = ([m]; X_1, \dots, X_m)$.

Lemma 11.5. *Let b_1, \dots, b_p be elements of $\mathcal{P}(\Theta_{n-1})$, and let $0 \leq r \leq s \leq p$. Let A and B be defined as follows:*

$$\begin{aligned} A &= \sigma_1^{\{0, \dots, s\}}(b_1, \dots, b_s) \cup \sigma_1^{\{r, \dots, s\}}(b_{r+1}, \dots, b_s) \sigma_1^{\{r, \dots, p\}}(b_{r+1}, \dots, b_p), \text{ and} \\ B &= \sigma_1^{[p]}(b_1, \dots, b_p). \end{aligned}$$

Then the natural map $A \rightarrow B$ is in T_{Θ_n} .

Proof. This follows from [35, Pr. 6.4], but the proof given there (even in the corrected version) relies on the false proposition [35, Pr. 2.19]. However it is straightforward to supply an alternate proof (along the lines of [35, Pr. 5.3]). First note that if each of the b_i were a representable presheaf, then the map $A \rightarrow B$ may be written as a pushout of the generating morphism $\mathcal{T}_{n, \infty}$, hence is manifestly an element of T_{Θ_n} (we leave this as an exercise). The general case, however, reduces to this case as every presheaf is (canonically) a colimit of representables, the functors $\sigma_1^{[l]}$ commute with these colimits separately in each variable, and T_{Θ_n} , being a saturated class, is closed under colimits. \square

Notation 11.6. We now define three additional classes of morphisms of $\mathcal{P}(\Theta_n)$. Let J_a be the set of all morphisms $H \rightarrow C_i$ ($0 \leq i \leq n$) of Υ_n ; let J_b be the set of all nondegenerate morphisms $H \rightarrow C_i$ ($0 \leq i \leq n$) of Θ_n ; and let J_c be set of all inclusions $C_j \hookrightarrow C_i$ ($0 \leq j \leq i \leq n$) of \mathbb{G}_n . Now, for $x \in \{a, b, c\}$, set:

$$T_{\Theta_n}^{(x)} := \left\{ [f : U \rightarrow V] \in T_{\Theta_n} \left| \begin{array}{l} \text{for any } [H \rightarrow C_i] \in J_x \text{ and } [V \rightarrow C_i] \in \mathcal{P}(\Theta_n), \\ \text{one has } f \times_{C_i} \nu H \in T_{\Theta_n} \end{array} \right. \right\}$$

Lemma 11.7. *Each of the three classes $T_{\Theta_n}^{(x)}$ ($x \in \{a, b, c\}$) is a strongly saturated class.*

Proof. As colimits are universal in $\mathcal{P}(\Theta_n)$, the functors $(-) \times_{C_i} \nu H$ preserves all small colimits. Thus the class $[(-) \times_{C_i} \nu H]^{-1}(T_{\Theta_n})$ is a saturated class in $\mathcal{P}(\Theta_n)$. Taking appropriate intersections of these classes and T_{Θ_n} yields the three classes in question. \square

We aim to show that the quasicategory $\text{CSS}(\Theta_n)$ is a homotopy theory of (∞, n) -categories. For this we need to prove Axioms (R.1-4) of Th. 9.2. The most difficult property, (R.1), would follow from Lm. 9.4 and Lm. 11.2 if we also knew the identity $T_{\Theta_n} = T_{\Theta_n}^{(a)}$. As these are saturated classes and $T_{\Theta_n}^{(a)} \subseteq T_{\Theta_n}$, it is enough to show

that the generators Segal_{Θ_n} and Comp_{Θ_n} of T_{Θ_n} are contained in $T_{\Theta_n}^{(a)}$. We will ultimately prove this by an inductive argument, but first we need some preliminaries.

First, we note the following.

Lemma 11.8. *One has $T_{\Theta_n} = T_{\Theta_n}^{(c)}$.*

Proof. By Lm. 11.7, it is enough to check that the generators of Segal_{Θ_n} and Comp_{Θ_n} of T_{Θ_n} are contained in $T_{\Theta_n}^{(c)}$. For that assume that $[U \rightarrow V]$ is a generator of T_{Θ_n} and let $C_j \hookrightarrow C_i$ be an inclusion. We wish to demonstrate that for all $V \rightarrow C_i$ we have that

$$(11.8.1) \quad U \times_{C_i} C_j \rightarrow V \times_{C_i} C_j$$

is in T_{Θ_n} . Recall that

$$\text{Comp}_{\Theta_n} = \iota_! \text{Comp}_{\Delta} \cup \sigma_! \text{Comp}_{\Theta_{n-1}} \quad \text{and} \quad \text{Segal}_{\Theta_n} = \text{Segal}_{\Theta_n} \cup \sigma_! \text{Segal}_{\Theta_{n-1}}.$$

There are several cases:

- (a) $i = j$. In this trivial case (11.8.1) reduces to $[U \rightarrow V] \in T_{\Theta_n}$.
- (b) The morphism $V \rightarrow C_i$ factors as $V \rightarrow C_0 \rightarrow C_i$. In this case the fiber product $C_j \times_{C_i} C_0$ is either C_0 or empty. In the latter case (11.8.1) is an isomorphism, and in the former case it is $[U \rightarrow V] \in T_{\Theta_n}$. Notice that this case covers $\iota_! \text{Comp}_{\Delta}$.
- (c) $j = 0 < i$ and the map $V \rightarrow C_i$ does not factor through C_0 . In this case it follows that $[U \rightarrow V]$ is not in $\iota_! \text{Comp}_{\Delta}$, and hence

$$V = ([m]; o_1, \dots, o_m)$$

is representable with $m \neq 0$. Moreover the map

$$V \rightarrow C_i = ([1]; C_{i-1})$$

consists of a surjective map $[m] \rightarrow [1]$ (which specifies a unique $0 < k \leq m$; the inverse image of $0 \in [1]$ consists of all elements strictly less than k) together with map $o_k \rightarrow C_{i-1}$. A direct calculation shows that in this situation (11.8.1) is either an isomorphism or a generator of T_{Θ_n} .

- (d) $0 < j \leq i$, the map

$$[U \rightarrow V] \in \sigma_! \text{Comp}_{\Theta_{n-1}} \cup \sigma_! \text{Segal}_{\Theta_{n-1}}$$

is a suspension, and the map $V \rightarrow C_i$ does not factor through C_0 . It follows that $V \rightarrow C_i$ is the suspension of a map. This case then follows by induction and Lm. 11.3.

- (e) The final case is when $0 < j < i$, the map $[U \rightarrow V] \in \text{Se}_{\Theta_n}$ is not a suspension, nor in Comp_{Θ_n} , and the map $V \rightarrow C_i$ does not factor through C_0 . In this case (11.8.1) is a map of the form described in Lm. 11.5. \square

Armed with this, we now reduce the problem to verifying that $T_{\Theta_n} = T_{\Theta_n}^{(b)}$.

Lemma 11.9. *If $T_{\Theta_n}^{(b)}$ coincides with T_{Θ_n} , then so does the class $T_{\Theta_n}^{(a)}$.*

Proof. First note that as Θ_n strongly generates $\mathcal{P}(\Theta_n)$, and T_{Θ_n} is strongly saturated, it is enough to consider $H \in \Theta_n$ representable. Let $f: U \rightarrow V$ be a morphism in T_{Θ_n} , let $V \rightarrow C_i$ be given, and let $H \rightarrow C_i$ be arbitrary. There exists a unique factorization $H \rightarrow C_k \hookrightarrow C_i$, with $H \rightarrow C_k$ nondegenerate. Consider the following diagram of pullbacks in $\mathcal{P}(\Theta_n)$:

$$\begin{array}{ccccc}
 U'' & \xrightarrow{\quad} & U' & \xrightarrow{\quad} & U \\
 \downarrow & & \downarrow & & \downarrow f \\
 V'' & \xrightarrow{\quad} & V' & \xrightarrow{\quad} & V \\
 \downarrow & & \downarrow & & \downarrow \\
 H & \xrightarrow{\quad} & C_k & \hookrightarrow & C_i
 \end{array}$$

Since $T_{\Theta_n}^{(c)} = T_{\Theta_n}$, we have $[U' \rightarrow V'] \in T_{\Theta_n}$, and if $T_{\Theta_n}^{(b)} = T_{\Theta_n}$, then we also have $[U'' \rightarrow V''] \in T_{\Theta_n}$, as desired. \square

Lemma 11.10. *For each $x \in \{a, b\}$, we have*

$$T_{\Theta_n}^{(x)} = \left\{ [f: U \rightarrow V] \in T_{\Theta_n} \left| \begin{array}{l} \text{for any } [H \rightarrow C_i] \in J_x \text{ and any nondegenerate} \\ [V \rightarrow C_i] \in \mathcal{P}(\Theta_n), \text{ one has } f \times_{C_i} \nu H \in T_{\Theta_n} \end{array} \right. \right\}$$

In other words, to verify that $f: U \rightarrow V$ is in one of these classes, it suffices to consider only those fiber products $f \times_{C_i} \nu H$ with $V \rightarrow C_i$ nondegenerate.

Proof. Let us focus on the case $x = a$. Let $f: U \rightarrow V$ be in class given on the right-hand side of the asserted identity. We wish to show that $f \in T_{\Theta_n}^{(a)}$, that is for any pair of morphism $H \rightarrow C_i$ and $V \rightarrow C_i$ we have

$$U' = U \times_{C_i} H \rightarrow V \times_{C_i} H = V'$$

is in T_{Θ_n} . This follows as there exists a factorization $V \rightarrow C_k \rightarrow C_i$ and a diagram of pullbacks:

$$\begin{array}{ccccccc}
 U' & \xrightarrow{\quad} & V' & \xrightarrow{\quad} & H' & \xrightarrow{\quad} & H \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 U & \xrightarrow{\quad} & V & \xrightarrow{\quad} & C_k & \hookrightarrow & C_i
 \end{array}$$

such that $V \rightarrow C_k$ is nondegenerate. The analogous result for $T_{\Theta_n}^{(b)}$ follows by the same argument and the observation that $H' \rightarrow C_k$ is nondegenerate if $H \rightarrow C_i$ is such. \square

Now we settle an important first case of the equality $T_{\Theta_n} = T_{\Theta_n}^{(b)}$.

Lemma 11.11. *Let $[U \rightarrow V] \in \text{Segal}_{\Theta_n}$ be a morphism that is not contained in $\sigma_!(\text{Segal}_{\Theta_{n-1}})$. Let $V \rightarrow C_i$ be nondegenerate, and let $[H \rightarrow C_i] \in J_b$ be a nondegenerate map in Θ_n . Then the morphism $U \times_{C_i} H \rightarrow V \times_{C_i} H$ is contained in T_{Θ_n} .*

Proof. For the special case $i = 0$, a more general version of this statement was proven by Rezk [35, Pr. 6.6] and forms one of the cornerstone results of that work. Our current proof builds on Rezk's ideas.

The fundamental argument is to construct a category \mathcal{Q} along with a functorial assignment of commuting squares

$$\begin{array}{ccc}
 A_\alpha & \longrightarrow & U \times_{C_i} H \\
 \wr \downarrow & & \downarrow \\
 B_\alpha & \longrightarrow & V \times_{C_i} H
 \end{array}$$

for each $\alpha \in \mathcal{Q}$. This assignment is required to satisfy a host of conditions.

First, each of the functors $A, B : \mathcal{Q} \rightarrow \mathcal{P}(\Theta_n)$ is required to factor through $\tau_{\leq 0} \mathcal{P}(\Theta_n)$, the category of 0-truncated objects. The 0-truncated objects of $\mathcal{P}(\Theta_n)$ consist precisely of those presheaves of spaces taking values in the homotopically discrete spaces. There is no harm regarding such objects simply as ordinary set-valued presheaves, and we will do so freely.

Second, we require that for each $\alpha \in \mathcal{Q}$ the natural morphism $A_\alpha \rightarrow B_\alpha$ is in the class T_{Θ_n} . As T_{Θ_n} is saturated, this second condition implies that the natural map $\text{colim}_{\mathcal{Q}} A \rightarrow \text{colim}_{\mathcal{Q}} B$ is also in T_{Θ_n} , where these colimits are taken in the quasicategory $\mathcal{P}(\Theta_n)$ (hence are equivalently homotopy colimits for a levelwise model structure on simplicial presheaves, see Rk. 11.1).

Third and last, we require that the natural maps $\text{colim}_{\mathcal{Q}} A \rightarrow U \times_{C_i} H$ and $\text{colim}_{\mathcal{Q}} B \rightarrow V \times_{C_i} H$ are equivalences in $\mathcal{P}(\Theta_n)$ (i.e. levelwise weak equivalences of space-valued presheaves). If all of the above properties hold, then we obtain a natural commuting square

$$\begin{array}{ccc} \text{colim}_{\mathcal{Q}} A & \xrightarrow{\cong} & U \times_{C_i} H \\ \wr \downarrow & & \downarrow \\ \text{colim}_{\mathcal{Q}} B & \xrightarrow{\cong} & V \times_{C_i} H \end{array}$$

in which the indicated morphisms are in the class T_{Θ_n} . As this class is saturated it follows that $U \times_{C_i} H \rightarrow V \times_{C_i} H$ is also in this class. Thus if such a \mathcal{Q} and associated functors can be produced, we will have completed the proof.

However, the construction of a \mathcal{Q} , A , and B demonstrably satisfying the above properties appears to be somewhat delicate. In the case $i = 0$ the original published proof [35, Pr. 6.6] was incorrect, and a corrected proof has been supplied by Rezk in [36, Pr. 2.1]. At this point we deviate from Rezk's treatment. Specifically our category \mathcal{Q} and associated functors will differ from his. We will focus on the more complicated case $i > 0$, and leave the necessary simplifications in the case $i = 0$ to the reader (or simply refer the reader to [35, Pr. 6.6]).

Under the assumptions of the statement of the lemma we have the following identifications of presheaves:

$$\begin{aligned} U &= j(\{0, \dots, k\}; o_1, \dots, o_k) \cup^{j(\{k\})} j(\{k, k+1, \dots, m\}; o_{k+1}, \dots, o_m) \\ V &= j([m]; o_1, \dots, o_m) \\ H &= j([n]; u_1, \dots, u_n) \end{aligned}$$

where $0 \leq k \leq m$ and $o_\alpha, u_\beta \in \Theta_{n-1}$ are given. If $i > 0$, then the i -cell is the representable presheaf $j([1]; C_{i-1})$. A nondegenerate map $V \rightarrow C_i$ includes a nondegenerate map $f : [m] \rightarrow [1]$, and likewise a nondegenerate map $H \rightarrow C_i$ includes a nondegenerate map $g : [n] \rightarrow [1]$. Let m' be the fiber over $0 \in [1]$, and let m'' be the fiber over 1 . Then $[m] = [m'] \cdot [m'']$ is the ordered concatenation of $[m']$ and $[m'']$. Similarly $[n] = [n'] \cdot [n'']$ is the ordered concatenation of the preimages of 0 and 1 under g .

Let $\delta = (\delta', \delta'') : [p] \rightarrow [m] \times_{[1]} [n]$ be a map which is an inclusion. There is a unique $-1 \leq r \leq p$ such that under the composite $[p] \rightarrow [m] \times_{[1]} [n] \rightarrow [1]$, an element s maps to 0 if and only if $s \leq r$ (hence maps to 1 if and only if $s > r$).

As each layer of B_\bullet is a disjoint union of certain C_δ , it is sufficient to show that the map

$$C_\delta \times_{(V \times_{C_i} H)} (U \times_{C_i} H) \rightarrow C_\delta$$

is in T_Θ for each C_δ . As in the previous construction, there exist unique $0 \leq r \leq s \leq p$ such that $\delta'(t) < k$ if and only if $t < r$, and $k < \delta'(t)$ if and only if $s < t$. The interval $\{r, \dots, s\} \subset [p]$ is precisely the preimage of $\{k\}$ under δ' . We then have

$$\begin{aligned} & C_\delta \times_{(V \times_{C_i} H)} (U \times_{C_i} H) \\ & \cong \sigma_!^{\{0, \dots, s\}}(b_1, \dots, b_s) \cup \sigma_!^{\{r, \dots, s\}}(b_{r+1}, \dots, b_s) \sigma_!^{\{r, r+1, \dots, p\}}(b_{r+1}, \dots, b_p) \end{aligned}$$

and so the desired result follows from Lm. 11.5. \square

Remark 11.12. It is possible to give an alternative proof of Lm. 11.11 which builds directly on Rezk's proof in the case $i = 0$. Let \mathcal{Q} be Rezk's category $\mathcal{Q}_{m,n}$ as defined in [36], and following Rezk define functors $A^{(i=0)}$ and $B^{(i=0)}$ as the objects

$$\begin{aligned} A_\delta^{(i=0)} & := V_{(\delta_1, \delta_2)^{-1}(M \times N)}(D_1, \dots, D_p), \quad \text{and} \\ B_\delta^{(i=0)} & := V[p](D_1, \dots, D_p), \end{aligned}$$

where we are also using the notation of [36]. Then these choices satisfy the requisite properties for the case $i = 0$, the most difficult being [36, Pr. 2.3].

For general i , notice that we have inclusions of subobjects

$$\begin{aligned} V \times_{C_i} H & \subseteq V \times H, \\ U \times_{C_i} H & \subseteq U \times H. \end{aligned}$$

We may define A_δ and B_δ as pullbacks

$$\begin{aligned} A_\delta & = A_\delta^{(i=0)} \times_{(U \times H)} (U \times_{C_i} H), \quad \text{and} \\ B_\delta & = B_\delta^{(i=0)} \times_{(V \times H)} (V \times_{C_i} H). \end{aligned}$$

Since colimits in ∞ -topoi are universal we have

$$\begin{aligned} \operatorname{colim}_{\mathcal{Q}} A & \simeq \operatorname{colim}_{\mathcal{Q}} A^{(i=0)} \times_{(U \times H)} (U \times_{C_i} H) \simeq U \times_{C_i} H, \\ \operatorname{colim}_{\mathcal{Q}} B & \simeq \operatorname{colim}_{\mathcal{Q}} B^{(i=0)} \times_{(V \times H)} (V \times_{C_i} H) \simeq V \times_{C_i} H, \end{aligned}$$

and so all that remains is to verify that $A_\delta \rightarrow B_\delta$ is indeed in T_{Θ_n} . This can be accomplished by explicitly computing A_δ and B_δ in terms of the functors $\sigma_!^{[\ell]}$ and invoking Lm. 11.5.

We may now complete the proof of (R.1) for complete Segal Θ_n -spaces.

Theorem 11.13. *The triple $(\Theta_n, T_{\Theta_n}, i)$ satisfies axiom (R.1), namely $i^*(S) \subset T_{\Theta_n}$.*

Proof. By Lm. 11.9, it is enough to show that the strongly saturated classes $T_{\Theta_n}^{(b)}$ contains the generating sets $\operatorname{Segal}_{\Theta_n}$ and $\operatorname{Comp}_{\Theta_n}$. By Lm. 11.10, it suffices to show that for any $[U \rightarrow V] \in \operatorname{Segal}_{\Theta_n} \cup \operatorname{Comp}_{\Theta_n}$, any nondegenerate morphism $[H \rightarrow C_i] \in \mathcal{J}_b$, and any nondegenerate morphism $V \rightarrow C_i$ of $\mathcal{P}(\Theta_n)$, we must show that

$$U' = U \times_{C_i} \nu H \rightarrow V \times_{C_i} \nu H = V'$$

is contained in T_{Θ_n} . Observe the following:

- If $[U \rightarrow V] \in \text{Segal}_{\Theta_n}$ is not in the image of $\sigma_!$, then $U \rightarrow V$ is contained in $T_{\Theta_n}^{(b)}$ by Lm. 11.11.
- If $[U \rightarrow V] \in \text{Comp}_{\Theta_n}$ is not in the image of $\sigma_!$, then $V = C_0$, and the only nondegenerate map $V \rightarrow C_i$ occurs when $i = 0$. In this case $U \rightarrow V$ is in $T_{\Theta_n}^{(b)}$ by [35, Pr. 6.1].

Thus we may restrict our attention to those generators $U \rightarrow V$ that lie in the image of $\sigma_!$. We proceed by induction. When $n = 1$, the set of generators in the image of $\sigma_!$ is empty.

Assume that

$$T_{\Theta_{n-1}} = T_{\Theta_{n-1}}^{(a)} = T_{\Theta_{n-1}}^{(b)} = T_{\Theta_{n-1}}^{(c)},$$

and let $U \rightarrow V$ be an element of $\text{Segal}_{\Theta_n} \cup \text{Comp}_{\Theta_n}$ that lies in the image of $\sigma_!$. Now note that if $C_i = C_0$, then $U' \rightarrow V'$ lies in T_{Θ_n} , again by [35, Pr. 6.1]. If $i \neq 0$, then by Lm. 11.4, the map $V \rightarrow C_i$ is also in the image of $\sigma_!$. In this case, if we have a factorization $H \rightarrow C_0 \rightarrow C_i$ (which, since $H \rightarrow C_i$ is nondegenerate, can only happen if $H = C_0$), then $U' \rightarrow V'$ is an equivalence (as both are empty). Hence it $U \rightarrow V$ lies in T_{Θ_n} .

This leaves the final case, where both $[U \rightarrow V]$ and $[V \rightarrow C_i]$ lie in the image of $\sigma_!$, and $[H \rightarrow C_i]$ is nondegenerate with $H = j([m]; o_1, \dots, o_m) \neq C_0$, for some $m \geq 1$, $o_i \in \Theta_{n-1}$. The nondegenerate map $H \rightarrow C_i = ([1]; C_{i-1})$ is given explicitly by the following data (see also the proof of Lm. 11.3): a map $i_k : [m] \rightarrow [1]$ for some $1 \leq k \leq m$ such that $i_k(i) = 0$ if $i < k$ and $i_k(i) = 1$ otherwise, together with a single (nondegenerate) map $o_k \rightarrow C_{i-1}$. In this case we may explicitly compute the pullback

$$U' = U \times_{C_i} H \rightarrow V \times_{C_i} H = V'$$

and deduce that it is contained in the class T_{Θ_n} .

As $[U \rightarrow V]$ is in the image of $\sigma_!^{[1]}$, it is of the form $([1]; U'') \rightarrow ([1]; V'')$ for some $[U'' \rightarrow V'']$ in $\text{Segal}_{\Theta_{n-1}}$. The pullback is then given explicitly as:

$$\begin{aligned} U' &= ([m]; o_1, \dots, o_k \times_{C_{i-1}} U'', o_{k+1}, \dots, o_m) \\ &\rightarrow ([m]; o_1, \dots, o_k \times_{C_{i-1}} V'', o_{k+1}, \dots, o_m) = V'. \end{aligned}$$

This map arises as the right-most vertical map in the following (oddly drawn) commuting square:

$$\begin{array}{ccc} j([k-1]; o_1, \dots, o_{k-1}) \cup^{j(\{k-1\})} (\{k-1, k\}; o_k \times_{C_{i-1}} U'') \cup^{j(\{k\})} j(\{k, \dots, m\}; o_{k+1}, \dots, o_m) & & \\ \downarrow & \searrow & \downarrow \\ & ([m]; o_1, \dots, o_k \times_{C_{i-1}} U'', o_{k+1}, \dots, o_m) & \\ & \searrow & \\ & ([m]; o_1, o_2, \dots, o_k \times_{C_{i-1}} V'', o_{k+1}, \dots, o_m) & \\ \downarrow & \swarrow & \\ j([k-1]; o_1, \dots, o_{k-1}) \cup^{j(\{k-1\})} (\{k-1, k\}; o_k \times_{C_{i-1}} V'') \cup^{j(\{k\})} j(\{k, \dots, m\}; o_{k+1}, \dots, o_m) & & \end{array}$$

The left-most vertical map is a pushout of identities and (by induction) a map in $\sigma_!(T_{\Theta_{n-1}})$. Thus by Lm. 11.3 it is contained in T_{Θ_n} . Both horizontal maps are contained in T_{Θ_n} by [35, Pr. 6.4], whence the right-most vertical map $[U' \rightarrow V']$ is also contained in T_{Θ_n} , as desired. \square

Lemma 11.14. *The triple $(\Theta_n, T_{\Theta_n}, i)$ satisfies axiom (R.2), namely $i_!(T_{\Theta_n}) \subseteq S$.*

Proof. As $i_!$ commutes with colimits, to show that $i_!(T_{\Theta_n}) \subseteq S$ it is sufficient to show this property for a subset which strongly generates the class T_{Θ_n} . The maps in Comp_{Θ_n} are clearly mapped into S . This leaves the maps Segal_{Θ_n} . We now write $S = S_n$ and induct on n . When $n = 0$, one has $\Upsilon_0 = \Theta_0 = \text{pt}$.

Assume that $i_!(T_{\Theta_{n-1}}) \subseteq S_{n-1}$. The suspension functor $\sigma_! : \mathcal{P}(\Upsilon_{n-1}) \rightarrow \mathcal{P}(\Upsilon_n)$ preserves colimits and sends the generators S_{n-1} into S_n . Hence the suspensions of maps in $i_!(T_{\Theta_{n-1}})$ are in S_n . Moreover, by construction the image under $i_!$ of the following map

$$j(\{0, 1\}; C_i) \cup^{j(\{1\})} j(\{1, 2\}; C_i) \rightarrow j(\{0, 1, 2\}; C_i, C_i)$$

is in S_n for all cells C_i . By induction, it follows that all the Segal generators are mapped into S_n except possibly the following

$$(11.14.1) \quad j(\{0, \dots, k\}; o_1, \dots, o_k) \cup^{j(\{k\})} j(\{k, \dots, m\}; o_{k+1}, \dots, o_m) \\ \rightarrow j(\{0, \dots, m\}; o_1, \dots, o_m)$$

where $o_i \in \Theta_{n-1}$. To show that $i_!$ maps the above morphism to a morphism in S_n , we observe that the above map may be rewritten as follows. The source may be written as

$$j(\{0, \dots, k\}; o_1, \dots, o_k) \times_{j(\{k-1, k\}; C_0)} \left[j(\{k-1, k\}; C_0) \cup^{j(\{k\})} j(\{k, k+1\}; C_0) \right] \\ \times_{j(\{k, k+1\}; C_0)} j(\{k, k+1, m\}; o_{k+1}, \dots, o_m)$$

while the target is

$$j(\{0, \dots, k\}; o_1, \dots, o_k) \times_{j(\{k-1, k\}; C_0)} j(\{k-1, k, k+1\}; C_0, C_0) \\ \times_{j(\{k, k+1\}; C_0)} j(\{k, k+1, m\}; o_{k+1}, \dots, o_m).$$

Schematically then, the map of (11.14.1) is of the form

$$A \times_{C_1} U \times_{C_1} B \rightarrow A \times_{C_1} V \times_{C_1} B$$

for $U \rightarrow V$ in S and $A, B \in \Upsilon_n$. By property (C.2) of $\text{Cat}_{(\infty, n)}$ (cf. also Pr. 7.4) it follows that (11.14.1) lies in S_n also. \square

Theorem 11.15. *The triple $(\Theta_n, T_{\Theta_n}, i)$ satisfies the axioms (R.1-4); The quasicategory $\text{CSS}(\Theta_n)$ of complete Segal Θ_n -spaces is a homotopy theory of (∞, n) -categories.*

Proof. Condition (R.4) is clear, and the functor $i : \Theta_n \rightarrow \Upsilon_n$ is a fully-faithful inclusion, hence (R.3) is automatically satisfied. Conditions (R.1) and (R.2) follow from Th. 11.13 and Lm. 11.14. \square

12. n -FOLD COMPLETE SEGAL SPACES ARE A HOMOTOPY THEORY OF (∞, n) -CATEGORIES

We give the iterative construction of the quasicategory $\text{CSS}(\Delta^{\times n})$ of n -fold complete Segal spaces, following [1] and [30, § 1], and we show that $\text{CSS}(\Delta^{\times n})$ is a homotopy theory of (∞, n) -categories.

Definition 12.1 ([1]). Let $\text{CSS}(\Delta^0)$ be the quasicategory \mathcal{S} of Kan simplicial sets. Suppose now that n is a positive integer; assume that both a presentable quasicategory $\text{CSS}(\Delta^{\times n-1})$ and a fully faithful functor

$$c_{n-1}: \text{CSS}(\Delta^0) \hookrightarrow \text{CSS}(\Delta^{\times n-1})$$

that preserves all small colimits have been constructed. Let us call a simplicial object $X: \mathbb{N}\Delta^{\text{op}} \rightarrow \text{CSS}(\Delta^{\times n-1})$ an *n-fold Segal space* if it satisfies the following pair of conditions.

- (B.1) The object X_0 lies in the essential image of c_{n-1} .
- (B.2) For any integers $0 < k < m$, the object X_m is exhibited as the limit of the diagram

$$X(\{0, 1, \dots, k\}) \rightarrow X(\{k\}) \leftarrow X(\{k, k+1, \dots, m\}).$$

Now for any n -fold Segal space X , one may apply the right adjoint to the functor c_{n-1} objectwise to X to obtain a simplicial space $\iota_1 X$. Let us call X an *n-fold complete Segal space* if it satisfies the following additional condition.

- (B.3) The Kan complex $(\iota_1 X)_0$ is exhibited as the limit of the composite functor

$$\Delta_{/NE}^{\text{op}} \rightarrow \Delta^{\text{op}} \xrightarrow{\iota_1 X} \text{CSS}_0,$$

where the category E is as in Ex. 2.2. Denote by $\text{CSS}(\Delta^{\times n})$ the full subcategory of $\text{Fun}(\mathbb{N}\Delta^{\text{op}}, \text{CSS}(\Delta^{\times n-1}))$ spanned by the n -fold complete Segal spaces.

In order to make sense of the inductive definition above, it is necessary to show that $\text{CSS}(\Delta^{\times n})$ is a presentable quasicategory, and to construct a fully faithful, colimit-preserving functor $c_n: \text{CSS}(\Delta^0) \hookrightarrow \text{CSS}(\Delta^{\times n})$. To prove presentability, we demonstrate that the quasicategory $\text{CSS}(\Delta^{\times n})$ is in fact an accessible localization of $\text{Fun}(\mathbb{N}\Delta^{\text{op}}, \text{CSS}(\Delta^{\times n-1}))$; then the desired functor c_n will be the composite

$$\text{CSS}(\Delta^0) \xrightarrow{c_{n-1}} \text{CSS}(\Delta^{\times n-1}) \xrightarrow{c} \text{Fun}(\Delta^{\text{op}}, \text{CSS}(\Delta^{\times n-1})) \xrightarrow{L} \text{CSS}(\Delta^{\times n}),$$

where c denotes the constant functor and L denotes the purported localization.

Lemma 12.2. *For any positive integer n , the quasicategory $\text{CSS}(\Delta^{\times n})$ is an accessible localization of $\text{Fun}(\mathbb{N}\Delta^{\text{op}}, \text{CSS}(\Delta^{\times n-1}))$.*

Proof. Denote by j any Yoneda embedding (the context will always be made clear). Let K denote the simplicial set as in 10.3, which we regard as a simplicial space that is discrete in each degree. This is a pushout along an inclusion, hence this is also a (homotopy) pushout in the quasicategory of simplicial spaces. Now let T be the strongly saturated class of morphisms of $\text{Fun}(\mathbb{N}\Delta^{\text{op}}, \text{CSS}(\Delta^{\times n-1}))$ generated by the three sets

$$\begin{aligned} &\{j([0], \mathbf{m}) \rightarrow j(\mathbf{0}) \mid \mathbf{m} \in \Delta^{\times n-1}\}, \\ &\{\text{Segal}_\Delta \boxtimes \mathbf{m} \mid \mathbf{m} \in \Delta^{\times n-1}\}, \\ &\{c_{n-1}(K) \rightarrow j(\mathbf{0})\}, \end{aligned}$$

One deduces immediately that a simplicial object of $\text{CSS}(\Delta^{\times n-1})$ is a Segal space if and only if it is local with respect to each of the first two sets of morphisms. To show that $\text{CSS}(\Delta^{\times n})$ coincides with the localization $T^{-1} \text{Fun}(\mathbb{N}\Delta^{\text{op}}, \text{CSS}(\Delta^{\times n-1}))$, it is enough to show that a 1-fold Segal space X is complete if and only if the natural map

$$X_0 \rightarrow \text{Map}(K, X)$$

is an equivalence. By the Yoneda lemma, our claim is just a restatement of [35, Pr. 10.1]. \square

Corollary 12.3. *For any nonnegative integer n , the quasicategory $\text{CSS}(\Delta^{\times n})$ is an accessible localization of $\mathcal{P}(\Delta^{\times n-1})$.*

Proof. If $n = 0$, there is nothing to prove. If n is positive, then let us suppose that we have written CSS_{n-1} as a localization $T_{\Delta^{\times n-1}}^{-1} \mathcal{P}(\Delta^{\times n-1})$ for some strongly saturated class $T_{\Delta^{\times n-1}}$ of small generation. Denote by

$$\boxtimes: \mathcal{P}(\Delta) \times \mathcal{P}(\Delta^{\times n-1}) \rightarrow \mathcal{P}(\Delta^{\times n})$$

the essentially unique functor that carries pairs of the form $(j[k], j(\mathbf{m}))$ to $j([k], \mathbf{m})$ and preserves colimits separately in each variable. Now let $T_{\Delta^{\times n}}$ be the strongly saturated class generated by the class T above along with the set

$$\{j[k] \boxtimes U \rightarrow j[k] \boxtimes V \mid [U \rightarrow V] \in T_{\Delta^{\times n-1}}\}.$$

Now $\text{CSS}(\Delta^{\times n})$ coincides with $T_{\Delta^{\times n}}^{-1} \mathcal{P}(\Delta^{\times n})$. \square

Remark 12.4. The class $T_{\Delta^{\times n}}$ is precisely the strongly saturated class generated by the union of $\text{Segal}_{\Delta^{\times n}}$, $\text{Glob}_{\Delta^{\times n}}$, and $\text{Comp}_{\Delta^{\times n}}$ as in Cor 10.4.

Write $d: \Delta^{\times n} \rightarrow \Upsilon_n$ for the composite of the functor $\delta_n: \Delta^{\times n} \rightarrow \Theta_n$ described in [7, Df. 3.8] followed by the fully faithful functor $i: \Theta_n \hookrightarrow \Upsilon_n$ consider in the previous section. We will now show that the triple $(\Delta^{\times n}, T_{\Delta^{\times n}}, d)$ satisfies conditions (R.1-4) of Th. 9.2, hence $\text{CSS}(\Delta^{\times n})$ is a homotopy theory of (∞, n) -categories. In contrast to the previous section, the functor d is not fully-faithful and hence condition (R.3) is not automatic. We thus begin with this condition.

Lemma 12.5. *The triple $(\Delta^{\times n}, T_{\Delta^{\times n}}, d)$ satisfies condition (R.3), that is for all objects $\mathbf{m} \in \Delta^{\times n}$, the canonical map $\mathbf{m} \rightarrow \delta_n^* \delta_n(\mathbf{m}) \simeq d^* d(\mathbf{m})$ is in $T_{\Delta^{\times n}}$.*

Proof. We will proceed by induction on n , the base case $n = 0$ being trivial. As the functor $j[m] \boxtimes (-)$ preserves colimits and sends the generators of $T_{\Delta^{\times n-1}}$ into $T_{\Delta^{\times n}}$, we have a containment $j[m] \boxtimes T_{\Delta^{\times n-1}} \subseteq T_{\Delta^{\times n}}$. Thus by induction the canonical map,

$$j[m] \boxtimes \mathbf{m} \rightarrow j[m] \boxtimes \delta_{n-1}^* \delta_{n-1}(\mathbf{m})$$

is in $T_{\Delta^{\times n}}$. In particular when $m = 0$, the map $j[0] \boxtimes \mathbf{m} \rightarrow j[0] \boxtimes \mathbf{0}$ is a composite of maps in $\text{Glob}_{\Delta^{\times n}}$, whence the map

$$j[0] \boxtimes \delta_{n-1}^* \delta_{n-1}(\mathbf{m}) \rightarrow j[0] \boxtimes \mathbf{0}$$

is in $T_{\Delta^{\times n}}$.

We will first prove the lemma for objects of the form $j[1] \boxtimes \mathbf{m} \in \Delta \times \Delta^{\times n-1} \cong \Delta^{\times n}$. One may readily check that the following is a pushout square of presheaves of sets:

$$\begin{array}{ccc} (j\{0\} \boxtimes \delta_{n-1}^* \delta_{n-1}(\mathbf{m})) \sqcup (j\{1\} \boxtimes \delta_{n-1}^* \delta_{n-1}(\mathbf{m})) & \hookrightarrow & j[1] \boxtimes \delta_{n-1}^* \delta_{n-1}(\mathbf{m}) \\ \downarrow & & \downarrow \\ (j\{0\} \boxtimes \mathbf{0}) \sqcup (j\{1\} \boxtimes \mathbf{0}) & \longrightarrow & \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m}) \end{array}$$

Moreover, as the topmost map is an inclusion of sets and pushouts in $\mathcal{P}(\Delta^{\times n})$ are computed object-wise, this is also a (homotopy) pushout square in $\mathcal{P}(\Delta^{\times n})$. As we just observed, the left-most map is in the strongly saturated $T_{\Delta^{\times n}}$, whence the right-most map is also in $T_{\Delta^{\times n}}$. It follows that the composite,

$$j[1] \boxtimes \mathbf{m} \rightarrow j[1] \boxtimes \delta_{n-1}^* \delta_{n-1}(\mathbf{m}) \rightarrow \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m})$$

is in $T_{\Delta^{\times n}}$.

To prove the general case, i.e. that the map $j[k] \boxtimes \mathbf{m} \rightarrow \delta_n^* \delta_n(j[k] \boxtimes \mathbf{m})$ is in $T_{\Delta^{\times n}}$, we induct on k . Assume the result holds when $k \leq m$. We will prove it for $k = m + 1$. First consider the following commutative square:

$$\begin{array}{ccc} j[m] \boxtimes \mathbf{m} \cup^{j[0] \boxtimes \mathbf{m}} j[1] \boxtimes \mathbf{m} & \xrightarrow{\sim} & j[m+1] \boxtimes \mathbf{m} \\ \downarrow \wr & & \downarrow \\ \delta_n^* \delta_n(j[m] \boxtimes \mathbf{m}) \cup^{j[0] \boxtimes \mathbf{0}} \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m}) & \rightarrow & \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m}) \end{array}$$

The indicated maps are in $T_{\Delta^{\times n}}$; the topmost map is a generator and the leftmost vertical map by induction. As $T_{\Delta^{\times n}}$ is saturated, the rightmost vertical map is in $T_{\Delta^{\times n}}$ if and only if the bottommost map is as well. Thus it suffices to prove that the natural map

$$\delta_n^* \delta_n(j[m] \boxtimes \mathbf{m}) \cup^{j[0] \boxtimes \mathbf{0}} \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m}) \rightarrow \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m})$$

is in $T_{\Delta^{\times n}}$.

The Yoneda embedding strongly generates any presheaf quasicategory and hence the object $\delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m})$ may canonically be written as a colimit of representable presheaves. Let $\mathcal{D} = (\Delta^{\times n} \downarrow \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m}))$ denote the overcategory consisting of pairs $(j[p] \boxtimes \mathbf{p}, \phi)$ where ϕ is a map

$$\phi : j[p] \boxtimes \mathbf{p} \rightarrow \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m}).$$

Let $B : \mathcal{D} \rightarrow \mathcal{P}(\Delta^{\times n})$ denote the functor which forgets the map ϕ . We have a canonical equivalence in $\mathcal{P}(\Delta^{\times n})$:

$$\operatorname{colim}_{\mathcal{D}} B \simeq \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m}).$$

By adjunction, specifying a map $\phi : j[p] \boxtimes \mathbf{p} \rightarrow \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m})$ is equivalent to a specifying a map $\phi' : \delta_n(j[p] \boxtimes \mathbf{p}) \rightarrow \delta_n(j[m+1] \boxtimes \mathbf{m})$, i.e. a map in $\mathcal{P}(\Theta_n)$:

$$\phi' : ([p]; \underbrace{\delta_{n-1}(\mathbf{p}), \dots, \delta_{n-1}(\mathbf{p})}_{p \text{ times}}) \rightarrow ([m+1]; \underbrace{\delta_{n-1}(\mathbf{m}), \dots, \delta_{n-1}(\mathbf{m})}_{m+1 \text{ times}}).$$

In particular every such map includes the data of a map $\bar{\phi} : [p] \rightarrow [m+1]$. To simplify notation we will denote the object $(j[p] \boxtimes \mathbf{p}, \phi)$ as B_ϕ or simply ϕ .

Let \mathcal{C} denote the full subcategory of \mathcal{D} consisting of the union of the following three types of objects:

- (a) those B_ϕ in which $\bar{\phi}$ factors as

$$\bar{\phi} : [p] \rightarrow \{0, \dots, m\} \subseteq [m+1],$$

- (b) those B_ϕ in which $\bar{\phi}$ factors as

$$\bar{\phi} : [p] \rightarrow \{m, m+1\} \subseteq [m+1], \quad \text{and}$$

- (c) those B_ϕ in which $(\bar{\phi})^{-1}(\{m\}) = \{r\} \subseteq [p]$ consists of a singleton for some $0 \leq r \leq p$.

For any object $D \in \mathcal{D}$, the under category $\mathcal{C}_{D/}$ actually has an initial object and is thus weakly contractible. Consequently (see, e.g., [29, Th. 4.1.3.1 and Pr. 4.1.1.8]), the induced morphism of (homotopy) colimits over these categories is an equivalence; in particular, it follows that the following canonical maps are equivalences in $\mathcal{P}(\Delta^{\times n})$:

$$\operatorname{colim}_{\mathcal{C}} B \simeq \operatorname{colim}_{\mathcal{D}} B \simeq \delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m}).$$

For each $\phi \in \mathcal{C}$, let A_ϕ denote the fiber product

$$A_\phi := B_\phi \times_{\delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m})} \left(\delta_n^* \delta_n(j[m] \boxtimes \mathbf{m}) \cup^{j[0] \boxtimes \mathbf{0}} \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m}) \right).$$

This gives rise to a new functor $A : \mathcal{C} \rightarrow \mathcal{P}(\Delta^{\times n})$, and as colimits in $\mathcal{P}(\Delta^{\times n})$ are universal, we have natural equivalences:

$$\begin{aligned} \operatorname{colim}_{\mathcal{C}} A &\simeq \left(\operatorname{colim}_{\mathcal{C}} B \right) \times_{\delta_n^* \delta_n(j[m+1] \boxtimes \mathbf{m})} \left(\delta_n^* \delta_n(j[m] \boxtimes \mathbf{m}) \cup^{j[0] \boxtimes \mathbf{0}} \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m}) \right) \\ &\simeq \delta_n^* \delta_n(j[m] \boxtimes \mathbf{m}) \cup^{j[0] \boxtimes \mathbf{0}} \delta_n^* \delta_n(j[1] \boxtimes \mathbf{m}). \end{aligned}$$

Thus the desired result follows if we can demonstrate that the natural map

$$\operatorname{colim}_{\mathcal{C}} A \rightarrow \operatorname{colim}_{\mathcal{C}} B$$

is in the class $T_{\Delta^{\times n}}$. This class, being saturated, is closed under colimits, and so it suffices to show that each of the maps

$$A_\phi \rightarrow B_\phi$$

is in $T_{\Delta^{\times n}}$. If $B_\phi \in \mathcal{C}$ is of type (a) or type (b), then $A_\phi \simeq B_\phi$ is an equivalence, hence in the desired class. If $B_\phi = (j[p] \boxtimes \mathbf{p}, \phi)$ is of type (c), so that $(\bar{\phi})^{-1}(\{m\}) = \{r\}$ for $0 \leq r \leq p$, then a direct calculation reveals:

$$A_\phi \simeq (j\{0, \dots, r\} \boxtimes \mathbf{p}) \cup^{(j\{r\} \boxtimes \mathbf{p})} (j\{r, r+1, \dots, p\} \boxtimes \mathbf{p}) \rightarrow j[p] \boxtimes \mathbf{p} \simeq B_\phi.$$

As this is one of the generators of $T_{\Delta^{\times n}}$, the result follows. \square

Theorem 12.6. *The quasicategory $\operatorname{CSS}(\Delta^{\times n})$ of n -fold complete Segal spaces is a homotopy theory of (∞, n) -categories.*

Proof. We will show that the triple $(\Delta^{\times n}, T_{\Delta^{\times n}}, d)$ satisfies conditions (R.1-4) of Th. 9.2. Condition (R.4) clearly holds. Condition (R.3) is the statement of Lm. 12.5.

For condition (R.2) we must show that $i_!(\delta_n)_!(T_{\Delta^{\times n}}) \subseteq S$. By Lm. 11.14 it is sufficient to show that $(\delta_n)_!(T_{\Delta^{\times n}}) \subseteq T_{\Theta_n}$, and as $(\delta_n)_!$ preserves colimits it is sufficient to check this on the generating classes $\operatorname{Segal}_{\Delta^{\times n}}$, $\operatorname{Glob}_{\Delta^{\times n}}$, and $\operatorname{Comp}_{\Delta^{\times n}}$. In each case this is clear: the set $\operatorname{Glob}_{\Delta^{\times n}}$ maps under $(\delta_n)_!$ to equivalences in $\mathcal{P}(\Theta_n)$, the set $\operatorname{Comp}_{\Delta^{\times n}}$ is constructed as the image of $\operatorname{Comp}_{\Delta^{\times n}}$ under $(\delta_n)_!$, and the image of $\operatorname{Segal}_{\Delta^{\times n}}$ under $(\delta_n)_!$ is a subset of $\operatorname{Segal}_{\Delta^{\Theta_n}}$.

For the final condition (R.1) we must show that $\delta_n^* i^*(S) \subseteq T_{\Delta^{\times n}}$. By Th. 11.13, it suffices to show that $\delta_n^*(T_{\Theta_n}) \subseteq T_{\Delta^{\times n}}$. As δ_n^* preserves colimits, it is sufficient to prove this for the generating class of T_{Θ_n} . As we previously mentioned, the set $\operatorname{Comp}_{\Delta^{\Theta_n}}$ consists of elements in the image of $(\delta_n)_!$. By Pr. 10.5 the remaining generators of T_{Θ_n} are retracts of maps in the image of $(\delta_n)_!$. Thus T_{Θ_n} is contained in the strongly saturated class generated from $(\delta_n)_!(T_{\Delta^{\times n}})$. Hence $\delta_n^*(T_{\Theta_n})$ is contained in the strongly saturated class generated by $\delta_n^*(\delta_n)_!(T_{\Delta^{\times n}})$. Using Lm. 12.5 one

readily deduces that the generators of $T_{\Delta^{\times n}}$ are mapped, via $\delta_n^*(\delta_n)_!$ back into $T_{\Delta^{\times n}}$. As the composite functor $\delta_n^*(\delta_n)_!$ preserves colimits, this implies that the saturated class generated by $\delta_n^*(\delta_n)_!(T_{\Delta^{\times n}})$ is contained in $T_{\Delta^{\times n}}$, whence $\delta_n^*(T_{\Theta_n}) \subseteq T_{\Delta^{\times n}}$. \square

Corollary 12.7. *The functor $\text{CSS}(\Theta_n) \rightarrow \text{CSS}(\Delta^{\times n})$ induced by δ_n is an equivalence of quasicategories.*

13. EPILOGUE: MODEL CATEGORIES OF (∞, n) -CATEGORIES

We conclude with a brief discussion of model categories of (∞, n) -categories, in which we describe some interactions between our results here and those of Bergner, Lurie, Rezk, and Simpson. We first note that a spate of further corollaries to our main results can be obtained by employing the following.

Construction 13.1. Suppose \mathcal{A} a category equipped with a subcategory $w\mathcal{A}$ that contains all the objects of \mathcal{A} (i.e., a *relative category* in the terminology of [4]). We call the morphisms of $w\mathcal{A}$ *weak equivalences*. In this situation, one may form the *hammock localization* $\mathbb{L}^H\mathcal{A}$ of Dwyer–Kan [17]; this is a simplicial category. One may apply to each mapping space a fibrant replacement R that preserves products (e.g., Ex^∞) to obtain a category enriched in *Kan complexes*, which we shall denote $\mathbb{L}_f^H\mathcal{A}$. We may now apply the simplicial nerve construction [29, 1.1.5.5] to obtain a quasicategory $\text{N}\mathbb{L}_f^H\mathcal{A}$, which we shall denote simply by $\text{N}^H\mathcal{A}$. We shall call $\text{N}^H\mathcal{A}$ the *quasicategory underlying* the relative category \mathcal{A} .

When \mathcal{A} is a simplicial model category, the simplicial localization $\mathbb{L}^H\mathcal{A}$ is equivalent [18] to the full sub-simplicial category \mathcal{A}° spanned by the cofibrant-fibrant objects. In this case, our $\text{N}^H\mathcal{A}$ is equivalent to $\text{N}\mathcal{A}^\circ$, as used by Lurie [29, A.2].

Remark 13.2. When \mathcal{A} and \mathcal{B} are model categories, and $F: \mathcal{A} \rightleftarrows \mathcal{B}: G$ is a Quillen equivalence between them, there is [18] an induced equivalence of hammock localizations $\mathbb{L}^H\mathcal{A} \simeq \mathbb{L}^H\mathcal{B}$, and thus of underlying quasicategories $\text{N}^H\mathcal{A} \simeq \text{N}^H\mathcal{B}$.

Example 13.3. The quasicategory underlying the relative category of n -relative categories [3] is a homotopy theory of (∞, n) -categories.

Definition 13.4. Let us call a model category \mathcal{A} a *model category of (∞, n) -categories* if its underlying quasicategory $\text{N}^H\mathcal{A}$ is a homotopy theory of (∞, n) -categories.

Example 13.5. By [27], the Joyal model category of simplicial sets is a model category of $(\infty, 1)$ -categories. More generally, all of the model categories listed in Fig. 1 are model categories of $(\infty, 1)$ -categories.

Example 13.6. By Th. 11.15 and Th. 12.6, both Rezk’s model category $\Theta_n \text{Sp}$ of complete Θ_n -spaces [35] and the model category of n -fold complete Segal spaces [1, 30] are model categories of (∞, n) -categories.

We may now use the construction above to find more examples of theories of (∞, n) -categories.

Example 13.7 ([30, Pr. 1.5.4]). Let \mathcal{M} be a left proper simplicial combinatorial model category which is an absolute distributor ([30, Df. 1.5.1]). Then the category $\text{Fun}(\Delta^{\text{op}}, \mathcal{M})$, of simplicial objects in \mathcal{M} , admits the \mathcal{M} -enriched complete Segal model structure $\text{CSS}_{\mathcal{M}}$, which is again left proper, simplicial, combinatorial, and

an absolute distributor. If \mathcal{M} is a model category of $(\infty, n-1)$ -categories, then $\text{CSS}_{\mathcal{M}}$ is a model category of (∞, n) -categories.

The condition of being an *absolute distributor* is needed in order to formulate the correct notion of *complete* \mathcal{M} -enriched Segal object. We refer the reader to [30] for details, but note that being an absolute distributor is a property of the underlying quasicategory of the given model category. In particular it is preserved under any Quillen equivalence.

Example 13.8. Suppose that \mathcal{M} is a model category satisfying the following list of conditions.

- (M.1) The class of weak equivalences of \mathcal{M} are closed under filtered colimits.
- (M.2) Every monomorphism of \mathcal{M} is a cofibration.
- (M.3) For any object Y of \mathcal{M} , the functor $X \mapsto X \times Y$ preserves colimits.
- (M.4) For any cofibrations $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$, the *pushout product*

$$f \square f': (X \times Y') \cup^{(X \times X')} (Y \times X') \rightarrow Y \times Y'$$

is a cofibration that is trivial if either f or f' is.

- (M.5) The quasicategory $\text{N}^{\text{H}}\mathcal{M}$ is a homotopy theory of $(\infty, n-1)$ -categories.

Work of Bergner [10] and Lurie [30], combined with 12.6 above, shows that each of the following is an example of a model category of (∞, n) -categories:

- the projective or (equivalently) the injective model category [30, 2.2.16, 2.3.1, 2.3.9] $\text{Seg}_{\mathcal{M}}$ of \mathcal{M} -enriched preSegal categories, and
- the model category [29, A.3.2] $\text{Cat}_{\mathcal{M}}$ of categories enriched in \mathcal{M} .

Moreover, following Simpson [38] the injective (AKA Reedy) model category of Segal $(n-1)$ -categories [23, 33, 38] satisfies conditions (M.1-4); indeed, the most difficult of these to verify is (M.4), which Simpson does in [38, Th. 19.3.2 (using Cor. 17.2.6)].

Thus, for example, the injective and projective model categories of $\Theta_n \text{Sp}$ -enriched Segal categories $\text{Seg}_{\Theta_n \text{Sp}}$ as well as the model category $\text{Cat}_{\Theta_n \text{Sp}}$ of categories enriched in $\Theta_n \text{Sp}$ are seen to be model categories of (∞, n) -categories. Indeed very recent work of Bergner and Rezk [11] discusses these model categories in detail and links them by an explicit chain of Quillen equivalences.

Additionally, we see that the injective model category of Segal n -categories is also a model category of (∞, n) -categories, as is the model category of categories enriched in Segal $(n-1)$ -categories.

A partial converse to 13.2 holds, which allows one to deduce Quillen equivalences between these various model categories.

Lemma 13.9 ([29, A.3.7.7]). *Two combinatorial model categories \mathcal{A} and \mathcal{B} are connected by a chain of Quillen equivalences if and only if $\text{N}^{\text{H}}\mathcal{A}$ and $\text{N}^{\text{H}}\mathcal{B}$ are equivalent quasicategories.*

From this it follows that if \mathcal{A} and \mathcal{B} are combinatorial model categories with the property that both $\text{N}^{\text{H}}\mathcal{A}$ and $\text{N}^{\text{H}}\mathcal{B}$ are homotopy theories of (∞, n) -categories, then \mathcal{A} and \mathcal{B} are connected by a chain of Quillen equivalences. This applies to all of the model categories of (∞, n) -categories mentioned above.

A zig-zag of Quillen equivalences can be a troublesome gadget to work with. It is usually far more informative to have a single direct and explicit Quillen equivalence between competing model categories of (∞, n) -categories. While our techniques do

not generally provide such a direct Quillen equivalence, we do offer the following recognition principle.

Proposition 13.10. *Let \mathcal{A} and \mathcal{B} be two model categories of (∞, n) -categories and let $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$ be a Quillen adjunction between them. Then (L, R) is a Quillen equivalence if and only if the left derived functor $N^H L : N^H \mathcal{A} \rightarrow N^H \mathcal{B}$ preserves the cells up to weak equivalence.*

Proof. A Quillen equivalence induces an equivalence $N^H L : N^H \mathcal{A} \rightarrow N^H \mathcal{B}$ of quasicategories. By Cor. 8.5 and Lm. 4.5 any such equivalence necessarily preserves the cells up to equivalence. Conversely, as the left-derived functor $N^H L : N^H \mathcal{A} \rightarrow N^H \mathcal{B}$ preserves (homotopy) colimits and $N^H \mathcal{A}$ and $N^H \mathcal{B}$ are generated under (homotopy) colimits by the cells (Cor. 8.4), it follows that $N^H L$ induces an equivalence of quasicategories. In particular it induces an equivalence of homotopy categories, and hence (L, R) is a Quillen equivalence. \square

In particular the above applies when the cells are fibrant-cofibrant objects of \mathcal{A} and \mathcal{B} which are preserved by L itself.

Example 13.11. The standard Quillen adjunction (cf. [30, Lm. 2.3.13]) from Segal n -categories to n -fold complete Segal spaces is a Quillen equivalence.

Example 13.12. The functor δ_n induces a Quillen equivalence between the model category of complete Segal Θ_n -spaces [35] and the model category of n -fold complete Segal spaces [30, 1.5.4]. (See also Bergner–Rezk [12]).

A category with a specified subcategory of weak equivalences is a *relative category*, and hence gives rise to a homotopy theory. Thus any theory of (∞, n) -categories arising this way may, in principle, be compared using our axioms. We therefore end with the following.

Conjecture 13.13. The quasicategory underlying Verity’s n -trivial weak complicial sets [40, 41] is a homotopy theory of (∞, n) -categories. The relative category consisting of Batanin’s ω -categories [5] such that every k -cell is an equivalence for $k > n$, together with the class of morphisms which are essentially k -surjective for all k is a homotopy theory of (∞, n) -categories.

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